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S-REGULAR FUNCTIONS WHICH PRESERVE A COMPLEX SLICE

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ABSTRACT. We study global properties of quaternionic slice regular functions (also called *s-regular*) defined on symmetric slice domains. In particular, thanks to new techniques and points of view, we can characterize the property of being one-slice preserving in terms of the projectivization of the vectorial part of the function. We also define a "Hermitian" product on slice regular functions which gives us the possibility to express the $*$-product of two s-regular functions in terms of the scalar product of suitable functions constructed starting from f and g. Afterwards we are able to determine, under different assumptions, when the sum, the $*$-product and the $*$-conjugation of two slice regular functions preserve a complex slice. We also study when the $*$-power of a slice regular function has this property or when it preserves all complex slices. To obtain these results we prove two factorization theorems: in the first one, we are able to split a slice regular function into the product of two functions: one keeping track of the zeroes and the other which is never-vanishing; in the other one we give necessary and sufficient conditions for a slice regular function (which preserves all complex slices) to be the symmetrized of a suitable slice regular one.

1. INTRODUCTION

Since the seminal paper by Gentili and Struppa [12], several articles [4, 5, 10, 11, 14, 15, 16, 18, 20, 22, 23] and some monographs [7, 8, 14] have been published in the field of (quaternionic) slice regular functions. The theory was mainly built to allow quaternionic polynomials to be regular and to mime, in some sense, the theory of complex holomorphic functions. For this reason many works in this field address the search for analogies with the theory of holomorphicity.

This paper, and the results enclosed therein, points out some global behaviour of slice regular functions which are proper of the realm of quaternions and have not a "complex analogous". This fact is investigated by means of the new techniques concerning the $*$-product of slice regular functions partially introduced in [3] and developed in the present work.

To state some of these results we begin with some notation and known fact. The main reference for this part is the monograph [14].

The space of quaternions $\mathbb{H}$ is the four dimensional associative algebra generated by $1, i, j, k$ with usual relations $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The division algebra of quaternions can be split as $\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}\mathbb{H}$ and inside $\mathrm{Im}\mathbb{H} \simeq \mathbb{R}^3$ we identify the sphere of imaginary units

$$\mathbb{S} := \{q \in \mathbb{H} \,|\, q^2 = -1\} = \{q_1 i + q_2 j + q_3 k \,|\, q_1^2 + q_2^2 + q_3^2 = 1\} \simeq S^2 \subset \mathrm{Im}\mathbb{H}.$$

A quaternion $q \in \mathbb{H}$ will then be written in the following ways:

$$q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + I\beta = q_0 + \vec{q},$$

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where, $q_i \in \mathbb{R}$ for $i = 0, 1, 2, 3$, $\beta \in \mathbb{R}$, $I \in \mathbb{S}$ and $\vec{q} \in \mathrm{Im}\mathbb{H}$. Each of the previous notations will be useful for some purpose. In particular we point out that the product of two quaternions $q = q_0 + \vec{q}, p = p_0 + \vec{p}$ can be written as

$$qp = q_0 p_0 - \langle \vec{q}, \vec{p} \rangle + q_0 \vec{p} + p_0 \vec{q} + \vec{q} \wedge \vec{p},$$

where $\langle \vec{q}, \vec{p} \rangle$ and $\vec{q} \wedge \vec{p}$ denote the standard Euclidean and vectorial product of $\mathrm{Im}\mathbb{H} \simeq \mathbb{R}^3$, respectively. In $\mathbb{H}$ we will consider the usual conjugation $q = q_0 + \vec{q} \mapsto q^c = q_0 - \vec{q}$, so that $qq^c = q_0^2 + \langle \vec{q}, \vec{q} \rangle = q_0^2 + q_1^2 + q_2^2 + q_3^2 = |q|^2$. We also denote the open unitary ball as

$$\mathbb{B} = \{q \in \mathbb{H} \mid |q| < 1\}.$$

We now pass to quaternionic functions theory. The definition of slice regularity is based on the observation that is possible to *unfold* the space of quaternions in the following way:

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I, \quad \mathbb{C}_I := \mathrm{Span}_{\mathbb{R}}(1, I).$$

We call *slice* any complex line of the form $\mathbb{C}_J$, for $J \in \mathbb{S}$. From this point of view one can see that is possible to consider a non-constant complex structure over $\mathbb{H} \setminus \mathbb{R}$ that is *tautological* with respect to the slice (see for instance [2, 11]). Before recalling the notion of regularity we establish the family of domains where we will define our spaces of functions.

Assumption 1.1. In the whole paper $\Omega \subset \mathbb{H}$ will denote a symmetric slice domain, see [14], that is a domain such that

- for any $q = \alpha + I\beta \in \Omega$, the set $\mathbb{S}_q := \{\alpha + J\beta \mid J \in \mathbb{S}\}$ is contained in Ω;
- the intersection $\Omega \cap \mathbb{R}$ is non-empty.

Notice that $\mathbb{S}_q$ consists of the single point q is $q \in \mathbb{R}$ and it a 2-dimensional sphere if $q \notin \mathbb{R}$.

Definition 1.2. A function $f : \Omega \to \mathbb{H}$ is said to be slice regular if all its restriction $f_J := f|_{\mathbb{C}_J \cap \Omega}$ are holomorphic with respect to the tautological complex structure, i.e. for any $J \in \mathbb{S}$, it holds

$$\frac{1}{2}\left(\frac{\partial}{\partial \alpha} + J\frac{\partial}{\partial \beta}\right) f_J(\alpha + J\beta) \equiv 0.$$

The family of slice regular functions over a fixed domain Ω is a real vector space, a right $\mathbb{H}$-module and in this paper it will be denoted by $\mathcal{S}(\Omega)$ (in some of the references, see e.g. [16], this symbol denotes the space of continuous slice functions which are not necessarily regular).

Examples of slice regular functions are quaternionic polynomials and quaternionic power series (in their domain of convergence), with right coefficients.

Thanks to the *Representation Formula* (see [14, Theorem 1.15] and [4]), it is known that the hypothesis on the symmetry of the domain is not restrictive, while the one involving the role of the real line is included to avoid some degenerate cases, see e.g. [1]. Again, thanks to the Representation Formula, if $g : \Omega \cap \mathbb{C}_I \to \mathbb{H}$ is any holomorphic function, then it is possible to extend it in a unique way to a slice regular function $f : \Omega \to \mathbb{H}$, such f will be called *regular extension* of g, (see [14], p. 9).

The following two natural subsets of $\mathcal{S}(\Omega)$ are of particular interest for our research.

Definition 1.3. A slice regular function $f \in \mathcal{S}(\Omega)$ is said to be

- *slice preserving* if $f(\Omega \cap \mathbb{C}_I) \subset \mathbb{C}_I$, for all $I \in \mathbb{S}$;
- *one-slice preserving* if there exists $J \in \mathbb{S}$ such that $f(\Omega \cap \mathbb{C}_J) \subset \mathbb{C}_J$; for a fixed $J \in \mathbb{S}$, these functions will also be called *$\mathbb{C}_J$-preserving.*

The set of slice preserving functions will be denoted by $\mathcal{S}_{\mathbb{R}}(\Omega)$, while the set of $\mathbb{C}_J$-preserving functions by $\mathcal{S}_J(\Omega)$.

Examples of slice preserving functions are quaternionic polynomials and quaternionic power series (in their domain of convergence) with *real* coefficients; examples of $\mathbb{C}_J$-preserving functions are quaternionic polynomials and quaternionic power series (in their domain of convergence) with right coefficients which belong to $\mathbb{C}_J$.

Slice preserving and one-slice preserving functions are special slice regular functions which are more likely to be studied with classical complex methods. In particular, the last ones can give a deeper insight on the general case being, in some sense, the middle point between the theory of holomorphic functions and the genuine quaternionic case (see for instance [23]).

Thanks to the following result (see [5, 16]), it is possible to split a slice regular function into a linear combination of any basis of $\mathbb{H}$ with slice preserving regular functions as coefficients.

Proposition 1.4. *Let $\{1, I_0, J_0, K_0\}$ be a basis of $\mathbb{H}$. Then the map*

$$(\mathcal{S}_{\mathbb{R}}(\Omega))^4 \ni (f_0, f_1, f_2, f_3) \mapsto f_0 + f_1 I_0 + f_2 J_0 + f_3 K_0 \in \mathcal{S}(\Omega)$$

is bijective. In particular it follows that given any $f \in \mathcal{S}(\Omega)$ there exist and are unique $f_0, f_1, f_2, f_3 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that

$$f = f_0 + f_1 I_0 + f_2 J_0 + f_3 K_0.$$

Moreover if $I_0 \in \mathbb{S}$ then $f \in \mathcal{S}_{I_0}(\Omega)$ iff $f_2 \equiv f_3 \equiv 0$.

The previous result allows us to define in a more natural way the regular conjugate and the $*$-product, already presented in [14].

Definition 1.5. Given any basis $\{1, I_0, J_0, K_0\}$ of $\mathbb{H}$ and $f = f_0 + f_1 I_0 + f_2 J_0 + f_3 K_0 \in \mathcal{S}(\Omega)$ we define its conjugate as $f^c = f_0 + f_1 I_0^c + f_2 J_0^c + f_3 K_0^c$.

In the previous definition, if $\{I_0, J_0, K_0\}$ is an orthonormal basis of Im$\mathbb{H}$, then $f^c = f_0 - f_1 I_0 - f_2 J_0 - f_3 K_0$.

From the conjugation just defined we can isolate the *real* and *vectorial* parts of a slice regular function.

Definition 1.6. Given $f \in \mathcal{S}(\Omega)$, we define the slice-regular functions f_0 and f_v on Ω by $f_0 = \frac{f+f^c}{2}$, $f_v = \frac{f-f^c}{2}$.

Clearly, fixed $f \in \mathcal{S}(\Omega)$, the functions f_0 and f_v only depend on the conjugation, in particular $f = f_0 + f_v$, $f^c = f_0 - f_v$ and, according to the notation used in Proposition 1.4, $f_v = f_1 I_0 + f_2 J_0 + f_3 K_0$.

In this new language the $*$-product can be defined by associativity and distributivity over $\mathcal{S}_{\mathbb{R}}(\Omega)$ in the following way.

Definition 1.7. Given $f = f_0 + f_1 I_0 + f_2 J_0 + f_3 K_0, g = g_0 + g_1 I_0 + g_2 J_0 + g_3 K_0 \in \mathcal{S}(\Omega)$ their $*$-product is given by

$$\begin{aligned} f * g = {} & f_0 g_0 - f_1 g_1 - f_2 g_2 - f_3 g_3 + f_0(g_1 I_0 + g_2 J_0 + g_3 K_0) + g_0(f_1 I_0 + f_2 J_0 + f_3 K_0) \\ & + (f_2 g_3 - f_3 g_2) I_0 + (f_3 g_1 - f_1 g_3) J_0 + (f_1 g_2 - f_2 g_1) K_0, \end{aligned}$$

where the products in the right hand side of the equality are the pointwise products (functions $f_0, \dots, f_3$ are slice preserving functions, so our definition coincides with the one given in [14]).

Remark 1.8. Notice that for slice preserving and $\mathbb{C}_J$-preserving functions, the $*$-product has special features. First of all if $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ and $g \in \mathcal{S}(\Omega)$, then we have $f * g = g * f = fg$ (that is, the $*$-product $f * g$ coincides with the pointwise product fg). For any $J \in \mathbb{S}$ and any couple of functions $f, g \in \mathcal{S}_J(\Omega)$ we have $f * g = g * f$. Finally if $\rho_1, \rho_2 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ and $a_1, a_2 \in \mathbb{H}$ then $(\rho_1 a_1) * (\rho_2 a_2) = \rho_1 \rho_2 a_1 a_2$.

Remark 1.9. As $\mathcal{S}_{\mathbb{R}}(\Omega)$ is a unitary commutative ring, Proposition 1.4 can be interpreted as the fact that $\mathcal{S}(\Omega)$ is a free module of rank 4 over $\mathcal{S}_{\mathbb{R}}(\Omega)$ and $\mathcal{S}_{I_0}(\Omega)$ is a free submodule of rank 2, for any $I_0 \in \mathbb{S}$. Following this point of view, we provide characterizations of the desired functions in terms of cosets of suitable submodules: in some sense, results like Theorem 4.3, 5.4 and 5.6 can be seen as a parametric description, while Proposition 4.5 displays "bilinear" equations.

Given $f, g \in \mathcal{S}(\Omega)$, the formula in Definition 1.7 can be simplified using the operators

$$(f \wedge g) = \frac{(f * g) - (g * f)}{2}, \qquad \langle f, g \rangle_* = (f * g^c)_0. \tag{1.1}$$

In fact, in terms of the notation of Proposition 1.4 we can rewrite the above intrinsic expressions in the following form:

$$\begin{aligned} f \wedge g &= (f_2 g_3 - f_3 g_2) I_0 + (f_3 g_1 - f_1 g_3) J_0 + (f_1 g_2 - f_2 g_1) K_0 \\ \langle f, g \rangle_* &= f_0 g_0 + f_1 g_1 + f_2 g_2 + f_3 g_3. \end{aligned}$$

At last, with the use of the above operators, we can write

$$f * g = f_0 g_0 - \langle f_v, g_v \rangle_* + f_0 g_v + g_0 f_v + f_v \wedge g_v,$$

in complete analogy with the quaternionic product in $\mathbb{H}$.

