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On the difference between weak solution and boundary datum in some elliptic equations and systems

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Abstract
This paper is concerned with weak solutions of elliptic equations and of systems of the form

$$\begin{cases} -\sum_{i=1}^{n} D_i \left(A_i^{\alpha}\left(x, Du(x)\right)\right) = 0, & \text{in } \Omega, \quad \forall\, \alpha \in \{1, \cdots, m\}, \\ u(x) = u_*(x), & \text{on } \partial\Omega. \end{cases}$$

We show that, assuming some high degree of integrability of Du_* the gradient of the boundary datum, we can obtain bounds of the difference $|u^{\gamma} - u_*^{\gamma}|_{\infty}$, $\gamma \in \{1, \cdots, m\}$. In particular, when $m = 1$, things are easier and we can get a better result.

Keywords Elliptic systems · Elliptic equations · Difference between weak solution and boundary datum

Mathematics Subject Classification 35J47 · 35J60 · 35B65

1 Introduction

In this paper we consider the elliptic equation

$$-\sum_{i=1}^{n} D_i \left(A_i\left(x, Du(x)\right)\right) = 0, \tag{1.1}$$

To Antonia Passarelli di Napoli, on the occasion of her birthday.

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where $x \in \Omega$ and Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 2$, $u : \Omega \to \mathbb{R}$ and $u = u_*$ on $\partial\Omega$. We assume that there exist constants $c_1 > 0$, $c_2 \geq 0$, $c_3 \geq c_1$ and $1 < p < n$, such that the following coercivity condition and growth condition hold

$$c_1|z|^p - c_2 \leq \sum_{i=1}^{n} A_i(x, z)z_i, \tag{1.2}$$

$$|A(x, z)| \leq c_3\big(|z|^{p-1} + 1\big), \tag{1.3}$$

for almost every $x \in \Omega$ and for every $z \in \mathbb{R}^n$.

Under the above assumptions, weak solutions u satisfy the following maximum principle,

$$\inf_{\partial\Omega} u_* - \hat{c} \leq u(x) \leq \sup_{\partial\Omega} u_* + \hat{c},$$

for almost every $x \in \Omega$, where u_* is the boundary datum and

$$\hat{c} = \frac{(n-1)p}{n-p}\left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p}-\frac{1}{p^*}}\, 2^{\frac{p^*}{p^*-p}},$$

moreover, $p^* = \frac{np}{n-p}$ is the Sobolev exponent.

This means that the graph of the solution u, $\{(x, y) : x \in \Omega, y = u(x)\}$ is contained in the box $\left\{(x, y) : x \in \Omega, \inf_{\partial\Omega} u_* - \hat{c} \leq y \leq \sup_{\partial\Omega} u_* + \hat{c}\right\}$ (see Fig. 1).

Does the behaviour of the datum u_* inside Ω give us additional information on the solution u? More precisely, are we able to control the difference $u(x) - u_*(x)$ inside Ω? A first answer has been given in [2, 5, 9–12, 15, 16] and [18]. These results can be extended to our case and, under (1.2) and (1.3), we can prove the following estimate for the difference $u(x) - u_*(x)$, see Theorem 2.1 below:

$$|u(x) - u_*(x)| \leq \hat{\hat{c}},$$

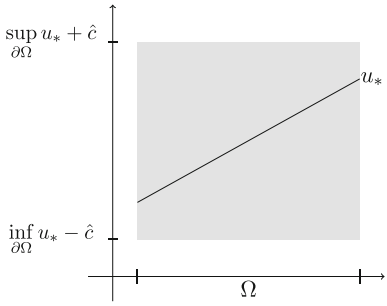

Fig. 1 The black line is the graph of u_*. In grey the region where the graph of u may stay

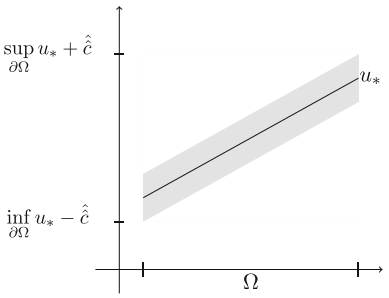

Fig. 2 The black line is the graph of u_*. In grey the region where the graph of u may stay

for almost every $x \in \Omega$, where

$$\hat{\hat{c}} = \frac{(n-1)p}{n-p}\left\{\left[p\left(\frac{c_2}{c_1}+1\right)\right]^{\frac{1}{p}}|\Omega|^{\frac{1}{q}} + \left(2^{\frac{1}{p}}\frac{c_3}{c_1}+1\right)\|Du_*\|_{L^q(\Omega)}\right\}$$
$$\times|\Omega|^{\left((\frac{1}{p}-\frac{1}{q})p^*-1\right)\frac{1}{p^*}} 2^{\frac{\left(\frac{1}{p}-\frac{1}{q}\right)p^*}{\left(\frac{1}{p}-\frac{1}{q}\right)p^*-1}},$$

provided $|Du_*| \in L^q(\Omega)$, where $p < n < q$.

This means that the graph of the solution u, $\{(x, y) : x \in \Omega, y = u(x)\}$ is contained in the strip $\left\{(x, y) : x \in \Omega, u_*(x) - \hat{\hat{c}} \le y \le u_*(x) + \hat{\hat{c}}\right\}$ (see Fig. 2).

In the second part, we are concerned with the difference between weak solution u and boundary datum u_* for some elliptic systems in divergence form of the following type

$$-\sum_{i=1}^{n} D_i\left(A_i^{\alpha}(x, Du(x))\right) = 0, \quad \alpha \in \{1, \cdots, m\}, \tag{1.4}$$

where $x \in \Omega$ and Ω is a bounded open subset in $\mathbb{R}^n$, $n \ge 2$. The function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, has components $\left(u^1, \cdots, u^m\right)$, then $Du(x)$ is the $m \times n$ matrix $\left(\frac{\partial u^{\alpha}}{\partial x_i}(x)\right)_{i=1,\cdots,n}^{\alpha=1,\cdots,m}$. Moreover, $u = u_*$ on $\partial\Omega$. Note that (1.4) is a system of m equations

$$\begin{cases} -\sum_{i=1}^{n} D_i\left(A_i^1(x, Du(x))\right) = 0, \\ -\sum_{i=1}^{n} D_i\left(A_i^2(x, Du(x))\right) = 0, \\ \qquad \cdots \\ -\sum_{i=1}^{n} D_i\left(A_i^m(x, Du(x))\right) = 0. \end{cases}$$

We assume that $A_i^\alpha(x,z)$ is measurable with respect to x and continuous with respect to z. Moreover, A satisfies coercivity and growth condition. We investigate whether it is possible to estimate the difference between the weak solution u and the boundary datum u_*, that is, to find a constant c such that

$$\left|u^\gamma(x) - u_*^\gamma(x)\right| \le c, \tag{1.5}$$

for all $\gamma \in \{1, \cdots, m\}$.

In the vectorial case, estimate (1.5) does not hold true any more: in [7], De Giorgi shows that $u(x) = \frac{x}{|x|^\sigma}$, with suitable $\sigma > 1$, is a weak solution to (1.4) in the linear case with suitable coefficients, where Ω is the unit ball. The boundary datum $u_*(x) = x$ is bounded, but $u(x)$ is unbounded near 0. Then, in order to prove (1.5), we need additional structure assumptions on $A_i^\alpha(x,z)$ in order to keep away De Giorgi's counterexample. In [1, 6, 14, 17, 19], componentwise coercivity is assumed: there exist constants $c_1 > 0$, $c_2 \ge 0$ and $1 < p < n$, such that

$$c_1|z^\alpha|^p - c_2 \le \sum_{i=1}^n A_i^\alpha(x,z) z_i^\alpha, \tag{1.6}$$

for every $\alpha \in \{1, \cdots, m\}$. This condition says that the α row of the system $A_i^\alpha(x,z)$ contains the gradient of all the components of u, but, when multiplied by the gradient of the α component of u, from below we can see only the gradient of the α component of u. In order to prove (1.5), we need growth condition: there exists constant $c_3 \ge c_1$, such that

$$\left|A^\alpha(x,z)\right| \le c_3\left(|z|^{p-1} + 1\right). \tag{1.7}$$

for every $\alpha = 1, \cdots, m$, where p is the same exponent in (1.6).

