Rend. Lincei Mat. Appl. 34 (2023), 491–529
DOI 10.4171/RLM/1016

Complex Variable(s) Functions. – *$*$-logarithm for slice regular functions*, by AMEDEO ALTAVILLA and CHIARA DE FABRITIIS, communicated on 10 February 2023.

ABSTRACT. – In this paper, we study the (possible) solutions of the equation $\exp_*(f) = g$, where g is a slice regular never vanishing function on a circular domain of the quaternions $\mathbb{H}$ and $\exp_*$ is the natural generalization of the usual exponential to the algebra of slice regular functions. Any function f which satisfies $\exp_*(f) = g$ is called a $*$-logarithm of g. We provide necessary and sufficient conditions, expressed in terms of the zero set of the "vector" part g_v of g, for the existence of a $*$-logarithm of g, under a natural topological condition on the domain Ω. By this way, we prove an existence result if g_v has no non-real isolated zeroes; we are also able to give a comprehensive approach to deal with more general cases. We are thus able to obtain an existence result when the non-real isolated zeroes of g_v are finite, the domain is either the unit ball, or $\mathbb{H}$, or $\mathbb{D}$ (the solid torus obtained by circularization in $\mathbb{H}$ of the disc contained in $\mathbb{C}$ and centered in $2\sqrt{-1}$ with radius 1), and a further condition on the "real part" g_0 of g is satisfied (see Theorem 6.19 for a precise statement). We also find some unexpected uniqueness results, again related to the zero set of g_v, in sharp contrast with the complex case. A number of examples are given throughout the paper in order to show the sharpness of the required conditions.

KEYWORDS. – Slice regular functions, quaternionic exponential, $*$-exponential, $*$-logarithm.

2020 MATHEMATICS SUBJECT CLASSIFICATION. – Primary 30G35; Secondary 30C25, 32A30, 33B10.

1. INTRODUCTION

The aim of this paper is to investigate the (possible) existence and/or uniqueness of solutions of

$$\exp_*(f) = g, \tag{1.1}$$

given a never vanishing function g which is slice regular on a circular domain Ω of the quaternions $\mathbb{H}$. A solution of this equation will be called $*$-logarithm of g.

The $*$-exponential operator $\exp_*$ on the space $\mathcal{SR}(\Omega)$ of slice regular functions was introduced by Colombo, Sabadini, and Struppa in [8] and was later studied by Altavilla and de Fabritiis in [3]. We underline the fact that its definition in the form $\exp_*(f) = \sum \frac{f^{*n}}{n!}$, where $f^{*n} = f * \cdots * f$ denotes the $*$-product of f with itself n times (i.e., the nth $*$-power), was due to the remark that in general the pointwise product (and thus the pointwise powers) of slice regular functions is not slice regular.

One of the features of $\exp_*(f)$ is that, as anyone would expect, it is a never vanishing slice regular function for any slice regular f, so equation (1.1) has no solution unless g is never vanishing. By an accurate investigation on the features of this operator, we will be able to understand under which conditions on Ω and g equation (1.1) admits a solution.

Similar results, suggested by different motivations and obtained by distinct techniques, were obtained at the same time by Gentili, Prezelij, and Vlacci (see [11]).

Slice regular functions on quaternions were introduced by Gentili and Struppa in 2006 in order to give a suitable notion of regularity for functions of a quaternionic variable which would provide a good balance between two requirements: the first one is the necessity of a smooth behavior, in the sense of existence of some kind of derivatives, while the second is the condition that the set of these functions is large enough to offer an interesting theory; for a detailed account on the path which led to this approach, see [12]. In particular, in our opinion, one of the key points of the theory is the definition of $*$-product which, together with the pointwise sum, gives to the space $\mathcal{SR}(\Omega)$ the structure of an associative $*$-algebra.

In this last 15 years, the theory quickly developed in several directions, creating many connections with differential geometry, algebraic geometry, functional analysis, operator theory, and applications to physics and engineering.

Before giving a description of the content of our paper, it is worthwhile looking a bit more thoroughly to the behavior of the operator $\exp_*$ with the aim to explain the reason of its definition and the analogies and dissimilarities with the exponential in the classical (i.e., complex analytic) sense.

Indeed, one can define an exponential map $\exp : \mathbb{H} \to \mathbb{H} \setminus \{0\}$ on the quaternions which turns out to be slice regular and slice preserving. Nonetheless, in general, the composition of slice regular functions is not slice regular (for a more detailed treatment of the composition in the setting of slice regular functions, see [9, 20, 21]). Thus, considering $\exp \circ f$ for any slice regular function f provides a never vanishing function which could be non-regular. Indeed, the equality $\exp \circ f = \exp_*(f)$ holds true when f is a slice preserving function and, on a suitable subset of Ω, for one-slice preserving functions, but in general, it does not for any slice regular function (see Definition 2.7, Remark 2.16, and Example 2.22 for a detailed comparison between $\exp \circ f$ and $\exp_*(f)$).

Thus, the topological approach used in the theory of holomorphic functions on complex numbers, where the logarithm of $g : \Omega \to \mathbb{C} \setminus \{0\}$ can be obtained by means of a lifting of the map g with respect to the covering $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, provided Ω is contractible, cannot be used on slice regular functions unless the function g has special properties (i.e., it preserves at least one slice). In antithesis with $\exp \circ f$, the definition of $\exp_*$ is the natural one since it relies on the $*$-product, it provides an operator which maps $\mathcal{SR}(\Omega)$ in $\mathcal{SR}(\Omega)$, and it coincides with $\exp \circ f$ on slice preserving functions.

We now give an outline of the paper.

Section 2 contains definitions and preliminary material: we recall the basic definitions of the theory of slice regular functions together with the main topological definition we need, that is, the notion of slice-contractible domain, and we cite the main properties of the $*$-exponential proven in [3]. Indeed, we can write any slice regular function f as a sum $f = f_0 + f_1 i + f_2 j + f_3 k$, where i, j, k are the standard basis of imaginary quaternions and f_ℓ, for $\ell = 0, 1, 2, 3$, are quaternionic valued, slice preserving regular functions (see [7, 14]). The function f_0 can be interpreted as the "real part" of f and $f_v = f_1 i + f_2 j + f_3 k$ as the "vector part" of f. Thanks to the splitting $f = f_0 + f_v$, we can give a useful rewriting of $\exp_*(f)$ in terms of the exponential of the real part of f and of two auxiliary slice preserving functions, namely, μ, ν, introduced in Definition 2.17. By means of μ and ν, we can compute the $*$-exponential in an easier way and thus compare $\exp_*(f)$ and $\exp \circ f$ for $f \in \mathcal{SR}(\Omega)$. Lastly, we prove that for any slice regular function g without non-real isolated zeroes and such that $g^s \not\equiv 0$ has a square root τ, the quotient $\tau^{-1} g$ is a well defined slice regular function.

In Section 3, we investigate in full detail the behavior of the functions μ and ν introduced in the previous section. For $n \in \mathbb{N}$, we define a family of circular domains $\mathcal{D}_n \subset \mathbb{H}$, showing that the restriction of the map μ to $\mathcal{D}_0$ is biregular onto $\mathbb{H} \setminus (-\infty, -1]$, while for any positive n, the restriction of the map μ to $\mathcal{D}_n$ is biregular onto $\mathbb{H} \setminus ((-\infty, -1] \cup [1, +\infty))$. This allows us to introduce the inverse $\varphi : \mathbb{H} \setminus (-\infty, -1] \to \mathcal{D}_0$ of $\mu|_{\mathcal{D}_0}$ which is a never vanishing function on $\mathcal{D}_0$ and will be extensively used in Section 6.

Section 4 contains a first existence result for the $*$-logarithm: namely, we prove that any one-slice preserving never vanishing function g defined on a slice-contractible domain Ω has a $*$-logarithm which preserves the same slice. Moreover, we show that if Ω is slice and g is positive on $\Omega \cap \mathbb{R}$, then there exists a *unique* slice-preserving $*$-logarithm of g, while if Ω is product, we can always find a slice preserving $*$-logarithm of a slice preserving function, but in this case, the $*$-logarithm is *never* unique. This statement allows us to prove the existence of a solution of a cosine-sine problem, in strong analogy with the case of holomorphic functions. Indeed, given a slice-contractible domain Ω and $a_0, a_1 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $a_0^2 + a_1^2 \equiv 1$, we show that there exists $\gamma \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $\cos_*(\gamma) = a_0$ and $\sin_*(\gamma) = a_1$. As a consequence of this proposition, we are able to classify zero divisors with identically zero real part on product domains (see [4, 5] for a detailed study of zero divisors).

In Section 5, we start by studying the uniqueness problem for the $*$-logarithm on slice preserving functions, and we then turn to the general case. The main point is that we have to consider two different situations: the first when we deal with slice preserving functions and the second one when we take into account non-slice preserving functions.

We underline that the results we obtain in this section do not depend on any topological condition on the domain Ω.

For the first case, we need to define the set

$$\mathfrak{N}(\Omega) := \left\{ f \in \mathcal{SR}(\Omega) \mid \exists m \in \mathbb{Z} \setminus \{0\},\ f_v^s = m^2\pi^2 \right\} \cup \mathcal{SR}_{\mathbb{R}}(\Omega),$$

whose elements have the property that the symmetrized function of their vectorial part always has a square root, and to anticipate the definition of the function $\mathcal{J} : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ given by $\mathcal{J}(q) = \frac{q-q^c}{|q-q^c|}$, where q^c denotes the usual quaternionic conjugation (see Definition 2.9). This allows us to state the following.

Theorem 1.1. *Let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that* $\exp_*(h) = \exp_*(\tilde{h}) \in \mathcal{SR}_{\mathbb{R}}(\Omega)$.

- *If Ω is a slice domain, then $h_0 \equiv \tilde{h}_0$, $h, \tilde{h} \in \mathfrak{N}(\Omega)$, and $\sqrt{h_v^s} \equiv \sqrt{\tilde{h}_v^s}$* $\pmod{2\pi}$.
- *If Ω is a product domain, then there exists $n \in \mathbb{Z}$ such that $h_0 = \tilde{h}_0 + \pi n \mathcal{J}$. Moreover, $h, \tilde{h} \in \mathfrak{N}(\Omega)$ and $\sqrt{h_v^s} \equiv \sqrt{\tilde{h}_v^s} + n\pi$* $\pmod{2\pi}$.

When the $*$-exponential of the functions we are considering does not belong to $\mathcal{SR}_{\mathbb{R}}(\Omega)$, the conclusion we obtain is quite different.

Theorem 1.2. *Let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that $h \neq \tilde{h}$ and* $\exp_*(h) = \exp_*(\tilde{h}) \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$.

- *If Ω is a slice domain, then $h_0 \equiv \tilde{h}_0$, both h_v and $\tilde{h}_v$ have no non-real isolated zeroes, both h_v^s and $\tilde{h}_v^s$ have a square root on Ω, and there exist $m \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, and $H_v \in \mathcal{SR}(\Omega)$ with $H_v^s \equiv 1$ such that $h_v = \alpha H_v$ and $\tilde{h}_v = (\alpha + 2\pi m)H_v = h_v + 2\pi m H_v$, so that*
$$\tilde{h} = h + 2\pi m H_v.$$
- *If Ω is a product domain, then one of the following holds:*
  (1) *There exists $n \in \mathbb{Z} \setminus \{0\}$ such that*
  $$\tilde{h} = h + 2\pi n \mathcal{J}.$$
  (2) *Both h_v and $\tilde{h}_v$ are not zero divisors in $\mathcal{SR}(\Omega)$ and have no non-real isolated zeroes, both h_v^s and $\tilde{h}_v^s$ have a square root on Ω, and there exists $n, m \in \mathbb{Z}$ such that $m \neq 0$ and $m \equiv n$* $\pmod 2$, *$\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, and $H_v \in \mathcal{SR}(\Omega)$ with $H_v^s \equiv 1$, such that $h_v = \alpha H_v$ and*
  $$\tilde{h} = h_0 + \pi n \mathcal{J} + (\alpha + \pi m)H_v = h + \pi(n\mathcal{J} + mH_v).$$

If the domain Ω is slice and the function h_v has a non-real isolated zero, the above theorem gives an unexpected uniqueness result for the $*$-logarithm.

COROLLARY 1.3. *Let Ω be slice and let $h \in \mathcal{SR}(\Omega)$ be such that* $\exp_*(h) \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$. *If h_v has a non-real isolated zero, then* $\exp_*(h) = \exp_*(\tilde{h})$ *if and only if $\tilde{h} \equiv h$.*

The last section contains the most important existence results of our paper. First of all, we show that if the domain is slice-contractible, we can always limit ourselves to look for the $*$-logarithm of a slice regular function g such that $g^s \equiv 1$.

We then get rid of the case when g_v is a zero divisor, showing that if Ω is a slice-contractible domain and g is a never vanishing function such that g_v is a zero divisor, then there exists a $*$-logarithm of g.

Next, we turn to the case when g_v is not a zero divisor (which is always the case when Ω is slice); under this hypothesis, we find a necessary condition for the existence of a $*$-logarithm of a function g whose symmetrized function is identically equal to 1. Indeed, if there exists a $*$-logarithm of such a g, we have that

(1) if Ω is a slice domain and q_0 is a non-real isolated zero of g_v, then $g(q_0) = 1$;

(2) if Ω is a product domain and q_0, q_1 are non-real isolated zeroes of g_v, then either $g(q_0) = g(q_1) = 1$ or $g(q_0) = g(q_1) = -1$.

Thus, the non-real isolated zeroes of g_v are the true obstruction we have to overcome in order to get the existence of a $*$-logarithm. We notice that, in the case of a slice domain, the above conclusion gives the following unexpected negative results for the existence of a $*$-logarithm of a function.

(1) Let Ω be a slice domain and let $f \in \mathcal{SR}(\Omega)$ be such that f_v has a non-real isolated zero. Then, $-\exp_*(f)$ has no $*$-logarithm.

(2) Let Ω be a slice-contractible slice domain and let $g \in \mathcal{SR}(\Omega)$ be a never vanishing function such that g_v has a non-real isolated zero. Then, at least one between g and $-g$ has no $*$-logarithm.

The fact that the non-real isolated zeroes of g_v are the genuine obstruction for the existence of a global $*$-logarithm of a function is confirmed by the following statement.

THEOREM 1.4. *Let Ω be slice-contractible. Then, any never vanishing $g \in \mathcal{SR}(\Omega)$ such that g_v has no non-real isolated zeroes has a $*$-logarithm.*

We now consider the problem near the non-real isolated zeroes of g_v: by means of the inverse of the function μ, we are able to build a $*$-logarithm of g near these points (to be more precise, afar from the points where g_0 takes values in $(-\infty, -1]$).

THEOREM 1.5. *Let Ω be slice-contractible and $g \in \mathcal{SR}(\Omega)$. If $g^s \equiv 1$ and for any $q_0 \in \Omega$ that is a non-real isolated zero of g_v we have that $g_0(q_0) = 1$, then, on every connected component of $\Omega \setminus g_0^{-1}((-\infty, -1])$, there exists a $*$-logarithm of g.*

We conclude our investigation by gluing the solutions obtained in Theorem 1.4 where g_v has no non-real isolated zeroes with the “patches” given by Theorem 1.5, thus obtaining our main theorem. The topological difficulties we have to deal with prevent us from obtaining a clear statement in the general case of a slice-contractible domain, so we prefer to give a result which pays something to generality in order to obtain a simpler formulation.

Let us denote by $\mathbb{B}$ the open unit ball in $\mathbb{H}$, by $\mathbb{D}$ the solid torus obtained by circularization in $\mathbb{H}$ of the disc contained in $\mathbb{C}$ and centered in $2\sqrt{-1}$ with radius 1, by $\mathbb{C}_I^+$ the closed upper half plane in $\mathrm{Span}_{\mathbb{R}}(1, I)$, and by $\mathcal{SR}^1(\Omega)$ the set of slice regular functions with symmetrized function identically equal to 1 (see Definition 2.25).

