



Fredholm maps of index zero between real Banach manifolds from the viewpoint of covering spaces

Pierluigi Benevieri¹ · Alessandro Calamai² · Massimo Furi³ · Maria Patrizia Pera³

Received: 25 November 2024 / Accepted: 11 July 2025 / Published online: 23 August 2025
© The Author(s) 2025

Abstract

We give an exhaustive presentation of the concept of topological orientation for Fredholm maps of index zero between real Banach manifolds introduced by the first and third authors during the last three years of the twentieth century. An application of this orientation to the construction of the degree introduced by the same two authors will be given at the end of the paper.

Keywords Topological orientation · Topological degree · Fredholm operators · Fredholm maps

Mathematics Subject Classification 47H11 · 47A53 · 58C25

The second and fourth authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and are supported by PRIN 2022 - Progetti di Ricerca di rilevante Interesse Nazionale, “Nonlinear differential problems with applications to real phenomena” (Grant Number: 2022ZXZTN2). P. Benevieri is partially supported by FAPESP, project n. 2022/14913-7 (Brazil). A. Calamai is partially supported by GNAMPA - INdAM (Italy).

✉ Alessandro Calamai
a.calamai@univpm.it

Pierluigi Benevieri
pluigi@ime.usp.br

Massimo Furi
furimas.unifi@gmail.com

Maria Patrizia Pera
mpatrizia.pera@unifi.it

¹ Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP CEP 05508-090, Brazil

² Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

³ Dipartimento di Matematica e Informatica “Ulisse Dini”, Università degli Studi di Firenze, Via S. Marta 3, 50139 Florence, Italy

1 Introduction

The main purpose of this work is an exhaustive exposition of the concept of *topological orientation* (*top-orientation*, for short) for Fredholm maps of index zero between real Banach manifolds whose origins date back to the published works [2, 6, 7]. This concept is based on the purely algebraic notion of *companion* (a kind of “guardian angel”) of any Fredholm linear operator of index zero between vector spaces (hereafter called a Φ_0 -operator), as well as on the concept of *algebraic orientation* (*alg-orientation* for short) of such an operator. As we shall see, any Φ_0 -operator, invertible or non-invertible, admits exactly two alg-orientations. Roughly speaking, a top-orientation of a Fredholm map of index zero, $f: \mathcal{M} \rightarrow \mathcal{N}$, between two manifolds, is a continuous assignment, with respect to $p \in \mathcal{M}$, of an alg-orientation to the differential df_p .

In this section we report the main phases of the construction of the two notions of orientation (algebraic and topological). The exposition is necessarily approximate, but hopefully sufficiently understandable. Details and rigor will be found in subsequent sections.

Let $\mathbb{E}$ and $\mathbb{F}$ be two real vector spaces and denote by $\mathcal{L}(\mathbb{E}, \mathbb{F})$ the space of linear operators from $\mathbb{E}$ into $\mathbb{F}$. We shall possibly write $\mathcal{L}(\mathbb{E})$ instead of $\mathcal{L}(\mathbb{E}, \mathbb{E})$.

Recall that $L \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is a *Fredholm linear operator* (also called Φ -operator, for short) if

$$\text{ind} L := \dim(\text{Ker} L) - \dim(\mathbb{F}/L(\mathbb{E}))$$

is a well defined integer, called *index of L* . The set of Fredholm operators between $\mathbb{E}$ and $\mathbb{F}$ will be denoted by $\Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$ and, given $n \in \mathbb{Z}$, $\Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$ stands for its subset of the operators with index n (the Φ_n -operators).

Given $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, a *companion of L* (also called *L -companion*) is an operator $A \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ with finite dimensional image such that $L + A$ is an isomorphism. One can check that the set $\mathcal{C}(L)$ of the L -companions is always nonempty, regardless of whether L is invertible or not.

There exists an equivalence relation in $\mathcal{C}(L)$ with exactly two equivalence classes; the choice of any of these classes is an *alg-orientation* of L ; so that, L admits two alg-orientations, considered opposite to each other. As we shall see, some special Φ_0 -operators have an alg-orientation which is more relevant than the other. This is the case, for example, of the invertible linear operators, or of the endomorphisms of a finite dimensional real vector space.

Assume that $\mathbb{E}$ and $\mathbb{F}$ are, in addition, Banach spaces. By abuse of notation, the set of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$ will still be denoted by $\mathcal{L}(\mathbb{E}, \mathbb{F})$, while $\Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$ is its subset of the Fredholm operators. It is known that, if $L \in \Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$, then its image, $\text{Im} L := L(\mathbb{E})$, is closed in $\mathbb{F}$ (see e.g. [26, Chapter IV, Theorem 13.2]). Moreover, given any $n \in \mathbb{Z}$, $\Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$ is open in the Banach space $\mathcal{L}(\mathbb{E}, \mathbb{F})$. This implies that, if $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, then a companion of L is still a companion of any operator in a neighborhood of L . As we shall see, this kind of stability is a prelude of the concept of *top-orientation* of a given subset of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Let $f: U \rightarrow \mathbb{F}$ be a Fredholm map of index zero defined on an open set of $\mathbb{E}$. Recall that, by definition, such a map is C^1 and, at any $x \in U$, its Fréchet differential df_x is a Φ_0 -operator. The set of these maps will be denoted by $\Phi_0(U, \mathbb{F})$.

A *pre-orientation* of a map $f \in \Phi_0(U, \mathbb{F})$ is a function that to any $x \in U$ assigns an alg-orientation or, equivalently, a companion of df_x . In this “flat case”, a pre-orientation is a *top-orientation* if, locally, the assignment could be given by only one companion. Thus, locally, f admits exactly two top-orientations, opposite to each other. In this way one gets a two-fold covering space over the open set U , and f is *top-orientable* if and only if over U

one has two disjoint sheets. Each of these sheets is a top-orientation of f . As we shall see, as a consequence of the covering space theory, one gets important properties of the concept of topological orientation for Fredholm maps of index zero in the flat case.

Less trivial is the situation for a Fredholm map of index zero $f: \mathcal{M} \rightarrow \mathcal{N}$ acting between two real Banach manifolds (in this case we shall write $f \in \Phi_0(\mathcal{M}, \mathcal{N})$), meaning that f is a C^1 -map such that, given any $x \in \mathcal{M}$, the differential df_x between the tangent spaces $T_x\mathcal{M}$ and $T_{f(x)}\mathcal{N}$ is a Φ_0 -operator. Even in this non-flat case, the alg-orientation of a differential df_x induces, by a sort of continuity, a top-orientation of the restriction of f to a convenient neighborhood of x in $\mathcal{M}$. In fact, the appropriate use of charts reduces a local problem to the flat case.

Even in this less trivial case, the theory of covering spaces helps us to deduce important properties of the concept of top-orientation. For instance, one can easily find examples of not top-orientable maps. A simple one is when $\mathcal{M}$ and $\mathcal{N}$ have the same finite dimension, f is a constant map and $\mathcal{M}$ is compact and non-orientable (as the Klein bottle or the two-dimensional projective space). Still by applying the covering space theory, one gets that, if the domain $\mathcal{M}$ of a map $f \in \Phi_0(\mathcal{M}, \mathcal{N})$ is simply connected, then f is top-orientable and, taking into account that a simply connected space is assumed to be connected, f admits exactly two top-orientations.

Although the definitions of alg-orientation and top-orientation go back to the papers [6, 7], in these works just the term “orientability” is present. The new terminology has been introduced for the first time in the recent paper [4] and shed a new light on the subject. For this reason in the present paper we are able to provide a more systematic exposition of the notions of orientability and orientation and we are able to give more precise statements.

We wish to point out that other authors proposed different notions of orientability and orientation in the context of Fredholm maps of index zero between real Banach manifolds. Among them we cite the work of Elworthy and Tromba (see [12, 13]) and of Fitzpatrick, Pejsachowicz and Rabier (see [14, 23]); see also [7, 27] and references therein. However, we stress that these notions of orientability differ in a significant way. For example, concerning those in [6, 14, 23], we observe that such concepts are not equivalent in the non-flat framework of Banach manifolds, even if, as shown in [22] by J. Pejsachowicz, they agree in Banach spaces. However, regarding the orientation (and not merely the orientability), the concepts are not equivalent even in the flat case. For more details, we refer the interested reader to Sect. 5 of [7] where the different notions of orientability are compared in an exhaustive way.

The paper is structured as follows. In Sect. 2 we collect some useful notation, while Sect. 3 contains both algebraic preliminaries and topological preliminaries. In Sect. 4 we introduce the notion of alg-orientation of Φ_0 -operators between real vector spaces. In Sect. 5, the concept of top-orientation in the flat case is introduced, first for Φ_0 -operators and then for Φ_0 -maps, and also discussed by means of the covering space theory. Section 6 deals with the top-orientation in the non-flat case. As already pointed out, the theory of covering spaces plays a crucial role in this setting too.

Section 7 is devoted to the application of the top-orientation theory to the concept of topological degree for Φ_0 -maps between real Banach manifolds, introduced by the first and the third authors in [2, 6–8]. In particular we show that, under appropriate assumptions, such a degree extends both the Brouwer degree for maps between oriented manifolds of the same finite dimension, and the Leray–Schauder degree for compact vector fields. We close the paper with Sect. 8 in which we summarize some concluding remarks.

2 Some useful symbols

There are many mathematical symbols commonly used in literature, here are a few that we report just for the sake of clarity.

The closure of a subset X of a topological space is denoted by $\overline{X}$ and its boundary by ∂X . If f is a function whose domain contains X , then $f|_X$ denotes its restriction to X . Regarding a self-function of a given set, unless otherwise specified, the symbol I represents the identity function. Given a vector v in a normed space, if not differently preferable, its norm is denoted by $\|v\|$.

Given two sets X and Y , by $X \subset Y$ we mean that X is strictly contained into Y , while the notation $X \subseteq Y$ implies that either $X \subset Y$ or $X = Y$.

Moreover, we tacitly assume that any simply connected topological space is connected.

As in Sect. 1, $\mathbb{E}$ and $\mathbb{F}$ are real vector spaces and $\mathcal{L}(\mathbb{E}, \mathbb{F})$ denotes the space of the linear operators from $\mathbb{E}$ into $\mathbb{F}$.

For simplicity, when $\mathbb{E}$ and $\mathbb{F}$ have, in addition, a normable topology, we will still use the symbol $\mathcal{L}(\mathbb{E}, \mathbb{F})$ to indicate the space of the linear operators from $\mathbb{E}$ into $\mathbb{F}$ which are additionally continuous (hence, bounded, if the topology is Banachable).

If a pair of the same set appears in a symbol, this pair could be simply represented without repetition. For example, we could possibly write $\mathcal{L}(\mathbb{E})$ instead of $\mathcal{L}(\mathbb{E}, \mathbb{E})$.

By a *manifold* we shall always mean a boundaryless real differentiable Banach manifold and, hereafter, $\mathcal{M}$ and $\mathcal{N}$ will always denote nontrivial manifolds. Possibly, a manifold could be *flat*; meaning that it is a nonempty open subset of a real Banach space.

Given a point $p \in \mathcal{M}$, the symbol $T_p\mathcal{M}$ denotes the (Banachable) tangent space of $\mathcal{M}$ at p . Thus, if $\mathcal{M}$ is an open subset of $\mathbb{E}$, then $T_p\mathcal{M}$ can be identified with $\mathbb{E}$.

Given $\mathcal{M}$ and $\mathcal{N}$, we consider the following set of triples:

$$\mathcal{L}^+(\mathcal{M}, \mathcal{N}) = \{(p, q, L) : p \in \mathcal{M}, q \in \mathcal{N}, L \in \mathcal{L}(T_p\mathcal{M}, T_q\mathcal{N})\}.$$

So that, when $\mathcal{M} \subseteq \mathbb{E}$ and $\mathcal{N} \subseteq \mathbb{F}$ are open sets, we can identify $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ with the flat subset $\mathcal{M} \times \mathcal{N} \times \mathcal{L}(\mathbb{E}, \mathbb{F})$ of the Banachable space $\mathbb{G} = \mathbb{E} \times \mathbb{F} \times \mathcal{L}(\mathbb{E}, \mathbb{F})$.

Any alphanumeric prefix in front of $\mathcal{L}(\mathbb{E}, \mathbb{F})$ or $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ identifies a specific subset of the corresponding set $\mathcal{L}(\mathbb{E}, \mathbb{F})$ or $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$. For example, $LS\mathcal{L}(\mathbb{E})$ will stand for the affine subspace of $\mathcal{L}(\mathbb{E})$ made up of the compact linear perturbations of the identity of $\mathbb{E}$ (the prefix LS is in honor of J. Leray and J. Schauder).