Using the conjugate function and the $*$-product, it is possible to define the symmetrized of a slice regular function. In some sense, this function plays the same formal role of the square norm in the space of quaternions.

Definition 1.10. Given $f = f_0 + f_1 i + f_2 j + f_3 k \in \mathcal{S}(\Omega)$, we define its *symmetrized* function f^s as

$$f^s = f * f^c = \langle f, f \rangle_* = f_0^2 + f_1^2 + f_2^2 + f_3^2.$$

We remark that for any $f \in \mathcal{S}(\Omega)$ its symmetrized function f^s, which is called normal function and is denoted by $\mathcal{N}(f)$ by some other authors (see for instance [18]) because it gives the norm of f in the $*$-algebra of slice regular functions, belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$.

We now spend some words on the zero locus of a slice regular function (see [13, 14, 18]). We start with a notation: given any $q = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$ we set

$$\mathbb{S}_{\alpha+I\beta} := \{\alpha + J\beta \,|\, J \in \mathbb{S}\}.$$

It is known that, if $f \in \mathcal{S}(\Omega) \setminus \{0\}$, then its zero locus is closed with empty interior; moreover it consists of isolated points and isolated 2-spheres of the form $\mathbb{S}_{\alpha+I\beta}$. In the particular cases of slice preserving or one-slice preserving functions, the previous assertion specializes as follows. If f belongs to $\mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$, then its zero set consists of isolated *real* points and isolated 2-spheres of the form $\mathbb{S}_{\alpha+I\beta}$. If f belongs to $\mathcal{S}_J(\Omega) \setminus \{0\}$, for some $J \in \mathbb{S}$, then its zero set consists of isolated points belonging to $\mathbb{C}_J$ and isolated 2-spheres of the form $\mathbb{S}_{\alpha+I\beta}$.

As in the complex case, it is possible, for slice regular functions, to factor out a zero as follows. Let f be an element of $\mathcal{S}(\Omega) \setminus \{0\}$ and $\mathbb{S}_{\alpha+I\beta} \subset \Omega$. Then there exist $m, n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathbb{S}_{\alpha+I\beta}$, with $p_\nu \neq p_{\nu+1}^c$ for all $\nu = 1, \dots, n-1$, such that

$$f(q) = [(q-\alpha)^2 + \beta^2]^m (q - p_1) * \cdots * (q - p_n) * g(q), \tag{1.2}$$

for some $g \in \mathcal{S}(\Omega)$ which does not have zeros in $\mathbb{S}_{\alpha+I\beta}$. Thanks to this factorization it is possible to introduce a notion of multiplicity of a zero in the following sense (see [14]).

Definition 1.11. Let $f \in \mathcal{S}(\Omega) \setminus \{0\}$ and let $\mathbb{S}_{\alpha+I\beta} \subset \Omega$ with $\beta \neq 0$. Let $m, n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathbb{S}_{\alpha+I\beta}$, with $p_\nu \neq p_{\nu+1}^c$ for all $\nu = 1, \dots, n-1$, such that Equation (1.2) holds for f and some regular function g which never vanishes in $\mathbb{S}_{\alpha+I\beta}$. We then say that $2m$ is the *spherical multiplicity* of $\mathbb{S}_{\alpha+I\beta}$ and that n

is the *isolated multiplicity* of p_1. If $x \in \mathbb{R}$ is such that $f(x) = 0$, then we call *isolated multiplicity* of f at x the number $k \in \mathbb{N}$ such that

$$f(q) = (q - x)^k h(q),$$

for some $h \in \mathcal{S}(\Omega)$ such that $h(x) \neq 0$.

Remark 1.12. It is known that if $\alpha + I\beta$ is such that $f(\alpha + I\beta) = 0$, then any point in the set $\mathbb{S}_{\alpha+I\beta}$ is a zero for f^s. In particular $\mathcal{S}(\Omega)$ is an integral domain and $f_v^s \equiv 0$ if and only if $f_v \equiv 0$. Indeed, since Ω is a slice domain if $f \not\equiv 0$ and $g \not\equiv 0$, it is enough to choose a real point $x_0 \in \Omega$ at which neither f nor g vanish and $(f * g)(x_0) = f(x_0)g\left((f(x_0))^{-1}x_0 f(x_0)\right) = f(x_0)g(x_0) \neq 0$, where the first equality is due to Theorem 3.4 in [14].

We now have all the prerequisites to state the results contained in the paper. We list them by giving the essential structure of the paper.

Next section is devoted to give an intrinsic characterization of the family of one-slice preserving functions in terms of the projectivization of their *vectorial* part.

In Section 3 we prove two factorization results which will be used in Theorem 5.10. The first one (Proposition 3.1) is a Weierstrass-like factorization theorem for slice regular functions with no non-real isolated zeroes. In the second one (Theorem 3.2), we give necessary and sufficient condition for a slice preserving function μ to be the symmetrized of the function $h \in \mathcal{S}(\Omega)$, that is $\mu = h^s = h * h^c$.

In Section 4 we are able to provide necessary and sufficient conditions for two functions $f \in \mathcal{S}_{I_0}(\Omega)$ and $g \in \mathcal{S}_{J_0}(\Omega)$, in order to determine whether their sum or $*$-product is $\mathbb{C}_{K_0}$-preserving for some $K_0 \in \mathbb{S}$. Then, in Theorem 4.5 we deepen our study of the $*$-product of f and g dropping out the hypothesis on f and g to be one-slice preserving.

Afterwards, in Section 5 we study the conjugation of two slice regular functions f and g:

$$h * f * h^c = g.$$

We firstly impose the condition on f (Theorem 5.4) then on h (Theorem 5.6) to be one-slice preserving in order to obtain necessary and sufficient conditions which guarantee $h * f * h^c$ is one-slice preserving. Then in Theorem 5.10 we study again the conjugation prescribing $g \in \mathcal{S}(\Omega)$, $f \in \mathcal{S}_{I_0}(\Omega)$ and asking h to be one-slice preserving. In Proposition 5.11 we study the same problem, exchanging the requests on f and h.

Exploiting the new results obtained in Section 5, in Section 6 we come back to $*$-products and we give necessary and sufficient conditions on $f, h \in \mathcal{S}(\Omega)$ in order that both $f * h$ and $h * f$ are one-slice preserving.

In the last section we examine the case, of $*$-power of a slice regular function f. After ruling out the trivial cases we give necessary and sufficient conditions for the $*$-power of the function f to be either one-slice preserving or slice preserving, finding an interesting link with some non-trivial result in real algebraic geometry. In fact the function f^{*d} belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ if the two functions f_0 and f_v are the zeros of a particular binary form of degree $[(d-1)/2]$ with real zeroes.

All the listed results are enriched with explicit examples and remarks on the hypotheses.

At last we point out that, since the initial result (Proposition 1.4), holds for the theory of slice regularity over a generic alternative $*$-algebra (see [17, Lemma 2.4]), the new techniques we are going to introduce may be generalized to this wider context.

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2. Preliminary results

In this section we introduce a "Hermitian" product defined on $\mathcal{S}(\Omega)$ which allows us to read the $*$-product in terms of the scalar product introduced in (1.1). We also expose an intrinsic characterization of the family of one-slice preserving functions based on Proposition 1.4.

Definition 2.1. The "Hermitian" product $\mathcal{H}_* : \mathcal{S}(\Omega) \times \mathcal{S}(\Omega) \to \mathcal{S}(\Omega)$ is given by

$$\mathcal{H}_*(f, g) = f * g^c,$$

for any $f, g \in \mathcal{S}(\Omega)$.

A trivial computation shows that the map $\mathcal{S}(\Omega) \ni f \mapsto \mathcal{H}_*(f, g)$ is left-$\mathcal{S}(\Omega)$-linear for any fixed $g \in \mathcal{S}(\Omega)$ and that $(\mathcal{H}_*(g, f))^c = \mathcal{H}_*(f, g)$ for any $f, g \in \mathcal{S}(\Omega)$, ensuring that $\mathcal{H}_*$ is in some sense Hermitian.

For any orthonormal basis i, j, k of $\mathrm{Im}\mathbb{H}$, Proposition 1.4 extends the natural relation between scalar and Hermitian product on $\mathbb{H}$ to $\mathcal{S}(\Omega)$, giving an analogous to the formula in the complex case.

Proposition 2.2. *For any $f, g \in \mathcal{S}(\Omega)$ and any orthonormal basis i, j, k of $\mathrm{Im}\mathbb{H}$, we have*

$$\mathcal{H}_*(f, g) = \langle f, g\rangle_* + \langle f, i * g\rangle_* i + \langle f, j * g\rangle_* j + \langle f, k * g\rangle_* k. \tag{2.1}$$

Proof. According to Proposition 1.4, let us write $f = f_0 + f_1 i + f_2 j + f_3 k$ and $g = g_0 + g_1 i + g_2 j + g_3 k$. By direct computation, the left hand term of (2.1) amounts to

$$\begin{aligned} f_0 g_0 + f_1 g_1 + f_2 g_2 + f_3 g_3 &+ (-f_0 g_1 + f_1 g_0 - f_2 g_3 + f_3 g_2) i \\ &+ (-f_0 g_2 + f_1 g_3 + f_2 g_0 - f_3 g_1) j + (-f_0 g_3 - f_1 g_2 + f_2 g_1 + f_3 g_0) k. \end{aligned}$$

A straightforward application of the definition of $\langle \cdot, \cdot \rangle_*$ gives

$$\begin{aligned} f_0 g_0 + f_1 g_1 + f_2 g_2 + f_3 g_3 &= \langle f, g\rangle_* \\ -f_0 g_1 + f_1 g_0 - f_2 g_3 + f_3 g_2 &= \langle f, i * g\rangle_* \\ -f_0 g_2 + f_1 g_3 + f_2 g_0 - f_3 g_1 &= \langle f, j * g\rangle_* \\ -f_0 g_3 - f_1 g_2 + f_2 g_1 + f_3 g_0 &= \langle f, k * g\rangle_* \end{aligned}$$

gives the conclusion. □

Now we turn to the issue of giving an intrinsic description of one-slice preserving functions; the quest for this result is originated from the need to characterize a function in this class without explicitly indicating the slice it preserves.

Theorem 2.3. *Let $f \in \mathcal{S}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$. The following are equivalent:*

(i) f is one-slice preserving;
(ii) f_v^s has a square root $\sqrt{f_v^s} \in \mathcal{S}_{\mathbb{R}}(\Omega)$ and the map $\frac{f_v}{\sqrt{f_v^s}}$ is constant outside the zero set of f_v^s;
(iii) the map $\Omega \cap \mathbb{R} \ni x \mapsto [f_v(x)] \in \mathbb{P}(\mathrm{Im}\mathbb{H})$ is constant outside the zero set of f_v.

Proof. *(i)⇒(ii)* Denote by $\mathbb{C}_{I_0}$ the slice preserved by f. By Proposition 1.4, the function f can be written as $f = f_0 + f_1 I_0$ and hence $f_v = f_1 I_0$. This immediately entails $f_v^s = f_1^2$, so f_v^s has a square root in $\mathcal{S}_{\mathbb{R}}(\Omega)$ and $\frac{f_v}{\sqrt{f_v^s}} \equiv I_0$ or $\frac{f_v}{\sqrt{f_v^s}} \equiv -I_0$ outside the zero set of f_v^s according to the fact that the choosen square root is equal to f_1 or to $-f_1$.

(ii)⇒(iii) It is enough to observe that $[f_v(x)] = \left[\left(\frac{f_v}{\sqrt{f_v^s}}\right)(x)\right] \in \mathbb{P}(\mathrm{Im}\mathbb{H})$ for any $x \in \Omega \cap \mathbb{R}$ outside the zero set of f_v.

