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A Birkhoff–Kellogg Type Theorem for Discontinuous Operators with Applications

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Abstract. By means of fixed point index theory for multivalued maps, we provide an analogue of the classical Birkhoff–Kellogg Theorem in the context of discontinuous operators acting on affine wedges in Banach spaces. Our theory is fairly general and can be applied, for example, to eigenvalues and parameter problems for ordinary differential equations with discontinuities. We illustrate in detail this fact for a class of second-order boundary value problem with deviated arguments and discontinuous terms. In a specific example, we explicitly compute the terms that occur in our theory.

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1. Introduction

The celebrated invariant-direction theorem due to Birkhoff and Kellogg [2] is an abstract existence result that, roughly speaking, gives conditions for the existence of a "nonlinear" eigenvalue and eigenvector for compact maps in normed linear spaces. Among its various extensions, one is set in cones and is due to Krasnosel'skiĭ and Ladyženskiĭ [20]. These classical functional analytic tools find applications, e.g., to eigenvalue problems for ODEs and PDEs (see for example the book [1], the recent papers [18,19], and references therein); typically, the methodology in this context is to reformulate the given boundary value problem as a fixed point problem in a suitable Banach space.

Recently, the first two authors developed a Birkhoff–Kellogg type theorem in the framework of *affine cones* (cf. [4], see also [3,6,9]). The motivation for this new type of results is that the setting of affine cones seems to be helpful when dealing with equations with delay effects. A key ingredient in [4] is the *continuity* of the involved operator. On the other hand, there has been

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recently a rising attention towards *discontinuous* differential equations, that occur when modelling real-world phenomena. Here, we mention the classical books by Filippov [13], Carl and Heikkilä [7], and Heikkilä and Lakshmikantham [17] and the more recent book by Figueroa et al. [12].

In the present paper, we provide a discontinuous version of the Birkhoff–Kellogg type result in the setting of affine wedges in Banach spaces; see Theorem 2.13. The proof of Theorem 2.13 is based on the fixed point index theory for discontinuous operators developed in [12]. We stress that a crucial point in the construction of the index for discontinuous operators is its equivalence with the corresponding one of a suitable *multivalued* map, for which it is already defined; see [14]. Note that this newly constructed topological tool for discontinuous operators inherits the key properties of the classical one. This construction is sketched in Sect. 2 for completeness.

In Sect. 3, we illustrate the applicability of our results to boundary value problems; see Theorem 3.4. In more details, we consider the following second-order parameter-dependent differential equation with deviated argument:

$$u''(t) + \lambda\, f(t, u(t), u(\sigma(t))) = 0, \quad t \in [0,1], \tag{1.1}$$

with initial condition

$$u(t) = \omega(t), \quad t \in [-r, 0], \tag{1.2}$$

and the final homogeneous boundary condition

$$u(1) = 0, \tag{1.3}$$

where $\lambda \geq 0$ is a parameter, $r \geq 0$, $\sigma : [0,1] \to [-r,1]$ and $\omega : [-r,0] \to [0,\infty)$ are suitable continuous functions, while the nonlinearity $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ may be discontinuous with respect to the second argument in an appropriate sense. We employ a concept of *admissible discontinuity curve* as in [12]. We conclude the paper by illustrating the applicability of our theory by means of a toy model with delay; see Example 3.9.

As far as we know, our results extend and complement the previous literature. This is highlighted in more details in Remarks 2.12, 2.15 and 3.10.

2. Birkhoff–Kellogg Type Results via Fixed Point Index Theory

2.1. On Fixed Point Index Theory for Discontinuous Operators

Let K be a nonempty closed and convex subset of a real Banach space $(X, \|\cdot\|)$, $U \subset K$ a relatively open subset and $T : \overline{U} \subset K \longrightarrow K$ a mapping, not necessarily continuous.

Definition 2.1. The closed-convex envelope of an operator $T : \overline{U} \subset K \longrightarrow K$ is the multivalued mapping $\mathbb{T} : \overline{U} \longrightarrow 2^X$ given by

$$\mathbb{T}x = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\, T\left(\overline{B}_\varepsilon(x) \cap \overline{U}\right) \quad \text{for every } x \in \overline{U}, \tag{2.4}$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε, and $\overline{\mathrm{co}}$ means closed-convex hull.

Example 2.2. 1. Consider the real function $T : \mathbb{R} \to \mathbb{R}$ defined as $T(x) = x$, if $x \leq 0$, and $T(x) = x + 1$, if $x > 0$. Its closed-convex envelope is the multivalued map $\mathbb{T}$ given by $\mathbb{T}(x) = \{x\}$, if $x < 0$; $\mathbb{T}(x) = \{x + 1\}$, if $x > 0$; and $\mathbb{T}(0) = [0, 1]$.
2. The closed-convex envelope of any continuous map T is equal to T.

Now, we recall some useful properties of closed-convex envelopes (cc-envelopes for short) and the definition of the fixed point index that we will employ throughout this paper. The reader is referred to [11,12] for details.

Proposition 2.3. *Let $\mathbb{T}$ be the cc-envelope of an operator $T : \overline{U} \longrightarrow K$. Then, the following properties hold*:

1. *If $\tilde{\mathbb{T}} : \overline{U} \longrightarrow 2^X$ is an upper semicontinuous (usc) operator which assumes closed and convex values and $Tx \in \tilde{\mathbb{T}}x$ for all $x \in \overline{U}$, then $\mathbb{T}x \subset \tilde{\mathbb{T}}x$ for all $x \in \overline{U}$;*
2. *If T maps bounded sets into relatively compact sets, then $\mathbb{T}$ assumes compact values, and it is usc;*
3. *If $T\overline{U}$ is relatively compact, then $\mathbb{T}\overline{U}$ is relatively compact.*

The fixed point index for a not necessarily continuous operator T was introduced in [10] using the degree theory developed in [11] and a retraction trick, just as in the classical case. Both topological degree and fixed point index theories are based on the available results for the multivalued cc-envelope $\mathbb{T}$.

Definition 2.4. Let $T : \overline{U} \subset K \longrightarrow K$ be an operator, such that $T\overline{U}$ is relatively compact, T has no fixed points on ∂U and

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for every } x \in \overline{U} \cap \mathbb{T}\overline{U}, \tag{2.5}$$

where $\mathbb{T}$ is the cc-envelope of T.

We define the fixed point index of T in K over U as

$$i_K(T, U) = \deg(I - T \circ r, r^{-1}(U), 0),$$

where r is a continuous retraction of X onto K and deg is the degree introduced in [11].

Remark 2.5. Note that condition (2.5) means that the set of fixed points of $\mathbb{T}$ (i.e., the set of points x, such that $x \in \mathbb{T}x$) is contained in the set of fixed points of T. This is a weaker condition than the continuity of T; indeed, if T is continuous, then $\mathbb{T}x = \{Tx\}$ for all $x \in \overline{U}$, and thus, (2.5) is trivially satisfied.

We now recall a useful proposition from [10] that relates the fixed point index of the discontinuous operator T with that of its associated multivalued mapping $\mathbb{T}$.