THEOREM 1.6. *Let Ω be one among $\mathbb{B}$, $\mathbb{H}$, or $\mathbb{D}$. Let $g \in \mathcal{SR}^1(\Omega)$ be such that*

- *g_v has a finite number of non-real isolated zeros $\{q_1, \dots, q_N\}$;*
- *$g_0(q_\ell) = 1$ for all $\ell = 1, \dots, N$;*
- *the union $\mathbb{S}_{q_1} \cup \cdots \cup \mathbb{S}_{q_N}$ is contained in a unique connected component $\mathcal{U}$ of $\Omega \setminus g_0^{-1}((-\infty, -1])$.*

If for some $I \in \mathbb{S}$ (and hence for any) the set $\mathcal{U}_I^+ = \mathcal{U} \cap \mathbb{C}_I^+$ is convex and $\mathcal{U}$ is slice if Ω is, then there exists a $*$-logarithm of g.

2. Preliminary results

In this section, we recall the basic definitions and results we will use in the following. We also prove a couple of basic, yet new, results.

We start with some general facts about quaternions. The real skew algebra of quaternions is defined as

$$\mathbb{H} := \{q = q_0 + q_1 i + q_2 j + q_3 k \mid q_\ell \in \mathbb{R},\ \ell = 0, 1, 2, 3,\\ i^2 = j^2 = k^2 = -1,\ ij = -ji = k\};$$

hence, i, j, and k are imaginary units. From these three elements, it is possible to construct the sphere of imaginary units defined as

$$\mathbb{S} := \{\alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\} = \{I \in \mathbb{H} \mid I^2 = -1\}.$$

By means of the standard conjugation

$$q = q_0 + q_1 i + q_2 j + q_3 k \mapsto q^c = q_0 - (q_1 i + q_2 j + q_3 k),$$

we can split any quaternion in its real and vector part $q = q_0 + q_v$, where $q_v = (q - q^c)/2$. The set of purely imaginary quaternions will be denoted by

$$\mathrm{Im}(\mathbb{H}) = \{q \in \mathbb{H} \mid q_0 = 0\}.$$

Another possible splitting is given as follows: any $q \in \mathbb{H} \setminus \mathbb{R}$ can be uniquely written as $q = \alpha + I\beta$, where $\alpha = q_0$, $\beta = |q_v| > 0$, and $I = q_v/|q_v| \in \mathbb{S}$. Such a decomposition gives the fundamental description of $\mathbb{H}$ underlying the theory of slice regular functions. Before presenting it, we need to introduce a few more definitions to specify the class of domains, i.e., open connected subsets of $\mathbb{H}$, where slice regular functions are defined. For any imaginary unit $I \in \mathbb{S}$, we denote by $\mathbb{C}_I$ the complex *slice* spanned by 1 and I over the reals, i.e.,

$$\mathbb{C}_I := \mathrm{Span}_{\mathbb{R}}(1, I).$$

We will also consider closed *semislices*

$$\mathbb{C}_I^+ = \{\alpha + I\beta \mid \alpha, \beta \in \mathbb{R},\ \beta \geq 0\}, \quad \mathbb{C}_I^- = \{\alpha + I\beta \mid \alpha, \beta \in \mathbb{R},\ \beta \leq 0\},$$

and, for any $q \in \mathbb{H} \setminus \mathbb{R}$, we denote by $\mathbb{C}_q$ the unique slice containing q. Moreover, if $\Omega \subset \mathbb{H}$, then we write $\Omega_I = \Omega \cap \mathbb{C}_I$ and $\Omega_I^\pm = \Omega \cap \mathbb{C}_I^\pm$.

Given any $q \in \mathbb{H}$, we define its sphere $\mathbb{S}_q$ as

$$\mathbb{S}_q := \left\{q_0 + |q_v| J \mid J \in \mathbb{S}\right\},$$

where, trivially, if $q \in \mathbb{R}$, then $q_v = 0$ and $\mathbb{S}_q = \{q\}$. For any $\Omega \subset \mathbb{H}$, its *symmetric completion* is given by

$$\bigcup_{q \in \Omega} \mathbb{S}_q;$$

a domain $\Omega \subset \mathbb{H}$ will be called *circular* if it coincides with its symmetric completion. A circular domain is said to be *slice* if $\Omega \cap \mathbb{R} \neq \emptyset$, while it is called *product* if $\Omega \cap \mathbb{R} = \emptyset$. Notice that if Ω is a product domain, then Ω is homeomorphic to $\Omega_I^+ \times \mathbb{S}$, for any $I \in \mathbb{S}$, thus explaining the origin of its name.

We are now ready to give our main topological definition.

Definition 2.1. A circular subset $\Omega \subset \mathbb{H}$ is said to be *slice-contractible* if, for some $I \in \mathbb{S}$ (and then any), Ω_I is a simply connected domain if Ω is slice, and Ω_I^+ is a simply connected domain if Ω is product.

From now on, we denote by $\mathbb{B} \subset \mathbb{H}$ the open unit ball centered at the origin and by $\mathbb{D}$ the "solid torus"

$$\mathbb{D} = \left\{\alpha + I\beta \in \mathbb{H} \mid \alpha^2 + (\beta - 2)^2 < 1,\ I \in \mathbb{S}\right\}, \tag{2.1}$$

which are both slice-contractible domains, the first one slice and the second one product.

Remark 2.2. We notice that for a slice domain Ω being simply connected does not mean being slice-contractible, in general. Indeed, if $\Omega = \mathbb{B} \setminus \{0\}$, then Ω is simply connected because it retracts on S^3, while Ω_I is not simply connected because it is a "flat" punctured disc in $\mathbb{C}_I$.

The following definitions identify the functions we will work with (see [15] for an introduction to the topic).

DEFINITION 2.3. Let Ω be a circular domain in $\mathbb{H}$ and, for any $I \in \mathbb{S}$, set

$$D = \{\alpha + \sqrt{-1}\beta \mid \alpha + I\beta \in \Omega\} \subset \mathbb{C}.$$

We say that a function $F : D \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{H} \oplus \imath\mathbb{H}$ is a *stem* function if the equality $F(\bar{z}) = \overline{F(z)}$, where $\overline{p + \imath q} = p - \imath q$, holds for any $z \in D$. A *slice* function $f : \Omega \to \mathbb{H}$ is a function induced by a stem function $F = F_0 + \imath F_1 : D \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ in the following way:

$$f(\alpha + \beta I) = F_0(\alpha + \sqrt{-1}\beta) + IF_1(\alpha + \sqrt{-1}\beta).$$

Such a function will also be denoted by $f = \mathcal{I}(F)$.

Ghiloni and Perotti [15] proved that a slice function f is induced by a unique stem function $F = F_0 + \imath F_1$, given by $F_0(\alpha + \sqrt{-1}\beta) = \frac{1}{2}(f(\alpha + I\beta) + f(\alpha - I\beta))$ and $F_1(\alpha + \sqrt{-1}\beta) = -\frac{1}{2}I(f(\alpha + I\beta) - f(\alpha - I\beta))$ for any $I \in \mathbb{S}$. In particular, if $F = F_0 + \imath F_1$ and $f = \mathcal{I}(F)$, we define its *slice conjugate* as the slice function $f^c := \mathcal{I}(F^c) : \Omega \to \mathbb{H}$, where $F^c = F_0^c + \imath F_1^c$ and $F_\ell^c(q) = (F_\ell(q))^c$ for $\ell = 0, 1$.

DEFINITION 2.4. Let $\Omega \subset \mathbb{H}$ be a circular domain. A *slice* function $f = \mathcal{I}(F) : \Omega \to \mathbb{H}$ is slice *regular* if F is holomorphic with respect to the natural complex structures of $\mathbb{C}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. We denote by $\mathcal{SR}(\Omega)$ the set of all slice regular functions on Ω with its natural structure of right $\mathbb{H}$-module.

If Ω is slice, the notion of slice regularity coincides with *Cullen regularity* (see [12]). We underline that a detailed study of the relation between the slice regularity adopted in the present paper and original Cullen regularity can be found in [13, 16] and in [12, Section 10.3]. A useful result for slice regular functions is the following.

PROPOSITION 2.5 (Representation formula). *Let $f \in \mathcal{SR}(\Omega)$ and let $\alpha + \beta J \in \Omega$. For all $I \in \mathbb{S}$, we have*

$$f(\alpha + J\beta) = \frac{1 - JI}{2} f(\alpha + I\beta) + \frac{1 + JI}{2} f(\alpha - I\beta).$$

A useful consequence of the representation formula is the fact that if $f_I : \Omega_I \to \mathbb{H}$ is a holomorphic function with respect to the left multiplication by I, then there exists a unique slice regular function $f : \Omega \to \mathbb{H}$ such that $f_I = f_{|\Omega_I}$. Such a function $\mathrm{ext}(f_I)$ will be called the *regular extension of f_I* (see [12]). From now on, if $f \in \mathcal{SR}(\Omega)$, we denote by f_I its restriction to Ω_I.

The strong relation between holomorphicity and slice regularity appears also in the following result obtained by merging [12, Theorem 1.12] and [1, Theorem 3.6].

PROPOSITION 2.6 (Identity principle). *Given $f, g \in \mathcal{SR}(\Omega)$,*

- *if Ω is slice and for some $I \in \mathbb{S}$, the functions f and g coincide on a subset of Ω_I having an accumulation point, then $f \equiv g$;*
- *if Ω is product and for some $I \in \mathbb{S}$, the functions f and g coincide on a subset of Ω_I having an accumulation point in Ω_I^+ and an accumulation point in Ω_I^-, then $f \equiv g$.*

The following two classes of regular functions are of particular interest for the theory.

DEFINITION 2.7. Given $f \in \mathcal{SR}(\Omega)$, we say that f is *one slice preserving* if $f(\Omega_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$ and f is *slice preserving* if $f(\Omega_I) \subset \mathbb{C}_I$ for all $I \in \mathbb{S}$. The set of slice preserving function on Ω will be denoted by $\mathcal{SR}_{\mathbb{R}}(\Omega)$, while the set of functions that preserve Ω_I will be denoted by $\mathcal{SR}_I(\Omega)$.

REMARK 2.8. Notice that a function f is slice preserving if and only if it preserves two different slices, if and only if it is intrinsic, i.e., $f(q^c) = (f(q))^c$, for all q in its domain Ω. Moreover, if Ω is slice, f is slice preserving if and only if $f(\Omega \cap \mathbb{R}) \subset \mathbb{R}$ (see [14, Lemma 6.8]).

We now introduce an interesting function which will be widely used in the course of the paper.

DEFINITION 2.9. Let us define the regular (slice preserving) function $\mathcal{J} : \mathbb{H} \setminus \mathbb{R} \to \mathbb{S} \subset \mathbb{H}$ as

$$\mathcal{J}(q_0 + q_v) = \frac{q_v}{|q_v|}.$$

Equivalently, if $q = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$, with $\beta > 0$, then $\mathcal{J}(q) = I$.

We now turn to the algebraic structure of $\mathcal{SR}(\Omega)$. It is a well known fact that in general, the pointwise product of two slice regular functions is no more slice. Nonetheless, this problem can be overcome by defining the following non-commutative product (see [15]).

DEFINITION 2.10. Let $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ be two slice regular functions on Ω. We denote by $f * g$ their $*$-product defined by $f * g = \mathcal{I}(F \cdot G)$ where $F \cdot G$ is the pointwise product with values in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, i.e., $(p + \imath q)(p' + \imath q') = pp' - qq' + \imath(pq' + qp')$.

REMARK 2.11. In [6], the authors prove that the $*$-*product* of f and g can be computed as

$$(f * g)(q) := \begin{cases} 0, & \text{if } f(q) = 0, \\ f(q)g(f(q)^{-1}qf(q)), & \text{if } f(q) \neq 0. \end{cases}$$

In some specific case, the $*$-product coincides with the pointwise product and in some more it turns to be commutative.

REMARK 2.12. Let $f, g \in \mathcal{SR}(\Omega)$. If Ω is slice, then $f * g = f \cdot g$ on $\Omega \cap \mathbb{R}$. If f is slice preserving, then $f * g = g * f = fg$. If both functions preserve the same slice Ω_I, then $f * g = \mathrm{ext}(f_I \cdot g_I) = \mathrm{ext}(g_I \cdot f_I) = g * f$.

Given any slice regular function $f \in \mathcal{SR}(\Omega)$, we then introduce its *symmetrized function* $f^s \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ defined as $f^s = f * f^c = f^c * f$.

The importance of the symmetrized function relies mainly on its connection with the zero set of f: in particular, the function f is a zero divisor if and only if $f^s \equiv 0$ and the fact that f has a non-real isolated zero q entails $f^s \equiv 0$ on $\mathbb{S}_q$.

The symmetrized function allows us to define the $*$-inverse of a regular function. Given $f \in \mathcal{SR}(\Omega)$ with $f^s \not\equiv 0$, we define its $*$-*inverse* as $f^{-*} = (f^s)^{-1} f^c$ which is slice regular outside the zero set of f^s. In the following, we will extensively use the following zero-product property.

PROPOSITION 2.13 ([17, Proposition 5.18]). *Let $f, g \in \mathcal{SR}(\Omega)$ be such that $f^s \not\equiv 0$. Then, $f * g \equiv 0$ implies $g \equiv 0$. In particular,*

- *$f \in \mathcal{SR}(\Omega) \setminus \{0\}$ is a zero divisor if and only if $f^s \equiv 0$;*
- *if Ω is slice, then $\mathcal{SR}(\Omega)$ is an integral domain;*
- *if $f \in \mathcal{SR}_{\mathbb{R}}(\Omega) \setminus \{0\}$, then $fg \equiv 0$ implies $g \equiv 0$.*

Now, let us assume that $(1, I, J, K)$ is an orthonormal basis of $\mathbb{H}$. Thanks to [7, Proposition 3.12] and [14, Lemma 6.11], any slice regular function $f \in \mathcal{SR}(\Omega)$ can be uniquely written as a sum $f = f_0 + f_1 I + f_2 J + f_3 K$, where $f_0, \ldots, f_3 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, thus giving to $\mathcal{SR}(\Omega)$ the structure of a 4-rank free module on $\mathcal{SR}_{\mathbb{R}}(\Omega)$. For the convenience of what follows, we call f_0 the *"real part"* of f and $f_v = f - f_0$ the *"vector part"* of f. We also introduce the following two operators: let $f, g \in \mathcal{SR}(\Omega)$, then

$$\langle f, g\rangle_* := (f * g^c)_0, \quad f \mathbin{\wedge\!\!\!\!\wedge} g := \frac{f * g - g * f}{2}.$$

The following result summarizes a series of properties of the $*$-product obtained via the above interpretation of the multiplicative structure of $\mathcal{SR}(\Omega)$ (see [3, Proposition 2.7 and Remark 2.8]).

PROPOSITION 2.14. *Let $f = f_0 + f_1 I + f_2 J + f_3 K, g = g_0 + g_1 I + g_2 J + g_3 K \in \mathcal{SR}(\Omega)$. Then,*

- *$f^c = f_0 - (f_1 I + f_2 J + f_3 K)$, $f_0 = \frac{f + f^c}{2}$, and $f_v = \frac{f - f^c}{2}$;*
- *$f * g = f_0 g_0 - \langle f_v, g_v\rangle_* + f_0 g_v + g_0 f_v + f_v \mathbin{\wedge\!\!\!\!\wedge} g_v$;*

- $f^s = f_0^2 + f_1^2 + f_2^2 + f_3^2$; *in particular,* $f^s \geq 0$ *on* $\Omega \cap \mathbb{R}$;
- $f_v * f_v = -f_v * (-f_v) = -f_v * f_v^c = -f_v^s$.

Following [8], we define the following operators on $\mathcal{SR}(\Omega)$.

