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Research article

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Boundary value problems associated with singular strongly nonlinear equations with functional terms

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Abstract: We study boundary value problems associated with singular, strongly nonlinear differential equations with functional terms of type

$$\big(\Phi(k(t)\,x'(t))\big)' + f(t, \mathcal{G}_x(t))\,\rho(t, x'(t)) = 0,$$

on a compact interval $[a, b]$. These equations are quite general due to the presence of a strictly increasing homeomorphism Φ, the so-called Φ-Laplace operator, of a non-negative function k, which may vanish on a set of null measure, and moreover of a functional term $\mathcal{G}_x$. We look for solutions, in a suitable weak sense, which belong to the Sobolev space $W^{1,1}([a, b])$. Under the assumptions of the existence of a well-ordered pair of upper and lower solutions and of a suitable Nagumo-type growth condition, we prove an existence result by means of fixed point arguments.

Keywords: boundary-value problems; singular ODEs; Φ-Laplace operator; functional ODEs; upper/lower solutions

MSC: 34K10, 34B16, 34L30

1 Introduction

Boundary value problems for highly nonlinear differential equations in the whole real line, even governed by nonlinear differential operators, have been widely investigated in the last decade. Such problems are involved in many applications in several fields, such as non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, theory of capillary surfaces and, more recently, the modeling of glaciology (see, e.g., [9, 20, 24]). Starting from the simplest types of ODEs governed by the *p-Laplace operator* $\Phi_p(z) := |z|^{p-2}z$, that is,

$$(\Phi_p(x'))' = f(t, x, x'),$$

many authors have proposed generalizations in various directions, in particular considering a more general nonlinear differential operator, a *Φ-Laplacian type operator*, which can be a generic homeomorphism, a sin-

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gular or a non-surjective operator (see, e.g., [2–4, 12–14, 19]). We also refer the reader to the survey [11] and to the references therein included. Let us also mention equations with *mixed differential operators*, that is,

$$\big(a(t,x)\Phi(x')\big)' = f(t,x,x'), \tag{1.1}$$

where a is a continuous positive function (see, e.g., [5, 8, 15, 18]). In the autonomous case, namely,

$$a(t,x) \equiv a(x),$$

equation (1.1) also arises in some models, e.g. reaction-diffusion equations with non-constant diffusivity and porous media equations.

In some models intervening in the aforementioned applications, the dynamics may also depend an a *functional argument*, since it may present a delay or a non-local term (such as a convolution integral). For instance, in reaction-diffusion models, the reactive term may involve the whole domain. As far as we know, equations involving both the Φ-Laplacian operator and functional terms are less studied and understood due to technical difficulties, see [1, 21]. So, the main aim of this paper is to provide a quite general approach in order to treat functional differential equations governed by general nonlinear differential operators, covering various types of functional dependences (e.g., delayed ODEs and non local equations). We also allow a functional dependence of boundary conditions.

More in detail, in this paper we study the solvability (in a suitable sense) of the following general boundary value problem (BVP, in short):

$$\begin{cases} \big(\Phi(k(t)\,x'(t))\big)' + f(t, \mathcal{G}_x(t))\,\rho(t, x'(t)) = 0 \quad \text{a.e. on } I := [a,b], \\ x(a) = \mathcal{H}_a[x], \;\; x(b) = \mathcal{H}_b[x]. \end{cases} \tag{1.2}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$, the so-called *$\Phi$-Laplacian operator*, is a strictly increasing homeomorphism, $k : I \to \mathbb{R}$ is a bounded non-negative function satisfying

$$1/k \in L^1(I),$$

f and ρ are Carathéodory functions, and $\mathcal{G}_x$, $\mathcal{H}_a$, $\mathcal{H}_b$ are *functional* terms, i.e.,

- $\mathcal{G} : W^{1,1}(I) \to L^\infty(I)$ is a continuous operator which verifies suitable boundedness and monotonicity conditions;
- $\mathcal{H}_a, \mathcal{H}_b : W^{1,1}(I) \to \mathbb{R}$ are continuous and increasing operators.

The proposed framework is very general, since it contains, as particular cases, delayed and non-local differential equations; moreover, we point out that the function $k(t)$ inside the differential operator may vanish on a set having zero Lebesgue measure. As a consequence, the differential equation in BVP (1.2) may be singular, and this requires an accurate choice of the space of the solutions (since they present a low regularity). In particular, we look for solutions in the Sobolev space $W^{1,1}(I)$, and this justifies the choice of $W^{1,1}(I)$ as the domain of the involved functional operators. However, as we show in our existence result, a possible higher regularity of the solutions is related to the rate of integrability of the function $1/k$. More precisely, we shall prove that when $1/k \in L^\theta$ (with $1 < \theta \leq \infty$), then there exists a solution belonging to $W^{1,\theta}(I)$; in particular, when $1/k$ is continuous, we find C^1-solutions. Hence, from this point of view, our main result concerns both the *existence* and the *regularity* of the solutions.

In this framework, a typical approach to get existence results is given by the combination of fixed point techniques and the method of upper and lower solutions. A crucial tool which gives a priori bounds for the derivatives of the solutions is a *Nagumo-type* growth condition on the nonlinearity. Recently, in the paper [23] the authors obtained an existence result assuming a weak form of Wintner-Nagumo growth condition. The approach of [23] has been fruitfully extended to the context of singular equations: see [5–8, 17]. In our main result (see Theorem 2.6 below) we assume the following weak Nagumo growth condition:

$$|f(t,z)\rho(t,y)| \leq \psi(|\Phi(k(t)y)|) \cdot \left(\ell(t) + \mu(t)|y|^{\frac{q-1}{q}}\right),$$

where $\mu \in L^q(I)$ (for some $q > 1$), $\ell \in L^1(I)$ and $\psi : (0, \infty) \to (0, \infty)$ satisfies

$$\int_1^{+\infty} \frac{\mathrm{d}s}{\psi(s)} = +\infty.$$

This assumption allows to consider a very general operator Φ.

While we refer to Section 4 for some concrete examples illustrating the applicability of our results, here we limit ourselves to point out that our approach allows us to prove the solvability of, e.g.,

$$\begin{cases} \left(\Phi_p\left(|\sin(t)|^{1/\vartheta_0}\, x'(t)\right)\right)' + x_\tau(t)\, |x'(t)|^\delta = 0 & \text{a.e. on } [0, 2\pi], \\ x(0) = \sqrt[3]{x(\pi)}, \;\; x(2\pi) = \frac{1}{4\pi}\int_0^{2\pi}(x(s) + 2)\,\mathrm{d}s \end{cases}$$

where $\Phi_p(z) = |z|^{p-2}z$ is the usual p-Laplace operator, ϑ_0, δ are positive constants and the functional term $\mathcal{G}_x = x_\tau$ is of delay-type, that is,

$$x_\tau(t) := \begin{cases} x(t-\tau), & \text{for } t \in [0, 2\pi],\ t \geq \tau, \\ x(0), & \text{otherwise} \end{cases}$$

A brief plan of the paper is now in order.

- ◇ In Section 2 we fix some preliminary definitions and we state our main existence result, namely Theorem 2.6.
- ◇ In Section 3 we provide the proof of Theorem 2.6, which articulates into two steps: first, we perform a truncation argument and we introduce an auxiliary BVP to which suitable existence results do apply; then, we show that *any* solution of the 'truncated' problem is a solution of the original BVP. In doing this, we use in a crucial way the assumption of the existence of a well-ordered pair of lower and upper solutions of our problem.
- ◇ In Section 4 we present some examples to which our Theorem 2.6 applies.
- ◇ Finally, we close the paper with an Appendix containing the explicit proof of a technical Lemma, which in some previous papers was missing, and in other papers was either not complete or not correct.

2 Preliminaries and main results

Let $a, b \in \mathbb{R}$ satisfy $a < b$, and let $I := [a, b]$. As mentioned in the Introduction, throughout this paper we shall be concerned with BVPs of the following form

$$\begin{cases} \left(\Phi(k(t)\, x'(t))\right)' + f(t, \mathcal{G}_x(t))\, \rho(t, x'(t)) = 0 & \text{a.e. on } I, \\ x(a) = \mathcal{H}_a[x], \;\; x(b) = \mathcal{H}_b[x], \end{cases} \tag{2.1}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism, $f, \rho : I \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and k, $\mathcal{G}$, $\mathcal{H}_a$, $\mathcal{H}_b$ satisfy the following assumptions:

(H1) $k : I \to \mathbb{R}$ is a *nonnegative* function satisfying

$$k \in L^\infty(I) \qquad \text{and} \qquad 1/k \in L^1(I). \tag{2.2}$$

(H2) $\mathcal{G} : W^{1,1}(I) \to L^\infty(I)$ is continuous (with respect to the usual norms) and *bounded when $W^{1,1}(I)$ is thought of as a subspace* of $L^\infty(I)$; this means, precisely, that for every $r > 0$ there exists $\eta_r > 0$ such that

$$\|\mathcal{G}_x\|_{L^\infty(I)} \leq \eta_r \quad \text{for any } x \in W^{1,1}(I) \text{ with } \|x\|_{L^\infty(I)} \leq r. \tag{2.3}$$

(H3) There exists a constant $\kappa \geq 0$ such that

$$\begin{gathered} f(t, \mathcal{G}_x(t)) + \kappa\, x(t) \leq f(t, \mathcal{G}_y(t)) + \kappa\, y(t) \text{ a.e. on } I \\ \text{for every } x, y \in W^{1,1}(I) \text{ such that } x \leq y \text{ a.e. on } I. \end{gathered} \tag{2.4}$$

(H4) $\mathcal{H}_a, \mathcal{H}_b : W^{1,1}(I) \to \mathbb{R}$ are continuous (with respect to the usual topologies) and monotone increasing, that is,

$$\begin{gathered}\mathcal{H}_a[x] \le \mathcal{H}_b[y] \quad \text{and} \quad \mathcal{H}_b[x] \le \mathcal{H}_b[y] \\ \text{for every } x, y \in W^{1,1}(I) \text{ such that } x \le y \text{ a.e. on } I.\end{gathered} \tag{2.5}$$

Remark 2.1. We point out, for a future reference, that the continuity of $\mathcal{G}$ from $W^{1,1}(I)$ into $L^\infty(I)$ is ensured if $\mathcal{G}$ maps continuously $W^{1,1}(I)$ into some Banach space $(X, \|\cdot\|_X)$ which is *continuously embedded* into $L^\infty(I)$.

This is the case, e.g., of the following functional spaces:
(1) $X = W^{1,p}(I)$ (with $p \ge 1$ and the usual norm);
(2) $X = C^n(I, \mathbb{R})$ (with $n \in \mathbb{N}$ and the usual norm).

Moreover, we also notice that the monotonicity assumption (H3) seems very natural to get existence results for problem (2.1). We point out that a similar monotonicity assumption has already been considered by the authors in a different context in the paper [16].

Remark 2.2. We explicitly notice that, in view of assumption (H1), the function k can vanish on a set $E \subseteq \mathbb{R}$ of zero Lebesgue measure (in particular, E could be infinite). As a consequence, the ODE appearing in (2.1) may be singular.