Proposition 2.6 [10, Proposition 2.12] *Let T be a mapping that satisfies the conditions of Definition 2.4. Then, the fixed point index of T is such that*

$$i_K(T, U) = i_K(\mathbb{T}, U),$$

where the right-hand index is the fixed point index defined for multivalued mappings; see [14].

As a straightforward consequence of the fixed point index theory for usc multivalued mappings, the following properties can be derived (see [12]).

Theorem 2.7. *Let T be a mapping that satisfies the conditions of Definition 2.4. Then, the following properties hold*:

(i) (*Homotopy invariance*) *Let $H : \overline{U} \times [0,1] \longrightarrow K$ be a mapping, such that*
   (a) *for each $(x,t) \in \overline{U} \times [0,1]$ and all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x, t) > 0$, such that*

$$s \in [0,1],\ |t-s| < \delta \implies \|H(z,t) - H(z,s)\| < \varepsilon \quad \forall\, z \in \overline{B}_\delta(x) \cap \overline{U};$$

   (b) $H\left(\overline{U} \times [0,1]\right)$ *is relatively compact*;
   (c) $\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\}$ *for all $t \in [0,1]$ and all $x \in \overline{U} \cap \mathbb{H}_t\overline{U}$, where $H_t(\cdot) := H(\cdot, t)$ and $\mathbb{H}_t$ denotes the cc-envelope of H_t.*

   If $x \neq H(x,t)$ for all $(x,t) \in \partial U \times [0,1]$, then the index $i_K(H_t, U)$ does not depend on $t \in [0,1]$.

(ii) (*Additivity*) *Let U be the disjoint union of two open sets U_1 and U_2. If $0 \notin (I-T)\left(\overline{U} \backslash (U_1 \cup U_2)\right)$, then*

$$i_K(T,U) = i_K(T,U_1) + i_K(T,U_2).$$

(iii) (*Excision*) *Let $A \subset U$ be a closed set. If $0 \notin (I-T)(\partial U) \cup (I-T)(A)$, then*

$$i_K(T,U) = i_K(T, U \backslash A).$$

(iv) (*Existence*) *If $i_K(T,U) \neq 0$, then there exists $x \in U$, such that $Tx = x$.*

(v) (*Normalization*) *For every constant map T, such that $T\overline{U} \subset U$, $i_K(T,U) = 1$.*

2.2. Birkhoff–Kellogg Theorem and Discontinuous Operators

The following notions will be used along the text. A closed-convex subset K of a Banach space $(X, \|\cdot\|)$ is a *wedge* if $\mu x \in K$ for every $x \in K$ and for all $\mu \geq 0$. Furthermore, if a wedge K satisfies that $K \cap (-K) = \{0\}$, then it is said to be a *cone*. A cone K induces the partial order in X given by $u \preceq v$ if and only if $v - u \in K$. The cone K is called *normal* if there exists $c > 0$, such that $\|u\| \leq c\|v\|$ for all $u, v \in X$ with $0 \preceq u \preceq v$.

Let K be a wedge of a Banach space $(X, \|\cdot\|)$. For a given $y \in X$, the *translate* of the wedge K is defined as follows:

$$K_y := y + K = \{y + x : x \in K\}.$$

Given an open-bounded subset $D \subset X$ with $0 \in D$, we will denote $D_{K_y} := (y + D) \cap K_y$, which is a relatively open subset of K_y. By $\overline{D}_{K_y}$ and ∂D_{K_y}, we will mean, respectively, the closure and the boundary of D_{K_y} relative to K_y.

For the convenience of the reader, we recall here the classical Birkhoff–Kellogg Theorem [2] and a variant of it set in *cones*. The latter result is due to Krasnosel'skiĭ and Ladyženskiĭ [20] (see also [16, Theorem 2.3.6]).

Theorem 2.8 (Birkhoff–Kellogg) *Let U be a bounded open neighborhood of 0 in an infinite-dimensional normed linear space X, and $T : \partial U \longrightarrow X$ a compact map satisfying $\|Tx\| \geq \alpha > 0$ for all $x \in \partial U$. Then, there exist $x_0 \in \partial U$ and $\lambda_0 > 0$, such that $x_0 = \lambda_0 Tx_0$.*

Theorem 2.9 (Krasnosel'skiĭ–Ladyženskiĭ) *Let X be a real Banach space, $U \subset X$ be an open-bounded set with $0 \in U$, $K \subset X$ be a cone, $T : K \cap \overline{U} \longrightarrow K$ be compact, and suppose that*

$$\inf_{x \in K \cap \partial U} \|Tx\| > 0.$$

Then, there exist $x_0 \in K \cap \partial U$ and $\lambda_0 > 0$, such that $x_0 = \lambda_0 Tx_0$.

In the context of *affine cones*, a Birkhoff–Kellogg type result was recently proved in [4, Theorem 2]. It reads as follows.

Theorem 2.10. *Let $(X, \|\cdot\|)$ be a real Banach space, $K \subset X$ be a cone and $D \subset X$ be an open-bounded set with $y \in D_{K_y}$. Assume that $T : \overline{D}_{K_y} \longrightarrow K$ is a compact map and consider the operator*

$$T_{(y,\lambda)} := y + \lambda T, \quad (\lambda \in \mathbb{R}).$$

Assume that there exists $\bar{\lambda} > 0$, such that $i_{K_y}(T_{(y,\bar{\lambda})}, D_{K_y}) = 0$. Then, there exist $x^* \in \partial D_{K_y}$ and $\lambda^* \in (0, \bar{\lambda})$, such that $x^* = y + \lambda^* T(x^*)$.

Now, we present a discontinuous version of this Birkhoff–Kellogg type result in *affine wedges*.

Theorem 2.11. *Let $D \subset X$ be an open-bounded set with $0 \in D$, $y \in X$ be fixed and K be a wedge. Assume that $T : \overline{D}_{K_y} \longrightarrow K$ is a mapping, such that $T\,\overline{D}_{K_y}$ is relatively compact and consider the operator*

$$T_{(y,\lambda)} := y + \lambda T, \quad (\lambda \in \mathbb{R}).$$

Moreover, assume that there exists $\bar{\lambda} > 0$, such that $i_{K_y}(T_{(y,\bar{\lambda})}, D_{K_y}) \neq 1$ and for each $\lambda \in (0, \bar{\lambda}]$

$$\{x\} \cap \mathbb{T}_{(y,\lambda)}(x) \subset \left\{T_{(y,\lambda)}(x)\right\} \quad \text{for every } x \in \overline{D}_{K_y}, \tag{2.6}$$

where $\mathbb{T}_{(y,\lambda)}$ denotes the cc-envelope of $T_{(y,\lambda)}$.

Then, there exist $x^* \in \partial D_{K_y}$ and $\lambda^* \in (0, \bar{\lambda})$, such that $x^* = y + \lambda^* T(x^*)$.

Proof. If $T_{(y,\lambda)}$ has a fixed point on ∂D_{K_y} for some $\lambda \in (0, \bar{\lambda})$, we are done. Otherwise, suppose that $T_{(y,\lambda)}$ is fixed point free on ∂D_{K_y}. Now, observe that for each $\lambda \in (0, \bar{\lambda}]$, the operator $T_{(y,\lambda)} : \overline{D}_{K_y} \longrightarrow K_y$ satisfies condition (2.6) and that $T_{(y,\lambda)}\left(\overline{D}_{K_y}\right)$ is relatively compact, which implies that the fixed point index $i_{K_y}(T_{(y,\lambda)}, D_{K_y})$ is well defined according to Definition 2.4.

