Non-ridge-chordal complexes whose clique complex has shellable Alexander dual

# NON-RIDGE-CHORDAL COMPLEXES WHOSE CLIQUE COMPLEX HAS SHELLABLE ALEXANDER DUAL

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ABSTRACT. A recent conjecture that appeared in three papers by Bigdeli–Faridi, Dochtermann, and Nikseresht, is that every simplicial complex whose clique complex has shellable Alexander dual, is ridge-chordal. This strengthens the long-standing Simon's conjecture that the $k$-skeleton of the simplex is extendably shellable, for any $k$. We show that the stronger conjecture has a negative answer, by exhibiting an infinite family of counterexamples.

## 1. INTRODUCTION

Shellability is a property satisfied by two important families of objects in combinatorics, namely, polytope boundaries [27] and order complexes of geometric lattices [9]. Moreover, skeleta of shellable complexes are themselves shellable [12]. *Extendable shellability* is the stronger demand that any shelling of any full-dimensional subcomplex may be continued into a shelling of the whole complex. This property is less understood than shellability, and much less common. It is easy to construct polytopes that are not extendably shellable [27]. In 1994 Simon conjectured that, for any integer $0 \leq d \leq n$, the $d$-skeleton of the $n$-simplex is extendably shellable [23, Conjecture 4.2.1]. For $d \leq 2$ this was soon proven by Björner and Eriksson [11], but for $3 \leq d \leq n-3$ the conjecture remains open.

Recently Bigdeli et al. [8] and Dochtermann et al. [18] established Simon's conjecture for $d \geq n-2$, showing also that shellability is equivalent to extendable shellability for $d$-complexes with up to $d+3$ vertices [18]. Their approach is based on a higher-dimensional extension of the graph-theoretic notion of chordality, called *ridge–chordality*, which we recall below. Given a $d$-dimensional pure simplicial complex $\Delta$, any $(d-1)$-dimensional face of it is called a *ridge*. *"Deleting above a ridge"* of $\Delta$ means to consider the simplicial complex whose facets are the facets of $\Delta$ not containing that ridge. A *clique* of $\Delta$ is any subset $V \subseteq [n]$ such that all subsets of $V$ of size $d+1$ appear among the facets of $\Delta$. For example, if $\Delta$ is the graph $\{12, 23, 13, 14\}$, then 1, 12 and 123 are cliques, whereas 124 and 1234 are not. A pure $d$-dimensional simplicial complex $\Delta$ is called *ridge-chordal* if $\Delta = \emptyset$ or if it can be reduced to the empty set by repeatedly deleting above a ridge $r$ such that the vertices of the star of $r$ form a clique [7]. One can see that "ridge-chordal 1-complexes" are precisely the graphs admitting a perfect elimination ordering, i.e. graphs in which every minimal vertex cut is a clique; by Dirac's theorem, these are precisely the "chordal graphs", the graphs where every cycle of length at least four has a chord [14].

Now, let $\mathrm{Cl}(\Delta)$ be the "clique complex" of $\Delta$, i.e., the simplicial complex whose faces are the cliques of $\Delta$. This $\mathrm{Cl}(\Delta)$ is a simplicial complex of dimension at least $d$, with the same $d$-faces of $\Delta$ and the same $(d-1)$-faces of the $n$-simplex. The following conjecture appeared naturally, in several recent works:

**Conjecture A** ([6, Question 6.3], [15, Conjecture 4.8], [21, Statement A])**.**
If the Alexander dual of $\mathrm{Cl}(\Delta)$ is shellable, then $\Delta$ is ridge-chordal.

There are three reasons why Conjecture A is natural and of interest:

(1) As explained by Bigdeli et al. [8, Corollary 3.7] and [21, Corollary 4.16], Conjecture A directly implies Simon's conjecture, cf. Remark 6.

(2) The conjecture is true if one slightly strengthens the assumption "shellable" into "vertex-decomposable". This fact is proven in the pure case by Nikseresht [21, Theorem 3.10], and in full-generality by Bigdeli–Faridi [6, Theorem 5.2]; see also Remark 5 below. Also, Conjecture A holds for $\dim \Delta = 1$.

(3) Some partial converse holds: If $\Delta$ is ridge-chordal, then the Alexander dual of $\mathrm{Cl}(\Delta)$ is Cohen–Macaulay over any field [7, Theorem 3.2], although not necessarily shellable or constructible [7, Example 3.14].

