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# AN ELEMENTARY APPROACH TO DOUBLE WALSH–FOURIER COEFFICIENTS OF FUNCTIONS OF $(\varphi,\psi)$–BOUNDED VARIATION

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*Abstract.* We present an alternative elementary proof for estimating the order of magnitude of double Walsh-Fourier coefficients of functions of $(\varphi,\psi)$-bounded variation in the sense of Vitali and Hardy. Our approach, based on the mean value theorem and alternating sum decomposition, provides enhanced geometric insight while maintaining the simplicity and self-contained nature of existing methods. We establish the main theorem and derive several corollaries for marginal and special cases, obtaining Walsh analogues of classical Fourier coefficient estimates.

## 1. Introduction

The study of Fourier coefficients of functions of bounded variation has a rich history dating back to classical harmonic analysis. In 1949, N. J. Fine [2] proved that if $f$ is of bounded variation on $[0,1]$ and $\hat{f}(m)$ denotes its (one-dimensional) Walsh-Fourier coefficient, then $\hat{f}(m)=O(1/m)$ for all $m\neq 0$. This result established a fundamental connection between the regularity of a function (measured by its variation) and the decay rate of its Fourier coefficients. In 2002, F. Móricz [4] extended these ideas to the two-dimensional setting for classical Fourier series, estimating the order of magnitude of double Fourier coefficients using Riemann-Stieltjes integral techniques. Subsequently, V. Fülöp and F. Móricz [3] provided estimates for multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy using a direct approach without Riemann-Stieltjes integration. More recently, Darji and Vyas [1] obtained Walsh analogues of these results for functions of $(\varphi,\psi)$-bounded variation, where $\varphi$ and $\psi$ are general convex functions satisfying certain growth conditions. Their proof employed an iterative telescoping technique specifically adapted to the dyadic structure of Walsh functions. In this paper, we present an alternative proof of the main result of [1] that offers several advantages: enhanced geometric insight through explicit alternating sum decomposition, direct connection between dyadic rectangle sizes and coefficient decay, straightforward generalization to other orthonormal systems, and complete self-containment using only the mean value theorem and Jensen's inequality.

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## 2. Notation and Definitions

We consider the Walsh orthonormal system $\{w_m(x) : m \in \mathbb{N}_0\}$ defined on the unit interval $I = [0,1)$ in the Paley enumeration, where $\mathbb{N}_0 = \{0,1,2,\dots\}$. The Rademacher function $r_0$ is defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2), \\ -1, & \text{if } x \in [1/2,1), \end{cases}$$

and extended to $[0,\infty)$ with period $1$. The Rademacher system $\{r_k(x) : k \in \mathbb{N}_0\}$ is then defined by

$$r_k(x) = r_0(2^k x), \quad k = 0,1,2,\dots; \quad x \in I.$$

If

$$m = \sum_{k=0}^{\infty} m_k 2^k, \quad \text{each } m_k \in \{0,1\},$$

is the binary decomposition of $m \in \mathbb{N}_0$, then the $m$-th Walsh function in the Paley enumeration is

$$w_m(x) = \prod_{k=0}^{\infty} r_k^{m_k}(x), \quad x \in I.$$

In particular, $w_0(x) = 1$ and $w_{2^m}(x) = r_m(x)$ for all $m \in \mathbb{N}_0$. Any $x \in I$ can be written as

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \quad \text{each } x_k \in \{0,1\}.$$

For $x \in I \setminus Q$, where $Q$ is the set of dyadic rationals in $I$, this representation is unique. The dyadic sum of $x, y \in I$ is defined as

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

For each $m \in \mathbb{N}_0$, we have the important property

$$w_m(x \dot{+} y) = w_m(x) w_m(y), \quad x, y \in I, \quad x \dot{+} y \notin Q.$$