(iii)⇒(i) Since $f \notin \mathcal{S}_{\mathbb{R}}(\Omega)$, the function f_v is not identically zero. Now choose $x_0 \in \Omega \cap \mathbb{R}$ such that $f_v(x_0) \neq 0$ and B_0 a ball of center x_0 contained in Ω on which f_v is never-vanishing; thus the restriction of the function f_v^s to B_0 has no zeroes. By Corollary 3.2 in [3] there exists a square root $\sqrt{f_v^s} \in \mathcal{S}_{\mathbb{R}}(B_0)$. Since the map $\Omega \cap \mathbb{R} \ni x \mapsto [f_v(x)] \in \mathbb{P}(\mathrm{Im}\mathbb{H})$ is constant on $B_0 \cap \mathbb{R}$, then also the map $\Omega \cap \mathbb{R} \ni x \mapsto \left[\frac{f_v}{\sqrt{f_v^s}}(x)\right]$ is constant on $B_0 \cap \mathbb{R}$. Thus, since $B_0 \cap \mathbb{R}$ is connected and $\mathbb{S} \to \mathbb{P}(\mathrm{Im}\mathbb{H})$ is a double-covering, there exists $I_0 \in \mathbb{S}$ such that $f_v(x) = \sqrt{f_v^s}(x) I_0$ for any $x \in B_0 \cap \mathbb{R}$. Thanks to the Identity Principle this equality holds on B_0. Now choose a basis $\{I_0, J_0, K_0\}$ of $\mathrm{Im}\mathbb{H}$ and write $f_v = f_1 I_0 + f_2 J_0 + f_3 K_0$, with $f_1, f_2, f_3 \in \mathcal{S}_{\mathbb{R}}(\Omega)$. The uniqueness given by Proposition 1.4 shows that $f_2 \equiv f_3 \equiv 0$ on B_0 and a further application of the Identity Principle gives at last $f \in \mathcal{S}_{I_0}(\Omega)$. □

Applying Corollary 3.2 in [3] and the above theorem to the case when $\Omega_{I_0} = \Omega \cap \mathbb{C}_{I_0}$ is simply connected, i.e.: $\pi_1(\Omega_{I_0}) = 0$ for some, and then any, $I_0 \in \mathbb{S}$, gives the following

Corollary 2.4. *Let $f \in \mathcal{S}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$ with $\Omega_{I_0} = \Omega \cap \mathbb{C}_{I_0}$ simply connected. The following are equivalent:*

(i) f is one-slice preserving;

(ii) the zero set of f_v does not contain non real isolated zeroes of odd multiplicity and $\frac{f_v}{\sqrt{f_v^s}}$ is constant outside the zero set of f_v^s.

3. Factorization theorems

In this section we present a factorization theorem "à la Weierstrass" for slice regular functions without non-real isolated zeroes which generalizes the result obtained by Gentili and Vignozzi in [15]. This result allows us to give necessary and sufficient conditions, in terms of values taken on the real line, on a given function $\mu \in \mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$ in order that there exists a one-slice preserving function h such that $h^s = \mu$. Since we will need to define the analogous of the Weierstrass primary factors given on the unit disc in $\mathbb{C}$, for the present section the domain $\Omega \subset \mathbb{H}$ is such that $\Omega_{I_0} = \Omega \cap \mathbb{C}_{I_0}$ is simply connected, i.e.: $\pi_1(\Omega_{I_0}) = 0$ for some, and then any, $I_0 \in \mathbb{S}$.

For each $m \in \mathbb{N}$ we introduce an analogous of the Weierstrass primary factor $E_m : \mathbb{H} \to \mathbb{H}$ given by

$$E_m(q) = (1-q)\exp\left(q + \frac{q^2}{2} + \cdots + \frac{q^m}{m}\right).$$

For any $f \in \mathcal{S}(\Omega)$ we consider the regular function defined on Ω

$$(E_m \circledast f)(q) = (1 - f(q)) * \exp_*\left(f(q) + \frac{f(q)^{*2}}{2} + \cdots + \frac{(f(q))^{*m}}{m}\right).$$

where $\exp_*$ is the quaternionic $*$-exponential introduced in [8] and studied in [3]. The previous *composition* operator denoted by $\circledast$, is the one introduced in [20, Definition 4.1] (since E_m has real coefficients the two definitions appearing in the cited paper coincide). In particular if $f \in \mathcal{S}_{I_0}(\Omega)$, the factor $(E_m \circledast f)$ coincides with the regular extension of the function $z \mapsto E_m(f(z))$ defined on $\mathbb{C}_{I_0}$ and it belongs to $\mathcal{S}_{I_0}(\Omega)$, too. Notice that the E_m's are slice preserving regular functions and, in analogy with what happens in the complex case (see [21], Chapter XV), we have that, for any m and for any $|q| \leq 1$

$$|1 - E_m(q)| \leq |q|^{m+1}. \tag{3.1}$$

Consider now a one-slice preserving function $f \in \mathcal{S}_J(\Omega)$, such that $f(\Omega) \subset \overline{\mathbb{B}}$, then

$$|1 - (E_m \circledast f)(q)| \leq \max_{\mathbb{S}_q} |f(q)|^{m+1}, \tag{3.2}$$

for any $q \in \Omega$. Indeed, since $1 - E_m \circledast f$ belongs to $\mathcal{S}_J(\Omega)$, thanks to Proposition 2.6 in [9] for any $q = \alpha + I\beta \in \Omega$ we have that

$$\begin{aligned} |1 - (E_m \circledast f)(q)| &\leq \max\{|1 - (E_m(f(\alpha + J\beta))|, |1 - (E_m(f(\alpha - J\beta))|\} \\ &\leq \max\{|f(\alpha + J\beta)|^{m+1}, |f(\alpha - J\beta)|^{m+1}\} = \max_{\mathbb{S}_q} |f(q)|^{m+1} \end{aligned}$$

In particular, if f is not a constant of modulus 1, then $\max_{\mathbb{S}_q} |f(q)|$ is strictly less than 1.

Proposition 3.1. *Given $f \in \mathcal{S}(\Omega)$ with no non-real isolated zeroes, suppose that $m \in \mathbb{N}$ is the multiplicity of f at 0, then there exist $\mathcal{R}, \mathcal{S} \in \mathcal{S}_{\mathbb{R}}(\Omega)$, $h \in \mathcal{S}(\Omega)$ with h never vanishing, such that*

$$f(q) = q^m \mathcal{R}(q)\mathcal{S}(q)h(q),$$

where

- *$\mathcal{R}$ vanishes exactly at the real non-vanishing zeroes of f;*
- *$\mathcal{S}$ vanishes exactly at the spherical zeroes of f.*

Proof. If f has only a finite number of zeroes (both real and spherical), then the thesis is a direct consequence of a finite number of repeated applications of Theorem 3.36 in [14].

Now we perform the proof in the case when both real and spherical zeroes are infinite. The case in which one of these sets is finite is left to the reader.

We denote by $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{R} \setminus \{0\}$ the sequence of the real non-vanishing zeros of f and by $\{S_n\}_{n\in\mathbb{N}}$ the sequence of the spherical zeros of f, where all the zeros are listed according to their multiplicities.

If $\Omega = \mathbb{H}$, the statement is a particular case of the Weierstrass factorization theorem given in [15]. That is, there exists a never vanishing function $h \in \mathcal{S}(\mathbb{H})$, and for all $n \in \mathbb{N}$, there exist $c_n \in S_n$, such that

$$f(q) = q^m \mathcal{R}(q)\mathcal{S}(q)h(q),$$

where

$$\mathcal{R}(q) = \prod E_n\left(\frac{q}{b_n}\right), \quad \mathcal{S}(q) = \prod \left(E_n \circledast (qc_n^{-1})\right)^s.$$

If $\Omega \neq \mathbb{H}$, Corollary 3.7 in [10] allows us to restrict to the case in which $\Omega = \mathbb{B}$.

For any $q_0 \in \mathbb{B} \setminus \{0\}$, we set

$$M_{q_0}(q) := \left(q_0 - \frac{q_0}{|q_0|}\right) * \left(q - \frac{q_0}{|q_0|}\right)^{-*} = \left(q - \frac{q_0}{|q_0|}\right)^{-*} * \left(q_0 - \frac{q_0}{|q_0|}\right),$$

which is the regular Möbius transformation defined by Stoppato in [22]. Now we choose $c_n \in S_n$; thanks to Theorem 3.12 in [14], we have that any closed ball centered at the origin with radius strictly less than 1 contains only a finite number of real and spherical zeroes, so that $\lim |b_n| = 1$ and $\lim |c_n| = 1$. Thus $\left|\frac{b_n}{|b_n|} - b_n\right| = |b_n| \left|\frac{1}{|b_n|} - 1\right| \to 0$ and $\left|\frac{c_n}{|c_n|} - c_n\right| = |c_n| \left|\frac{1}{|c_n|} - 1\right| \to 0$ for $n \to \infty$.

Now consider the following factors

$$\mathcal{R}(q) = \prod \left(E_n \circledast M_{b_n}\right)(q), \quad \mathcal{S}(q) = \prod \left(E_n \circledast M_{c_n}\right)^s(q)$$

According to (3.2) and the estimates contained in Theorem 15.11 in [21], each factor is well defined on $\mathbb{B}$ and belongs to $\mathcal{S}_{\mathbb{R}}(\mathbb{B})$, moreover the product $q^m\mathcal{R}(q)\mathcal{S}(q)$ has the same zeroes of f. At this point, arguing as in Theorems 4.31 and 4.32 in [14], we can find a never-vanishing function $h \in \mathcal{S}(\mathbb{B})$ such that

$$f(q) = q^m \mathcal{R}(q)\mathcal{S}(q)h(q)$$

and this concludes the proof. □

We now give necessary and sufficient conditions for a slice preserving function to be the symmetrized of a one-slice preserving function.

Theorem 3.2. *Given $\mu \in \mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$, there exists $h \in \mathcal{S}_{I_0}(\Omega)$ such that $h^s = \mu$ if and only if $\mu \geq 0$ on $\Omega \cap \mathbb{R}$ and the order of the real zeros of μ is even.*

Notice that the statement is independent from $I_0 \in \mathbb{S}$. Indeed, if there exist $I_0 \in \mathbb{S}$ and $h = h_0 + h_1 I_0 \in \mathcal{S}_{I_0}(\Omega)$ such that $h^s = \mu$, then for any $J_0 \in \mathbb{S}$ the function $\tilde{h} = h_0 + h_1 J_0 \in \mathcal{S}_{J_0}(\Omega)$ satisfies $\tilde{h}^s = \mu$.

Proof. As usual we write $h = h_0 + h_1 I_0$ with $I_0 \in \mathbb{S}$ and $h_0, h_1 \in \mathcal{S}_{\mathbb{R}}(\Omega)$; then the equality $h^s = \mu$ becomes $\mu = h_0^2 + h_1^2$.

The condition $\mu \geq 0$ on $\Omega \cap \mathbb{R}$ is trivially necessary. If $\mu(x_0) = 0$ with $x_0 \in \Omega \cap \mathbb{R}$ then $h(x_0) = 0$, so the slice preserving function $(q - x_0)$ divides h and hence $(q - x_0)^2$ divides $h^s = \mu$ and the necessity of the second condition is also proved.

In order to prove the sufficiency of the above stated conditions, we denote by $2m$ the multiplicity of $q = 0$ as a zero of μ, by $\{b_n\}$ the real non-vanishing zeroes repeated accordingly to half their multiplicity, by $\{S_n\}$ the sequence of spherical zeroes repeated accordingly to half their multiplicity and by c_n the element of $S_n \cap \mathbb{C}^+_{I_0}$.

Thanks to Proposition 3.1 it is possible to factorize μ as follows

$$\mu(q) = q^{2m}\mathcal{R}^2(q)\mathcal{S}(q)\nu(q)$$

where

- $\mathcal{R}^2$ vanishes exactly at the real non-vanishing zeroes of f,
- $\mathcal{S}$ vanishes exactly at the spherical zeroes of f,

both with the appropriate multiplicities and $\nu \in \mathcal{S}_{\mathbb{R}}(\Omega)$ never vanishing.

If $\Omega = \mathbb{H}$ we have that

$$\mathcal{R}(q) = \prod E_n\left(\frac{q}{b_n}\right), \quad \mathcal{S}(q) = \prod \left(E_n \circledast (qc_n^{-1})\right)^s.$$

Since we chose c_n all lying in the same $\mathbb{C}_{I_0}$, then we can write

$$\mathcal{S}(q) = \prod_* \left(E_n \circledast (qc_n^{-1})\right) * \prod_* \left(E_n \circledast (qc_n^{-1})\right)^c.$$

Thanks to Proposition 3.1 in [3] there exists a square root $\sigma \in \mathcal{S}_{\mathbb{R}}(\mathbb{H})$ of ν and hence the function

$$h(q) = q^m \mathcal{R}(q)\sigma(q)\tilde{\mathcal{S}}(q)$$

where,

$$\tilde{\mathcal{S}}(q) = \prod_* \left(E_n \circledast (qc_n^{-1})\right)$$

belongs to $\mathcal{S}_{I_0}(\mathbb{H})$ and is such that $h^s = \mu$.