When neither $\mathcal{L}$ nor $\mathcal{L}^+$ appears before a pair, then any other prefix identifies a set of functions between the two elements of the pair.

Here are some examples: $C^1(\mathbb{E}, \mathbb{F})$ is the set of the C^1 -functions from $\mathbb{E}$ into $\mathbb{F}$; $C(\mathcal{M}, \mathcal{N})$ is the set of the continuous functions from $\mathcal{M}$ into $\mathcal{N}$; given $n \in \mathbb{N}$, $C^n(\mathcal{M})$ is the set of the self-maps of $\mathcal{M}$ of class C^n ; $LS(\mathbb{E})$ is the set of the compact vector fields defined on $\mathbb{E}$; given an open subset U of $\mathbb{E}$, $C^1LS(U, \mathbb{E})$ is the set of the compact vector fields of class C^1 defined on U .

Observe that, if $f \in C^1LS(U, \mathbb{E})$, then the differential df of f is a continuous map from U into $LS\mathcal{L}(\mathbb{E})$. In fact, denoting by k the compact C^1 -map perturbing I , and given any $x \in U$, the differential $dk_x \in \mathcal{L}(\mathbb{E})$ is a compact operator (see, for example, [11, Sect. 8]).

We admit two exceptions to our rule. They regard the following well known symbols: $\text{Iso}(\mathbb{E}, \mathbb{F})$ and $\text{GL}(\mathbb{E})$, representing, respectively, the subsets of the invertible operators of $\mathcal{L}(\mathbb{E}, \mathbb{F})$ and of $\mathcal{L}(\mathbb{E})$. Obviously, for the sake of coherence, we could write, respectively, $\text{Iso}\mathcal{L}(\mathbb{E}, \mathbb{F})$ and $\text{GL}\mathcal{L}(\mathbb{E})$, but we won't do that.

A notation that we shall frequently use is $\Phi_0(\mathcal{M}, \mathcal{N})$, which is the set of the Fredholm maps of index zero from $\mathcal{M}$ into $\mathcal{N}$ (see Definition 3.6 below). It should not be confused

with $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$, which is the subset of $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ consisting of the triples (p, q, L) , where $L: T_p\mathcal{M} \rightarrow T_q\mathcal{N}$ is a Φ_0 -operator.

Other symbols will be introduced in Sect. 3. Some of them are $F\mathcal{L}(\mathbb{E}, \mathbb{F})$, $A\mathcal{L}(\mathbb{E})$, $\Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$, $\Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$, $\Phi_0(\mathcal{M})$.

3 Preliminaries

In this section we recall some important notions present in the literature and add some notation that we will need later.

For the convenience of the reader, the topics in this section are divided into two subsections: Algebraic preliminaries and Topological preliminaries.

3.1 Algebraic preliminaries

In this subsection $\mathbb{E}$ and $\mathbb{F}$ are two real vector spaces, no additional structure being required.

The following notion is of basic importance (see, for example, [10, 26]).

Definition 3.1 (*Fredholm operator*) A linear operator $L \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is said to be *Fredholm*, or a Φ -operator, if its kernel and cokernel are finite dimensional, where the cokernel of L (written $\text{coKer}L$) is the quotient space $\mathbb{F}/L(\mathbb{E})$. The *index* of L is the integer

$$\text{ind}L := \dim(\text{Ker}L) - \dim(\text{coKer}L).$$

A Φ -operator of index $n \in \mathbb{Z}$ is called a Φ_n -operator.

Observe that, if $\mathbb{E}$ and $\mathbb{F}$ have finite dimension, then any $L \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is a Fredholm operator of index $n = \dim \mathbb{E} - \dim \mathbb{F}$.

The notations $\Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$ and $\Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$ represent, respectively, the subsets of $\mathcal{L}(\mathbb{E}, \mathbb{F})$ of the Φ -operators and the Φ_n -operators.

Here are two fundamental algebraic properties of the Fredholm operators.

Proposition 3.2 (Algebraic properties of Φ -operators) *Concerning the Fredholm operators between vector spaces we have the following properties:*

1. *The composition of Fredholm operators is still Fredholm and its index is the sum of the indices of the composite operators;*
2. *If $L \in \Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$ and $A \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ has finite dimensional image, then $L + A \in \Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$.*

Notice that, if k is a positive integer and $\mathbb{E}_1$ is a k -codimensional subspace of $\mathbb{E}$, then the inclusion operator $\mathbb{E}_1 \hookrightarrow \mathbb{E}$ is Fredholm of index $-k$. Consequently, from Property 1 of Proposition 3.2 one gets that, if $L \in \Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$, then the restriction of L to $\mathbb{E}_1$ belongs to $\Phi_{n-k}\mathcal{L}(\mathbb{E}_1, \mathbb{F})$. Analogously, given L as above, if its image is contained in an s -codimensional subspace $\mathbb{F}_1$ of $\mathbb{F}$, then the restriction of L to the same domain and codomain $\mathbb{F}_1$ is an element of $\Phi_{n+s}\mathcal{L}(\mathbb{E}, \mathbb{F}_1)$.

Hereafter, by $F\mathcal{L}(\mathbb{E}, \mathbb{F})$ we mean the subspace of $\mathcal{L}(\mathbb{E}, \mathbb{F})$ of the operators with finite dimensional image.

Any $L \in \mathcal{L}(\mathbb{E})$ such that $I - L \in F\mathcal{L}(\mathbb{E})$ is said to be *admissible for the determinant*, or *det-admissible* for short. The symbol $A\mathcal{L}(\mathbb{E})$ denotes the affine subspace of $\mathcal{L}(\mathbb{E})$ of the det-admissible endomorphisms. Obviously, if $\mathbb{E}$ is finite dimensional, then $A\mathcal{L}(\mathbb{E}) = \mathcal{L}(\mathbb{E})$.

Since the identity of $\mathbb{E}$, as any isomorphism, is Fredholm of index 0, from Property 2 of Proposition 3.2 one gets $AL(\mathbb{E}) \subseteq \Phi_0\mathcal{L}(\mathbb{E})$.

In [18], the *extended determinant* of any $L \in AL(\mathbb{E})$ is defined as the number $\det(L|_{\mathbb{E}_1})$, where $L|_{\mathbb{E}_1}$ is the restriction of L , both as domain and codomain, to any finite dimensional subspace $\mathbb{E}_1$ of $\mathbb{E}$ containing the image of $I - L$; with the usual convention that the determinant of the unique endomorphism acting on the trivial vector space is 1.

Hereafter, for the sake of clarity, the extended determinant will be denoted by $\det_e$, while the classical notation “det” will be reserved exclusively for the finite dimensional case. So that, $\det_e(L) = \det(L)$ only when L acts on a finite dimensional vector space.

Here are three basic properties of the extended determinant, whose proof can be easily deduced from the analogous ones in the finite dimensional case. See, for example, [9] for a discussion about other properties.

Proposition 3.3 (Fundamental properties of the extended determinant) *Let $L, L_1, L_2 \in AL(\mathbb{E})$ be given. Then,*

1. $\det_e(L) \neq 0$ if and only if L is invertible;
2. $R \in \text{Iso}(\mathbb{E}, \mathbb{F})$ implies $RLR^{-1} \in AL(\mathbb{F})$ and $\det_e(RLR^{-1}) = \det_e(L)$;
3. $L_2L_1 \in AL(\mathbb{E})$ and $\det_e(L_2L_1) = \det_e(L_2)\det_e(L_1)$.

The following remark is a consequence of the definition of extended determinant.

Remark 3.4 Let $\mathbb{E}_1 \oplus \mathbb{E}_2$ be a splitting of $\mathbb{E}$ with $\dim \mathbb{E}_1 < \infty$. Suppose that, according to this splitting, $L \in \mathcal{L}(\mathbb{E})$ can be represented in a block matrix form as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ 0 & I_{22} \end{pmatrix},$$

where I_{22} is the identity operator on $\mathbb{E}_2$. Then $(I - L)(\mathbb{E}) \subseteq \mathbb{E}_1$ and, consequently, $\det_e(L) = \det(L_{11})$.

3.2 Topological preliminaries

In this subsection, any vector space has the additional property of being Banach (or, more generally, Banachable). Recall that, in this case, any operator in $\mathcal{L}(\mathbb{E}, \mathbb{F})$ is assumed to be bounded.

Proposition 3.5 below collects some important properties of the Fredholm operators between Banach spaces (see [10, 26]). Recall that a continuous map between metric spaces is said to be *proper* if the pre-image of any compact set is a compact set. It can be verified that such a map is *closed*, meaning that the image of any closed set is a closed set.

Proposition 3.5 (Topological properties of $\Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$) *In the context of Banach spaces one has the following properties:*

1. if $L \in \Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$, then its image is closed in $\mathbb{F}$;
2. if $L \in \Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$ and $K \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is compact, then $L + K \in \Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$;
3. for any $n \in \mathbb{Z}$, the set $\Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$ is open in $\mathcal{L}(\mathbb{E}, \mathbb{F})$;
4. if $L \in \Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$, then it is proper on any bounded and closed subset of $\mathbb{E}$.

The following notion of Fredholm map between manifolds is of basic importance for us (see [25]).

Definition 3.6 (*Fredholm maps*) A C^1 -map $f: \mathcal{M} \rightarrow \mathcal{N}$ is said to be *Fredholm of index* $n \in \mathbb{Z}$ (or a Φ_n -map, for short) if, for any $p \in \mathcal{M}$, the differential of f at p , $df_p: T_p\mathcal{M} \rightarrow T_{f(p)}\mathcal{N}$, is a Fredholm operator of index n . The map f is *Fredholm* (or a Φ -map) if it is Fredholm of index n for some n .

The symbol $\widehat{\Phi}(\mathcal{M}, \mathcal{N})$ denotes the set of the Fredholm maps from $\mathcal{M}$ into $\mathcal{N}$, while $\Phi_n(\mathcal{M}, \mathcal{N})$ is its subset of the Φ_n -maps. Special flat cases are $\Phi(\mathbb{E}, \mathbb{F})$ and $\Phi_n(\mathbb{E}, \mathbb{F})$, not to be confused with $\Phi\mathcal{L}(\mathbb{E}, \mathbb{F})$ and $\Phi_n\mathcal{L}(\mathbb{E}, \mathbb{F})$, which are open subsets of the Banach space $\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Theorems 3.7 and 3.8 below are two important results in covering space theory that we will need later. A comprehensible exposition of this theory can be found, for example, in [15, 16].

We recall (see e.g. [16, Chapter 1]) that a *covering space* of a topological space Y is a topological space $\widehat{Y}$ together with a continuous map $\mathcal{P}: \widehat{Y} \rightarrow Y$, satisfying the following condition: every point in Y has an open neighborhood U such that $\mathcal{P}^{-1}(U)$ is a disjoint union of open sets in $\widehat{Y}$ and each of these sets is mapped homeomorphically onto U by $\mathcal{P}$. In the sequel, for the sake of simplicity, we will also call covering space the map $\mathcal{P}$.

Theorem 3.7 (*Lifting property*) Let $\mathcal{P}: \widehat{Y} \rightarrow Y$ be a covering space and let $\Gamma: \mathcal{X} \rightarrow Y$ be a continuous map defined on a connected topological space. Then, given two lifts of Γ , either they coincide or they have disjoint images. Moreover, if $\mathcal{X}$ is simply connected and locally path connected, then, given x_0 in $\mathcal{X}$ and $\widehat{y}_0$ in the fiber $\mathcal{P}^{-1}(\Gamma(x_0))$, Γ admits a lift $\widehat{\Gamma}$ such that $\widehat{\Gamma}(x_0) = \widehat{y}_0$.

Theorem 3.8 (*Homotopy lifting*) Let $\mathcal{P}: \widehat{Y} \rightarrow Y$ be a covering space and let $\mathcal{X}$ be a topological space. Consider a continuous map $\Gamma: \mathcal{X} \rightarrow Y$ which admits a lift $\widehat{\Gamma}$. Then, given a homotopy $\mathcal{H}: \mathcal{X} \times [0, 1] \rightarrow Y$ with $\mathcal{H}(\cdot, 0) = \Gamma$, there exists a unique lift $\widehat{\mathcal{H}}$ of $\mathcal{H}$ such that $\widehat{\mathcal{H}}(\cdot, 0) = \widehat{\Gamma}$.

4 Algebraic orientation of Φ_0 -operators

This section is dedicated to the notion of algebraic orientation of Fredholm operators of index zero between real vector spaces, as well as to some consequences of such a notion (see [6, 7]). Here, no additional property of $\mathbb{E}$ and $\mathbb{F}$ is needed.

Let $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. In [6] an operator $A \in F\mathcal{L}(\mathbb{E}, \mathbb{F})$ was called a *corrector* of L if $L + A \in \text{Iso}(\mathbb{E}, \mathbb{F})$. Since in the case when L is an isomorphism the word “corrector” could be misleading as L needs not to be corrected to become invertible, here we will use the more appropriate expression *companion* of L or, briefly, *L -companion*.