The aim of this paper is to study the solvability of (2.1) in a weak sense, according to the following definition.

Definition 2.3. We say that a function $x \in W^{1,1}(I)$ is a *solution* of problem (2.1) if it satisfies the following two properties:
(1) the map $t \mapsto \Phi(k(t)\,x'(t))$ is in $W^{1,1}(I)$ and

$$\big(\Phi(k(t)\,x'(t))\big)' + f(t, \mathcal{G}_x(t))\,\rho(t, x'(t)) = 0 \qquad \text{for a.e. } t \in I;$$

(2) $x(a) = \mathcal{H}_a[x]$ and $x(b) = \mathcal{H}_b[x]$.

If x satisfies only property (1), we say that x is a *solution of the ODE*

$$\big(\Phi(k(t)\,x'(t))\big)' + f(t, \mathcal{G}_x(t))\,\rho(t, x'(t)) = 0. \tag{2.6}$$

A fundamental notion for our investigation of the solvability of (2.1) is that of lower/upper solution, which is contained in the next definition.

Definition 2.4. We say that a function $x \in W^{1,1}(I)$ is a *lower* [resp. *upper*] *solution* of problem (2.1) if it satisfies the following two properties:
(1) the map $t \mapsto \Phi(k(t)\,x'(t))$ is in $W^{1,1}(I)$ and

$$\big(\Phi(k(t)\,x'(t))\big)' + f(t, \mathcal{G}_x(t))\,\rho(t, x'(t)) \ge \ [\le]\ 0 \qquad \text{for a.e. } t \in I;$$

(2) $x(a) \le [\ge]\ \mathcal{H}_a[x]$ and $x(b) \le [\ge]\ \mathcal{H}_b[x]$.

If the function x satisfies only property (1), we say that it is a *lower* [resp. *upper*] *solution of the ODE* (2.6).

Remark 2.5. If $u \in W^{1,1}(I)$ is any function such that

$$t \mapsto \Phi(k(t)\,u'(t)) \in W^{1,1}(I)$$

(this is the case, e.g., of any lower/upper solution of (2.6)), the continuity of Φ^{-1} implies the existence of a (unique) continuous function $\mathcal{K}_u$ such that

$$\mathcal{K}_u(t) = k(t)\,u'(t) \text{ for a.e. } t \in I \quad \text{and} \quad \Phi \circ \mathcal{K}_u \in W^{1,1}(I).$$

In particular, this is true if u is a *solution* of (2.6).

We are now ready to state the main result of the paper.

Theorem 2.6. *Let the structural assumptions* (H1)*-to-*(H4) *be in force. Moreover, let us suppose that the following additional hypotheses are satisfied:*

(H5) there exist a lower solution α and an upper solution β of problem (2.1) *which are* well-ordered *on I, that is, $\alpha(t) \le \beta(t)$ for every $t \in I$;*

(H6) for every $R > 0$ and every non-negative function $\gamma \in L^1(I)$ there exists a non-negative function $h = h_{R,\gamma} \in L^1(I)$ such that

$$|f(t,z)\,\rho(t,y(t))| \le h_{R,\gamma}(t) \qquad \text{for a.e. } t \in I, \text{ every } z \in \mathbb{R} \text{ with } |z| \le R \text{ and every } y \in L^1(I) \text{ such that } |y(s)| \le \gamma(s) \text{ for a.e. } s \in I. \tag{2.7}$$

(H7) for every $R > 0$ there exist a constant $H = H_R > 0$, a non-negative function $\mu = \mu_R \in L^q(I)$ (with $1 < q \le \infty$), a non-negative function $l = l_R \in L^1(I)$ and a measurable function $\psi = \psi_R : (0,\infty) \to (0,\infty)$ such that

$$(\star)\ 1/\psi \in L^1_{\mathrm{loc}}(0,\infty) \quad \text{and} \quad \int^{\infty} \frac{1}{\psi(t)}\,\mathrm{d}t = \infty; \tag{2.8}$$

$$(\star\star)\ \begin{array}{l} |f(t,z)\,\rho(t,y)| \le \psi\big(|\Phi(k(t)y)|\big) \cdot \big(l(t) + \mu(t)\,|y|^{\frac{q-1}{q}}\big); \\ \text{for a.e. } t \in I, \text{ any } z \in [-R,R] \text{ and any } y \in \mathbb{R} \text{ with } |k(t)y| \ge H. \end{array} \tag{2.9}$$

Then, there exists a solution $x_0 \in W^{1,1}(I)$ of problem (2.1)*, further satisfying*

$$\alpha(t) \le x_0(t) \le \beta(t) \qquad \text{for every } t \in I. \tag{2.10}$$

Moreover, the following higher-regularity properties hold:

(1) if $1/k \in L^{\vartheta}(I)$ for some $1 < \vartheta \le \infty$, one also has that $x_0 \in W^{1,\vartheta}(I)$;

(2) if $k \in C(I,\mathbb{R})$ and $k > 0$ on I, one also has that $x_0 \in C^1(I,\mathbb{R})$.

Finally, if $M > 0$ is any real number such that $\|\alpha\|_{L^\infty(I)}, \|\beta\|_{L^\infty(I)} \le M$, there exists a constant $L_M > 0$, only depending on M, such that

$$\|x_0\|_{L^\infty(I)} \le M \qquad \text{and} \qquad \|\mathcal{K}_{x_0}\|_{L^\infty(I)} \le L_M. \tag{2.11}$$

Remark 2.7. It is worth highlighting that the existence of a well-ordered pair of lower and upper solutions α, β for (2.1) is far from being obvious (see, e.g., [11, 22] and the reference therein for general results on this topic). Here, we limit ourselves to observe that, if $\rho(t,0) = 0$ for every $t \in I$, then *any constant function* is both a lower and an upper solution *for the ODE* (2.6).

As a positive counterpart of the previous comment, we shall present in the next Section 4 a couple of examples of BVPs to which Theorem 2.6 applies.

3 Proof of Theorem 2.6

The proof of Theorem 2.6 is rather technical and long; for this reason, after having introduced some constants and parameters used throughout, we shall proceed by establishing several claims. Roughly put, our approach consists of two steps.

Step I: As a first step, by crucially exploiting the existence of a well-ordered pair of lower and upper solutions α, β for (2.1) (see, precisely, assumption (H5)), we perform a truncation argument and we introduce a new problem, say $(\mathrm{P})_\tau$, to which some abstract results do apply.

Step II: Then, we show that *any* solution of $(\mathrm{P})_\tau$ is actually a solution of (2.1). In doing this, we use again in a crucial way the fact that α and β are, respectively, a lower and an upper solution for (2.1).

We then begin by fixing some quantities which shall be used all over the proof.

First of all, we choose a real $M > 0$ in such a way that $\|\alpha\|_{L^\infty(I)} \leq M$ and $\|\beta\|_{L^\infty(I)} \leq M$. Moreover, using assumption (H2), we let $\eta_M > 0$ be such that

$$\|\mathcal{G}u\|_{L^\infty(I)} \leq \eta_M \qquad \text{for all } u \in W^{1,1}(I) \text{ with } \|u\|_{L^\infty(I)} \leq M. \tag{3.1}$$

With reference to assumption (H7), we then set

$$H_M := H_{\eta_M}, \qquad \mu_M := \mu_{\eta_M}, \qquad l_M := l_{\eta_M}, \qquad \psi_M := \psi_{\eta_M}.$$

Now, since Φ is strictly increasing, we can choose $N > 0$ such that

$$\begin{gathered} \Phi(N) > 0, \quad \Phi(-N) < 0 \qquad \text{and} \\ N > \max\left\{H_M, \frac{2M}{b-a} \cdot \|k\|_{L^\infty(I)}\right\}; \end{gathered} \tag{3.2}$$

accordingly, owing to (2.8), we fix $L = L_M \geq N > 0$ in such a way that

$$\begin{gathered} \min\left\{ \int_{\Phi(N)}^{\Phi(L_M)} \frac{1}{\psi_M}\, \mathrm{d}s, \int_{-\Phi(-N)}^{-\Phi(-L_M)} \frac{1}{\psi_M}\, \mathrm{d}s \right\} \\ > \|l_M\|_{L^1(I)} + \|\mu_M\|_{L^q(I)} \cdot (2M)^{\frac{q-1}{q}}, \end{gathered} \tag{3.3}$$

and we consider the function $y_L \in L^1(I)$ defined as:

$$y_L(t) := \frac{L_M}{k(t)} + |\alpha'(t)| + |\beta'(t)|. \tag{3.4}$$

Following the notation in Appendix A, we also define the truncating operators

$$\mathcal{T} := \mathcal{T}^{\alpha,\beta} \qquad \text{and} \qquad \mathcal{D} := \mathcal{T}^{-y_L, y_L}. \tag{3.5}$$

Given any $x \in W^{1,1}(I)$, we then consider the function F_x defined by

$$F_x(t) := -f\big(t, \mathcal{G}_{\mathcal{T}_x}(t)\big)\, \rho\big(t, \mathcal{D}_{\mathcal{T}_x'}(t)\big) + \arctan\big(x(t) - \mathcal{T}_x(t)\big). \tag{3.6}$$

Finally, we consider the operators $\mathcal{B}_a, \mathcal{B}_b : W^{1,1}(I) \to \mathbb{R}$ defined as

$$\mathcal{B}_a := \mathcal{H}_a \circ \mathcal{T}, \qquad \mathcal{B}_b := \mathcal{H}_b \circ \mathcal{T}. \tag{3.7}$$

Thanks to all these preliminaries, we can finally introduce the following BVP (which can be thought of as a truncated version of problem (2.1)):

$$\begin{cases} \big(\Phi(k(t)\, x'(t))\big)' = F_x(t) & \text{a.e. on } I, \\ x(a) = \mathcal{B}_a[x], \ \ x(b) = \mathcal{B}_b[x]. \end{cases} \tag{3.8}$$

We now proceed by following the steps described above.

Step I. In this first step we prove the following result: *there exists* (*at least*) *one solution* $u \in W^{1,1}(I)$ *of problem* (3.8)*; this means, precisely, that*
- *the map* $t \mapsto \Phi(k(t)\, u'(t))$ *is in* $W^{1,1}(I)$ *and*

$$\big(\Phi(k(t)\, u'(t))\big)' = F_u(t) \qquad \textit{for a.e. } t \in I;$$

- $u(a) = \mathcal{B}_a[u]$ *and* $u(b) = \mathcal{B}_b[u]$.

Furthermore, the following higher-regularity assertions hold:

(i) if $1/k \in L^\vartheta(I)$ for some $1 < \vartheta \leq \infty$, one also has that $x_0 \in W^{1,\vartheta}(I)$;
(ii) if $k \in C(I, \mathbb{R})$ and $k > 0$ on I, then $u \in C^1(I, \mathbb{R})$.