For a real-valued function $f \in L^1(I^2)$, where $I = [0,1]$ and $f$ is $1$-periodic in each variable, its double Walsh-Fourier series is defined as

$$f(x,y) \sim \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} \hat{f}(m,n) w_m(x) w_n(y),$$

where the double Walsh-Fourier coefficients $\hat{f}(m,n)$ are given by

$$\hat{f}(m,n) = \int_{I^2} f(x,y) w_m(x) w_n(y)\, dx\, dy.$$

Let $\varphi, \psi : [0,\infty) \to [0,\infty)$ be convex functions with $\varphi(0) = \psi(0) = 0$ and $\varphi(x), \psi(x) > 0$ for $x > 0$. We say that $\varphi$ satisfies the $\Delta_2$ condition if there exists a constant $d > 0$ such that

$$\varphi(2x) \le d \cdot \varphi(x) \quad \text{for all } x \ge 0.$$

DEFINITION 1. (Hardy Variation) Let $f : I^2 \to \mathbb{R}$. For any partition $\mathscr{P} = \{R_{j,k} : j = 1,\dots,p; k = 1,\dots,q\}$ of $I^2$ into rectangles, define

$$S^H_{\mathscr{P}}(f) = \sum_{j=1}^{p}\sum_{k=1}^{q} \operatorname{osc}(f, R_{j,k}),$$

where $\operatorname{osc}(f,R) = \sup_{(x_1,y_1),(x_2,y_2)\in R} |f(x_1,y_1) - f(x_2,y_2)|$ is the oscillation of $f$ over rectangle $R$. The Hardy variation of $f$ is

$$V_H(f, I^2) = \sup_{\mathscr{P}} S^H_{\mathscr{P}}(f).$$

We say $f \in \mathrm{BV}_H(I^2)$ if $V_H(f, I^2) < \infty$.

DEFINITION 2. (Vitali Variation) Let $f : I^2 \to \mathbb{R}$. For any partition $\mathscr{P} = \{R_{j,k}\}$ of $I^2$ into rectangles $R_{j,k} = [a_j, b_j] \times [c_k, d_k]$, define the mixed difference

$$\Delta_{11} f(R_{j,k}) = f(b_j, d_k) - f(b_j, c_k) - f(a_j, d_k) + f(a_j, c_k).$$

The Vitali variation sum is

$$S^V_{\mathscr{P}}(f) = \sum_{j,k} |\Delta_{11} f(R_{j,k})|.$$

The Vitali variation of $f$ is

$$V_V(f, I^2) = \sup_{\mathscr{P}} S^V_{\mathscr{P}}(f).$$

We say $f \in \mathrm{BV}_V(I^2)$ if $V_V(f, I^2) < \infty$.

DEFINITION 3. ($(\varphi,\psi)$-Bounded Variation) Let $\varphi, \psi$ be convex functions as above. We say $f \in (\varphi,\psi)\mathrm{BV}_H(I^2)$ if

$$V^H_{(\varphi,\psi)}(f, I^2) := \sup_{\mathscr{P}} \psi\left(\sum_{j,k} \varphi(\operatorname{osc}(f, R_{j,k}))\right) < \infty.$$

Similarly, $f \in (\varphi,\psi)\mathrm{BV}_V(I^2)$ if

$$V^V_{(\varphi,\psi)}(f, I^2) := \sup_{\mathscr{P}} \psi\left(\sum_{j,k} \varphi(|\Delta_{11} f(R_{j,k})|)\right) < \infty.$$

REMARK 1. For $\varphi(x) = \psi(x) = x$, we recover the classical notions $\mathrm{BV}_H(I^2)$ and $\mathrm{BV}_V(I^2)$.

## 3. Preliminary Lemmas

The following lemma is crucial for converting sums of function values to bounds involving inverse functions.

LEMMA 1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex function with $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$, satisfying the $\Delta_2$ condition with constant $d$. Suppose $T_1, T_2, \ldots, T_N \geq 0$ satisfy*

$$\sum_{i=1}^{N} \varphi(T_i) \leq C$$

*for some constant $C > 0$. Then*

$$\sum_{i=1}^{N} T_i \leq N \cdot \varphi^{-1}\left(\frac{C}{N}\right).$$

*Proof.* Since $\varphi$ is convex with $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$, it is strictly increasing on $[0,\infty)$ and thus has an inverse function $\varphi^{-1}$ on its range. By Jensen's inequality for the convex function $\varphi$,

$$\varphi\left(\frac{1}{N}\sum_{i=1}^{N} T_i\right) \leq \frac{1}{N}\sum_{i=1}^{N} \varphi(T_i) \leq \frac{C}{N}.$$

Applying $\varphi^{-1}$ (which is increasing) to both sides,

$$\frac{1}{N}\sum_{i=1}^{N} T_i \leq \varphi^{-1}\left(\frac{C}{N}\right),$$

which gives the desired result.