If $\Omega \neq \mathbb{H}$, again Corollary 3.7 in [10] allows us to restrict to the case in which $\Omega = \mathbb{B}$. In this case we have that

$$\mathcal{R}(q) = \prod (E_n \circledast M_{b_n})(q), \quad \mathcal{S}(q) = \prod (E_n \circledast M_{c_n})^s(q)$$

Again, since we chose c_n all lying in the same $\mathbb{C}_{I_0}$, we then have

$$\mathcal{S}(q) = \prod_* (E_n \circledast M_{c_n}) * \prod_* (E_n \circledast M_{c_n})^c$$

and the existence of a square root $\sigma \in \mathcal{S}_{\mathbb{R}}(\mathbb{B})$ of ν allows us to conclude with the same argument as above. ∎

Remark 3.3. Notice that, given $\mu \in \mathcal{S}_{\mathbb{R}}(\Omega)\setminus\{0\}$, if there exists $\hat{h} \in \mathcal{S}(\Omega)$ such that $\hat{h}^s = \mu$, then trivially $\mu \geq 0$ on $\Omega \cap \mathbb{R}$ and the order of the real zeros of μ is even. Thus the previous result shows that the following conditions are equivalent:

- there exists $\hat{h} \in \mathcal{S}_{I_0}(\Omega)$ such that $\hat{h}^s = \mu$,
- there exists $k \in \mathcal{S}(\Omega)$ such that $k^s = \mu$.

Example 3.4. Consider $\mu : \mathbb{H} \to \mathbb{H}$ given by $\mu(q) = q^2 + 1$. On the real line μ is always strictly positive and it can be written as $\mu = h^s$, where $h(q) = q + I_0$, for any $I_0 \in \mathbb{S}$. Nonetheless we can also write $\mu = \hat{h}^s$, where $\hat{h}(q) = \cos(q) + \sin(q)i + q\cos(q)j + q\sin(q)k$ and thanks to Theorem 2.3 it is not difficult to show that $\hat{h}$ preserves no slice.

4. Sum and $*$-product

Let $f, h : \Omega \to \mathbb{H}$ be two slice regular functions such that f is $\mathbb{C}_{I_0}$-preserving and h is $\mathbb{C}_{J_0}$-preserving for some $I_0, J_0 \in \mathbb{S}$. We want to understand when their sum and $*$-product is a $\mathbb{C}_{K_0}$-preserving regular function, for a suitable $K_0 \in \mathbb{S}$. If $I_0 = \pm J_0$ then the question is trivial, so in this section we suppose that $I_0 \neq \pm J_0$; for the same reason we assume that f and h are not slice-preserving functions.

Proposition 4.1. *Let $f, h : \Omega \to \mathbb{H}$ be two slice regular functions such that $f = f_0 + f_1 I_0$ is $\mathbb{C}_{I_0}$-preserving and $h = h_0 + h_1 J_0$ is $\mathbb{C}_{J_0}$-preserving with f_0, f_1, h_0, h_1 slice preserving functions. Then there exists $K_0 \in \mathbb{S}$ such that $f + h$ is $\mathbb{C}_{K_0}$-preserving if and only if there exist $a, b \in \mathbb{R}\setminus\{0\}$ such that $K_0 = aI_0 + bJ_0$ and $bf_1 - ah_1 \equiv 0$.*

Proof. The sufficiency of the condition is trivial. In order to prove its necessity, notice that, as I_0 and J_0 are linearly independent, they can be completed to a basis I_0, J_0, L_0 of $\mathrm{Im}\mathbb{H}$ and we can write K_0 as $aI_0 + bJ_0 + \varepsilon L_0$ for suitable $a, b, \varepsilon \in \mathbb{R}$. Then $f + h$ is equal to $f_0 + h_0 + f_1 I_0 + h_1 J_0$. Now $f + h$ is $\mathbb{C}_{K_0}$-preserving if and only if there exist two slice preserving functions m_0, m_1 such that

$$f_0 + h_0 + f_1 I_0 + h_1 J_0 = f + h = m_0 + m_1 K_0 = m_0 + am_1 I_0 + bm_1 J_0 + \varepsilon m_1 L_0.$$

The bijectivity guaranteed by Proposition 1.4 entails that $f_1 = am_1$, $h_1 = bm_1$ and $\varepsilon m_1 = 0$. Since neither f nor h are slice preserving, then the function m_1 cannot be identically zero and a and b are both different from zero. This implies that $\varepsilon = 0$, $K_0 = aI_0 + bJ_0$ and $bf_1 - ah_1 \equiv 0$. □

We start the discussion on the $*$-product of two functions with a preliminary remark that sets the question in the case their $*$-product belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$.

Remark 4.2. Let $f, h \in \mathcal{S}(\Omega)\setminus\{0\}$. Then $f*h$ belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ if and only if also $h*f$ belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$. In fact if $f*h \in \mathcal{S}_{\mathbb{R}}(\Omega)$, then $f^s h^s = (f^c*f)*(h*h^c) = f^c*(f*h)*h^c = (f*h)*f^c*h^c = (f*h)*(h*f)^c$. As both $f^s h^s$ and $f*h$ belong to $\mathcal{S}_{\mathbb{R}}(\Omega)$, then $(h*f)^c$ also lies in $\mathcal{S}_{\mathbb{R}}(\Omega)$ and therefore $h*f \in \mathcal{S}_{\mathbb{R}}(\Omega)$.

Moreover $f*h$ belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ if and only if f, h^c are linearly dependent over $\mathcal{S}_{\mathbb{R}}(\Omega)$. In fact if there exist $\alpha, \beta \in \mathcal{S}_{\mathbb{R}}(\Omega)\setminus\{0\}$, such that $\alpha f + \beta h^c \equiv 0$, then $\alpha f * h + \beta h^s \equiv 0$. This implies that $\alpha f * h$ belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ and therefore $f * h \in \mathcal{S}_{\mathbb{R}}(\Omega)$. Vice versa if $f * h \in \mathcal{S}_{\mathbb{R}}(\Omega)$, then $(f*h)*h^c = f*(h*h^c) = h^s f$. As both $f*h$ and h^s are not identically zero because $\mathcal{S}(\Omega)$ is an integral domain, we are done.

Now we turn to the non-trivial case. We first characterize, giving an explicit parametric description, the sets of regular functions which preserve two different slices whose $*$-product also preserves a slice.

Theorem 4.3. *Let $f, h : \Omega \to \mathbb{H}$ be two slice regular functions such that $f = f_0 + f_1 I_0$ is $\mathbb{C}_{I_0}$-preserving and $h = h_0 + h_1 J_0$ is $\mathbb{C}_{J_0}$-preserving with f_0, f_1, h_0, h_1 slice preserving functions and I_0, J_0 linearly independent. Then there exists $K_0 \in \mathbb{S}$ such that $f * h$ is $\mathbb{C}_{K_0}$-preserving if and only if there exist $a, b \in \mathbb{R}$, $\varepsilon \in \mathbb{R}\setminus\{0\}$ such that $K_0 = aI_0 + bJ_0 + \varepsilon I_0 \wedge J_0$, $f = f_1\left(\frac{b}{\varepsilon} + I_0\right)$ and $h = h_1\left(\frac{a}{\varepsilon} + J_0\right)$.*

Proof. As above, the sufficiency of the condition is obtained by direct computation. In order to prove its necessity, first of all we compute

$$f * h = (f_0 + f_1 I_0) * (h_0 + h_1 J_0) = f_0 h_0 + h_0 f_1 I_0 + f_0 h_1 J_0 + f_1 h_1 I_0 J_0.$$

As $I_0 J_0 = -\langle I_0, J_0\rangle + I_0 \wedge J_0$ we have that

$$f * h = f_0 h_0 - f_1 h_1 \langle I_0, J_0\rangle + h_0 f_1 I_0 + f_0 h_1 J_0 + f_1 h_1 I_0 \wedge J_0.$$

Then $f * h$ is $\mathbb{C}_{K_0}$-preserving for some $K_0 \in \mathbb{S}$ if and only if there exist two slice preserving functions m_0, m_1 and $a, b, \varepsilon \in \mathbb{R}$ such that

$$f_0 h_0 - f_1 h_1 \langle I_0, J_0\rangle + h_0 f_1 I_0 + f_0 h_1 J_0 + f_1 h_1 I_0 \wedge J_0 = m_0 + m_1 K_0 = m_0 + a m_1 I_0 + b m_1 J_0 + \varepsilon m_1 I_0 \wedge J_0;$$

this because $I_0, J_0, I_0 \wedge J_0$ is a basis of $\mathrm{Im}\mathbb{H}$. Again, the bijectivity guaranteed by Proposition 1.4 entails that

$$\begin{cases} f_0 h_0 - f_1 h_1 \langle I_0, J_0\rangle = m_0, \\ f_1 h_0 = a m_1, \\ h_1 f_0 = b m_1, \\ f_1 h_1 = \varepsilon m_1. \end{cases}$$

Since neither f nor h are slice preserving, the functions f_1 and h_1 are not identically zero. As $\mathcal{S}(\Omega)$ is an integral domain, then the function m_1 cannot be identically zero and ε has to be different from zero. Thus $m_1 = \frac{f_1 h_1}{\varepsilon}$ and therefore $f_0 h_1 = b\frac{f_1 h_1}{\varepsilon}$ and $h_0 f_1 = a\frac{f_1 h_1}{\varepsilon}$. These equalities can be written as $h_1\left(f_0 - \frac{b}{\varepsilon} f_1\right) \equiv 0$ and $f_1\left(h_0 - \frac{a}{\varepsilon} h_1\right) \equiv 0$. Again, since f_1 and h_1 are not identically zero, we obtain $f_0 = \frac{b}{\varepsilon} f_1$ and $h_0 = \frac{a}{\varepsilon} h_1$ that is $f = f_1\left(\frac{b}{\varepsilon} + I_0\right)$ and $h = h_1\left(\frac{a}{\varepsilon} + J_0\right)$.

□

Remark 4.4. We underline that, chosen a basis $I_0, J_0, K_0 \in \mathbb{S}$, the above result locates two real directions in the planes $\mathbb{C}_{I_0}$ and $\mathbb{C}_{J_0}$, respectively generated by $\frac{b}{\varepsilon} + I_0$ and by $\frac{a}{\varepsilon} + J_0$, which "give the angles" of the rotations needed to obtain f and h from the slice preserving functions f_1 and h_1. That is, for any $p_0 \in \mathbb{H}$ real multiple of $\frac{b}{\varepsilon} + I_0$ and $q_0 \in \mathbb{H}$ real multiple of $\frac{a}{\varepsilon} + J_0$ and for any $f_1, h_1 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ the $*$-product of the functions $f = f_1 p_0 \in \mathcal{S}_{I_0}(\Omega)$ and $h = h_1 q_0 \in \mathcal{S}_{J_0}(\Omega)$ belongs to $\mathcal{S}_{K_0}(\Omega)$ and vice versa.

We now pass to another result related to the $*$-product of two regular functions. In this case, given functions f and g we write explicit "bilinear" equations which characterize the fact that the $*$-product $f * g$ preserves a given slice.

Proposition 4.5. *Given $I_0 \in \mathbb{S}$ and $f, g \in \mathcal{S}(\Omega)$, the following are equivalent*

(i) the $*$-product $f * g$ belongs to $\mathcal{S}_{I_0}(\Omega)$;
(ii) $\langle f, M_0 * g^c\rangle_* \equiv 0$, for all $M_0 \in \mathbb{S}$ orthogonal to I_0;
(iii) $\langle f^c, g * M_0\rangle_* \equiv 0$, for all $M_0 \in \mathbb{S}$ orthogonal to I_0.

Proof. *(i)* ⇔ *(ii)* Choose an orthonormal basis $\{I_0, J_0, K_0\}$ of $\mathrm{Im}\mathbb{H}$ and notice that, thanks to Definition 2.1, condition *(i)* is equivalent to $\mathcal{H}_*(f, g^c) \in \mathcal{S}_{I_0}(\Omega)$. Now Equality (2.1) ensures that $\mathcal{H}_*(f, g^c) \in \mathcal{S}_{I_0}(\Omega)$ if and only if $\langle f, J_0 * g^c\rangle_* \equiv 0$ and $\langle f, K_0 * g^c\rangle_* \equiv 0$ which, by linearity on $\mathbb{R}$, holds if and only if $\langle f, M_0 * g^c\rangle_* \equiv 0$ for all $M_0 \in \mathbb{S}$ orthogonal to I_0.

(ii) ⇔ *(iii)* Since $\langle f, M_0 * g^c\rangle_* = \langle f^c, (M_0 * g^c)^c\rangle_* = \langle f^c, g * (-M_0)\rangle_* = -\langle f^c, g * M_0\rangle_*$ the equivalence of the two conditions is immediately proven. □

5. Conjugates

We first establish a convention for following reference. This will simplify the presentation of the forecoming results.

Notation 5.1. If $I_0, M_0 \in \mathbb{S}$ are linearly independent, throughout the rest of the paper we will keep the following notation. We denote by I_0, J_0, K_0 the orthonormal basis of $\mathrm{Im}\mathbb{H}$ such that K_0 is a positive multiple of $I_0 \wedge M_0$ and $J_0 = K_0 I_0$. This gives that, up to the substitution of I_0, J_0, K_0 with $I_0, -J_0, -K_0$, we have $M_0 = aI_0 + bJ_0$ for some $b > 0$ with $a^2 + b^2 = 1$.