The purpose of this short note is to strongly disprove Conjecture A:

**Theorem A.** For any $k \geq 2$ there is a constructible 2-dimensional complex $\Delta_k$ that is not ridge-chordal, such that the Alexander dual of $\mathrm{Cl}(\Delta_k)$ is pure $(5k-2)$-dimensional, shellable, and even 4-decomposable.

Theorem A provides a non-trivial class of complexes that are 4- but not 0-decomposable (cf. Remark 5) in arbitrarily high dimension. This infinite family does not disprove Simon's conjecture, because the shelling of the Alexander dual of $\mathrm{Cl}(\Delta_k)$, which is $(5k-2)$-dimensional on $5k+2$ vertices, does extend to a shelling of the $(5k-2)$-skeleton of the $(5k+1)$-simplex, as we will see in Remark 8.

## 2. Construction of the counterexamples

Recall that the link and the deletion of a face $\sigma \in \Delta$ are defined respectively by

$$\mathrm{link}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \sigma \subseteq F \supseteq \tau \text{ for some facet } F\} \quad \text{and} \quad \mathrm{del}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \not\subseteq \tau\}.$$

We say that a face $\sigma$ in a pure simplicial complex $\Delta$ is *shedding* if $\mathrm{del}_\Delta(\sigma)$ is pure of dimension $\dim \Delta$. An equivalent formulation (see for instance [26, Definition 3.1]) is the following: $\sigma$ is shedding if and only if for every face $F \in \Delta$ such that $\sigma \subseteq F$ and for every $v \in \sigma$, there exists $w \notin F$ such that $(F \setminus \{v\}) \cup \{w\} \in \Delta$. A pure simplicial complex $\Delta$ is *$k$-decomposable* if $\Delta$ is a simplex or if there exists a shedding face $\sigma \in \Delta$ with $\dim \sigma \leq k$ such that $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{del}_\Delta(\sigma)$ are both $k$-decomposable [22]. It is easy to see that if $\Delta$ is $k$-decomposable then it is also $t$-decomposable, for every $k \leq t \leq \dim \Delta$. The notion of $k$-decomposable interpolates between vertex-decomposable complexes (which are the same as 0-decomposable complexes) and shellable complexes (which are the same as $d$-decomposable complexes, where $d$ is their dimension).

We start with a Lemma that is implicit in the work of Bigdeli–Faridi [6]. Recall that a *free face* in a simplicial complex $\Delta$ is a face strictly contained in only one facet of $\Delta$.

**Lemma 1.** *Let $r$ be a ridge of a pure $d$-dimensional simplicial complex $\Delta$, with $d \geq 1$. Let $S$ be the set of vertices of* $\mathrm{Star}(r, \Delta)$. *Then $S \in \mathrm{Cl}(\Delta) \iff r$ is a free face in* $\mathrm{Cl}(\Delta)$.

*Proof.* $\Rightarrow$**:** If $r$ lies in two facets $F_1$ and $F_2$ of $\mathrm{Cl}(\Delta)$, then $F_i = r \cup S_i$ for some $S_i \subseteq [n]$. Since $F_1, F_2 \in \mathrm{Cl}(\Delta)$, for every $s \in S_1 \cup S_2$ we have $r \cup \{s\} \in \Delta$. So $r \cup (S_1 \cup S_2) \subseteq S$ is a clique of $\Delta$. Since $r \cup S_1$ and $r \cup S_2$ are both facets of $\mathrm{Cl}(\Delta)$, we have $S_1 = S_2$, whence $F_1 = F_2$.

$\Leftarrow$**:** Let $F$ be the unique facet of $\mathrm{Cl}(\Delta)$ that contains $r$. Were there a vertex $s$ of $S$ outside $F$, we would have $r \cup \{s\} \in \Delta \subseteq \mathrm{Cl}(\Delta)$; so there would be $G \in \mathrm{Cl}(\Delta)$, $G \neq F$, such that $r \cup \{s\} \subseteq G$, a contradiction. Hence $S \subseteq F$ and $S \in \mathrm{Cl}(\Delta)$. ∎

**Lemma 2.** *Let* $\Delta$ *be a pure simplicial complex. If* $\Delta$ *is ridge-chordal and* $\dim \Delta = \dim \mathrm{Cl}(\Delta)$*, then* $\Delta$ *has at least one free ridge.*