DEFINITION 2.15. Let $f \in \mathcal{SR}(\Omega)$. We set

$$\exp_*(f) = \sum_{n\in\mathbb{N}} \frac{f^{*n}}{n!},$$
$$\cos_*(f) = \sum_{n\in\mathbb{N}} \frac{(-1)^n f^{*(2n)}}{(2n)!},$$
$$\sin_*(f) = \sum_{n\in\mathbb{N}} \frac{(-1)^n f^{*(2n+1)}}{(2n+1)!}.$$

REMARK 2.16. If Ω is slice, then $\exp_*(f) = \exp\circ f$ on $\Omega\cap\mathbb{R}$ and the same holds for $\cos_*$ and $\sin_*$, i.e., $\cos_*(f) = \cos\circ f$ and $\sin_*(f) = \sin\circ f$ on $\Omega\cap\mathbb{R}$. If $f \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, then $\exp_*(f) = \exp\circ f = \exp(f)$, $\cos_*(f) = \cos\circ f = \cos(f)$, and $\sin_*(f) = \sin\circ f = \sin(f)$. Moreover, if $f \in \mathcal{SR}_I(\Omega)$, then $\exp_*(f) = \mathrm{ext}(\exp(f_I))$, $\cos_*(f) = \mathrm{ext}(\cos(f_I))$, and $\sin_*(f) = \mathrm{ext}(\sin(f_I))$.

We introduce the following definition in order to restate some of the contents of [3].

DEFINITION 2.17. We denote by $\mu, \nu : \mathbb{H} \to \mathbb{H}$ the following slice preserving entire functions:

$$\mu(q) = \sum_{m\in\mathbb{N}} \frac{(-1)^m q^m}{(2m)!}, \quad \nu(q) = \sum_{m\in\mathbb{N}} \frac{(-1)^m q^m}{(2m+1)!}.$$

REMARK 2.18. We notice that

$$\mu(q^2) = \cos(q), \quad \nu(q^2)q = \sin(q),$$

for all $q \in \mathbb{H}$. In particular,

$$\mu(\pi^2 n^2) = (-1)^n, \text{ for all } n \in \mathbb{Z}, \quad \nu(0) = 1, \quad \text{and}$$
$$\{q \in \mathbb{H} \mid \nu(q) = 0\} = \{\pi^2 n^2 \in \mathbb{H} \mid n \in \mathbb{N}\setminus\{0\}\}.$$

Moreover, for any $q \in \mathbb{H}$, the following equality holds:

$$\mu^2(q) + \nu^2(q)q \equiv 1. \tag{2.2}$$

Indeed, for any $q \in \mathbb{R}^+$, choose $t \in \mathbb{R}\setminus\{0\}$ such that $q = t^2$, and then $\mu(q) = \mu(t^2) = \cos(t)$, $\nu(q) = \nu(t^2) = \frac{\sin(t)}{t}$. Thus, $\mu^2(q) + \nu^2(q)q = \cos^2(t) + \frac{\sin^2(t)}{t^2}t^2 = \cos^2(t) + \sin^2(t) = 1$. By the identity principle, we are done.

The next proposition collects several features of the $*$-exponential (see [3, Propositions 4.3, 4.5, and 4.13], where a slightly different notation is used).

Proposition 2.19. *Let $f, g \in \mathcal{SR}(\Omega)$. Then, we have the following equalities:*

$$\exp_*(-f) = \big(\exp_*(f)\big)^{-*}, \tag{2.3}$$

$$\big(\exp_*(f)\big)^s = \exp(2f_0), \tag{2.4}$$

$$\exp_*(f) = \exp_*(f_0)\big(\mu(f_v^s) + \nu(f_v^s)f_v\big), \tag{2.5}$$

$$\exp_*(f+g) = \exp_*(f) * \exp_*(g), \quad \textit{if } f * g = g * f. \tag{2.6}$$

In particular, equality (2.3) shows that $\exp_*(f)$ is never vanishing on Ω.

The following examples elucidate the behavior of the $*$-exponential in some notable cases.

Example 2.20. As $\mathcal{J}$ defined in 2.9 is slice preserving, then a straightforward computation gives $\mathcal{J}^s = \mathcal{J}^2 \equiv -1$. Therefore, we have

$$\begin{aligned}\exp_*(\pi\mathcal{J}) = \exp(\pi\mathcal{J}) &= \sum_{n\in\mathbb{N}} \frac{(\pi\mathcal{J})^n}{n!} = \sum_{m\in\mathbb{N}} \frac{(-1)^m\pi^{2m}}{(2m)!} + \sum_{m\in\mathbb{N}} \frac{(-1)^m\pi^{2m+1}}{(2m+1)!}\mathcal{J} \\ &= \cos(\pi) + \sin(\pi)\mathcal{J} = -1.\end{aligned}$$

Example 2.21. If g_v is a zero divisor, then $g_v^s \equiv 0$ by Proposition 2.13, and thus

$$\exp_*(g_v) = \mu(g_v^s) + \nu(g_v^s)g_v = \mu(0) + \nu(0)g_v = 1 + g_v.$$

Example 2.22. Let $f(q) = i + qj$. As $f_0 \equiv 0$ and $f_v^s = 1 + q^2$, we have

$$\exp_*(f) = \mu(q^2+1) + \nu(q^2+1)(i+qj).$$

In particular, for any $q \in \mathbb{S}$, we have $q^2 + 1 = 0$, so $(\exp_*(f))(q) = 1 + i + qj$, giving thus $(\exp_*(f))(j) = i$. Nonetheless, $(\exp \circ f)(j) = e^{f(j)} = e^{i-1}$.

Example 2.23. If $f(q) = \pi\cos(q)i + \pi\sin(q)j$, again $f_0 \equiv 0$ and $f_v^s \equiv \pi^2$, so

$$\exp_*(f) = \mu(\pi^2) + \nu(\pi^2)\pi(\cos(q)i + \sin(q)j) \equiv -1.$$

Lastly, we compute $\exp_*$ on one-slice preserving functions.

Remark 2.24. Notice that if f is $\mathbb{C}_I$-preserving for some $I \in \mathbb{S}$, then we have $f_v = f_1 I$ with f_1 slice preserving. This entails

$$\exp_*(f) = \exp(f_0)\big(\cos(f_1) + \sin(f_1)I\big).$$

Indeed, $\mu(f_v^s) = \mu(f_1^2) = \cos(f_1)$, $\nu(f_v^s)f_v = \nu(f_1^2)f_1 I = \sin(f_1)I$.

We now introduce the notion of $*$-logarithm.

DEFINITION 2.25. We set

$$\mathcal{SR}^*(\Omega) := \{g \in \mathcal{SR}(\Omega) \mid g \text{ is never vanishing}\},$$
$$\mathcal{SR}^1(\Omega) := \{g \in \mathcal{SR}(\Omega) \mid g^s \equiv 1\}.$$

Given $g \in \mathcal{SR}^*(\Omega)$, a function $f \in \mathcal{SR}(\Omega)$ such that $\exp_*(f) = g$ is said to be a $*$-*logarithm* of g.

Notice that the elements in $\mathcal{SR}^*(\Omega)$ act by conjugation both on $\mathcal{SR}^1(\Omega)$ and on the set of functions having a $*$-logarithm.

REMARK 2.26. If $h \in \mathcal{SR}^1(\Omega)$ and $\chi \in \mathcal{SR}^*(\Omega)$, since $(\chi^{-*})^s = \chi^{-s} = (\chi^s)^{-1}$, we have $(\chi^{-*} * h * \chi)^s = \chi^{-s} * h^s * \chi^s = \chi^{-s} * \chi^s = 1$. Moreover, if $f \in \mathcal{SR}(\Omega)$ is a $*$-logarithm of h, then a trivial computation shows that $\chi^{-*} * f * \chi$ is a $*$-logarithm of $\chi^{-*} * h * \chi$ as $\exp_*(\chi^{-*} * f * \chi) = \chi^{-*} * \exp_*(f) * \chi$.

We conclude this section by showing that a slice regular function without non-real isolated zeroes can be factorized as the product of a slice preserving function and a slice regular function in $\mathcal{SR}^1(\Omega)$.

PROPOSITION 2.27. *Let $g \in \mathcal{SR}(\Omega)$ be such that g has no non-real isolated zeroes and g^s has a square root $\tau \in \mathcal{SR}_{\mathbb{R}}(\Omega) \setminus \{0\}$. Then, $\tau^{-*} * g = \tau^{-1} g = g/\tau$ is a well-defined slice regular function on Ω which belongs to $\mathcal{SR}^1(\Omega)$.*

PROOF. Since τ is a slice preserving function, outside the zero set of τ, the function $\tau^{-*} * g$ is a well defined slice regular function which coincides with the pointwise product $\tau^{-1} g = g/\tau$.

Then, we are left to define the function $\tau^{-1} g$ at the zeroes of τ. Since g is not a zero divisor, then it only has real isolated zeroes and spherical isolated zeroes (non-real isolated zeroes are ruled out by the assumption), so the zero set of τ coincides with the zero set of g (and of g^s).

If $x_0 \in \Omega \cap \mathbb{R}$ is a zero of g, we choose a ball $U_{x_0} \subset \Omega$ centered at x_0 on which g vanishes at x_0 only. By [12, Theorem 3.38], we can write $g(q) = (q - x_0)^m \tilde{g}(q)$ for a suitable $\tilde{g}$ slice regular and never vanishing on U_{x_0}. Then, we have $g^s(q) = (q - x_0)^{2m} \tilde{g}^s(q)$. As $\tilde{g}^s$ is never vanishing on U_{x_0} and it is strictly positive on $U_{x_0} \cap \mathbb{R}$, then there exists $\alpha \in \mathcal{SR}_{\mathbb{R}}(U_{x_0})$ such that $\alpha^2 = \tilde{g}^s$. A trivial computation shows that the slice regular function $\beta(q) = (q - x_0)^m \alpha(q)$ is a square root of g^s on U_{x_0}. By the identity principle, there are only two slice-preserving square roots of g^s on U_{x_0}, so either $\beta = \tau$ or $\beta = -\tau$; up to a change of sign of α (and thus of β), we can suppose $\beta = \tau$.

On U_{x_0}, we consider the function $\tilde{g}(q)/\alpha(q)$ (which is well defined because α is never vanishing and slice preserving on U_{x_0}). Moreover, on $U_{x_0} \setminus \{x_0\}$, we have $\tilde{g}(q)/\alpha(q) = ((q - x_0)^m \tilde{g}(q))/((q - x_0)^m \alpha(q)) = g(q)/\tau(q)$, so that the two functions truly coincide on $U_{x_0} \setminus \{x_0\}$ and define a slice regular function on U_{x_0}.

Analogously, if g has a spherical zero at $\mathbb{S}_{q_0}$, then in a "toric" neighborhood U_{q_0} of such a sphere, we can write $g(q) = \Delta_{q_0}(q)^m \tilde{g}(q)$ with $\tilde{g}$ slice regular and never vanishing on U_{q_0}. Again, we find that $g^s(q) = \Delta_{q_0}(q)^{2m} \tilde{g}^s(q)$; as $\tilde{g}^s$ is never vanishing on U_{q_0}, it has a slice-preserving square root α on that neighborhood. Thus, on U_{q_0}, the function $\beta(q) = \Delta_{q_0}(q)^m \alpha(q)$ is a square root of g^s which coincides with τ, up to a change of sign of α (and thus of β). On U_{q_0} we consider $\tilde{g}(q)/\alpha(q)$ (which as above is well defined) which on $U_{q_0} \setminus \{\mathbb{S}_{q_0}\}$ coincides with g/τ, and we are done. ∎

If the domain Ω is slice-contractible and g is not a zero divisor and has no non-real isolated zeroes, then the existence of a square root $\tau \not\equiv 0$ of g^s is a consequence of [3, Corollary 3.2].

Corollary 2.28. *Let Ω be a slice-contractible domain. For any $g \in \mathcal{SR}(\Omega)$ which is not a zero divisor and has no non-real isolated zeroes, let us denote by $\tau \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ a square root of g^s. Then, $\tau^{-*} * g = g/\tau$ is a well-defined slice regular-function on Ω which belongs to $\mathcal{SR}^1(\Omega)$.*

3. Behavior of the entire function μ

We now investigate the behavior of the slice preserving map μ on the quaternions. In particular, we will prove an invertibility result for the restriction of function μ to a family of subdomains of $\mathbb{H}$ that will be used in Section 6.

Definition 3.1. We define the following slice domains (see Figure 1):

$$\mathcal{D}_0 := \left\{x + yJ \mid x < \pi^2 - \frac{y^2}{4\pi^2},\ J \in \mathbb{S}\right\} \subset \mathbb{H},$$

and for any positive $n \in \mathbb{N}$,

$$\mathcal{D}_n := \left\{x + yJ \mid n^2\pi^2 - \frac{y^2}{4n^2\pi^2} < x < (n+1)^2\pi^2 - \frac{y^2}{4(n+1)^2\pi^2},\ J \in \mathbb{S}\right\} \subset \mathbb{H}.$$

For $n > 0$, we also denote by

$$\Gamma_n := \left\{x + yJ \mid x = n^2\pi^2 - \frac{y^2}{4n^2\pi^2},\ J \in \mathbb{S}\right\} \subset \mathbb{H}$$

the boundaries of the above domains. Indeed, we have $\partial\mathcal{D}_0 = \Gamma_1$ and $\partial\mathcal{D}_n = \Gamma_n \cup \Gamma_{n+1}$, for any positive $n \in \mathbb{N}$.

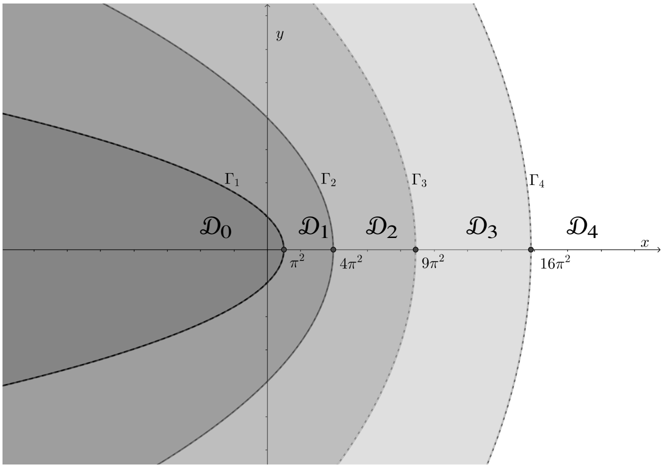

FIGURE 1. The section of the sets $\mathcal{D}_n$ and Γ_n on a fixed slice.

DEFINITION 3.2. Let Ω be a slice domain and f a non-constant slice preserving function on Ω. We say that f is *biregular* on Ω if there exists $g \in \mathcal{SR}_{\mathbb{R}}(f(\Omega))$, such that $g \circ f = \mathrm{id}_\Omega$ and $f \circ g = \mathrm{id}_{f(\Omega)}$. In such a case, we call g the *biregular inverse* of f.

THEOREM 3.3. *The restriction of the map μ to $\mathcal{D}_0$ is biregular onto $\mathbb{H} \setminus (-\infty, -1]$. For any positive $n \in \mathbb{N}$, the restriction of the map μ to $\mathcal{D}_n$ is biregular onto the domain $\mathbb{H} \setminus ((-\infty, -1] \cup [1, +\infty))$.*

PROOF. We first carry out the proof in the case of $\mathcal{D}_0$. Consider the set

$$\widetilde{\mathcal{D}_0} := \{x + Jy \in \mathbb{H} \mid 0 < x < \pi,\ J \in \mathbb{S}\}$$

(see Figure 2). It is easily seen (working slice by slice) that the restriction to $\widetilde{\mathcal{D}_0}$ of the function $s : \mathbb{H} \to \mathbb{H}$, defined as $s(q) = q^2$, gives a bijection onto $\mathcal{D}_0 \setminus (-\infty, 0]$. Furthermore, working again slice by slice, the restriction to $\widetilde{\mathcal{D}_0}$ of the cosine function gives a bijection onto $\mathbb{H} \setminus ((-\infty, -1] \cup [1, +\infty))$.