Consider the map $H : \overline{D}_{K_y} \times [0,1] \to K_y$ defined as

$$H(x,t) = y + t\,\bar{\lambda}\,T(x).$$

Note that H satisfies conditions (a)–(c) in Theorem 2.7, *i*. Hence, if $x \neq H(x,t)$ for all $(x,t) \in \partial D_{K_y} \times [0,1]$, then

$$i_{K_y}(T_{(y,\bar{\lambda})}, D_{K_y}) = i_{K_y}(H(\cdot,1), D_{K_y}) = i_{K_y}(H(\cdot,0), D_{K_y}) = i_{K_y}(y, D_{K_y}).$$

By the normalization property, since $y \in D_{K_y}$, we have

$$i_{K_y}(T_{(y,\bar{\lambda})}, D_{K_y}) = i_{K_y}(y, D_{K_y}) = 1,$$

a contradiction. In conclusion, there exist $t^* \in (0,1)$ and $x^* \in \partial D_{K_y}$, such that $x^* = y + t^* \bar{\lambda}\, T(x^*)$. □

Remark 2.12. Note that Theorem 2.11 has a twofold interest: not only is a generalization of Theorem 2.10 to the context of discontinuous operators, but also an improvement in the continuous case, since the conditions on the index are weakened and the result is extended to the setting of wedges.

We now prove a result in the setting of normal cones which can be of a more direct applicability due to the use of the norm, as in the classical Birkhoff–Kellogg Theorem.

Theorem 2.13. *Let $K \subset X$ be a normal cone with normal constant $c > 0$ in a Banach space X, $D \subset X$ be an open-bounded set with $0 \in D$, and $y \in X$ be fixed. Assume that $T : \overline{D}_{K_y} \longrightarrow K$ is a mapping, such that $T\,\overline{D}_{K_y}$ is relatively compact and*

$$\inf\{\|v\| : v \in \mathbb{T}x,\ x \in \partial D_{K_y}\} > 0.$$

If there exists a positive number

$$\bar{\lambda} > \frac{c \sup_{x \in \partial D} \|x\|}{\inf\{\|v\| : v \in \mathbb{T}x,\ x \in \partial D_{K_y}\}},$$

such that the operator $T_{(y,\lambda)}$ satisfies condition (2.6) for each $\lambda \in (0, \bar{\lambda}]$, then there exist $x^* \in \partial D_{K_y}$ and $\lambda^* \in (0, \bar{\lambda})$, such that $x^* = y + \lambda^* T(x^*)$.

Proof. We shall show that $i_{K_y}(T_{(y,\bar{\lambda})}, D_{K_y}) = 0$, and so, the conclusion is obtained as a consequence of Theorem 2.11.

Take $x_0 \in K \backslash \{0\}$ and let us see that

$$x \notin y + \bar{\lambda}\, \mathbb{T}x + \beta\, x_0 \quad \text{for all } \beta \geq 0 \text{ and } x \in \partial D_{K_y}.$$

Indeed, suppose that there exist $x_1 \in \partial D_{K_y}$, $v \in \mathbb{T}x_1$ and $\beta_0 \geq 0$, such that

$$x_1 = y + \bar{\lambda}\, v + \beta_0\, x_0.$$

Then, $\bar{\lambda}\, v \preceq \bar{\lambda}\, v + \beta_0\, x_0 = x_1 - y$ and since K is normal

$$\bar{\lambda}\, \|v\| \leq c\, \|x_1 - y\|.$$

Observe that $x_1 - y \in \partial D$ and so

$$\bar{\lambda}\, \|v\| \leq c \left(\sup_{x \in \partial D} \|x\| \right),$$

a contradiction with the choice of $\bar{\lambda}$.

On the other hand, since $\overline{D}_{K_y}$ and $\mathbb{T}\left(\overline{D}_{K_y}\right)$ are bounded, there exists $\bar{\beta} > 0$, such that

$$x \notin y + \bar{\lambda}\, \mathbb{T}x + \bar{\beta}\, x_0 \quad \text{for all } x \in \overline{D}_{K_y}.$$

Consider the multivalued homotopy $\mathcal{H} : \overline{D}_{K_y} \times [0,1] \to 2^{K_y}$ defined as

$$\mathcal{H}(x,t) = y + \bar{\lambda}\, \mathbb{T}x + t\, \bar{\beta}\, x_0.$$

By the homotopy invariance property of the index for usc multivalued maps [14]

$$i_{K_y}(\mathbb{T}_{(y,\bar{\lambda})}, D_{K_y}) = i_{K_y}(\mathcal{H}(\cdot,0), D_{K_y}) = i_{K_y}(\mathcal{H}(\cdot,1), D_{K_y}) = 0.$$

Therefore, it follows from Proposition 2.6 that $i_{K_y}(T_{(y,\bar{\lambda})}, D_{K_y}) = i_{K_y}(\mathbb{T}_{(y,\bar{\lambda})}, D_{K_y}) = 0$. □

The following corollary can be seen as an analogue of the classical result of Krasnosel'skiĭ and Ladyženskiĭ.

Corollary 2.14. *Let $K \subset X$ be a normal cone in a Banach space X and $D \subset X$ be an open-bounded set with $y \in D_{K_y}$. Assume that $T : \overline{D}_{K_y} \longrightarrow K$ is a mapping, such that $T\,\overline{D}_{K_y}$ is relatively compact and, for each $\lambda > 0$, the operator $T_{(y,\lambda)}$ satisfies condition (2.6). If*

$$\inf\{\|v\| : v \in \mathbb{T}x,\ x \in \partial\, D_{K_y}\} > 0,$$

then there exist $x^* \in \partial\, D_{K_y}$ and $\lambda^* > 0$, such that $x^* = y + \lambda^*\, T(x^*)$.

Remark 2.15. Note that, in the non-affine case, Corollary 2.14 extends Theorem 2.9 within the setting of discontinuous operators in normal cones. We stress that, in the non-affine case, Corollary 2.14 can also be deduced as a consequence of the multivalued generalization of the Birkhoff–Kellogg theorem given in [15].

3. Applications

Consider the second-order parameter-dependent differential equation

$$u''(t) + \lambda\, f(t, u(t), u(\sigma(t))) = 0, \quad t \in [0,1], \tag{3.7}$$

with initial conditions of the form

$$u(t) = \omega(t), \quad t \in [-r, 0], \tag{3.8}$$

and the final homogeneous boundary condition

$$u(1) = 0, \tag{3.9}$$

where λ is a positive parameter, $r \geq 0$, and $\sigma : [0,1] \to [-r,1]$ and $\omega : [-r,0] \to [0,\infty)$ are continuous functions. The nonlinearity $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ may be discontinuous with respect to the second argument in a sense which will be specified later.

