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*Original*

Non-ridge-chordal complexes whose clique complex has shellable Alexander dual / Benedetti, Bruno; Bolognini, Davide. - In: JOURNAL OF COMBINATORIAL THEORY. SERIES A. - ISSN 0097-3165. - ELETTRONICO. - 180:(2021). [10.1016/j.jcta.2021.105430]

*Availability:*

This version is available at: 11566/331184 since: 2024-06-04T16:04:10Z

*Publisher:*

*Published*

DOI:10.1016/j.jcta.2021.105430

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(Article begins on next page)

# NON-RIDGE-CHORDAL COMPLEXES WHOSE CLIQUE COMPLEX HAS SHELLABLE ALEXANDER DUAL

BRUNO BENEDETTI AND DAVIDE BOLOGNINI

ABSTRACT. A recent conjecture that appeared in three papers by Bigdeli–Faridi, Dochtermann, and Nikseresht, is that every simplicial complex whose clique complex has shellable Alexander dual, is ridge-chordal. This strengthens the long-standing Simon’s conjecture that the  $k$ -skeleton of the simplex is extendably shellable, for any  $k$ . We show that the stronger conjecture has a negative answer, by exhibiting an infinite family of counterexamples.

## 1. INTRODUCTION

Shellability is a property satisfied by two important families of objects in combinatorics, namely, polytope boundaries [27] and order complexes of geometric lattices [9]. Moreover, skeleta of shellable complexes are themselves shellable [12]. *Extendable shellability* is the stronger demand that any shelling of any full-dimensional subcomplex may be continued into a shelling of the whole complex. This property is less understood than shellability, and much less common. It is easy to construct polytopes that are not extendably shellable [27]. In 1994 Simon conjectured that, for any integer  $0 \leq d \leq n$ , the  $d$ -skeleton of the  $n$ -simplex is extendably shellable [23, Conjecture 4.2.1]. For  $d \leq 2$  this was soon proven by Björner and Eriksson [11], but for  $3 \leq d \leq n - 3$  the conjecture remains open.

Recently Bigdeli et al. [8] and Dochtermann et al. [18] established Simon’s conjecture for  $d \geq n - 2$ , showing also that shellability is equivalent to extendable shellability for  $d$ -complexes with up to  $d + 3$  vertices [18]. Their approach is based on a higher-dimensional extension of the graph-theoretic notion of chordality, called *ridge-chordality*, which we recall below. Given a  $d$ -dimensional pure simplicial complex  $\Delta$ , any  $(d - 1)$ -dimensional face of it is called a *ridge*. “*Deleting above a ridge*” of  $\Delta$  means to consider the simplicial complex whose facets are the facets of  $\Delta$  not containing that ridge. A *clique* of  $\Delta$  is any subset  $V \subseteq [n]$  such that all subsets of  $V$  of size  $d + 1$  appear among the facets of  $\Delta$ . For example, if  $\Delta$  is the graph  $\{12, 23, 13, 14\}$ , then 1, 12 and 123 are cliques, whereas 124 and 1234 are not. A pure  $d$ -dimensional simplicial complex  $\Delta$  is called *ridge-chordal* if  $\Delta = \emptyset$  or if it can be reduced to the empty set by repeatedly deleting above a ridge  $r$  such that the vertices of the star of  $r$  form a clique [7]. One can see that “ridge-chordal 1-complexes” are precisely the graphs admitting a perfect elimination ordering, i.e. graphs in which every minimal vertex cut is a clique; by Dirac’s theorem, these are precisely the “chordal graphs”, the graphs where every cycle of length at least four has a chord [14].

Now, let  $\text{Cl}(\Delta)$  be the “clique complex” of  $\Delta$ , i.e., the simplicial complex whose faces are the cliques of  $\Delta$ . This  $\text{Cl}(\Delta)$  is a simplicial complex of dimension at least  $d$ , with the same  $d$ -faces of  $\Delta$  and the same  $(d - 1)$ -faces of the  $n$ -simplex. The following conjecture appeared naturally, in several recent works:

**Conjecture A** ([6, Question 6.3], [15, Conjecture 4.8], [21, Statement A]).

If the Alexander dual of  $\text{Cl}(\Delta)$  is shellable, then  $\Delta$  is ridge-chordal.