Under (1.6) and (1.7), we can prove (1.5), with

$$\begin{aligned} c = \frac{(n-1)p}{n-p} \Bigg\{ \Bigg\{ \frac{c_2}{c_1} |\Omega|^{1-\left(\frac{1}{p}-\frac{1}{q}\right)} + \frac{c_3}{c_1} \Bigg[\Bigg((2+p) \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p \|Du_*\|_{L^p(\Omega)}^p \\ + p \left(1 + m^{1+\frac{p}{2}} \frac{c_2}{c_1} \right) |\Omega| \Bigg)^{\frac{p-1}{p}} + |\Omega|^{\frac{p-1}{p}} \Bigg] \|Du_*\|_{L^q(\Omega)} \Bigg\}^{\frac{1}{p}} \\ + \|Du_*\|_{L^q(\Omega)} |\Omega|^{\left(\frac{1}{p}-\frac{1}{q}\right)\left(1-\frac{1}{p}\right)} \Bigg\} |\Omega|^{\left(\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}-1\right)\frac{1}{p^*}} 2^{\frac{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}}{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}-1}}, \end{aligned}$$

provided $|Du_*| \in L^q(\Omega)$, where $\left(\frac{1}{p} - \frac{1}{q}\right)\frac{p^*}{p} > 1$. See Theorem 4.1 below.

We remark that the condition on p and q in the case of one equation $m = 1$ is $p < n < q$: it is equivalent to $\left(\frac{1}{p} - \frac{1}{q}\right) p^* > 1$. Thus, it is weaker than the condition for systems $m \ge 2$ given by $\left(\frac{1}{p} - \frac{1}{q}\right)\frac{p^*}{p} > 1$.

This paper is organized as follows: Sections 2 and 3 are devoted to one equation (1.1); systems (1.4) are dealt with in Sections 4 and 5.

2 Assumptions and results for equations

In Sections 2 and 3 we are dealing with the difference between weak solution u and boundary datum u_* for some elliptic equations

$$-\sum_{i=1}^{n} D_i \left(A_i\left(x, Du(x)\right)\right) = 0, \tag{2.1}$$

where $x \in \Omega$ and Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 2$; moreover, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $u = u_*$ on $\partial\Omega$.

We assume that $A_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq n$ are Carathéodory functions satisfying for every $x \in \Omega$ and for every $z \in \mathbb{R}^n$ the following inequalities:

$$c_1|z|^p - c_2 \leq \sum_{i=1}^{n} A_i(x, z) z_i, \tag{2.2}$$

$$|A(x, z)| \leq c_3\left(|z|^{p-1} + 1\right), \tag{2.3}$$

where $1 < p < n$, c_1, c_2 and c_3 are constants, with $c_1 > 0$, $c_2 \geq 0$ and $c_3 \geq c_1$.

Let us recall that $u \in u_* + W_0^{1,p}(\Omega)$ is a weak solution of (2.1) if

$$\int_\Omega \sum_{i=1}^{n} A_i\left(x, Du(x)\right) D_i\varphi(x)\mathrm{d}x = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.4}$$

As far as the boundary datum u_* is concerned, we assume that $u_* \in W^{1,p}(\Omega)$.

As usual, the Sobolev exponent is $p^* = \frac{np}{n-p}$ if $p < n$. The Hölder conjugate exponent is $p' = \frac{p}{p-1}$.

We have the following result about the difference between weak solution u and boundary datum u_*.

Theorem 2.1 *Under the assumptions (2.2) and (2.3), let $u_* \in W^{1,p}(\Omega)$ be the boundary datum. Let $u \in u_* + W_0^{1,p}(\Omega)$ be a weak solution of (2.1). Moreover, let $|Du_*| \in L^q(\Omega)$, $q > p$, where p, q satisfy*

$$\left(\frac{1}{p} - \frac{1}{q}\right) p^* > 1. \tag{2.5}$$

Then

$$|u(x) - u_*(x)| \leq c|\Omega|^{\left(\left(\frac{1}{p}-\frac{1}{q}\right)p^*-1\right)\frac{1}{p^*}} 2^{\frac{\left(\frac{1}{p}-\frac{1}{q}\right)p^*}{\left(\frac{1}{p}-\frac{1}{q}\right)p^*-1}},$$

where

$$c = \frac{(n-1)p}{n-p}\left\{\left[p\left(\frac{c_2}{c_1}+1\right)\right]^{\frac{1}{p}}|\Omega|^{\frac{1}{q}} + \left(2^{\frac{1}{p}}\frac{c_3}{c_1}+1\right)\|Du_*\|_{L^q(\Omega)}\right\},$$

for almost every $x \in \Omega$.

Remark 2.1 We remark that condition (2.5) is less restrictive than (4.5). Moreover, condition (2.5) is equivalent to $q > n$.

3 Proof of Theorem 2.1

In order to prove Theorem 2.1 and Theorem 4.1, we need the following Stampacchia Lemma, which can be found, for example, in [20].

Lemma 3.1 *Let* c, α, β *be positive constants and* $k_0 \in \mathbb{R}$. *Let* $\varphi : [k_0, +\infty) \to [0, +\infty)$ *be nonincreasing and such that*

$$\varphi(h) \le \frac{c}{(h-k)^\alpha}[\varphi(k)]^\beta, \tag{3.1}$$

for every h, k *with* $h > k \ge k_0$. *If* $\beta > 1$ *then*

$$\varphi(k_0 + d) = 0,$$

where

$$d^\alpha = c[\varphi(k_0)]^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}. \tag{3.2}$$

We now proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1 We take $L \ge 0$ and we define $\varphi = G_L(u - u_*)$, where

$$G_L(u-u_*) = \begin{cases} (u-u_*) - L, & \text{if } u - u_* > L, \\ 0, & \text{if } -L \le u - u_* \le L, \\ (u-u_*) + L, & \text{if } u - u_* < -L. \end{cases}$$

Note that

$$D_i\varphi = (D_iu - D_iu_*)\,1_{\{|u-u_*|>L\}},$$

where $1_E(x)$ is the characteristic function of the set E, that is, $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise. Since $u = u_*$ on $\partial\Omega$, then

$$u - u_* \in W_0^{1,p}(\Omega) \Rightarrow G_L(u-u_*) \in W_0^{1,p}(\Omega) \Rightarrow \varphi \in W_0^{1,p}(\Omega). \tag{3.3}$$

We insert such a test function φ into (2.4) and we get

$$0 = \int_{\Omega} \sum_{i=1}^{n} A_i\,(x, Du)\, D_i \varphi \mathrm{d}x = \int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} A_i\,(x, Du)\,(D_i u - D_i u_*)\,\mathrm{d}x,$$

thus

$$\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} A_i\,(x, Du)\, D_i u \mathrm{d}x = \int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} A_i\,(x, Du)\, D_i u_* \mathrm{d}x. \quad (3.4)$$

For the left hand-side of (3.4), by using (2.2), we have

$$\begin{aligned} &\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} A_i\,(x, Du)\, D_i u \mathrm{d}x \\ &\geq \int_{\{|u-u_*|>L\}} \Big(c_1 |Du|^p - c_2\Big)\mathrm{d}x \\ &= c_1 \int_{\{|u-u_*|>L\}} |Du|^p \mathrm{d}x - c_2\,|\{|u-u*|>L\}|\,. \end{aligned} \quad (3.5)$$

We now estimate the right-hand side of (3.4) by using (2.3) and Hölder inequality

$$\begin{aligned} &\int_{\{|u-u_*|>L\}} \sum_{i=1}^{n} A_i\,(x, Du)\, D_i u_* \mathrm{d}x \\ &\leq c_3 \int_{\{|u-u_*|>L\}} \Big(|Du|^{p-1} + 1\Big)|Du_*|\mathrm{d}x \\ &\leq c_3 \Big(\int_{\{|u-u_*|>L\}} \big(|Du|^{p-1} + 1\big)^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\{|u-u_*|>L\}} |Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}. \end{aligned} \quad (3.6)$$

Note that

$$\begin{aligned} &\Big(\int_{\{|u-u_*|>L\}} \big(|Du|^{p-1} + 1\big)^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{p-1}{p}} \\ &= \Big\| |Du|^{p-1} + 1 \Big\|_{L^{\frac{p}{p-1}}\{|u-u_*|>L\}} \\ &\leq \Big\| |Du|^{p-1} \Big\|_{L^{\frac{p}{p-1}}\{|u-u_*|>L\}} + \|1\|_{L^{\frac{p}{p-1}}\{|u-u_*|>L\}} \\ &= \Big(\int_{\{|u-u_*|>L\}} |Du|^p \mathrm{d}x\Big)^{\frac{p-1}{p}} + |\{|u-u_*|>L\}|^{\frac{p-1}{p}}. \end{aligned} \quad (3.7)$$

