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Robust synchronization problems for max-plus linear systems with polytopic uncertainties

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ABSTRACT

In this paper, the robust system synchronization problems for max-plus linear systems with polytopic uncertainties affecting all system matrices are addressed. Uncertainties in discrete event systems can arise from variations in timing constraints or processing delays. This work provides a methodology to tackle synchronization problems by leveraging a geometric approach in the algebraic max-plus framework. Solution methods aim to synthesize a control law to make the system replicate a reference model, forcing the system's output to equal the model's output. An illustrative example supports the proposed theoretical results.

1. Introduction

Max-plus algebra serves as a fundamental tool for modeling and analyzing synchronization constraints without concurrency in event-driven systems, enabling comprehensive studies of scheduling, resource allocation, and flow control across various fields [1,2]. This framework has proven highly effective in describing industrial and manufacturing processes [3,4], as well as traffic systems [5], robotic patrols [6–8], and blended teaching environments [9], where timing and concurrency constraints are critical.

Synchronization problems in conventional linear systems have been systematically addressed through Model Matching Problems (MMPs) using geometric control methods based on controlled and conditioned invariance [10–18]. MMPs represent a fundamental paradigm in control theory, providing a systematic strategy for enforcing prescribed output behaviors via the structural properties of dynamical systems. Within the geometric control framework, MMPs encompass a wide range of control objectives, including disturbance decoupling, output regulation, tracking, and synchronization, and admit exact solutions when suitable invariance conditions are satisfied [10,11,19,20]. This viewpoint has motivated extensive developments of MMPs across different system classes, including nonlinear, time-varying, switching, and uncertain systems [16,18,21], highlighting the effectiveness of geometric methods in deriving solvability conditions and constructive control laws. Similar structural approaches have been successfully adapted to max-plus algebra for scheduling and control applications [3,22–24]. Indeed, in the max-plus algebra framework, synchronization and scheduling objectives can be naturally formulated as MMPs, where geometric invariance concepts play a key role in characterizing structural solvability properties [3,25,26].

Existing work on MMPs in max-plus systems is limited to uncertainty-free cases [25–30]. However, real-world systems invariably experience variations in dynamic, input, and output matrices due to modeling approximations, structural changes, or external disturbances. This has driven, e.g., the development of interval max-plus approaches for reachability analysis, optimal state-feedback control, and set-membership estimation [31–33].

Since interval max-plus linear systems can be reformulated as max-plus linear systems with polytopic uncertainties, this latter formalization offers greater analytical tractability through structural methods grounded in the geometric framework [10,11], and it makes it possible to handle synchronization problems in which polytopic uncertainties affect all the system matrices.

Specifically, building on this model matching perspective, this paper addresses two novel control problems for max-plus linear systems with polytopic uncertainties, namely the Robust System Synchronization Problem (RSSP) and the Robust Feedback System Synchronization Problem (RFSSP). Both problems aim to enforce specified reference behaviors while handling parameter variations. The comprehensive

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extension of robust geometric techniques for both problems is then addressed.

Motivated by the above considerations, the proposed approach differs from existing methods in the literature by explicitly addressing robust synchronization problems in which polytopic uncertainties simultaneously affect the system, input, and output matrices. Several contributions on max-plus systems have focused on robustness properties under specific uncertainty structures and interval descriptions, also within optimal or performance-oriented control frameworks [31–34]. In contrast, the approach pursued in this paper is geometric in nature and aims to provide exact solutions in a structural sense by characterizing solvability via invariance-based conditions and deriving constructive synthesis procedures. Other works have instead analyzed uncertainty-free model matching formulations by means of geometric control techniques [25,26,30]. Despite these contributions, robust synchronization problems formulated as model matching problems within a unified geometric framework that account for polytopic uncertainties affecting all system matrices remain largely unexplored. In this context, the main contributions of this paper can be summarized as follows:

1. the formulation of the Robust System Synchronization Problem (RSSP) and the Robust Feedback System Synchronization Problem (RFSSP) for max-plus linear systems with polytopic uncertainties affecting all system matrices;
2. the extension of the geometric controlled invariance approach to the robust framework, leading to verifiable solvability conditions;
3. the derivation of constructive solvability conditions for both robust synchronization problems.

In this way, through a robust, controlled-invariance perspective adapted to the max-plus semiring, we derive explicit procedures for managing event synchronization under significant parameter fluctuations, effectively extending classical geometric insights to a more general, practically relevant class of max-plus systems.

A preliminary contribution in the same direction has been given in [35], where, however, the uncertainty was limited to affect only the dynamics matrix A of the system. This work is motivated by the fact that the presence of uncertainty in the input and output matrices, respectively, B and C, gives rise to additional non-trivial difficulties.

The paper is structured as follows. Section 2 introduces max-plus linear systems with polytopic uncertainties and establishes the analytical framework. Section 3 formulates the RSSP and RFSSP for this system class. Section 4 presents the structural geometric background, including robust controlled invariance and robust feedback controlled invariance concepts. Section 5 delivers the main solvability conditions with supporting proofs. Section 6 illustrates the RFSSP approach through a numerical example, and Section 7 summarizes the key contributions.

Notations. The max-plus algebra, or *tropical* algebra, denoted as $\mathbb{R}_{\max}$, is defined over the extended set of real numbers $\mathbb{R} \cup \{-\infty\}$, equipped with two binary operations. The *tropical sum* is represented by $\oplus$, with $a \oplus b = \max\{a, b\}$ where $a, b \in \mathbb{R}_{\max}$. The *tropical product*, denoted by $\otimes$, follows the rule $a \otimes b = a + b$ when $a \in \mathbb{R}$, $b \in \mathbb{R}$. Moreover, multiplication with $-\infty$ results in $(-\infty)\otimes a = a\otimes(-\infty) = -\infty$, with $a \in \mathbb{R}_{\max}$. For the tropical sum $\oplus$, the neutral element is $\epsilon = -\infty$, while for the tropical product it is $e = 0$. When there is no risk of ambiguity, the symbol $\otimes$ may be omitted. Since the additive inverse does not exist and the product $\otimes$ distributes over the sum $\oplus$, the algebraic structure defined by $\{\mathbb{R} \cup \{-\infty\}, \otimes, \oplus\}$ is a semiring.