Claim 1. *There exists a non-negative function $\psi \in L^1(I)$ such that*

$$|F_x(t)| \leq \psi(t) \qquad \text{for a.e. } t \in I \text{ and every } x \in W^{1,1}(I). \tag{3.9}$$

First of all, by the choice of M and the very definition of $\mathcal{T}_x$ we have

$$-M \leq \alpha(t) \leq \mathcal{T}_x(t) \leq \beta(t) \leq M \qquad \text{for all } x \in W^{1,1}(I) \text{ and any } t \in I;$$

as a consequence, owing to the choice of η_M in (3.1), we get

$$\|\mathcal{G}_{\mathcal{T}_x}\|_{L^\infty(I)} \leq \eta_M \qquad \text{for every } x \in W^{1,1}(I). \tag{3.10}$$

Moreover, owing to the very definition of $\mathcal{D}$, we also have

$$|\mathcal{D}_{\mathcal{T}'_x}(t)| \leq y_L(t) \qquad \text{for any } x \in W^{1,1}(I) \text{ and a.e. } t \in I. \tag{3.11}$$

Gathering together (3.10) and (3.11), we deduce from assumption (H6) that there exists a non-negative function $h = h_{\eta_M, y_L} \in L^1(I)$ such that

$$|F_x(t)| \leq \left| f\big(t, \mathcal{G}_{\mathcal{T}_x}(t)\big)\, \rho\big(t, \mathcal{D}_{\mathcal{T}'_x}(t)\big)\right| + \frac{\pi}{2} \leq h_{\eta_M, y_L}(t) + \frac{\pi}{2}, \tag{3.12}$$

and this estimate holds for every $x \in W^{1,1}(I)$ and a.e. $t \in I$. Since

$$\psi := h_{\eta_M, y_L} + \pi/2 \in L^1(I),$$

we conclude at once that $F_x \in L^1(I)$ for every $x \in W^{1,1}(I)$ (hence, F maps $W^{1,1}(I)$ into $L^1(I)$) and that F satisfies estimate (3.9).

Claim 2. *F is continuous from $W^{1,1}(I)$ into $L^1(I)$.*

Let $x_0 \in W^{1,1}(I)$ be fixed, and let $\{x_n\}_n \subseteq W^{1,1}(I)$ be a sequence converging to x_0 as $n \to \infty$. Moreover, let $\{u_k := x_{n_k}\}_k$ be any sub-sequence of $\{x_n\}_n$.

To demonstrate the continuity of F it suffices to show that, by choosing a further sub-sequence if necessary, one has

$$\lim_{k\to\infty} F_{u_k} = F_{x_0} \qquad \text{in } L^1(I). \tag{3.13}$$

First of all we observe that, since $u_k \to x_0$ in $W^{1,1}(I)$ as $k \to \infty$, we have

$$\lim_{k\to\infty} u_k(t) = x_0(t) \qquad \text{uniformly for } t \in I; \tag{3.14}$$

moreover, by Lemma A.1 we also have

$$\lim_{k\to\infty} \mathcal{T}_{u_k} = \mathcal{T}_{x_0} \qquad \text{in } W^{1,1}(I). \tag{3.15}$$

In particular, since (3.15) implies that $\mathcal{T}'_{u_k} \to \mathcal{T}'_{x_0}$ in $L^1(I)$ as $k \to \infty$, by possibly choosing a sub-sequence we can assume that

$$\lim_{k\to\infty} \mathcal{T}'_{u_k}(t) = \mathcal{T}'_{x_0}(t) \qquad \text{for a.e. } t \in I. \tag{3.16}$$

Now, since $\mathcal{G}$ is continuous from $W^{1,1}(I)$ to $L^\infty(I)$, from (3.15) we get

$$\lim_{k\to\infty} \mathcal{G}_{\mathcal{T}_{u_k}}(t) = \mathcal{G}_{\mathcal{T}_{x_0}}(t) \qquad \text{for every } t \in I. \tag{3.17}$$

Moreover, from (3.16) we get that

$$\lim_{k\to\infty} \mathcal{D}_{\mathcal{T}'_{u_k}}(t) = \mathcal{D}_{\mathcal{T}'_{x_0}}(t) \qquad \text{for a.e. } t \in I. \tag{3.18}$$

Gathering together (3.17), (3.18) and (3.14), we then obtain (remind that, by assumptions, f and ρ are Carathéodory functions on $I \times \mathbb{R}$)

$$\begin{aligned}\lim_{k\to\infty} F_{u_k}(t) &= \lim_{k\to\infty} \Big(-f\big(t, \mathcal{G}_{\mathcal{T}_{u_k}}(t)\big)\,\rho\big(t, \mathcal{D}_{\mathcal{T}'_{u_k}}(t)\big) + \arctan\big(u_k(t) - \mathcal{T}_{u_k}(t)\big)\\ &= -f\big(t, \mathcal{G}_{\mathcal{T}_{x_0}}(t)\big)\,\rho\big(t, \mathcal{D}_{\mathcal{T}'_{x_0}}(t)\big) + \arctan\big(u_0(t) - \mathcal{T}_{u_0}(t)\big)\\ &= F_{x_0}(t) \qquad \text{for a.e. } t \in I.\end{aligned}$$

From this, a standard dominated-convergence based on (3.12) allows us to conclude that $F_{u_k} \to F_{x_0}$ in $L^1(I)$ as $k \to \infty$, which is exactly the desired (3.13).

Claim 3. *$\mathcal{B}_a$ and $\mathcal{B}_b$ are continuous and bounded (from $W^{1,1}(I)$ to $\mathbb{R}$).*

As regards the continuity, since $\mathcal{H}_a$, $\mathcal{H}_b$ are continuous from $W^{1,1}(I)$ to $\mathbb{R}$ (by assumption (H4)) and since $\mathcal{T}$ is continuous on $W^{1,1}(I)$ (by Lemma A.1), we deduce that $\mathcal{B}_a = \mathcal{H}_a \circ \mathcal{T}$ and $\mathcal{B}_b = \mathcal{H}_b \circ \mathcal{T}$ are continuous.

As regards the boundedness, since $\mathcal{H}_a$, $\mathcal{H}_b$ are monotone increasing (see (2.5)), for every fixed $x \in W^{1,1}(I)$ we have

$$\mathcal{H}_a[\alpha] \le \mathcal{H}_a[\mathcal{T}_x] \le \mathcal{H}_a[\beta] \qquad \text{and} \qquad \mathcal{H}_b[\alpha] \le \mathcal{H}_b[\mathcal{T}_x] \le \mathcal{H}_b[\beta]$$

(remind that, by definition, $\alpha \le \mathcal{T}_x \le \beta$ for all $x \in W^{1,1}(I)$). From this, we deduce that $\mathcal{B}_a$, $\mathcal{B}_b$ are globally bounded, and the claim is proved.

Using the results established in the above claims, one can prove the existence of solutions for (3.8) (and the higher-regularity assertions (i)-(ii)) by arguing essentially as in [17, Lem. 2.1 and Thm. 2.2]. The key points are the following.

- Thanks to Claim 3, it can be proved that for every $x \in W^{1,1}(I)$ there exists a *unique* real number $z = z_x \in \mathbb{R}$ such that

$$\mathcal{B}_b[x] - \mathcal{B}_a[x] = \int_a^b \frac{1}{k(t)}\,\Phi^{-1}\big(z_x + \mathcal{F}_x(t)\big)\,\mathrm{d}t,$$

  where $\mathcal{F}_x(t) := \int_a^t F_x(s)\,\mathrm{d}s$. Moreover, the map $x \mapsto z_x$ is *bounded*, i.e.,

$$|z_x| \le \mathbf{c}_0 \qquad \text{for every } x \in W^{1,1}(I),$$

  where $\mathbf{c}_0 > 0$ is a universal constant which is independent of x.
- The solutions of (3.8) are precisely the fixed points (in $W^{1,1}(I)$) of the operator $\mathcal{A} : W^{1,1}(I) \to W^{1,1}(I)$ defined as follows:

$$\mathcal{A}x(t) := \mathcal{B}_a[x] + \int_a^t \frac{1}{k(t)}\,\Phi^{-1}\big(z_x + \mathcal{F}_x(t)\big)\,\mathrm{d}t \qquad (t \in I).$$

- Using all the above claims, it can be proved that $\mathcal{A}$ is continuous, bounded and compact on $W^{1,1}(I)$; thus, Schauder's Fixed-Point theorem ensures that $\mathcal{A}$ possesses (at least) one fixed point $x_0 \in W^{1,1}(I)$.
- Finally, the higher-regularity assertions (i)-(ii) are straightforward consequences of the following simple observations:
  - ◇ $\mathcal{A}(W^{1,1}(I)) \subseteq W^{1,\vartheta}(I)$ if $1/k \in L^{\vartheta}(I)$ (for some $1 < \vartheta \le \infty$);
  - ◇ $\mathcal{A}(W^{1,1}(I)) \subseteq C^1(I, \mathbb{R})$ if $k \in C(I, \mathbb{R})$ and $k > 0$ on I.

We proceed with the second step.

Step II. In this second step we establish the following result: *if $u \in W^{1,1}(I)$ is* any *solution of* (3.8)*, then u is also a solution of* (2.1).

Claim 1. *$\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$, so that*

$$\mathcal{T}_u \equiv u \text{ and } \mathcal{G}_{\mathcal{T}_u} \equiv \mathcal{G}_u \text{ on } I.$$

We argue by contradiction and, to fix ideas, we assume that the (continuous) function $v := u - \alpha$ attains a strictly negative minimum on I.

Since u solves (3.8), α is a lower solution of problem (2.1) and the operators $\mathcal{H}_a$, $\mathcal{H}_b$ are monotone increasing (see assumption (H4)), we get

$$u(a) = \mathcal{H}_a[\mathcal{T}_u] \geq \mathcal{H}_a[\alpha] \geq \alpha(a) \quad \text{and} \quad u(b) = \mathcal{H}_b[\mathcal{T}_u] \geq \mathcal{H}_b[\alpha] \geq \alpha(b)$$

(remind that, by definition, $\mathcal{T}_u \geq \alpha$ on I). As a consequence, it is possible to find three points $t_1, t_2, \theta \in \operatorname{int}(I)$, with $t_1 < \theta < t_2$, such that

(1) $u(t_i) - \alpha(t_i) = 0$ for $i = 1, 2$;
(2) $u(t) - \alpha(t) < 0$ for all $t \in (t_1, t_2)$;
(3) $u(\theta) - \alpha(\theta) = \min_{t \in I}(u(t) - \alpha(t)) < 0$.