Substituting (3.5), (3.6) and (3.7) into (3.4), one obtains

$$c_1 \int_{\{|u-u_*|>L\}} |Du|^p \mathrm{d}x - c_2\,|\{|u-u_*|>L\}|$$

$$\leq c_3\Big(\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\left(\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\right)^{\frac{p-1}{p}}$$
$$+c_3\Big(\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}|\{|u-u_*|>L\}|^{\frac{p-1}{p}}.$$

Therefore

$$\begin{aligned}&\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\\&\leq\frac{c_2}{c_1}|\{|u-u_*|>L\}|+\frac{c_3}{c_1}\Big(\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\left(\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\\&+\frac{c_3}{c_1}\Big(\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}|\{|u-u_*|>L\}|^{\frac{p-1}{p}}.\end{aligned}\tag{3.8}$$

Using Young inequality with exponents p and $\frac{p}{p-1}$, we get

$$\begin{aligned}&\frac{c_3}{c_1}\Big(\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\left(\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\\&\leq\frac{1}{p}\left(\frac{c_3}{c_1}\right)^p\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x+\frac{p-1}{p}\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\frac{c_3}{c_1}\Big(\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}|\{|u-u_*|>L\}|^{\frac{p-1}{p}}\\&\leq\frac{1}{p}\left(\frac{c_3}{c_1}\right)^p\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x+\frac{p-1}{p}|\{|u-u_*|>L\}|.\end{aligned}$$

So that (3.8) becomes

$$\begin{aligned}&\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\\&\leq\left(\frac{c_2}{c_1}+\frac{p-1}{p}\right)|\{|u-u_*|>L\}|+\frac{2}{p}\left(\frac{c_3}{c_1}\right)^p\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x\\&+\frac{p-1}{p}\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x.\end{aligned}\tag{3.9}$$

Shifting the last term to the left-hand side, we can get

$$\begin{aligned}&\int_{\{|u-u_*|>L\}}|Du|^p\mathrm{d}x\\&\leq p\left(\frac{c_2}{c_1}+1\right)|\{|u-u_*|>L\}|+2\left(\frac{c_3}{c_1}\right)^p\int_{\{|u-u_*|>L\}}|Du_*|^p\,\mathrm{d}x.\end{aligned}\tag{3.10}$$

By the definition of $G_L(u-u_*)$ in (3.3) and Sobolev inequality, we have,

$$\begin{aligned}
&\left(\int_\Omega |G_L(u-u_*)|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\\
&\le S\left(\int_\Omega \left|D\big(G_L(u-u_*)\big)\right|^p \mathrm{d}x\right)^{\frac{1}{p}}\\
&= S\left(\int_{\{|u-u_*|>L\}} \left|Du-Du_*\right|^p \mathrm{d}x\right)^{\frac{1}{p}}\\
&\le S\left(\int_{\{|u-u_*|>L\}} |Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + S\left(\int_{\{|u-u_*|>L\}} |Du_*|^p\,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned} \tag{3.11}$$

where $S=\frac{(n-1)p}{n-p}$ is the Sobolev constant.

Combining (3.10) and (3.11), one has

$$\begin{aligned}
&\left(\int_\Omega |G_L(u-u_*)|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\\
&\le S\left\{p\left(\frac{c_2}{c_1}+1\right)|\{|u-u_*|>L\}| + 2\left(\frac{c_3}{c_1}\right)^p \int_{\{|u-u_*|>L\}} |Du_*|^p\,\mathrm{d}x\right\}^{\frac{1}{p}}\\
&\quad + S\left(\int_{\{|u-u_*|>L\}} |Du_*|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le S\left[p\left(\frac{c_2}{c_1}+1\right)\right]^{\frac{1}{p}} |\{|u-u_*|>L\}|^{\frac{1}{p}} + S\left(2^{\frac{1}{p}}\frac{c_3}{c_1}+1\right)\left(\int_{\{|u-u_*|>L\}} |Du_*|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

We use Hölder inequality to derive that, for $q>p$,

$$\begin{aligned}
&\left(\int_{\{|u-u_*|>L\}} |Du_*|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le \left[\left(\int_{\{|u-u_*|>L\}} |Du_*|^q\,\mathrm{d}x\right)^{\frac{p}{q}} |\{|u-u_*|>L\}|^{1-\frac{p}{q}}\right]^{\frac{1}{p}}\\
&\le \|Du_*\|_{L^q(\Omega)}\,|\{|u-u_*|>L\}|^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

therefore

$$\begin{aligned}
&\left(\int_\Omega |G_L(u-u_*)|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\\
&\le S\left[p\left(\frac{c_2}{c_1}+1\right)\right]^{\frac{1}{p}} |\{|u-u_*|>L\}|^{\frac{1}{p}}\\
&\quad + S\left(2^{\frac{1}{p}}\frac{c_3}{c_1}+1\right)\|Du_*\|_{L^q(\Omega)}\,|\{|u-u_*|>L\}|^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

Since

$$\begin{aligned}|\{|u-u_*|>L\}|^{\frac{1}{p}} &= |\{|u-u_*|>L\}|^{\frac{1}{p}-\frac{1}{q}} \cdot |\{|u-u_*|>L\}|^{\frac{1}{q}} \\ &\le |\Omega|^{\frac{1}{q}} \cdot |\{|u-u_*|>L\}|^{\frac{1}{p}-\frac{1}{q}},\end{aligned}$$

we get

$$\left(\int_\Omega |G_L(u-u_*)|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \le c_4\,|\{|u-u_*|>L\}|^{\frac{1}{p}-\frac{1}{q}}, \tag{3.12}$$

where

$$c_4 = S\left\{\left[p\left(\frac{c_2}{c_1}+1\right)\right]^{\frac{1}{p}}|\Omega|^{\frac{1}{q}} + \left(2^{\frac{1}{p}}\frac{c_3}{c_1}+1\right)\|Du_*\|_{L^q(\Omega)}\right\}.$$

For $H > L \ge 0$, we have

$$\begin{aligned}&\left(\int_\Omega |G_L(u-u_*)|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\ &= \left(\int_{\{|u-u_*|>L\}} |G_L(u-u_*)|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\ &= \left(\int_{\{|u-u_*|>L\}} \Big(|u-u_*|-L\Big)^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\ &\ge \left(\int_{\{|u-u_*|>H\}} \Big(|u-u_*|-L\Big)^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\ &\ge \left(\int_{\{|u-u_*|>H\}} (H-L)^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\ &= \Big((H-L)^{p^*}\,|\{|u-u_*|>H\}|\Big)^{\frac{1}{p^*}}.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), one has

$$\Big((H-L)^{p^*}\,|\{|u-u_*|>H\}|\Big)^{\frac{1}{p^*}} \le c_4\,|\{|u-u_*|>L\}|^{\frac{1}{p}-\frac{1}{q}},$$

from which we derive

$$|\{|u-u_*|>H\}| \le \frac{(c_4)^{p^*}}{(H-L)^{p^*}}\,|\{|u-u_*|>L\}|^{\left(\frac{1}{p}-\frac{1}{q}\right)p^*}.$$

The assumption (3.1) of Lemma 3.1 holds with

$$\varphi(h) = |\{|u-u_*|>h\}|,$$

$$c = (c_4)^{p^*},$$

$$\alpha = p^*,$$
$$\beta = \left(\frac{1}{p} - \frac{1}{q}\right) p^*,$$

and

$$k_0 = 0.$$

The assumption (2.5) implies $\beta > 1$. Lemma 3.1 tells us that for the constant (3.2), we have

$$|\{|u - u_*| > d\}| = 0,$$

that is

$$\begin{aligned} |u - u_*| \le d &= \left[(c_4)^{p^*} |\{|u - u_*| > 0\}|^{\beta - 1} 2^{\frac{\alpha\beta}{\beta-1}} \right]^{\frac{1}{\alpha}} \\ &\le c_4 |\Omega|^{(\beta-1)\frac{1}{p^*}} 2^{\frac{\beta}{\beta-1}} \\ &= c_4 |\Omega|^{\left(\left(\frac{1}{p} - \frac{1}{q}\right)p^* - 1\right)\frac{1}{p^*}} 2^{\frac{\left(\frac{1}{p} - \frac{1}{q}\right)p^*}{\left(\frac{1}{p} - \frac{1}{q}\right)p^* - 1}}. \end{aligned}$$