For any $\xi \in \mathbb{H} \setminus ((-\infty, -1] \cup [1, +\infty))$, there exists a unique $t \in \widetilde{\mathcal{D}_0}$ such that $\xi = \cos(t)$. Now, $q = s(t) = t^2 \in \mathcal{D}_0 \setminus (-\infty, 0]$ is the unique element of $\mathcal{D}_0 \setminus (-\infty, 0]$ which is mapped in ξ by μ; indeed, we have $\mu(q) = \mu(t^2) = \cos(t) = \xi$.

Now, consider $\xi \in [1, +\infty)$. If $t \in \mathbb{H}$ is such that $\cos(t) = \xi$, then there exist $k \in \mathbb{N}$ and $I \in \mathbb{S}$ such that $a = 2k\pi$, $b = \operatorname{arccosh}(\xi)$, and $t = a + Ib$. Thus, $t^2 = 4k^2\pi^2 - (\operatorname{arccosh}(\xi))^2 + 4k\pi \operatorname{arccosh}(\xi)I$ belongs to $\mathcal{D}_0$ if and only if $k = 0$. In this particular case, $q = t^2$ is equal to $-(\operatorname{arccosh}(\xi))^2 \in (-\infty, 0]$ which is the unique element in $q \in \mathcal{D}_0$ such that $\mu(q) = \xi$.

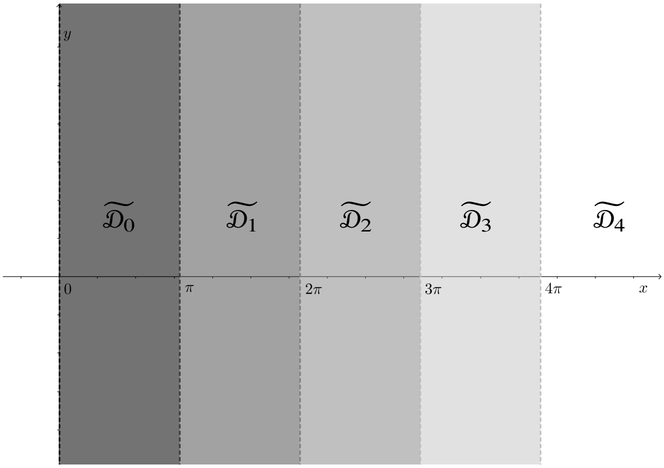

FIGURE 2. The section of the sets $\widetilde{\mathcal{D}_n}$ on a fixed slice.

Then, the function μ is a slice preserving bijection from $\mathcal{D}_0$ onto $\mathbb{H} \setminus (-\infty, -1]$. Let now $I \in \mathbb{S}$. Since μ is slice preserving, then its restriction $\mu_I : \mathcal{D}_0 \cap \mathbb{C}_I \to \mathbb{C}_I \setminus (-\infty, -1]$ is a holomorphic bijection and thus a biholomorphism. Then, the inverse of μ_I is a holomorphic function from $\mathbb{C}_I \setminus (-\infty, -1]$ to $\mathcal{D}_0 \cap \mathbb{C}_I$, thus showing that the inverse of μ is slice regular.

By using

$$\widetilde{\mathcal{D}_n} := \{x + Jy \in \mathbb{H} \mid n\pi < x < (n+1)\pi,\ J \in \mathbb{S}\},$$

with $n > 0$ in place of $\widetilde{\mathcal{D}_0}$, the above argument is easily adapted to prove the last part of the assertion. ∎

REMARK 3.4. Notice that the boundaries of $\mathcal{D}_0$ and $\mathcal{D}_n$ are mapped by μ in either the left half line $(-\infty, -1]$ or the right half line $[1, +\infty)$. Indeed, we have

$$\mu(\Gamma_n) = \begin{cases} (-\infty, -1], & \text{for } n \text{ odd}, \\ [1, +\infty), & \text{for } n \text{ even}. \end{cases}$$

Theorem 3.3 allows us to give the following definition.

DEFINITION 3.5. We denote by $\varphi : \mathbb{H} \setminus (-\infty, -1] \to \mathcal{D}_0$ the biregular inverse of $\mu|_{\mathcal{D}_0}$.

REMARK 3.6. As an immediate consequence of the definition of φ, we have

$$\varphi \circ (\mu|_{\mathcal{D}_0}) = \mathrm{id}_{\mathcal{D}_0}, \quad (\mu|_{\mathcal{D}_0}) \circ \varphi = \mu \circ \varphi = \mathrm{id}_{\mathbb{H}\setminus(-\infty,-1]}.$$

REMARK 3.7. In particular, we observe that $\varphi(1) = 0$ and that the function ν is never vanishing on $\mathcal{D}_0$. The first assertion is trivial because 0 is the unique point in $\mathcal{D}_0$ whose image via μ is 1. Moreover, $q \in \mathbb{H}$ is a zero of ν if and only if $q \in \{n^2\pi^2 \mid n \in \mathbb{Z} \setminus \{0\}\}$ and none of these points belong to $\mathcal{D}_0$.

4. Initial existence results

We now start our discussion on the solvability of equation (1.1). The first case we consider is when the function g preserves one slice; in this case, under suitable topological hypothesis on Ω, namely, the fact that Ω is a slice-contractible domain, the solvability of equation (1.1) follows almost immediately from the complex holomorphic case. Nonetheless, differences with the complex case arise when looking to the case of a product domain and of a slice preserving function.

Proposition 4.1. *Let Ω be a slice-contractible domain and let $g \in \mathcal{SR}^*(\Omega)$. If g is one-slice preserving, then there exists $f \in \mathcal{SR}(\Omega)$ which preserves the same slice as g and such that $\exp_*(f) = g$. Moreover,*

- *if Ω is a slice domain and g is slice preserving and positive on $\Omega \cap \mathbb{R}$, then there exists a unique slice preserving $*$-logarithm of g;*
- *if Ω is a product domain and g is slice preserving, then there exists a slice preserving $*$-logarithm of g.*

Proof. Let us denote by $\mathbb{C}_I$ the preserved slice and consider the never vanishing restriction $g_I : \Omega_I \to \mathbb{C}_I$. Since either Ω_I is simply connected (whether Ω contains real points) or its two connected components are (whether Ω contains no real points), then we can find a logarithm of g_I that is a holomorphic function $f_I : \Omega_I \to \mathbb{C}_I$ such that $\exp(f_I) = g_I$. Let f be the slice regular extension of f_I to Ω. Then, we have

$$\left(\exp_*(f)\right)_I = \exp(f_I) = g_I,$$

and the identity principle entails the first part of the statement.

If Ω is a slice domain and g is slice preserving and positive on $\Omega \cap \mathbb{R}$, then $g_I(\Omega_I \cap \mathbb{R}) \subset (0, +\infty)$ and therefore there exists a unique logarithm $f_I : \Omega_I \to \mathbb{C}_I$ of g_I such that $f_I(\Omega_I \cap \mathbb{R}) \subset \mathbb{R}$; the extension of such a function to Ω is the required slice preserving solution f.

Now, suppose that Ω is a product domain and that g is slice preserving. Fix any $I \in \mathbb{S}$ and take $h = h_0 + h_1 I$ as a $\mathbb{C}_I$-preserving slice function such that $\exp_*(h) = g$. Formula (2.5) entails that $\exp_*(h_0)\nu(h_1^2)h_1 I \equiv 0$, as g is slice preserving. If $h_1 \equiv 0$, then setting $f = h = h_0$ gives the required function. Otherwise, $\nu(h_1^2)$ must be identically zero and hence there exists $n \in \mathbb{N} \setminus \{0\}$ such that $h_1^2 \equiv \pi^2 n^2$ since $\nu(q) = 0$ if and only if $q \in \mathbb{R} \setminus \{0\}$ and $q = \pi^2 n^2$ for $n \in \mathbb{N} \setminus \{0\}$. Thus, $(h_1 - \pi n) \cdot (h_1 + \pi n) \equiv 0$ and Proposition 2.13 entails that either $h_1 \equiv \pi n$ or $h_1 = -\pi n$.

If n is even, then $\exp_*(\pm\pi n I) \equiv 1$ on Ω, and by taking $f = h_0$, we are done; if n is odd, then $\exp_*(\pm\pi n I) \equiv -1$ on Ω, and thus we obtain the thesis by taking $f = h_0 + \pi\mathcal{J}$. ∎

Corollary 4.2. *Let Ω be a slice-contractible domain and let g be a one-slice preserving function. Then, g has a $*$-logarithm if and only if it is never vanishing. In particular, this holds for $g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$.*

Trivially, if Ω is slice, g is slice preserving and never vanishing on Ω, and it is negative on $\Omega \cap \mathbb{R}$, then there exists no slice preserving $*$-logarithm of g. Nonetheless, the family of $*$-logarithms of a slice preserving function with this feature is quite large and displays an unexpected behavior.

Example 4.3. For any $a \in \mathbb{R} \setminus \{0\}$ and $I \in \mathbb{S}$, we have that

$$\exp_*(aI) \equiv \cos(a) + \sin(a)I.$$

In particular, $\exp_*(\pi i) = \exp_*(\pi j) \equiv -1$, while

$$\begin{aligned}\exp_*(\pi i + \pi j) &= \exp_*\left(\sqrt{2}\pi \cdot \frac{i+j}{\sqrt{2}}\right)\\ &= \cos(\sqrt{2}\pi) + \sin(\sqrt{2}\pi)\frac{i+j}{\sqrt{2}}\\ &\neq 1 = \exp_*(\pi i) * \exp_*(\pi j),\end{aligned}$$

giving an explicit example of the application of [3, Theorem 4.14] (here $(\pi i)^s_v = (\pi j)^s_v = \pi^2$ and $2\langle \pi i, \pi j\rangle_* = 0$, so that $\exp_*(\pi i + \pi j) \not\equiv \exp_*(\pi i) * \exp_*(\pi j)$).

Corollary 4.4. *Let $g \in \mathcal{SR}^*(\Omega)$ be such that there exists $\chi \in \mathcal{SR}^*(\Omega)$ for which $\chi^{-*} * g * \chi$ is one-slice preserving. Then, g has a $*$-logarithm.*

Remark 4.5. Notice that if g is conjugated to a one-slice preserving function via a never vanishing χ, then g^s_v has a square root. Nonetheless, the existence of a square root of g^s_v is not a sufficient condition for the existence of a $*$-logarithm of a never vanishing function (see Example 6.9).

Proposition 4.1 allows us to prove a natural generalization to the quaternions of a classical result in the theory of holomorphic functions that will be used later in the search for a solution of equation (1.1). The second part of the statement gives a uniqueness result obtained accordingly to the structure of the domain Ω: indeed, if the domain is slice, uniqueness up to a constant integer multiple of 2π holds, while in the case of a product domain, uniqueness up to a constant integer multiple of 2π holds for slice preserving functions only.

PROPOSITION 4.6. *Let Ω be a slice-contractible domain. Given $a_0, a_1 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $a_0^2 + a_1^2 \equiv 1$, there exists $\gamma \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that*

$$\begin{cases} \cos_*(\gamma) = a_0 \\ \sin_*(\gamma) = a_1. \end{cases} \tag{4.1}$$

Moreover,

- *if Ω is a slice domain and $\widetilde{\gamma} \in \mathcal{SR}(\Omega)$ is another solution of (4.1), then there exists $k \in \mathbb{Z}$ such that $\widetilde{\gamma} \equiv \gamma + 2k\pi$;*
- *if Ω is a product domain and $\widetilde{\gamma} \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ is another solution of (4.1), then there exists $k \in \mathbb{Z}$ such that $\widetilde{\gamma} \equiv \gamma + 2k\pi$.*

PROOF. Fix $I \in \mathbb{S}$ and consider the $\mathbb{C}_I$-slice preserving function given by $a = a_0 + a_1 I$. Clearly, $a^s \equiv 1$ so that a is never vanishing. Proposition 4.1 gives $f = f_0 + f_1 I \in \mathcal{SR}_I(\Omega)$ such that $\exp_*(f) = a$; the last equality can also be written as

$$\exp(f_0)(\mu(f_v^s) + \nu(f_v^s) f_v) = \exp(f_0)(\mu(f_1^2) + \nu(f_1^2) f_1 I) = a_0 + a_1 I. \tag{4.2}$$

Since $a^s \equiv 1$, we obtain that $(\exp_*(f))^s \equiv \exp(2f_0) \equiv 1$, that is, $(\exp(f_0))^2 \equiv 1$. As $\mathcal{SR}_{\mathbb{R}}(\Omega)$ is an integral domain, then either $\exp(f_0) \equiv 1$ or $\exp(f_0) \equiv -1$. In the first case, the equalities

$$\begin{cases} \mu(f_1^2) = \cos(f_1), \\ \nu(f_1^2) f_1 = \sin(f_1), \end{cases}$$

together with (4.2), ensure that $\cos(f_1) = a_0$ and $\sin(f_1) = a_1$, so that $\gamma = f_1$ gives the required function. In the second case, performing the same computations as above gives that the function $\gamma = f_1 + \pi$ is a solution of system (4.1).

Now, suppose that Ω is slice and that $\widetilde{\gamma}$ is another solution of system (4.1). The fact that a_0 is slice preserving and $-1 \le a_0 \le 1$ on $\Omega \cap \mathbb{R}$ implies that

$$\big(\cos_*(\widetilde{\gamma})\big)(t) = \cos\big(\widetilde{\gamma}(t)\big)$$

belongs to the interval $[-1, 1]$ for any $t \in \Omega \cap \mathbb{R}$. As $\cos(\alpha + I\beta) = \cos(\alpha)\cosh(\beta) - \sin(\alpha)\sinh(\beta) I \in [-1, 1]$ if and only if $\beta = 0$, we have $\widetilde{\gamma}(t) \in \mathbb{R}$ for any $t \in \Omega \cap \mathbb{R}$, showing that $\widetilde{\gamma}$ is slice preserving. Considering the functions on $\Omega \cap \mathbb{R}$, formula (4.1) trivially entails that $\widetilde{\gamma} \equiv \gamma + 2k\pi$, for some $k \in \mathbb{Z}$.

We are left to consider the case in which Ω is a product domain and $\widetilde{\gamma}$ is another slice preserving solution of (4.1). Let $I \in \mathbb{S}$, and consider the restrictions of γ and $\widetilde{\gamma}$ on $\Omega_I = \Omega_I^+ \cup \Omega_I^-$. Trivially, there exist $n_+, n_- \in \mathbb{Z}$ such that $\widetilde{\gamma} = \gamma + 2\pi n_+$ on Ω_I^+ and $\widetilde{\gamma} = \gamma + 2\pi n_-$ on Ω_I^-. By the representation formula, we have, for $\beta > 0$

and $J \in \mathbb{S}$,

$$\begin{aligned}\widetilde{\gamma}(\alpha + J\beta) &= \frac{1 - JI}{2}\widetilde{\gamma}(\alpha + I\beta) + \frac{1 + JI}{2}\widetilde{\gamma}(\alpha - I\beta)\\ &= \frac{1 - JI}{2}\big(\gamma(\alpha + I\beta) + 2\pi n_+\big) + \frac{1 + JI}{2}\big(\gamma(\alpha - I\beta) + 2\pi n_-\big)\\ &= \gamma(\alpha + J\beta) + \pi\big(n_+ + n_- + J(n_- - n_+)I\big)\end{aligned}$$

and this function is slice preserving if and only if $n_- - n_+ = 0$. Thus, $\widetilde{\gamma} = \gamma + 2\pi n_+$ on Ω_I and the identity principle entails the proof. ∎

Notice that the hypothesis on the sliceness of the domain contained in the statement of Proposition 4.6 cannot be removed without adding a requirement on the behavior of the function $\widetilde{\gamma}$, as it is shown in the following example.