To study the problem (3.7)–(3.9), we shall use a superposition principle as in [5]. To do so, first consider the Dirichlet BVP

$$\begin{cases} u''(t) + y(t) = 0, & t \in [0,1], \\ u(0) = u(1) = 0, \end{cases}$$

whose unique solution is given by

$$u(t) = \int_0^1 G(t,s) y(s)\, \mathrm{d}s,$$

where G is the corresponding Green's function. It is well known that

$$G(t,s) = \begin{cases} t(1-s), & \text{if } 0 \le t \le s \le 1, \\ (1-t)s, & \text{if } 0 \le s < t \le 1, \end{cases}$$

and moreover (see [21])

$$G(t,s) \le \Phi(s), \quad t,s \in [0,1],$$
$$\frac{1}{4}\Phi(s) \le G(t,s), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \ s \in [0,1],$$

with $\Phi(s) := s(1-s)$. Associated to the Green's function, we consider the kernel $k : [-r,1] \times [0,1] \to \mathbb{R}$ defined as

$$k(t,s) = \begin{cases} G(t,s), & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{3.10}$$

On the other hand, note that the function $\hat{y}(t) = 1-t$ solves the Dirichlet BVP

$$\begin{cases} u''(t) = 0, & t \in [0,1], \\ u(0) = 1, \ u(1) = 0, \end{cases}$$

so we define the function

$$y(t) = \begin{cases} \omega(t), & \text{if } t \le 0, \\ \hat{y}(t)\,\omega(0), & \text{if } t > 0, \end{cases} \tag{3.11}$$

which will be the *vertex* of our affine cone.

To apply the theory of the previous section, we will work in the Banach space of continuous functions $X = \mathcal{C}([-r,1])$, endowed with the usual sup-norm, $\|\cdot\|_{[-r,1]}$, and the cone

$$K = \left\{ u \in \mathcal{C}([-r,1],[0,\infty)) : u(t) = 0 \text{ for all } t \in [-r,0], \min_{t \in [1/4,3/4]} u(t) \ge \frac{1}{4}\|u\|_{[0,1]} \right\}.$$

Observe that K is a normal cone with normal constant $c = 1$ and that $\|u\|_{[-r,1]} = \|u\|_{[0,1]}$ for all $u \in K$. Now, for the vertex y defined in (3.11), we consider the translate of the cone K given by

$$K_y := y + K = \{y + u : u \in K\},$$

and for each $\rho > 0$, we denote by $K_{y,\rho}$, the relatively open-bounded set

$$K_{y,\rho} := \left\{ y + u : u \in K, \ \|u\|_{[0,1]} < \rho \right\}.$$

We will look for solutions of the following perturbed Hammerstein integral equation:

$$u(t) = y(t) + \lambda \int_0^1 k(t,s) f(s,u(s),u(\sigma(s)))\,\mathrm{d}s =: y(t) + \lambda T u(t), \quad t \in [-r,1], \tag{3.12}$$

located in the affine cone K_y.

Definition 3.1. By a solution of the problem (3.7)–(3.9), we mean a solution $u \in \mathcal{C}([-r,1],\mathbb{R})$ of the integral equation (3.12).

Before doing so, we need to define the type of regions where f is allowed to be discontinuous. The concept of *admissible discontinuity curve* used here has been widely employed in [12].

Definition 3.2. A λ-admissible discontinuity curve for the second-order parameter-dependent differential equation $u'' + \lambda f(t, u, u(\sigma)) = 0$ is a $W^{2,1}$ function $\gamma : [a, b] \subset [0, 1] \to [0, \infty)$ satisfying that there exist $\varepsilon > 0$ and $\psi \in L^1(a, b)$, $\psi(t) > 0$ for a.a. $t \in [a, b]$, such that either

$$\begin{aligned} -\gamma''(t) + \psi(t) < \lambda f(t, y, z) \quad & \text{for a.a. } t \in [a, b], \\ \text{all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon] \text{ and all } z \in [0, \infty) \quad & \end{aligned} \tag{3.13}$$

or

$$\begin{aligned} -\gamma''(t) - \psi(t) > \lambda f(t, y, z) \quad & \text{for a.a. } t \in [a, b], \\ \text{all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon] \text{ and all } z \in [0, \infty). \quad & \end{aligned} \tag{3.14}$$

Remark 3.3. Since f is non-negative, to have that γ is a λ-admissible discontinuity curve for the differential equation (3.7) and any $\lambda > 0$, it suffices that

$$0 < \gamma''(t) \quad \text{for a.a. } t \in [a, b].$$

Indeed, one may check that condition (3.13) holds with $\psi(t) = \gamma''(t)/2$, $t \in [a, b]$.

Let us now state and prove the main result of this section.

Theorem 3.4. *Let $\rho > 0$ and assume that the following conditions hold*:

(H_1) *any composition $t \in [0, 1] \mapsto f(t, u(t), v(t))$ is measurable provided that $u, v \in \mathcal{C}([0, 1], [0, \infty))$;*

(H_2) *there exists $M_\rho \in L^1([0, 1])$, such that*

$$f(t, u, v) \le M_\rho(t) \quad \text{for a.a. } t \in [0, 1] \quad \text{and all } (u, v) \text{ with } 0 \le u, v \le \rho + \|\omega\|_{[-r,0]};$$

(H_3) *there exists $\delta_\rho \in L^1([1/4, 3/4])$, such that*

$$f(t, u, v) \ge \delta_\rho(t) \quad \text{for a.a. } t \in [1/4, 3/4] \quad \text{and all } (u, v) \text{ with } 0 \le u, v \le \rho + \|\omega\|_{[-r,0]}$$

and

$$\bar{\delta} := \sup_{t \in [1/4, 3/4]} \int_{1/4}^{3/4} k(t, s) \delta_\rho(s) \, \mathrm{d}s > 0;$$

(H_4) *there exists a countable number of curves $\gamma_n : I_n = [a_n, b_n] \to [0, \infty)$, $n \in \mathbb{N}$, such that for a.a. $t \in [0, 1]$, the function $f(t, \cdot, \cdot)$ is continuous on $\big([0, \infty) \setminus \bigcup_{n : t \in I_n} \{\gamma_n(t)\}\big) \times [0, \infty)$ and, moreover, each γ_n is a λ-admissible discontinuity curve for each $\lambda \in (0, \bar{\lambda}]$ and some $\bar{\lambda} > \rho/\bar{\delta}$.*

Then, there exist $\lambda_\rho \in (0, \bar{\lambda})$ and $u_\rho \in \partial K_{y,\rho}$ that satisfy the integral equation (3.12).

Proof. Let us divide the proof in several steps.

Step 1. The operator T, defined in (3.12), maps the set $\overline{K}_{y,\rho}$ into the cone K and, moreover, $T\,\overline{K}_{y,\rho}$ is relatively compact.