The estimate holds true almost everywhere in Ω, as desired. □

4 Assumptions and results about systems

In Sections 4 and 5 we are concerned with the difference between weak solution u and boundary datum u_* for some elliptic systems of the following kind

$$-\sum_{i=1}^{n} D_i \left(A_i^\alpha (x, Du(x))\right) = 0, \quad \alpha \in \{1, \cdots, m\}, \tag{4.1}$$

where $x \in \Omega$ and Ω is a bounded open subset in $\mathbb{R}^n$, $n \ge 2$. Moreover, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, has components $\left(u^1, \cdots, u^m\right)$, then $Du(x)$ is the $m \times n$ matrix $\left(\frac{\partial u^\alpha}{\partial x_i}(x)\right)_{i=1,\cdots,n}^{\alpha=1,\cdots,m}$. In addition, $u = u_*$ on $\partial\Omega$.

We assume that $A_i^\alpha : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $1 \le i \le n$, $1 \le \alpha \le m$ are Carathéodory functions satisfying for every $x \in \Omega$ and for every $z = \left(z^1, \cdots, z^m\right)^T \in \mathbb{R}^{m\times n}$ the following inequalities:

$$c_1 |z^\alpha|^p - c_2 \le \sum_{i=1}^{n} A_i^\alpha (x, z) z_i^\alpha, \tag{4.2}$$
$$\left|A^\alpha (x, z)\right| \le c_3 \left(|z|^{p-1} + 1\right), \tag{4.3}$$

where, as in the case of equations, $1 < p < n$, $c_1 > 0$, $c_2 \ge 0$ and $c_3 \ge c_1$. These two inequalities hold true for every $\alpha \in \{1, \cdots, m\}$.

Let us recall that $u \in u_* + W_0^{1,p}(\Omega;\mathbb{R}^m)$ is a weak solution of (4.1) if

$$\int_\Omega \sum_{\alpha=1}^{m}\sum_{i=1}^{n} A_i^\alpha(x, Du(x))\, D_i\varphi^\alpha(x)\mathrm{d}x = 0, \quad \forall \varphi \in W_0^{1,p}\left(\Omega;\mathbb{R}^m\right). \tag{4.4}$$

As far as the boundary datum u_* is concerned, we assume that $u_* \in W^{1,p}(\Omega;\mathbb{R}^m)$.

Our result is the following:

Theorem 4.1 *Under the previous assumptions (4.2) and (4.3), let $u_* \in W^{1,p}(\Omega;\mathbb{R}^m)$ be the boundary datum. Let $u = \left(u^1, \cdots, u^m\right) \in u_* + W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a weak solution of (4.1). Moreover, let $|Du_*| \in L^q(\Omega)$, $q > p$, where p, q satisfy*

$$\left(\frac{1}{p} - \frac{1}{q}\right)\frac{p^*}{p} > 1. \tag{4.5}$$

Then, every component u^γ of $u = \left(u^1, \cdots, u^m\right)$ enjoys

$$\left|u^\gamma(x) - u_*^\gamma(x)\right| \le c|\Omega|^{\left(\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}-1\right)\frac{1}{p^*}} 2^{\frac{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}}{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}-1}},$$

for almost every $x \in \Omega$, where

$$\begin{aligned} c = \frac{(n-1)p}{n-p}\Bigg\{ & \frac{c_2}{c_1}|\Omega|^{1-\left(\frac{1}{p}-\frac{1}{q}\right)} + \frac{c_3}{c_1}\Bigg[\left((2+p)\left(m^{1+\frac{p}{2}}\frac{c_3}{c_1}\right)^p \|Du_*\|_{L^p(\Omega)}^p \right. \\ & \left. + p\left(1 + m^{1+\frac{p}{2}}\frac{c_2}{c_1}\right)|\Omega|\right)^{\frac{p-1}{p}} + |\Omega|^{\frac{p-1}{p}}\Bigg]\|Du_*\|_{L^q(\Omega)}\Bigg\}^{\frac{1}{p}} \\ & + \frac{(n-1)p}{n-p}\|Du_*\|_{L^q(\Omega)}|\Omega|^{\left(\frac{1}{p}-\frac{1}{q}\right)\left(1-\frac{1}{p}\right)}. \end{aligned}$$

Remark 4.1 We remark that when p, q satisfy (4.5), it is equivalent to p, q satisfying the following relationship

(i) If $n = 2 \cup n = 3$, $p \in (1, n)$, $q > \frac{np}{p^2+n-np}$, and

(ii) If $n \ge 4$, $p \in \left(1, \frac{n-\sqrt{n^2-4n}}{2}\right) \cup \left(\frac{n+\sqrt{n^2-4n}}{2}, n\right)$, $q > \frac{np}{p^2+n-np}$.

Example 4.1 Componentwise coercivity holds true when we have the Euler Lagrange systems of a functional with Uhlenbeck structure

$$\int_\Omega g(x, |Du(x)|)\mathrm{d}x,$$

where $\frac{\frac{\partial g}{\partial t}(x,t)}{t} \geq \nu t^{p-2}$ with $p \geq 2$ and $\nu > 0$. Indeed,

$$A_i^\alpha(x,z) = \frac{\partial g}{\partial t}(x,|z|)\frac{z_i^\alpha}{|z|},$$

so that

$$\sum_i A_i^\alpha(x,z) z_i^\alpha = \sum_i \frac{\partial g}{\partial t}(x,|z|)\frac{|z_i^\alpha|^2}{|z|} \geq \nu |z|^{p-2}\left|z^\alpha\right|^2 \geq \nu\left|z^\alpha\right|^p,$$

where we used $p \geq 2$ when controlling $|z|^{p-2}$ by $|z^\alpha|^{p-2}$ in the last inequality. Then, componentwise coercivity holds true for this structure.

Remark 4.2 Componentwise coercivity might fail in the vectorial case $m \geq 2$ when $1 < p < 2$, see [13]. Indeed, standard p-Laplacian system

$$\begin{cases} -\sum_i^n D_i\left[p|Du|^{p-2}D_i u^1\right] = 0, \\ -\sum_i^n D_i\left[p|Du|^{p-2}D_i u^2\right] = 0, \\ \qquad \cdots \\ -\sum_i^n D_i\left[p|Du|^{p-2}D_i u^m\right] = 0, \end{cases}$$

does not enjoy componentwise coercivity.

Here, $1 < p < 2$, $m \geq 2$ and $A_i^\alpha(x,z) = p|z|^{p-2}z_i^\alpha$. We argue by contradiction: let us assume that, for some $c_1, \tilde{p} \in (0,+\infty)$ and $c_2 \in [0,+\infty)$ we have

$$c_1|z^\alpha|^{\tilde{p}} - c_2 \leq \sum_i^n A_i^\alpha(x,z)z_i^\alpha = p|z|^{p-2}|z^\alpha|^2. \tag{4.6}$$

Since $m \geq 2$, there exists $\beta \in \{1,\cdots,m\}\setminus\{\alpha\}$ and we take z as follows

$$z_1^\alpha = t > 0, \quad z_1^\beta = t^s, \ s > 1, \quad z_i^\gamma = 0 \text{ otherwise},$$

then, for large t,

$$|z^\alpha| = t, \quad |z| = \left(t^2 + t^{2s}\right)^{\frac{1}{2}} \simeq t^s, \quad |z|^{p-2}|z^\alpha|^2 \simeq t^{s(p-2)+2}.$$

Taking $s > 1$ large, we have $s(p-2)+2 < 0$, so, as $t \to +\infty$, $|z|^{p-2}|z^\alpha|^2 \to 0$, but $|z^\alpha|^{\tilde{p}} \to +\infty$, and (4.6) turns out to be

$$+\infty \leq 0,$$

this is false. Then, componentwise coercivity (4.6) fails.

Now we show a system with subquadratic growth $1 < p < 2$ that enjoys componentwise coercivity.