The n-dimensional free semimodule over $\mathbb{R}_{\max}$, with $n \in \mathbb{N}$, is denoted by $\mathbb{R}^n_{\max}$. The zero vector of this space, defined by all entries equal to ϵ, is also written as ϵ when the context is clear.

For a given subsemimodule $\mathcal{V} \subseteq \mathbb{R}^n_{\max}$, the notation $\mathcal{V}^{[N]}$, with $N \in \mathbb{N}$, represents the subsemimodule of $\mathbb{R}^{n\times N}_{\max}$ formed by vectors of the form $[v_1^\top, \dots, v_N^\top]^\top$, where each $v_i \in \mathcal{V}$, with $i \in \mathcal{I} = \{1, \dots, N\}$. If a matrix V is composed of columns forming a basis of $\mathcal{V}$, then the columns of the corresponding block-diagonal matrix $V^{[N]} = \operatorname{diag}\{\underbrace{V, \dots, V}_{N \text{ times}}\}$ are a basis for $\mathcal{V}^{[N]}$.

Considering the set of matrices $\{M_i \in \mathbb{R}^{r\times s}_{\max}$, with $i \in \mathcal{I}\}$, the notation $\operatorname{co}_{i\in\mathcal{I}}(M_i)$ represents the $(Nr)\times s$ matrix obtained by vertically concatenating the matrices M_i for all $i \in \mathcal{I}$, i.e.,

$$\operatorname{co}_{i\in\mathcal{I}}(M_i) = \begin{bmatrix} M_1 \\ \vdots \\ M_N \end{bmatrix}.$$

Furthermore, the matrix $\operatorname{co}_{i\in\mathcal{I}}(M_i)$ defines the following linear mapping, denoted by the same symbol:

$$\operatorname{co}_{i\in\mathcal{I}}(M_i) : \mathbb{R}^s_{\max} \to \mathbb{R}^{Nr}_{\max}, \quad w \mapsto \begin{bmatrix} M_1 w \\ \vdots \\ M_N w \end{bmatrix}.$$

2. Max-plus linear systems with polytopic uncertainties

In order to define a max-plus linear system subject to polytopic uncertainties, it is first essential to present the standard formulation of a max-plus linear system.

2.1. Max-plus linear systems

The evolution of a max-plus linear system, representing a discrete-event dynamical system, is described as follows:

$$\Sigma \equiv \left\{ \begin{array}{rcl} x(k+1) & = & Ax(k) \oplus Bu(k+1) \\ y(k) & = & Cx(k). \end{array} \right. \tag{1}$$

In this setting, $k \in \mathbb{N}$ is the index of events. At the same time, $x(k)$, $u(k)$, and $y(k)$ are sequences taking values in $\mathcal{X} = \mathbb{R}^n_{\max}$, $\mathcal{U} = \mathbb{R}^m_{\max}$, and $\mathcal{Y} = \mathbb{R}^p_{\max}$, representing the internal, input, and output event daters, respectively.

Daters describe the timing of the associated events. In particular, each component of the internal, input, and output event daters corresponds to a specific type of internal, input, or output event, respectively. As for the interpretation of $x_i(k)$ in $x(k) \in \mathbb{R}^n_{\max}$, written as $[x_1(k), \dots, x_n(k)]^\top$, it denotes the time at which the kth internal event of type i occurs. The same applies to $u(k)$ and $y(k)$ for input and output events. The matrices $A \in \mathbb{R}^{n\times n}_{\max}$, $B \in \mathbb{R}^{n\times m}_{\max}$, and $C \in \mathbb{R}^{p\times n}_{\max}$ define the system coefficients, specifying the dependencies between successive occurrences of the various events.

To ensure physical consistency, event daters must be non-decreasing functions. A sufficient condition for guaranteeing this property for any initial condition is to impose that the system matrix A satisfies $A \geq I_n$ component-wise, with I_n denoting the max-plus identity of size $n \times n$, which has e on the diagonal and ϵ elsewhere. In the paper, it is assumed that the condition $A \geq I_n$ is satisfied.

Remark 1. The assumption $A \geq I_n$ reflects the nonanticipative nature of max-plus linear systems, where the $(k+1)$-th occurrence of an event cannot take place before its kth occurrence. This condition is standard in synchronization models formulated in the max-plus algebra (see, e.g., [25]), and it is inherent to the system description.

2.2. Polytopic representation of uncertainties

The systems we will take into account are subject to polytopic uncertainties, and they can be described by

$$\Sigma \equiv \left\{ \begin{array}{rcl} x(k+1) & = & A(\mu)x(k) \oplus B(\mu)u(k+1) \\ y(k) & = & C(\nu)x(k) \end{array} \right. \tag{2}$$

with k, $x(.)$, $u(.)$, and $y(.)$ as in (1), and with the coefficient matrices defined as convex combinations over $\mathbb{R}_{max}$ of the form

$$A(\mu) = \bigoplus_{i\in\mathcal{I}} \mu_i A_i$$
$$B(\mu) = \bigoplus_{i\in\mathcal{I}} \mu_i B_i$$
$$C(\nu) = \bigoplus_{i\in\mathcal{I}} \nu_i C_i$$

in which

- $\mathcal{I} = \{1, \dots, N\}$ is a set of indices;
- $\mu = [\mu_1, \dots, \mu_N]^\top \in \mathbb{R}^N_{\max}$ is a vector of independent parameters taking value in $\mathbb{R}_{\max}$ such that