First, let $u \in \overline{K}_{y,\rho}$ be arbitrarily fixed and let us show that $Tu \in K$. By definition

$$Tu(t) = \int_0^1 k(t,s) f(s, u(s), u(\sigma(s)))\,\mathrm{d}s, \quad t \in [-r, 1].$$

The continuity of the kernel k, jointly with hypothesis (H_1), (H_2) and the constant sign of f and k, imply that $Tu \in \mathcal{C}([-r,1],[0,\infty))$. Moreover, since $k(t,s) = 0$ for all $t \le 0$, we have that $Tu(t) = 0$ for all $t \in [-r, 0]$. Now, for $t \in [1/4, 3/4]$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s) f(s, u(s), u(\sigma(s)))\,\mathrm{d}s \\ &\ge \frac{1}{4}\int_0^1 \Phi(s) f(s, u(s), u(\sigma(s)))\,\mathrm{d}s \ge \frac{1}{4}\,\|Tu\|_{[0,1]}, \end{aligned}$$

as a consequence of the properties of the Green's function G stated above. In conclusion, $Tu \in K$.

On the other hand, the compactness of the set $\overline{T\,\overline{K}_{y,\rho}}$ follows from assumption (H_2) and the continuity of the kernel k, combined with a careful use of the Arzelà–Ascoli theorem (see [22]).

Step 2. For each $\lambda \in (0, \bar{\lambda}]$, the operator $y + \lambda T$ satisfies that

$$\{u\} \cap \{y + \lambda\,\mathbb{T}(u)\} \subset \{y + \lambda\,T(u)\} \quad \text{for every } u \in \overline{K}_{y,\rho}, \tag{3.15}$$

where $\bar{\lambda}$ is fixed by hypothesis (H_4).

Fix arbitrary $\lambda \in (0, \bar{\lambda}]$ and $u \in \overline{K}_{y,\rho}$. Now, consider two different cases:

Case 1. $m\left(\{t \in I_n : u(t) = \gamma_n(t)\}\right) = 0$ for all $n \in \mathbb{N}$ (where m denotes Lebesgue measure).

Let us prove that T is continuous at u, which implies that $\mathbb{T}(u) = \{T(u)\}$, and thus, condition (3.15) holds for such u. Indeed, in this case, we have that for a.a. $t \in [0,1]$, the function $f(t,\cdot,\cdot)$ is continuous at $(u(t), u(\sigma(t)))$. Hence, if $u_k \to u$ uniformly in $[-r,1]$, then

$$f(t, u_k(t), u_k(\sigma(t))) \to f(t, u(t), u(\sigma(t))) \quad \text{for a.a. } t \in [0,1],$$

which implies, due to Lebesgue's dominated convergence theorem, that $Tu_k \to Tu$ in $\mathcal{C}([-r,1])$.

Case 2. $m\left(\{t \in I_n : u(t) = \gamma_n(t)\}\right) > 0$ for some $n \in \mathbb{N}$.

In this case, one can show that $u \notin y + \lambda\,\mathbb{T}(u)$, which implies that condition (3.15) holds for such u. The proof is based on condition (H_4) and the fact that the function γ_n is a λ-admissible discontinuity curve for the problem. It can be replicated following the reasoning in the proof of Proposition 4.7, Case 2, in [10].

Step 3. *It holds that*

$$\inf\left\{\|v\|_{[-r,1]} : v \in \mathbb{T}u,\ u \in \partial\, K_{y,\rho}\right\} \ge \bar{\delta} > 0.$$

For $u \in \partial K_{y,\rho}$ and $\varepsilon > 0$, take $u_i \in \overline{B}_\varepsilon(u) \cap \overline{K}_{y,\rho}$ and $\lambda_i \geq 0$ with $\sum \lambda_i = 1$, $i = 1, 2, \ldots, m$. Then, by assumption (H_3), we have for $t \in [1/4, 3/4]$

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i \, Tu_i(t) &= \sum_{i=1}^{m} \lambda_i \int_0^1 k(t,s) f(s, u_i(s), u_i(\sigma(s))) \, \mathrm{d}s \\ &\geq \sum_{i=1}^{m} \lambda_i \int_{1/4}^{3/4} k(t,s) f(s, u_i(s), u_i(\sigma(s))) \, \mathrm{d}s \\ &\geq \sum_{i=1}^{m} \lambda_i \int_{1/4}^{3/4} k(t,s) \delta_\rho(s) \, \mathrm{d}s \\ &= \int_{1/4}^{3/4} k(t,s) \delta_\rho(s) \, \mathrm{d}s.\end{aligned}$$

Hence, for any $v \in \operatorname{co} T\left(\overline{B}_\varepsilon(u) \cap \overline{K}_{y,\rho}\right)$, we have

$$\|v\|_{[-r,1]} \geq \|v\|_{[1/4,3/4]} \geq \sup_{t \in [1/4,3/4]} \int_{1/4}^{3/4} k(t,s) \delta_\rho(s) \, \mathrm{d}s = \bar{\delta}.$$

Since $\mathbb{T}u \subset \overline{\operatorname{co}} T\left(\overline{B}_\varepsilon(u) \cap \overline{K}_{y,\rho}\right)$, it follows that $\|v\|_{[-r,1]} \geq \bar{\delta} > 0$ for any $v \in \mathbb{T}u$, as wished.

Therefore, the conclusion follows from Theorem 2.13. □

Remark 3.5. We emphasize that hypotheses (H_1), (H_2), and (H_4) do not imply that f be a Carathéodory map, since, due to (H_4), the function f can be discontinuous with respect to the last variables.

Furthermore, note that if, for each $(x, y) \in [0, \infty) \times [0, \infty)$, the map $t \in [0,1] \mapsto f(t, x, y)$ is measurable and, for a.a. $t \in [0,1]$, the map $(x, y) \mapsto f(t, x, y)$ is continuous, then condition (H_1) holds. However, the measurability of the map $t \in [0,1] \mapsto f(t, x, y)$ together with (H_4) does not imply necessarily that condition (H_1) holds. More information about the measurability of compositions in this setting can be found in [8, Section 3.1].

Corollary 3.6. *Let $\rho > 0$ and assume that conditions (H_1)–(H_3) hold and, moreover,*

(H_4^*) *there exist a countable number of curves $\gamma_n : I_n = [a_n, b_n] \to [0, \infty)$, $n \in \mathbb{N}$, such that $\gamma_n'' > 0$ and for a.a. $t \in [0,1]$, the function $f(t, \cdot, \cdot)$ is continuous on $\left([0, \infty) \setminus \bigcup_{n : t \in I_n} \{\gamma_n(t)\}\right) \times [0, \infty)$.*

Then, there exist $\lambda_\rho > 0$ and $u_\rho \in \partial K_{y,\rho}$ that satisfy the integral equation (3.12).

Proof. It follows from Theorem 3.4 together with Remark 3.3. □

Consider the special case of (3.7) where the nonlinearity can be seen as a discontinuous perturbation of a Carathéodory function, that is

$$u''(t) + \lambda \left(g(t, u(\sigma(t))) + h(u(t))\right) = 0, \quad t \in [0,1], \tag{3.16}$$

where $g : [0,1] \times [0, \infty) \to [0, \infty)$ is a Carathéodory function and $h : [0, \infty) \to [0, \infty)$ is locally bounded and continuous except at most at a countable number of points.