Example 4.2 Let us consider the variational integral

$$\mathcal{F}_1 = \int_\Omega f_1(x, z)\mathrm{d}x,$$

where

$$f_1(x, z) = \sum_{\beta=1}^{m} |z^\beta|^p + |z|^p$$

and $1 < p < 2, m \geq 2$. Then,

$$A_i^\alpha(x, z) = \frac{\partial f_1}{\partial z_i^\alpha}(x, z) = p|z^\alpha|^{p-2} z_i^\alpha + p|z|^{p-2} z_i^\alpha.$$

The Euler system of functional $\mathcal{F}_1$ is

$$\begin{cases} -\sum_i^n D_i \left[p|Du^1|^{p-2} D_i u^1 + p|Du|^{p-2} D_i u^1 \right] = 0, \\ -\sum_i^n D_i \left[p|Du^2|^{p-2} D_i u^2 + p|Du|^{p-2} D_i u^2 \right] = 0, \\ \qquad\qquad \cdots \\ -\sum_i^n D_i \left[p|Du^m|^{p-2} D_i u^m + p|Du|^{p-2} D_i u^m \right] = 0. \end{cases}$$

Let us check componentwise coercivity

$$\begin{aligned} \sum_i^n A_i^\alpha(x, z) z_i^\alpha &= p|z^\alpha|^{p-2} \sum_i^n z_i^\alpha z_i^\alpha + p|z|^{p-2} \sum_i^n z_i^\alpha z_i^\alpha \\ &= p|z^\alpha|^{p-2}|z^\alpha|^2 + p|z|^{p-2}|z^\alpha|^2 \\ &= p|z^\alpha|^p + p|z|^{p-2}|z^\alpha|^2 \\ &\geq p|z^\alpha|^p. \end{aligned}$$

Now, let us check growth condition. Since $p > 1$, we have

$$\begin{aligned} \left|A^\alpha(x, z)\right| &= \left| p|z^\alpha|^{p-2} z^\alpha + p|z|^{p-2} z^\alpha \right| \\ &\leq p|z^\alpha|^{p-2}|z^\alpha| + p|z|^{p-2}|z^\alpha| \\ &\leq p|z|^{p-1} + p|z|^{p-1} \\ &= 2p|z|^{p-1}. \end{aligned}$$

Then componentwise coercivity (4.2) and growth condition (4.3) are satisfied for $c_1 = p$, $c_2 = 0$ and $c_3 = 2p$.

Now we write a system, with quadratic or superquadratic growth $p \geq 2$, that enjoys componentwise coercivity.

Example 4.3 Let us consider the variational integral

$$\mathcal{F}_2(u) = \int_\Omega \left\{ |Du|^p + a(x) \left(D_1 u^1(x) - D_2 u^1(x) \right)^2 \right\} \mathrm{d}x,$$

where $p \geq 2$, $0 \leq a(x) \leq M$.

This functional has been inspired by an example in [3]; see also [8, 13].

Let us consider the density of $\mathcal{F}_2$:

$$f_2(x, z) = |z|^p + a(x) \left(z_1^1 - z_2^1 \right)^2,$$

then

$$A_i^\alpha(x, z) = \frac{\partial f_2}{\partial z_i^\alpha}(x, z) = p|z|^{p-2} z_i^\alpha + a(x) \delta^{\alpha 1} 2 \left(z_1^1 - z_2^1 \right) [\delta_{i1} - \delta_{i2}],$$

here and after, δ_{ij} is Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise; in a similar manner δ^{ij} is defined.

The system is

$$\begin{cases} -\sum_i^n D_i \left[p|Du|^{p-2} D_i u^1 + a(x) 2 \left(D_1 u^1 - D_2 u^1 \right) (\delta_{i1} - \delta_{i2}) \right] = 0, \\ -\sum_i^n D_i \left[p|Du|^{p-2} D_i u^2 \right] = 0, \\ \qquad \cdots \\ -\sum_i^n D_i \left[p|Du|^{p-2} D_i u^m \right] = 0. \end{cases}$$

Let us check componentwise coercivity:

$$\begin{aligned} \sum_i A_i^\alpha(x, z) z_i^\alpha &= \sum_i \left[p|z|^{p-2} |z_i^\alpha|^2 + a(x) \delta^{\alpha 1} 2 \left(z_1^1 - z_2^1 \right) [\delta_{i1} - \delta_{i2}] z_i^\alpha \right] \\ &= p|z|^{p-2} |z^\alpha|^2 + 2a(x) \delta^{\alpha 1} \left(z_1^1 - z_2^1 \right) z_1^1 - 2a(x) \delta^{\alpha 1} \left(z_1^1 - z_2^1 \right) z_2^1 \\ &= p|z|^{p-2} |z^\alpha|^2 + 2a(x) \delta^{\alpha 1} \left(z_1^1 - z_2^1 \right)^2 \\ &\geq p|z|^{p-2} |z^\alpha|^2 \\ &\geq p|z^\alpha|^{p-2} |z^\alpha|^2 \\ &= p|z^\alpha|^p, \end{aligned}$$

where we have lowered $|z|^{p-2}$ with $|z^\alpha|^{p-2}$ since $p \geq 2$.

We notice that

$$\left|A^\alpha(x,z)\right| = \left(\sum_i^n \left|A_i^\alpha(x,z)\right|^2\right)^{\frac{1}{2}} \leq \left[\left(\sum_i^n \left|A_i^\alpha(x,z)\right|\right)^2\right]^{\frac{1}{2}} = \sum_i^n \left|A_i^\alpha(x,z)\right|. \tag{4.7}$$

Let us check (4.3),

$$\begin{aligned}
|A^\alpha(x,z)| &\leq \sum_i^n \left| p|z|^{p-2} z_i^\alpha + a(x)\delta^{\alpha 1} 2\left(z_1^1 - z_2^1\right)[\delta_{i1} - \delta_{i2}] \right| \\
&\leq np|z|^{p-2}|z^\alpha| + 2a(x)\delta^{11} 2\left|z_1^1 - z_2^1\right| \\
&\leq n\Big(p|z|^{p-1} + 2a(x)\left(\left|z_1^1\right| + \left|z_2^1\right|\right)\Big) \\
&\leq n\left(p|z|^{p-1} + 4M|z|\right) \\
&\leq n(p+4M)\left(|z|^{p-1} + 1\right),
\end{aligned}$$

where the last inequality holds true since $p \geq 2$. Then conditions (4.2) and (4.3) are satisfied for $c_1 = p$, $c_2 = 0$ and $c_3 = n(p+4M)$.

Now we recall an example in [4]; see also [6].

Example 4.4 For $n = 3$, $m = 2$ the tensor A is defined by

$$A_i^\alpha(x,z) = \sigma^\alpha(x) z_i^\alpha + \delta_{i1} \frac{z_1^1 z_1^2}{\sqrt{1 + (z_1^1 z_1^2)^2}} z_1^{\tilde{\alpha}}, \quad \text{for } \alpha = 1,2 \text{ and } i \in 1,2,3,$$

where

$$\sigma^1(x) = 3 + \frac{x_1}{|x|}, \ \sigma^2(x) = 6 + \frac{x_1}{|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\}$$

and

$$\tilde{\alpha} = \begin{cases} 2, & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha = 2. \end{cases}$$

Let us write explicitly the system: we have

$$\begin{cases} -\sum_i^3 D_i \left[\left(3 + \frac{x_1}{|x|}\right) D_i u^1 + \delta_{i1} \frac{D_1 u^1 D_1 u^2}{\sqrt{1 + (D_1 u^1 D_1 u^2)^2}} D_1 u^2\right] = 0, \\ -\sum_i^3 D_i \left[\left(6 + \frac{x_1}{|x|}\right) D_i u^2 + \delta_{i1} \frac{D_1 u^1 D_1 u^2}{\sqrt{1 + (D_1 u^1 D_1 u^2)^2}} D_1 u^1\right] = 0. \end{cases}$$

Condition (4.2) and (4.3) are satisfied with $c_1 = 2$, $p = 2$, $c_2 = 0$ and $c_3 = 8$.