Example 4.7. Let $\Omega = \mathbb{H} \setminus \mathbb{R}$, fix any $I \in \mathbb{S}$, and consider $\widetilde{\gamma} : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined as $\widetilde{\gamma} = 2\pi \mathcal{J} I$. Clearly, $\widetilde{\gamma}_0 \equiv 0$, and hence $\widetilde{\gamma}^{*2} = -\widetilde{\gamma}^s \equiv 4\pi^2$. Thus, we have

$$\begin{aligned}\cos_*(\widetilde{\gamma}) &= \sum_{n\in\mathbb{N}} \frac{(-1)^n \widetilde{\gamma}^{*2n}}{(2n)!} = \sum_{n\in\mathbb{N}} \frac{(-1)^n (4\pi^2)^n}{(2n)!} = \sum_{n\in\mathbb{N}} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}\\ &= \cos(2\pi) = 1,\end{aligned}$$

and analogously, $\sin_*(\widetilde{\gamma}) = 0$. This gives a continuous family of functions parametrized by $I \in \mathbb{S}$ defined on $\mathbb{H} \setminus \mathbb{R}$ solving system (4.1) with $a_0 \equiv 1$ and $a_1 \equiv 0$, in sharp contrast with the "discrete" uniqueness behavior which holds in the case of slice domains.

The above proposition allows us to give a more precise description of the zero divisors whose "real" part is identically zero in the case when the product domain Ω is slice-contractible (see the third paragraph of [4] for a comprehensive investigation of such functions). We denote by $\mathcal{SEM}(\Omega)$ the algebra of slice semi-regular functions (see [18, 19] for the definitions and a detailed study of the singularities and [5] for an investigation in the flavor of a vector space structure over the field of slice preserving semi-regular functions).

Proposition 4.8. *Let Ω be a slice-contractible product domain and let $f \in \mathcal{SEM}(\Omega)$ be a zero divisor such that $f_0 \equiv 0$. Then, there exists an orthonormal basis (i, j, k) of* $\mathrm{Im}(\mathbb{H})$, *$\alpha \in \mathcal{SEM}_{\mathbb{R}}(\Omega)$ and $\vartheta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, such that*

$$f = \alpha i + \alpha \mathcal{J} \cos(\vartheta) j + \alpha \mathcal{J} \sin(\vartheta) k.$$

Proof. Since $f = f_v$ is not identically zero, we can find an orthonormal basis (i, j, k) of $\mathrm{Im}(\mathbb{H})$, such that $f = f_1 i + f_2 j + f_3 k$, with $f_1 \not\equiv 0$. Then, thanks to [5, Proposition 2.14], we can write $f = -2(fi)_0 i * \rho = 2 f_1 i * \rho$, for a suitable idempotent

$\rho = \frac{1}{2} + \rho_1 i + \rho_2 j + \rho_3 k \in \mathcal{SR}(\Omega)$, that is, $\rho * \rho = \rho$. Now, we have

$$\begin{aligned} f &= 2f_1 i * \rho = 2f_1 i * \left(\frac{1}{2} + \rho_1 i + \rho_2 j + \rho_3 k\right) \\ &= f_1 i - 2f_1\rho_1 + 2f_1\rho_2 k - 2f_1\rho_3 j, \end{aligned}$$

and therefore, the fact that $f = f_v$ implies $\rho_1 \equiv 0$, that is, $\rho = \frac{1}{2} + \rho_2 j + \rho_3 k$. As ρ is an idempotent, we obtain $\rho_2^2 + \rho_3^2 \equiv -\frac{1}{4}$, and hence the functions $\varphi_2 = 2\mathcal{J}\rho_2$ and $\varphi_3 = 2\mathcal{J}\rho_3$ satisfy $\varphi_2^2 + \varphi_3^2 \equiv 1$. Thus, Proposition 4.6 implies we can find $\vartheta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $\varphi_2 = \cos(\vartheta)$ and $\varphi_3 = \sin(\vartheta)$. Setting $\alpha = f_1$ gives the required equality

$$f = \alpha i + \alpha\mathcal{J}\cos(\vartheta) j + \alpha\mathcal{J}\sin(\vartheta) k. \qquad \blacksquare$$

5. Uniqueness results

We now begin a detailed investigation on the possible family of solutions of equation (1.1) starting from the case of slice preserving functions. Our first statement classifies slice preserving functions which share the same exponential, giving a first uniqueness result for the exponential problem in the case of slice domains.

Proposition 5.1. *Let $h_0, \tilde{h}_0 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ be such that*

$$\exp_*(h_0) \equiv \exp_*(\tilde{h}_0).$$

- *If Ω is a slice domain, then $h_0 \equiv \tilde{h}_0$.*
- *If Ω is a product domain, then there exists $n \in \mathbb{Z}$ such that $h_0 = \tilde{h}_0 + 2\pi n\mathcal{J}$.*

Proof. Since both h_0 and $\tilde{h}_0$ are slice preserving, Proposition 4.3 in [3] gives us the possibility to work on the difference $f_0 = h_0 - \tilde{h}_0$ which is a solution of

$$\exp(f_0) \equiv 1. \tag{5.1}$$

If Ω is a slice domain, equality (5.1) gives $f \equiv 0$ on $\Omega \cap \mathbb{R}$. The identity principle entails $f_0 \equiv 0$ on Ω, that is, $h_0 \equiv \tilde{h}_0$ on Ω and hence the thesis.

Assume now that $\Omega \cap \mathbb{R} = \emptyset$. Fix $I \in \mathbb{S}$ and restrict equality (5.1) to Ω_I. Since f_0 is slice preserving and Ω_I has two connected components, $\exp(f_0) \equiv 1$ implies the existence of $n_+, n_- \in \mathbb{Z}$ such that

$$f_0(x + Iy) = \begin{cases} 2\pi n_+ I, & \text{for } y > 0, \\ 2\pi n_- I, & \text{for } y < 0. \end{cases}$$

For $y > 0$ and $J \in \mathbb{S}$, the representation formula yields

$$\begin{aligned} f_0(x+Jy) &= \frac{1-JI}{2} f_0(x+Iy) + \frac{1+IJ}{2} f_0(x-Iy) \\ &= (1-JI)\pi n_+ I + (1+I)\pi n_- I \\ &= \pi\big((n_+ + n_-)I + (n_+ - n_-)J\big). \end{aligned}$$

Since $f_0 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, this equality gives $n_+ + n_- = 0$ and thus $f_0(x+Jy) = 2\pi n_+ J$ for any $y > 0$ and $J \in \mathbb{S}$, that is, $h_0 = \tilde{h}_0 + 2\pi n_+ \mathcal{J}$. ∎

Corollary 5.2. *Let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that*

$$\exp_*(h) \equiv \exp_*(\tilde{h}). \tag{5.2}$$

- *If Ω is a slice domain, then $h_0 \equiv \tilde{h}_0$ and $\exp_*(h_v) \equiv \exp_*(\tilde{h}_v)$.*
- *If Ω is a product domain, then there exists $n \in \mathbb{Z}$ such that $h_0 = \tilde{h}_0 + \pi n \mathcal{J}$. In this case, $\exp_*(h_v) \equiv \exp_*(\tilde{h}_v)$ if n is even and $\exp_*(h_v) \equiv -\exp_*(\tilde{h}_v)$ if n is odd.*

Proof. Using formula (2.4), equality (5.2) implies $\exp(2h_0) = \exp(2\tilde{h}_0)$. As $h_0, \tilde{h}_0$ are slice preserving functions, Proposition 5.1 gives that either $h_0 \equiv \tilde{h}_0$ or Ω is a product domain and

$$2h_0 = 2\tilde{h}_0 + 2\pi n \mathcal{J}, \tag{5.3}$$

for a suitable $n \in \mathbb{Z}$. As $h_0, \tilde{h}_0$ and $\mathcal{J}$ are slice preserving functions, by formula (2.6), we have that $\exp_*(h) = \exp_*(h_0)\exp_*(h_v)$ and $\exp_*(\tilde{h}) = \exp_*(\tilde{h}_0)\exp_*(\tilde{h}_v)$. A straightforward application of formula (5.3) yields the thesis. ∎

We now study when two functions $h, \tilde{h}$ give the same $*$-exponential. The first case we analyze is when $\exp_*(h)$ is slice preserving. In order to simplify notations, we set

$$\mathfrak{N}(\Omega) := \left\{ f \in \mathcal{SR}(\Omega) \mid \exists m \in \mathbb{Z} \setminus \{0\},\ f_v^s = m^2\pi^2 \right\} \cup \mathcal{SR}_{\mathbb{R}}(\Omega).$$

We strongly underline that if $f \in \mathfrak{N}(\Omega)$, then either $f_v \equiv 0$ or f_v is never vanishing; in both cases, f_v has no non-real isolated zeroes and f_v^s always admits a square root (which, by the way, is constant).

Theorem 5.3. *Let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that $\exp_*(h) = \exp_*(\tilde{h}) \in \mathcal{SR}_{\mathbb{R}}(\Omega)$.*

- *If Ω is a slice domain, then $h_0 \equiv \tilde{h}_0$, $h, \tilde{h} \in \mathfrak{N}(\Omega)$, and $\sqrt{h_v^s} \equiv \sqrt{\tilde{h}_v^s} \pmod{2\pi}$.*
- *If Ω is a product domain, then there exists $n \in \mathbb{Z}$ such that $h_0 = \tilde{h}_0 + \pi n \mathcal{J}$. Moreover, $h, \tilde{h} \in \mathfrak{N}(\Omega)$ and $\sqrt{h_v^s} \equiv \sqrt{\tilde{h}_v^s} + n\pi \pmod{2\pi}$.*

PROOF. By Corollary 5.2, we have that either $h_0 = \tilde{h}_0$ or Ω is a product domain and there exists $n \in \mathbb{Z}$ such that $h_0 = \tilde{h}_0 + \pi n \mathcal{J}$. In particular, these equalities imply

$$\exp(h_0) \equiv \exp(\tilde{h}_0), \qquad \text{for } n \text{ even,} \tag{5.4}$$

$$\exp(h_0) \equiv -\exp(\tilde{h}_0), \qquad \text{for } n \text{ odd,} \tag{5.5}$$

where the second one can occur only if Ω is a product domain. We start by considering the case of a slice domain. In this case, we have $\exp_*(h_v) = \exp_*(\tilde{h}_v) \in \mathcal{SR}_{\mathbb{R}}(\Omega)$. If $h_v = \tilde{h}_v \equiv 0$, we are done. Otherwise, we can suppose $h_v \not\equiv 0$. As $\exp_*(h) = \exp_*(h_0)(\mu(h_v^s) + \nu(h_v^s)h_v) \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, we have $\exp_*(h_0)\nu(h_v^s)h_v \equiv 0$, which implies $\nu(h_v^s) \equiv 0$ and therefore guarantees that there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $h_v^s = m^2\pi^2$. If $\tilde{h}_v \equiv 0$, we have

$$\cos(m\pi) = \mu(h_v^s) = \exp_*(h_v) = \exp_*(\tilde{h}_v) = \exp_*(0) \equiv 1,$$

thus showing that m is even and that

$$\sqrt{h_v^s} = \sqrt{m^2\pi^2} \equiv 0 = \sqrt{\tilde{h}_v^s} \pmod{2\pi}.$$

If $\tilde{h}_v \not\equiv 0$, as $\nu(h_v^s)h_v = \nu(\tilde{h}_v^s)\tilde{h}_v \equiv 0$, we can find $n \in \mathbb{Z} \setminus \{0\}$ such that $\tilde{h}_v^s \equiv n^2\pi^2$; as $\mu(h_v^s) = \mu(\tilde{h}_v^s)$, we find that m and n have the same parity, thus showing again $\sqrt{h_v^s} \equiv \sqrt{\tilde{h}_v^s} \pmod{2\pi}$.

The case of a product domain is obtained following the same lines of reasoning, by studying separately the two cases given by formulas (5.4) and (5.5). ■

We now turn to the case when $\exp_*(h)$ is not slice preserving; under this hypothesis, if Ω is slice, we find a dichotomy: either h_v and $\tilde{h}_v$ have no non-real isolated zeroes and are such that h_v^s and $\tilde{h}_v^s$ have a square root (independently from the fact that Ω is slice-contractible), in which case we find a "discrete" family of functions producing the same $*$-exponential, or $h = \tilde{h}$; that is, there is an unexpected uniqueness result. An analogous, more refined statement can be obtained also in the case of a product domain.

THEOREM 5.4. *Let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that $h \neq \tilde{h}$ and $\exp_*(h) = \exp_*(\tilde{h}) \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$.*

- *If Ω is a slice domain, then $h_0 \equiv \tilde{h}_0$, both h_v and $\tilde{h}_v$ have no non-real isolated zeroes, both h_v^s and $\tilde{h}_v^s$ have a square root on Ω, and there exist $m \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, and $H_v \in \mathcal{SR}(\Omega)$ with $H_v^s \equiv 1$ such that $h_v = \alpha H_v$ and $\tilde{h}_v = (\alpha + 2\pi m)H_v = h_v + 2\pi m H_v$, so that*
$$\tilde{h} = h + 2\pi m H_v.$$

- *If Ω is a product domain, then one of the following holds:*
  (1) *There exists $n \in \mathbb{Z} \setminus \{0\}$ such that*
$$\tilde{h} = h + 2\pi n \mathcal{J}.$$
  (2) *Both h_v and $\tilde{h}_v$ are not zero divisors and have no non-real isolated zeroes, both h_v^s and $\tilde{h}_v^s$ have a square root on Ω, and there exist $n, m \in \mathbb{Z}$ such that $n \equiv m \pmod 2$ and $m \neq 0$, $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, and $H_v \in \mathcal{SR}(\Omega)$ with $H_v^s \equiv 1$, such that $h_v = \alpha H_v$ and*
$$\tilde{h} = h_0 + \pi n \mathcal{J} + (\alpha + \pi m) H_v = h + \pi(n\mathcal{J} + m H_v).$$

Proof. If $h_v = \tilde{h}_v$, then $\exp(h_0) = \exp(\tilde{h}_0)$, so the hypothesis $h \neq \tilde{h}$ and Proposition 5.1 give that Ω is a product domain and there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $h_0 = \tilde{h}_0 + 2\pi n \mathcal{J}$. Thus, from now on, we assume that $h_v \neq \tilde{h}_v$.

If Ω is a slice domain, Corollary 5.2 gives $h_0 = \tilde{h}_0$ and $\exp_*(h_v) = \exp_*(\tilde{h}_v) \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$. We have $\nu(h_v^s)h_v \equiv \nu(\tilde{h}_v^s)\tilde{h}_v \not\equiv 0$. This implies that h_v and $\tilde{h}_v$ are linearly dependent on $\mathcal{SR}_{\mathbb{R}}(\Omega)$ and thus commute (see [3, Proposition 2.10]). Now, choose $p \in \Omega \setminus \mathbb{R}$ such that $\nu(h_v^s)h_v$ is never vanishing on $\mathbb{S}_p$, and denote by $\widetilde{\Omega}$ a slice-contractible product domain contained in Ω such that $\nu(h_v^s)h_v$ is never vanishing on $\widetilde{\Omega}$.