Corollary 3.7. *Assume that the following conditions hold*:

(C_1) *g satisfies the Carathéodory conditions, namely,*
- (a) *$g(\cdot, v)$ is measurable for each fixed $v \in [0, \infty)$;*
- (b) *$g(t, \cdot)$ is continuous for a.a. $t \in [0, 1]$;*
- (c) *for each $R > 0$, there exists $M_R \in L^1([0, 1])$, such that*

$$g(t, v) \leq M_R(t) \quad \text{for a.a. } t \in [0, 1] \text{ and all } v \in [0, R];$$

(C_2) *h is locally bounded, $t \mapsto h(u(t))$ is measurable for each non-negative continuous function u, and there exists a countable set A, such that h is continuous in $[0, \infty) \backslash A$;*

(C_3) *there exists $\delta \in L^1([0, 1])$, $\delta(t) > 0$ for a.a. $t \in [0, 1]$, such that*

$$g(t, v) \geq \delta(t) \quad \text{for a.a. } t \in [0, 1] \text{ and all } v \geq 0.$$

Then, for each $\rho > 0$, there exist $\lambda_\rho > 0$ and $u_\rho \in \partial K_{y,\rho}$ that satisfy the BVP (3.16)–(3.8)–(3.9).

Proof. Observe that, for each $\rho > 0$, Theorem 3.4 can be applied to the function

$$f(t, u, v) = g(t, v) + h(u).$$

Note that hypotheses (H_1)–(H_3) are satisfied.

Now, consider the countable set A where h may be discontinuous and denote $A = \{a_k : k \in \mathbb{N}\}$. Define the constant functions $\gamma_n : [0, 1] \to [0, \infty)$ given by $\gamma_n(t) = a_n$, $t \in [0, 1]$, $n \in \mathbb{N}$. For each $\lambda > 0$ fixed, choose the L^1-function $\psi(t) = \lambda\, \delta(t)$, $t \in [0, 1]$. Then, each function γ_n satisfies condition (3.13), and so, it is a λ-admissible discontinuity curve. □

Now, let us restrict our efforts to the particular case of problem (3.7)–(3.9) in which the deviated argument is given by a continuously differentiable function σ with constant derivative equal to 1 or -1. Notice that it covers the meaningful situations of equations with delay (where $\sigma(t) = t - r$) or with reflection of the argument (where, for instance, $\sigma(t) = 1 - r - t$).

In this case, we are able to prove another version of Theorem 3.4 where the nonlinearity f may be *more* discontinuous. More precisely, we weaken assumption (H_4) allowing f to be discontinuous w.r.t. the second and third variables over the graphs of two countable families of functions.

Theorem 3.8. *Let $\rho > 0$ and assume conditions (H_1)–(H_3) in Theorem 3.4 hold. Moreover, suppose that $\sigma : [0, 1] \to [-r, 1]$ is a continuously differentiable function with constant derivative $\sigma' = \pm 1$ and the following assumption holds*:

(D) *there exist two countable families of curves $\gamma_n : I_n = [a_n, b_n] \to [0, \infty)$, $n \in \mathbb{N}$, and $\Gamma_j : \mathcal{I}_j = [c_j, d_j] \to [0, \infty)$, $j \in \mathbb{N}$, such that for a.a. $t \in [0, 1]$, the function*

$$f(t, \cdot, \cdot) \text{ is continuous on } \left([0, \infty) \backslash \bigcup_{n : t \in I_n} \{\gamma_n(t)\}\right) \times \left([0, \infty) \backslash \bigcup_{j : t \in \mathcal{I}_j} \{\Gamma_j(t)\}\right).$$

For each $\lambda \in (0, \bar{\lambda}]$ with $\bar{\lambda} > \rho/\bar{\delta}$, each function γ_n is a λ-admissible discontinuity curve and each function Γ_j satisfies that

(a) $\Gamma_j(t) \neq \omega(\sigma(t))$ *for a.a.* $t \in \mathcal{I}_j \cap \sigma^{-1}([-r, 0])$;

(b) *the restriction of* Γ_j *to* $\mathcal{I}_j \cap \sigma^{-1}([0, 1])$ *satisfies either of the following conditions*: *there exists* $\varepsilon_j > 0$ *and* $\psi_j \in L^1(\mathcal{I}_j)$, $\psi_j(t) > 0$ *for a.a.* $t \in \mathcal{I}_j \cap \sigma^{-1}([0, 1])$, *such that*

(a) $-\Gamma_j''(t) + \psi_j(t) < \lambda f(\sigma(t), y, z)$ *for a.a.* $t \in \mathcal{I}_j \cap \sigma^{-1}([0, 1])$, *all* $y \in [\Gamma_j(t) - \varepsilon_j, \Gamma_j(t) + \varepsilon_j]$ *and all* $z \in [0, \infty)$; *or*

(b) $-\Gamma_j''(t) - \psi_j(t) > \lambda f(\sigma(t), y, z)$ *for a.a.* $t \in \mathcal{I}_j \cap \sigma^{-1}([0, 1])$, *all* $y \in [\Gamma_j(t) - \varepsilon_j, \Gamma_j(t) + \varepsilon_j]$ *and all* $z \in [0, \infty)$.

Then, there exist $\lambda_\rho \in (0, \bar{\lambda})$ *and* $u_\rho \in \partial K_{y,\rho}$ *that satisfy the integral equation* (3.12), *that is, they solve the problem* (3.7)–(3.9).

Proof. It follows in line of the proof of Theorem 3.4 as a consequence of Theorem 2.13. Observe that it suffices to rewrite Step 2. Let us prove that for each $\lambda \in (0, \bar{\lambda}]$, the operator $y + \lambda T$ satisfies that

$$\{u\} \cap \{y + \lambda \mathbb{T}(u)\} \subset \{y + \lambda T(u)\} \quad \text{for every } u \in \overline{K}_{y,\rho}, \tag{3.17}$$

where $\bar{\lambda}$ is fixed by assumption (D).

Fix arbitrary $\lambda \in (0, \bar{\lambda}]$ and $u \in \overline{K}_{y,\rho}$. Now, consider three different cases:

Case 1. $m(\{t \in I_n : u(t) = \gamma_n(t)\} \cup \{t \in \mathcal{I}_j : u(\sigma(t)) = \Gamma_j(t)\}) = 0$ for all $j, n \in \mathbb{N}$. Then, for a.a. $t \in [0, 1]$, the function $f(t, \cdot, \cdot)$ is continuous at $(u(t), u(\sigma(t)))$, and so, T is continuous at u.

Case 2. $m(\{t \in \mathcal{I}_j : u(\sigma(t)) = \Gamma_j(t)\}) > 0$ for some $j \in \mathbb{N}$. Let us prove that $u \notin y + \lambda \mathbb{T}u$, which can be justified as in the proof of [10, Proposition 4.7], but we include the reasoning here again for completeness.