5 Proof of Theorem 4.1

We now proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1 We fix $\gamma \in \{1, \cdots, m\}$ and we take $L \geq 0$, where u^γ is the γ-th component of $u = \left(u^1, \cdots, u^m\right)$; we define $\varphi = \left(\varphi^1, \cdots, \varphi^m\right)$ as follows:

$$\varphi^\alpha = \begin{cases} 0, & \text{if } \alpha \neq \gamma, \\ G_L(u^\gamma - u_*^\gamma), & \text{if } \alpha = \gamma, \end{cases}$$

where

$$G_L(u^\gamma - u_*^\gamma) = \begin{cases} (u^\gamma - u_*^\gamma) - L, & \text{if } u^\gamma - u_*^\gamma > L, \\ 0, & \text{if } -L \leq u^\gamma - u_*^\gamma \leq L, \\ (u^\gamma - u_*^\gamma) + L, & \text{if } u^\gamma - u_*^\gamma < -L. \end{cases}$$

Note that

$$D_i\varphi^\alpha = \begin{cases} 0, & \text{if } \alpha \neq \gamma, \\ \left(D_i u^\gamma - D_i u_*^\gamma\right) 1_{\left\{|u^\gamma - u_*^\gamma| > L\right\}}, & \text{if } \alpha = \gamma. \end{cases}$$

Since $u = u_*$ on $\partial\Omega$, then

$$u^\gamma - u_*^\gamma \in W_0^{1,p}(\Omega) \Rightarrow G_L\left(u^\gamma - u_*^\gamma\right) \in W_0^{1,p}(\Omega) \Rightarrow \varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m). \quad (5.1)$$

We insert such a test function φ into (4.4) and we get

$$\begin{aligned} 0 &= \int_\Omega \sum_{\alpha=1}^m \sum_{i=1}^n A_i^\alpha\,(x, Du)\, D_i\varphi^\alpha \mathrm{d}x = \int_\Omega \sum_{i=1}^n A_i^\gamma\,(x, Du)\, D_i\varphi^\gamma \mathrm{d}x \\ &= \int_{\left\{|u^\gamma - u_*^\gamma| > L\right\}} \sum_{i=1}^n A_i^\gamma\,(x, Du)\left(D_i u^\gamma - D_i u_*^\gamma\right) \mathrm{d}x, \end{aligned}$$

thus

$$\int_{\left\{|u^\gamma - u_*^\gamma| > L\right\}} \sum_{i=1}^n A_i^\gamma\,(x, Du)\, D_i u^\gamma \mathrm{d}x = \int_{\left\{|u^\gamma - u_*^\gamma| > L\right\}} \sum_{i=1}^n A_i^\gamma\,(x, Du)\, D_i u_*^\gamma \mathrm{d}x. \quad (5.2)$$

For the left hand-side of (5.2), by using componentwise coercivity (4.2), we have

$$\int_{\left\{|u^\gamma - u_*^\gamma| > L\right\}} \sum_{i=1}^n A_i^\gamma\,(x, Du)\, D_i u^\gamma \mathrm{d}x$$

$$\geq \int_{\{|u^\gamma - u_*^\gamma|>L\}} \left(c_1 |Du^\gamma|^p - c_2\right) \mathrm{d}x$$
$$= c_1 \int_{\{|u^\gamma - u_*^\gamma|>L\}} |Du^\gamma|^p \mathrm{d}x - c_2 \left|\left\{\left|u^\gamma - u_*^\gamma\right| > L\right\}\right|. \tag{5.3}$$

We now estimate the right-hand side of (5.2) by using (4.3) and Hölder inequality

$$\begin{aligned}
&\int_{\{|u^\gamma - u_*^\gamma|>L\}} \sum_{i=1}^n A_i^\gamma (x, Du) D_i u_*^\gamma \mathrm{d}x \\
&\leq c_3 \int_{\{|u^\gamma - u_*^\gamma|>L\}} \left(|Du|^{p-1} + 1\right) |Du_*^\gamma| \mathrm{d}x \\
&\leq c_3 \left(\int_{\{|u^\gamma - u_*^\gamma|>L\}} \left(|Du|^{p-1} + 1\right)^{\frac{p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\{|u^\gamma - u_*^\gamma|>L\}} |Du_*|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{5.4}$$

Remark 5.1 We compare (5.4) with (3.6): in (5.4) we have Du the gradient of every component not only Du^γ. On the other hand, in (3.6) we are dealing with the gradient of only one component, like in (3.5).

Note that

$$\begin{aligned}
&\left(\int_{\{|u^\gamma - u_*^\gamma|>L\}} \left(|Du|^{p-1} + 1\right)^{\frac{p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&= \left\| |Du|^{p-1} + 1 \right\|_{L^{\frac{p}{p-1}}\{|u^\gamma - u_*^\gamma|>L\}} \\
&\leq \left\| |Du|^{p-1} \right\|_{L^{\frac{p}{p-1}}\{|u^\gamma - u_*^\gamma|>L\}} + \|1\|_{L^{\frac{p}{p-1}}\{|u^\gamma - u_*^\gamma|>L\}} \\
&= \left(\int_{\{|u^\gamma - u_*^\gamma|>L\}} |Du|^p \mathrm{d}x\right)^{\frac{p-1}{p}} + \left|\left\{\left|u^\gamma - u_*^\gamma\right| > L\right\}\right|^{\frac{p-1}{p}} \\
&\leq \|Du\|_{L^p(\Omega)}^{p-1} + |\Omega|^{\frac{p-1}{p}}.
\end{aligned} \tag{5.5}$$

Substituting (5.3), (5.4) and (5.5) into (5.2), one obtains

$$\begin{aligned}
&c_1 \int_{\{|u^\gamma - u_*^\gamma|>L\}} |Du^\gamma|^p \mathrm{d}x - c_2 \left|\left\{\left|u^\gamma - u_*^\gamma\right| > L\right\}\right| \\
&\leq c_3 \left(\|Du\|_{L^p(\Omega)}^{p-1} + |\Omega|^{\frac{p-1}{p}}\right) \left(\int_{\{|u^\gamma - u_*^\gamma|>L\}} |Du_*|^p \, \mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned}$$

then, $\forall L \geq 0$, we get

$$\begin{aligned}
&\int_{\left\{\left|u^\gamma - u_*^\gamma\right|>L\right\}} |Du^\gamma|^p \mathrm{d}x \\
&\leq \frac{c_2}{c_1} \left|\left\{\left|u^\gamma - u_*^\gamma\right| > L\right\}\right| + \frac{c_3}{c_1} \left(\|Du\|_{L^p(\Omega)}^{p-1} + |\Omega|^{\frac{p-1}{p}}\right) \left(\int_{\left\{\left|u^\gamma - u_*^\gamma\right|>L\right\}} |Du_*|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{5.6}$$

Remark 5.2 Let us compare (5.6) with (3.9): on the right-hand side of (3.9), we have Du, which can be shifted to the left-hand side. However, this is not possible in (5.6), because its right-hand side contains Du while the left-hand side only involves Du^γ.

According to the above estimate (5.6) with $L = 0$, we have

$$\begin{aligned}&\int_{\{|u^\gamma - u_*^\gamma|>0\}} |Du^\gamma|^p \mathrm{d}x \\ &\le \frac{c_2}{c_1}\left|\left\{\left|u^\gamma - u_*^\gamma\right| > 0\right\}\right| + \frac{c_3}{c_1}\left(\|Du\|_{L^p(\Omega)}^{p-1} + |\Omega|^{\frac{p-1}{p}}\right)\left(\int_{\{|u^\gamma - u_*^\gamma|>0\}} |Du_*|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned} \tag{5.7}$$

On the set $\left\{\left|u^\gamma - u_*^\gamma\right| = 0\right\}$, we have $D\left(u^\gamma - u_*^\gamma\right) = 0$ almost everywhere, that is

$$Du^\gamma = Du_*^\gamma \text{ a.e. in } \left\{\left|u^\gamma - u_*^\gamma\right| = 0\right\}. \tag{5.8}$$