Observe that any $L \in \text{Iso}(\mathbb{E}, \mathbb{F})$ has a *natural companion*: the zero element of the space $F\mathcal{L}(\mathbb{E}, \mathbb{F})$. This fact was of fundamental importance for the concept of *orientation* of L introduced in [6]: it was shown that any $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ admits exactly two orientations, hereafter called *algebraic orientations* or, simply, *alg-orientations*.

Given $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, denote by $\mathcal{C}(L)$ the set of L -companions. One can check that, whatever is L , invertible or not, such a set is nonempty.

If A and B are two L -companions, then the automorphism $(L + B)^{-1}(L + A)$ of $\mathcal{L}(\mathbb{E})$ is det-admissible. In fact, one has

$$(L + B)^{-1}(L + A) = (L + B)^{-1}(L + B + A - B) = I - (L + B)^{-1}(B - A).$$

Thus, the following relation in $\mathcal{C}(L)$ makes sense.

Definition 4.1 (*Equivalent L -companions*) Two companions A and B of an operator $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ are said to be *L -equivalent* if $\det_e((L + B)^{-1}(L + A)) > 0$.

Observe that, given two companions A and B of an operator $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, Proposition 3.3 implies that the extended determinants of the following four operators are either all positive or all negative:

$$(L + B)^{-1}(L + A), (L + A)^{-1}(L + B), (L + A)(L + B)^{-1}, (L + B)(L + A)^{-1}.$$

Therefore one can deduce the following result.

Lemma 4.2 (*L -equivalence relation*) Given an operator $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, the relation in $\mathcal{C}(L)$ introduced in Definition 4.1 is an equivalence relation.

Proof The relation is clearly reflexive. Property 3 of Proposition 3.3 implies that it is as well symmetric and transitive. $\square$

We can now deduce a more exhaustive result.

Proposition 4.3 (*Two equivalence classes of the L -equivalence relation*) Given any $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, the relation in $\mathcal{C}(L)$ introduced in Definition 4.1 is an equivalence relation with two equivalence classes.

Proof Because of Lemma 4.2 we need only to show that the equivalence classes of $\mathcal{C}(L)$ are two. In fact, given three companions of L , say A , B_1 and B_2 , Proposition 3.3 implies that, if both B_1 and B_2 are not L -equivalent to A , then they are L -equivalent to each other. $\square$

Proposition 4.3 allows us to introduce the following concept of algebraic orientation of an arbitrary operator $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Definition 4.4 (*Algebraic orientation of a Φ_0 -operator*) Let $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ be given. An *algebraic orientation* (or *alg-orientation*) of L is any of the two equivalence classes of $\mathcal{C}(L)$, also referred as the *alg-orienting class* of L , whose elements are regarded as the *alg-orienting companions* of L .

Formally, an *algebraic oriented* operator $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is a pair (L, α) , where α is the alg-orienting class of L . However, if clarity is preserved without writing any class symbol, we could informally refer to L as an alg-oriented operator.

Sometimes, the nature of an operator $L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ suggests that one of the two possible algebraic orientations of L is more appropriate than its opposite. Here we introduce three significative definitions.

Definition 4.5 (*Natural alg-orientation*) Any $L \in \text{Iso}(\mathbb{E}, \mathbb{F})$ admits the *natural alg-orientation*: given by considering the null operator of $F\mathcal{L}(\mathbb{E}, \mathbb{F})$ as an orienting companion of L .

For the next concept of associated orientation we need the classical notion of orientation of a finite-dimensional vector space, that we recall here for completeness.

Definition 4.6 An orientation of a finite-dimensional vector space is a choice of one of the two equivalence classes of ordered bases.

Definition 4.7 (*Associated alg-orientation*) Assume that $\mathbb{E}$ and $\mathbb{F}$ have the same finite dimension and are oriented up to an inversion of both the orientations (or, equivalently, suppose that $\mathbb{E} \times \mathbb{F}$ is oriented). Then any $L \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ admits the alg-orientation *associated* to the orientations of $\mathbb{E}$ and $\mathbb{F}$ (or to the orientation of $\mathbb{E} \times \mathbb{F}$): the one given by regarding as an orienting L -companion any $A \in \mathcal{C}(L)$ such that $L + A$ is orientation preserving.

Observe that if $\mathbb{E}$ is finite dimensional, then $\mathbb{E} \times \mathbb{E}$ has a canonical orientation. Thus, according to Definition 4.7, the following concept is a special case of the associated alg-orientation.

Definition 4.8 (*Canonical alg-orientation*) Let $\mathbb{E}$ be finite dimensional. Then any endomorphism of $\mathbb{E}$ admits the *canonical* alg-orientation: the one obtained by considering as an orienting L -companion any $A \in \mathcal{C}(L)$ such that $\det(L + A) > 0$.

The next definition regards the concept of algebraic orientation of the composition of two alg-oriented Φ_0 -operators (see also [3, Definition 2.9]).

Definition 4.9 (*Composition of alg-oriented operators*) Let $L_1: \mathbb{E}_1 \rightarrow \mathbb{E}_2$ and $L_2: \mathbb{E}_2 \rightarrow \mathbb{E}_3$ be two alg-oriented Φ_0 -operators. Their *alg-oriented composition* is $L_2 L_1$ endowed with the alg-orientation whose orienting companions are the operators of the type

$$(L_2 + A_2)(L_1 + A_1) - L_2 L_1,$$

where A_1 and A_2 are alg-orienting companions of L_1 and L_2 , respectively.

One can check that Definition 4.9 is well posed.

The alg-oriented composition is associative. In fact, if $L_1 \in \Phi_0 \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$, $L_2 \in \Phi_0 \mathcal{L}(\mathbb{E}_2, \mathbb{E}_3)$, $L_3 \in \Phi_0 \mathcal{L}(\mathbb{E}_3, \mathbb{E}_4)$, and A_1, A_2 and A_3 are, respectively, alg-orienting companions of L_1, L_2 and L_3 , one has

$$\begin{aligned} & \left((L_3 + A_3)(L_2 + A_2) \right) (L_1 + A_1) - (L_3 L_2) L_1 \\ &= (L_3 + A_3) \left((L_2 + A_2)(L_1 + A_1) \right) - L_3 (L_2 L_1). \end{aligned}$$

From now on, unless otherwise indicated, in a composition of two or more alg-oriented operators, any isomorphism will be considered naturally alg-oriented (according to Definition 4.5).

Recall that, if $\dim \mathbb{E} < \infty$ and $L \in \mathcal{L}(\mathbb{E})$, the (classical) sign of L is simply the sign of $\det(L)$. More generally, if $\mathbb{E}$ and $\mathbb{F}$ have the same finite dimension and are oriented, the sign of L is defined as follows (see, for example, [21]):

$$\text{sign} L = \begin{cases} 0 & \text{if } L \text{ is non-invertible,} \\ +1 & \text{if } L \text{ is orientation preserving,} \\ -1 & \text{if } L \text{ is orientation reversing.} \end{cases}$$

Obviously, whenever it makes sense, the (classical) sign of a composition is the product of the signs of the composite operators.

As in [2, 6, 7], we have the following concept of sign of an alg-oriented operator, denoted here sign_{bf} to distinguish it from the finite dimensional classical one.

Definition 4.10 (*sign_{bf} of an alg-oriented operator*) The sign_{bf} of an algebraic oriented operator $L \in \Phi_0 \mathcal{L}(\mathbb{E}, \mathbb{F})$ is

$$\text{sign}_{bf} L = \begin{cases} 0 & \text{if } L \text{ is non-invertible,} \\ +1 & \text{if } L \text{ is invertible and naturally alg-oriented,} \\ -1 & \text{if } L \text{ is invertible and not naturally alg-oriented.} \end{cases}$$

As in the classical case, whenever it makes sense, the sign_{bf} of a composition is the product of the sign_{bf} of the composite operators.

The easy proof of the following remark is left to the reader.

Remark 4.11 (Classical sign equals sign_{bf}) Assume that $\mathbb{E}$ and $\mathbb{F}$ have the same finite dimension and are oriented, then, given $L \in \mathcal{L}(\mathbb{E}, \mathbb{F})$, one has $\text{sign}L = \text{sign}_{bf}L$, provided that L has the associated alg-orientation.

From Definitions 4.1, 4.5 and 4.10 one gets the following result (see [3] for further details).

Proposition 4.12 (Sign test of an invertible operator) *Let $L \in \text{Iso}(\mathbb{E}, \mathbb{F})$ be alg-oriented. Then, given any alg-orienting companion A of L , one has*

$$\text{sign}_{bf}L = \text{signdet}_e(I + L^{-1}A).$$

Proof Denoting by $\mathbf{0}$ the null operator of $F\mathcal{L}(\mathbb{E}, \mathbb{F})$, we get

$$\text{sign}_{bf}L = \text{signdet}_e((L + \mathbf{0})^{-1}(L + A)),$$

which implies the assertion. $\square$

5 Flat topological orientation

In this section we collect some results stated in [2, 6, 7] and discussed in subsequent papers. Moreover, we will present the concept of topological orientation in an exhaustive way including new proofs.

Assume, from now on, that the spaces $\mathbb{E}$ and $\mathbb{F}$ are Banach or, more generally, Banachable. As pointed out in Sect. 3, in this topological context, $\mathcal{L}(\mathbb{E}, \mathbb{F})$ denotes the Banachable space of the bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$; so that, its subset $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is open (see Property 3 of Proposition 3.5). Furthermore, as the reader can check, if $\text{Iso}(\mathbb{E}, \mathbb{F})$ is nonempty and nontrivial, one has the following two inclusions:

$$\text{Iso}(\mathbb{E}, \mathbb{F}) \subset \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F}) \subseteq \overline{\text{Iso}(\mathbb{E}, \mathbb{F})}.$$

the second one being an equality if (and only if) $\mathbb{E}$ and $\mathbb{F}$ are finite dimensional. Thus, we will tacitly assume that $\text{Iso}(\mathbb{E}, \mathbb{F})$ is nonempty and nontrivial; so that, some operators of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ are non-invertible.

We point out that, if $\mathbb{E}$ and $\mathbb{F}$ are infinite dimensional, then the vector space $F\mathcal{L}(\mathbb{E}, \mathbb{F})$ is not a closed subset of $\mathcal{L}(\mathbb{E}, \mathbb{F})$. Consequently, in this case, it is simply an incomplete normable space, but, important for us, contained in a Banach space.

The following result (compare with Lemma 3.1 in [7]) is a prodrome of the concept of topological orientation for subsets of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ given in Definition 5.4 below. The proof here takes into account that $\text{Iso}(\mathbb{E}, \mathbb{F})$ is an open subset of $\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Lemma 5.1 (Double stability property) *Let $A_* \in F\mathcal{L}(\mathbb{E}, \mathbb{F})$ be a companion of an operator $L_* \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. Then, there exist two neighborhoods, U of A_* in $F\mathcal{L}(\mathbb{E}, \mathbb{F})$ and W of L_* in $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, such that A is an L -companion for any $(A, L) \in U \times W$.*

Proof Since the spaces $\mathbb{E}$ and $\mathbb{F}$ are Banach, the set $\text{Iso}(\mathbb{E}, \mathbb{F})$ is open in $\mathcal{L}(\mathbb{E}, \mathbb{F})$. Therefore, there exists a neighborhood $\widehat{\Omega}$ of (A_*, L_*) in $\mathcal{L}(\mathbb{E}, \mathbb{F})^2$, such that, for any $(A, L) \in \widehat{\Omega}$, $L + A$ is invertible. Thus,

$$\Omega = \widehat{\Omega} \cap (F\mathcal{L}(\mathbb{E}, \mathbb{F}) \times \mathcal{L}(\mathbb{E}, \mathbb{F}))$$

is a neighborhood of (A_*, L_*) in $FL(\mathbb{E}, \mathbb{F}) \times \mathcal{L}(\mathbb{E}, \mathbb{F})$. We claim that Ω is actually contained in $FL(\mathbb{E}, \mathbb{F}) \times \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. In fact, take $(A, L) \in \Omega$. Since $L + A$ is invertible, it is, in particular, Fredholm of index zero. Thus, Property 2 of Proposition 3.2 implies that $(L + A) - A$ is as well a Φ_0 -operator.

The assumption implies $(A_*, L_*) \in \Omega$, thus there exists a neighborhood $U \times W$ of (A_*, L_*) such that, for any $(A, L) \in U \times W$, A is an L -companion. $\square$

The following property of the sign of the extended determinant holds despite the fact that the function $\det_e : \mathcal{AL}(\mathbb{E}) \rightarrow \mathbb{R}$ is possibly discontinuous (see [9] for an example in the real Hilbert space ℓ^2).