In particular, from (2) we infer that $\mathcal{T}_u \equiv \alpha$ on (t_1, t_2), and thus

$$\mathcal{D}_{\mathcal{T}'_u}(t) = \mathcal{D}_{\alpha'}(t) = \alpha'(t) \qquad \text{for a.e. } t \in (t_1, t_2).$$

By using once again the fact that u solves (3.8), and since α is a lower solution of problem (2.1), from assumption (H3) we then obtain (for a.e. $t \in (t_1, t_2)$)

$$\begin{aligned}
\big(\Phi(k(t)\,u'(t))\big)' &= -f(t, \mathcal{G}_{\mathcal{T}_u}(t))\,\rho(t, \alpha'(t)) + \arctan(u(t) - \alpha(t)) \\
&< -f(t, \mathcal{G}_{\mathcal{T}_u}(t))\,\rho(t, \alpha'(t)) \\
&= -\big(f(t, \mathcal{G}_{\mathcal{T}_u}(t)) + \kappa\,\mathcal{T}_u(t)\big)\rho(t, \alpha'(t)) + \kappa\,\mathcal{T}_u(t)\,\rho(t, \alpha'(t)) \\
&\quad \big(\text{by (2.4), since } \mathcal{T}_u \geq \alpha \text{ on } I \text{ and } \rho \geq 0 \text{ on } \mathbb{R}\big) \\
&\leq -\big(f(t, \mathcal{G}_\alpha(t)) + \kappa\,\alpha(t)\big)\rho(t, \alpha'(t)) + \kappa\,\mathcal{T}_u(t)\,\rho(t, \alpha'(t)) \\
&\quad \big(\text{since } \mathcal{T}_u \equiv \alpha \text{ on } (t_1, t_2)\big) \\
&= -f(t, \mathcal{G}_\alpha(t))\,\rho(t, \alpha'(t)) \leq \big(\Phi(k(t)\,\alpha'(t))\big)'.
\end{aligned} \tag{3.19}$$

We now consider the sets $A_1, A_2 \subseteq I$ defined as follows:

$$A_1 := \big\{t \in (t_1, \theta) : \exists\, u'(t),\ \alpha'(t) \text{ and } u'(t) < \alpha'(t)\big\} \qquad \text{and}$$
$$A_2 := \big\{t \in (\theta, t_2) : \exists\, u'(t),\ \alpha'(t) \text{ and } u'(t) > \alpha'(t)\big\}.$$

Since $u, \alpha \in W^{1,1}(I)$ and since $u < \alpha$ on (t_1, t_2), it follows that both A_1 and A_2 have *positive* Lebesgue measure; as a consequence, there exist $\tau_1 \in A_1$ and $\tau_2 \in A_2$ such that (see also Remarks 2.2 and 2.5)

(a) $k(\tau_i) > 0$ for $i = 1, 2$;
(b) $\mathcal{K}_u(\tau_i) = k(\tau_i)\,u'(\tau_i)$ for $i = 1, 2$;
(c) $\mathcal{K}_\alpha(\tau_i) = k(\tau_i)\,\alpha'(\tau_i)$ for $i = 1, 2$.

By integrating both sides of (3.19) on $[\tau_1, \theta]$, and using (b)-(c), we then get

$$\Phi\big(\mathcal{K}_u(\theta)\big) - \Phi\big(k(\tau_1)\,u'(\tau_1)\big) < \Phi\big(\mathcal{K}_\alpha(\theta)\big) - \Phi\big(k(\tau_1)\,\alpha'(\tau_1)\big).$$

Since Φ is strictly increasing, by (a) and the choice of τ_1 we obtain

$$\Phi\big(\mathcal{K}_u(\theta)\big) - \Phi\big(\mathcal{K}_\alpha(\theta)\big) < 0. \tag{3.20}$$

On the other hand, by integrating both sides of inequality (3.19) on $[\theta, \tau_2]$ (and using once again (b)-(c)), we derive that

$$\Phi\big(k(\tau_2)\,u'(\tau_2)\big) - \Phi\big(\mathcal{K}_u(\theta)\big) < \Phi\big(k(\tau_2)\,\alpha'(\tau_2)\big) - \Phi\big(\mathcal{K}_\alpha(\theta)\big);$$

Since Φ is strictly increasing, by (a) and the choice of τ_2 we obtain

$$\Phi\big(\mathcal{K}_u(\theta)\big) - \Phi\big(\mathcal{K}_\alpha(\theta)\big) > 0.$$

This is clearly in contradiction with (3.20), and thus $u(t) - \alpha(t) \geq 0$ for every $t \in I$. By arguing exactly in the same way one can also prove that $u(t) - \beta(t) \leq 0$ for every $t \in I$, and the claim is completely demonstrated.

Claim 2. *$|u(t)| \leq M$ and $|\mathcal{G}_u(t)| \leq \eta_M$ for every $t \in I$.*

By statement (i) and the choice of $M \geq \|\alpha\|_{L^\infty(I)}, \|\beta\|_{L^\infty(I)}$, we get

$$-M \leq \alpha(t) \leq u(t) \leq \beta(t) \leq M \qquad \text{for every } t \in I,$$

and this proves that $|u(t)| \leq M$ for all $t \in I$. From this, by taking into account the choice of η_M in (3.1), we derive that $|\mathcal{G}_u(t)| \leq \eta_M$, as desired.

Claim 3. *If $N > 0$ is as in* (3.2), *then*

$$\min_{t \in I} |\mathcal{K}_u(t)| \leq N. \tag{3.21}$$

By contradiction, let us assume that (3.21) does not hold; moreover, to fix ideas (and taking into account the continuity of $\mathcal{K}_u$), let us suppose that

$$\mathcal{K}_u(t) > N \qquad \text{for every } t \in I. \tag{3.22}$$

By integrating on I both sides of the above inequality, we obtain

$$N(b-a) < \int_a^b \mathcal{K}_u(t)\,\mathrm{d}t = \int_a^b k(t)\,u'(t)\,\mathrm{d}t = (\bigstar);$$

from this, since (3.22) implies that $u'(t) > N/k(t)$ for a.e. $t \in I$, we then get

$$\begin{aligned}(\bigstar) &\leq \|k\|_{L^\infty(I)} \int_a^b u'(t)\,\mathrm{d}t = \|k\|_{L^\infty(I)}\,(u(b) - u(a)) \\ &\quad \big(\text{by statement (ii) and the choice of } N, \text{ see (3.2)}\big) \\ &\leq (2M) \cdot \|k\|_{L^\infty(I)} < N(b-a).\end{aligned}$$

This is clearly a contradiction, and thus $\min_I \mathcal{K}_u \leq N$. By arguing exactly in the same way one can also show that $\sup_I \mathcal{K}_u \geq -N$, and this proves (3.21).

Claim 4. *$|\mathcal{K}_u(t)| \leq L_M$ for every $t \in I$.*

Arguing again by contradiction, we assume that there exists some point τ in I such that $|\mathcal{K}_u(\tau)| > L_M$; moreover, to fix ideas, we suppose that

$$\mathcal{K}_u(\tau) > L_M > 0.$$

Since $L_M > N$ (see (3.3)), by (3.21) (and the continuity of $\mathcal{K}_u$) we deduce the existence of two points $t_1, t_2 \in I$, with (to fix ideas) $t_1 < t_2$, such that

(a) $\mathcal{K}_u(t_1) = N$ and $\mathcal{K}_u(t_2) = L_M$;

(b) $N < \mathcal{K}_u(t) < L_M$ for all $t \in (t_1, t_2) \subseteq I$.

In particular, from (b), (3.4) and the choice of N in (3.2) we derive that

$$k(t)u'(t) \geq H_M \quad \text{and} \quad 0 < u'(t) < \frac{L_M}{k(t)} \leq y_L(t) \tag{3.23}$$

for almost every $t \in (t_1, t_2)$. Now, on account of (3.23) and of the very definition of $\mathcal{D}$, we deduce that $\mathcal{D}_{u'} \equiv u'$ on (t_1, t_2); as a consequence, since u is a solution of (3.8) and $\mathcal{T}_u \equiv u$ on I (by Claim 1.), we have (a.e. on (t_1, t_2))

$$\begin{aligned}
\left|\left(\Phi(\mathcal{K}_u(t))\right)'\right| &= \left|\left(\Phi(k(t)\,u'(t))\right)'\right| = \left|f(t, \mathcal{G}_u(t))\,\rho(t, u'(t))\right| \\
&\quad \left(\text{by (2.9), since } k(t)u'(t) \geq H_M \text{ and } \|\mathcal{G}_u\| \leq \eta_M, \text{ see (ii)}\right) \\
&\leq \psi_M\left(|\Phi(k(t)\,u'(t))|\right) \cdot \left(l_M(t) + \mu_M(t)\,|u'(t)|^{\frac{q-1}{q}}\right) \\
&= \psi_M\left(|\Phi(\mathcal{K}_u(t))|\right) \cdot \left(l_M(t) + \mu_M(t)\,|u'(t)|^{\frac{q-1}{q}}\right).
\end{aligned}$$

In particular, since $u' > 0$ a.e. on (t_1, t_2) (see (3.23)) and since

$$\Phi(\mathcal{K}_u(t)) > \Phi(N) > 0 \qquad \text{for any } t \in (t_1, t_2)$$

(by (b), the monotonicity of Φ and the choice of N in (3.2)), we obtain

$$\left|\left(\Phi(\mathcal{K}_u(t))\right)'\right| \leq \psi_M\left(\Phi(\mathcal{K}_u(t))\right) \cdot \left(l_M(t) + \mu_M(t)\,(u'(t))^{\frac{q-1}{q}}\right) \tag{3.24}$$

for almost every $t \in (t_1, t_2)$. Using this last inequality, we then get (remind that $\Phi \circ \mathcal{K}_u$ is absolutely continuous, see Remark 2.5)

$$\begin{aligned}
\int_{\Phi(N)}^{\Phi(L_M)} \frac{1}{\psi_M}\,\mathrm{d}s &= \int_{\Phi(\mathcal{K}_u(t_1))}^{\Phi(\mathcal{K}_u(t_2))} \frac{1}{\psi_M}\,\mathrm{d}s = \int_{t_1}^{t_2} \frac{\left(\Phi(\mathcal{K}_u(t))\right)'}{\psi_M\left(\Phi(\mathcal{K}_u(t))\right)}\,\mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \left(l_M(t) + \mu_M(t)\,(u'(t))^{\frac{q-1}{q}}\right)\mathrm{d}t \\
&\leq \|l_M\|_{L^1(I)} + \int_{t_1}^{t_2} \mu_M(t)\,(u'(t))^{\frac{q-1}{q}}\,\mathrm{d}t \\
&\quad \left(\text{by Hölder's inequality, since } \mu_M \in L^q(I)\right) \\
&\leq \|l_M\|_{L^1(I)} + \|\mu_M\|_{L^q(I)}\,(u(t_2) - u(t_1))^{\frac{q-1}{q}} \\
&\quad \left(\text{since } |u(t)| \leq M \text{ for all } t \in I, \text{ see Claim 2.}\right) \\
&\leq \|l_M\|_{L^1(I)} + \|\mu_M\|_{L^q(I)}\,(2M)^{\frac{q-1}{1}}.
\end{aligned}$$

This is in contradiction with the choice of L_M in (3.3), and thus $\mathcal{K}_u(t) \leq L_M$ for every $t \in I$. By arguing exactly in the same way one can also show that $\mathcal{K}_u(t) \geq -L_M$ for all $t \in I$, and the claim is completely proved.

Claim 5. *$|u'(t)| \leq L_M/k(t)$ for a.e. $t \in I$, so that*

$$\mathcal{D}_{u'} \equiv u' \textit{ a.e. on } I.$$

By Claim 4 and the very definition of $\mathcal{K}_u$ we get

$$|u'(t)| = \frac{|\mathcal{K}_u(t)|}{k(t)} \leq \frac{L_M}{k(t)} \qquad \text{for a.e. } t \in I;$$

as a consequence, since $L_M/k(t) \leq y_L(t)$ a.e. on I (see (3.4)), from the very definition of $\mathcal{D}$ in (3.5) we conclude that $\mathcal{D}_{u'} \equiv u'$ on I.