Then, according to (5.7) and (5.8), we arrive at

$$\begin{aligned}&\int_\Omega |Du^\gamma|^p \mathrm{d}x \\ &= \int_{\{|u^\gamma - u_*^\gamma|=0\}} |Du^\gamma|^p \mathrm{d}x + \int_{\{|u^\gamma - u_*^\gamma|>0\}} |Du^\gamma|^p \mathrm{d}x \\ &\le \int_{\{|u^\gamma - u_*^\gamma|=0\}} |Du_*^\gamma|^p \mathrm{d}x + \frac{c_2}{c_1}\left|\left\{\left|u^\gamma - u_*^\gamma\right| > 0\right\}\right| \\ &\quad + \frac{c_3}{c_1}\left(\|Du\|_{L^p(\Omega)}^{p-1} + |\Omega|^{\frac{p-1}{p}}\right)\left(\int_{\{|u^\gamma - u_*^\gamma|>0\}} |Du_*|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\le \|Du_*\|_{L^p(\Omega)}^p + \frac{c_2}{c_1}|\Omega| + \frac{c_3}{c_1}\|Du\|_{L^p(\Omega)}^{p-1}\|Du_*\|_{L^p(\Omega)} \\ &\quad + \frac{c_3}{c_1}|\Omega|^{\frac{p-1}{p}}\|Du_*\|_{L^p(\Omega)}.\end{aligned} \tag{5.9}$$

Now we estimate $|Du|^p$,

$$\begin{aligned}|Du|^p &= \left(\sum_{\alpha=1}^m |Du^\alpha|^2\right)^{\frac{p}{2}} \le \left(m \max_\alpha |Du^\alpha|^2\right)^{\frac{p}{2}} = m^{\frac{p}{2}}\left(\max_\alpha |Du^\alpha|^2\right)^{\frac{p}{2}} \\ &= m^{\frac{p}{2}}\left(\left(\max_\alpha |Du^\alpha|\right)^2\right)^{\frac{p}{2}} = m^{\frac{p}{2}}\left(\max_\alpha |Du^\alpha|\right)^p \le m^{\frac{p}{2}}\sum_{\alpha=1}^m |Du^\alpha|^p.\end{aligned}$$

By integrating the above estimate and (5.9), we have

$$
\begin{aligned}
&\int_{\Omega} |Du|^p \mathrm{d}x \\
&\leq m^{\frac{p}{2}} \sum_{\gamma=1}^{m} \int_{\Omega} |Du^{\gamma}|^p \mathrm{d}x \\
&\leq m^{\frac{p}{2}} \cdot m \left[\|Du_*\|^p_{L^p(\Omega)} + \frac{c_2}{c_1} |\Omega| + \frac{c_3}{c_1} \|Du\|^{p-1}_{L^p(\Omega)} \|Du_*\|_{L^p(\Omega)} \right. \\
&\quad \left. + \frac{c_3}{c_1} |\Omega|^{\frac{p-1}{p}} \|Du_*\|_{L^p(\Omega)} \right].
\end{aligned} \tag{5.10}
$$

We use Young inequality to get

$$
\begin{aligned}
&m^{1+\frac{p}{2}} \frac{c_3}{c_1} \|Du\|^{p-1}_{L^p(\Omega)} \|Du_*\|_{L^p(\Omega)} \\
&\leq \frac{1}{p} \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \|Du_*\|_{L^p(\Omega)} \right)^p + \frac{p-1}{p} \|Du\|^p_{L^p(\Omega)}.
\end{aligned} \tag{5.11}
$$

We use Young inequality again, and we obtain

$$
\begin{aligned}
&m^{1+\frac{p}{2}} \frac{c_3}{c_1} |\Omega|^{\frac{p-1}{p}} \|Du_*\|_{L^p(\Omega)} \\
&\leq \frac{1}{p} \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \|Du_*\|_{L^p(\Omega)} \right)^p + \frac{p-1}{p} |\Omega|.
\end{aligned} \tag{5.12}
$$

Substituting the estimates (5.11) and (5.12) into (5.10), we arrive at

$$
\begin{aligned}
\|Du\|^p_{L^p(\Omega)} \leq\ & m^{1+\frac{p}{2}} \|Du_*\|^p_{L^p(\Omega)} + m^{1+\frac{p}{2}} \frac{c_2}{c_1} |\Omega| + \frac{p-1}{p} \|Du\|^p_{L^p(\Omega)} \\
&+ \frac{2}{p} \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p \|Du_*\|^p_{L^p(\Omega)} + \frac{p-1}{p} |\Omega|.
\end{aligned}
$$

Therefore

$$
\|Du\|^p_{L^p(\Omega)} \leq \left[2 \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p + p m^{1+\frac{p}{2}} \right] \|Du_*\|^p_{L^p(\Omega)} + \left[p - 1 + p m^{1+\frac{p}{2}} \frac{c_2}{c_1} \right] |\Omega|.
$$

Let us remark that $m \geq 1$, $p > 1$, $c_3 \geq c_1$ allow us to estimate

$$
m^{1+\frac{p}{2}} \leq \left(m^{1+\frac{p}{2}} \right)^p \leq \left(m^{1+\frac{p}{2}} \right)^p \left(\frac{c_3}{c_1} \right)^p = \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p,
$$

that is

$$
\|Du\|^p_{L^p(\Omega)} \leq (2+p) \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p \|Du_*\|^p_{L^p(\Omega)} + p \left(1 + m^{1+\frac{p}{2}} \frac{c_2}{c_1} \right) |\Omega|. \tag{5.13}
$$

Substituting (5.13) into (5.6), $\forall L \geq 0$, we get

$$\begin{aligned}
&\int_{\{|u^\gamma - u_*^\gamma| > L\}} |Du^\gamma|^p \mathrm{d}x \\
&\leq \frac{c_2}{c_1} \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right| + \frac{c_3}{c_1} \left\{ \left[(2+p) \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p \|Du_*\|^p_{L^p(\Omega)} \right.\right. \\
&\quad \left.\left. + p \left(1 + m^{1+\frac{p}{2}} \frac{c_2}{c_1} \right) |\Omega| \right]^{\frac{p-1}{p}} + |\Omega|^{\frac{p-1}{p}} \right\} \cdot \left(\int_{\{|u^\gamma - u_*^\gamma| > L\}} |Du_*|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned} \tag{5.14}$$

We use Hölder inequality to derive that, for $q > p$, we have

$$\begin{aligned}
&\left(\int_{\{|u^\gamma - u_*^\gamma| > L\}} |Du_*|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\leq \left[\left(\int_{\{|u^\gamma - u_*^\gamma| > L\}} |Du_*|^q \, \mathrm{d}x \right)^{\frac{p}{q}} \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{1-\frac{p}{q}} \right]^{\frac{1}{p}} \\
&\leq \|Du_*\|_{L^q(\Omega)} \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned} \tag{5.15}$$

Combining (5.14) and (5.15), we obtain

$$\int_{\{|u^\gamma - u_*^\gamma| > L\}} |Du^\gamma|^p \mathrm{d}x \leq c_5 \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right| + c_6 \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}},$$

where

$$c_5 = \frac{c_2}{c_1},$$

$$\begin{aligned}
c_6 = \frac{c_3}{c_1} &\left\{ \left[(2+p) \left(m^{1+\frac{p}{2}} \frac{c_3}{c_1} \right)^p \|Du_*\|^p_{L^p(\Omega)} \right.\right. \\
&\left.\left. + p \left(1 + m^{1+\frac{p}{2}} \frac{c_2}{c_1} \right) |\Omega| \right]^{\frac{p-1}{p}} + |\Omega|^{\frac{p-1}{p}} \right\} \|Du_*\|_{L^q(\Omega)}.
\end{aligned}$$

Since

$$\begin{aligned}
\left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right| &= \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{1-\left(\frac{1}{p}-\frac{1}{q}\right)} \cdot \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}} \\
&\leq |\Omega|^{1-\left(\frac{1}{p}-\frac{1}{q}\right)} \cdot \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

We then have

$$\int_{\{|u^\gamma - u_*^\gamma| > L\}} |Du^\gamma|^p \mathrm{d}x \le c_7 \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}}, \tag{5.16}$$

where $c_7 = c_5 |\Omega|^{1-\left(\frac{1}{p}-\frac{1}{q}\right)} + c_6$.