There are three reasons why Conjecture A is natural and of interest:

- (1) As explained by Bigdeli et al. [8, Corollary 3.7] and [21, Corollary 4.16], Conjecture A directly implies Simon’s conjecture, cf. Remark 6.
- (2) The conjecture is true if one slightly strengthens the assumption “shellable” into “vertex-decomposable”. This fact is proven in the pure case by Nikseresht [21, Theorem 3.10], and in full-generality by Bigdeli–Faridi [6, Theorem 5.2]; see also Remark 5 below. Also, Conjecture A holds for  $\dim \Delta = 1$ .
- (3) Some partial converse holds: If  $\Delta$  is ridge-chordal, then the Alexander dual of  $\text{Cl}(\Delta)$  is Cohen–Macaulay over any field [7, Theorem 3.2], although not necessarily shellable or constructible [7, Example 3.14].

The purpose of this short note is to strongly disprove Conjecture A:

**Theorem A.** For any  $k \geq 2$  there is a constructible 2-dimensional complex  $\Delta_k$  that is not ridge-chordal, such that the Alexander dual of  $\text{Cl}(\Delta_k)$  is pure  $(5k - 2)$ -dimensional, shellable, and even 4-decomposable.

Theorem A provides a non-trivial class of complexes that are 4- but not 0-decomposable (cf. Remark 5) in arbitrarily high dimension. This infinite family does not disprove Simon’s conjecture, because the shelling of the Alexander dual of  $\text{Cl}(\Delta_k)$ , which is  $(5k - 2)$ -dimensional on  $5k + 2$  vertices, does extend to a shelling of the  $(5k - 2)$ -skeleton of the  $(5k + 1)$ -simplex, as we will see in Remark 8.

## 2. CONSTRUCTION OF THE COUNTEREXAMPLES

Recall that the link and the deletion of a face  $\sigma \in \Delta$  are defined respectively by

$$\text{link}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \sigma \subseteq F \supseteq \tau \text{ for some facet } F\} \quad \text{and} \quad \text{del}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \not\subseteq \tau\}.$$

We say that a face  $\sigma$  in a pure simplicial complex  $\Delta$  is *shedding* if  $\text{del}_\Delta(\sigma)$  is pure of dimension  $\dim \Delta$ . An equivalent formulation (see for instance [26, Definition 3.1]) is the following:  $\sigma$  is shedding if and only if for every face  $F \in \Delta$  such that  $\sigma \subseteq F$  and for every  $v \in \sigma$ , there exists  $w \notin F$  such that  $(F \setminus \{v\}) \cup \{w\} \in \Delta$ . A pure simplicial complex  $\Delta$  is *k-decomposable* if  $\Delta$  is a simplex or if there exists a shedding face  $\sigma \in \Delta$  with  $\dim \sigma \leq k$  such that  $\text{link}_\Delta(\sigma)$  and  $\text{del}_\Delta(\sigma)$  are both *k-decomposable* [22]. It is easy to see that if  $\Delta$  is *k-decomposable* then it is also *t-decomposable*, for every  $k \leq t \leq \dim \Delta$ . The notion of *k-decomposable* interpolates between vertex-decomposable complexes (which are the same as 0-decomposable complexes) and shellable complexes (which are the same as *d-decomposable* complexes, where *d* is their dimension).

We start with a Lemma that is implicit in the work of Bigdeli–Faridi [6]. Recall that a *free face* in a simplicial complex  $\Delta$  is a face strictly contained in only one facet of  $\Delta$ .