If I_0 and M_0 are not orthogonal and we are interested only in the slices $\mathbb{C}_{I_0}, \mathbb{C}_{M_0}$, up to substituting I_0, J_0, K_0 with $-I_0, J_0, -K_0$ we can also suppose that $a > 0$.

In this section we study the behaviour of conjugates $h * f * h^c$ of slice regular maps, in the cases when either the conjugator h or the conjugated f is one-slice preserving. In order to obtain more information on this sort of functions, first of all we compute $h * f * h^c$ by means of the decomposition in real and vectorial part as introduced in Section 1. Setting $f = f_0 + f_v$ and $h = h_0 + h_v$ we have $h^c = h_0 - h_v$ and we can state the following lemma.

Lemma 5.2. *Given $f, h \in \mathcal{S}(\Omega)$ it holds*

$$h * f * h^c = [h * f * h^c]_0 + \langle h_v, f_v \rangle_* h_v + h_0^2 f_v + 2h_0 h_v \pitchfork f_v - (h_v \pitchfork f_v) \pitchfork h_v. \tag{5.1}$$

Proof. The thesis is obtained thanks to the following chain of equalities

$$\begin{aligned} h * f * h^c &= (h_0 + h_v) * (f_0 + f_v) * (h_0 - h_v) = (h_0 f_0 - \langle h_v, f_v \rangle_* + h_0 f_v + f_0 h_v + h_v \pitchfork f_v) * (h_0 - h_v) \\ &= [h * f * h^c]_0 - h_0 f_0 h_v + \langle h_v, f_v \rangle_* h_v + h_0^2 f_v + h_0 f_0 h_v + h_0 h_v \pitchfork f_v - h_0 f_v \pitchfork h_v - (h_v \pitchfork f_v) \pitchfork h_v \\ &= [h * f * h^c]_0 + \langle h_v, f_v \rangle_* h_v + h_0^2 f_v + 2h_0 h_v \pitchfork f_v - (h_v \pitchfork f_v) \pitchfork h_v. \end{aligned}$$

□

Using the previous lemma, the first result we can prove is a complete classification of the regular functions which satisfy the equality $h * f * h^c = h^c * f * h$.

Proposition 5.3. *Let $f, g \in \mathcal{S}(\Omega)$, then $h * f * h^c = h^c * f * h$ if and only if either $h_0 \equiv 0$ or f_v and h_v are linearly dependent over $\mathcal{S}_{\mathbb{R}}(\Omega)$.*

Proof. Since $h^c = h_0 - h_v$, from Equation (5.1) we have

$$h^c * f * h = [h^c * f * h]_0 + \langle h_v, f_v \rangle_* h_v + h_0^2 f_v - 2h_0 h_v \pitchfork f_v - (h_v \pitchfork f_v) \pitchfork h_v.$$

A straightforward computation shows that $[h * f * h^c]_0 = h_0^2 f_0 + f_0 \langle h_v, h_v \rangle_* = [h^c * f * h]_0$, hence $h * f * h^c = h^c * f * h$ is equivalent to

$$4h_0(h_v \pitchfork f_v) \equiv 0. \tag{5.2}$$

As $\mathcal{S}(\Omega)$ is an integral domain, (5.2) is equivalent to either $h_0 \equiv 0$ or $h_v \pitchfork f_v \equiv 0$. Thanks to Proposition 2.8 in [3] the statement follows. □

We carry on our investigation on the behaviour of the conjugate by showing under which (non-trivial) conditions on h the function $h * f * h^c$ is one-slice preserving when f is.

Theorem 5.4. *Let $f \in \mathcal{S}_{I_0}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$ and $h \in \mathcal{S}(\Omega) \setminus \mathcal{S}_{I_0}(\Omega)$. Then there exists $M_0 \in \mathbb{S}$ such that $h * f * h^c \in \mathcal{S}_{M_0}(\Omega)$ if and only if*

(i) in the case $\mathbb{C}_{I_0} \neq \mathbb{C}_{M_0}$, by means of Notation 5.1, we can find $g \in \mathcal{S}_{I_0}(\Omega) \setminus \{0\}$ such that

$$h = \left(1 - \frac{a \pm 1}{b} K_0\right) * g;$$

(ii) in the case $\mathbb{C}_{I_0} = \mathbb{C}_{M_0}$, there exist $J_0 \in \mathbb{S}$ with $I_0 \perp J_0$ and $g \in \mathcal{S}_{I_0}(\Omega) \setminus \{0\}$ such that $h = J_0 * g$.

Proof. We first consider the case in which I_0 and M_0 are linearly independent and adopt Notation 5.1, according to the statement.

With respect to the chosen basis we can decompose f_v and h_v as $f_1 I_0$ and $h_1 I_0 + h_2 J_0 + h_3 K_0$, respectively. Since $h \notin \mathcal{S}_{I_0}(\Omega)$ then h_2 and h_3 are not both identically zero. As

$$h_v \wedge f_v = f_1 h_3 J_0 - f_1 h_2 K_0 \quad \text{and} \quad (h_v \wedge f_v) \wedge h_v = f_1[(h_2^2 + h_3^2) I_0 - h_1 h_2 J_0 - h_1 h_3 K_0]$$

from (5.1), we have

$$h * f * h^c = [h * f * h^c]_0 + f_1[(h_0^2 + h_1^2 - h_2^2 - h_3^2) I_0 + 2(h_1 h_2 + h_0 h_3) J_0 + 2(h_1 h_3 - h_0 h_2) K_0].$$

This function belongs to $\mathcal{S}_{M_0}(\Omega)$ if and only if there exists $m_1 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that

$$\begin{cases} h_0^2 + h_1^2 - h_2^2 - h_3^2 = a m_1, \\ 2(h_1 h_2 + h_0 h_3) = b m_1, \\ 2(h_1 h_3 - h_0 h_2) = 0. \end{cases}$$

As $b \neq 0$ we obtain

$$h_0^2 + h_1^2 - h_2^2 - h_3^2 = 2\frac{a}{b}(h_1 h_2 + h_0 h_3) \qquad \text{and} \qquad h_1 h_3 = h_0 h_2. \tag{5.3}$$

If $h_2 \not\equiv 0$, we multiply first equation in (5.3) by h_2^2 and find, thanks to second equality in (5.3),

$$(h_1^2 - h_2^2)(h_2^2 + h_3^2) = 2\frac{a}{b} h_1 h_2 (h_2^2 + h_3^2).$$

The facts that $h \notin \mathcal{S}_{I_0}(\Omega)$ and that Ω contains real points, ensure that $h_2^2 + h_3^2 \not\equiv 0$. Since $\mathcal{S}(\Omega)$ is an integral domain, we therefore have

$$h_1^2 - 2\frac{a}{b} h_1 h_2 - h_2^2 \equiv 0.$$

Now choose $x_0 \in \Omega \cap \mathbb{R}$ such that $h_2(x_0) \neq 0$; in a suitable neighborhood of x_0 the function h_2 is never-vanishing and hence we can consider the quotient $\tau = \frac{h_1}{h_2}$ which satisfies $\tau^2 - 2\frac{a}{b}\tau - 1 = 0$. Thus, as $a^2 + b^2 = 1$, we find $\tau = \frac{a \pm 1}{b} \neq 0$. So in the same neighbourhood we have that $h_1 = \frac{a \pm 1}{b} h_2 \not\equiv 0$ that is $h_2 = -\frac{a \mp 1}{b} h_1$ since $a^2 + b^2 = 1$. By the Identity Principle (see [14], Theorem 1.12) we obtain that either $h_2 \equiv -\frac{a+1}{b} h_1$ or $h_2 \equiv -\frac{a-1}{b} h_1$. Using again the second equality in (5.3) we find $h_1 h_3 = -h_0 \frac{a \pm 1}{b} h_1$ and hence $h_3 = -h_0 \frac{a \pm 1}{b}$.

At last, we obtain

$$\begin{aligned} h &= h_0 + h_1 I_0 - \frac{a \pm 1}{b} h_1 J_0 - h_0 \frac{a \pm 1}{b} K_0 = h_0 \left(1 - \frac{a \pm 1}{b} K_0\right) + h_1 \left(I_0 - \frac{a \pm 1}{b} K_0 I_0\right) \\ &= h_0 \left(1 - \frac{a \pm 1}{b} K_0\right) + h_1 \left(1 - \frac{a \pm 1}{b} K_0\right) I_0 = \left(1 - \frac{a \pm 1}{b} K_0\right) * (h_0 + h_1 I_0). \end{aligned}$$

If $h_2 \equiv 0$, then $h_3 \not\equiv 0$, as $h \notin \mathcal{S}_{I_0}(\Omega)$. The second equality in (5.3) now becomes $h_1 h_3 \equiv 0$, which gives $h_1 \equiv 0$. Thus we obtain $h_0^2 - h_3^2 = 2\frac{a}{b} h_0 h_3$ and the same reasoning as above yields $h_3 = -h_0 \frac{a \pm 1}{b}$ that is $h = \left(1 - \frac{a \pm 1}{b} K_0\right) h_0$. Thus, in both cases, setting $g = h_0 + h_1 I_0$, we have been able to find a function in $\mathcal{S}_{I_0}(\Omega)$ such that $h = \left(1 - \frac{a \pm 1}{b} K_0\right) * g$.

Vice versa, suppose that $h = \left(1 - \frac{a \pm 1}{b} K_0\right) * g$ for some $g \in \mathcal{S}_{I_0}(\Omega)$. Then $h * f * h^c = \left(1 - \frac{a \pm 1}{b} K_0\right) * g * f * g^c * \left(1 - \frac{a \pm 1}{b} K_0\right)^c$. Since $g * f * g^c$ belongs to $\mathcal{S}_{I_0}(\Omega)$ it is enough to show that for any $m = m_0 + m_1 I_0$ the function $\left(1 - \frac{a \pm 1}{b} K_0\right) * (m_0 + m_1 I_0) * \left(1 - \frac{a \pm 1}{b} K_0\right)^c$ also belongs to $\mathcal{S}_{I_0}(\Omega)$. The vectorial part of the above function is given by $m_1\left(1-\frac{a\pm1}{b}K_0\right)I_0\left(1+\frac{a\pm1}{b}K_0\right)$, so the following chain of equality gives the assertion,

$$\begin{aligned}\left(1-\frac{a\pm1}{b}K_0\right)I_0\left(1+\frac{a\pm1}{b}K_0\right)&=\left(I_0-\frac{a\pm1}{b}J_0\right)\left(1+\frac{a\pm1}{b}K_0\right)\\&=I_0-\frac{a\pm1}{b}J_0+\frac{a\pm1}{b}I_0K_0-\left(\frac{a\pm1}{b}\right)^2J_0K_0\\&=\left(1-\left(\frac{a\pm1}{b}\right)^2\right)I_0-2\frac{a\pm1}{b}J_0\\&=\frac{b^2-a^2\mp2a-1}{b^2}I_0-2\frac{a\pm1}{b}J_0=-2a\frac{a\pm1}{b^2}I_0-2\frac{a\pm1}{b}J_0\\&=-\frac{2(a\pm1)}{b^2}(aI_0+bJ_0)=-\frac{2(a\pm1)}{b^2}M_0.\end{aligned}$$

Now we turn to the case when I_0 and M_0 are linearly dependent, so that we can suppose $I_0=M_0$. Choosen any orthonormal basis I_0,J_0,K_0 of $\mathrm{Im}\mathbb{H}$, we again use the expression for $h*f*h^c$ given by Equation (5.1) obtaining that $h*f*h^c$ belongs to $\mathcal{S}_{I_0}(\Omega)$ if and only if

$$\begin{cases}h_1h_2+h_0h_3=0,\\h_1h_3-h_0h_2=0.\end{cases}$$

We multiply the first equation by h_2, the second one by h_3 and sum up, thus obtaining $h_1(h_2^2+h_3^2)\equiv0$. Again, the facts that $h\notin\mathcal{S}_{I_0}(\Omega)$ and that Ω contains real points, imply $h_1\equiv0$ and thus also h_0 has to be zero. This gives

$$h=h_2J_0+h_3K_0=h_2J_0-h_3J_0I_0=J_0*(h_2-h_3I_0);$$

setting $g=h_2-h_3I_0$ gives the assertion. A direct inspection shows that all the functions of this form satisfy the condition $h*f*h^c\in\mathcal{S}_{I_0}(\Omega)$. □

Remark 5.5. If $I_0,J_0\in\mathbb{S}$ with $I_0\perp J_0$ and $g\in\mathcal{S}_{I_0}(\Omega)$ then $J_0*g=g^c*J_0$. Thus the functions which appear in part *(ii)* of the statement of the previous theorem can also be seen as products of a $\mathbb{C}_{I_0}$-preserving function for a suitable quaternion orthogonal to I_0.

Now we turn to the "dual" problem, that is under which (non-trivial) conditions on f the function $h*f*h^c$ is one-slice preserving when h is.