*Proof.* If $\Delta$ is ridge-chordal, then it must have a ridge $r$ such that the vertices of $\mathrm{Star}(r, \Delta)$ form a clique. By Lemma 1, this $r$ is a free face of $\mathrm{Cl}(\Delta)$. But since $\dim\Delta = \dim \mathrm{Cl}(\Delta)$, the set of free ridges of $\Delta$ coincides with the set of $(\dim \Delta - 1)$-dimensional free faces of $\mathrm{Cl}(\Delta)$, because any ridge we add when passing from $\Delta$ to $\mathrm{Cl}(\Delta)$ belongs to no face of dimension equal to $\dim \Delta$. □

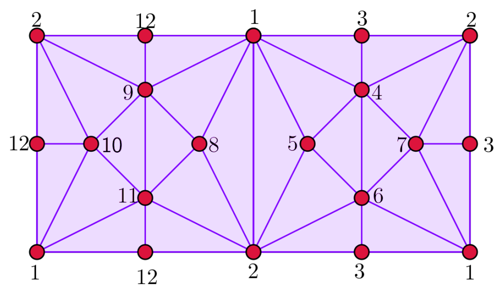


FIGURE 1. The constructible contractible 2-complex $\Delta^2_2$ without free edges, constructed by Barmak in [4, Example 11.2.9], is not ridge-chordal: See Proposition 3 below.

A pure $d$-dimensional complex $\Delta$ with $N$ facets is *constructible* if either $d(N-1) = 0$, or if $\Delta$ splits as $\Delta = \Delta_1 \cup \Delta_2$, with $\Delta_1$, $\Delta_2$ constructible $d$-dimensional complexes and $\Delta_1 \cap \Delta_2$ constructible $(d-1)$-dimensional. All shellable complexes are constructible, but the converse is false, cf. e.g. [5, Prop. 6.7].

**Proposition 3.** *For any integers* $d, k \geq 2$ *there is a constructible, contractible* $d$*-dimensional complex on* $k2^d + k + d$ *vertices that is not ridge-chordal.*

*Proof.* For any $d \geq 2$, there exists a shellable contractible simplicial $d$-complex $C_d$ on $2^d + d + 1$ vertices that has only one free ridge [2]. Let $\Delta^d_k$ be the $d$-complex obtained by glueing together $k$ copies of the complex $C_d$ via the identification of their free ridges; the case $d = k = 2$ is illustrated in Figure 1, and appeared also in Barmak's book [4, Example 11.2.9]. By definition, $\Delta^d_k$ is constructible and has $k(2^d + d + 1) - (k-1)d = k2^d + k + d$ vertices. By van Kampen's theorem, $\Delta^d_k$ is contractible. We claim that $\dim \Delta^d_k = \dim \mathrm{Cl}(\Delta^d_k)$. In fact, were there a face in $\mathrm{Cl}(\Delta^d_k)$ of dimension $t > d$, then $\Delta^d_k$ would contain an induced subcomplex $S$ on $t+1$ vertices with the same $d$-skeleton of the $t$-simplex. So $S$ would have nontrivial $d$-th homology, against the contractibility of $\Delta^d_k$. This proves the claim. But since $\Delta^d_k$ has no free ridge, it is neither ridge-chordal (by Lemma 2) nor shellable (because all shellable contractible complexes are collapsible, cf. [16, Lemma 17]). □

All minimal non-faces of $\mathrm{Cl}(\Delta^d_k)$ have dimension $d$. So the Alexander dual of $\mathrm{Cl}(\Delta^d_k)$ is pure $(k2^d+k-2)$-dimensional, with $k2^d+k+d$ vertices and $\binom{k2^d+d+k}{d+1} - f_d(\Delta^d_k)$ facets. To disprove Conjecture A, it remains to find values of $d$ and $k$ for which the Alexander dual of $\mathrm{Cl}(\Delta^d_k)$ is shellable.

**Lemma 4.** *Let* $\Delta$ *be a pure simplicial complex on* $[n]$*. Suppose that the minimal non-faces* $N_1, \ldots, N_t$ *of* $\Delta$ *have the property that* $N_j \cap N_h = \emptyset$ *for every* $j \neq h$*. Then* $\Delta$ *is vertex-decomposable.*

*Proof.* Let $m := \max\{|N_i|\}_{1 \leq i \leq t}$ and $V := [n] \setminus \bigcup_{i=1}^t N_i$. If $m = 1$, then $|N_i| = 1$ for every $1 \leq i \leq t$. So

$$\Delta = \begin{cases} \{\emptyset\} & \text{if } V = \emptyset \\ \text{a simplex} & \text{if } V \neq \emptyset. \end{cases}$$