$$\bigoplus_{i \in \mathcal{I}} \mu_i = e;$$

- $\nu = [\nu_1, \dots, \nu_N]^\top \in \mathbb{R}^N_{\max}$ is a vector of independent parameters taking value in $\mathbb{R}_{\max}$ such that

$$\bigoplus_{i \in \mathcal{I}} \nu_i = e;$$

- $\{A_1, \dots, A_N\} = \{A_i\}_{i \in \mathcal{I}}$ is a family of matrices of size $n \times n$, that is $A_i \in \mathbb{R}^{n \times n}_{\max}$, with $A_i \geq I_n$ verified for every $i \in \mathcal{I}$
- $\{B_1, \dots, B_N\} = \{B_i\}_{i \in \mathcal{I}}$ is a family of $n \times m$ matrices with entries in $\mathbb{R}_{\max}$, i.e. $B_i \in \mathbb{R}^{n \times m}_{\max}$
- $\{C_1, \dots, C_N\} = \{C_i\}_{i \in \mathcal{I}}$ represents a family of $p \times n$ matrices with entries in $\mathbb{R}_{\max}$, that is $C_i \in \mathbb{R}^{p \times n}_{\max}$.

Throughout the paper, the condition $A_i \geq I_n$ is assumed to be satisfied for every $i \in \mathcal{I}$. Consequently, also $A(\mu) \geq I_n$ is guaranteed for all $\mu = [\mu_1, \dots, \mu_N]$ with $\bigoplus_{i \in \mathcal{I}} \mu_i = e$.

Note that considering a single set of indices for all the above combinations does not affect the generality of the representation.

Remark 2. In modeling real dynamical systems, the polytopic form introduced above can be used to model uncertainty on the coefficient matrices when only a maximum and a minimum value for each entry of them is known. To see this, let us consider the family of systems of the form (1) in which the dynamic matrix A belongs to the family P_A defined by

$$P_A = \Big\{ A \in \mathbb{R}^{n \times n}_{\max}, \text{ such that } a^{min}_{hk} \leq a_{hk} \leq a^{max}_{hk}, \text{ with } a^{min}_{hk}, a^{max}_{hk} \in \mathbb{R}_{\max}, \text{ for } h = 1, \dots, n; k = 1, \dots, n \Big\},$$

and analogously B and C belong to P_B and P_C, respectively, with

$$P_B = \Big\{ B \in \mathbb{R}^{n \times m}_{\max}, \text{ such that } b^{min}_{hk} \leq b_{hk} \leq b^{max}_{hk}, \text{ with } b^{min}_{hk}, b^{max}_{hk} \in \mathbb{R}_{\max}, \text{ for } h = 1, \dots, n; k = 1, \dots, m \Big\},$$

and

$$P_C = \Big\{ C \in \mathbb{R}^{p \times n}_{\max}, \text{ such that } c^{min}_{hk} \leq c_{hk} \leq c^{max}_{hk}, \text{ with } c^{min}_{hk}, c^{max}_{hk} \in \mathbb{R}_{\max}, \text{ for } h = 1, \dots, p; k = 1, \dots, n \Big\}.$$

Let us denote:

- A_0 the $n \times n$ matrix belonging to P_A, with entries $a_{hk} = a^{min}_{hk}$ for all $h = 1, \dots, n$, and all $k = 1, \dots, n$;
- A_{hk} the $n \times n$ matrix belonging to P_A with $a_{hk} = a^{\max}_{hk}$ and all other entries equal to their minimum value;
- B_0 the $n \times m$ matrix belonging to P_B, with entries $b_{hk} = b^{min}_{hk}$ for all $h = 1, \dots, n$, and all $k = 1, \dots, m$;
- B_{hk} the $n \times m$ matrix belonging to P_B with $b_{hk} = b^{\max}_{hk}$ and all other entries equal to their minimum value;
- C_0 the $p \times n$ matrix belonging to P_C, with entries $c_{hk} = c^{min}_{hk}$ for all $h = 1, \dots, p$, and all $k = 1, \dots, n$;
- C_{hk} the $p \times n$ matrix belonging to P_C with $c_{hk} = c^{\max}_{hk}$ and all other entries equal to their minimum value.

Then, for any system $\bar{\Sigma}$ in the considered family, the coefficient matrices, denoted by $\bar{A}$, $\bar{B}$ and $\bar{C}$, whose entries are respectively $\bar{a}_{i,j}$, $\bar{b}_{i,j}$ and $\bar{c}_{i,j}$, can be represented as convex combinations over $\mathbb{R}_{max}$ of the form

$$\bar{\Sigma} \equiv \begin{cases} x(k+1) &= A(\mu')x(k) \oplus B(\mu'')u(k+1) \\ y(k) &= C(\nu)x(k) \end{cases}$$

where matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ are as follows:

$$\begin{aligned} \bar{A} &= eA_0 \oplus \left(\bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,n}} (\bar{a}_{hk} - \bar{a}^{max}_{hk}) A_{hk} \right) \\ &= eA_0 \oplus \bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,n}} \mu'_{hk} A_{hk} \text{ with } \mu'_{hk} = (\bar{a}_{hk} - \bar{a}^{max}_{hk}) \end{aligned}$$

$$\begin{aligned} \bar{B} &= eB_0 \oplus \left(\bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,m}} (\bar{b}_{hk} - \bar{b}^{max}_{hk}) B_{hk} \right) \\ &= eB_0 \oplus \bigoplus_{\substack{h=1,\dots,n; \\ k=1,\dots,m}} \mu''_{hk} B_{hk} \text{ with } \mu''_{hk} = (\bar{b}_{hk} - \bar{b}^{max}_{hk}) \end{aligned}$$

$$\begin{aligned} \bar{C} &= eC_0 \oplus \left(\bigoplus_{\substack{h=1,\dots,p \\ k=1,\dots,n}} (\bar{c}_{hk} - \bar{c}^{max}_{hk}) C_{hk} \right) \\ &= eC_0 \oplus \bigoplus_{\substack{h=1,\dots,p \\ k=1,\dots,n}} \nu_{hk} C_{hk} \text{ with } \nu_{hk} = (\bar{c}_{hk} - \bar{c}^{max}_{hk}) \end{aligned}$$