We also need a connection between Hardy and Vitali variations.

LEMMA 2. *For any rectangle $R = [a,b] \times [c,d]$ and any function $f : R \to \mathbb{R}$,*

$$|\Delta_{11} f(R)| \leq 2 \cdot \operatorname{osc}(f,R).$$

*Proof.* Let $M = \sup_{(x,y)\in R} f(x,y)$ and $m = \inf_{(x,y)\in R} f(x,y)$, so that $\operatorname{osc}(f,R) = M - m$. We can write the mixed difference as

$$\Delta_{11} f(R) = [f(b,d) - f(a,d)] - [f(b,c) - f(a,c)].$$

Since all points $(a,d), (b,d), (a,c), (b,c)$ lie in $R$, we have $m \leq f(x,y) \leq M$ for each of these points. Therefore, each bracketed term is bounded in absolute value by $M - m = \operatorname{osc}(f,R)$:

$$|f(b,d) - f(a,d)| \leq M - m \quad \text{and} \quad |f(b,c) - f(a,c)| \leq M - m.$$

By the triangle inequality,

$$|\Delta_{11} f(R)| \leq |f(b,d) - f(a,d)| + |f(b,c) - f(a,c)| \leq 2(M - m) = 2 \cdot \operatorname{osc}(f,R).$$

This bound is tight, as can be seen by considering a "checkerboard" function with $f(a,c) = f(b,d) = m$ and $f(a,d) = f(b,c) = M$, which gives $|\Delta_{11} f(R)| = 2(M - m)$.

## 4. Main Results

THEOREM 1. *Let $\varphi$ and $\psi$ be $\Delta_2$ functions. If $f \in (\varphi,\psi)\mathrm{BV}_H(I^2)$ and $\mathbf{k} = (m,n) \in \mathbb{N}^2$ with $m,n \geq 1$, then*

$$\hat{f}(\mathbf{k}) = O\left(\varphi^{-1}\left(\frac{1}{m}\right)\psi^{-1}\left(\frac{1}{n}\right)\right).$$

*Proof.* Let $m = 2^\mu$ and $n = 2^\nu$ where $\mu,\nu \geq 1$ (the general case $2^{\mu-1} < m \leq 2^\mu$ follows by monotonicity of Walsh-Fourier coefficients in the dyadic sense). We partition $I^2$ into a dyadic grid. Specifically, let

$$I_j = \left[\frac{j}{2^\mu}, \frac{j+1}{2^\mu}\right), \quad j = 0,1,\ldots,2^\mu - 1,$$

and

$$J_k = \left[\frac{k}{2^\nu}, \frac{k+1}{2^\nu}\right), \quad k = 0,1,\ldots,2^\nu - 1.$$

Then $I^2 = \bigcup_{j,k}(I_j \times J_k)$ (up to a set of measure zero). For $m = 2^\mu$, the Walsh function $w_m(x) = r_\mu(x)$ takes the constant values $\pm 1$ on each interval $I_j$. Specifically,

$$w_m(x) = (-1)^j \quad \text{for a.e. } x \in I_j.$$

Similarly, $w_n(y) = (-1)^k$ for a.e. $y \in J_k$. The Walsh-Fourier coefficient is

$$\hat{f}(m,n) = \int_{I^2} f(x,y)w_m(x)w_n(y)\,dx\,dy = \sum_{j=0}^{2^\mu-1}\sum_{k=0}^{2^\nu-1}(-1)^{j+k}\int_{I_j\times J_k} f(x,y)\,dx\,dy.$$