**Lemma 1.** *Let  $r$  be a ridge of a pure  $d$ -dimensional simplicial complex  $\Delta$ , with  $d \geq 1$ . Let  $S$  be the set of vertices of  $\text{Star}(r, \Delta)$ . Then  $S \in \text{Cl}(\Delta) \iff r$  is a free face in  $\text{Cl}(\Delta)$ .*

*Proof.*  $\implies$ : If  $r$  lies in two facets  $F_1$  and  $F_2$  of  $\text{Cl}(\Delta)$ , then  $F_i = r \cup S_i$  for some  $S_i \subseteq [n]$ . Since  $F_1, F_2 \in \text{Cl}(\Delta)$ , for every  $s \in S_1 \cup S_2$  we have  $r \cup \{s\} \in \Delta$ . So  $r \cup (S_1 \cup S_2) \subseteq S$  is a clique of  $\Delta$ . Since  $r \cup S_1$  and  $r \cup S_2$  are both facets of  $\text{Cl}(\Delta)$ , we have  $S_1 = S_2$ , whence  $F_1 = F_2$ .

$\impliedby$ : Let  $F$  be the unique facet of  $\text{Cl}(\Delta)$  that contains  $r$ . Were there a vertex  $s$  of  $S$  outside  $F$ , we would have  $r \cup \{s\} \in \Delta \subseteq \text{Cl}(\Delta)$ ; so there would be  $G \in \text{Cl}(\Delta)$ ,  $G \neq F$ , such that  $r \cup \{s\} \subseteq G$ , a contradiction. Hence  $S \subseteq F$  and  $S \in \text{Cl}(\Delta)$ .  $\square$

**Lemma 2.** *Let  $\Delta$  be a pure simplicial complex. If  $\Delta$  is ridge-chordal and  $\dim \Delta = \dim \text{Cl}(\Delta)$ , then  $\Delta$  has at least one free ridge.*

*Proof.* If  $\Delta$  is ridge-chordal, then it must have a ridge  $r$  such that the vertices of  $\text{Star}(r, \Delta)$  form a clique. By Lemma 1, this  $r$  is a free face of  $\text{Cl}(\Delta)$ . But since  $\dim \Delta = \dim \text{Cl}(\Delta)$ , the set of free ridges of  $\Delta$  coincides with the set of  $(\dim \Delta - 1)$ -dimensional free faces of  $\text{Cl}(\Delta)$ , because any ridge we add when passing from  $\Delta$  to  $\text{Cl}(\Delta)$  belongs to no face of dimension equal to  $\dim \Delta$ .  $\square$

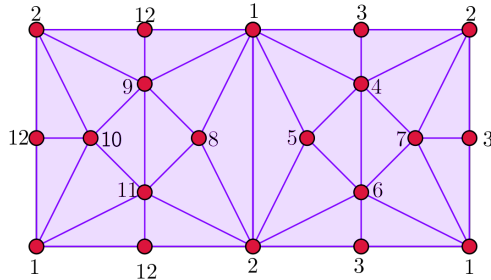


FIGURE 1. The constructible contractible 2-complex  $\Delta_2^2$  without free edges, constructed by Barmak in [4, Example 11.2.9], is not ridge-chordal: See Proposition 3 below.

A pure  $d$ -dimensional complex  $\Delta$  with  $N$  facets is *constructible* if either  $d(N - 1) = 0$ , or if  $\Delta$  splits as  $\Delta = \Delta_1 \cup \Delta_2$ , with  $\Delta_1, \Delta_2$  constructible  $d$ -dimensional complexes and  $\Delta_1 \cap \Delta_2$  constructible  $(d - 1)$ -dimensional. All shellable complexes are constructible, but the converse is false, cf. e.g. [5, Prop. 6.7].

**Proposition 3.** *For any integers  $d, k \geq 2$  there is a constructible, contractible  $d$ -dimensional complex on  $k2^d + k + d$  vertices that is not ridge-chordal.*