Note that, in each of the above combinations, the sum of the indices equals e. Now, adding irrelevant terms to the above convex combinations, we can write

$$\bar{A} = eA_0 \oplus \bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,n}} \mu'_{hk} A_{hk} \oplus \bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,m}} \mu''_{hk} A_0$$

and

$$\bar{B} = eB_0 \oplus \bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,m}} \mu'_{hk} B_0 \oplus \bigoplus_{\substack{h=1,\dots,n \\ k=1,\dots,m}} \mu''_{hk} B_{hk}.$$

Renaming the matrices and the coefficients which appear in the rightmost member of the above equations and unifying the sets of indices, we can represent $\bar{\Sigma}$ in the form (2).

3. Problem formulations

A possible strategy for controlling a max-plus linear system affected by polytopic uncertainties is to compel its output to follow the output of a predefined model. This leads to the definition of the Robust System Synchronization Problem (RSSP) and its feedback counterpart, the Robust Feedback System Synchronization Problem (RFSSP), for this class of systems. This approach represents a generalization of the Model Matching Problem found in literature to max-plus linear systems [25] and to switching max-plus systems [26]. In this case, the problem is expanded here to account for polytopic uncertainties affecting the system, which were not previously considered in the two works cited.

Problem 1 (*Robust System Synchronization Problem*)**.** Let us consider a max-plus linear system Σ_P expressed as in (2) affected by polytopic uncertainties, referred to as the *plant* and described, with an obvious use of symbols, by

$$\Sigma_P \equiv \begin{cases} x_P(k+1) = A_P(\mu)x_P(k) \oplus B_P(\mu)u_P(k+1) \\ y_P(k) = C_P(\nu)x_P(k) \end{cases} \tag{3}$$

and a max-plus linear system Σ_M expressed as in (1), referred to as the *model*, and described by

$$\Sigma_M \equiv \begin{cases} x_M(k+1) = A_M x_M(k) \oplus B_M u_M(k+1) \\ y_M(k) = C_M x_M(k) \end{cases} \tag{4}$$

with $\dim y_P = \dim y_M$, the Robust System Synchronization Problem (RSSP) involves determining, for each admissible non-decreasing model input $\{u_M(k+1)\}_{k\in\mathbb{N}}$, a corresponding non-decreasing plant input $\{u_P(k+1)\}_{k\in\mathbb{N}}$ which, starting from initial conditions $x_P(0) = \epsilon$ and $x_M(0) = \epsilon$, allows to satisfy, for each $\mu = [\mu_1, \dots, \mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \mu_i = e$ and for all $\nu = [\nu_1, \dots, \nu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \nu_i = e$, the equality of the plant output $y_P(.)$ with the model output $y_M(.)$, that is

$$y_P(k+1) = y_M(k+1)$$

for every $k \in \mathbb{N}$.

This approach is suited to scenarios where the plant behavior, subject to uncertainties, needs to be controlled, ensuring it follows a predefined schedule through an appropriate control input. The model, in contrast, is assumed to be free of uncertainty, as its output represents the intended system behavior. However, this simplification does not affect the generality of the description if the model was also affected by polytopic uncertainties.

The feedback version of the RSSP is formulated as follows.

Problem 2 (*Robust Feedback System Synchronization Problem*). Let us consider a plant (3) together with a model (4), as above. The Robust Feedback System Synchronization Problem (RFSSP) involves determining a matrix $F \in \mathbb{R}_{\max}^{m_P \times (n_P+n_M)}$ and a matrix $G \in \mathbb{R}_{\max}^{m_P \times m_M}$ ensuring that

$$u_P(k+1) = F \begin{pmatrix} x_P(k) \\ x_M(k) \end{pmatrix} \oplus G\, u_M(k+1) \oplus u_P(k) \text{ for } k \geq 0. \tag{5}$$

The feedback control input expressed in (5), with $u_P(0) = \epsilon$, is a solution of the corresponding RSSP.

Since the elements of F and G are not constrained to be nonnegative, the resulting solution to the RFSSP may involve an anticipative feedback. This implies that implementing the control law may require prior knowledge of a portion of the input sequence $\{u_M(.)\}$ (see Remark 2 in [25]).

In summary, while the RSSP addresses synchronization through suitably designed open-loop input sequences, the RFSSP enforces synchronization via a feedback control law.

To facilitate the analysis of these problems, we reformulate them equivalently. To do this, starting from Σ_P as in (3) to express the plant and from Σ_M as in (4) for the model, we define the joint internal event dater as:

$$x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathcal{X}_E = \mathbb{R}_{\max}^{(n_P+n_M)}.$$

The joint system dynamics governing $x_E(.)$ can be expressed through

$$x_E(k+1) = A_E(\mu) x_E(k) \oplus B_1(\mu) u_P(k+1) \oplus B_2 u_M(k+1), \tag{6}$$

where the coefficient matrices are defined as

$$A_E(\mu) = \begin{pmatrix} A_P(\mu) & \epsilon \\ \epsilon & A_M \end{pmatrix}, \quad B_1(\mu) = \begin{pmatrix} B_P(\mu) \\ \epsilon \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \epsilon \\ B_M \end{pmatrix}, \quad \text{and} \quad x_E(0) = \epsilon.$$

Then, Problem 1 can be reformulated as that of finding, for any given non-decreasing input $\{u_M(k+1)\}_{k\in\mathbb{N}}$ for the model, a corresponding non-decreasing input $\{u_P(k+1)\}_{k\in\mathbb{N}}$ for the plant, which lets the trajectory $x_E(k)$ remains confined within the *output equalizer* subsemimodule $\mathcal{K} \subseteq \mathcal{X}_E = \mathbb{R}_{\max}^{(n_P+n_M)}$, starting from the initial conditions $x_E(0) = \epsilon$, for all $\mu = [\mu_1, \dots, \mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \mu_i = e$. This output equalizer subsemimodule $\mathcal{K}$ is defined as:

$$\mathcal{K} = \left\{ x_E = \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathbb{R}_{\max}^{(n_P+n_M)} \;\middle|\; C_P(\nu) x_P = C_M x_M \right\} \tag{7}$$

for all $k \in \mathbb{N}$.