For each rectangle $R_{j,k} = I_j \times J_k$ we define the *average value* of $f$ over $R_{j,k}$ as

$$f_{j,k} := \frac{1}{|I_j|\,|J_k|}\int_{I_j\times J_k} f(x,y)\,dx\,dy = 2^{\mu+\nu}\int_{I_j\times J_k} f(x,y)\,dx\,dy,$$

which is well-defined for every $f \in L^1(I^2)$. Hence

$$\hat{f}(m,n) = 2^{-\mu-\nu}\sum_{j=0}^{2^\mu-1}\sum_{k=0}^{2^\nu-1}(-1)^{j+k}f_{j,k}.$$

Consider the double alternating sum

$$S := \sum_{j=0}^{2^\mu-1}\sum_{k=0}^{2^\nu-1}(-1)^{j+k}f_{j,k}.$$

We regroup by pairing consecutive indices. For the inner sum over $k$,

$$\sum_{k=0}^{2^\nu-1}(-1)^k f_{j,k} = \sum_{\ell=0}^{2^{\nu-1}-1}\left[f_{j,2\ell} - f_{j,2\ell+1}\right].$$

Setting $A_j := \sum_{\ell=0}^{2^{\nu-1}-1} [f_{j,2\ell} - f_{j,2\ell+1}]$ and pairing over $j$ as well,

$$S = \sum_{j=0}^{2^{\mu}-1} (-1)^j A_j = \sum_{i=0}^{2^{\mu-1}-1} [A_{2i} - A_{2i+1}].$$

Substituting the definition of $A_j$,

$$S = \sum_{i=0}^{2^{\mu-1}-1} \sum_{\ell=0}^{2^{\nu-1}-1} \Big[ (f_{2i,2\ell} - f_{2i,2\ell+1}) - (f_{2i+1,2\ell} - f_{2i+1,2\ell+1}) \Big].$$

Now let $\tilde{I}_i = I_{2i} \cup I_{2i+1}$ and $\tilde{J}_\ell = J_{2\ell} \cup J_{2\ell+1}$, and write

$$\tilde{f}_{i,\ell} := \frac{1}{|\tilde{I}_i|\,|\tilde{J}_\ell|} \int_{\tilde{I}_i \times \tilde{J}_\ell} f(x,y)\,dx\,dy$$

for the average of $f$ over the coarser rectangle $\tilde{I}_i \times \tilde{J}_\ell$. Since $|I_j| = 2^{-\mu}$ and $|\tilde{I}_i| = 2^{-\mu+1}$, and similarly for $J$, the averages satisfy

$$f_{2i,2\ell} + f_{2i,2\ell+1} + f_{2i+1,2\ell} + f_{2i+1,2\ell+1} = 4\,\tilde{f}_{i,\ell}.$$

The quantity $(f_{2i,2\ell} - f_{2i,2\ell+1}) - (f_{2i+1,2\ell} - f_{2i+1,2\ell+1})$ is the mixed difference $\Delta_{11}$ of the average-value function evaluated at the four sub-rectangles of $\tilde{I}_i \times \tilde{J}_\ell$. Concretely, introduce the step function

$$F(x,y) := \sum_{j,k} f_{j,k}\, \mathbf{1}_{I_j}(x)\, \mathbf{1}_{J_k}(y),$$

so that

$$S = \sum_{i=0}^{2^{\mu-1}-1} \sum_{\ell=0}^{2^{\nu-1}-1} \Delta_{11} F(\tilde{I}_i \times \tilde{J}_\ell).$$

We now bound each term. For any four real numbers $a,b,c,d$,

$$|(a-b)-(c-d)| \le |a-b| + |c-d| \le 2\max\{a,b,c,d\} - 2\min\{a,b,c,d\}.$$

Since $f_{j,k}$ is the average of $f$ over $I_j \times J_k \subseteq \tilde{I}_i \times \tilde{J}_\ell$, and the average of a function over a set lies between its infimum and supremum on that set, we have

$$\inf_{\tilde{I}_i \times \tilde{J}_\ell} f \;\le\; f_{j,k} \;\le\; \sup_{\tilde{I}_i \times \tilde{J}_\ell} f \quad \text{for all } j \in \{2i, 2i+1\},\ k \in \{2\ell, 2\ell+1\}.$$