Either way, $\Delta$ is vertex-decomposable and we are done. Now suppose $m > 1$ and denote by $\partial N_i$ the boundary of a simplex on the vertices of $N_i$. Then

$$\Delta = \begin{cases} \partial N_1 * \cdots * \partial N_t & \text{if } V = \emptyset \\ V * \partial N_1 * \cdots * \partial N_t & \text{if } V \neq \emptyset, \end{cases}$$

where $*$ denotes the join of simplicial complexes on disjoint sets of vertices. Either way, $\Delta$ is the join of vertex-decomposable complexes, hence vertex-decomposable. □

**Proof of Theorem A.** Let $k \geq 2$ and let $A_k$ be the Alexander dual of $\mathrm{Cl}(\Delta^2_k)$. Since all minimal non-faces of $\mathrm{Cl}(\Delta^2_k)$ have dimension 2, this $A_k$ is pure $(5k-2)$-dimensional, with $n := 5k+2$ vertices and $\binom{5k+2}{3} - 13k$ facets. Let $\gamma_j$ be the set of vertices in the $j$-th copy of $C_2$ that do not belong to the free face. Then $[n] \setminus \gamma_j$ is not in $\mathrm{Cl}(\Delta^2_k)$, because $\dim([n] \setminus \gamma_j) = 5(k-1)+1 > 2 = \dim \mathrm{Cl}(\Delta^2_k)$. So $\gamma_j \in A_k$, for all $1 \leq j \leq k$. Define

$$D^k_0 := A_k, \quad D^k_j := \mathrm{del}_{D^k_{j-1}}(\gamma_j), \quad \text{ and } L^k_j := \mathrm{link}_{D^k_{j-1}}(\gamma_j), \quad \text{ for } 1 \leq j \leq k.$$

If $j > 1$ and $t \geq j$, we have $\gamma_t \in D^k_{j-1}$, because $\gamma_h \not\subseteq \gamma_t$, for every $h \leq j-1$. Moreover, if $k > 2$, $\gamma_{j-1} \cup \gamma_j \in D^k_{j-2}$, i.e. $\gamma_{j-1} \in \mathrm{link}_{D^k_{j-2}}(\gamma_j)$, because $\dim([n] \setminus (\gamma_{j-1} \cup \gamma_j)) = 5(k-2)+1 > 2 = \dim \mathrm{Cl}(\Delta^2_k)$.

We are going to show that $A_k$ is 4-decomposable by induction on $k \geq 2$. Let $k = 2$. We checked using [13] that $D^2_1$ and $D^2_2$ are pure 8-dimensional. Moreover, we checked that $L^2_1 \simeq L^2_2 \simeq A_1$, where $A_1$ is the Alexander dual of $\mathrm{Cl}(C_2)$. The reader may verify that a shelling for such 3-complex is

$$[4,5,6,7], [3,5,6,7], [2,4,6,7], [1,4,6,7], [1,3,6,7], [1,2,6,7], [3,4,5,7], [1,3,5,7],$$
$$[1,2,5,7], [2,3,5,7], [2,3,4,7], [1,2,4,7], [3,4,5,6], [2,3,4,6], [2,3,5,6], [1,2,5,6],$$
$$[1,3,4,6], [1,2,4,6], [1,2,3,6], [1,3,4,5], [1,2,4,5], [1,2,3,4].$$

Since $D^2_2$ is vertex-decomposable, it follows that $A_2$ is 4-decomposable.
Now let $k > 2$. Notice that $\mathrm{link}_{A_k}(\gamma_j) \simeq A_{k-1}$, for every $j$, where '$\simeq$' stands for 'combinatorially equivalent'. In particular, $L^k_1 \simeq A_{k-1}$. In general, we have $L^k_j \simeq D^{k-1}_{j-1}$. We proceed by induction on $j$. Let $j > 1$. We have

$$L^k_j = \mathrm{link}_{D^k_{j-1}}(\gamma_j) = \mathrm{link}_{\mathrm{del}_{D^k_{j-2}}(\gamma_{j-1})}(\gamma_j) = \mathrm{del}_{\mathrm{link}_{D^k_{j-2}}(\gamma_j)}(\gamma_{j-1}) \simeq \mathrm{del}_{D^{k-1}_{j-2}}(\gamma_{j-1}) = D^{k-1}_{j-1},$$

where the combinatorial equivalence is ensured by $\mathrm{link}_{D^k_{j-2}}(\gamma_j) \simeq L^k_{j-1} \simeq D^{k-1}_{j-2}$. Moreover, the third equality holds because, for every $G \in \Delta$ and $F \in \mathrm{link}_\Delta(G)$, we have $\mathrm{link}_{\mathrm{del}_\Delta(F)}(G) = \mathrm{del}_{\mathrm{link}_\Delta(G)}(F)$. We have to verify that for $j = 1, 2, 3$, $\gamma_j$ is a shedding face of $D^k_{j-1}$. Here is a proof:

- Let $F = [n] \setminus S$ be a facet of $A_k = D^k_0$ containing $\gamma_1$. Let $w \in \gamma_1$. We claim that there exists $s \in S$ such that $\{s, w\} \notin \Delta^2_k$. In fact, $S \cap \gamma_j \neq \emptyset$ for some $j \geq 2$, otherwise $\bigcup_{j=2}^k \gamma_j \subseteq F$. Let $r$ be the free ridge of $C_2$. Hence $S \subseteq r \cup \gamma_1$ and $S \cap \gamma_1 \neq \emptyset$, a contradiction. Let $v \in S \setminus \{s\}$ and we have $(F \setminus \{w\}) \cup \{v\} \in A_k$, because $(S \setminus \{v\}) \cup \{w\} \notin \mathrm{Cl}(\Delta^2_k)$.

- Let $F = [n] \setminus S$ be a facet in $D_1^k$ containing $\gamma_2$. Let $w \in \gamma_2$. Notice that $S \cap \gamma_1 \neq \emptyset$. Let $s \in S \cap \gamma_1$ and consider $v \in S \setminus \{s\}$. We have $(F \setminus \{w\}) \cup \{v\} \in D_1^k$. In fact, $(S \setminus \{v\}) \cup \{w\} \notin \mathrm{Cl}(\Delta_k^2)$, because $\{s, w\} \notin \Delta_k^2$, and $[(S \setminus \{v\}) \cup \{w\}] \cap \gamma_1 \neq \emptyset$.
- Let $F = [n] \setminus S$ be a facet in $D_2^k$ containing $\gamma_3$. Let $w \in \gamma_3$. Notice that $S \cap \gamma_1 \neq \emptyset$ and $S \cap \gamma_2 \neq \emptyset$. Let $s_i \in S \cap \gamma_i$, for $i = 1, 2$, and consider $v \in S \setminus \{s_1, s_2\}$. We have $(F \setminus \{w\}) \cup \{v\} \in D_2^k$. In fact, $(S \setminus \{v\}) \cup \{w\} \notin \mathrm{Cl}(\Delta_k^2)$, because $\{s_1, s_2\} \notin \Delta_k^2$, and $[(S \setminus \{v\}) \cup \{w\}] \cap \gamma_i \neq \emptyset$, for $i = 1, 2$.

Now we are ready to conclude.

Since $L_j^k \simeq D_{j-1}^{k-1}$, the complexes $L_j^k$ are 4-decomposable for $1 \leq j \leq 3$, by the inductive assumption. The unique minimal non-faces of $D_3^k$ are $\{\gamma_1, \gamma_2, \gamma_3\}$, because the set of facets of $D_3^k$ is $\{[n] \setminus S \in A_k : |S| = 3, |S \cap \gamma_j| = 1, j = 1, 2, 3\}$. Since $\{\gamma_1, \gamma_2, \gamma_3\}$ are disjoint, then $D_3^k$ is vertex-decomposable by Lemma 4. Hence $A_k$ is 4-decomposable, as desired. □

**Remark 5.** By the work of Bidgeli, Faridi [6] and Nikseresht [21] there cannot be any 0-decomposable counterexample to Conjecture A. To see this, recall that the $d$-closure of a pure $d$-dimensional simplicial complex $\Delta$ (see [6, Definition 2.1]) is exactly the clique complex $\mathrm{Cl}(\Delta)$. Hence, by [6, Proposition 2.7] and [6, Theorem 3.4], the following properties are equivalent:

- $\Delta$ is ridge-chordal;
- $\mathrm{Cl}(\Delta)$ is $d$-chordal, in the sense of Bigdeli-Faridi [6, Definition 2.6];
- $\mathrm{Cl}(\Delta)$ is $d$-collapsible, in the sense of Wegner [25].