*Proof.* For any  $d \geq 2$ , there exists a shellable contractible simplicial  $d$ -complex  $C_d$  on  $2^d + d + 1$  vertices that has only one free ridge [2]. Let  $\Delta_k^d$  be the  $d$ -complex obtained by glueing together  $k$  copies of the complex  $C_d$  via the identification of their free ridges; the case  $d = k = 2$  is illustrated in Figure 1, and appeared also in Barmak's book [4, Example 11.2.9]. By definition,  $\Delta_k^d$  is constructible and has  $k(2^d + d + 1) - (k - 1)d = k2^d + k + d$  vertices. By van Kampen's theorem,  $\Delta_k^d$  is contractible. We claim that  $\dim \Delta_k^d = \dim \text{Cl}(\Delta_k^d)$ . In fact, were there a face in  $\text{Cl}(\Delta_k^d)$  of dimension  $t > d$ , then  $\Delta_k^d$  would contain an induced subcomplex  $S$  on  $t + 1$  vertices with the same  $d$ -skeleton of the  $t$ -simplex. So  $S$  would have nontrivial  $d$ -th homology, against the contractibility of  $\Delta_k^d$ . This proves the claim. But since  $\Delta_k^d$  has no free ridge, it is neither ridge-chordal (by Lemma 2) nor shellable (because all shellable contractible complexes are collapsible, cf. [16, Lemma 17]).  $\square$

All minimal non-faces of  $\text{Cl}(\Delta_k^d)$  have dimension  $d$ . So the Alexander dual of  $\text{Cl}(\Delta_k^d)$  is pure  $(k2^d + k - 2)$ -dimensional, with  $k2^d + k + d$  vertices and  $\binom{k2^d + d + k}{d+1} - f_d(\Delta_k^d)$  facets. To disprove Conjecture A, it remains to find values of  $d$  and  $k$  for which the Alexander dual of  $\text{Cl}(\Delta_k^d)$  is shellable.

**Lemma 4.** *Let  $\Delta$  be a pure simplicial complex on  $[n]$ . Suppose that the minimal non-faces  $N_1, \dots, N_t$  of  $\Delta$  have the property that  $N_j \cap N_h = \emptyset$  for every  $j \neq h$ . Then  $\Delta$  is vertex-decomposable.*

*Proof.* Let  $m := \max\{|N_i|\}_{1 \leq i \leq t}$  and  $V := [n] \setminus \bigcup_{i=1}^t N_i$ . If  $m = 1$ , then  $|N_i| = 1$  for every  $1 \leq i \leq t$ . So

$$\Delta = \begin{cases} \{\emptyset\} & \text{if } V = \emptyset \\ \text{a simplex} & \text{if } V \neq \emptyset. \end{cases}$$

Either way,  $\Delta$  is vertex-decomposable and we are done. Now suppose  $m > 1$  and denote by  $\partial N_i$  the boundary of a simplex on the vertices of  $N_i$ . Then

$$\Delta = \begin{cases} \partial N_1 * \cdots * \partial N_t & \text{if } V = \emptyset \\ V * \partial N_1 * \cdots * \partial N_t & \text{if } V \neq \emptyset, \end{cases}$$

where  $*$  denotes the join of simplicial complexes on disjoint sets of vertices. Either way,  $\Delta$  is the join of vertex-decomposable complexes, hence vertex-decomposable.  $\square$

**Proof of Theorem A.** Let  $k \geq 2$  and let  $A_k$  be the Alexander dual of  $\text{Cl}(\Delta_k^2)$ . Since all minimal non-faces of  $\text{Cl}(\Delta_k^2)$  have dimension 2, this  $A_k$  is pure  $(5k - 2)$ -dimensional, with  $n := 5k + 2$  vertices and  $\binom{5k+2}{3} - 13k$  facets. Let  $\gamma_j$  be the set of vertices in the  $j$ -th copy of  $C_2$  that do not belong to the free face. Then  $[n] \setminus \gamma_j$  is not in  $\text{Cl}(\Delta_k^2)$ , because  $\dim([n] \setminus \gamma_j) = 5(k - 1) + 1 > 2 = \dim \text{Cl}(\Delta_k^2)$ . So  $\gamma_j \in A_k$ , for all  $1 \leq j \leq k$ . Define

$$D_0^k := A_k, \quad D_j^k := \text{del}_{D_{j-1}^k}(\gamma_j), \quad \text{and } L_j^k := \text{link}_{D_{j-1}^k}(\gamma_j), \quad \text{for } 1 \leq j \leq k.$$

If  $j > 1$  and  $t \geq j$ , we have  $\gamma_t \in D_{j-1}^k$ , because  $\gamma_h \not\subseteq \gamma_t$ , for every  $h \leq j - 1$ . Moreover, if  $k > 2$ ,  $\gamma_{j-1} \cup \gamma_j \in D_{j-2}^k$ , i.e.  $\gamma_{j-1} \in \text{link}_{D_{j-2}^k}(\gamma_j)$ , because  $\dim([n] \setminus (\gamma_{j-1} \cup \gamma_j)) = 5(k - 2) + 1 > 2 = \dim \text{Cl}(\Delta_k^2)$ .