Since $u \in K_y$, we have $u(s) = y(s) = \omega(s)$ for all $s \in [-r, 0]$, and thus, $u(\sigma(t)) = \omega(\sigma(t))$ for all $t \in \sigma^{-1}([-r, 0])$. Now, condition (D), (a), implies that

$$m\left(\left\{t \in \mathcal{I}_j \cap \sigma^{-1}([-r, 0]) : u(\sigma(t)) = \Gamma_j(t)\right\}\right) = 0 \quad \text{for all } j \in \mathbb{N}.$$

Hence, we can fix some $j \in \mathbb{N}$, such that

$$m\left(\left\{t \in \mathcal{I}_j \cap \sigma^{-1}([0, 1]) : u(\sigma(t)) = \Gamma_j(t)\right\}\right) > 0.$$

By condition (D), (b), we can assume that there exist $\varepsilon_j > 0$ and $\psi_j \in L^1(\mathcal{I}_j)$, $\psi_j(t) > 0$ on $\mathcal{I}_j$, such that

$$\begin{aligned} -\Gamma_j''(t) + \psi_j(t) < \lambda f(\sigma(t), y, z) \quad &\text{for a.a. } t \in \mathcal{I}_j \cap \sigma^{-1}([0, 1]), \\ &\text{all } y \in [\Gamma_j(t) - \varepsilon_j, \Gamma_j(t) + \varepsilon_j] \text{ and all } z \in [0, \infty). \end{aligned} \tag{3.18}$$

In what follows, let us denote $J := \left\{t \in \mathcal{I}_j \cap \sigma^{-1}([0, 1]) : u(\sigma(t)) = \Gamma_j(t)\right\}$ and $M := \lambda(M_\rho \circ \sigma)$. By technical results of Lebesgue measure (see [10, Lemma 4.2 and Corollary 4.3]), we know that there exists a measurable set $J_0 \subset J$ with $m(J) = m(J_0)$, such that for all $\tau_0 \in J_0$

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \setminus J} M(s)\, \mathrm{d}s}{\int_{\tau_0}^t \psi_j(s)\, \mathrm{d}s} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \setminus J} M(s)\, \mathrm{d}s}{\int_t^{\tau_0} \psi_j(s)\, \mathrm{d}s} \tag{3.19}$$

and, moreover, there is $J_1 \subset J_0$ with $m(J_0 \backslash J_1) = 0$, such that for all $\tau_0 \in J_1$

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \cap J_0} \psi_j(s)\, \mathrm{d}s}{\int_{\tau_0}^{t} \psi_j(s)\, \mathrm{d}s} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \cap J_0} \psi_j(s)\, \mathrm{d}s}{\int_{t}^{\tau_0} \psi_j(s)\, \mathrm{d}s}. \tag{3.20}$$

Now, fix $\tau_0 \in J_1$. By (3.19) and (3.20), there exists $\bar{t} > 0$ sufficiently close to 0, such that for all $t \in [\bar{t}, 2\bar{t}]$, the following inequalities hold:

$$2 \int_{[\tau_0, \tau_0 + t] \backslash J} M(s)\, \mathrm{d}s < \frac{1}{4} \int_{\tau_0}^{\tau_0 + t} \psi_j(s)\, \mathrm{d}s,$$
$$\int_{[\tau_0, \tau_0 + t] \cap J} \psi_j(s)\, \mathrm{d}s > \frac{1}{2} \int_{\tau_0}^{\tau_0 + t} \psi_j(s)\, \mathrm{d}s$$

and

$$2 \int_{[\tau_0 - t, \tau_0] \backslash J} M(s)\, \mathrm{d}s < \frac{1}{4} \int_{\tau_0 - t}^{\tau_0} \psi_j(s)\, \mathrm{d}s,$$
$$\int_{[\tau_0 - t, \tau_0] \cap J} \psi_j(s)\, \mathrm{d}s > \frac{1}{2} \int_{\tau_0 - t}^{\tau_0} \psi_j(s)\, \mathrm{d}s.$$

Define the positive number

$$r = \frac{\bar{t}}{8} \min \left\{ \int_{\tau_0 - \bar{t}}^{\tau_0} \psi_j(s)\, \mathrm{d}s, \int_{\tau_0}^{\tau_0 + \bar{t}} \psi_j(s)\, \mathrm{d}s \right\}.$$

Let $\varepsilon_j > 0$ be given above. Let us show that for every finite family $u_i \in \overline{B}_{\varepsilon_j}(u) \cap \overline{K}_{y,\rho}$ and $\mu_i \in [0,1]$ $(i = 1, 2, \ldots, m)$, with $\sum \mu_i = 1$, we have

$$\left\| u - \left(y + \lambda \sum_{i=1}^{m} \mu_i T u_i \right) \right\|_{[0,1]} \geq r, \tag{3.21}$$

which implies $u \notin y + \lambda\, \mathbb{T} u$. Notice that we can suppose without loss of generality that the restriction of u to $[0,1]$, denoted also as u, satisfies that $u \in y + \lambda\, Q$ where Q is the subset of $\mathcal{C}([0,1])$ defined as

$$Q = \left\{ u \in \mathcal{C}^1([0,1]) : \left| u'(t) - u'(s) \right| \leq \int_s^t M_\rho(r)\, dr \ \text{ whenever } 0 \leq s \leq t \leq 1 \right\}.$$

Indeed, due to assumption (H_2), then

$$Tu(t) = \int_0^1 k(t,s) f(s, u(s), u(\sigma(s)))\, \mathrm{d}s \leq \int_0^1 M_\rho(s)\, \mathrm{d}s.$$

Therefore, $T\left(\overline{K}_{y,\rho}\right) \subset Q$ and Q is a closed-convex subset of $\mathcal{C}([0,1])$ (see [10, Lemma 4.5]), which implies that $\mathbb{T}\left(\overline{K}_{y,\rho}\right) \subset Q$.

To prove (3.21), for simplicity, let us denote $z = \lambda \sum_{i=1}^{m} \mu_i T u_i$ and $v = u - y$. For a.a. $t \in J$, we have by the chain rule that

$$(z \circ \sigma)''(t) = z''(\sigma(t)) \left(\sigma'(t)\right)^2 + z'(\sigma(t)) \sigma''(t)$$

and, since $\sigma' = \pm 1$,

$$(z \circ \sigma)''(t) = z''(\sigma(t)) = -\lambda \sum_{i=1}^{m} \mu_i f(\sigma(t), u_i(\sigma(t)), u_i(\sigma(\sigma(t)))).$$

On the other hand, for every $i \in \{1, \dots, m\}$ and $t \in J$, we deduce from $u_i \in \overline{B}_{\varepsilon_j}(u)$ that

$$|u_i(\sigma(t)) - \Gamma_j(t)| = |u_i(\sigma(t)) - u(\sigma(t))| \leq \varepsilon_j,$$

and then, condition (3.18) ensures that for a.a. $t \in J$

$$\begin{aligned} z''(\sigma(t)) &= -\lambda \sum_{i=1}^{m} \mu_i f(\sigma(t), u_i(\sigma(t)), u_i(\sigma(\sigma(t)))) \\ &< \sum_{i=1}^{m} \mu_i \left(\Gamma_j''(t) - \psi_j(t)\right) = \Gamma_j''(t) - \psi_j(t) = u''(\sigma(t)) - \psi_j(t). \end{aligned}$$

Note that $y''(s) = 0$ for all $s > 0$, so we obtain that for a.a. $t \in J$

$$z''(\sigma(t)) < u''(\sigma(t)) - y''(\sigma(t)) - \psi_j(t) = v''(\sigma(t)) - \psi_j(t).$$