Lemma 5.2 (*Sign stability of the extended determinant*) *The following subsets of $\mathcal{AL}(\mathbb{E})$ are relatively open:*

$$\mathcal{AL}(\mathbb{E})_+ = \{L : \det_e(L) > 0\} \text{ and } \mathcal{AL}(\mathbb{E})_- = \{L : \det_e(L) < 0\}.$$

Proof Since $\mathbb{E}$ is Banach, the subset $GL(\mathbb{E})$ of $\mathcal{L}(\mathbb{E})$ is open. Take $L_* \in \mathcal{AL}(\mathbb{E})_+$. This implies $L_* \in GL(\mathbb{E})$. Hence, there exists an open ball B , centered at L_* and contained in $GL(\mathbb{E})$. Consider any $L \in B$. It is enough to prove that $\det_e(L) > 0$.

Let $\mathbb{E}'$ be a finite dimensional subspace of $\mathbb{E}$ containing both the images of $I - L_*$ and $I - L$. So that, for any $t \in [0, 1]$, the (nonzero) extended determinant of $\Lambda_t = tL + (1 - t)L_*$ coincides with $\det(\Lambda_t|_{\mathbb{E}'})$, and the assertion follows from the continuity of the classical determinant. For $\mathcal{AL}(\mathbb{E})_-$ the argument is the same. $\square$

The next result (obtained in [7, Lemma 3.1]) shows that, in the context of Banach spaces, the L -equivalence relation introduced in Definition 4.1 enjoys a sort of complete stability. We present here a new proof.

Proposition 5.3 (*Triple stability property*) *Assume that A_* and B_* are two equivalent [non-equivalent] companions of an operator $L_* \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. Then, there exist three neighborhoods, U and V respectively of A_* and B_* in $FL(\mathbb{E}, \mathbb{F})$, and W of L_* in $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, such that, for any $A \in U$, $B \in V$ and $L \in W$, A and B are equivalent [non-equivalent] L -companions.*

Proof As a consequence of Lemma 5.1 there exist three neighborhoods, U and V of A_* and B_* in $FL(\mathbb{E}, \mathbb{F})$, and W of L_* in $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, such that, for any $A \in U$, $B \in V$ and $L \in W$, the operators $L + A$ and $L + B$ are invertible. Thus, for these operators, $(L + B)^{-1}(L + A)$ is a well defined det-admissible automorphism of $\mathbb{E}$.

Consequently, although the function $\det_e$ could be discontinuous, the assertion follows from Lemma 5.2 regarding the stability of the sign of such a function. $\square$

We introduce now the basic notion of *topological orientation* of a subset of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. Recall Definition 4.4 for the concept of algebraic orientation.

Definition 5.4 (*Pre-orientation and top-orientation of a subset of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$*) *A pre-orientation of a subset $\mathcal{Y}$ of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is a function ω that to any $L \in \mathcal{Y}$ assigns an algebraic orientation $\omega(L)$. Such a function is a topological orientation (or a top-orientation) of $\mathcal{Y}$ if, for any $L_* \in \mathcal{Y}$, there exists $A_* \in FL(\mathbb{E}, \mathbb{F})$ which belongs to $\omega(L)$ for any $L \in \mathcal{Y}$ sufficiently close to L_* . The set $\mathcal{Y}$ is top-orientable if it admits a top-orientation, and top-oriented if a top-orientation of $\mathcal{Y}$ has been given.*

From Lemma 5.1 we get an equivalent notion of top-orientation which, in some sense, is invariant under diffeomorphisms.

Remark 5.5 (Alternative notion of top-orientation of a subset of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$) Let ω be a pre-orientation of a subset $\mathcal{Y}$ of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. Such a set-valued function is a top-orientation of $\mathcal{Y}$ if, for any $L \in \mathcal{Y}$, ω admits a continuous selection defined on a neighborhood of L .

Observe that, if $\mathcal{Y} \subseteq \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is top-oriented, then any subset of $\mathcal{Y}$ receives an *induced top-orientation*.

Given $A \in F\mathcal{L}(\mathbb{E}, \mathbb{F})$, taking into account Lemma 5.1 and the assumption that $\text{Iso}(\mathbb{E}, \mathbb{F})$ is nontrivial and nonempty, we get that

$$\mathfrak{D}(A) := \{L \in \mathcal{L}(\mathbb{E}, \mathbb{F}) : L + A \in \text{Iso}(\mathbb{E}, \mathbb{F})\}$$

is a nonempty open subset of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, hereafter called *operator domain of A*.

Notice that a topological orientation of a subset $\mathcal{Y}$ of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ can be given by a covering of $\mathcal{Y}$ consisting of operator domains, as long as the covering is *consistent*; meaning that, whenever $L \in \mathcal{Y}$ belongs to the intersection of two operator domains, say $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$, then A and B are L -equivalent.

The following definition regards some relevant top-orientable sets of Φ_0 -operators. Their attributes are inherited from the special algebraic orientations in Definitions 4.5, 4.7 and 4.8.

Definition 5.6 (*Three special top-orientations*)

- (1) Whatever is the (non-zero) dimension of $\mathbb{E}$, the sets $\text{GL}(\mathbb{E})$ and $\text{Iso}(\mathbb{E}, \mathbb{F})$ are top-orientable and admit the *natural top-orientation*, obtained by assigning the natural alg-orientation to any operator (recall Definition 4.5).
- (2) If $\mathbb{E}$ and $\mathbb{F}$ have the same finite dimension, then $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, which in this case coincides with $\mathcal{L}(\mathbb{E}, \mathbb{F})$, is top-orientable. Indeed, choosing one of the two possible orientations of $\mathbb{E} \times \mathbb{F}$, one can assign, according to Definition 4.7, the associated alg-orientation to any operator of $\mathcal{L}(\mathbb{E}, \mathbb{F})$. This *top-orientation* will inherit the attribute of being *associated*.
- (3) Analogously, if $\mathbb{E}$ is finite dimensional, then $\mathcal{L}(\mathbb{E})$ admits the *canonical top-orientation*, inherited from Definition 4.8.

In spite of Definition 5.6, in which it is shown that the nonempty set $\text{Iso}(\mathbb{E}, \mathbb{F})$ is top-orientable, this could be untrue for the subset $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ of $\text{Iso}(\mathbb{E}, \mathbb{F})$. In fact, in [7, Theorem 3.15] it is proved that, when $\text{GL}(\mathbb{E})$ is connected, the containing set $\Phi_0\mathcal{L}(\mathbb{E})$ is as well connected but not top-orientable. Therefore, in the infinite dimensional situation, the closure in $\Phi_0\mathcal{L}(\mathbb{E})$ of a top-orientable subset of $\Phi_0\mathcal{L}(\mathbb{E})$ may not be top-orientable.

The connectedness of $\text{GL}(\mathbb{E})$ seems a peculiar situation. However, a surprising result of N. Kuiper (see [10, 19]) asserts that the subset $\text{GL}(\ell^2)$ of the real Hilbert space ℓ^2 is not only connected, but it is actually contractible.

As we shall see, an important convex subset of $\Phi_0\mathcal{L}(\mathbb{E})$ is top-orientable and admits a special top-orientation that we call *standard* (see Definition 5.17 below). It is the Leray–Schauder set $L\mathcal{S}\mathcal{L}(\mathbb{E})$ consisting of the compact linear perturbations of the identity (recall Property 2 of Proposition 3.5).

The following result regards an important property of the function sign_{bf} .

Lemma 5.7 (*Topological stability of sign_{bf}*) If $\mathcal{Y}$ is any top-oriented connected subset of $\text{Iso}(\mathbb{E}, \mathbb{F})$, then all its operators have the same non-zero sign_{bf} .

Proof If $\mathcal{Y}$ is open, the statement is a straightforward consequence of Definitions 4.10 and 5.4. Thus, the assertion follows from the fact that the top-orientation of $\mathcal{Y}$ can be extended to a connected open subset of $\text{Iso}(\mathbb{E}, \mathbb{F})$. $\square$

Observe that Lemma 5.7 implies the following assertion regarding the existence of at least one non-invertible Φ_0 -operator:

- If a connected top-oriented subset of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ has two operators with opposite sign_{bf} , then it admits at least one non-invertible operator.

To show this, let $\mathcal{Y}$ denote the given set and assume that the assertion is false. Thus, $\mathcal{Y}$ is contained in $\text{Iso}(\mathbb{E}, \mathbb{F})$. Therefore, Lemma 5.7 applies, getting a contradiction.

We proceed in this section with the following three subsections:

- (1) Maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$;
- (2) Flat Φ_0 -maps;
- (3) The two-fold covering space over $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ and applications.

The first subsection concerns continuous maps into the open subset $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ of the Banachable space $\mathcal{L}(\mathbb{E}, \mathbb{F})$. Many concepts of a map of this type will be defined in this subsection. An important one is the topological orientation.

The second deals with Φ_0 -maps of the type $f: U \rightarrow V$, where U and V are nonempty open subsets of $\mathbb{E}$ and $\mathbb{F}$, respectively. By definition, the attributes of such a map are those of the differential $df: U \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, according to the notions introduced in the first subsection.

The last one is dedicated to the construction of a two-fold covering space over $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. This turns out to be an useful tool for dealing with the top-orientation of the maps contemplated in the previous two subsections. As we shall see, an important application is the homotopic transport of the top-orientation.

5.1 Maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$

The following basic definition extends the notions of pre-orientation and top-orientation previously given for subsets of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Definition 5.8 (*Pre-oriented and top-oriented maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$*) Given a topological space $\mathcal{X}$ and a continuous map $\Gamma: \mathcal{X} \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, a *pre-orientation* of Γ is a function ω that to any $x \in \mathcal{X}$ assigns to $\Gamma(x)$ an alg-orientation $\omega(x)$. Such a set-valued function is a *top-orientation* if any $x \in \mathcal{X}$ admits a neighborhood which is the domain of a continuous selection of ω .

Observe that a subset $\mathcal{Y}$ of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is *top-orientable* or *top-oriented* (according to the two equivalent Definitions 5.4 and 5.5) if and only if so is the inclusion map $\mathcal{Y} \hookrightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. Moreover, if $\mathcal{Y} \subseteq \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is top-orientable, then so is any continuous map from a topological space into $\mathcal{Y}$. In particular, any constant map $\Gamma: \mathcal{X} \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is top-orientable.

We introduce now the concept of homotopy of maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Definition 5.9 ($\Phi_0\mathcal{L}$ -homotopy) Given a topological space $\mathcal{X}$, a continuous map $\mathcal{H}$ from $\mathcal{X} \times [0, 1]$ into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ is called a $\Phi_0\mathcal{L}$ -homotopy. Given a parameter $t \in [0, 1]$, $\mathcal{H}_t$ denotes the *partial map* $\mathcal{H}(\cdot, t): \mathcal{X} \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ of $\mathcal{H}$ at t . Moreover, if ω is a top-orientation of $\mathcal{H}$, ω_t is the *partial top-orientation* of $\mathcal{H}$ at t (also regarded as the restriction of ω to the set $\mathcal{X} \times \{t\}$). Any two partial maps are said to be $\Phi_0\mathcal{L}$ -homotopic.

5.2 Flat Φ_0 -maps

By a flat Φ_0 -map we mean a function $f \in \Phi_0(U, V)$, where U and V are open sets of $\mathbb{E}$ and $\mathbb{F}$, respectively.

Since the differential of f is a continuous map into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, the following definition makes sense.

Definition 5.10 (*Pre-orientation and top-orientation of a flat Φ_0 -map*)

A *pre-orientation* or a *top-orientation* of a map $f \in \Phi_0(U, V)$ is, respectively, a pre-orientation or a top-orientation of $df: U \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, according to Definition 5.8.

An important example of a flat Φ_0 -map is a *compact vector field of class C^1* defined on an open subset U of $\mathbb{E}$. Namely, a C^1 -map $f: U \rightarrow \mathbb{E}$ with the property that the *perturbing map* $I - f$ is compact; meaning that it sends bounded sets inside compact sets. In fact, as already pointed out in Sect. 2, the differential at one point of a compact map of class C^1 is a compact linear operator. Consequently, $df: U \rightarrow \mathbb{E}$ is a Φ_0 -map, because of Property 2 of Proposition 3.5.

The following important concept regarding homotopies of flat Φ_0 -maps is related to Definition 5.9.