Using the results established in the above claims, we can complete the proof of this step. Indeed, by Claim 1. we have $\mathcal{T}_u \equiv u$ and $\mathcal{G}_{\mathcal{T}_u} \equiv \mathcal{G}_u$ on I; moreover, by Claim 5. we know that $\mathcal{D}_{u'} \equiv u'$ a.e. on I. Gathering together all these facts (and since u is a solution of (3.8)), for almost every $t \in I$ we get

$$\left(\Phi(\mathcal{K}_u(t))\right)' = -f(t, \mathcal{G}_{\mathcal{T}_u}(t))\,\rho(t, \mathcal{D}_{\mathcal{T}_u'}(t)) + \arctan\left(u(t) - \mathcal{T}_u(t)\right)$$

$$= -f(t, \mathcal{G}_u(t))\,\rho(t, u'(t)),$$

and thus u solves the ODE (2.6). Furthermore, by (3.7) we have

$$u(a) = \mathcal{B}_a[u] = \mathcal{H}_a[\mathcal{T}_u] = \mathcal{H}_a[u] \qquad \text{and}$$
$$u(b) = \mathcal{B}_b[u] = \mathcal{H}_b[\mathcal{T}_u] = \mathcal{H}_b[u],$$

and this proves that u is a solution of the BVP (2.1).

Thanks to the results in Steps I and II, we are finally in a position to conclude the proof of Theorem 2.6. Indeed, by Step I we know that there exists (at least) one solution $x_0 \in W^{1,1}(I)$ of the truncated BVP (3.8); on the other hand, we derive from Step II that x_0 is actually a solution of (2.1).

To proceed further we observe that, owing to Claim 1. in Step II, we get that x_0 satisfies (2.10); moreover, the result in Step I ensures that

- if $1/k \in L^\vartheta(I)$ for some $1 < \vartheta \le \infty$, then $x_0 \in W^{1,\vartheta}(I)$;
- if $k \in C(I, \mathbb{R})$ and $k > 0$ on I, then $x_0 \in C^1(I, \mathbb{R})$.

Finally, by combining Claims 2. and 5. in Step II, we conclude that x_0 satisfies the 'a-priori' estimate (2.11), and the proof is complete. □

Remark 3.1. By carefully scrutinizing the proof Theorem 2.6, one can recognize that estimate (2.7) in assumption (H6) has been used only for demonstrating the result in Step I, and with the specific choice

$$R = \eta_M \qquad \text{and} \qquad y(t) = y_L(t) = \frac{L_M}{k(t)} + |\alpha'(t)| + |\beta'(t)|.$$

As a consequence, if we know that

$$y_L \in L^\vartheta(I) \qquad (\text{for some } \vartheta > 1), \tag{3.25}$$

assumption (H6) can be replaced by the following weaker one:

(H6)' for every $R > 0$ and every non-negative function $y \in L^\vartheta(I)$ there exists a non-negative function $h = h_{R,y} \in L^1(I)$ such that

$$\begin{gathered} |f(t,z)\,\rho(t,y(t))| \le h_{R,y}(t) \\ \text{for a.e. } t \in I, \text{ every } z \in \mathbb{R} \text{ with } |z| \le R \\ \text{and every } y \in L^\vartheta(I) \text{ such that } |y(s)| \le y(s) \text{ for a.e. } s \in I \end{gathered} \tag{3.26}$$

Notice that (3.25) is certainly satisfied if $1/k \in L^\vartheta(I)$ and if the lower/upper solutions α, β in assumption (H5) can be chosen in $W^{1,\vartheta}(I)$.

4 Some examples

In this last section of the paper we present some 'model BVPs' illustrating the applicability of our existence result in Theorem 2.6.

Example 4.1. Let us consider the following BPV on $I = [0, 1]$

$$\begin{cases} \left(\sinh\left(\sqrt{t(1-t)}\cdot x'(t)\right)\right)' + a\left(\int_0^t x^3(s)\,\mathrm{d}s\right)|x'(t)|^\varrho = 0 \quad \text{a.e. on } I, \\ x(0) = \max\left\{x(1), 1\right\}, \;\; x(1) = \varepsilon\int_0^1 x(s)\,\mathrm{d}s, \end{cases} \tag{4.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a general continuous non-decreasing function and $\varepsilon, \varrho \in (0, 1)$. Problem (4.1) takes the form (2.1), with

(*) $k : I \to \mathbb{R}$, $k(t) := \sqrt{t(1-t)}$;
(*) $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(z) := \sinh(z)$;
(*) $f : I \times \mathbb{R} \to \mathbb{R}$, $f(t,z) := a(z)$;
(*) $\rho : I \times \mathbb{R} \to \mathbb{R}$, $\rho(t,y) := |y|^{\varrho}$;
(*) $\mathcal{G}_x(t) := \int_0^t x^3(s)\,\mathrm{d}s$ (for $x \in W^{1,1}(I)$);
(*) $\mathcal{H}_0[x] := \max\{x(1), 1\}$ and $\mathcal{H}_1[x] := \varepsilon \int_0^1 x(s)\,\mathrm{d}s$ (for $x \in W^{1,1}(I)$).

We aim to show that *all* the assumptions of Theorem 2.6 are satisfied in this case, so that problem (4.1) possesses (at least) one solution $x_0 \in W^{1,1}(I)$. We explicitly point out that, in view of the boundary conditions, x_0 *cannot be constant*.

To begin with, we observe that assumption (H1) is trivially satisfied, since

$$1/k \in L^{\vartheta}(I) \text{ for all } \vartheta \in [1,2). \tag{4.2}$$

As regards assumptions (H2)-(H3), we first notice that $\mathcal{G}$ is a continuous operator mapping $W^{1,1}(I)$ into $C^1(I,\mathbb{R})$; as a consequence, owing to Remark 2.1, we know that $\mathcal{G}$ is continuous from $W^{1,1}(I)$ into $L^{\infty}(I)$ (with the usual norms).

Furthermore, if $r > 0$ is any fixed positive number, we have

$$\|\mathcal{G}_x\|_{L^{\infty}(I)} \le \int_0^1 |x(t)|^3\,\mathrm{d}t \le r^3 \quad \text{for all } x \in W^{1,1} \text{ with } \|x\|_{L^{\infty}} \le r,$$

and thus (2.3) is satisfied with $\eta_r := r^3$. Finally, since a is non-decreasing and $\mathcal{G}$ is increasing (with respect to the point-wise order), it follows that

$$W^{1,1}(I) \ni x \mapsto f(t, \mathcal{G}_x(t)) = a\Big(\int_0^t x^3(s)\,\mathrm{d}s\Big)$$

is monotone increasing, so that (2.4) holds with $\kappa = 0$.

As regards assumption (H4), it is easy to check that $\mathcal{H}_0$, $\mathcal{H}_1$ are continuous from $W^{1,1}(I)$ to $\mathbb{R}$ (remind that $W^{1,1}(I)$ is continuously embedded into $C(I,\mathbb{R})$); moreover, if $x, y \in W^{1,1}(I)$ are such that $x \le y$ point-wise on I, then

$$\mathcal{H}_0[x] = \max\{x(1), 1\} \le \max\{y(1), 1\} = \mathcal{H}_0[y] \qquad \text{and}$$
$$\mathcal{H}_1[x] = \varepsilon \int_0^1 x(s)\,\mathrm{d}s \le \varepsilon \int_0^1 y(s)\,\mathrm{d}s = \mathcal{H}_1[y],$$

so that $\mathcal{H}_0$, $\mathcal{H}_1$ are also monotone increasing (w.r.t. to the point-wise order).

We now turn to prove the validity of assumptions (H5)-to-(H7).

Assumption (H5). We claim that the constant functions

$$\alpha(t) := -1 \qquad \text{and} \qquad \beta(t) := 1$$

are, respectively, a lower and an upper solution of problem (4.1).

In fact, since $\rho(t,0) = 0$ for all $t \in I$, we know from Remark 2.7 that α and β are both lower and upper solutions of the *differential equation*

$$\big(\sqrt{t(1-t)} \cdot x'(t)\big)' + a\Big(\int_0^t x^3(s)\,\mathrm{d}s\Big)\,|x'(t)|^{\varrho} = 0;$$

moreover, owing to the very definitions of $\mathcal{H}_0$ and $\mathcal{H}_1$ we have

(a) $\alpha(0) = -1 \le 1 = \mathcal{H}_0[\alpha]$ and $\alpha(1) = -1 \le -\varepsilon = \mathcal{H}_1[\alpha]$;

(b) $\beta(0) = 1 = \mathcal{H}_0[\beta]$ and $\beta(1) = 1 \ge \varepsilon = \mathcal{H}_1[\beta]$.

On account of Definition 2.4, from (a)-(b) we get that α is a lower solution and β is an upper solution of *problem* (4.1).

ASSUMPTION (H6). Let $R > 0$ be fixed and let y be a non-negative function belonging to $L^1(I)$. Since, by assumption, $a \in C(\mathbb{R}, \mathbb{R})$, we have

$$|f(t,z)\,\rho(t,y(t))| = |a(z)| \cdot |y(t)|^{\varrho} \le \big(\max_{|z|\le R}|a(z)|\big) \cdot \mathrm{y}(t)^{\varrho} =: h_{R,\mathrm{y}}(t)$$
$$\text{for a.e. } t \in I, \text{ every } z \in \mathbb{R} \text{ with } |z| \le R$$
$$\text{and every } y \in L^1(I) \text{ such that } |y(s)| \le \mathrm{y}(s) \text{ for a.e. } s \in I.$$

As a consequence, since $h_{R,\mathrm{y}} \in L^1(I)$ (remind that, by assumption, $0 < \varrho < 1$), we get that estimate (2.7) is satisfied.

ASSUMPTION (H7). Let $R > 0$ be arbitrarily fixed. Since, by assumption, $a \in C(\mathbb{R}, \mathbb{R})$, we have the following estimate

$$|f(t,z)\,\rho(t,y)| = |a(z)| \cdot |y|^{\varrho} \le \big(\max_{|z|\le R}|a(z)|\big) \cdot |y|^{\varrho},$$

holding true for a.e. $t \in I$, every $z \in [-R, R]$ and every $y \in \mathbb{R}$. As a consequence, we conclude that estimate (2.9) is satisfied with the choice

$$H_R = 1, \quad \psi_R \equiv 1, \quad l_R(t) \equiv 0, \quad \mu_R := \big(\max_{|z|\le R}|a(z)|\big), \quad q = \frac{1}{1-\varrho}.$$

Since all the assumptions of Theorem 2.6 are fulfilled, we can conclude that there exists (at least) one solution $x_0 \in W^{1,1}(I)$ of problem (4.1), further satisfying

$$-1 \le x_0(t) \le 1 \qquad \text{for every } t \in I.$$

Moreover, from (4.2) we deduce that $x_0 \in W^{1,\vartheta}(I)$ for all $\vartheta \in [1, 2)$.