By the definition of $G_L\left(u^\gamma - u_*^\gamma\right)$ in (5.1) and Sobolev inequality, we obtain

$$\begin{aligned}
&\left(\int_\Omega \left|G_L\left(u^\gamma - u_*^\gamma\right)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\le S\left(\int_\Omega \left|D\left(G_L\left(u^\gamma - u_*^\gamma\right)\right)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&= S\left(\int_{\{|u^\gamma - u_*^\gamma| > L\}} \left|Du^\gamma - Du_*^\gamma\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\le S\left(\int_{\{|u^\gamma - u_*^\gamma| > L\}} \left|Du^\gamma\right|^p \mathrm{d}x\right)^{\frac{1}{p}} + S\left(\int_{\{|u^\gamma - u_*^\gamma| > L\}} \left|Du_*^\gamma\right|^p \mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{5.17}$$

Substituting (5.16) and (5.15) into (5.17), we get

$$\begin{aligned}
&\left(\int_\Omega \left|G_L\left(u^\gamma - u_*^\gamma\right)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\le S\left(c_7\right)^{\frac{1}{p}} \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{p}} + S\left\|Du_*\right\|_{L^q(\Omega)} \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\frac{1}{p}-\frac{1}{q}} \\
&= \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\left(\frac{1}{p}-\frac{1}{q}\right)\left(1-\frac{1}{p}\right)} \cdot \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{p}} \\
&\le |\Omega|^{\left(\frac{1}{p}-\frac{1}{q}\right)\left(1-\frac{1}{p}\right)} \cdot \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{p}},
\end{aligned}$$

therefore

$$\left(\int_\Omega \left|G_L\left(u^\gamma - u_*^\gamma\right)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \le c_8 \left|\left\{|u^\gamma - u_*^\gamma| > L\right\}\right|^{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{p}}, \tag{5.18}$$

where $c_8 = S\left(c_7\right)^{\frac{1}{p}} + S\left\|Du_*\right\|_{L^q(\Omega)} |\Omega|^{\left(\frac{1}{p}-\frac{1}{q}\right)\left(1-\frac{1}{p}\right)}$.

For $H > L \ge 0$, we obtain

$$\left(\int_\Omega \left|G_L\left(u^\gamma - u_*^\gamma\right)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}}$$

$$
\begin{aligned}
&= \left(\int_{\left\{ \left| u^\gamma - u_*^\gamma \right| > L \right\}} \left| G_L \left(u^\gamma - u_*^\gamma \right) \right|^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \\
&= \left(\int_{\left\{ \left| u^\gamma - u_*^\gamma \right| > L \right\}} \left(\left| u^\gamma - u_*^\gamma \right| - L \right)^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \\
&\geq \left(\int_{\left\{ \left| u^\gamma - u_*^\gamma \right| > H \right\}} \left(\left| u^\gamma - u_*^\gamma \right| - L \right)^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \\
&\geq \left(\int_{\left\{ \left| u^\gamma - u_*^\gamma \right| > H \right\}} (H - L)^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \\
&= \left((H - L)^{p^*} \left| \left\{ \left| u^\gamma - u_*^\gamma \right| > H \right\} \right| \right)^{\frac{1}{p^*}}. \qquad (5.19)
\end{aligned}
$$

Combining (5.18) and (5.19), one has

$$
\left((H - L)^{p^*} \left| \left\{ \left| u^\gamma - u_*^\gamma \right| > H \right\} \right| \right)^{\frac{1}{p^*}} \leq c_8 \left| \left\{ \left| u^\gamma - u_*^\gamma \right| > L \right\} \right|^{\left(\frac{1}{p} - \frac{1}{q} \right) \frac{1}{p}},
$$

from which we derive

$$
\left| \left\{ \left| u^\gamma - u_*^\gamma \right| > H \right\} \right| \leq \frac{(c_8)^{p^*}}{(H - L)^{p^*}} \left| \left\{ \left| u^\gamma - u_*^\gamma \right| > L \right\} \right|^{\left(\frac{1}{p} - \frac{1}{q} \right) \frac{p^*}{p}}.
$$

The assumption (3.1) of Lemma 3.1 holds with

$$
\varphi(h) = \left| \left\{ \left| u^\gamma - u_*^\gamma \right| > h \right\} \right|,
$$

$$
c = (c_8)^{p^*},
$$

$$
\alpha = p^*,
$$

$$
\beta = \left(\frac{1}{p} - \frac{1}{q} \right) \frac{p^*}{p},
$$

and

$$
k_0 = 0.
$$

The assumption (4.5) implies $\beta > 1$. Lemma 3.1 tells us that for the constant (3.2), we obtain

$$
\left| \left\{ \left| u^\gamma - u_*^\gamma \right| > d \right\} \right| = 0,
$$

that is

$$\begin{aligned}\left|u^\gamma - u_*^\gamma\right| \le d &= \left[(c_8)^{p^*}\left|\left\{\left|u^\gamma - u_*^\gamma\right| > 0\right\}\right|^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}\right]^{\frac{1}{\alpha}} \\ &\le c_8\,|\Omega|^{(\beta-1)\frac{1}{p^*}} 2^{\frac{\beta}{\beta-1}} \\ &= c_8|\Omega|^{\left(\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}-1\right)\frac{1}{p^*}} 2^{\frac{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}}{\left(\frac{1}{p}-\frac{1}{q}\right)\frac{p^*}{p}-1}}.\end{aligned}$$

The estimate holds true almost everywhere in Ω, for every $\gamma = 1, \cdots, m$. □

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Declarations

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

1. Bjorn, J.: Boundedness and differentiability for nonlinear elliiptic systems. Trans. Amer. Math. Soc. **353**, 4545–4565 (2001)
2. Carozza, M., Giannetti, F., Leonetti, F., Passarelli di Napoli, A.: Pointwise bounds for minimizers of some anisotropic functionals. Nonlinear Anal. **117**, 254–269 (2018)
3. Cianchi, A.: Local boundedness of minimizers of anisotropic functionals Ann. Inst. H. Poincare Anal. Non Lineaire **17**(2), 147–168 (2000)
4. Cirmi, G.R., D'Asero, S., Leonardi, S., Leonetti, F., Rocha, E., Staicu, V.: Existence and boundedness of weak solutions to some vectorial Dirichlet problems. Nonlinear Anal. **254**, 113751 (2025)
5. Colucci, R., Leonetti, F., Treu, G.: An estimate concerning the difference between minimizer and boundary value in some polyconvex problems. Nonlinear Anal. **215**, 112635 (2022)
6. Cupini, G., Leonetti, F., Mascolo, E.: Local boundedness for solutions of a class of nonlinear elliptic systems. Calc. Var. Partial Differential Equations **61**(3), 94 (2022)
7. De Giorgi, E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Boll. Un. Mat. Ital. **4**, 135–137 (1968)
8. Esposito, A., Leonetti, F., Petricca, P.V.: Absence of Lavrentiev gap for non-autonomous functionals with (p, q)-growth Adv. Nonlinear Anal. **8**, 73–78 (2019)

9. Gao, H., Di, Q., Ma, D.: Integrability for solutions to some anisotropic obstacle problems. Manuscripta Math. 11371050 (2014)
10. Gao, H., Liu, C., Tian, H.: Remarks on a paper by Leonetti and Siepe. J. Math. Anal. Appl. **401**, 881–887 (2013)
11. Gao, H., Zhang, Y., Li, S.: Integrability for solutions of anisotropic obstacle problems. Int. J. Math. Math. Sci. 549285 (2012)
12. Innamorati, A., Leonetti, F.: Global integrability for weak solutions to some anisotropic elliptic equations. Nonlinear Anal. **113**, 430–434 (2015)
13. Leonetti, F.: Elliptic systems with bounded solutions and double phase functionals. Bruno Pini Math. Anal. Semin. **15**, 138–163 (2024)
14. Leonetti, F., Petricca, P.V.: Regularity for solutions to some nonlinear elliptic systems. Complex Var. Elliptic. Equ. **56**, 1099–1113 (2011)
15. Leonetti, F., Siepe, F.: Integrability for solutions to some anisotropic elliptic equations. Nonlinear Anal. **75**, 2867–2873 (2012)
16. Leonetti, F., Siepe, F.: Global integrability for minimizers of anisotropic functionals. Manuscripta Math. **144**, 91–98 (2014)
17. Palagachev, D., Softova, L.: Boundedness of solutions to a class of coercive systems with Morrey data. Nonlinear Anal. **191**, 111630 (2020)
18. Shan, T., Gao, H., Liu, Q.: Summability properties under nonstandard growth conditions. Georgian Math. J. https://doi.org/10.1515/gmj-2025-2066
19. Softova, L.: Boundedness of the solutions to non-linear systems with Morrey data. Complex Var. Elliptic Equ. **63**(11), 1581–1594 (2018)
20. Stampacchia, G.: Équations elliptiques du second ordre á cofficients discontinus [Second order elliptic equations with discontinuous coefficients] Séminaire Jean Leray. **3**, 1-77 (1963)

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