Definition 5.11 (*Φ_0 -homotopy*) Let U and V be two open sets of $\mathbb{E}$ and $\mathbb{F}$, respectively, and assume that $\mathcal{K}: U \times [0, 1] \rightarrow V$ is a continuous map which is C^1 with respect to the first variable. Given a parameter $t \in [0, 1]$, let $\mathcal{K}_t$ denote the partial map $\mathcal{K}(\cdot, t)$. The function $\mathcal{K}$ is called a Φ_0 -homotopy if its partial differential with respect to the first variable, defined by

$$\partial_1 \mathcal{K}: U \times [0, 1] \rightarrow \mathcal{L}(\mathbb{E}, \mathbb{F}), \quad (x, t) \mapsto d(\mathcal{K}_t)_x,$$

is a $\Phi_0\mathcal{L}$ -homotopy. Any two partial maps of $\mathcal{K}$ are said to be Φ_0 -homotopic.

Observe that two partial maps of $\mathcal{K}$ are Φ_0 -homotopic if and only if the corresponding two partial maps of $\mathcal{H} := \partial_1 \mathcal{K}$ are $\Phi_0\mathcal{L}$ -homotopic.

5.3 Flat two-fold covering space and applications

We need the following straightforward consequence of Proposition 5.3.

Corollary 5.12 (*Double local top-orientability of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$*) Let A_* and B_* be two non-equivalent companions of an operator $L_* \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$. Then, Proposition 5.3 ensures the existence of a neighborhood W of L_* such that, for any $L \in W$, A_* and B_* are non-equivalent L -companions. Therefore, one gets two opposite top-orientations of W : ω_a and ω_b , having the property that $A_* \in \omega_a(L)$ and $B_* \in \omega_b(L)$, for any $L \in W$.

Remark 5.13 (*Alternative double local top-orientability of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$*) Notice that, in the assertion of Corollary 5.12, the L_* -companions A_* and B_* may be regarded as non-equivalent (constant) selections of the set-valued function $\mathcal{C}|_W$ that to any $L \in W$ assigns the set $\mathcal{C}(L)$ of the L -companions. An equivalent and less restrictive assertion, which, in some sense, is invariant under diffeomorphisms of open sets of $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, can again be deduced from Proposition 5.3 simply replacing A_* and B_* with two pointwise non-equivalent continuous selections of $\mathcal{C}|_W$. In this way we get two different orientations of W in the spirit of Remark 5.5.

Hereafter $\widehat{\Phi}_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ denotes the set of the alg-oriented Φ_0 -operators of $\mathcal{L}(\mathbb{E}, \mathbb{F})$. That is,

$$\widehat{\Phi}_0\mathcal{L}(\mathbb{E}, \mathbb{F}) = \{(L, \alpha) : L \in \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F}), \alpha \text{ is an alg-orientation of } L\}.$$

Corollary 5.12 implies the following assertion (compare with [7, Section 3] where, however, a precise statement is not given).

Proposition 5.14 (The two-fold covering space over $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$)

The set $\widehat{\Phi}_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ has a unique topology which makes the natural projection

$$\mathcal{P}: \widehat{\Phi}_0\mathcal{L}(\mathbb{E}, \mathbb{F}) \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F}), (L, \alpha) \mapsto L,$$

a two-fold covering space.

Thus, a remark follows.

Remark 5.15 (Top-oriented maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ by means of the covering space) Given a topological space $\mathcal{X}$ and a continuous map $\Gamma: \mathcal{X} \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$, the second component ω of a lift

$$x \in \mathcal{X} \mapsto \widehat{\Gamma}(x) = (\Gamma(x), \omega(x)) \in \widehat{\Phi}_0\mathcal{L}(\mathbb{E}, \mathbb{F})$$

of Γ is a top-orientation of Γ according to Definition 5.8.

The top-orientability criterion below can be deduced from Theorem 3.7 on the theory of covering spaces. Recall that any simply connected space is assumed to be connected.

Proposition 5.16 (Top-orientability criterion for the maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$) *If a topological space $\mathcal{X}$ is simply connected and locally path connected, then any continuous map $\Gamma: \mathcal{X} \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ admits exactly two top-orientations. Moreover, given $x \in \mathcal{X}$, if α is any of the two alg-orientations of $\Gamma(x)$, then there exists one and only one top-orientation ω of Γ such that $\omega(x) = \alpha$.*

For example, the above consequence of the theory of covering spaces can be applied to the Leray–Schauder set $LS\mathcal{L}(\mathbb{E})$ of the compact linear perturbations of the identity. In fact, this set, being convex (it is actually an affine space), is as well simply connected and locally path connected. Therefore it is top-orientable and admits exactly two top-orientations. One of them, according to the following definition, will be considered more relevant than the other.

Definition 5.17 (Standard top-orientation of $LS\mathcal{L}(\mathbb{E})$) The unique topological orientation of $LS\mathcal{L}(\mathbb{E})$, which coincides with the natural algebraic orientation at the identity, will be called *standard*. Consequently, an important subset of $LS\mathcal{L}(\mathbb{E})$ admits the standard orientation, and this is $\mathcal{AL}(\mathbb{E})$.

The next result (see also [4, Proposition 2.13]) regards an important property of the function sign_{bf} on the set of det-admissible linear operators.

Lemma 5.18 (The function sign_{bf} on $\mathcal{AL}(\mathbb{E})$ with the standard top-orientation) *If $\mathcal{AL}(\mathbb{E})$ has the standard top-orientation, then, given any $L = I - H \in \mathcal{AL}(\mathbb{E})$, one has $\text{sign}_{bf} L = \text{sign}(\det_e(L))$. Therefore, as in the finite dimensional case, if the eigenvalues of the perturbing operator H are counted with their multiplicity and L is invertible, then its sign_{bf} is $(-1)^\rho$ where ρ is the number of the eigenvalues bigger than 1 of the perturbing operator H or, equivalently, the number of the characteristic values of H in the interval $(0, 1)$.*

Proof The assertions are easy consequences of the analogous ones in the finite dimensional case, taking into account the definition of extended determinant. $\square$

Recalling Definition 5.9, from Theorem 3.8 one can deduce the following result, given in [7], on the homotopic transport of the top-orientation for maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$.

Proposition 5.19 (Homotopic transport of the top-orientation for maps into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$) *Given a topological space $\mathcal{X}$, let $\mathcal{H}: \mathcal{X} \times [0, 1] \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ be a homotopy. Assume that a partial map $\mathcal{H}_t: \mathcal{X} \rightarrow \Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ of $\mathcal{H}$ has a top-orientation α . Then, there exists one and only one top-orientation ω of $\mathcal{H}$ whose partial top-orientation ω_t coincides with α . In particular, if two maps from $\mathcal{X}$ into $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ are $\Phi_0\mathcal{L}$ -homotopic, then they are both top-orientable or both not top-orientable.*

Regarding the flat Φ_0 -maps, we get some useful results related to the theory of covering spaces. As before U and V are open subsets of real Banach spaces.

The following result is a consequence of Theorem 3.7 and Definition 5.10.

Theorem 5.20 (Top-orientation of a flat Φ_0 -map) *If the domain U of a function $f \in \Phi_0(U, V)$ is simply connected, then f admits exactly two top-orientations, opposite to each other.*

The next notion makes sense because of Definitions 5.10 and 5.17.

Definition 5.21 (Standard top-orientation of a C^1 compact vector field) *Given an open subset U of $\mathbb{E}$ and a C^1 compact vector field f on U , the standard top-orientation of f is the one induced on $df: U \rightarrow L\mathcal{S}\mathcal{L}(\mathbb{E})$ by the standard top-orientation of $L\mathcal{S}\mathcal{L}(\mathbb{E})$.*

As a straightforward consequence of Proposition 5.19 we get a result on the homotopic transport of the top-orientation for flat Φ_0 -maps. Recall Definition 5.11 for the notion of Φ_0 -homotopy.

Theorem 5.22 (Homotopic transport of the top-orientation for flat Φ_0 -maps) *Assume that a partial map $\mathcal{K}_t: U \rightarrow V$ of a Φ_0 -homotopy $\mathcal{K}$ has a top-orientation α . Then, there exists one and only one top-orientation ω of $\mathcal{K}$ whose partial top-orientation ω_t coincides with α . In particular, if two flat Φ_0 -maps from U into V are Φ_0 -homotopic, then they are both top-orientable or both not top-orientable.*

6 Non-flat topological orientation

In this section we present the concept of topological orientation in the non-flat setting (see [2, 6, 7]).

Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, we consider the set of triples

$$\mathcal{L}^+(\mathcal{M}, \mathcal{N}) = \{(p, q, L) : p \in \mathcal{M}, q \in \mathcal{N}, L \in \mathcal{L}(T_p\mathcal{M}, T_q\mathcal{N})\},$$

that, in some sense, can be regarded as a non-flat extension of the space $\mathcal{L}(\mathbb{E}, \mathbb{F})$.

This set will be given the structure of real Banach manifold modeled on the Banachable space $\mathbb{G} = \mathbb{E} \times \mathbb{F} \times \mathcal{L}(\mathbb{E}, \mathbb{F})$. In fact, let $\phi: U \rightarrow \mathbb{E}$ be a chart of $\mathcal{M}$ and $\psi: V \rightarrow \mathbb{F}$ a chart of $\mathcal{N}$, then the map

$$(\phi, \psi)^+: \mathcal{L}^+(U, V) \rightarrow \mathbb{G}, \quad (p, q, L) \mapsto (\phi(p), \psi(q), d\psi_q L (d\phi_p)^{-1}),$$

is a *chart* of $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$. The collection of these maps is an *atlas* giving $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ the structure of a manifold.

From now on, we will assume that the open subset

$$\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}) = \{(p, q, L) : p \in \mathcal{M}, q \in \mathcal{N}, L \in \Phi_0\mathcal{L}(T_p\mathcal{M}, T_q\mathcal{N})\}$$

of $\mathcal{L}^+(\mathcal{M}, \mathcal{N})$, considered in [7], is nonempty. This set may be regarded as the non-flat analogous of the flat manifold $\Phi_0\mathcal{L}(\mathbb{E}, \mathbb{F})$ previously introduced.

Here is the basic notion of topological orientation of a subset of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$.

Definition 6.1 (*Pre-orientation and top-orientation of a subset of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$*) Let $\mathcal{Y}^+ \subseteq \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$. A *pre-orientation* of $\mathcal{Y}^+$ is a function ω that to any $r^+ = (p, q, L) \in \mathcal{Y}^+$ assigns an algebraic orientation $\omega(r^+)$ of L . Such a set-valued function is, in addition, a *topological orientation* (or a *top-orientation*) of $\mathcal{Y}^+$, if any element of $\mathcal{Y}^+$ admits a relative neighborhood which is the domain of a continuous selection of ω . The set $\mathcal{Y}^+$ is *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation of $\mathcal{Y}^+$ has been given.

Hereafter any nonempty open subset of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ will be called a *patch* (of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$), so that a patch could be top-oriented or not top-oriented, top-orientable or not top-orientable. Obviously, any nonempty open subset V^+ of a top-oriented patch U^+ inherits a top-orientation from U^+ . Equivalently, we may say that, if V^+ is connected, the top-orientation of V^+ is *induced* by that of U^+ . A collection of top-oriented patches will be said *compatible* if any two patches either are disjoint or induce the same top-orientation on the intersection. Observe that $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is top-orientable if and only if it admits a compatible covering of top-oriented patches.

Analogously to the above section, also in this one we proceed, for the convenience of the reader, with three subsections:

- (1) Maps into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$;
- (2) Non-flat Φ_0 -maps;
- (3) Non-flat two-fold covering space and applications.

The first concerns continuous maps into the manifold $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ and the related notions of pre-orientation and top-orientation for these maps. The most significant results concerning such maps will be obtained in the last subsection by means of the covering space theory. An important one is the homotopic transport of the top-orientation.

The second deals with Φ_0 -maps between manifolds (recall Definition 3.6), whose notion of top-orientation will be given in this subsection by using charts (see [6]). This simple approach has the advantage of reducing the problem to the flat case, but the disadvantage of not utilizing the powerful covering space theory in order to obtain important properties of the concept of top-orientation that will be discussed in the third subsection.

Finally, in the last subsection we apply the covering space theory to get useful results (some of them obtained in [7]) on the top-orientation and top-orientability of the maps considered in the previous two subsections.

6.1 Maps into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$

The following is an extension of the concept of topological orientation given in Definition 6.1 for subsets of the manifold $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$.

Definition 6.2 (*Pre-orientation and top-orientation of maps into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$*) Let $\mathcal{X}$ be a topological space and $\Gamma^+ : \mathcal{X} \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ a continuous map. Given any $x \in \mathcal{X}$,

consider the triple $\Gamma^+(x) = (p_x, q_x, L_x)$. A *pre-orientation* of Γ^+ is a function ω that for any $x \in \mathcal{X}$ assigns to L_x an alg-orientation $\omega(x)$. Such a set-valued function is a *top-orientation* if any $x \in \mathcal{X}$ admits a neighborhood which is the domain of a continuous selection of ω .