Theorem 5.6. *Let $h=h_0+h_1I_0\in\mathcal{S}_{I_0}(\Omega)\setminus\mathcal{S}_{\mathbb{R}}(\Omega)$ and $f\in\mathcal{S}(\Omega)\setminus\mathcal{S}_{I_0}(\Omega)$. Then there exists $M_0\in\mathbb{S}$ such that $h*f*h^c\in\mathcal{S}_{M_0}(\Omega)$ if and only if*

(i) in the case $I_0\perp M_0$, by setting $K_0=I_0M_0$, there exists $\rho\in\mathcal{S}_{\mathbb{R}}(\Omega)$ such that $(h^s)^2|\rho(h_0^2-h_1^2)$, $(h^s)^2|\rho h_0h_1$ and

$$f=f_0+\frac{\rho(h_0^2-h_1^2)}{(h^s)^2}M_0-2\frac{\rho h_0h_1}{(h^s)^2}K_0$$

(ii) in the case $\mathbb{C}_{I_0}\neq\mathbb{C}_{M_0}$ and $I_0\not\perp M_0$, by means of Notation 5.1, there exists $\rho\in\mathcal{S}_{\mathbb{R}}(\Omega)$ such that $h^s|\rho(h_0^2-h_1^2)$, $h^s|h_0h_1\rho$ and

$$f=f_0+a\rho I_0+\frac{b\rho(h_0^2-h_1^2)}{h^s}J_0-2\frac{b\rho h_0h_1}{h^s}K_0.$$

Proof. We first consider the case in which I_0 and M_0 are linearly independent. Using the basis given in Notation 5.1, we can decompose f_v and h_v as $f_1I_0+f_2J_0+f_3K_0$ and h_1I_0 respectively. Since $f\notin\mathcal{S}_{I_0}(\Omega)$

then f_2 and f_3 are not both identically zero. In order to apply Equality (5.1), we need to compute $h_v \wedge f_v = h_1(-f_3J_0 + f_2K_0)$ and $(h_v \wedge f_v) \wedge h_v = h_1^2(f_2J_0 + f_3K_0)$. So we obtain

$$h * f * h^c = [h * f * h^c]_0 + f_1 h^s I_0 + [f_2(h_0^2 - h_1^2) - 2h_0h_1f_3]J_0 + [f_3(h_0^2 - h_1^2) + 2h_0h_1f_2]K_0,$$

where $h^s = h_0^2 + h_1^2 \not\equiv 0$. Then the function $h * f * h^c$ is $\mathbb{C}_{M_0}$-preserving if and only if there exists $m_1 \in \mathcal{S}_\mathbb{R}(\Omega)$ such that

$$\begin{cases} f_1h^s = am_1, \\ f_2(h_0^2 - h_1^2) - 2h_0h_1f_3 = bm_1, \\ f_3(h_0^2 - h_1^2) + 2h_0h_1f_2 = 0. \end{cases}$$

Multiplying the second equation by $h_0^2 - h_1^2$, the third one by $2h_0h_1$ and summing up, we obtain $[(h_0^2 - h_1^2)^2 + 4h_0^2h_1^2]f_2 = bm_1(h_0^2 - h_1^2)$ that can also be written as $(h^s)^2 f_2 = bm_1(h_0^2 - h_1^2)$; analogously we find $(h^s)^2 f_3 = -2bm_1h_0h_1$.

If I_0 and M_0 are orthogonal, then $a = 0$, $b = 1$ and therefore $f_1 \equiv 0$ and $(h^s)^2$ divides both $m_1(h_0^2 - h_1^2)$ and $m_1h_0h_1$. Setting $\rho = m_1$ we can write

$$f = f_0 + \frac{\rho(h_0^2 - h_1^2)}{(h^s)^2} M_0 - \frac{2\rho h_0 h_1}{(h^s)^2} K_0$$

and we are done.

If I_0 and M_0 are not orthogonal, then $f_1h^s = am_1$; since $a \neq 0$ the function h^s divides m_1 and hence there exists $\rho \in \mathcal{S}_\mathbb{R}(\Omega)$ such that $m_1 = h^s\rho$. Thus $f_1 = a\rho$, moreover h^s divides both $\rho(h_0^2 - h_1^2)$ and $h_0h_1\rho$, so that we have

$$f = f_0 + a\rho I_0 + \frac{b\rho(h_0^2 - h_1^2)}{h^s} J_0 - 2\frac{b\rho h_0 h_1}{h^s} K_0.$$

In both cases, the converse is easily checked by direct inspection.

Now we are left to deal with the case $\mathbb{C}_{I_0} = \mathbb{C}_{M_0}$ showing that it cannot take place. Chosen any orthonormal basis I_0, J_0, K_0 of $\mathrm{Im}\mathbb{H}$, we again compute $h * f * h^c$ by means of the decomposition in real and vectorial part as above obtaining that $h * f * h^c$ belongs to $\mathcal{S}_{I_0}(\Omega)$ if and only if

$$\begin{cases} f_2(h_0^2 - h_1^2) - 2h_0h_1f_3 = 0, \\ f_3(h_0^2 - h_1^2) + 2h_0h_1f_2 = 0. \end{cases}$$

The same row reduction as above entails $f_2 = f_3 \equiv 0$ that contradicts the fact that $f \notin \mathcal{S}_{I_0}(\Omega)$. □

The following three examples give general, explicit applications of the previous result, clarifying the role of the divisibility conditions. In particular Example 5.7 completely describes the case when h is never-vanishing and Example 5.9 does the same when h is a polynomial.

Example 5.7. If h is never-vanishing, then h^s is never-vanishing and in particular it is invertible in $\mathcal{S}_\mathbb{R}(\Omega)$. In this case the conditions on the divisibility by h^s or $(h^s)^2$ are always satisfied, so if I_0 and M_0 are orthogonal then f is given by

$$f = f_0 + \rho\left(\frac{h_0^2 - h_1^2}{(h^s)^2} M_0 - 2\frac{h_0h_1}{(h^s)^2} K_0\right)$$

for any $f_0, \rho \in \mathcal{S}_\mathbb{R}(\Omega)$ and if I_0 and M_0 are lineraly independent but not orthogonal then f is given by

$$f = f_0 + \rho\left(aI_0 + \frac{b(h_0^2 - h_1^2)}{h^s} J_0 - 2\frac{bh_0h_1}{h^s} K_0\right)$$

for any $f_0, \rho \in \mathcal{S}_\mathbb{R}(\Omega)$.

Example 5.8. If $h_0 = h_1$, that is $h = h_0(1+I_0)$, then $h^s = 2h_0^2$ automatically divides both $\rho(h_0^2 - h_1^2) \equiv 0$ and $h_0 h_1 \rho = h_0^2 \rho$ for any $\rho \in \mathcal{S}_{\mathbb{R}}(\Omega)$. Thus if I_0 and M_0 are linearly independent and not orthogonal, then the divisibility conditions are always satisfied and hence $f = f_0 + \rho\,(aI_0 - bK_0)$ for any $f_0, \rho \in \mathcal{S}_{\mathbb{R}}(\Omega)$. If I_0 and M_0 are orthogonal, the first divisibility condition is trivial and the second becomes $h_0^4 | \rho h_0^2$ which is equivalent to $h_0^2 | \rho$. In this case the function ρ must be a multiple of h_0^2, so there exists $\mu \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that $\rho = -2h_0^2 \mu$ and hence f can be written as $f_0 + \mu K_0$ for any $f_0, \mu \in \mathcal{S}_{\mathbb{R}}(\Omega)$.

Example 5.9. If h_0, h_1 are polynomials in q (with real coefficients since they are slice preserving), then we can factor out their GCD α and write them as $h_0 = \alpha\beta_0$, $h_1 = \alpha\beta_1$ with β_0 and β_1 coprime. Now, $h^s = \alpha^2(\beta_0^2 + \beta_1^2)$. If I_0 and M_0 are linearly independent and not orthogonal, then the divisibility conditions become

$$\alpha^2(\beta_0^2 + \beta_1^2) | \rho\alpha^2(\beta_0^2 - \beta_1^2) \qquad \text{and} \qquad \alpha^2(\beta_0^2 + \beta_1^2) | \rho\alpha^2\beta_0\beta_1$$

which are equivalent to

$$(\beta_0^2 + \beta_1^2) | \rho(\beta_0^2 - \beta_1^2) \qquad \text{and} \qquad (\beta_0^2 + \beta_1^2) | \rho\beta_0\beta_1.$$

As $\beta_0^2 + \beta_1^2$ and $\beta_0^2 - \beta_1^2$ are coprime in the ring $\mathbb{R}[q]$, then the ideal they generate in $\mathbb{R}[q]$ coincides with $\mathbb{R}[q]$. This entails that $\beta_0^2 + \beta_1^2$ divides ρ which therefore can be written as $(\beta_0^2 + \beta_1^2)\mu$ for a suitable $\mu \in \mathcal{S}_{\mathbb{R}}(\Omega)$. The second divisibility condition now becomes trivial and hence we can write f as

$$\begin{aligned} f &= f_0 + a(\beta_0^2 + \beta_1^2)\mu I_0 + b\mu(\beta_0^2 - \beta_1^2)J_0 - 2b\mu\beta_0\beta_1 K_0 \\ &= f_0 + \mu\left(a(\beta_0^2 + \beta_1^2)I_0 + b(\beta_0^2 - \beta_1^2)J_0 - 2b\beta_0\beta_1 K_0\right) \end{aligned}$$

for suitable $f_0, \mu \in \mathcal{S}_{\mathbb{R}}(\Omega)$.

If I_0 and M_0 are orthogonal, then the divisibility conditions become

$$\alpha^2(\beta_0^2 + \beta_1^2)^2 | \rho(\beta_0^2 - \beta_1^2) \qquad \text{and} \qquad \alpha^2(\beta_0^2 + \beta_1^2)^2 | \rho\beta_0\beta_1.$$

Again $(\beta_0^2 + \beta_1^2)^2$ and $\beta_0^2 - \beta_1^2$ are coprime in the ring $\mathbb{R}[q]$, then $(\beta_0^2 + \beta_1^2)^2$ divides ρ which therefore can be written as $(\beta_0^2 + \beta_1^2)^2\mu$ for a suitable $\mu \in \mathcal{S}_{\mathbb{R}}(\Omega)$. The above relations become

$$\alpha^2 | \mu(\beta_0^2 - \beta_1^2) \qquad \text{and} \qquad \alpha^2 | \mu\beta_0\beta_1.$$

Since $\beta_0^2 - \beta_1^2$ and $\beta_0\beta_1$ are coprime in the ring $\mathbb{R}[q]$, then the ideal generated by $\mu(\beta_0^2 - \beta_1^2)$ and $\mu\beta_0\beta_1$ in $\mathcal{S}_{\mathbb{R}}(\Omega)$ coincides with the ideal generated by μ and therefore α^2 divides μ in $\mathcal{S}_{\mathbb{R}}(\Omega)$. Thus we can write $\mu = \alpha^2\nu$ for a suitable slice preserving function ν. Hence we can write f as

$$\begin{aligned} f &= f_0 + \frac{\alpha^2\nu(\beta_0^2 + \beta_1^2)^2\alpha^2(\beta_0^2 - \beta_1^2)}{\alpha^4(\beta_0^2 + \beta_1^2)^2} M_0 - 2\frac{\alpha^2\nu(\beta_0^2 + \beta_1^2)^2\alpha^2\beta_0\beta_1}{\alpha^4(\beta_0^2 + \beta_1^2)^2} K_0 \\ &= f_0 + \nu\left((\beta_0^2 - \beta_1^2)M_0 - 2\beta_0\beta_1 K_0\right) \end{aligned}$$

for suitable $f_0, \nu \in \mathcal{S}_{\mathbb{R}}(\Omega)$. As $(\beta_0^2 - \beta_1^2)M_0 - 2\beta_0\beta_1 K_0 = (\beta_0 - \beta_1 I_0)^2 M_0 = \left(\frac{h^c}{\alpha}\right)^2 M_0$, last equality can also be written as

$$f = f_0 + \nu\left(\frac{h^c}{\alpha}\right)^2 M_0$$

for some $f_0, \nu \in \mathcal{S}_{\mathbb{R}}(\Omega)$.

We now change slightly our point of view by studying the existence of a one-slice preserving solution of the equation $h * f * h^c = g$, given the function g and one between f and h which is chosen to be one-slice preserving. In Theorem 5.10, we answer this question when f is one-slice preserving and g is given: we look for the solvability of $g = h * f * h^c$, where the solution h should be again one-slice preserving; Theorem 5.11 answers the same issue exchanging the role of f and h.

From now until the end of this section we ask that $\Omega \subset \mathbb{H}$ is such that $\Omega_{I_0} = \Omega \cap \mathbb{C}_{I_0}$ has trivial first fundamental group, i.e.: $\pi_1(\Omega_{I_0}) = 0$ for one, and then any, $I_0 \in \mathbb{S}$.