We are going to show that  $A_k$  is 4-decomposable by induction on  $k \geq 2$ . Let  $k = 2$ . We checked using [13] that  $D_1^2$  and  $D_2^2$  are pure 8-dimensional. Moreover, we checked that  $L_1^2 \simeq L_2^2 \simeq A_1$ , where  $A_1$  is the Alexander dual of  $\text{Cl}(C_2)$ . The reader may verify that a shelling for such 3-complex is

$$\begin{aligned} & [4, 5, 6, 7], [3, 5, 6, 7], [2, 4, 6, 7], [1, 4, 6, 7], [1, 3, 6, 7], [1, 2, 6, 7], [3, 4, 5, 7], [1, 3, 5, 7], \\ & [1, 2, 5, 7], [2, 3, 5, 7], [2, 3, 4, 7], [1, 2, 4, 7], [3, 4, 5, 6], [2, 3, 4, 6], [2, 3, 5, 6], [1, 2, 5, 6], \\ & [1, 3, 4, 6], [1, 2, 4, 6], [1, 2, 3, 6], [1, 3, 4, 5], [1, 2, 4, 5], [1, 2, 3, 4]. \end{aligned}$$

Since  $D_2^2$  is vertex-decomposable, it follows that  $A_2$  is 4-decomposable.

Now let  $k > 2$ . Notice that  $\text{link}_{A_k}(\gamma_j) \simeq A_{k-1}$ , for every  $j$ , where ‘ $\simeq$ ’ stands for ‘combinatorially equivalent’. In particular,  $L_1^k \simeq A_{k-1}$ . In general, we have  $L_j^k \simeq D_{j-1}^{k-1}$ . We proceed by induction on  $j$ . Let  $j > 1$ . We have

$$L_j^k = \text{link}_{D_{j-1}^k}(\gamma_j) = \text{link}_{\text{del}_{D_{j-2}^k}(\gamma_{j-1})}(\gamma_j) = \text{del}_{\text{link}_{D_{j-2}^k}(\gamma_{j-1})}(\gamma_{j-1}) \simeq \text{del}_{D_{j-2}^{k-1}}(\gamma_{j-1}) = D_{j-1}^{k-1},$$

where the combinatorial equivalence is ensured by  $\text{link}_{D_{j-2}^k}(\gamma_j) \simeq L_{j-1}^k \simeq D_{j-2}^{k-1}$ . Moreover, the third equality holds because, for every  $G \in \Delta$  and  $F \in \text{link}_{\Delta}(G)$ , we have  $\text{link}_{\text{del}_{\Delta}(F)}(G) = \text{del}_{\text{link}_{\Delta}(G)}(F)$ . We have to verify that for  $j = 1, 2, 3$ ,  $\gamma_j$  is a shedding face of  $D_{j-1}^k$ . Here is a proof:

- Let  $F = [n] \setminus S$  be a facet of  $A_k = D_0^k$  containing  $\gamma_1$ . Let  $w \in \gamma_1$ . We claim that there exists  $s \in S$  such that  $\{s, w\} \notin \Delta_k^2$ . In fact,  $S \cap \gamma_j \neq \emptyset$  for some  $j \geq 2$ , otherwise  $\bigcup_{j=2}^k \gamma_j \subseteq F$ . Let  $r$  be the free ridge of  $C_2$ . Hence  $S \subseteq r \cup \gamma_1$  and  $S \cap \gamma_1 \neq \emptyset$ , a contradiction. Let  $v \in S \setminus \{s\}$  and we have  $(F \setminus \{w\}) \cup \{v\} \in A_k$ , because  $(S \setminus \{v\}) \cup \{w\} \notin \text{Cl}(\Delta_k^2)$ .