In particular, this means that h_v^s is never vanishing on $\widetilde{\Omega}$, and therefore there exists a square root α of h_v^s which is never vanishing on $\widetilde{\Omega}$. Moreover, the equality $\nu(h_v^s)h_v \equiv \nu(\tilde{h}_v^s)\tilde{h}_v$ gives that $\nu(\tilde{h}_v^s)$ is never vanishing on $\widetilde{\Omega}$, and thus there exists $\beta \in \mathcal{SR}_{\mathbb{R}}(\widetilde{\Omega})$ such that $\tilde{h}_v = \beta h_v$. As $h \not\equiv \tilde{h}$ on Ω, the identity principle gives that β is not identically equal to 1 on $\widetilde{\Omega}$. Since h_v and $\tilde{h}_v$ commute, we can apply [3, Proposition 4.3], and we are left to study $\exp_*((1-\beta)h_v) \equiv 1$. As $h_v \not\equiv 0$, this implies that there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $((1-\beta)h_v)^s = 4m^2\pi^2$, that is, $(1-\beta)^2 h_v^s = 4m^2\pi^2$, therefore giving $\beta = \frac{2\pi m}{\alpha} + 1$, up to a possible change of sign of m. We then obtain
$$\tilde{h}_v = \left(\frac{2\pi m}{\alpha} + 1\right) h_v,$$
on $\widetilde{\Omega}$, which can also be written as
$$\alpha \cdot \tilde{h}_v = (2\pi m + \alpha) h_v. \tag{5.6}$$
By computing the symmetrized function of both members of equation (5.6), we obtain $h_v^s \tilde{h}_v^s = (2\pi m + \alpha)^2 h_v^s$. As h_v^s is never vanishing on $\widetilde{\Omega}$, we also have $\tilde{h}_v^s = (2\pi m + \alpha)^2$, from which we infer
$$\alpha = \frac{\tilde{h}_v^s - h_v^s - 4\pi^2 m^2}{4\pi m},$$

on $\widetilde{\Omega}$, thanks to $\alpha^2 = h_v^s$. By squaring both members, we thus get

$$h_v^s = \frac{1}{16\pi^2 m^2}(\tilde{h}_v^s - h_v^s - 4\pi^2 m^2)^2.$$

The last equality was obtained on $\widetilde{\Omega}$, but since both members are defined and regular on Ω, by the identity principle, we have that it holds on the whole Ω, thus showing that h_v^s has a square root in Ω. Up to a change of sign, we can suppose that it agrees with the previous one on $\widetilde{\Omega}$, so we still denote it by $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$. A further application of the identity principle shows that equality (5.6) continues on Ω.

Now, suppose that q_0 is a non-real isolated zero of h_v; thus, h_v^s and therefore α are identically zero on $\mathbb{S}_{q_0}$. The left hand side of equality (5.6) is then identically zero on $\mathbb{S}_{q_0}$, while the right hand side is equal to $2\pi m \cdot h_v$; as $m \neq 0$, the last function has an isolated zero in q_0. This contradiction shows that h_v cannot have non-real isolated zeroes.

Thanks to Proposition 2.27, the function $H_v := \frac{h_v}{\alpha}$ is a well-defined slice regular function on Ω with $H_v^s \equiv 1$; as $h_v = \alpha H_v$, by the zero-product property (see Proposition 2.13), equality (5.6) can also be written in the form $\tilde{h}_v = (2\pi m + \alpha)H_v$, which holds on the whole of Ω by a further application of the identity principle.

Now, since $\tilde{h}_v$ is the product of the never vanishing function H_v by the slice-preserving factor $(2\pi m + \alpha)$, we have that the only zeroes of $\tilde{h}_v$ are the zeroes of $(2\pi m + \alpha)$. Since $(2\pi m + \alpha)$ is slice preserving, then $\tilde{h}_v$ has no non-real isolated zeroes as well. Lastly, $\tilde{h}_v^s = (2\pi m + \alpha)^2$, which ensures that $\tilde{h}_v^s$ has a square root on the domain Ω.

Now, we turn our attention to the case in which Ω is a product domain. Corollary 5.2 gives that there exists $n \in \mathbb{Z}$ such that $h_0 = \tilde{h}_0 + \pi n \mathcal{J}$, and moreover, either $\exp_*(h_v) \equiv \exp_*(\tilde{h}_v)$ if n is even or $\exp_*(h_v) \equiv -\exp_*(\tilde{h}_v)$ if n is odd. We first deal with the case when n is even; in particular, this gives $\nu(h_v^s)h_v \equiv \nu(\tilde{h}_v^s)\tilde{h}_v \not\equiv 0$.

We first look at the case in which h_v is a zero divisor. As $h_v^s \equiv 0$, we have $\nu(h_v^s) \equiv 1$; thus, the above equality becomes $h_v = \nu(\tilde{h}_v^s)\tilde{h}_v \not\equiv 0$, which in particular implies that $\nu(\tilde{h}_v^s) \not\equiv 0$. By taking the symmetrized function of both members of the above equality, we obtain $0 \equiv h_v^s = \nu(\tilde{h}_v^s)^2\tilde{h}_v^s$, showing that $\tilde{h}_v^s \equiv 0$, too. Hence, $\nu(\tilde{h}_v^s) \equiv 1$, and therefore $h_v \equiv \tilde{h}_v$ which is a contradiction to the assumption $h_v \neq \tilde{h}_v$.

We now turn to the case in which h_v is not a zero divisor. As n is even, $\nu(h_v^s)h_v \equiv \nu(\tilde{h}_v^s)\tilde{h}_v$ is not a zero divisor, and $h_v \neq \tilde{h}_v$, we can argue as in the case of a slice domain obtaining that both h_v and $\tilde{h}_v$ have no non-real isolated zeroes and there exist an even $m \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, and $H_v \in \mathcal{SR}(\Omega)$ with $H_v^s \equiv 1$ such that $h_v = \alpha H_v$ and $\tilde{h}_v = (\alpha + \pi m)H_v = h_v + \pi m H_v$.

Finally, we consider the case in which n is odd. The only difference with the above reasoning is due to the fact that $\exp_*(h_v) \equiv -\exp_*(\tilde{h}_v)$. In the case when h_v is a zero divisor, again we get that $\tilde{h}_v$ is a zero divisor. Thus, $\nu(\tilde{h}_v^s) \equiv 1$, and we obtain $1 + h_v = -(1 + \tilde{h}_v)$ which is equivalent to $h_v + \tilde{h}_v = -2$. This is a contradiction because -2 is slice preserving and different from 0, while $h_v + \tilde{h}_v$ coincides with its vector part. Finally, in the case when h_v is not a zero divisor, the above reasoning gives that $\tilde{h}_v$ is not a zero divisor as well; moreover, both h_v and $\tilde{h}_v$ have no non-real isolated zeroes, and there exists $m \in \mathbb{Z}$ odd, $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, $H_v \in \mathcal{SR}(\Omega)$ with $H_v^s \equiv 1$ such that $h_v = \alpha H_v$ and $\tilde{h}_v = (\alpha + \pi m)H_v = h_v + \pi m H_v$. ■

Remark 5.5. Notice that the function $H_v \in \mathcal{SR}(\Omega)$ such that $H_v^s \equiv 1$ which appears in the previous statement is unique up to a change of sign and can be interpreted as the quotient of h_v by a square root of h_v^s. Indeed, if $\alpha H_v = \beta L_v = h_v$ for $\alpha, \beta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ and $H_v^s = L_v^s \equiv 1$, we have $\alpha^2 = \beta^2$; thus, either $\alpha = \beta$ (which gives $L_v = H_v$) or $\alpha = -\beta$ (which gives $L_v = -H_v$). Moreover, $h_v^s = (\alpha H_v)^s = \alpha^2 H_v^s = \alpha^2$, so that α is a square root of h_v^s.

To stress the relevance, and also the unexpectedness, of the above theorem, we give a couple of partial restatements which underline the uniqueness result when Ω is slice and the “vector” part of the function has a non-real isolated zero.

Corollary 5.6. *Let Ω be slice and let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that* $\exp_*(h) \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$. *If h_v has a non-real isolated zero, then* $\exp_*(h) = \exp_*(\tilde{h})$ *if and only if $h \equiv \tilde{h}$.*

Corollary 5.7. *Let Ω be slice and let $h, \tilde{h} \in \mathcal{SR}(\Omega)$ be such that* $\exp_*(h) = \exp_*(\tilde{h}) \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$. *If there exists $q \in \Omega$ such that $h(q) = \tilde{h}(q)$, then $h \equiv \tilde{h}$.*

Proof. As Ω is slice, if $h \not\equiv \tilde{h}$, then there exist $m \in \mathbb{Z} \setminus \{0\}$ and H_v with $H_v^s \equiv 1$ such that $\tilde{h} = h + 2\pi m H_v$. Thus, $\tilde{h}(q) = h(q) + 2\pi m H_v(q)$ gives $m H_v(q) = 0$. Since $H_v^s \equiv 1$, this entails $m = 0$, which is a contradiction. ■

6. Existence results for the $*$-logarithm

As a first consequence of the results obtained in Section 4, Proposition 4.1 allows us to restrict our attention to a particular class of never vanishing functions.

Remark 6.1. Let Ω be a slice-contractible domain. For any $g \in \mathcal{SR}^*(\Omega)$, we have that g^s belongs to $\mathcal{SR}^*(\Omega) \cap \mathcal{SR}_{\mathbb{R}}(\Omega)$, and it is positive on the reals if $\Omega \cap \mathbb{R} \neq \emptyset$. Then, Proposition 4.1 entails the existence of a $\psi_g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $\exp(\psi_g) = g^s$ (and ψ_g is unique if Ω is slice). A trivial computation shows that $\exp(-\psi_g/2)g$ belongs to $\mathcal{SR}^1(\Omega)$. Moreover, since $\exp(-\psi_g/2)$ is slice preserving, by [3, Corollary 4.4], we have that g has a $*$-logarithm if and only if $\exp(-\psi_g/2)g$ has. Thanks to these considerations, without loss of generality, we can reduce ourselves to study equation (1.1) in the case when $g^s \equiv 1$.

ASSUMPTION 6.2. Since Corollary 4.2 gives the existence of a $*$-logarithm for all never vanishing slice preserving regular functions, from now on, we consider equation (1.1) only in the case when $g \notin \mathcal{SR}_{\mathbb{R}}(\Omega)$, that is, $g_v \not\equiv 0$.

PROPOSITION 6.3. *Let $g \in \mathcal{SR}^1(\Omega)$ and suppose that f is a $*$-logarithm of g. If Ω is slice, then $f_0 \equiv 0$; if Ω is product, then there exists $n \in \mathbb{Z}$ such that $f_0 \equiv n\pi\mathcal{J}$. In particular,*

- *if Ω is slice, then any $*$-logarithm of g has "real part" identically zero;*
- *if Ω is product, then, up to substituting g with $-g$, we can find a $*$-logarithm of g whose real part is identically zero.*

PROOF. Let us assume that f is a solution of equation (1.1). Thanks to Proposition 2.19, as $g^s \equiv 1$, we have that $\exp(2f_0) \equiv 1 \equiv \exp(0)$. Thus, Corollary 5.2 ensures that $f_0 \equiv 0$ if Ω is a slice domain, while there exists $n \in \mathbb{Z}$ such that $f_0 \equiv n\pi\mathcal{J}$ if Ω is a product domain. ∎

The above proposition tells us that we can limit ourselves to look for solutions of $\exp_*(f_v) = g$ if Ω is slice or Ω is product and n is even and $\exp_*(f_v) = -g$ if Ω is product and n is odd.

The following result sets the existence of a $*$-logarithm for a never vanishing function whose "vector part" is a zero divisor (obviously, this case can occur only if Ω is a product domain).

PROPOSITION 6.4. *Let Ω be a slice-contractible domain and let $g \in \mathcal{SR}^*(\Omega)$ be such that g_v is a zero divisor. Then, there exists $f \in \mathcal{SR}(\Omega)$ such that $\exp_*(f) = g$.*

PROOF. By Remark 6.1, we can suppose that $g^s = g_0^2 + g_v^s \equiv 1$. As g_v is a zero divisor, we have $g_v^s \equiv 0$ and therefore $g_0^2 \equiv 1$, which entails that either $g_0 \equiv 1$ or $g_0 \equiv -1$. In the first case, a trivial computation gives

$$\exp_*(g_v) = \mu(g_v^s) + \nu(g_v^s)g_v = \mu(0) + \nu(0)g_v = 1 + g_v = g_0 + g_v = g,$$

and in the second,

$$\begin{aligned}\exp_*(\pi\mathcal{J} - g_v) &= \exp(\pi\mathcal{J}) * \big(\mu(g_v^s) + \nu(g_v^s)(-g_v)\big) = -\big(\mu(0) - \nu(0)g_v\big) \\ &= -1 + g_v = g_0 + g_v = g.\end{aligned}$$

∎

ASSUMPTION 6.5. Thanks to Proposition 6.4, we can refine 6.2 by assuming that g_v is neither identically zero nor a zero divisor.

Thanks to formula (2.5), if $g = \exp_*(f_v)$, then

$$\begin{cases} \mu(f_v^s) = g_0, \\ \nu(f_v^s) f_v = g_v. \end{cases} \tag{6.1}$$

A first simple necessary condition in order to ensure the solvability of equation (1.1) entails the behavior of g at non-real isolated zeroes of g_v. As a surprising consequence, we obtain that the presence of non-real isolated zeroes of g_v could be an obstruction to the existence of a $*$-logarithm of g. This feature underlines the strong difference between the complex and the quaternionic case for the exponential function.

PROPOSITION 6.6. *If $g \in \mathcal{SR}^1(\Omega)$ has a $*$-logarithm, we have that*

(1) *if Ω is a slice domain and q_0 is a non-real isolated zero of g_v, then $g(q_0) = 1$;*

(2) *if Ω is a product domain and q_0, q_1 are non-real isolated zeroes of g_v, then either $g(q_0) = g(q_1) = 1$ or $g(q_0) = g(q_1) = -1$.*

PROOF. Let Ω be a slice domain and f a $*$-logarithm of g. By Proposition 6.3, we have that $f = f_v$. Then, if q_0 is a non-real isolated zero of g_v, the second equation of system (6.1) implies that $\nu(f_v^s) f_v$ has a non-real isolated zero at q_0. Since $\nu(f_v^s)$ is slice preserving, we have that q_0 is a non-real isolated zero of f_v and thus $f_v^s(q_0) = 0$. Therefore, the first equation gives $g(q_0) = g_0(q_0) = \mu(0) = 1$.

If Ω is a product domain, let $f = f_v$ be a $*$-logarithm of either g or $-g$. Again, $\nu(f_v^s) f_v$ has non-real isolated zeroes at q_0 and q_1, so that f_v has non-real isolated zeroes at q_0 and q_1 and thus $f_v^s(q_0) = f_v^s(q_1) = 0$. The first equation of system (6.1) thus gives $g(q_0) = g(q_1) = 1$, if f is a $*$-logarithm of g, and $-g(q_0) = -g(q_1) = 1$, if f is a $*$-logarithm of $-g$. ■

In particular, if Ω is a slice domain, the previous proposition gives a strong obstruction to the existence of a $*$-logarithm. The following two corollaries give explicit restraints to the existence of a $*$-logarithm: the first one applies to any slice domain, while the function must have a special form, and the second one holds in a smaller class of domains, but for a larger class of functions.