Now, let $\Delta$ be a complex such that the Alexander dual of $\mathrm{Cl}(\Delta)$ is 0-decomposable. By [6, Theorem 5.2], the complex $\mathrm{Cl}(\Delta)$ is $d$-chordal; so by the equivalence above, $\Delta$ is ridge-chordal and Conjecture A holds. En passant, this also explains why Conjecture A is equivalent to [6, Question 6.3]. Our complex $\Delta_2^2$ of Figure 1 is not ridge-chordal, so in particular $\mathrm{Cl}(\Delta_2^2)$ is not 2-chordal.

**Remark 6.** In the literature, the problems we discussed are often phrased in terms of "clutters". Let $d \geq 1$ be an integer. A *$d$-uniform clutter* $\mathcal{C}$ is the collection of the facets of a pure $(d-1)$-dimensional simplicial complex $\Gamma_{\mathcal{C}}$. Denote by $I(\mathcal{C})$ the *edge ideal* of $\mathcal{C}$. Let $\overline{\mathcal{C}}$ be the clutter with vertices $1, \ldots, n$ whose edges are the $(d-1)$-dimensional non-faces of $\Gamma_{\mathcal{C}}$. It is easy to see that the edge ideal of $\overline{\mathcal{C}}$ is the Stanley–Reisner ideal of $\mathrm{Cl}(\Gamma_{\mathcal{C}})$. Moreover, the ridge-chordality of $\Gamma_{\mathcal{C}}$ is equivalent to the chordality of $\mathcal{C}$, as defined in [7]. With this terminology, Conjecture A can be rephrased as

"*For $d \geq 2$, if $\mathcal{C}$ is a $d$-uniform clutter such that $I(\overline{\mathcal{C}})$ has linear quotients, then $\mathcal{C}$ is chordal.*"

Theorem A, forgetting the constructibility and the 4-decomposability claims, could be then stated as

"*Infinitely many 3-uniform clutters $\mathcal{C}$ such that $I(\overline{\mathcal{C}})$ has linear quotients, are not chordal.*"

**Remark 7.** Ridge-chordality was introduced in [7] with the goal to extend Fröberg's characterization of the squarefree monomial ideals with 2-linear resolutions [20]. This notion was also implicit in [3, Section 6.2] and [17]. Several other higher-dimensional extensions of graph chordality exist in the literature: see for instance [1], [19], [24], [26]. A weakening of ridge-chordality is the demand that $I(\overline{\Delta})$ have a linear resolution over any field [7, Theorem 3.2], where $\overline{\Delta}$ is the complex whose facets are the $d$-dimensional non-faces of $\Delta$. As shown by [6, Example 4.8] or by our complex $\Delta_2^2$ of Figure 1, some complexes satisfying this property are not ridge-chordal. En passant, this clarifies what is new in Proposition 3: examples of constructible and even shellable non-ridge-chordal complexes were previously known, but they are not contractible, see for instance [10, Exercise 7.37, pag. 277]. Examples of contractible non-ridge-chordal complexes were also known, like [6, Example 4.8], but they are not constructible.

**Remark 8.** Let $\Delta$ be a pure $d$-complex on $n+1$ vertices such that $\dim \Delta = \dim \mathrm{Cl}(\Delta)$. We claim that if the Alexander dual of $\mathrm{Cl}(\Delta)$ is shellable, then the shelling extends to the $(n-d-1)$-skeleton of the $n$-simplex. In fact, all the minimal non-faces of $\mathrm{Cl}(\Delta)$ have cardinality $d+1$. Hence the Alexander dual $A$ of $\mathrm{Cl}(\Delta)$ has dimension $k-1$, where $k = n-d$. Moreover, the $(k-2)$-skeleton of $A$ is the $(k-2)$-skeleton of the $n$-simplex. By contradiction, let $N$ be a minimal non-face of $A$, with $|N| < k$. Then $[n+1] \setminus N$ is a facet of $\mathrm{Cl}(\Delta)$ of cardinality $|[n+1] - N| = n+1-|N| > n+1-k = d+1$ and $\dim \mathrm{Cl}(\Delta) > d$. This implies that all the missing facets of $A$ of dimension $k-1$ can be attached along their whole boundary to extend the shelling.

## 3. Open problems

We conclude proposing two questions:

**Question 9.** Is it true that the Alexander dual of $\mathrm{Cl}(\Delta_k^d)$ is $2^d$-decomposable?

**Question 10.** If both $\Delta$ and the Alexander dual of $\mathrm{Cl}(\Delta)$ are shellable, is it true that $\Delta$ is ridge-chordal?


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