- Let  $F = [n] \setminus S$  be a facet in  $D_1^k$  containing  $\gamma_2$ . Let  $w \in \gamma_2$ . Notice that  $S \cap \gamma_1 \neq \emptyset$ . Let  $s \in S \cap \gamma_1$  and consider  $v \in S \setminus \{s\}$ . We have  $(F \setminus \{w\}) \cup \{v\} \in D_1^k$ . In fact,  $(S \setminus \{v\}) \cup \{w\} \notin \text{Cl}(\Delta_k^2)$ , because  $\{s, w\} \notin \Delta_k^2$ , and  $[(S \setminus \{v\}) \cup \{w\}] \cap \gamma_1 \neq \emptyset$ .
- Let  $F = [n] \setminus S$  be a facet in  $D_2^k$  containing  $\gamma_3$ . Let  $w \in \gamma_3$ . Notice that  $S \cap \gamma_1 \neq \emptyset$  and  $S \cap \gamma_2 \neq \emptyset$ . Let  $s_i \in S \cap \gamma_i$ , for  $i = 1, 2$ , and consider  $v \in S \setminus \{s_1, s_2\}$ . We have  $(F \setminus \{w\}) \cup \{v\} \in D_2^k$ . In fact,  $(S \setminus \{v\}) \cup \{w\} \notin \text{Cl}(\Delta_k^2)$ , because  $\{s_1, s_2\} \notin \Delta_k^2$ , and  $[(S \setminus \{v\}) \cup \{w\}] \cap \gamma_i \neq \emptyset$ , for  $i = 1, 2$ .

Now we are ready to conclude.

Since  $L_j^k \simeq D_{j-1}^{k-1}$ , the complexes  $L_j^k$  are 4-decomposable for  $1 \leq j \leq 3$ , by the inductive assumption. The unique minimal non-faces of  $D_3^k$  are  $\{\gamma_1, \gamma_2, \gamma_3\}$ , because the set of facets of  $D_3^k$  is  $\{[n] \setminus S \in A_k : |S| = 3, |S \cap \gamma_j| = 1, j = 1, 2, 3\}$ . Since  $\{\gamma_1, \gamma_2, \gamma_3\}$  are disjoint, then  $D_3^k$  is vertex-decomposable by Lemma 4. Hence  $A_k$  is 4-decomposable, as desired.  $\square$

**Remark 5.** By the work of Bidgeli, Faridi [6] and Nikseresht [21] there cannot be any 0-decomposable counterexample to Conjecture A. To see this, recall that the  $d$ -closure of a pure  $d$ -dimensional simplicial complex  $\Delta$  (see [6, Definition 2.1]) is exactly the clique complex  $\text{Cl}(\Delta)$ . Hence, by [6, Proposition 2.7] and [6, Theorem 3.4], the following properties are equivalent:

- $\Delta$  is ridge-chordal;
- $\text{Cl}(\Delta)$  is  $d$ -chordal, in the sense of Bidgeli-Faridi [6, Definition 2.6];
- $\text{Cl}(\Delta)$  is  $d$ -collapsible, in the sense of Wegner [25].

Now, let  $\Delta$  be a complex such that the Alexander dual of  $\text{Cl}(\Delta)$  is 0-decomposable. By [6, Theorem 5.2], the complex  $\text{Cl}(\Delta)$  is  $d$ -chordal; so by the equivalence above,  $\Delta$  is ridge-chordal and Conjecture A holds. En passant, this also explains why Conjecture A is equivalent to [6, Question 6.3]. Our complex  $\Delta_2^2$  of Figure 1 is not ridge-chordal, so in particular  $\text{Cl}(\Delta_2^2)$  is not 2-chordal.