COROLLARY 6.7. *Let Ω be a slice domain and let $f \in \mathcal{SR}(\Omega)$ be such that f_v has a non-real isolated zero. Then, $-\exp_*(f)$ has no $*$-logarithm.*

PROOF. As $-\exp_*(f) = -\exp(f_0)\exp_*(f_v)$, then $-\exp_*(f)$ has a $*$-logarithm if and only if $-\exp_*(f_v)$ has. Now, notice that $(-\exp_*(f_v))^s \equiv 1$, so, if $-\exp_*(f_v)$ has a $*$-logarithm, then it fulfills the hypotheses of Proposition 6.6. Let us denote by q_0 a non-real isolated zero of f_v. On $\mathbb{S}_{q_0}$, we have $f_v^s \equiv 0$; hence, $\mu(f_v^s) = 1$ on $\mathbb{S}_{q_0}$, and thus, $-\exp_*(f_v)(q_0) = -1$ which is a contradiction to Proposition 6.6. ■

COROLLARY 6.8. *Let Ω be a slice-contractible slice domain and $g \in \mathcal{SR}(\Omega)$ a never vanishing function such that g_v has a non-real isolated zero. Then, at least one between g and $-g$ has no $*$-logarithm.*

PROOF. By using the notation contained in Remark 6.1, we have that $\psi_g = \psi_{-g}$, and thus $\exp_*(-\psi_g/2)g$ and $\exp_*(-\psi_{-g}/2)(-g)$ are opposite one another. Proposition 6.6 ensures that one of these two functions has no $*$-logarithm, and thus at least one of the two functions g and $-g$ has no $*$-logarithm, too. ■

EXAMPLE 6.9. In view of Corollary 6.7, we may give a large family of examples of never vanishing functions without $*$-logarithm. Take, for instance, the polynomials $(q-i)*j$, $(q-i)^{*2}*j$, or $(q-i)*(q-2j)*(-2i+j)$. It is not difficult to check that these three polynomials have no "real part" and have only non-real isolated zeros. Therefore, the functions $-\exp_*((q-i)*j)$, $-\exp_*((q-i)^{*2}*j)$, and $-\exp_*((q-i)*(q-2j)*(-2i+j))$ have no $*$-logarithm. In particular, notice that

$$-\exp_*\big((q-i)^{*2}*j\big)_v = -\nu\big((q^2+1)^2\big)(q-i)^{*2}*j$$

and its symmetrized function is given by $\nu((q^2+1)^2)^2(q^2+1)^2$ which trivially has the square root $\nu((q^2+1)^2)(q^2+1)$. This provides the example of a function $g \in \mathcal{SR}^1(\mathbb{H})$ such that g_v^s has a square root but g has no $*$-logarithm as we were referring in Remark 4.5.

REMARK 6.10. Notice that if g is one-slice preserving, then g_v cannot have non-real isolated zeroes. Indeed, if the preserved slice lies in $\mathbb{C}_I$, then $g_v = g_1 I$, where $g_1 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ and hence g_v has only real and spherical isolated zeroes.

If Ω is slice-contractible, as g_v is neither identically zero nor a zero divisor by Assumption 6.5, following the outline of the proof of [2, Proposition 3.1] (and taking Ω_I^+ as the unitary disc centered at $2i$ and h without spherical zeroes in the case of a product domain), we can find $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ and $W \in \mathcal{SR}(\Omega)$, such that

$$g_v = \alpha W, \tag{6.2}$$

where $W \not\equiv 0$ is not a zero divisor and has neither real nor spherical zeroes. In particular, we notice that $W_0 \equiv 0$. The second equation of system (6.1) becomes now

$$\nu(f_v^s) f_v = \alpha W. \tag{6.3}$$

LEMMA 6.11. *Let f be a slice regular function satisfying equation (6.3), where $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega) \setminus \{0\}$ and W has neither real nor spherical zeroes. Then, there exists $\beta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, such that $f_v = \beta W$.*

Proof. Equation (6.3) ensures that $\nu(f_v^s)$ is not identically zero. If q_0 is a real isolated zero of $\nu(f_v^s)$ of multiplicity n, then it is a real isolated zero of αW of multiplicity greater than or equal to n; as W has no real zeroes, then q_0 is a real isolated zero of α with multiplicity greater than or equal to n. The same holds for the spherical zeroes of $\nu(f_v^s)$. Choose $I \in \mathbb{S}$ and consider the restriction of both $\nu(f_v^s)$ and α to Ω_I. The above considerations on the multiplicities of these functions entail that there exists an intrinsic holomorphic function β_I on Ω_I such that $\alpha = \beta_I \nu(f_v^s)$ on Ω_I; we denote by β the regular extension of β_I to Ω. As both α and $\nu(f_v^s)$ are slice preserving, then β is slice preserving too and $\alpha = \beta\nu(f_v^s)$ on Ω by the identity principle. Now, write $\nu(f_v^s) f_v = \beta\nu(f_v^s) W$; as $\nu(f_v^s)$ is a non-identically zero slice preserving function, we obtain that $f_v = \beta W$. ∎

Thanks to system (6.1) and Lemma 6.11, we obtain $\nu(W^s\beta^2)\beta W = \alpha W$ that entails the following result.

Lemma 6.12. *Let $g \in \mathcal{SR}^1(\Omega)$ be such that $g_v = \alpha W$ as in equation (6.2). If f_v is a $*$-logarithm of g, then there exists $\beta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, such that $f_v = \beta W$ and β satisfies*

$$\begin{cases} \mu(W^s\beta^2) = g_0, \\ \nu(W^s\beta^2)\beta = \alpha. \end{cases} \tag{6.4}$$

Contrarily, if β is a solution of the previous system, then $f_v = \beta W$ is a $*$-logarithm of g.

Proof. Suppose f_v is a $*$-logarithm of g, i.e., a solution of $\exp_*(f_v) = g$. Then, Lemma 6.11 ensures that there exists $\beta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $f_v = \beta W$. Thus, $\exp_*(f_v) = g$ is equivalent to

$$\begin{cases} \mu(W^s\beta^2) = g_0, \\ \nu(W^s\beta^2)\beta W = \alpha W. \end{cases}$$

As W is not identically zero and $\nu(W^s\beta^2)\beta, \alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, we can cancel W and hence we obtain system (6.4).

Contrarily, suppose that β is a solution of system (6.4) and set $f = f_v = \beta W$. Thus,

$$\begin{aligned} \exp_*(f_v) &= \mu(f_v^s) + \nu(f_v^s) f_v = \mu(W^s\beta^2) + \nu(W^s\beta^2)\beta W \\ &= g_0 + \alpha W = g. \end{aligned}$$

∎

This result allows us to prove that any function with $*$-logarithm carries along a whole family of functions with $*$-logarithm, thus generalizing Remark 2.26.

Corollary 6.13. *Let $g \in \mathcal{SR}^1(\Omega)$ be such that $g_v = \alpha W$, where $\alpha \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ and W has neither real nor spherical zeroes. If g has a $*$-logarithm, for any $U \in \mathcal{SR}(\Omega)$ such that $U_0 \equiv 0$ and $U^s \equiv W^s$, the function $\tilde{g} = g_0 + \alpha U$ has a $*$-logarithm as well.*

PROOF. Let $f_0 + f_v$ be a $*$-logarithm of g. By Proposition 6.3, either $\exp_*(f_v) = g$ or $\exp_*(f_v) = -g$. In the first case, Lemma 6.12 shows that there exists $\beta \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $f_v = \beta W$ and β satisfies (6.4). A straightforward computation shows that $\exp_*(\beta U) = \tilde{g}$. If $\exp_*(f_v) = -g$, we apply the above reasoning to $-g$, obtaining that $\exp_*(f_0 + \beta U) = \tilde{g}$. ∎

Our first positive result on the solvability of equation (1.1) deals with the more manageable case in which g_v has no non-real isolated zeroes; that is, the function W appearing in equation (6.3) is never vanishing. The next theorem provides the existence of a $*$-logarithm for this class of functions.

THEOREM 6.14. *Let Ω be slice-contractible. Then, any $g \in \mathcal{SR}^*(\Omega)$ such that g_v has no non-real isolated zeroes has a $*$-logarithm.*

PROOF. By Remark 6.1, we can limit ourselves to the case $g^s \equiv 1$. As W is never vanishing, then [3, Corollary 3.2] guarantees the existence of a square root $\tau \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ of W^s and system (6.4) becomes

$$\begin{cases} \mu(\tau^2\beta^2) = g_0, \\ \nu(\tau^2\beta^2)\beta = \alpha, \end{cases}$$

or, equivalently,

$$\begin{cases} \mu(\tau^2\beta^2) = g_0, \\ \nu(\tau^2\beta^2)\beta\tau = \alpha\tau. \end{cases}$$

Using the relation between the power series of μ and cos and ν and sin, the last system can be written as

$$\begin{cases} \cos(\tau\beta) = g_0, \\ \sin(\tau\beta) = \alpha\tau. \end{cases}$$

Since $g_0^2 + \alpha^2\tau^2 = g_0^2 + \alpha^2 W^s = g_0^2 + g_v^s = 1$, by Proposition 4.6, there exists $\gamma \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ which solves

$$\begin{cases} \cos(\gamma) = g_0, \\ \sin(\gamma) = \alpha\tau. \end{cases}$$

Now, setting $\beta = \gamma/\tau$, where the second term is well defined since $\mathcal{SR}_{\mathbb{R}}(\Omega)$ is abelian and τ is never vanishing, we have that β is a solution of system (6.4), and hence, thanks to Lemma 6.12, $f = f_v = \beta W$ is a $*$-logarithm of g. ∎

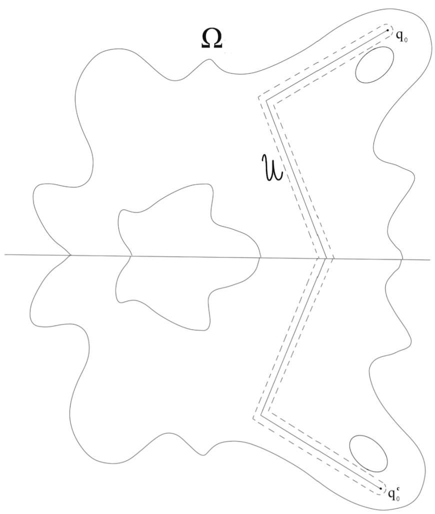

FIGURE 3. The domains involved in the first part of the proof of Proposition 6.16.

Theorem 6.14 allows us to give a first local existence result for the $*$-logarithm.

COROLLARY 6.15. *Let $g \in \mathcal{SR}(\Omega)$ and $q_0 \in \Omega$ be such that $g_{|\mathbb{S}_{q_0}}$ is never vanishing and g_v has no non-real isolated zeros on $\mathbb{S}_{q_0}$. Then, there exists a circular slice-contractible neighborhood Ω_0 of $\mathbb{S}_{q_0}$ such that $g_{|\Omega_0}$ admits a $*$-logarithm.*

PROOF. Theorem 6.14 guarantees the existence of a $*$-logarithm of g on any circular neighborhood Ω_0 of q_0 such that Ω_0 is slice-contractible, provided g_v has no non-real isolated zeroes on Ω_0. Such Ω_0 exists because the sets of spheres where g vanishes and those where g_v has a non-real isolated zero are discrete and do not contain $\mathbb{S}_{q_0}$. ■

When Ω is a slice domain, the previous local existence result can be improved to a suitable slice subdomain.

PROPOSITION 6.16. *Let Ω be a slice domain, $g \in \mathcal{SR}^*(\Omega)$, and let $q_0 \in \Omega \setminus \mathbb{R}$ be such that the sphere $\mathbb{S}_{q_0}$ does not contain any non-real isolated zero of g_v. Then, there exists a slice neighborhood Ω_0 of q_0 which is slice-contractible where g has a $*$-logarithm; i.e., there exists $f \in \mathcal{SR}(\Omega_0)$ such that* $\exp_*(f) = g|_{\Omega_0}$.

PROOF. Again, Theorem 6.14 yields the proof provided we construct Ω_0 as in the thesis of the statement.

Since Ω is slice and $\Omega \cap \mathbb{C}_{q_0}$ is connected by arcs, we can find a piecewise linear path joining $x_0 \in \Omega \cap \mathbb{R}$ with q_0 which touches the real line at x_0 only and is such that an ε-neighborhood $\mathcal{U}$ of this path in $\mathbb{C}_{q_0}$ is contractible and contained in Ω_I (see Figure 3). By replacing Ω with the symmetric completion of this domain, we can suppose that Ω is slice and slice-contractible.

Thanks to [10, Corollary 3.7], we can suppose that either Ω equals $\mathbb{H}$ or the unitary ball $\mathbb{B} \subset \mathbb{H}$ centered in 0. Let $\Omega_1 \Subset \Omega$ be a ball centered at the origin containing q_0. Lemma 3.12 in [12] entails that the set of non-real isolated zeroes of g_v contained in Ω_1 is finite. Let us denote by $S_1, \dots, S_N$ the spheres containing the non-real isolated zeroes of g_v. Take a closed interval $\ell \subset \Omega_1 \cap \mathbb{R}$ and consider the infinitely many segments joining q_0 to the points of ℓ. As $\mathcal{F} := (S_1 \cup \dots \cup S_N) \cap \mathbb{C}_{q_0}$ is finite, we can find a segment $M \subset \Omega_1 \cap \mathbb{C}_{q_0}$ joining q_0 to a point in ℓ which does not intersect $\mathcal{F}$. As $\mathcal{F}$ is symmetric with respect to conjugation in $\mathbb{C}_{q_0}$ and M is compact, we can find a simply connected neighborhood $\mathcal{V}$ of M in $\mathbb{C}_{q_0}$ symmetric with respect to conjugation in $\mathbb{C}_{q_0}$ which does not intersect $\mathcal{F}$. Then, the symmetric completion of $\mathcal{V}$ is the required Ω_0. ∎

We now continue our investigation in search of a $*$-logarithm of g in $\mathcal{SR}^1(\Omega)$. By Proposition 6.3, up to a change of sign of g if Ω is a product domain, we can limit ourselves to look for solutions of $\exp_*(f_v) = g$, with the necessary condition that $g_0(q_0) = 1$ for any q_0 that is a non-real isolated zero of g_v.

Before stating the theorem, we notice that for any $g \in \mathcal{SR}^*(\Omega)$, the set $g_0^{-1}((-\infty, -1])$ is a circular set because it is a union of pre-images of real points by the slice preserving function g_0.

THEOREM 6.17. *Let Ω be slice-contractible and let $g \in \mathcal{SR}^1(\Omega)$ be such that for any $q_0 \in \Omega$ that is a non-real isolated zero of g_v, we have $g_0(q_0) = 1$. Then, on every connected component of $\Omega \setminus g_0^{-1}((-\infty, -1])$, there exists a $*$-logarithm of g.*

PROOF. Let us denote by $\mathcal{U}$ a connected component of $\Omega \setminus g_0^{-1}((-\infty, -1])$. Notice that as g_0 is slice preserving, then $\mathcal{U}$ is a circular domain. We claim that $\mathcal{U}$ is a domain where equation (1.1) admits a solution.

Let us write $g_v = \alpha W$ on Ω as in formula (6.3). Our choice of $\mathcal{U}$ entails that $g_0(\mathcal{U}) \subset \mathbb{H} \setminus (-\infty, -1]$. Since the function φ given in Definition 3.5 and g_0 are slice preserving, then $\varphi \circ g_0 : \mathcal{U} \to \mathcal{D}_0$ is a well defined slice preserving function. Thanks to Remark 3.7, the function $\nu \circ \varphi \circ g_0$ is a never vanishing slice preserving regular function on $\mathcal{U}$. Now, set

$$\beta = \frac{\alpha}{\nu \circ \varphi \circ g_0}. \tag{6.5}$$

We claim that β is a solution of system (6.4) on $\mathcal{U}$.