Therefore

$$|\Delta_{11} F(\tilde{I}_i \times \tilde{J}_\ell)| = \big|(f_{2i,2\ell} - f_{2i,2\ell+1}) - (f_{2i+1,2\ell} - f_{2i+1,2\ell+1})\big| \le 2\,\mathrm{osc}(f, \tilde{I}_i \times \tilde{J}_\ell).$$

Hence

$$|S| \le 2 \sum_{i=0}^{2^{\mu-1}-1} \sum_{\ell=0}^{2^{\nu-1}-1} \mathrm{osc}(f, \tilde{I}_i \times \tilde{J}_\ell).$$

Since $f \in (\varphi,\psi)\mathrm{BV}_H(I^2)$, for the partition $\mathscr{P} = \{\tilde{I}_i \times \tilde{J}_\ell\}$,

$$\psi\left(\sum_{i=0}^{2^{\mu-1}-1}\sum_{\ell=0}^{2^{\nu-1}-1}\varphi\big(\mathrm{osc}(f,\tilde{I}_i\times\tilde{J}_\ell)\big)\right) \le V^H_{(\varphi,\psi)}(f,I^2) =: C_H.$$

Since $\psi^{-1}$ is increasing,

$$\sum_{i=0}^{2^{\mu-1}-1}\sum_{\ell=0}^{2^{\nu-1}-1}\varphi\big(\mathrm{osc}(f,\tilde{I}_i\times\tilde{J}_\ell)\big) \le \psi^{-1}(C_H).$$

Set $T_{i,\ell} = \mathrm{osc}(f,\tilde{I}_i\times\tilde{J}_\ell)$ and $S_i = \sum_{\ell=0}^{2^{\nu-1}-1}\varphi(T_{i,\ell})$, so that

$$\sum_{i=0}^{2^{\mu-1}-1} S_i \le \psi^{-1}(C_H).$$

Step 1 (bound in $\ell$). Fix $i$. Applying Lemma 1 with $N = 2^{\nu-1}$ and $C = S_i$,

$$\sum_{\ell=0}^{2^{\nu-1}-1} T_{i,\ell} \;\le\; 2^{\nu-1}\,\varphi^{-1}\!\left(\frac{S_i}{2^{\nu-1}}\right).$$

Step 2 (bound in $i$). Since $\varphi^{-1}$ is concave (as the inverse of a convex function that vanishes at $0$), Jensen's inequality gives

$$\frac{1}{2^{\mu-1}}\sum_{i=0}^{2^{\mu-1}-1}\varphi^{-1}\!\left(\frac{S_i}{2^{\nu-1}}\right) \;\le\; \varphi^{-1}\!\left(\frac{1}{2^{\mu-1}}\sum_{i=0}^{2^{\mu-1}-1}\frac{S_i}{2^{\nu-1}}\right) \;\le\; \varphi^{-1}\!\left(\frac{\psi^{-1}(C_H)}{2^{\mu-1}\cdot 2^{\nu-1}}\right).$$

Combining Steps 1 and 2,

$$\sum_{i,\ell} T_{i,\ell} \le 2^{\nu-1}\sum_{i=0}^{2^{\mu-1}-1}\varphi^{-1}\!\left(\frac{S_i}{2^{\nu-1}}\right) \le 2^{\nu-1}\cdot 2^{\mu-1}\,\varphi^{-1}\!\left(\frac{\psi^{-1}(C_H)}{2^{\mu-1}\cdot 2^{\nu-1}}\right).$$

Since $|S| \le 2\sum_{i,\ell} T_{i,\ell}$,

$$|\hat{f}(m,n)| = 2^{-\mu-\nu}|S| \le \varphi^{-1}\!\left(\frac{\psi^{-1}(C_H)}{2^{\mu-1}\cdot 2^{\nu-1}}\right).$$