By integration, for $t \in [\bar{t}, 2\,\bar{t}]$

$$\begin{aligned} z'(\sigma(\tau_0)) - z'(\sigma(\tau_0 - t)) &= \int_{\tau_0 - t}^{\tau_0} z''(\sigma(s))\,\mathrm{d}s = \int_{[\tau_0 - t, \tau_0] \cap J} z''(\sigma(s))\,\mathrm{d}s \\ &\quad + \int_{[\tau_0 - t, \tau_0] \setminus J} z''(\sigma(s))\,\mathrm{d}s \\ &< \int_{[\tau_0 - t, \tau_0] \cap J} v''(\sigma(s))\,\mathrm{d}s - \int_{[\tau_0 - t, \tau_0] \cap J} \psi_j(s)\,\mathrm{d}s \\ &\quad + \int_{[\tau_0 - t, \tau_0] \setminus J} M(s)\,\mathrm{d}s \\ &= v'(\sigma(\tau_0)) - v'(\sigma(\tau_0 - t)) - \int_{[\tau_0 - t, \tau_0] \setminus J} v''(\sigma(s))\,\mathrm{d}s \\ &\quad - \int_{[\tau_0 - t, \tau_0] \cap J} \psi_j(s)\,\mathrm{d}s + \int_{[\tau_0 - t, \tau_0] \setminus J} M(s)\,\mathrm{d}s \\ &\leq v'(\sigma(\tau_0)) - v'(\sigma(\tau_0 - t)) - \int_{[\tau_0 - t, \tau_0] \cap J} \psi_j(s)\,\mathrm{d}s \\ &\quad + 2 \int_{[\tau_0 - t, \tau_0] \setminus J} M(s)\,\mathrm{d}s \\ &< v'(\sigma(\tau_0)) - v'(\sigma(\tau_0 - t)) - \frac{1}{4} \int_{\tau_0 - t}^{\tau_0} \psi_j(s)\,\mathrm{d}s. \end{aligned}$$

Hence, we have for all $t \in [\bar{t}, 2\,\bar{t}]$ that

$$z'(\sigma(\tau_0 - t)) - v'(\sigma(\tau_0 - t)) > z'(\sigma(\tau_0)) - v'(\sigma(\tau_0)) + \frac{1}{4} \int_{\tau_0 - t}^{\tau_0} \psi_j(s)\,\mathrm{d}s.$$

In case $z'(\sigma(\tau_0)) \geq v'(\sigma(\tau_0))$, then

$$z'(\sigma(\tau_0 - t)) - v'(\sigma(\tau_0 - t)) > \frac{1}{4} \int_{\tau_0 - \bar{t}}^{\tau_0} \psi_j(s)\,\mathrm{d}s \quad \text{for all } t \in [\bar{t}, 2\,\bar{t}],$$

so by integration

$$z(\sigma(\tau_0 - \bar{t})) - v(\sigma(\tau_0 - \bar{t})) > z(\sigma(\tau_0 - 2\,\bar{t})) - v(\sigma(\tau_0 - 2\,\bar{t})) + \frac{\bar{t}}{4} \int_{\tau_0 - \bar{t}}^{\tau_0} \psi_j(s)\,\mathrm{d}s$$

$$\geq z(\sigma(\tau_0 - 2\,\bar{t})) - v(\sigma(\tau_0 - 2\,\bar{t})) + 2\,r.$$

Then, either $z(\sigma(\tau_0-2\,\bar{t}))-v(\sigma(\tau_0-2\,\bar{t})) < -r$ or $z(\sigma(\tau_0-\bar{t}))-v(\sigma(\tau_0-\bar{t})) > r$, and thus, $\|v - z\| > r$, as wished.

It can be seen in a similar way that

$$z'(\sigma(\tau_0)) - v'(\sigma(\tau_0)) > z'(\sigma(\tau_0 + t)) - v'(\sigma(\tau_0 + t)) + \frac{1}{4}\int_{\tau_0}^{\tau_0+t} \psi_j(s)\,\mathrm{d}s, \quad t \in [\bar{t}, 2\,\bar{t}],$$

which ensures that $\|v - z\| > r$ if $z'(\sigma(\tau_0)) < v'(\sigma(\tau_0))$.

Case 3. $m\left(\{t \in I_n : u(t) = \gamma_n(t)\}\right) > 0$ for some $n \in \mathbb{N}$. It follows as in Case 2.

Finally, Theorem 2.13 gives the conclusion. □

Example 3.9. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x) = \sum_{n:q_n<x} 2^{-n},$$

where $\{q_n\}_{n\in\mathbb{N}}$ is an enumeration of the rational numbers. Observe that ϕ is discontinuous at the rational numbers and continuous at the irrational ones.

We study the existence of solutions for the following BVP with delay:

$$\begin{cases} -u''(t) = \lambda\left(\dfrac{2 - \phi(u(t) - t^2)}{\sqrt{t}} + \phi(u(t - 1/2) - t^2)(u(t))^3\right), & t \in [0, 1], \\ u(t) = \sqrt{1 + 2\,t}, & t \in [-1/2, 0], \\ u(1) = 0. \end{cases} \tag{3.22}$$

In this case

$$f(t, u, v) = \frac{2 - \phi(u - t^2)}{\sqrt{t}} + \phi(v - t^2)u^3, \quad (t, u, v) \in [0, 1] \times [0, +\infty) \times [0, +\infty).$$

For a fixed $\rho > 0$, we can choose

$$M_\rho(t) = \frac{2}{\sqrt{t}} + (\rho + 1)^3 \quad \text{and} \quad \delta_\rho(t) = \delta(t) = \frac{1}{\sqrt{t}} \qquad (t \in [0, 1]),$$

to check hypotheses (H_2) and (H_3). On the other hand, for each rational q_n, we define the function $\gamma_n : [0, 1] \to \mathbb{R}$ as

$$\gamma_n(t) = t^2 + q_n,$$

and so, for a.a. $t \in [0, 1]$, $f(t, \cdot, \cdot)$ is continuous on $([0, \infty)\setminus\bigcup_n\{\gamma_n(t)\}) \times ([0, \infty)\setminus\bigcup_n\{\gamma_n(t)\})$. Note that for each $n \in \mathbb{N}$ and each $\lambda > 0$

$$-\gamma_n''(t) = -2 < 0 \leq \lambda\, f(t, u, v) \quad \text{for a.a. } t \in [0, 1] \text{ and all } u, v \in [0, +\infty),$$

and thus, condition (D) in Theorem 3.8 holds. Therefore, this result ensures that the BVP (3.22) has uncountable many pairs of solutions and parameters (u_ρ, λ_ρ).

Remark 3.10. We stress that the theory presented so far is applicable and represents a novelty even in the *special case* of eigenvalue problems for ODEs, in presence of discontinuities. To illustrate this fact, one may consider the eigenvalue problem

$$u''(t) = \lambda \tilde{f}(u(t)), \quad t \in (0,1);\ u(0) = u(1) = 0, \tag{3.23}$$

a classical problem studied in the book of Guo and Lakshmikantham [16, Example 2.3.2], where in our case, the nonlinearity can be allowed to be discontinuous.

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Declarations

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