Similarly to the flat case, a subset $\mathcal{Y}^+$ of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is *top-orientable* or *top-oriented*, according to Definition 6.1, if and only if so is the inclusion map $\mathcal{Y}^+ \hookrightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$.

Here is the notion of homotopy of maps into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$.

Definition 6.3 ($\Phi_0\mathcal{L}^+$ -homotopy) Given a topological space $\mathcal{X}$, a continuous map $\mathcal{H}^+$ from $\mathcal{X} \times [0, 1]$ into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is called a $\Phi_0\mathcal{L}^+$ -homotopy. Given a parameter $t \in [0, 1]$, $\mathcal{H}_t^+$ denotes the *partial map* $\mathcal{H}^+(\cdot, t): \mathcal{X} \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ of $\mathcal{H}^+$ at t . Moreover, if ω is the top-orientation of $\mathcal{H}^+$, ω_t denotes the *partial top-orientation* of $\mathcal{H}^+$ at t (also regarded as the restriction of ω to the set $\mathcal{X} \times \{t\}$). Any two partial maps are said to be $\Phi_0\mathcal{L}^+$ -homotopic.

6.2 Non-flat Φ_0 -maps

Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be a Φ_0 -map between manifolds. The following definition regards the simple notion of pre-orientation of f .

Definition 6.4 (*Pre-orientation of a non-flat Φ_0 -map*) A *pre-orientation* of f is a function that to any $p \in \mathcal{M}$ assigns an alg-orientation to the differential $df_p: T_p\mathcal{M} \rightarrow T_{f(p)}\mathcal{N}$.

The notion of pre-oriented composition for Φ_0 -maps is as follows.

Definition 6.5 (*Pre-oriented composition of two pre-oriented Φ_0 -maps*) Given two pre-oriented Φ_0 -maps, $f: \mathcal{M} \rightarrow \mathcal{N}$ and $g: \mathcal{N} \rightarrow \mathcal{O}$, the pre-orientation of the composite map $g \circ f$ is given pointwise; that is, by assigning to any $p \in \mathcal{M}$ the alg-orientation, according to Definition 4.9, of the composite operator $dg_q df_p \in \mathcal{L}(T_p\mathcal{M}, T_r\mathcal{O})$, where $q = f(p)$ and $r = g(q)$.

According to the comment after Definition 4.9, the above composition is associative, so it makes sense the pre-oriented composition of more than two pre-oriented non-flat Φ_0 -maps.

Whenever it makes sense, a pre-orientation of f inherits some additional properties from the corresponding properties of the alg-orientation of its differential. More precisely, from Definitions 4.5, 4.7 and 4.8 one gets three particular pre-oriented non-flat Φ_0 -maps.

Definition 6.6 (*Three special pre-oriented non-flat Φ_0 -maps*) A pre-orientation of f will be called *natural*, *associated* and *canonical* if the differential of f has the corresponding property. In particular, the last one, which is a special case of the associated pre-orientation, concerns only the case of a self-map of a connected, orientable, finite dimensional manifold.

Unless otherwise specified, in the composition of pre-oriented Φ_0 -maps, any diffeomorphism will be tacitly considered naturally pre-oriented.

Definition 6.7 (*Top-orientation of a non-flat Φ_0 -map by means of charts*) Let f be a Φ_0 -map between two manifolds $\mathcal{M}$ and $\mathcal{N}$. A pre-orientation of f is a *top-orientation* if, given any two naturally pre-oriented charts $\psi: V \rightarrow \mathbb{F}$ of $\mathcal{N}$ and $\phi: U \rightarrow \mathbb{E}$ of $\mathcal{M}$ such that $f(U) \subseteq V$, the flat pre-oriented composition

$$\psi \circ f \circ \phi^{-1}: \phi(U) \rightarrow V$$

is a top-oriented map, according to Definition 5.10.

The reader can check that an equivalent notion of top-orientation of a Φ_0 -map $f: \mathcal{M} \rightarrow \mathcal{N}$ can be given by considering the map

$$df^+: \mathcal{M} \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}), \text{ defined by } p \mapsto (p, f(p), df_p),$$

and applying Definition 6.2.

The next definition regards three types of top-orientations for non-flat Φ_0 -maps. They derive from Definition 6.6 and are still called *natural*, *associated* and *canonical*. The latter is a special case of the associated orientation, and will be considered only if f is a self-map of a connected, finite dimensional, orientable manifold $\mathcal{M}$. The name “canonical” is due to the fact that the associated top-orientation of $f \in \Phi_0(\mathcal{M})$ does not depend on the chosen orientation of $\mathcal{M}$.

Definition 6.8 (*Three special top-oriented Φ_0 -maps*)

(Natural) A special top-orientation of a non-flat Φ_0 -map is the *natural* one, which makes sense whenever $f: \mathcal{M} \rightarrow \mathcal{N}$ is a diffeomorphism (or, more generally, a local diffeomorphism): given any $x \in \mathcal{M}$, according to Definition 4.5, one assigns the natural alg-orientation to the differential df_x .

(Associated) Given a C^1 map $f: \mathcal{M} \rightarrow \mathcal{N}$ between two finite dimensional oriented manifolds of the same dimension, one can assign to f the top-orientation which is *associated to the orientations of $\mathcal{M}$ and $\mathcal{N}$* : to any differential df_x one assigns the alg-orientation associated to the orientations of $T_x\mathcal{M}$ and $T_{f(x)}\mathcal{N}$, according to Definition 4.7.

(Canonical) If f is a self-map of a connected, orientable, finite dimensional manifold $\mathcal{M}$, one can assign to f the *canonical top-orientation*: the one which is associated to any orientation of $\mathcal{M}$, provided that it is the same as domain and as codomain of f .

6.3 Non-flat two-fold covering space and applications

In this subsection we define a two-fold covering space whose *base space* is the manifold $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ and the *total space* the set $\widehat{\Phi}_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ given by

$$\{(p, q, L, \alpha) : (p, q, L) \in \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}), \alpha \text{ is an alg-orientation of } L\}.$$

We will show that this set is as well a manifold and the *natural projection* $\mathcal{P}^+: \widehat{\Phi}_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}) \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is a two-fold covering space.

For this purpose we need the following non-flat local orientability result, which is a consequence of Corollary 5.12, taking into account Remark 5.13, Definition 6.1, and the fact that $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is locally diffeomorphic to a flat manifold.

Corollary 6.9 (*Double local top-orientability of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$*) Consider the set-valued map $\mathcal{C}$ that to any $r^+ = (p, q, L) \in \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ assigns the set $\mathcal{C}(L)$ of the companions of L . Then, each $r^+ \in \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ admits a neighborhood which is the domain of two pointwise non-equivalent continuous selections of $\mathcal{C}$ (in the sense of Remark 5.13).

Corollary 6.9 implies the following assertion (see [7, Section 3]).

Proposition 6.10 (The two-fold covering space over $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$)

The set $\widehat{\Phi}_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is a manifold whose corresponding natural projection

$$\mathcal{P}^+: \widehat{\Phi}_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}) \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}), \quad (p, q, L, \alpha) \mapsto (p, q, L)$$

is a two-fold covering space.

Because of Proposition 6.10, the following notion of topological orientation for maps into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ makes sense, and is equivalent to the one given in Definition 6.2.

Definition 6.11 (*Covering notion of top-orientation for a Γ^+ -map*) Let Γ^+ be a continuous map from a topological space $\mathcal{X}$ into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$. A *top-orientation* of Γ^+ is the second component ω of a lift $\widehat{\Gamma}^+ = (\Gamma^+, \omega): \mathcal{X} \rightarrow \widehat{\Phi}_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$. The map Γ^+ is called *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen. A subset $\mathcal{Y}^+$ of $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ is *top-orientable* or *top-oriented* if so is the inclusion map $\mathcal{Y}^+ \hookrightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$.

The following orientability criterion can be deduced from Theorem 3.7 on the theory of covering spaces.

Proposition 6.12 (*Top-orientability criterion for a Γ^+ -map*) Let Γ^+ be a map into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ defined on a simply connected and locally path connected topological space. Then Γ^+ admits exactly two (opposite) top-orientations.

Recalling Definition 6.3, Theorem 3.8 implies the following result on the homotopic transport of the top-orientation for maps into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$.

Proposition 6.13 ($\Phi_0\mathcal{L}^+$ -homotopic transport of the top-orientation) Given a topological space $\mathcal{X}$, consider the homotopy $\mathcal{H}^+: \mathcal{X} \times [0, 1] \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$. Assume that a partial map $\mathcal{H}_t^+: \mathcal{X} \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ of $\mathcal{H}^+$ has a top-orientation α . Then, there exists one and only one top-orientation ω of $\mathcal{H}^+$ whose partial top-orientation ω_t coincides with α . In particular, if two maps from $\mathcal{X}$ into $\Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N})$ are $\Phi_0\mathcal{L}^+$ -homotopic, then they are both top-orientable or both not top-orientable.

The following notion of top-orientation of a non-flat Φ_0 -map is equivalent to Definition 6.7 given by means of charts. However, this alternative point of view turns out to be useful for deducing important properties based on the theory of covering spaces.

Definition 6.14 (*Top-orientation of a non-flat Φ_0 -map by means of covering spaces*) Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be a Φ_0 -map between two manifolds. A *top-orientation* of f is, according to Definition 6.11, a top-orientation of the map

$$df^+: \mathcal{M} \rightarrow \Phi_0\mathcal{L}^+(\mathcal{M}, \mathcal{N}), \quad x \mapsto (x, f(x), df_x).$$

As a direct consequence of Proposition 6.12 and Definition 6.14 we get the following top-orientability criterion for Φ_0 -maps between manifolds. Recall that a simply connected topological space is assumed to be connected.

Proposition 6.15 (*Top-orientability criterion for Φ_0 -maps between manifolds*) Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be a Φ_0 -map between two manifolds. If $\mathcal{M}$ is simply connected, then f admits exactly two (opposite) top-orientations.

Perhaps, the simplest example of a not top-orientable Φ_0 -map is a constant self-map f of the 2-dimensional real projective space $\mathbb{P}^2$ (see [7]). This is a consequence of the fact that even dimensional real projective spaces are not orientable. Observe that, unlike in the flat case, here the map that to each $x \in \mathbb{P}^2$ assigns the differential df_x is not constant.

Now we introduce the concept of Φ_0 -homotopy between two manifolds, as well as the notion of its top-orientation.

Definition 6.16 (*Homotopy of Φ_0 -maps between manifolds*) Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, a homotopy $\mathcal{H}: \mathcal{M} \times [0, 1] \rightarrow \mathcal{N}$ is a Φ_0 -homotopy if it is continuously differentiable with respect to the first variable, and such that, given any parameter $t \in [0, 1]$, the partial map $\mathcal{H}_t$ belongs to $\Phi_0(\mathcal{M}, \mathcal{N})$. Any two partial maps of a Φ_0 -homotopy are said to be Φ_0 -homotopic. A top-orientation of $\mathcal{H}$ is, according to Definition 6.11, a top-orientation of the map

$$(\partial_1 \mathcal{H})^+ : \mathcal{M} \times [0, 1] \rightarrow \Phi_0 \mathcal{L}^+(\mathcal{M}, \mathcal{N}), \quad (x, t) \mapsto (x, \mathcal{H}(x, t), d(\mathcal{H}_t)_x),$$

where ∂_1 denotes the partial differential with respect to the first variable.

Let ω be a top-orientation of a Φ_0 -homotopy $\mathcal{H}: \mathcal{M} \times [0, 1] \rightarrow \mathcal{N}$ and observe that, given any parameter $t \in [0, 1]$, $\omega_t := \omega(\cdot, t)$ is a top-orientation of the partial map $\mathcal{H}_t$, called the *partial top-orientation of $\mathcal{H}$ at t* . Conversely, one has the following consequence of Theorem 3.8 on the theory of covering spaces (see [7]).

Proposition 6.17 (*Homotopic transport of the top-orientation for Φ_0 -maps*) *Let α be a top-orientation of a partial map $\mathcal{H}_t: \mathcal{M} \rightarrow \mathcal{N}$ of a Φ_0 -homotopy $\mathcal{H}$. Then, there exists one and only one top-orientation ω of $\mathcal{H}$ whose partial top-orientation ω_t coincides with α . Consequently, if two maps from $\mathcal{M}$ into $\mathcal{N}$ are Φ_0 -homotopic, then they are either both top-orientable or both not top-orientable.*

Recall that the identity map of any manifold is top-orientable. In fact, it admits the natural top-orientation (see Definition 6.8). Therefore, Proposition 6.17 implies that any self-map of a manifold which is Φ_0 -homotopic to the identity is top-orientable. This property agrees with a well known fact: if a compact connected manifold is not orientable, then it is not contractible.