Example 4.2. Let $\vartheta_0 \in (1, \infty)$ be fixed, and let $\tau \in (0, 2\pi)$. Moreover, let $p, \delta \in \mathbb{R}$ be two positive real numbers satisfying the following relation

$$1 < p < \vartheta_0 + 1 \qquad \text{and} \qquad 0 < \delta < p - \frac{p-1}{\vartheta_0}. \tag{4.3}$$

Finally, let $\Phi_p(z) := |z|^{p-2}z$ be usual p-Laplace operator on $\mathbb{R}$. We then consider the following BVP on $I = [0, 2\pi]$

$$\begin{cases} \big(\Phi_p\big(|\sin(t)|^{1/\vartheta_0}\,x'(t)\big)\big)' + x_\tau(t)\,|x'(t)|^{\delta} = 0 & \text{a.e. on } I, \\ x(0) = \sqrt[3]{x(\pi)}, \ \ x(2\pi) = \frac{1}{4\pi}\int_0^{2\pi}(x(s)+2)\,\mathrm{d}s \end{cases} \tag{4.4}$$

where x_τ is the delay-type function defined as

$$x_\tau(t) := \begin{cases} x(t-\tau), & \text{if } \tau \le t \le 2\pi, \\ x(0), & \text{if } 0 \le t < \tau. \end{cases}$$

Problem (4.4) takes the form (2.1), with

(⋆) $k : I \to \mathbb{R}, \quad k(t) := |\sin(t)|^{1/\vartheta_0}$;

(⋆) $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(z) = \Phi_p(z) = |z|^{p-2}z$;

(⋆) $f : I \times \mathbb{R} \to \mathbb{R}, \quad f(t,z) := z$;

(⋆) $\rho : I \times \mathbb{R} \to \mathbb{R}, \quad \rho(t,y) := |y|^{\delta}$;

(⋆) $\mathcal{G}x(t) := x_\tau$ (for $x \in W^{1,1}(I)$);

(⋆) $\mathcal{H}_0[x] := \sqrt[3]{x(\pi)}$ and $\mathcal{H}_{2\pi}[x] := \frac{1}{4\pi}\int_0^{2\pi}(x(s)+2)\,\mathrm{d}s$ (for $x \in W^{1,1}(I)$).

We aim to show that *all* the assumptions of Theorem 2.6 are satisfied in this case, so that problem (4.4) possesses (at least) one solution $x_0 \in W^{1,1}(I)$. We explicitly point out that, in view of the boundary conditions, x_0 *cannot be constant*.

To begin with, we observe that assumption (H1) is trivially satisfied, since

$$1/k \in L^{\vartheta}(I) \text{ for all } \vartheta \in [1, \vartheta_0). \tag{4.5}$$

As regards assumptions (H2)-(H3), we first notice that $\mathcal{G}$ is a well-defined linear operator mapping $W^{1,1}(I)$ into $L^\infty(I)$; as a consequence, since

$$\|\mathcal{G}x\|_{L^\infty(I)} \le \|x\|_{L^\infty(I)} \le C\,\|x\|_{W^{1,1}(I)} \qquad \text{for every } x \in W^{1,1}(I),$$

we get that $\mathcal{G}$ is continuous from $W^{1,1}(I)$ into $L^\infty(\mathbb{R})$. Furthermore, if $r > 0$ is any fixed positive number, we also have

$$\|\mathcal{G}x\|_{L^\infty(I)} \le \|x\|_{L^\infty(I)} \le r \quad \text{for all } x \in W^{1,1} \text{ with } \|x\|_{L^\infty} \le r,$$

and thus (2.3) is satisfied with $\eta_r := r$. Finally, since $\mathcal{G}$ is monotone increasing with respect to the point-wise order (as it is easy to check) and since $f(t,z) = z$, one straightforwardly derives that (2.4) holds with $\kappa = 0$.

As regards assumption (H4), it is easy to check that $\mathcal{H}_0$, $\mathcal{H}_{2\pi}$ are continuous from $W^{1,1}(I)$ to $\mathbb{R}$ (remind that $W^{1,1}(I)$ is continuously embedded into $C(I,\mathbb{R})$); moreover, if $x, y \in W^{1,1}(I)$ are such that $x \le y$ point-wise on I, then

$$\mathcal{H}_0[x] = \sqrt[3]{x(\pi)} \le \sqrt[3]{y(\pi)} = \mathcal{H}_0[y] \qquad \text{and}$$

$$\mathcal{H}_{2\pi}[x] = \frac{1}{4\pi}\int_0^{2\pi}(x(s)+2)\,\mathrm{d}s \le \frac{1}{4\pi}\int_0^{2\pi}(y(s)+2)\,\mathrm{d}s = \mathcal{H}_{2\pi}[y],$$

so that $\mathcal{H}_0$, $\mathcal{H}_{2\pi}$ are also monotone increasing (w.r.t. to the point-wise order).

We now turn to prove the validity of assumptions (H5)-to-(H7).

Assumption (H5). We claim that the constant functions

$$\alpha(t) := 1 \qquad \text{and} \qquad \beta(t) := 2$$

are, respectively, a lower and an upper solution of problem (4.1).

In fact, since $\rho(t,0) = 0$ for all $t \in I$, we know from Remark 2.7 that α and β are both lower and upper solutions of the *differential equation*

$$\left(\Phi_p\left(|\sin(t)|^{1/\vartheta_0}\, x'(t)\right)\right)' + x_\tau(t)\,|x'(t)|^{\delta} = 0;$$

moreover, owing to the very definitions of $\mathcal{H}_0$ and $\mathcal{H}_{2\pi}$ we have

(a) $\alpha(0) = 1 = \mathcal{H}_0[\alpha]$ and $\alpha(2\pi) = 1 < 3/2 = \mathcal{H}_{2\pi}[\alpha]$;

(b) $\beta(0) = 2 > \mathcal{H}_0[\beta]$ and $\beta(2) = 2 = \mathcal{H}_{2\pi}[\beta]$.

On account of Definition 2.4, from (a)-(b) we get that α is a lower solution and β is an upper solution of *problem* (4.4).

Assumption (H6). We first observe that, by (4.3), we have

$$0 < \delta < \vartheta_0;$$

thus, setting $\vartheta := \max\{1, \delta\} \in [1, \vartheta_0)$, by (4.5) we have $1/k \in L^{\vartheta}(I)$. On the other hand, since α, β are constant, one has

$$\alpha, \beta \in W^{1,\vartheta}(I);$$

as a consequence, according to Remark 3.1, it suffices to demonstrate that assumption (H6) holds in the weaker form (H6)' (with $\vartheta = \max\{1, \delta\}$).

Let then $R > 0$ be fixed and let y be a non-negative function belonging to the space $L^{\vartheta}(I)$. Reminding that $f(t, z) = z$, we have the following computation

$$|f(t, z)\,\rho(t, y(t))| = |z| \cdot |y(t)|^{\delta} \le R \cdot y(t)^{\delta} =: h_{R,y}(t)$$

$$\text{for a.e. } t \in I, \text{ every } z \in \mathbb{R} \text{ with } |z| \le R$$
$$\text{and every } y \in L^1(I) \text{ such that } |y(s)| \le y(s) \text{ for a.e. } s \in I.$$

From this, since $h_{R,y} \in L^1(I)$ (remind that, by definition, $\delta \le \vartheta$), we get that estimate (3.26) is satisfied.

Assumption (H7). Let $R > 0$ be arbitrarily fixed. Since $k \in C(I, \mathbb{R})$ (and since $f(t, z) = z$), we have the following computation

$$\begin{aligned}|f(t, z)\,\rho(t, y)| &= |z| \cdot |y|^{\delta} \le \frac{R}{k(t)^{\delta}} \cdot |k(t)y|^{\delta}\\ &\Big(\text{by (4.3), setting } q := \frac{\vartheta_0}{p-1} > 1\Big)\\ &\le |k(t)y|^{p-1}\Big(\frac{R}{k(t)^{\delta}} \cdot |k(t)y|^{\frac{q-1}{q}}\Big)\\ &= \Phi_p\big(|k(t)y|\big) \cdot \Big(\frac{R}{k(t)^{\delta+1/q-1}}\Big)\,|y|^{\frac{q-1}{q}}\end{aligned}$$

holding true for a.e. $t \in I$, every $z \in [-R, R]$ and every $y \in \mathbb{R}$ with $|k(t)y| \ge 1$. As a consequence, if we are able to demonstrate that

$$t \mapsto \frac{R}{k(t)^{\delta+1/q-1}} \in L^q(I), \tag{4.6}$$

we conclude that estimate (2.9) is satisfied with the choice

$$H_R = 1, \quad \psi_R(s) = s, \quad l_R(t) \equiv 0, \quad \mu_R(t) = \frac{R}{k(t)^{\delta+1/q-1}}, \quad q = \frac{\vartheta_0}{p-1}.$$

In its turn, the needed (4.6) follows from (4.5) and from the fact that

$$q(\delta + 1/q - 1) = \frac{\vartheta_0}{p-1}\Big(\delta + \frac{p-1}{\vartheta_0} - 1\Big) < \frac{\vartheta_0}{p-1} \cdot (p-1) = \vartheta_0.$$

Since all the assumptions of Theorem 2.6 are fulfilled, we can conclude that there exists (at least) one solution $x_0 \in W^{1,1}(I)$ of problem (4.4), further satisfying

$$1 \le x_0(t) \le 2 \qquad \text{for every } t \in I.$$

Moreover, from (4.2) we deduce that $x_0 \in W^{1,\vartheta}(I)$ for all $\vartheta \in [1, \vartheta_0)$.

Example 4.3. Let $d_1, d_2 \in (0, \infty)$ be arbitrarily fixed, and let $I = [-1, 1]$. Denoting by χ_A the indicator function of a set $A \subseteq \mathbb{R}$, we define

$$\kappa(t) := d_1 \cdot \chi_{[-1,0]}(t) + d_2 \cdot \chi_{[0,1]}(t). \tag{4.7}$$

We then consider the following BVP:

$$\begin{cases} \Big(\big(\kappa(t)\,x'(t)\big)^3\Big)' + \big(\max\limits_{s\in[-1,t]} x(s)\big)\cdot\log\big(1+|\sqrt[3]{t}\,x'(t)|^2\big) = 0 \quad \text{a.e. on } I, \\ x(-1)=0,\;\; x(1)=1. \end{cases} \tag{4.8}$$

Problem (4.8) takes the form (2.1), with

(⋆) $k: I \to \mathbb{R}, \quad k(t) := \kappa(t) = d_1 \cdot \chi_{[-1,0]}(t) + d_2 \cdot \chi_{[0,1]}(t)$;
(⋆) $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(z) := z^3$;
(⋆) $f : I \times \mathbb{R} \to \mathbb{R}, \quad f(t,z) := z$;
(⋆) $\rho : I \times \mathbb{R} \to \mathbb{R}, \quad \rho(t,y) := \log\big(1+|\sqrt[3]{t}\,y|^2\big)$;
(⋆) $\mathcal{G}_x(t) := \max_{s\in[-1,t]} x(s)$ (for $x \in W^{1,1}(I)$);
(⋆) $\mathcal{H}_{-1}[x] := 0$ and $\mathcal{H}_1[x] := 1$ (for $x \in W^{1,1}(I)$).