Similarly, Problem 2 can be expressed with an additional requirement that $\{u_P(k+1)\}_{k\in\mathbb{N}}$ follows the control law (5) with some appropriately chosen matrices F and G.

4. Structural geometric background

Structural and geometric methods provide effective notions for addressing control problems whose solutions can be obtained by constraining the state evolution in specific subsets of the state space. For max-plus systems, this approach has been investigated by many authors [23,25,26,30,36].

Before introducing the formal definitions, we recall that invariance-based geometric methods aim to constrain the system's evolution to suitable subsets of the state space, enforcing prescribed output behaviors through the system's structural properties. Here, in order to tackle the RSSP and the RFSSP stated in the previous section, we introduce some structural and geometric background notions that apply to the case of max-plus linear systems with polytopic uncertainties expressed as in (2).

Definition 1. Let us consider a max-plus linear system with polytopic uncertainties Σ expressed as in (2). Then, a subsemimodule $\mathcal{V} \subseteq \mathcal{X}$ is referred to as a *robust invariant subsemimodule* for Σ, if and only if

$$A(\mu)\mathcal{V} = \left(\bigoplus_{i\in\mathcal{I}} \mu_i A_i \right) \mathcal{V} \subseteq \mathcal{V}$$

for all $\mu = [\mu_1, \dots, \mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \mu_i = e$.

Proposition 1. *A subsemimodule $\mathcal{V} \subseteq \mathcal{X}$ is a robust invariant subsemimodule for a max-plus linear system with polytopic uncertainties Σ expressed as in (2) if and only if*

$$A_i \mathcal{V} \subseteq \mathcal{V}$$

for each $i \in \mathcal{I} = \{1, \dots, N\}$.

Definition 2. Let us consider a max-plus linear system with polytopic uncertainties Σ expressed as in (2). Then, a subsemimodule $\mathcal{V} \subseteq \mathcal{X}$ is referred to as a *robust controlled invariant subsemimodule*, or a *robust $(A(\mu), B(\mu))$-invariant subsemimodule*, for Σ if and only if for all $x \in \mathcal{V}$ there exists $u \in \mathbb{R}_{\max}^m$ satisfying

$$A(\mu)x \oplus B(\mu)u = \left(\bigoplus_{i\in\mathcal{I}} \mu_i A_i \right) x \oplus \left(\bigoplus_{i\in\mathcal{I}} \mu_i B_i \right) u \subseteq \mathcal{V}$$

for all $\mu = [\mu_1, \dots, \mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \mu_i = e$.

Proposition 2. *A subsemimodule $\mathcal{V} \subseteq \mathcal{X}$ is a robust controlled invariant subsemimodule for a max-plus linear system with polytopic uncertainties Σ expressed as in (2) if and only if, for all $x \in \mathcal{V}$, there exists $u \in \mathbb{R}_{\max}^m$, satisfying*

$$A_i x \oplus B_i u \in \mathcal{V}$$

for each $i \in \mathcal{I} = \{1, \dots, N\}$, or, equivalently, satisfying

$$co_{i\in\mathcal{I}} A_i x \oplus co_{i\in\mathcal{I}} B_i u \in \mathcal{V}.$$

Proposition 3. *Let us consider a max-plus linear system with polytopic uncertainties Σ expressed as in (2) and a subsemimodule $\mathcal{K} \subseteq \mathcal{X}$. Then, the set of all robust controlled invariant subsemimodules for Σ included in $\mathcal{K}$ has a maximum element, denoted by $\mathcal{V}^\star(\mathcal{K})$.*

The maximum element $\mathcal{V}^\star(\mathcal{K})$ is computed as $\mathcal{V}_\infty$, resulting from the convergence of the recursively defined sequence of semimodules $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$, constructed as follows:

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{K}, \\ \mathcal{V}_k &= \mathcal{V}_{k-1} \cap \left(\mathrm{co}_{i\in\mathcal{I}}(A_i)\right)^{-1} \left(\mathcal{V}_{k-1}^{[N]} \ominus \mathrm{Im}\,\mathrm{co}_{i\in\mathcal{I}}(B_i) \right) \end{aligned} \tag{8}$$

where the notation $\left(\mathrm{co}_{i\in\mathcal{I}}(A_i)\right)^{-1}(\mathcal{Y})$ represents the inverse image of $\mathcal{Y}$ under the linear map defined by $\mathrm{co}_{i\in\mathcal{I}}(A_i)$ and the subsemimodule $\mathcal{V}^{[N]} \ominus \mathrm{Im}\,\mathrm{co}_{i\in\mathcal{I}}(B_i)$ is defined by

$$\mathcal{V}^{[N]} \ominus \mathrm{Im}\,\mathrm{co}_{i\in\mathcal{I}}(B_i) = \Big\{ x \in \mathbb{R}_{\max}^{n\times N} \text{ for which } \exists u \in \mathbb{R}_{\max}^m \text{ s. t. } x \oplus (\mathrm{co}_{i\in\mathcal{I}}(B_i))u \in \mathcal{V}^{[N]} \Big\}.$$

The convergence of the sequence $\{\mathcal{V}_k$, with $k \in \mathbb{N}\}$ is not necessarily guaranteed within a finite number of iterations. Consequently, Eq. (8) does not define an algorithm for computing $\mathcal{V}^\star(\mathcal{K})$ in general. However, if there exists some index $\bar{k} \in \mathbb{N}$ such that $\mathcal{V}_{\bar{k}+1} = \mathcal{V}_{\bar{k}}$, then the sequence stabilizes, yielding $\mathcal{V}_\infty = \mathcal{V}_{\bar{k}} = \mathcal{V}^\star(\mathcal{K})$. This behavior agrees with that of the analogous sequence for systems over a ring [37]. However, it contrasts with that of the analogous sequence of subspaces, which reaches the maximum controlled invariant subspace within a given one, in a finite number of iterations in the context of classical linear systems over a field [10,11].