7 Topological degree

Here we present some applications of the concept of top-orientation to a special topological degree for Φ_0 -maps between real Banach manifolds, whose construction goes back to the articles [6, 7]. Such a degree is an integer-valued function, here called *bf-degree* to distinguish it from other degrees, whose domain is a family of triples, called *admissible* (see Definition 7.1 below).

As in the previous sections, $\mathcal{M}$ and $\mathcal{N}$ are real Banach manifolds, while $\mathbb{E}$ and $\mathbb{F}$ are real Banach spaces, possibly considered as flat manifolds. Obviously, the symbols $\mathcal{M}$ and $\mathcal{N}$ are generic: they could represent open subsets of manifolds, or open subsets of real Banach spaces. As before, we assume that, given any $x \in \mathcal{M}$ and any $y \in \mathcal{N}$, the set $\text{Iso}(T_x \mathcal{M}, T_y \mathcal{N})$ is nonempty.

Definition 7.1 (*admissible triples*) Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, a triple (f, U, y) is *admissible* (for the *bf-degree*), if the following conditions are satisfied:

- f is a continuous $\mathcal{N}$ -valued map defined on a subset $\mathcal{D}(f)$ of $\mathcal{M}$;
- U is a possibly empty open subset of $\mathcal{D}(f)$;
- the restriction of f to U is a top-oriented Φ_0 -map;
- $y \in \mathcal{N}$ is such that $U \cap f^{-1}(y)$ is a possibly empty compact set.

7.1 The axiomatic approach

As in the work of Amann–Weiss [1] regarding the uniqueness of the Leray–Schauder degree (hereafter called LS-degree) satisfying three axioms (Normalization, Additivity and Homotopy Invariance), in [8, Section 5] it is shown that as well the bf-degree is the unique integer valued function, $\deg_{bf}$, defined on the class of the admissible triples and satisfying the following three *Fundamental Properties*:

- (Normalization) If $f: U \rightarrow \mathcal{N}$ is a naturally top-oriented diffeomorphism onto an open subset of $\mathcal{N}$, then

$$\deg_{bf}(f, U, y) = 1, \quad \forall y \in f(U).$$

- (Additivity) Let (f, U, y) be an admissible triple. If U_1 and U_2 are two disjoint open subsets of U such that $U \cap f^{-1}(y) \subseteq U_1 \cup U_2$, then

$$\deg_{bf}(f, U, y) = \deg_{bf}(f|_{U_1}, U_1, y) + \deg_{bf}(f|_{U_2}, U_2, y).$$

- (Homotopy Invariance) Suppose that $\mathcal{H}: U \times [0, 1] \rightarrow \mathcal{N}$ is a top-oriented Φ_0 -homotopy and $\gamma: [0, 1] \rightarrow \mathcal{N}$ is a continuous path. If the set

$$\{(x, t) \in U \times [0, 1] : \mathcal{H}(x, t) = \gamma(t)\}$$

is compact, then $\deg_{bf}(\mathcal{H}(\cdot, t), U, \gamma(t))$ does not depend on $t \in [0, 1]$.

7.2 Consequences of the axiomatic approach

Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be a Φ_0 -map. It is folklore to call *point* any element of the domain $\mathcal{M}$ of f and *value* any element of the codomain $\mathcal{N}$. A point x is said to be *regular* if the differential $df_x: T_x \mathcal{M} \rightarrow T_{f(x)} \mathcal{N}$ is an isomorphism, and *critical* otherwise. An element $y \in \mathcal{N}$ is a *regular value* if $f^{-1}(y)$ does not contain critical points, otherwise it is a *critical value*. The celebrated Sard's Lemma [24] implies that, in the finite dimensional case, the set of regular values of f is dense in $\mathcal{N}$. As pointed out in [6], the same statement can be deduced, by an appropriate finite dimensional reduction, also in the infinite dimensional situation (see [25] for some particular infinite dimensional cases).

Here are some useful consequences of the three *Fundamental Properties* of the bf-degree (see [8] for details).

- (Localization) If (f, U, y) is admissible, then so is $(f|_U, U, y)$ and

$$\deg_{bf}(f|_U, U, y) = \deg_{bf}(f, U, y).$$

- (Excision) If (f, U, y) is admissible and V is an open subset of U such that $U \cap f^{-1}(y) \subseteq V$, then

$$\deg_{bf}(f, V, y) = \deg_{bf}(f, U, y).$$

- (Existence) If (f, U, y) is admissible and $\deg_{bf}(f, U, y) \neq 0$, then the equation $f(x) = y$ admits at least one solution in U .
- (Computation Formula) If (f, U, y) is admissible and y is a regular value for f on U , then the set $U \cap f^{-1}(y)$ is finite and

$$\deg_{bf}(f, U, y) = \sum_{x \in U \cap f^{-1}(y)} \text{sign}_{bf} df_x, \quad (7.1)$$

with the convention that the sum is zero if $U \cap f^{-1}(y)$ is empty.

Observe that, if $U \cap f^{-1}(y) = \emptyset$, the above convention regarding the sum agrees with the following two facts: y is a regular value for f in U ; the integer $\deg_{bf}(f, U, y)$ is zero as a consequence of the Existence Property.

Sometimes the class of the admissible triples is too broad for some applications. Therefore, a distinguished subclass is defined as follows. Recall that a proper map between metric spaces sends closed sets onto closed sets.

Definition 7.2 (*Strictly admissible triples*) An admissible triple (f, U, y) is *strictly admissible* if it satisfies the following three additional properties:

- the closure $\overline{U}$ of U is contained in the domain $\mathcal{D}(f)$ of f ;
- the map f is proper on $\overline{U}$;
- the target value y belongs to the open set $\mathcal{N} \setminus f(\partial U)$.

Observe that, if (f, U, y) is strictly admissible, then the compact set $\overline{U} \cap f^{-1}(y)$ is contained in U , therefore the integer $\deg_{bf}(f, U, y)$ is well defined.

For the strictly admissible triples one gets the following straightforward consequence of the Homotopy Invariance Property.

- (Continuous Dependence) The integer valued function $\deg_{bf}(f, U, \cdot)$ is constant on any connected component of the open set $\mathcal{N} \setminus f(\partial U)$.

Moreover, as one can check, for the strictly admissible triples, the Homotopy Invariance Property admits the following special formulation (recall Definition 6.16).

- (Special Homotopy Invariance) Let U be an open subset of $\mathcal{M}$ and $y \in \mathcal{N}$. Assume that $\mathcal{H}: \overline{U} \times [0, 1] \rightarrow \mathcal{N}$ is a proper map whose restriction to $U \times [0, 1]$ is a top-oriented Φ_0 -homotopy. If

$$\mathcal{H}(x, t) \neq y, \text{ for all } (x, t) \in \partial U \times [0, 1],$$

then $\deg_{bf}(\mathcal{H}(\cdot, t), U, y)$ does not depend on $t \in [0, 1]$.

As a simple example of an admissible triple which is not strictly admissible consider $(\exp, \mathbb{R}, y)$, where the exponential function is canonically top-oriented (recall Definition 6.8) and $y \in \mathbb{R}$ is arbitrary. Observe that $\exp$ is not a proper map and $\deg_{bf}(\exp, \mathbb{R}, \cdot)$ is discontinuous at $0 \in \mathbb{R}$.

Suppose that an admissible triple (f, U, y) satisfies the following properties:

- $U = \mathcal{M} = \mathcal{N}$,
- f is a proper map,
- $\mathcal{M}$ is connected.

Then, $(f, U, y) = (f, \mathcal{M}, y)$ is strictly admissible, $\partial \mathcal{M} = \emptyset$, and the above Continuous Dependence Property implies that $\deg_{bf}(f, \mathcal{M}, y)$ does not depend on y . Therefore, in this case, the integer $\deg_{bf}(f, \mathcal{M}, y)$ could simply be denoted by $\deg_{bf}(f)$. With this notation, no information is lost, since, formally, a map is a triple: domain, codomain and graph.

A significant example of the above simplified notation is when $\mathcal{M}$ is the complex plane $\mathbb{C}$ regarded as $\mathbb{R}^2$ and f is a complex polynomial P with positive algebraic degree that, being P a self map of a connected orientable finite dimensional space, we tacitly assume to be canonically top-oriented (recall Definition 6.8). In this case we have the following result whose proof, given in [5, Section 2.6], does not require the Fundamental Theorem of Algebra.

Theorem 7.3 (Equality between the bf-degree and the algebraic degree of a non-constant polynomial) *Let P be a complex polynomial of positive algebraic degree. Then, regarding $\mathbb{C}$ as $\mathbb{R}^2$, P is a proper map and $\deg_{bf}(P)$ is the same as its algebraic degree. Consequently, because of the existence property of the bf-degree, P has at least one root.*

Proof The map P is proper, since $\lim_{|z| \rightarrow \infty} |P(z)| = \infty$. Hence, $\deg_{bf}(P)$ is a well defined integer. Let $n > 0$ be the algebraic degree of P . Given any $z \in \mathbb{C}$, we have $P(z) = az^n + Q(z)$, with $a \in \mathbb{C}$ different from zero and Q a (possibly null) polynomial of algebraic degree less than n .

The same proof showing that P is a proper map can be adapted to the homotopy $\mathcal{H}: \mathbb{C} \times [0, 1] \rightarrow \mathbb{C}$, given by $\mathcal{H}(z, t) = az^n + tQ(z)$. Thus, the Homotopy Invariance Property of the bf-degree implies $\deg_{bf}(P) = \deg_{bf}(\mathcal{H}(\cdot, 0))$. From the Computation Formula (7.1) we easily get $\deg_{bf}(\mathcal{H}(\cdot, 0)) = n$. In fact, the equation $az^n = a$ has exactly n solutions (the well-known roots of the monomial z^n), each of them being regular and having orientation preserving differential. Thus, the Computation Formula implies the assertion. $\square$

The following result shows that, under appropriate assumptions, the bf-degree for Φ_0 -maps between oriented manifolds of the same finite dimension is the same as the Brouwer degree (hereafter denoted Br-degree). A key hypothesis is that the top-orientation of the considered map is the associated one (recall Definition 6.8).

Proposition 7.4 (Equality between the Brouwer degree and the bf-degree) *Let $\mathcal{M}$ and $\mathcal{N}$ be two oriented real manifolds of the same finite dimension. Assume that $f: \overline{U} \rightarrow \mathcal{N}$ is a proper map defined on the closure of an open subset U of $\mathcal{M}$. If f is C^1 on U , it has the associated orientation, and the target value $y \in \mathcal{N}$ does not belong to the closed set $f(\partial U)$, then $\deg_{bf}(f, U, y) = \deg_{Br}(f, U, y)$.*

Proof The assertion follows easily by comparing the Computation Formulas for the bf-degree (in Sect. 7.2 above) and for the Br-degree (see, for example, [17, 21]), and by taking into account the following three facts:

- since $f|_U$ has the associated top-orientation, then $\text{sign}_{bf} df_x = \text{sign} df_x$ for any $x \in U$ (see Remark 4.11);
- because of Sard's Lemma, the set of regular values of $f|_U$ is dense in $\mathcal{N}$;
- both degrees are locally constant as functions of the target value.

$\square$

7.3 Comparison of degrees for compact vector fields

Let f be a compact vector field defined on the closure $\overline{U}$ of an open subset U of $\mathbb{E}$. Assume that the restriction $f|_U$ of f to U is of class C^1 . Thus, as pointed out in Sect. 2, the differential $d(f|_U)$ is a continuous map from U into the affine subspace $LS\mathcal{L}(\mathbb{E})$ of $\mathcal{L}(\mathbb{E})$. Therefore, in this case, we shall tacitly assume that $f|_U$ inherits the standard top-orientation according to Definition 5.21.

Suppose, in addition, that f is a proper map and that $y \in \mathbb{E}$ does not belong to the closed set $f(\partial U)$. Thus, the triple (f, U, y) is strictly admissible and, consequently, $\deg_{bf}(f, U, y)$ is a well defined integer, depending only on the connected component of the open set $\mathbb{E} \setminus f(\partial U)$ containing y (recall the above Continuous Dependence Property for strictly admissible triples).

Obviously, under these assumptions, the open set U may coincide with $\mathbb{E}$, so that $\partial U = \emptyset$, which implies that $\deg_{bf}(f, U, y)$ does not depend on $y \in \mathbb{E}$. As already pointed out, in this case, such an integer could simply be denoted by $\deg_{bf}(f)$, and this includes the situation in which f is a non-constant complex polynomial regarded as a self-map of $\mathbb{R}^2$.