Given $f \in \mathcal{S}_{I_0}(\Omega)$, a slice $\mathbb{C}_{M_0}$ with $M_0 \in \mathbb{S}$, and $g \in \mathcal{S}(\Omega)$, if I_0, M_0 are linearly independent we follow Notation 5.1 writing $f = f_0 + f_1 I_0$, and $g = g_0 + g_1 I_0 + g_2 J_0 + g_3 K_0$, with $f_0, f_1, g_0, g_1, g_2, g_3 \in \mathcal{S}_{\mathbb{R}}(\Omega)$.

Theorem 5.10. *Given f, M_0 and g as above, there exists $h \in \mathcal{S}_{M_0}(\Omega)$, such that $g = h * f * h^c$ if and only if*

(i) in the case $\mathbb{C}_{M_0} = \mathbb{C}_{I_0}$, then $g \in \mathcal{S}_{I_0}(\Omega)$, f divides g, the quotient g/f belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$, it is non-negative over the reals and all its real zeros have even multiplicity;

(ii) in the case $\mathbb{C}_{M_0} \neq \mathbb{C}_{I_0}$ and $M_0 \not\perp I_0$, then f_0 divides g_0 and f_1 divides g_v; denoted by α_0 the quotient g_0/f_0 and α_l the quotient g_l/f_1, for $l = 1, 2, 3$, we have that α_0 and α_2 are non-negative over the reals and their real zeros have even multiplicity; moreover α_2 has spherical zeroes of multiplicity which is a multiple of 4; the multiplicities of the zeroes of α_3 are at least equal to half the multiplicities of the zeroes of α_2 and

$$a(\alpha_0 - \alpha_1) = b\alpha_2, \tag{5.4}$$

$$2ab\alpha_1\alpha_2 = a^2\alpha_3^2 + (2a^2 - 1)\alpha_2^2. \tag{5.5}$$

(iii) in the case $M_0 \perp I_0$, then f_0 divides g_0, the quotient $\alpha_0 = g_0/f_0$ is non-negative over the reals and its real zeros have even multiplicity; $g_2 \equiv 0$; f_1 divides g_v and denoted by α_l the quotient g_l/f_1, for $l = 1, 3$, we have

$$\alpha_0^2 = \alpha_1^2 + \alpha_3^2. \tag{5.6}$$

Proof. *(i)* In this case $h * f * h^c = h^s f$. Thus $g \in \mathcal{S}_{I_0}(\Omega)$, the function f divides g and the quotient g/f belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$, it is non-negative over the reals and has real zeroes of even multiplicities. Vice versa if $g \in \mathcal{S}_{I_0}(\Omega)$ can be written as $g = \alpha f$, for $\alpha \in \mathcal{S}_{\mathbb{R}}(\Omega)$ with α non-negative over the reals and having real zeroes of even multiplicity, thanks to Theorem 3.2, we can find $h \in \mathcal{S}_{I_0}(\Omega)$ such that $h^s = \alpha$ and hence $g = h^s f = h * f * h^c$.

If $\mathbb{C}_{I_0} \neq \mathbb{C}_{M_0}$ we use Notation 5.1 and we write $h = h_0 + h_1 M_0$ for suitable $h_0, h_1 \in \mathcal{S}_{\mathbb{R}}(\Omega)$. According to (5.1) we have

$$h * f * h^c = f_0(h_0^2 + h_1^2) + f_1[(h_0^2 + (2a^2 - 1)h_1^2)I_0 + 2abh_1^2 J_0 - 2bh_0h_1K_0].$$

Then $h * f * h^c = g$ is equivalent to the following system of equations:

$$\begin{cases} g_0 = f_0(h_0^2 + h_1^2) \\ g_1 = f_1(h_0^2 + (2a^2 - 1)h_1^2) \\ g_2 = 2abf_1h_1^2 \\ g_3 = -2bf_1h_0h_1. \end{cases} \tag{5.7}$$

(ii) In this case both a and b are positive, so if $g = h * f * h^c$, we have (5.7) and the necessity is straightforward. Vice versa the conditions on α_2 guarantee the existence of $h_1 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that $\alpha_2 = 2abh_1^2$; in particular the multiplicities of the zeroes of h_1 are equal to half of the multiplicities of the zeroes of α_2. The relations on the multiplicities of the zeroes of α_2 and α_3 allow us to find $h_0 = \frac{\alpha_3}{-2bh_1} \in \mathcal{S}_{\mathbb{R}}(\Omega)$. Equality (5.5) gives that

$$\alpha_1 = \frac{a^2\alpha_3^2 + (2a^2 - 1)\alpha_2^2}{2ab\alpha_2}.$$

Substituting the formulas for α_2 and α_3 in terms of h_0 and h_1 in the previous equality we find $\alpha_1 = h_0^2 + (2a^2 - 1)h_1^2$. Now (5.4) gives $\alpha_0 = \alpha_1 + \frac{b}{a}\alpha_2$. By substituting α_1 and α_2 in terms of h_0 and h_1 and

recalling that $a^2+b^2=1$ we have $\alpha_0=h_0^2+h_1^2$ and thus we find $h=h_0+h_1M_0$ such that $h*f*h^c=g$.
(iii) In this case $a=0$ and $b=1$, so if $g=h*f*h^c$, we have $g_2\equiv 0$ and

$$\begin{cases} g_0=f_0(h_0^2+h_1^2) \\ g_1=f_1(h_0^2-h_1^2) \\ g_3=-2f_1h_0h_1 \end{cases}$$

and the necessity of the conditions is trivial. Vice versa suppose that f_0 divides g_0, the quotient $\alpha_0=g_0/f_0$ is non-negative over the reals and has real zeros of even multiplicity; $g_2\equiv 0$; f_1 divides g_v and, by denoting by α_l the quotient g_l/f_1, for $l=1,3$, Equality (5.6) is satisfied. Thanks to Proposition 3.1 we can get rid of the common spherical zeroes of α_0 and α_1 and so of α_3. Now the hypothesis on the sign of α_0 over the reals, together with (5.6), entail that $u=\frac{\alpha_0+\alpha_1}{2}$ and $v=\frac{\alpha_0-\alpha_1}{2}$ both belong to $\mathcal{S}_{\mathbb{R}}(\Omega)$, are non-negative over the reals and thus their real zeroes must have even multiplicity. Since α_0 and α_1 have no common spherical zeroes, then the same holds for u and v. If S_0 is a spherical zero for u, then it is also a spherical zero for α_3^2 and hence for α_3. The fact that it is not a spherical zero for v ensures that it is a spherical zero of u with multiplicity which is a multiple of 4; the same property on the multiplicities of spherical zeroes holds for v. Thus, thanks to Proposition 3.1 in [3], we can find $h_0,h_1\in\mathcal{S}_{\mathbb{R}}(\Omega)$, such that $u=h_0^2$ and $v=h_1^2$. The definitions of u and v entail that $\alpha_0=h_0^2+h_1^2$ and $\alpha_1=h_0^2-h_1^2$. Thus (5.6) gives $\alpha_3^2=4h_0^2h_1^2$; up to a change of sign for h_1 we obtain $\alpha_3=-2h_0h_1$, that is $g=h*f*h^c$ for $h=h_0+h_1M_0$. □

Now we deal with the "dual" problem of giving appropriate conditions on a one-slice preserving function h and a function g in order to find a one-slice preserving map f such that $h*f*h^c=g$. Given $h\in\mathcal{S}_{M_0}(\Omega)\setminus\mathcal{S}_{\mathbb{R}}(\Omega)$, $\mathbb{C}_{I_0}$ a slice in $\mathbb{H}$ with $I_0\in\mathbb{S}$ and $g\in\mathcal{S}(\Omega)$, if I_0 and M_0 are linearly independent, we choose an orthonormal basis I_0,J_0,K_0 of Im$\mathbb{H}$ as in Notation 5.1. Moreover, we write $h=h_0+h_1M_0$, and $g=g_0+g_1I_0+g_2J_0+g_3K_0$, with h_0,h_1 and g_0,g_1,g_2,g_3 belonging to $\mathcal{S}_{\mathbb{R}}(\Omega)$.

Proposition 5.11. *Given h, I_0 and g as above, there exists $f\in\mathcal{S}_{I_0}(\Omega)$, such that $g=h*f*h^c$ if and only if*

(i) in the case $\mathbb{C}_{M_0}=\mathbb{C}_{I_0}$, we have $g\in\mathcal{S}_{I_0}(\Omega)$ and h^s divides g.

(ii) in the case $\mathbb{C}_{M_0}\neq\mathbb{C}_{I_0}$ and $M_0\not\perp I_0$, we have that h^s divides g_0, $h_0^2+(2a^2-1)h_1^2$ divides g_1, h_1^2 divides g_2, h_0h_1 divides g_3 and

$$h_0g_2+ah_1g_3=0, \tag{5.8}$$

$$(h_0^2+(2a^2-1)h_1^2)g_2=2abh_1^2g_1. \tag{5.9}$$

(iii) in the case $M_0\perp I_0$, we have that h^s divides g_0, $h_0^2-h_1^2$ divides g_1, h_0h_1 divides g_3, $g_2\equiv 0$ and

$$2h_0h_1g_1+(h_0^2-h_1^2)g_3=0. \tag{5.10}$$

Proof. *(i)* In this case $g=h^sf$, therefore the necessity of the conditions holds trivially. Vice versa if g belongs to $\mathcal{S}_{I_0}(\Omega)$ and h^s divides g then the quotient g/h^s is in $\mathcal{S}_{I_0}(\Omega)$ because h^s is slice preserving. The thesis is obtained by taking $f=g/h^s$.
If $\mathbb{C}_{I_0}\neq\mathbb{C}_{M_0}$, we write $f=f_0+f_1I_0$ for suitable $f_0,f_1\in\mathcal{S}_{\mathbb{R}}(\Omega)$. The computations performed in the proof of Theorem 5.10 entail the system of conditions (5.7).
(ii) In this case the necessity of conditions is again trivial from system (5.7). Vice versa, setting $f_0=g_0/h^s$, $f_1=g_2/(2abh_1^2)$, we obtain, thanks to (5.8) and (5.9) that the equality $h*f*h^c=g$ holds thanks to (5.7).
(iii) Again, the necessity of the conditions is straightforward. If $h_0\not\equiv 0$, then setting $f_0=g_0/h^s$ and $f_1=g_3/(-2h_0h_1)$ gives the thesis, thanks to (5.10) and (5.7). If $h_0\equiv 0$ we then have $h^s=h_1^2$ and (5.10) entails $g_3\equiv 0$. Then, setting $f_0=g_0/h_1^2$ and $f_1=-g_1/h_1^2$ ends the proof again thanks to (5.7). □

Remark 5.12. The fact that statement and proof of Proposition 5.11 are neater than the ones of Theorem 5.10 can be seen as a consequence that in a certain sense the equality $h * f * h^c = g$ is "linear" in f and "quadratic" in h.

6. More products

Given f and h such that $f*h$ is one-slice preserving, it is not always true that $h*f$ is one-slice preserving as well. The results obtained so far for the conjugate of a given function give us a better understanding of the behaviour of the $*$-product. In particular we are able to give necessary and sufficient conditions on the two factors in order that the two $*$-products in different orders are both one-slice preserving. The first result explicitly describes the two factors in terms of functions which are one-slice preserving, showing that if the products of two functions in the two possible orders are both one-slice preserving, then the two factors are obtained by suitably "twisting" two one-slice preserving function which preserve the same slice for a fixed quaternion.

Theorem 6.1. *Let $f, h \in \mathcal{S}(\Omega) \setminus \{0\}$. There exist $I_0, M_0 \in \mathbb{S}$ such that $f * h \in \mathcal{S}_{I_0}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$ and $h * f \in \mathcal{S}_{M_0}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$ if and only if*

(i) in the case $\mathbb{C}_{I_0} = \mathbb{C}_{M_0}$, either $f, h \in \mathcal{S}_{I_0}(\Omega)$ or there exist $J_0, K_0 \in \mathbb{S}$ both orthogonal to I_0 and $\tilde{f}, \tilde{h} \in \mathcal{S}_{I_0}(\Omega) \setminus \{0\}$ such that

$$f = \tilde{f} * K_0, \qquad h = J_0 * \tilde{h}. \tag{6.1}$$

(ii) in the case $\mathbb{C}_{I_0} \neq \mathbb{C}_{M_0}$, there exist $\tilde{f}, \tilde{h} \in \mathcal{S}_{I_0}(\Omega) \setminus \{0\}$, such that

$$f = \tilde{f} * \left(1 + \frac{a \pm 1}{b} K_0\right), \qquad h = \left(1 - \frac{a \pm 1}{b} K_0\right) * \tilde{h}, \tag{6.2}$$

where we follow Notation 5.1.