First of all, recall that $\mu \circ \varphi = \mathrm{id}|_{\mathbb{H} \setminus (-\infty, -1]}$. Thanks to this relation, the first equality in system (6.4) is satisfied if $\varphi \circ g_0 = \beta^2 W^s$. By squaring equality (6.5), we have

$$\beta^2 W^s = \frac{\alpha^2 W^s}{(\nu \circ \varphi \circ g_0)^2}. \tag{6.6}$$

If $q \in \mathcal{U}$ is such that $(\varphi \circ g_0)(q) = 0$, then $g_0(q) = 1$ (see Remark 3.7); as $g^s \equiv 1$, we then have $\alpha^2(q)W^s(q) = 0$. Formula (6.6) implies $\beta^2(q)W^s(q) = 0 = (\varphi \circ g_0)(q)$. Suppose now that $q \in \mathcal{U}$ is such that $(\varphi \circ g_0)(q) \neq 0$. Then, the following chain of equalities is due to the fact that $g^s \equiv 1$, to formula (2.2), and to the fact that $\mu \circ \varphi = \mathrm{id}|_{\mathbb{H}\setminus(-\infty,-1]}$:

$$\begin{aligned}\beta^2(q)W^s(q) &= \frac{\alpha^2(q)W^s(q)}{\big(\nu\big(\varphi\big(g_0(q)\big)\big)\big)^2} = \frac{1-g_0^2(q)}{\big(\nu\big(\varphi\big(g_0(q)\big)\big)\big)^2} = \frac{\big(1-g_0^2(q)\big)\cdot\varphi\big(g_0(q)\big)}{\big(\nu\big(\varphi\big(g_0(q)\big)\big)\big)^2\cdot\varphi\big(g_0(q)\big)}\\ &= \frac{\big(1-g_0^2(q)\big)\cdot\varphi\big(g_0(q)\big)}{1-\big(\mu\big(\varphi\big(g_0(q)\big)\big)\big)^2} = \frac{\big(1-g_0^2(q)\big)\cdot\varphi\big(g_0(q)\big)}{1-g_0^2(q)} = \varphi\big(g_0(q)\big).\end{aligned}$$

Now, since $\beta^2 W^s = \varphi \circ g_0$, equality (6.5) immediately gives

$$\nu(\beta^2 W^s)\cdot\beta = \alpha,$$

which is the second equation of system (6.4). Finally, thanks to Lemma 6.12, the assertion follows by setting $f = f_v = \beta \cdot W|_{\mathcal{U}}$. ∎

Corollary 6.18. *Let Ω be a slice-contractible product domain and $g \in \mathcal{SR}^1(\Omega)$. Assume that for any $q_0 \in \Omega$ that is a non-real isolated zero of g_v, we have that $g_0(q_0) = -1$. Then, on every connected component of $\Omega \setminus g_0^{-1}([1,+\infty))$, there exists a $*$-logarithm of g.*

Proof. Set $\tilde{g} = \exp_*(\pi\mathcal{J})g = -g$. Then, $\tilde{g}$ satisfies the hypotheses of Theorem 6.17, and, therefore, there exists f such that $\exp_*(f) = \tilde{g}$. A trivial computation gives $\exp_*(\pi\mathcal{J} + f) = g$. ∎

It is worth observing that the difficulty of the proof of Proposition 6.16 is of a purely topological nature since the existence of a circular neighborhood of q_0 where W^s is never vanishing is trivial, but the key point is that we are looking for a *slice* circular neighborhood of q_0 whose intersection with any slice is simply connected. On the contrary, the proof of Theorem 6.17 has to overcome a problem of analytical nature: indeed, the existence of a circular neighborhood $\mathcal{U}$ of q_0 such that $g_0((-\infty,-1]) \cap \mathcal{U} = \emptyset$ immediately follows by the continuity of the function g_0, while the construction of the function that gives the logarithm of g on $\mathcal{U}$ requires the sharp analytical properties of the function μ obtained in Section 3.

In particular, we are able to overcome this double kind of difficulties when suitable topological hypotheses allow us to succeed in glueing three different solutions: one which is defined near the non-real isolated zeroes of g_v and two which are given on suitable slice-contractible domains which do not contain the non-real isolated zero.

The idea of the proof is to provide a solution (uniquely if the domain is slice) near the "bad points" (i.e., the non-real isolated zeroes of g_v) and to use this solution to select two suitable solutions in two appropriate (i.e., slice-contractible) domains whose union is exactly given by Ω minus the spheres containing the bad points. In order for this kind of reasoning to work, the key problems should show two aspects. First of all, the non-real isolated zeroes of g_v could belong to different connected components of $\Omega \setminus g_0^{-1}((-\infty,-1])$, and thus we could not be sure that the leaves we selected around a point agree around a different zero of g_v as well. Secondly, even if all the non-real isolated zeroes of g_v belong to the same connected component $\mathcal{U}$ of $\Omega \setminus g_0^{-1}((-\infty,-1])$, we have no information on the topology of $\mathcal{U}$ itself. Therefore, the construction of the two slice simply connected domains whose union is Ω minus the spheres where g_v has non-real isolated zeroes could give a domain which does not allow to apply analytic continuation around each of such zeroes.

The following statement describes a situation in which the existence of a $*$-logarithm holds. Recall the definition of $\mathbb{D}$ given in formula (2.1).

THEOREM 6.19. *Let Ω be one among $\mathbb{B}$, $\mathbb{H}$, or $\mathbb{D}$. Let $g \in \mathcal{SR}^1(\Omega)$ be such that*

- *g_v has a finite number of non-real isolated zeros $\{q_1, \ldots, q_N\}$;*
- *$g_0(q_\ell) = 1$ for all $\ell = 1, \ldots, N$;*
- *the union $\mathbb{S}_{q_1} \cup \cdots \cup \mathbb{S}_{q_N}$ is contained in a unique connected component $\mathcal{U}$ of $\Omega \setminus g_0^{-1}((-\infty,-1])$.*

If for some $I \in \mathbb{S}$ (and hence for any) the set $\mathcal{U}_I^+ = \mathcal{U} \cap \mathbb{C}_I^+$ is convex and $\mathcal{U}$ is slice if Ω is, then there exists a $*$-logarithm of g.

PROOF. First of all, choose any imaginary unit $I \in \mathbb{S}$ and denote $q'_\ell = \mathbb{S}_{q_\ell} \cap \mathbb{C}_I^+$.

Moreover, for $\ell = 1, \ldots, N$, choose 2 outwarding segments (or rays in the case $\Omega = \mathbb{H}$) $s_\ell, \sigma_\ell, \subset \mathbb{C}_I^+ \setminus \mathbb{R}$ starting from q'_ℓ, such that

- $(s_\ell \cup \sigma_\ell) \cap (s_{\ell'} \cup \sigma_{\ell'}) = \emptyset$ if $\ell \neq \ell'$,
- $\Omega_I \setminus (s_1 \cup \bar{s}_1 \cup \cdots \cup s_N \cup \bar{s}_N)$ and $\Omega_I \setminus (\sigma_1 \cup \bar{\sigma}_1 \cup \cdots \cup \sigma_N \cup \bar{\sigma}_N)$ are contractible if Ω is slice and have two contractible connected components if Ω is product.

We denote by $\widehat{\Omega}$ and $\widetilde{\Omega}$ the symmetric completions of $\Omega_I \setminus (s_1 \cup \bar{s}_1 \cup \cdots \cup s_N \cup \bar{s}_N)$ and $\Omega_I \setminus (\sigma_1 \cup \bar{\sigma}_1 \cup \cdots \cup \sigma_N \cup \bar{\sigma}_N)$, respectively. By Theorem 6.17, we can find a $*$-logarithm $f_{\mathcal{U}}$ of g on $\mathcal{U}$, while by Theorem 6.14, we can find $\hat{h} \in \mathcal{SR}(\widehat{\Omega})$ and $\tilde{h} \in \mathcal{SR}(\widetilde{\Omega})$ which are $*$-logarithms of g on $\widehat{\Omega}$ and $\widetilde{\Omega}$, respectively.

As $\mathcal{U}_I^+$ is convex, then both $\mathcal{U}_I^+ \setminus (s_1 \cup \cdots \cup s_N))$ and $\mathcal{U}_I^+ \setminus (\sigma_1 \cup \cdots \cup \sigma_N)$ are connected; we will denote them by $\widehat{\mathcal{U}}_I^+$ and $\widetilde{\mathcal{U}}_I^+$, respectively. Thus, also their symmetric completions $\mathcal{U} \cap \widehat{\Omega}$ and $\mathcal{U} \cap \widetilde{\Omega}$ are connected and will be denoted by $\widehat{\mathcal{U}}$

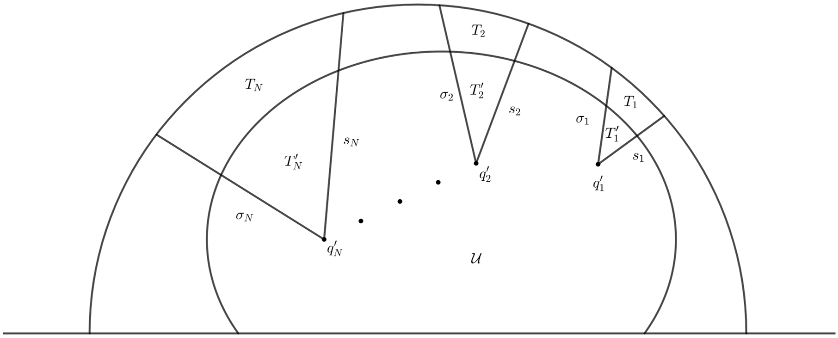

FIGURE 4. An overview of the above geometric construction.

and $\widetilde{\mathcal{U}}$. Moreover, $(\widehat{\Omega} \cap \widetilde{\Omega}) \cap \mathbb{C}_I^+ = \Omega_I^+ \setminus (s_1 \cup \cdots \cup s_N \cup \sigma_1 \cup \cdots \cup \sigma_N)$ is the union of $N+1$ connected components which are given by N "triangles" T_ℓ with vertex in q'_ℓ and whose boundary in Ω_I^+ is given by $s_\ell \cup \sigma_\ell$ and a connected component which is the complement of these triangles and will be denoted by Ω_I^0 (see Figure 4).

Again, the convexity of $\mathcal{U}_I^+$ gives that

$$(\mathcal{U} \cap \widehat{\Omega} \cap \widetilde{\Omega}) \cap \mathbb{C}_I^+ = \mathcal{U}_I^+ \setminus (s_1 \cup \cdots \cup s_N \cup \sigma_1 \cup \cdots \cup \sigma_N)$$

is the union of $N+1$ connected components which are given by N smaller "triangles" $T'_\ell = T_\ell \cap \mathcal{U}_I^+$ with vertex in q'_ℓ and whose boundary in $\mathcal{U}_I^+$ is given by $(s_\ell \cup \sigma_\ell) \cap \mathcal{U}_I^+$ and a connected component which is the complement of these smaller triangles in $\mathcal{U}_I^+$ and will be denoted by $\mathcal{U}_I^0$.

The slice-contractibility of $\widehat{\Omega}$ and $\widetilde{\Omega}$ and the fact that g_v has no non-real isolated zeroes in both these domains ensure that both $\hat{h}_v$ and $\tilde{h}_v$ have no non-real isolated zeroes and thus imply the existence of a square root $\sqrt{\hat{h}_v^s}$ on $\widehat{\Omega}$ and $\sqrt{\tilde{h}_v^s}$ on $\widetilde{\Omega}$. We also set $\widehat{H}_v = \frac{\hat{h}_v}{\sqrt{\hat{h}_v^s}}$ and $\widetilde{H}_v = \frac{\tilde{h}_v}{\sqrt{\tilde{h}_v^s}}$.

We first perform the proof in the case when Ω is slice; by our assumptions, $\mathcal{U}$ is slice too.

Theorem 5.4 and Remark 5.5 entail the existence of $\hat{m}, \tilde{m} \in \mathbb{Z}$ such that

$$f_{\mathcal{U}} - \hat{h} = 2\hat{m}\pi \widehat{H}_v \quad \text{on } \widehat{\mathcal{U}}, \tag{6.7}$$

$$f_{\mathcal{U}} - \tilde{h} = 2\tilde{m}\pi \widetilde{H}_v \quad \text{on } \widetilde{\mathcal{U}}. \tag{6.8}$$

Now, set $\hat{f} = \hat{h} + 2\pi\hat{m}\widehat{H}_v$ on $\widehat{\Omega}$ and $\tilde{f} = \tilde{h} + 2\pi\tilde{m}\widetilde{H}_v$ on $\widetilde{\Omega}$. Equality (6.7) entails $\hat{f} = f_{\mathcal{U}}$ on $\widehat{\mathcal{U}}$, while equality (6.8) gives $\tilde{f} = f_{\mathcal{U}}$ on $\widetilde{\mathcal{U}}$. Thus, $\hat{f} = \tilde{f}$ on $\widehat{\mathcal{U}} \cap \widetilde{\mathcal{U}}$. As $\widehat{\mathcal{U}} \cap \widetilde{\mathcal{U}}$ contains accumulation points in any of the $N+1$ connected components of $\widehat{\Omega} \cap \widetilde{\Omega}$, then the identity principle implies $\hat{f} = \tilde{f}$ on $\widehat{\Omega} \cap \widetilde{\Omega}$.

Setting

$$f(q) = \begin{cases} \hat{f}(q), & \text{if } q \in \widehat{\Omega}, \\ \tilde{f}(q), & \text{if } q \in \widetilde{\Omega}, \\ f_{\mathcal{U}}(q), & \text{if } q \in \mathcal{U}, \end{cases}$$

gives a well-defined slice regular function which is a $*$-logarithm of g on Ω.

We now turn our attention to the case in which Ω is product, which of course entails that $\mathcal{U}$ is product, too.

Theorem 5.4 entails the existence of $\hat{n}, \hat{m}, \tilde{n}, \tilde{m} \in \mathbb{Z}$ with $\hat{n} \equiv \hat{m} \pmod 2$ and $\tilde{n} \equiv \tilde{m} \pmod 2$, such that

$$f_{\mathcal{U}} - \hat{h} = \pi\hat{n}\mathcal{J} + \pi\hat{m}\widehat{H}_v \quad \text{on } \widehat{\mathcal{U}},$$
$$f_{\mathcal{U}} - \tilde{h} = \pi\tilde{n}\mathcal{J} + \pi\tilde{m}\widetilde{H}_v \quad \text{on } \widetilde{\mathcal{U}}.$$

Setting again $\hat{f} = \hat{h} + \pi\hat{n}\mathcal{J} + \pi\hat{m}\widehat{H}_v$ on $\widehat{\Omega}$ and $\tilde{f} = \tilde{h} + \pi\tilde{n}\mathcal{J} + \pi\tilde{m}\widetilde{H}_v$ on $\widetilde{\Omega}$ and reasoning as above gives the existence of a $*$-logarithm of g on Ω. ∎

In the case when Ω is product, the second condition of the previous theorem can be changed.

Corollary 6.20. *Let $g \in \mathcal{SR}^1(\mathbb{D})$ be such that*

- *g_v has a finite number of non-real isolated zeros $\{q_1, \dots, q_N\}$;*
- *$g_0(q_\ell) = -1$ for all $\ell = 1, \dots, N$;*
- *the union $\mathbb{S}_{q_1} \cup \dots \cup \mathbb{S}_{q_N}$ is contained in a unique connected component $\mathcal{U}$ of $\Omega \setminus g_0^{-1}((-\infty, -1])$.*

If for some $I \in \mathbb{S}$ (and hence for any) the set $\mathcal{U}_I^+ = \mathcal{U} \cap \mathbb{C}_I^+$ is convex, then there exists a $*$-logarithm of g.

Proof. By applying Theorem 6.19 to $-g$, we find a $*$-logarithm f of $-g$, and then the function $f + \pi\mathcal{J}$ is a $*$-logarithm of g. ∎

Remark 6.21. We notice that the statement of Theorem 6.19 can be generalized to a larger variety of domains and functions. Indeed, the techniques we use in the proof can be applied when Ω is slice-contractible, all the non-real isolated zeroes of g_v belong to the same connected component $\mathcal{U}$ of $\Omega \setminus g_0^{-1}((-\infty, -1])$, and for any non-real isolated zero, we can "draw" two paths issuing from the non-real isolated zeroes of g_v which give two contractible subdomains of Ω_I and do not disconnect $\mathcal{U}_I$.

Acknowledgments. – The authors warmly thank the anonymous referee for their careful reading and valuable remarks.

Funding. – This work was partially supported by GNSAGA of INdAM and the INdAM project "Teoria delle funzioni ipercomplesse e applicazioni". The authors acknowledge with pleasure the support of FBK-Cirm, Trento, and the Clifford Research Group at Ghent University where parts of this paper were written. The first author was also financially supported by a INdAM fellowship "*Mensilità di borse di studio per l'estero a.a. 2018–2019*" spent at Ghent University.

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Received 5 August 2021,
and in revised form 19 January 2023

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