As observed in [7], if U is bounded and the compact vector field $f: \bar{U} \rightarrow \mathbb{E}$ is C^1 on U , then the bf-degree is the same as the Leray–Schauder degree (see Theorem 7.7 below). The proof of such a result requires two preliminary lemmas.

Lemma 7.5 (Uniform C^1 -approximation by finite vector fields) *Let $f: \bar{U} \rightarrow \mathbb{E}$ be a compact vector field defined on the closure of a bounded open subset of $\mathbb{E}$. Assume that f is C^1 on U . Then, given $\varepsilon > 0$, there exists a compact vector field $g: \bar{U} \rightarrow \mathbb{E}$ with the following properties:*

- $I - g$ maps $\bar{U}$ into a finite dimensional subspace of $\mathbb{E}$;
- $\|f(x) - g(x)\| \leq \varepsilon$ for all $x \in \bar{U}$;
- the restriction of g to U is of class C^1 .

Proof Put $k := I - f$ and let $\mathcal{E}$ be a finite ε -net of the totally bounded subset $k(\bar{U})$ of $\mathbb{E}$. Let $\mathbb{E}_1$ be a finite dimensional subspace of $\mathbb{E}$ containing $\mathcal{E}$ and consider any closed complement $\mathbb{E}_2$ of $\mathbb{E}_1$. So that, we have the splitting $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ and the associated projections Π_1 and Π_2 .

Put $h := \Pi_1 \circ k$ and define $g := I - h$; so that the first statement of the thesis is satisfied. To prove the second one, take any $x \in U$ and consider the equality

$$\|f(x) - g(x)\| = \|\Pi_1(k(x)) - k(x)\|.$$

Choosing $x_1 \in \mathcal{E}$ such that $\|k(x) - x_1\| \leq \varepsilon$ and taking into account that the condition $\mathcal{E} \subset \mathbb{E}_1$ implies $x_1 - \Pi_1(x_1) = 0$, we get

$$\|f(x) - g(x)\| = \|\Pi_1(k(x)) - \Pi_1(x_1) + x_1 - k(x)\| \leq \varepsilon \|\Pi_1\| + \varepsilon \leq 2\varepsilon \|\Pi_1\|,$$

and the second statement follows from the arbitrariness of ε . The last statement of the thesis is a consequence of the fact that Π_1 , as a linear operator, is a C^∞ -map. $\square$

The following result regarding the bf-degree is, in a certain sense, the homologue of what for the LS-degree is the well-posed definition of degree given by Leray and Schauder in [20] via finite dimensional approximation.

Lemma 7.6 (Finite dimensional reduction of the bf-degree) *Let f and g be as in Lemma 7.5. Consider a finite dimensional subspace $\mathbb{E}_1$ of $\mathbb{E}$ containing both y and the image of $I - g$. Denote by g_1 the restriction of g to $U_1 := U \cap \mathbb{E}_1$, as domain, and to $\mathbb{E}_1$, as codomain. Then, if g is sufficiently close to f , one has $\deg_{bf}(f, U, y) = \deg_{Br}(g_1, U_1, y)$.*

Proof If g is sufficiently close to f , the Homotopy Invariance Property of the bf-degree implies $\deg_{bf}(f, U, y) = \deg_{bf}(g, U, y)$. Hence, we need to prove that

$$\deg_{bf}(g, U, y) = \deg_{Br}(g_1, U_1, y).$$

Because of Proposition 7.4, we get $\deg_{Br}(g_1, U_1, y) = \deg_{bf}(g_1, U_1, y)$. Therefore, it remains to show that

$$\deg_{bf}(g, U, y) = \deg_{bf}(g_1, U_1, y). \quad (7.2)$$

Taking into account that the bf-degree is locally constant with respect to the target value y , we may assume, because of Sard's Lemma, that y is a regular value for the finite dimensional map $g_1: U_1 \rightarrow \mathbb{E}_1$.

Observe that the set $g^{-1}(y)$ is contained in the finite dimensional space $\mathbb{E}_1$ and, consequently, it coincides with $g_1^{-1}(y)$, and y is as well a regular value for g .

Finally, the equality (7.2) is a consequence of the Computation Formula for the bf-degree. In fact, consider any $x \in g^{-1}(y) = g_1^{-1}(y)$. If x is regarded as a point of U , then dg_x is a det-admissible operator and its bf-sign is $\text{sign}(\det_e(dg_x))$. Taking into account the definition of the extended determinant, we get $\det_e(dg_x) = \det(d(g_1)_x)$. Otherwise, if x is regarded as a point of U_1 , then $\text{sign}_{bf}d(g_1)_x = \text{signdet}(d(g_1)_x)$. Hence, $\text{sign}_{bf}dg_x = \text{sign}_{bf}d(g_1)_x$, $\forall x \in g^{-1}(y) = g_1^{-1}(y)$, and this implies the equality (7.2). $\square$

We conclude this Section with the following result.

Theorem 7.7 (Equality between the bf-degree and the LS-degree) *Let $f: \overline{U} \rightarrow \mathbb{E}$ be a compact vector field defined on the closure of a bounded open subset U of $\mathbb{E}$. Assume that the restriction of f to U is of class C^1 and has the standard top-orientation. Then, given any target value $y \notin f(\partial U)$, one has*

$$\deg_{bf}(f, U, y) = \deg_{LS}(f, U, y).$$

Proof Let g be an ε -approximation of f as in Lemma 7.5. Consider a finite dimensional subspace $\mathbb{E}_1$ of $\mathbb{E}$ containing both the image of $I - g$ and the target value y . Denote by g_1 the restriction of g to $U_1 = U \cap \mathbb{E}_1$ and regard it as a map with codomain $\mathbb{E}_1$. According to the definition of the Leray-Schauder degree given in [20], if ε is sufficiently small, one has $\deg_{LS}(f, U, y) := \deg_{Br}(g_1, U_1, y)$. The assertion now follows from Lemma 7.6. $\square$

8 Concluding remarks

As stated in the Introduction, this work is mainly a comprehensive exposition of the concept of orientation for Fredholm maps of index zero between real Banach manifolds and its application to the construction of a topological degree, namely, the bf-degree, for this kind of maps.

This theory was first presented in [6, 7] in the late 1990s. Over the following years, various aspects were explored and clarified. In [9], new results were obtained on the orientation of Fredholm linear operators between Banach spaces, while in [8] it was shown that, as for the Brouwer degree and the Leray-Schauder degree, the bf-degree is unique starting from three suitable conditions and could be obtained from them axiomatically. Finally, in the three recent works [3–5], new properties have been observed and discussed, which had not been dealt with in previous years.

This paper presents in a homogeneous and orderly way the topics of orientation and degree by putting together results that have appeared over the years in a partial way and not connected to each other, in some cases re-demonstrated in a clearer and more coherent way. In this section, we summarize the relationship of this work with the papers cited above and throughout the previous sections.

First of all, let us say that the use of the terminology algebraic and topological orientation has been adopted in [4], while the notion of pre-orientation appears in [3].

The concept of algebraic orientation of a Fredholm linear operator between vector spaces has been presented in [6]. In [3] the not immediate construction of the orientation of a

composition of Fredholm operators has been clarified. In the present work, this concept is exposed with more details.

The flat and non-flat topological orientations for nonlinear Fredholm maps between Banach spaces or Banach manifolds have been introduced in [6], while in [7] the construction is given by using the covering spaces theory. The approach followed here is in the general spirit of [7]. However, the alternative notions of topological orientation as the choice of continuous selections in the multivalued maps of pre-orientations as seen in the previous sections—Remark 5.5, Definition 5.8, Remark 5.13, Definitions 6.1 and 6.2—is a novelty of this work. In addition, the results regarding the double and triple stability properties, respectively in Lemma 5.1 and Proposition 5.3, are reformulated here with more careful statements and proofs than the original versions given in [7].

Finally, concerning the degree, in [7] it is studied the comparison of the bf-degree with the degree introduced by Fitzpatrick, Pejsachowicz and Rabier as well as with that given by Elworthy and Tromba. In this paper, we analyze differences and similarities between the bf-degree and the Brouwer degree as well as the Leray–Schauder degree, recalling the notion of standard orientation for compact vector fields, introduced in [4, 5].

Acknowledgements The authors would like to thank the anonymous referee for the careful reading of the manuscript and the useful comments and suggestions.

Funding Open access funding provided by Università Politecnica delle Marche within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Amann, H., Weiss, S.A.: On the uniqueness of the topological degree. *Math. Z.* **130**, 39–54 (1973)
2. Benevieri, P.: Recenti sviluppi nella teoria del grado topologico. Ph.D. Thesis, University of Florence (1997)
3. Benevieri, P., Calamai, A., Furi, M., Pera, M.P.: A degree associated to linear eigenvalue problems in Hilbert spaces and applications to nonlinear spectral theory. *J. Dyn. Differ. Equ.* **34**(1), 555–581 (2022)
4. Benevieri, P., Calamai, A., Furi, M., Pera, M.P.: An infinite dimensional version of the intermediate value theorem. *J. Fixed Point Theory Appl.* **25**(3), 70 (2023)
5. Benevieri, P., Calamai, A., Pera, M.P.: An infinite dimensional version of the Kronecker index and its relation with the Leray–Schauder degree. *Z. Anal. Anwend.* **43**(1–2), 169–197 (2024)
6. Benevieri, P., Furi, M.: A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory. *Ann. Sci. Math. Québec* **22**, 131–148 (1998)
7. Benevieri, P., Furi, M.: On the concept of orientability for Fredholm maps between real Banach manifolds. *Topol. Methods Nonlinear Anal.* **16**, 279–306 (2000)
8. Benevieri, P., Furi, M.: On the uniqueness of the degree for nonlinear Fredholm maps of index zero between Banach manifolds. *Commun. Appl. Anal.* **15**, 203–216 (2011)
9. Benevieri, P., Furi, M., Pera, M.P., Spadini, M.: About the sign of oriented Fredholm operators between Banach spaces. *Z. Anal. Anwendungen* **22**(3), 619–645 (2003)
10. Bleecker, D., Booß-Bavnek, B.: *Index Theory: With Applications to Mathematics and Physics*. International Press, Somerville (2013)
11. Deimling, K.: *Nonlinear Functional Analysis*. Springer, Berlin (1985)

12. Elworthy, K.D., Tromba, A.J.: Differential structures and Fredholm maps on Banach manifolds. In: S.S. Chern, S. Smale (eds.) *Global Analysis. Proceedings of Symposium on Pure Mathematics*, vol. 15, pp. 45–94 (1970)
13. Elworthy, K.D., Tromba, A.J.: Degree theory on Banach manifolds. In: F.E. Browder (ed.) *Nonlinear Functional Analysis. Proceedings of Symposium Pure Mathematics (Part 1)*, vol. 18, pp. 86–94 (1970)
14. Fitzpatrick, M.P., Pejsachowicz, J., Rabier, P.J.: Orientability of Fredholm families and topological degree for orientable nonlinear Fredholm mappings. *J. Funct. Anal.* **124**, 1–39 (1994)
15. Greenberg, M.J.: *Lectures on Algebraic Topology*. Benjamin, London (1967)
16. Hatcher, A.: *Algebraic Topology*. Cambridge University Press, Cambridge (2002)
17. Hirsch, M.W.: *Differential Topology*. Graduate Texts in Mathematics, vol. 33. Springer, Berlin (1976)
18. Kato, T.: *Perturbation Theory for Linear Operators*, Classics in Mathematics. Springer, Berlin (1995)
19. Kuiper, N.H.: The homotopy type of the unitary group of a Hilbert space. *Topology* **3**, 19–30 (1965)
20. Leray, J., Schauder, J.: Topologie et équations fonctionnelles. (French). *Ann Sci. École Norm. Sup.* **3**(51), 45–78 (1934)
21. Milnor, J.W.: *Topology from the Differentiable Viewpoint*. Univ. Press of Virginia, Charlottesville (1965)
22. Pejsachowicz, J.: Orientation of Fredholm maps. *J. Fixed Point Theory Appl.* **2**, 97–116 (2007)
23. Pejsachowicz, J., Rabier, P.J.: Degree theory for C^1 -Fredholm mappings of index 0. *J. Anal. Math.* **76**, 289–319 (1998)
24. Sard, A.: The measure of the critical values of differentiable maps. *Bull. Am. Math. Soc.* **48**, 883–890 (1942)
25. Smale, S.: An infinite dimensional version of Sard's theorem. *Am. J. Math.* **87**, 861–866 (1965)
26. Taylor, A.E., Lay, D.C.: *Introduction to Functional Analysis*. Wiley, New York (1980)
27. Văth, M.: *Topological Analysis: From the Basics to the Triple Degree for Nonlinear Fredholm Inclusions*. de Gruyter, Berlin (2012)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.