We aim to show that *all* the assumptions of Theorem 2.6 are satisfied in this case, so that problem (4.8) possesses (at least) one solution $x_0 \in W^{1,1}(I)$. We explicitly point out that, in view of the boundary conditions, x_0 *cannot be constant*.

To begin with, we observe that assumption (H1) is trivially satisfied, since

$$1/k \in L^{\vartheta}(I) \text{ for all } 1 \le \vartheta \le \infty. \tag{4.9}$$

As regards assumptions (H2)-(H3), we first notice that $\mathcal{G}$ is a well-defined operator mapping $W^{1,1}(I)$ into $L^\infty(I)$; as a consequence, since we have

$$\|\mathcal{G}_x - \mathcal{G}_y\|_{L^\infty(I)} \le \|x-y\|_{L^\infty(I)} \le C\|x-y\|_{W^{1,1}(I)},$$

we immediately derive that $\mathcal{G}$ is continuous from $W^{1,1}(I)$ into $L^\infty(I)$ (with the usual norms). Furthermore, if $r > 0$ is any fixed positive number, we have

$$\|\mathcal{G}_x\|_{L^\infty(I)} \le \|x\|_{L^\infty(I)} \le r \quad \text{for all } x \in W^{1,1} \text{ with } \|x\|_{L^\infty} \le r,$$

and thus (2.3) is satisfied with $\eta_r := r$. Finally, since $\mathcal{G}$ is monotone increasing with respect to the point-wise order and since $f(t,z) = z$, by arguing as in Example 4.2 we derive that (2.4) holds with $\kappa = 0$.

As regards assumption (H4), since $\mathcal{H}_{-1}$ and $\mathcal{H}_1$ are *constant*, it follows that these operators are continuous (from $W^{1,1}(I)$ to $\mathbb{R}$) and monotone increasing (w.r.t. to the point-wise order).

We now turn to prove the validity of assumptions (H5)-to-(H7).

Assumption (H5). We claim that the constant functions

$$\alpha(t) := 0 \qquad \text{and} \qquad \beta(t) := 1$$

are, respectively, a lower and an upper solution of problem (4.1).

In fact, since $\rho(t,0) = 0$ for all $t \in I$, we know from Remark 2.7 that α and β are both lower and upper solutions of the *differential equation*

$$\Big(\big(\kappa(t)\,x'(t)\big)^3\Big)' + \big(\max_{s\in[-1,t]} x(s)\big)\cdot\log\big(1+|\sqrt[3]{t}\,x'(t)|^2\big) = 0;$$

moreover, since $\mathcal{H}_{-1} \equiv 0$ and $\mathcal{H}_1 \equiv 1$, we immediately derive that α is a lower solution and β is an upper solution of *problem* (4.8).

Assumption (H6). We first observe that, on account of (4.9), we have (in particular) $1/k \in L^2(I)$; moreover, since α, β are constant, one also has

$$\alpha, \beta \in W^{1,2}(I);$$

As a consequence, according to Remark 3.1, it suffices to demonstrate that assumption (H6) holds in the weaker form (H6)' (with $\vartheta = 2$).

Let then $R > 0$ be fixed and let y be a non-negative function belonging to $L^2(I)$. Since $\log(1+\tau) \le \tau$ for every $\tau \ge 0$, we have

$$|f(t,z)\,\rho(t,y(t))| = |z| \cdot \log\left(1 + |\sqrt[3]{t}\,y(t)|^2\right) \le R\,|\sqrt[3]{t}\,y(t)|^2 \le R\,y(t)^2 =: h_{R,y}(t)$$

$$\text{for a.e. } t \in I = [-1,1], \text{ every } z \in \mathbb{R} \text{ with } |z| \le R$$
$$\text{and every } y \in L^1(I) \text{ such that } |y(s)| \le y(s) \text{ for a.e. } s \in I.$$

As a consequence, since we have $h_{R,y} = Ry^2 \in L^1(I)$ (as $y \in L^2(I)$), we immediately conclude that estimate (2.7) is satisfied.

Assumption (H7). Let $R > 0$ be arbitrarily fixed. Using once again the fact that $\log(1+\tau) \le \tau$ for all $\tau \ge 0$, and since $z^3 \ge z^2$ if $z \ge 1$, we get the estimate

$$|f(t,z)\,\rho(t,y)| = |z| \cdot \log\left(1 + |\sqrt[3]{t}\,y|^2\right) \le R\,|\sqrt[3]{t}\,y|^2 \le R\,|y|^2$$
$$(\text{setting } d := \min\{d_1, d_2\} > 0)$$
$$\le \frac{R}{d^2} \cdot |k(t)y|^2 \le \frac{R}{d^2} \cdot \Phi\big(|k(t)y|\big),$$

holding true for a.e. $t \in I$, every $z \in [-R, R]$ and every $y \in \mathbb{R}$ with $|k(t)\,y| \ge 1$. As a consequence, we conclude that estimate (2.9) is satisfied with the choice

$$H_R = 1, \quad \psi_R(s) = s, \quad l_R(t) := \frac{R}{d^2}, \quad \mu_R \equiv 0.$$

Since all the assumptions of Theorem 2.6 are fulfilled, we can conclude that there exists (at least) one solution $x_0 \in W^{1,1}(I)$ of problem (4.8), further satisfying

$$0 \le x_0(t) \le 1 \qquad \text{for every } t \in I.$$

Moreover, from (4.9) we deduce that $x_0 \in W^{1,\vartheta}(I)$ for all $\vartheta \in [1,\infty]$. In particular, x_0 is Lipschitz-continuous on I, but not of class C^1 (if $d_1 \ne d_2$).

A Continuity of truncating operators

In this Appendix we prove in detail some properties of the truncating operator. Despite these results are probably very classical, we were not be able to locate a precise reference in the literature; thus, we present here a complete demonstration for the sake of completeness.

To begin with, we fix a pair of functions $\omega, \zeta \in L^1(I)$ satisfying the ordering relation $\omega(t) \le \zeta(t)$ a.e. in I, and we introduce the truncating operator

$$\mathcal{T}^{\omega,\zeta} : L^1(I) \to L^1(I), \qquad \mathcal{T}^{\omega,\zeta}_x(t) = \max\left\{\omega(t), \min\{x(t), \zeta(t)\}\right\}.$$

We then prove the following result.

Lemma A.1. *For every $x, y \in L^1(I)$, one has*

$$\left|\mathcal{T}^{\omega,\zeta}_x(t) - \mathcal{T}^{\omega,\zeta}_y(t)\right| \le |x(t) - y(t)|. \tag{A.1}$$

Moreover, if we further assume that $\omega, \zeta \in W^{1,1}(I)$, we have

(i) $\mathcal{T}^{\omega,\zeta}\big(W^{1,1}(I)\big) \subseteq W^{1,1}(I)$.

(ii) $\mathcal{T}^{\omega,\zeta}$ *is continuous from* $W^{1,1}(I)$ *into itself (with respect to the usual norm).*

Proof. We limit ourselves to prove only assertion (ii), since (A.1) is trivial and (i) is an immediate consequence of (A.1) and the well-known characterization of $W^{1,1}(I)$ in terms of absolutely continuous functions (see, e.g., [10]).

First of all we observe that, if we introduce the operators

$$\begin{aligned} &\mathrm{M} : W^{1,1}(I) \to W^{1,1}(I), \qquad \mathrm{M}_x(t) := \max\{\omega(t), x(t)\}, \\ &\mathrm{m} : W^{1,1}(I) \to W^{1,1}(I), \qquad \mathrm{m}_x(t) := \min\{\zeta(t), x(t)\}, \end{aligned} \tag{A.2}$$

the operator $\mathcal{T}^{\omega,\zeta}$ is the composition between m and M, that is,

$$\mathcal{T}_x^{\omega,\zeta} = (\mathrm{M} \circ \mathrm{m})(x) \qquad \text{for all } x \in W^{1,1}(I).$$

As a consequence, to prove the lemma it suffices to show that *both* M and m are continuous on $W^{1,1}(I)$. Here we limit ourselves to demonstrate this fact only for the operator M, since the case of m goes along the same lines.

Let then $x_0 \in W^{1,1}(I)$ be fixed, and let $\{x_n\}_n \subseteq W^{1,1}(I)$ be a sequence converging to x_0 as $n \to \infty$ in $W^{1,1}(I)$. Moreover, let $\{y_k := x_{n_k}\}_k$ be an arbitrary sub-sequence of $\{x_n\}_n$. To prove the continuity of M we show that, by choosing a further sub-sequence if necessary, one has

$$\lim_{k\to\infty} \mathrm{M}_{y_k} = x_0 \qquad \text{in } W^{1,1}(I). \tag{A.3}$$

To ease the readability, we split the demonstration of (A.3) into some steps.

Step I. In this step we show that

$$\lim_{k\to\infty} \|\mathrm{M}_{y_k} - \mathrm{M}_{x_0}\|_{L^1(I)} = 0. \tag{A.4}$$

To this end, we first notice that, since $y_k \to x_0$ in $W^{1,1}(I)$ as $k \to \infty$, we also have that y_k converges *uniformly on* I to x_0 as $k \to \infty$ (see, e.g., [10, Theorem 8.8]); as a consequence, from the estimate

$$\|\mathrm{M}_{y_k} - \mathrm{M}_{x_0}\|_{L^\infty(I)} \le \|y_k - x_0\|_{L^\infty(I)},$$

we deduce that $\mathrm{M}_{y_k} \to \mathrm{M}_{x_0}$ uniformly on I as $k \to \infty$, and (A.4) follows.

Step II. In this step we show that, up to a sub-sequence, one has

$$\lim_{k\to\infty} \mathrm{M}'_{y_k}(t) = \mathrm{M}'_{x_0}(t) \qquad \text{a.e. on } I. \tag{A.5}$$

To this end, we first fix some notation which shall be useful in the sequel. Given any point $t_0 \in (a, b)$ and any $\rho > 0$, we set

$$I(t_0, \rho) := [t_0 - \rho, t_0 + \rho];$$

moreover, given any function $\xi \in W^{1,1}(I)$, we define

$$\mathcal{N}_\xi := \{t \in (a, b) : \xi \text{ is not differentiable at } t\}. \tag{A.6}$$

Notice that, since $\xi \in W^{1,1}(I)$, the set $\mathcal{N}_\xi$ has zero Lebesgue measure.

We now start with the proof of (A.5). First of all, since $y_k \to x_0$ in $W^{1,1}(I)$ as $k \to \infty$, we clearly have that $y'_k \to x'_0$ in $L^1(I)$ (as $k \to \infty$); as a consequence, it is possible to find a non-negative function $g \in L^1(I)$ and a set $\mathcal{Z} \subseteq I$, with vanishing Lebesgue measure, such that (up to a sub-sequence)

(i) $y'_k(t) \to x'_0(t)$ as $k \to \infty$ for every $t \in I \setminus \mathcal{Z}$;

(ii) $|y_k'(t)| \leq g(t)$ for every $k \in \mathbb{N}$ and every $t \in I \setminus \mathcal{Z}$.