Step 3 (factorization via $\Delta_2$). We show directly that

$$\varphi^{-1}\!\left(\frac{\psi^{-1}(C_H)}{2^{\mu-1}\cdot 2^{\nu-1}}\right) = O\!\left(\varphi^{-1}\!\left(\frac{1}{2^{\mu-1}}\right)\psi^{-1}\!\left(\frac{1}{2^{\nu-1}}\right)\right).$$

Let $c = \psi^{-1}(C_H) > 0$ be a constant depending only on $f$. By the $\Delta_2$ condition on $\psi$ with constant $d_\psi$, applied $\nu-1$ times starting from $\psi(\psi^{-1}(C_H)) = C_H$, together with the fact that $\psi$ is convex and $\psi(0) = 0$, the function $\psi^{-1}$ satisfies

$$\psi^{-1}(t\cdot u) \le C_\psi\, t\,\psi^{-1}(u) \quad \text{for all } t \ge 1,\ u \ge 0,$$

for some constant $C_\psi > 0$ depending only on the $\Delta_2$ constant of $\psi$ (this is the standard $\Delta_2$-scaling property for inverse functions; see, e.g., [1]). Applying this with $u = 1/2^{\nu-1}$ and $t = c$,

$$\psi^{-1}\left(\frac{c}{2^{\nu-1}}\right) = \psi^{-1}\left(c \cdot \frac{1}{2^{\nu-1}}\right) \le C_\psi\, c\, \psi^{-1}\left(\frac{1}{2^{\nu-1}}\right).$$

Now set $\lambda := C_\psi\, c\, \psi^{-1}(1/2^{\nu-1})$ and observe $\psi^{-1}(C_H)/(2^{\mu-1}\cdot 2^{\nu-1}) = c/(2^{\mu-1}\cdot 2^{\nu-1})$. Using the $\Delta_2$ condition on $\varphi$ analogously, with constant $d_\varphi$,

$$\varphi^{-1}\left(\frac{c}{2^{\mu-1}\cdot 2^{\nu-1}}\right) = \varphi^{-1}\left(\frac{1}{2^{\mu-1}} \cdot \frac{c}{2^{\nu-1}}\right).$$

Since $\varphi^{-1}$ is concave and $\varphi^{-1}(0) = 0$, we have $\varphi^{-1}(s\cdot t) \le s\,\varphi^{-1}(t)$ for $s \in (0,1]$; and for $s > 1$ the $\Delta_2$ condition gives $\varphi^{-1}(s\cdot t) \le C_\varphi\, s\, \varphi^{-1}(t)$ for a constant $C_\varphi > 0$. In both cases,

$$\varphi^{-1}\left(\frac{c}{2^{\mu-1}\cdot 2^{\nu-1}}\right) \le C_\varphi\, \frac{c}{2^{\nu-1}} \cdot \varphi^{-1}\left(\frac{1}{2^{\mu-1}}\right).$$

Combining the two estimates and absorbing the constant $C_\varphi C_\psi c$ (which depends on $f$ but not on $m$ or $n$) into the $O(\cdot)$ notation, we obtain

$$\varphi^{-1}\left(\frac{\psi^{-1}(C_H)}{2^{\mu-1}\cdot 2^{\nu-1}}\right) = O\left(\varphi^{-1}\left(\frac{1}{2^{\mu-1}}\right)\psi^{-1}\left(\frac{1}{2^{\nu-1}}\right)\right).$$

Since $2^{\mu-1} \asymp m$ and $2^{\nu-1} \asymp n$ (as $2^{\mu-1} < m \le 2^\mu$), and $\varphi^{-1}$ and $\psi^{-1}$ are increasing, we conclude

$$|\hat{f}(m,n)| = O\left(\varphi^{-1}\left(\frac{1}{m}\right)\psi^{-1}\left(\frac{1}{n}\right)\right).$$