Remark 3. Considering $\mathcal{Z}$ and $\mathcal{Y}$ as two finitely generated semimodules and A as a matrix of suitable dimensions, the semimodules resulting from operations such as $\mathcal{Y}\cap\mathcal{Z}$, $A^{-1}(\mathcal{Y})$, and $\mathcal{Y}\ominus\mathcal{Z}$ remain finitely generated, as stated in Corollary 86 of [38]. Consequently, if $\mathcal{K}$ is finitely generated, each semimodule $\mathcal{V}_k$ in $\mathcal{V}_{k\,k\in\mathbb{N}}$ inherits this property. Moreover, as highlighted in Remark 1 of [23], the generators of such semimodules can be determined through the resolution of by solving systems of linear equations in the max-plus algebra. These computations can be efficiently performed using the Max-plus Toolbox in the `ScicosLab` software [39].

Definition 3. Let us consider a max-plus linear system with polytopic uncertainties Σ expressed as in (2). Then, a subsemimodule $\mathcal{V}\subseteq\mathcal{X}$ is referred to as a *robust feedback controlled invariant subsemimodule*, or a *robust* $(A(\mu),B(\mu))$*-invariant subsemimodule* of feedback type, for Σ if there exists a matrix $F\in\mathbb{R}_{\max}^{m\times n}$ satisfying

$$(A(\mu)\oplus B(\mu)F)\,\mathcal{V}=\left(\left(\bigoplus_{i\in\mathcal{I}}\mu_iA_i\right)\oplus\left(\bigoplus_{i\in\mathcal{I}}\mu_iB_i\right)F\right)\mathcal{V}\subseteq\mathcal{V},$$

for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$.

Proposition 4. *A subsemimodule $\mathcal{V}\subseteq\mathcal{X}$ is a robust feedback controlled invariant subsemimodule for a max-plus linear system with polytopic uncertainties Σ expressed as in (2) if there exists a matrix $F\in\mathbb{R}_{\max}^{m\times n}$ satisfying*

$$\left(A_i\oplus B_iF\right)\mathcal{V}\subseteq\mathcal{V},$$

for each $i\in\mathcal{I}=\{1,\dots,N\}$, or, equivalently, satisfying

$$(co_{i\in I}A_i\oplus co_{i\in I}B_i)\mathcal{V}\subseteq V^{[N]}.$$

For systems defined over a field, controlled invariance and its feedback counterpart are known to be equivalent [10,11]. However, this equivalence does not hold when dealing with systems defined over a ring [40] or a semiring [23], as is the case with max-plus systems. Nevertheless, it remains true that robust feedback controlled invariance necessarily implies robust controlled invariance.

5. Problem solutions

The structural/geometric notions previously introduced are now used to provide a sufficient solvability condition to the RSSP and the RFSSP. For the RSSP, we have the following theorem.

Theorem 1. *Let us consider a plant Σ_P expressed as in (3), a model Σ_M expressed as in (4), and the joint dynamics defined by (6). Then, the related RSSP is solvable if for any $u_M\in\mathbb{R}_{\max}^{m_M}$ there exists $u_P\in\mathbb{R}_{\max}^{m_P}$ satisfying*

$$B_1(\mu)u_P\oplus B_2u_M\in\mathcal{V}^*(\mathcal{K}),$$

for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$, in which $\mathcal{V}^*(\mathcal{K})\subseteq\mathcal{X}_E$ is the maximum robust $(A_E(\mu),B_1(\mu))$-invariant semimodule included within the output equalizer semimodule $\mathcal{K}\subseteq\mathcal{X}_E$ expressed as in (7).

Proof. Let us start from the robust $(A_E(\mu),B_1(\mu))$-invariance of $\mathcal{V}^*(\mathcal{K})$. From this notion, we have that for any $x_E\in\mathcal{V}^*(\mathcal{K})$, there exists $u_P\in\mathbb{R}_{\max}^{m_P}$ such that

$$A_E(\mu)x_E\oplus B_1(\mu)u_P\subseteq\mathcal{V}^*(\mathcal{K}),$$

for all possible $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. Similarly, it is assumed that, for any given $u_M\in\mathbb{R}_{\max}^{m_M}$, there exists u_P such that

$$B_1(\mu)u_P\oplus B_2u_M\subseteq\mathcal{V}^*(\mathcal{K}).$$

Now, starting from any non-decreasing input $\{u_M(k+1)\}_{k\in\mathbb{N}}$, the sequence $\{u_P(k+1)\}_{k\in\mathbb{N}}$ is constructed by defining

$$u_P(k+1)=\begin{cases}u_2(1), & \text{for}\quad k=0\\ u_1(k+1)\oplus u_2(k+1)\oplus u_P(k), & \text{for}\quad k>0\end{cases}\tag{9}$$

where $u_1(k)$ and $u_2(k)$ are appropriately chosen to ensure that the following inclusions are verified:

$$A_E(\mu)x_E(k)\oplus B_1(\mu)u_1(k+1)\in\mathcal{V}^*(\mathcal{K})\tag{10}$$

$$B_1(\mu)u_2(k+1)\oplus B_2u_M(k+1)\in\mathcal{V}^*(\mathcal{K}).\tag{11}$$

for every $k\in\mathbb{N}$ and for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. To this aim, given $u_M(1)$, we take $u_P(1)=u_2(1)$ with $u_2(1)$ such that the condition (11) is satisfied for $k=0$. Then, $x_E(1)$ is computed through Eq. (6), using $x_E(0)=\epsilon$, $u_P(1)$ and $u_M(1)$, namely

$$\begin{aligned}x_E(1)&=A_E(\mu)x_E(0)\oplus B_1(\mu)u_P(1)\oplus B_2u_M(1)\\&=B_1(\mu)u_2(1)\oplus B_2u_M(1)\end{aligned}$$

and, by construction, $x(1)$ is contained in $\mathcal{V}^*(\mathcal{K})$. At this point, for $k=1$ we take $u_1(2)$ and $u_2(2)$ such that the conditions (10) and (11) are verified and we define $u_P(2)=u_1(2)\oplus u_2(2)\oplus u_P(1)$, as in (9). The procedure is iterated incrementing k by 1 for each iteration, obtaining a non-decreasing input $\{u_P(k+1)\}_{k\in\mathbb{N}}$, thanks to $u_P(k)$ in (9). Note that, for $k>0$, we have

$$\begin{aligned}x_E(k+1)\;=\;&A_E(\mu)x_E(k)\oplus B_1(\mu)u_1(k+1)\oplus\\&B_1(\mu)u_2(k+1)\oplus B_1(\mu)u_P(k)\oplus\\&B_2u_M(k+1).\end{aligned}\tag{12}$$

The term $B_1u_P(k)$ in (12) can be removed, since $x_E(k)\geq B_1(\mu)u_P(k)$ for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. Moreover, the assumptions $A_i\geq I_{n_P}$ for $i\in\mathcal{I}$, $A_M\geq I_{n_M}$ imply $A_E(\mu)\geq I_{n_P+n_M}$ for all $\mu=[\mu_1,\dots,\mu_N]$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. Hence, since $x_M(1)$ is contained in $\mathcal{V}^*(\mathcal{K})$, using induction it is possible to state that $x_M(k+1)$ is contained in $\mathcal{V}^*$ by construction.

The condition $\mathcal{V}^*(\mathcal{K})\subseteq\mathcal{K}$, then assures that $\{y_P(k+1)\}_{k\in\mathbb{N}}$, obtained from $\{u_P(k+1)\}_{k\in\mathbb{N}}$ for the plant, constructed through (9), coincides with $\{y_M(k+1)\}_{k\in\mathbb{N}}$, corresponding to the $\{u_M(k+1)\}_{k\in\mathbb{N}}$ of the model, for all values of $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. Therefore, the RSSP is solved. □

For the RFSSP, we have the following theorem.

Theorem 2. *Let us consider a plant Σ_P expressed as in (3), a model Σ_M expressed as in (4), and the joint dynamics defined by (6). Then, the RFSSP is solvable if there exists a robust $(A_E(\mu),B_1(\mu))$-invariant subsemimodule of feedback type $\mathcal{V}\subseteq\mathcal{X}_E$ within the output equalizer subsemimodule $\mathcal{K}\subseteq\mathcal{X}_E$, defined by (7), with the property that for any $u_M\in\mathbb{R}_{\max}^{m_M}$ there exists $u_P\in\mathbb{R}_{\max}^{m_P}$ satisfying*

$$B_1(\mu)u_P\oplus B_2u_M\in\mathcal{V}$$

for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$.

Proof. To give proof of the theorem, we can start by considering a robust $(A_E(\mu),B_1(\mu))$-invariant subsemimodule of feedback type $\mathcal{V}\subseteq\mathcal{K}$ satisfying the theorem's conditions. Consequently, it is assumed that a feedback matrix F exists and satisfies, for every $x_E(k)\in\mathcal{V}$, the condition

$$\left(A_E(\mu)\oplus B_1(\mu)F\right)x_E(k)\in\mathcal{V}$$

for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. Additionally, there exists an $m_P\times m_M$ matrix G which guarantees that the matrix

$$\begin{pmatrix}\epsilon\\B_M\end{pmatrix}I_{m_M}\oplus\begin{pmatrix}B_P(\mu)\\\epsilon\end{pmatrix}G=\begin{pmatrix}B_P(\mu)G\\B_M\end{pmatrix}$$

has its columns belonging to $\mathcal{V}$ for all $\mu=[\mu_1,\dots,\mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}}\mu_i=e$. At this point, an arbitrary non-decreasing input sequence $\{u_M(k+1)\}_{k\in\mathbb{N}}$ is chosen and the sequence $\{u_P(k+1)\}_{k\in\mathbb{N}}$ defined as in (5), i.e.,:

$$u_P(k+1)=Fx_E(k)\oplus Gu_M(k+1)\oplus u_P(k),\tag{13}$$

is constructed, with the initial condition $u_P(0)=\epsilon$. The sequence is generated iteratively as follows:

- set $u_P(1) = Fx_E(0) \oplus Gu_M(1) \oplus u_P(0) = Gu_M(1)$;
- calculate $x_E(1)$ using (6), given $x_E(0)$, $u_M(1)$, as well as $u_P(1)$;
- next, define $u_P(2) = Fx_E(1) \oplus Gu_M(2) \oplus u_P(1)$;
- this procedure is repeated for increasing values of k.