With reference to (A.6), we consider the following set:

$$\mathcal{N} := \bigcup_{k\in\mathbb{N}} \mathcal{N}_{y_k} \cup \bigcup_{k\in\mathbb{N}} \mathcal{N}_{\mathrm{M}_{y_k}} \cup \mathcal{N}_{x_0} \cup \mathcal{N}_{\mathrm{M}_{x_0}} \cup \mathcal{N}_\omega \cup \mathcal{Z}. \tag{A.7}$$

Since it is a countable union of sets with zero Lebesgue measure, the set $\mathcal{N}$ has zero Lebesgue measure as well; thus, to prove (A.5) it suffices to show that

$$\lim_{k\to\infty} \mathrm{M}_{y_k}'(t) = \mathrm{M}_{x_0}'(t) \qquad \text{for all } t \in (a,b) \setminus \mathcal{N}. \tag{A.8}$$

Let then $\theta_0 \in (a,b) \setminus \mathcal{N}$ be arbitrary but fixed. We demonstrate the claimed (A.8) by analyzing separately the following three possibilities.

(1) $x_0(\theta_0) < \omega(\theta_0)$. In this case, we let $\rho > 0$ be so small that

$$I(\theta_0,\rho) \subseteq (a,b) \text{ and } x_0 < \omega \text{ on } I(\theta_0,\rho). \tag{A.9}$$

Owing to the very definition of M in (A.2), we get $\mathrm{M}_{x_0} \equiv \omega$ on $I(\theta_0,\rho)$; moreover, since ω is differentiable in $\theta_0 \notin \mathcal{N}_\omega$, we have

$$\mathrm{M}_{x_0}'(\theta_0) = \omega'(\theta_0). \tag{A.10}$$

Now, since we know from Step I that y_k converges *uniformly on I* to x_0 as $k \to \infty$, by (A.9) we can find a natural number κ_0 such that

$$y_k(t) < \omega(t) \qquad \text{for } t \in I(\theta_0,\rho) \text{ and every } k \geq \kappa_0;$$

thus, again by definition of M we deduce that $\mathrm{M}_{y_k} \equiv \omega$ on $I(\theta_0,\rho)$ for all $k \geq \kappa_0$. In particular, ω being differentiable at θ_0 we have

$$\mathrm{M}_{y_k}'(\theta_0) = \omega'(\theta_0) \qquad \text{for every } k \geq \kappa_0. \tag{A.11}$$

Gathering together (A.10) and (A.11), we then obtain (A.8) in this case.

(2) $x_0(\theta_0) > \omega(\theta_0)$. In this case, we let $\rho > 0$ be so small that

$$I(\theta_0,\rho) \subseteq (a,b) \text{ and } x_0 > \omega \text{ on } I(\theta_0,\rho). \tag{A.12}$$

Owing to the very definition of M in (A.2), we get $\mathrm{M}_{x_0} \equiv x_0$ on $I(\theta_0,\rho)$; moreover, since x_0 is differentiable in $\theta_0 \notin \mathcal{N}_{x_0}$, we have

$$\mathrm{M}_{x_0}'(\theta_0) = x_0'(\theta_0). \tag{A.13}$$

Now, using (A.12) and arguing again as in (1), we can find $\kappa_0 \in \mathbb{N}$ such that

$$\mathrm{M}_{y_k} \equiv y_k \text{ on } I(\theta_0,\rho) \text{ for all } k \geq \kappa_0;$$

in particular, y_k being differentiable at θ_0 for all $k \in \mathbb{N}$, we have

$$\mathrm{M}_{y_k}'(\theta_0) = y_k'(\theta_0) \qquad \text{for every } k \geq \kappa_0. \tag{A.14}$$

Since $\theta_0 \notin \mathcal{Z}$ and since $y_k' \to x_0'$ as $k \to \infty$ on $I \setminus \mathcal{Z}$, by combining (A.13) with (A.14) we readily conclude that (A.8) holds also in this case.

(3) $x_0(\theta_0) = \omega(\theta_0)$. First of all, since for every $k \in \mathbb{N}$ the function M_{y_k} is differentiable at θ_0 (as $\theta_0 \notin \mathcal{N}$, see (A.7)), by the very definition of M we have

$$\mathrm{M}_{y_k}'(\theta_0) \in \{\omega'(\theta_0),\ y_k'(\theta_0)\}.$$

As a consequence, since we know that $y_k'(\theta_0) \to x_0'(\theta_0)$ as $k \to \infty$ (as $\theta_0 \notin \mathcal{Z}$, see (i) at the beginning of this step), to prove (A.8) it suffices to show that

$$\mathrm{M}_{x_0}'(\theta_0) = x_0'(\theta_0) = \omega'(\theta_0). \tag{A.15}$$

To establish (A.15) we need to consider three different sub-cases.

$(3)_1$ $\theta_0 \notin \partial\{x_0 > \omega\}$. In this case, since $\mathcal{O} := \{x_0 > \omega\}$ is open and $\theta_0 \notin \mathcal{O}$, there exists $\rho > 0$ such that $x_0 \leq \omega$ on $I(\theta_0, \rho)$; thus, by (A.2) we have

$$\mathrm{M}_{x_0} \equiv \omega \text{ on } I(\theta_0, \rho).$$

Since ω is differentiable at θ_0 (as $\theta_0 \notin \mathcal{N}_\omega$), we then obtain

$$\mathrm{M}'_{x_0}(\theta_0) = \omega'(\theta_0). \tag{A.16}$$

On the other hand, since $x_0 - \omega \leq 0$ on $I(\theta_0, \rho)$ and $x_0(\theta_0) = \omega(\theta_0)$, we see that θ_0 is an interior maximum point for $x_0 - \omega$ on $I(\theta_0, \rho)$; this function being differentiable at θ_0, we then conclude that

$$x'_0(\theta_0) = \omega'(\theta_0). \tag{A.17}$$

Gathering together (A.16) and (A.17), we obtain (A.15) in this case.

$(3)_2$ $\theta_0 \notin \partial\{x_0 < \omega\}$. In this case, since $\mathcal{O} := \{x_0 < \omega\}$ is open and $\theta_0 \notin \mathcal{O}$, there exists $\rho \in (0, \rho_0)$ such that $x_0 \geq \omega$ on $I(\theta_0, \rho)$; thus, by (A.2) we have

$$\mathrm{M}_{x_0} \equiv x_0 \text{ on } I(\theta_0, \rho_0).$$

From this, by arguing exactly as in case $(3)_1$, we obtain (A.15).

$(3)_3$ $\theta_0 \in \partial\{x_0 < \omega\} \cap \partial\{x_0 > \omega\}$. In this last case, both the open sets

$$\mathcal{O}^+ = \{x_0 > \omega\} \qquad \text{and} \qquad \mathcal{O}^- = \{x_0 < \omega\}$$

are non-empty and $\theta_0 \in \partial(\mathcal{O}^+) \cap \partial(\mathcal{O}^-)$; thus, by crucially exploiting the fact that the functions M_{x_0} and x_0 are differentiable at θ_0, we can write

$$\begin{aligned}
\mathrm{M}'_{x_0}(\theta_0) &= \lim_{\substack{t\to\theta_0 \\ t\in I}} \frac{\mathrm{M}_{x_0}(t) - \mathrm{M}_{x_0}(\theta_0)}{t - \theta_0} \\
&\quad \big(\text{since } \mathrm{M}_{x_0}(\theta_0) = x_0(\theta_0) = \omega(\theta_0), \text{ see (A.2)}\big) \\
&= \lim_{\substack{t\to\theta_0 \\ t\in \mathcal{O}^+}} \frac{\mathrm{M}_{x_0}(t) - x_0(\theta_0)}{t - \theta_0} \\
&\quad \big(\text{since } x_0 > \omega \text{ on } \mathcal{O}^+\big) \\
&= \lim_{\substack{t\to\theta_0 \\ t\in \mathcal{O}^+}} \frac{x_0(t) - x_0(\theta_0)}{t - \theta_0} = x'_0(\theta_0).
\end{aligned}$$

On the other hand, using the fact that ω is differentiable at θ_0, we also have

$$\begin{aligned}
\mathrm{M}'_{x_0}(\theta_0) &= \lim_{\substack{t\to\theta_0 \\ t\in I}} \frac{\mathrm{M}_{x_0}(t) - \mathrm{M}_{x_0}(\theta_0)}{t - \theta_0} \\
&= \lim_{\substack{t\to\theta_0 \\ t\in \mathcal{O}^-}} \frac{\mathrm{M}_{x_0}(t) - \omega(\theta_0)}{t - \theta_0} \\
&\quad \big(\text{since } x_0 < \omega \text{ on } \mathcal{O}^-\big) \\
&= \lim_{\substack{t\to\theta_0 \\ t\in \mathcal{O}^-}} \frac{\omega(t) - \omega(\theta_0)}{t - \theta_0} = \omega'(\theta_0).
\end{aligned}$$

Gathering together these two facts, we conclude that

$$\mathrm{M}'_{x_0}(\theta_0) = x'_0(\theta_0) = \omega'(\theta_0), \tag{A.18}$$

which is exactly the desired (A.15).

Step III. In this step we prove that, up to a sub-sequence, one has

$$\lim_{k\to\infty} \|\mathrm{M}'_{y_k} - \mathrm{M}'_{x_0}\|_{L^1(I)} = 0. \tag{A.19}$$

To begin with, by exploiting the results in Step II, we know that there exists a set $\mathcal{N} \subseteq (a, b)$, with zero Lebesgue measure, such that (up to a sub-sequence)

(a) $y'_k \to x'_0$ point-wise on $I \setminus \mathcal{N}$;

(b) $|y'_k| \le g$ on $I \setminus \mathcal{N}$ for a suitable function $g \in L^1(I)$;

(c) $\mathrm{M}'_{y_k} \to \mathrm{M}'_{x_0}$ point-wise on $I \setminus \mathcal{N}$.

In particular, since for every $k \in \mathbb{N}$ we have

$$\mathrm{M}'_{y_k} \in \{\omega'(t),\, y'_k(t)\} \qquad \text{a.e. on } I,$$

from (b) we obtain the following estimate

$$|\mathrm{M}'_{y_k}(t)| \le |\omega'(t)| + g(t) =: \xi(t), \qquad \text{for a.e. } t \in I. \tag{A.20}$$

By combining (A.20) with (c) we can perform a standard dominated-convergence argument, proving the claimed (A.19).

Step IV. In this last step we complete the demonstration of the lemma. By combining (A.4) in Step I with (A.19) in Step III, we straightforwardly get

$$\lim_{k\to\infty} \|\mathrm{M}_{y_k} - \mathrm{M}_{x_0}\|_{W^{1,1}(I)} = \lim_{k\to\infty} \big(\|\mathrm{M}_{y_k} - \mathrm{M}_{x_0}\|_{L^1(I)} + \|\mathrm{M}'_{y_k} - \mathrm{M}'_{x_0}\|_{L^1(I)}\big) = 0,$$

and this is exactly our starting goal (see (A.3)). This ends the proof. □

Remark A.2. As a matter of fact, in the recent paper [17] it is contained a proof of Lemma A.1; however, it seems that this proof is not correct. We thus take this occasion to correct the mistake in [17] by giving a new proof of Lemma A.1.

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