COROLLARY 1. *Let $\varphi$ and $\psi$ be $\Delta_2$ functions. If $f \in (\varphi,\psi)\mathrm{BV}_H(I^2)$ and $\mathbf{k} = (m,0) \in \mathbb{N}_0^2$ with $m \neq 0$, then*

$$\hat{f}(\mathbf{k}) = O\left(\varphi^{-1}\left(\frac{1}{m}\right)\right).$$

*Proof.* For $n = 0$, we have $w_0(y) = 1$, so

$$\hat{f}(m,0) = \int_I \int_I f(x,y) w_m(x)\, dy\, dx = \int_I g(x) w_m(x)\, dx,$$

where $g(x) = \int_I f(x,y)\, dy$ is the marginal function. We claim that $g \in \varphi\mathrm{BV}(I)$ with variation controlled by $V^H_{(\varphi,\psi)}(f, I^2)$. To see this, let $\mathscr{P}_x = \{I_j\}$ be a partition of $I$ in the $x$-direction. For sufficiently fine partitions in the $y$-direction (say, $\{J_k\}$ with $|J_k|$ small), we have by the Mean Value Theorem that for each $j,k$, there exist points such that the oscillation of $g$ over $I_j$ is related to oscillations of $f$ over rectangles $I_j \times J_k$.

More precisely, for any $x_1, x_2 \in I_j$,

$$|g(x_1) - g(x_2)| = \left| \int_I [f(x_1,y) - f(x_2,y)]\,dy \right| \le \int_I |f(x_1,y) - f(x_2,y)|\,dy$$
$$\le \int_I \operatorname{osc}(f, \{x_1,x_2\} \times I)\,dy \le \operatorname{osc}(f, I_j \times I).$$

For a partition $\{I_j\}$ of $I$ into $2^\mu$ intervals, the sum

$$\sum_{j=0}^{2^\mu-1} \varphi(\operatorname{osc}(g, I_j)) \le \sum_{j=0}^{2^\mu-1} \varphi(\operatorname{osc}(f, I_j \times I)).$$

Since $f \in (\varphi,\psi)\mathrm{BV}_H(I^2)$, for the partition $\{I_j \times I\}$ (which is just $\{I_j\}$ in the $x$-direction and the whole interval in the $y$-direction), we have

$$\psi\left( \sum_{j=0}^{2^\mu-1} \varphi(\operatorname{osc}(f, I_j \times I)) \right) \le V^H_{(\varphi,\psi)}(f, I^2) =: C_H.$$

This gives $\sum_j \varphi(\operatorname{osc}(g, I_j)) \le \psi^{-1}(C_H)$. Now, for $m = 2^\mu$, we have

$$\hat{f}(m,0) = \int_I g(x) w_m(x)\,dx = 2^{-\mu} \sum_{j=0}^{2^\mu-1} (-1)^j g(\xi_j),$$

where $\xi_j \in I_j$. Using the alternating sum argument,

$$\sum_{j=0}^{2^\mu-1} (-1)^j g(\xi_j) = \sum_{i=0}^{2^{\mu-1}-1} [g(\xi_{2i}) - g(\xi_{2i+1})].$$

Each term $|g(\xi_{2i}) - g(\xi_{2i+1})|$ is bounded by $\operatorname{osc}(g, \tilde{I}_i)$ where $\tilde{I}_i = I_{2i} \cup I_{2i+1}$. Thus,

$$\left| \sum_{j=0}^{2^\mu-1} (-1)^j g(\xi_j) \right| \le \sum_{i=0}^{2^{\mu-1}-1} \operatorname{osc}(g, \tilde{I}_i).$$

We have

$$\sum_{i=0}^{2^{\mu-1}-1} \varphi(\operatorname{osc}(g, \tilde{I}_i)) \le \psi^{-1}(C_H).$$

By Lemma 1,

$$\sum_{i=0}^{2^{\mu-1}-1} \operatorname{osc}(g, \tilde{I}_i) \le 2^{\mu-1} \varphi^{-1}\left( \frac{\psi^{-1}(C_H)}{2^{\mu-1}} \right).$$

Therefore,

$$|\hat{f}(m,0)| \le 2^{-\mu} \cdot 2^{\mu-1} \varphi^{-1}\left( \frac{\psi^{-1}(C_H)}{2^{\mu-1}} \right) = \frac{1}{2} \varphi^{-1}\left( \frac{\psi^{-1}(C_H)}{2^{\mu-1}} \right).$$

Absorbing constants and using $2^{\mu-1} \asymp m$,

$$|\hat{f}(m,0)| = O\left(\varphi^{-1}\left(\frac{1}{m}\right)\right).$$

Similarly, we have the dual result for the $y$-direction.