**Remark 6.** In the literature, the problems we discussed are often phrased in terms of “clutters”. Let  $d \geq 1$  be an integer. A  $d$ -uniform clutter  $\mathcal{C}$  is the collection of the facets of a pure  $(d-1)$ -dimensional simplicial complex  $\Gamma_{\mathcal{C}}$ . Denote by  $I(\mathcal{C})$  the *edge ideal* of  $\mathcal{C}$ . Let  $\bar{\mathcal{C}}$  be the clutter with vertices  $1, \dots, n$  whose edges are the  $(d-1)$ -dimensional non-faces of  $\Gamma_{\mathcal{C}}$ . It is easy to see that the edge ideal of  $\bar{\mathcal{C}}$  is the Stanley–Reisner ideal of  $\text{Cl}(\Gamma_{\mathcal{C}})$ . Moreover, the ridge-chordality of  $\Gamma_{\mathcal{C}}$  is equivalent to the chordality of  $\mathcal{C}$ , as defined in [7]. With this terminology, Conjecture A can be rephrased as

“For  $d \geq 2$ , if  $\mathcal{C}$  is a  $d$ -uniform clutter such that  $I(\bar{\mathcal{C}})$  has linear quotients, then  $\mathcal{C}$  is chordal.”

Theorem A, forgetting the constructibility and the 4-decomposability claims, could be then stated as

“Infinitely many 3-uniform clutters  $\mathcal{C}$  such that  $I(\bar{\mathcal{C}})$  has linear quotients, are not chordal.”

**Remark 7.** Ridge-chordality was introduced in [7] with the goal to extend Fröberg’s characterization of the squarefree monomial ideals with 2-linear resolutions [20]. This notion was also implicit in [3, Section 6.2] and [17]. Several other higher-dimensional extensions of graph chordality exist in the literature: see for instance [1], [19], [24], [26]. A weakening of ridge-chordality is the demand that  $I(\bar{\Delta})$  have a linear resolution over any field [7, Theorem 3.2], where  $\bar{\Delta}$  is the complex whose facets are the  $d$ -dimensional non-faces of  $\Delta$ . As shown by [6, Example 4.8] or by our complex  $\Delta_2^2$  of Figure 1, some complexes satisfying this property are not ridge-chordal. En passant, this clarifies what is new in Proposition 3: examples of constructible and even shellable non-ridge-chordal complexes were previously known, but they are not contractible, see for instance [10, Exercise 7.37, pag. 277]. Examples of contractible non-ridge-chordal complexes were also known, like [6, Example 4.8], but they are not constructible.

**Remark 8.** Let  $\Delta$  be a pure  $d$ -complex on  $n + 1$  vertices such that  $\dim \Delta = \dim \text{Cl}(\Delta)$ . We claim that if the Alexander dual of  $\text{Cl}(\Delta)$  is shellable, then the shelling extends to the  $(n - d - 1)$ -skeleton of the  $n$ -simplex. In fact, all the minimal non-faces of  $\text{Cl}(\Delta)$  have cardinality  $d + 1$ . Hence the Alexander dual  $A$  of  $\text{Cl}(\Delta)$  has dimension  $k - 1$ , where  $k = n - d$ . Moreover, the  $(k - 2)$ -skeleton of  $A$  is the  $(k - 2)$ -skeleton of the  $n$ -simplex. By contradiction, let  $N$  be a minimal non-face of  $A$ , with  $|N| < k$ . Then  $[n + 1] \setminus N$  is a facet of  $\text{Cl}(\Delta)$  of cardinality  $|[n + 1] - N| = n + 1 - |N| > n + 1 - k = d + 1$  and  $\dim \text{Cl}(\Delta) > d$ . This implies that all the missing facets of  $A$  of dimension  $k - 1$  can be attached along their whole boundary to extend the shelling.

### 3. OPEN PROBLEMS

We conclude proposing two questions:

**Question 9.** Is it true that the Alexander dual of  $\text{Cl}(\Delta_k^d)$  is  $2^d$ -decomposable?

**Question 10.** If both  $\Delta$  and the Alexander dual of  $\text{Cl}(\Delta)$  are shellable, is it true that  $\Delta$  is ridge-chordal?

### ACKNOWLEDGMENTS

We are grateful to Anton Dochtermann and Joseph Doolittle for useful comments, and to the referees for useful corrections and references. The first author is supported by NSF Grant 1855165. The second author was supported by INdAM.

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