By following an argument analogous to the proof of Theorem 1, we can state that $x_E(k)$ remains within $\mathcal{V}$ for each k, ensuring that the condition holds for every $\mu = [\mu_1, \dots, \mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \mu_i = e$. Since $\mathcal{V}$ is a subsemimodule of the output equalizer semimodule $\mathcal{K}$, it follows that the plant output sequence $\{y_P(k+1)\}_{k\in\mathbb{N}}$ produced by $\{u_P(k+1)\}_{k\in\mathbb{N}}$ – the plant control input expressed as in (13) –coincides with the model output sequence $\{y_M(k+1)\}_{k\in\mathbb{N}}$ for all values of $\mu = [\mu_1 \dots \mu_N]^\top$ with $\bigoplus_{i\in\mathcal{I}} \mu_i = e$. Consequently, the RFSSP is solved. □

Remark 4. It is possible to check the conditions of Theorem 1 and implement (9) by solving systems of linear equations in the max-plus algebra, similarly to what is outlined in Remark 7 in [25]. This process involves the coefficient matrices $A_{E_1}, \dots, A_{E_N}$, the input matrices $B_{1_1}, \dots, B_{1_N}, B_2$, and the generators of $\mathcal{V}^*(\mathcal{K})$. A comparable procedure applies to Theorem 2 for computing the feedback matrices F and G in (13). Several computational strategies and related challenges for handling these problems in the max-plus framework have been explored in [23,38,41].

6. Example

This section presents an illustrative, theoretical example aimed at describing the application of the devised methodology, which provides exact solutions to the considered problems. The example is introduced to address the RFSSP for a max-plus linear system affected by polytopic uncertainties, computed with the ScicosLab software [39], to show how the proposed invariance-based conditions can be applied, and how a synchronizing feedback law can be constructed in the presence of polytopic uncertainties affecting all system matrices.

A plant Σ_P expressed as in (3) is taken into account, with $\mathcal{I} = \{1, 2\}$,

$$A_{P1} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \quad A_{P2} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix},$$

$$B_{P1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad B_{P2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

$$C_{P1} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad C_{P2} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

and a model Σ_M of the form (4) with

$$A_M = 4, \quad B_M = 6, \quad C_M = 1.$$

Please note that in the matrices, the value $e = 0$ is replaced by 0. This choice is dictated simply to allow a better comprehension of the results for the reader, due to the graphical similarity of e with $\epsilon = -\inf$.

The matrices of the joint dynamics described by (6), with $\mathcal{I} = \{1, 2\}$, are as follows:

$$A_{E_1} = \begin{pmatrix} 0 & 0 & \epsilon \\ 3 & 0 & \epsilon \\ \epsilon & \epsilon & 4 \end{pmatrix}, \quad A_{E_2} = \begin{pmatrix} 2 & 0 & \epsilon \\ 4 & 0 & \epsilon \\ \epsilon & \epsilon & 4 \end{pmatrix}$$

$$B_{1_1} = \begin{pmatrix} 3 \\ 0 \\ \epsilon \end{pmatrix}, \quad B_{1_2} = \begin{pmatrix} 3 \\ 1 \\ \epsilon \end{pmatrix}, \quad B_2 = \begin{pmatrix} \epsilon \\ \epsilon \\ 6 \end{pmatrix}.$$

The output equalizer subsemimodule $\mathcal{K}$ turns out to be given by

$$\mathcal{K} = \operatorname{Im} K = \operatorname{Im} \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \\ 0 & 0 \end{pmatrix}$$

and it is possible to construct the sequence of semimodules defined by (8). Such sequence converges in two steps, showing that $\mathcal{V}^\star(\mathcal{K})$ and $\mathcal{K}$ coincide, i.e.,

$$\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}^\star(\mathcal{K}) = \mathcal{K} = \operatorname{Im} K = \operatorname{Im} \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \\ 0 & 0 \end{pmatrix}.$$

The subsemimodule $\mathcal{V}^*(\mathcal{K})$ is a robust $(A_E(\mu), B_1(\mu))$-invariant subsemimodule of feedback type, since, taking $F = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$, we have that

$$(A_{E_i} \oplus B_{1_i} F)\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$$

for all $i \in \mathcal{I} = 1, 2$. Therefore, Proposition 4 applies. Moreover, for any $u_M \in \mathbb{R}_{\max}^{m_M} = \mathbb{R}_{\max}$ and $u_p = Gu_M$ taking $G = 3$, we have that

$$B_{1_1} u_P \oplus B_2 u_M \in \mathcal{V}^*(\mathcal{K})$$

$$B_{1_2} u_P \oplus B_2 u_M \in \mathcal{V}^*(\mathcal{K}).$$

or, in other words, we have that the matrix

$$\begin{pmatrix} \epsilon \\ B_M \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_{P_i} \\ \epsilon \end{pmatrix} G = \begin{pmatrix} B_{P_i} G \\ B_M \end{pmatrix}$$

has all its columns contained in $\mathcal{V}^*(\mathcal{K})$ for all $i \in \mathcal{I} = \{1, 2\}$. Consequently, Theorem 2 is satisfied, and both the RFSSP and RSSP are solvable. A solution to both problems is obtained by defining the plant input $\{u_p(k+1)\}_{k\in\mathbb{N}}$ as in (13) with $F = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ and $G = 3$, for a model input sequence $\{u_M(k+1)\}_{k\in\mathbb{N}}$, both evolving in a non-decreasing manner.

The repository containing the ScicosLab script to execute the presented example is available at the following link:

https://github.com/LabMACS/MaxPlusPolytopic-RFSSP.

7. Conclusions

This paper has investigated the problems of Robust System Synchronization and Robust Feedback System Synchronization for max-plus linear systems with polytopic uncertainties affecting all system matrices. A structural geometric approach is introduced to establish solvability conditions for these problems, leveraging the concept of robust controlled invariance for the framework of the max-plus algebra. A numerical example illustrates the proposed theoretical results and validates the effectiveness of the approach.

CRediT authorship contribution statement

David Scaradozzi: Writing – original draft, Validation, Supervision, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Veronica Bartolucci:** Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Anna Maria Perdon:** Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Giuseppe Conte:** Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Elena Zattoni:** Writing – original draft, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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