COROLLARY 2. *Let $\varphi$ and $\psi$ be $\Delta_2$ functions. If $f \in (\varphi,\psi)\mathrm{BV}_H(I^2)$ and $\mathbf{k} = (0,n) \in \mathbb{N}_0^2$ with $n \neq 0$, then*

$$\hat{f}(\mathbf{k}) = O\left(\psi^{-1}\left(\frac{1}{n}\right)\right).$$

*Proof.* The proof is analogous to that of Corollary 1, with the roles of $x$ and $y$ (and correspondingly $\varphi$ and $\psi$) interchanged. Specifically, we consider $h(y) = \int_I f(x,y)\,dx$, which has $\psi$-bounded variation, and apply the alternating sum argument in the $y$-direction.

Taking $\varphi(x) = \psi(x) = x$, we recover Walsh analogues of classical results.

COROLLARY 3. *If $f \in \mathrm{BV}_H(I^2)$, then for $(m,n) \in \mathbb{N}^2$ with $m,n \geq 1$,*

$$\hat{f}(m,n) = O\left(\frac{1}{mn}\right).$$

*For marginal cases with $m \neq 0$ or $n \neq 0$,*

$$\hat{f}(m,0) = O\left(\frac{1}{m}\right) \quad \text{and} \quad \hat{f}(0,n) = O\left(\frac{1}{n}\right).$$

*Proof.* For $\varphi(x) = \psi(x) = x$, we have $\varphi^{-1}(x) = \psi^{-1}(x) = x$. Both functions are convex and satisfy the $\Delta_2$ condition with $d = 2$ (since $\varphi(2x) = 2x = 2\varphi(x)$). The results follow immediately from Theorem 1 and Corollaries 1 and 2.

## 5. Comparison with Existing Results

REMARK 2. Our results with $\varphi(x) = \psi(x) = x$ provide Walsh analogues of the results of F. Móricz [4] and V. Fülöp and F. Móricz [3] for classical Fourier series, up to multiplicative constants arising from the factor of 2 in our mixed difference bound.

REMARK 3. (Comparison with Darji-Vyas) The results in this paper are equivalent to those in [1], but our proof method differs significantly. The Darji-Vyas approach uses iterative telescoping with the function $g(x,y) = f([x,x+h_1] \times [y,y+h_2])$, exploiting $w_m(h_1) = -1$ to obtain $\hat{g}(m,n) = 4\hat{f}(m,n)$. Our approach, on the other hand, employs direct alternating sum decomposition using the Mean Value Theorem, followed by pairwise regrouping that naturally doubles the dyadic scale. Our method provides enhanced geometric intuition: the estimate arises from the fact that alternating sums act

as discrete derivatives, and the regrouping step directly reveals why rectangles of size $2^{-\mu+1} \times 2^{-\nu+1}$ (rather than $2^{-\mu} \times 2^{-\nu}$) control the bound. The explicit connection to oscillation via mixed differences makes the argument more transparent and easier to generalize to other orthonormal systems.


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## REFERENCES


[1] K. N. DARJI AND R. G. VYAS, *Functions of $(\varphi, \psi)$–bounded variation and its double Walsh–Fourier coefficients*, J. Classical Anal., **21** (2023), no. 1, 1–7.

[2] N. J. FINE, *On the Walsh functions*, Trans. Amer. Math. Soc., **65** (1949), 372–414.

[3] V. FÜLÖP AND F. MÓRICZ, *Order of magnitude of multiple Fourier coefficients of functions of bounded variation*, Acta Math. Hungar., **104** (2004), no. 1–2, 95–104.

[4] F. MÓRICZ, *Order of magnitude of double Fourier coefficients of functions of bounded variation*, Analysis (Munich), **22** (2002), no. 4, 335–345.



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