Proof. First of all we notice that

$$\begin{aligned} &h * (f * h) * h^c = h * f * h^s = h^s (h * f), \\ &f^c * (f * h) * f = f^s (h * f). \end{aligned} \tag{6.3}$$

As $f * h \in \mathcal{S}_{I_0}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$ and $h * f \in \mathcal{S}_{M_0}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$, Theorem 5.4 drives the remainder of the proof. *(i)* For the sufficiency of the conditions it is enough to perform direct computations keeping in mind that:

- if $J_0, K_0 \in \mathbb{S}$ are both orthogonal to I_0, then $J_0 K_0 \in \mathbb{C}_{I_0}$;
- if $g \in \mathcal{S}_{I_0}(\Omega)$ and $J_0 \in \mathbb{S}$ is orthogonal to I_0, then $J_0 * g = g^c * J_0$.

Vice versa, if at least one between f and h does not belong to $\mathcal{S}_{I_0}(\Omega)$, then, both f and h do not belong to $\mathcal{S}_{I_0}(\Omega)$ because both their $*$-products do. Thus Theorem 5.4 allows us to find $J_0, K_0 \in \mathbb{S}$ both orthogonal to I_0 and $g, \tilde{h} \in \mathcal{S}_{I_0}(\Omega)$ such that

$$f^c = K_0 * g, \qquad h = J_0 * \tilde{h}.$$

Setting $\tilde{f} = -g^c$ gives (6.1).
(ii) Again, the sufficiency of the conditions is proved by direct inspection. Vice versa, we observe that by first equality in (6.3), h cannot belong to $\mathcal{S}_{I_0}(\Omega)$. Then, by adopting Notation 5.1, we have that Theorem 5.4 ensures the existence of $g, \tilde{h} \in \mathcal{S}_{I_0}(\Omega)$ such that

$$f^c = \left(1 - \frac{a \pm 1}{b} K_0\right) * g, \qquad h = \left(1 - \frac{a \pm 1}{b} K_0\right) * \tilde{h}.$$

Setting $\tilde{f} = g^c$ gives (6.2). □

In the case one of the two factors appearing in the previous result is one-slice preserving itself, the special form of the two factors obtained in the statement becomes even more special.

Proposition 6.2. *Let* $f \in \mathcal{S}_{I_0}(\Omega)\backslash\mathcal{S}_{\mathbb{R}}(\Omega)$ *and* $h \in \mathcal{S}(\Omega)\backslash\{0\}$ *such that* $f*h \in \mathcal{S}_{M_0}(\Omega)$ *and* $h*f \in \mathcal{S}_{N_0}(\Omega)$*, then*

(i) in the case $\mathbb{C}_{M_0} = \mathbb{C}_{N_0}$*, either* $\mathbb{C}_{I_0} = \mathbb{C}_{M_0}$ *and* $f, h \in \mathcal{S}_{M_0}(\Omega)$ *or* $I_0 \perp M_0$ *and there exist* $\alpha \in \mathcal{S}_{\mathbb{R}}(\Omega)$*,* $\tilde{h} \in \mathcal{S}_{M_0}(\Omega)$ *and* $J_0 \perp M_0$ *such that* $f = \alpha I_0$ *and* $h = J_0 * \tilde{h}$*;*

(ii) in the case $\mathbb{C}_{M_0} \neq \mathbb{C}_{N_0}$*, setting* $K_0 = \frac{M_0 \wedge N_0}{|M_0 \wedge N_0|}$*,* $L_0 = -M_0K_0$ *(so that* M_0, L_0, K_0 *is a positive orthonormal basis) and* $N_0 = aM_0 + bL_0$*, we can write* $I_0 = lM_0 + mL_0 + nK_0$ *with* $bm + l(a \pm 1) = 0$ *and there exist* $\alpha \in \mathcal{S}_{\mathbb{R}}(\Omega)$*,* $\tilde{h} \in \mathcal{S}_{M_0}(\Omega)$ *such that*

$$f = \alpha\left(-\frac{n}{m} + M_0\right) * \left(1 + \frac{a \pm 1}{b}K_0\right), \qquad h = \left(1 - \frac{a \pm 1}{b}K_0\right) * \tilde{h}.$$

Proof. *(i)* In this case Theorem 6.1 entails that either $f, h \in \mathcal{S}_{M_0}(\Omega)$ so that $\mathbb{C}_{I_0} = \mathbb{C}_{M_0}$ or there exist $J_0, K_0 \in \mathbb{S}$ both orthogonal to M_0 and $\tilde{f}, \tilde{h} \in \mathcal{S}_{M_0}(\Omega) \setminus \{0\}$ such that $f = \tilde{f} * K_0$ and $h = J_0 * \tilde{h}$. Setting $\tilde{f} = \tilde{f}_0 + \tilde{f}_1 M_0$, a trivial computation shows that $f = \tilde{f}_0 K_0 + \tilde{f}_1 M_0 K_0$. Now consider the orthonormal basis $M_0, K_0, L_0 = M_0K_0$ of $\mathrm{Im}\mathbb{H}$ and write $I_0 = aM_0 + bK_0 + cL_0$. Since $f \in \mathcal{S}_{I_0}(\Omega)$ there exist $\alpha_0, \alpha_1 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that $f = \tilde{f}_0 K_0 + \tilde{f}_1 L_0 = \alpha_0 + \alpha_1 I_0 = \alpha_0 + a\alpha_1 M_0 + b\alpha_1 K_0 + c\alpha_1 L_0$. Uniqueness given by Proposition 1.4 entails $\alpha_0 = 0$, $a = 0$, $\tilde{f}_0 = b\alpha_1$, $\tilde{f}_1 = c\alpha_1$, so that $I_0 \perp M_0$ and $f = \alpha_1 I_0$.
(ii) Again by Theorem 6.1 there exist $\tilde{f}, \tilde{h} \in \mathcal{S}_{M_0}(\Omega) \setminus \{0\}$, such that

$$f = \tilde{f} * \left(1 + \frac{a \pm 1}{b}K_0\right), \qquad h = \left(1 - \frac{a \pm 1}{b}K_0\right) * \tilde{h}.$$

Setting $\tilde{f} = \tilde{f}_0 + \tilde{f}_1 M_0$ we find that $f_0 = \tilde{f}_0$ and

$$f_v = \tilde{f}_1 M_0 + \tilde{f}_0 \frac{a \pm 1}{b} K_0 + \tilde{f}_1 \frac{a \pm 1}{b} M_0 K_0 = \tilde{f}_1 M_0 - \tilde{f}_1 \frac{a \pm 1}{b} L_0 + \tilde{f}_0 \frac{a \pm 1}{b} K_0.$$

As $f \in \mathcal{S}_{I_0}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$ there exists $\gamma \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that $f_v = \gamma I_0$ which is equivalent to

$$\begin{cases} \tilde{f}_1 = \gamma l \\ -\frac{a \pm 1}{b}\tilde{f}_1 = \gamma m \\ \frac{a \pm 1}{b}\tilde{f}_0 = \gamma n \end{cases} \tag{6.4}$$

where $I_0 = lM_0 + mL_0 + nK_0$. Since $\tilde{f}_1 \not\equiv 0$ the comparison between the first two equations in (6.4) gives $bm + l(a \pm 1) = 0$, while last two equations entail $\tilde{f}_0 = -\frac{n}{m}\tilde{f}_1$, that is

$$\tilde{f} = -\frac{n}{m}\tilde{f}_1 + \tilde{f}_1 M_0 = \tilde{f}_1\left(-\frac{n}{m} + M_0\right).$$

Setting $\alpha = \tilde{f}_1$ ends the proof. □

Remark 6.3. We point out that, again by direct computation, we have that conditions *(i)* and *(ii)* in Proposition 6.2 are also sufficient in order to obtain $f * h \in \mathcal{S}_{M_0}(\Omega)$ and $h * f \in \mathcal{S}_{N_0}(\Omega)$.

7. ∗-Powers

In order to conclude our investigation on the structure of one-slice preserving functions, we turn our attention to the problem of classifying slice regular functions whose ∗-powers preserve one single slice or all of them. To rule out trivial cases, in this section we will always consider $f \in \mathcal{S}(\Omega) \setminus \mathcal{S}_{\mathbb{R}}(\Omega)$, which means that the vectorial part of f is not identically zero.

The first tool we need is the following computation of the $*$-powers of f in terms of the components of the splitting $f = f_0 + f_v$.

Lemma 7.1. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$, then*

$$f^{*d} = \sum_{n=0}^{[d/2]} (-1)^n \binom{d}{2n} f_0^{d-2n}(f_v^s)^n + \left(\sum_{n=0}^{[(d-1)/2]} (-1)^n \binom{d}{2n+1} f_0^{d-(2n+1)}(f_v^s)^n \right) f_v.$$

Proof. Since $f_v * f_v = -f_v * f_v^c = -f_v^s$, we have

$$\begin{aligned} f^{*d} = (f_0 + f_v)^{*d} &= \sum_{m=0}^{d} \binom{d}{m} f_0^{d-m}(f_v)^{*m} \\ &= \sum_{n=0}^{[d/2]} (-1)^n \binom{d}{2n} f_0^{d-2n}(f_v^s)^n + \left(\sum_{n=0}^{[(d-1)/2]} (-1)^n \binom{d}{2n+1} f_0^{d-(2n+1)}(f_v^s)^n \right) f_v. \end{aligned}$$

□

Remark 7.2. Notice that the above computation for $d = 2$ entails that $f^{*2} \in \mathcal{S}_{\mathbb{R}}(\Omega)$ if and only if $f_0 \equiv 0$.

Because of the above remark, from now on we take into account only functions $f = f_0 + f_v$ with $f_0 \not\equiv 0$. Moreover, in order to avoid trivial statements, if we are looking for a suitable $*$-power $d > 2$ of f which is one-slice preserving, we rule out the case in which f itself is one-slice preserving. In particular, thanks to Lemma 7.1, under this hypothesis we have the following

Remark 7.3. Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$ which preserves no slice. Then f^{*d} is one-slice preserving if and only if f^{*d} is slice preserving. This last condition is equivalent to

$$\sum_{n=0}^{[(d-1)/2]} (-1)^n \binom{d}{2n+1} f_0^{d-(2n+1)}(f_v^s)^n \equiv 0. \tag{7.1}$$

To simplify the statement of the results, from now on we only consider functions $f = f_0 + f_v \in \mathcal{S}(\Omega)$ such that f preserves no slice. In order to carry on our investigation we need to set some notation and quote a result on the real roots of a binary form.

We denote by $Q_d(x, y)$ the homogeneous polynomial of degree d given by

$$Q_d(x, y) = \sum_{n=0}^{[(d-1)/2]} (-1)^n \binom{d}{2n+1} x^{d-(2n+1)} y^{2n+1}.$$

We denote by $\Sigma_d \subset \mathbb{P}(\mathbb{R}^2) \sim \mathbb{R} \cup \{\infty\}$ the set of roots of Q_d different from 0 and ∞; a straightforward computation shows that $0 = [0 : 1]$ is a root of Q_d and only if d is even and that $\infty = [1 : 0]$ is always a root of Q_d. Due to Proposition 41 in [19], see also [6], Q_d has d real distinct roots and therefore Σ_d contains $d - 2$ elements if d is even and $d - 1$ if d is odd.

Now choose $q_0 \in \Omega \cap \mathbb{R}$ such that $f_0(q_0) \neq 0$ and $f_v(q_0) \neq 0$. Then we can choose a suitable spherical neighborhood $U = B_{q_0}(r)$ of the point q_0 where both f_0 and f_v never vanish, which entails that also f_v^s is never-vanishing on U. Thus Corollary 3.2 in [3] gives the existence of $\rho \in \mathcal{S}_{\mathbb{R}}(U)$ such that $\rho^2 = f_v^s$ on U.

Equality (7.1) thus implies that on U we have

$$\sum_{n=0}^{[(d-1)/2]} (-1)^n \binom{d}{2n+1} f_0^{d-(2n+1)} \rho^{2n+1} \equiv 0.$$

In particular, at any $q \in U$ the point $[f_0(q) : \rho(q)] \in \mathbb{R} \setminus \{0\}$ is a root of Q_d and thus there exists $\xi(q) \in \Sigma_d$ such that $[f_0(q) : \rho(q)] = \xi(q)$. As U is connected, Σ_d is a finite subset of $\mathbb{R}$ and the map $U \ni q \mapsto (f_0(q), \rho(q)) \in \mathbb{R}^2 \setminus \{(0,0)\}$ is continuous, then $\xi(q)$ is constant and hence $f_0 = \xi\rho$ on U. The Identity Principle then entails that $f_0^2 = \xi^2\rho^2 = \xi^2 f_v^s$ on all Ω. The above argument can be summarized as follows:

Proposition 7.4. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$. There exists $d > 2$ such that f^{*d} is slice preserving if and only if there exists $\xi \in \Sigma_d$ such that $f_0^2 \equiv \xi^2 f_v^s$, that is f_0 is a square root of $\xi^2 f_v^s$.*

If the first fundamental group of $\Omega_I = \Omega \cap \mathbb{C}_I$ is trivial, then the set of slice regular functions whose $d-th$ $*$-power belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ can be characterized even more precisely. Indeed we have

Corollary 7.5. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$ and suppose $\pi_1(\Omega_I) = \{0\}$ for some $I \in \mathbb{S}$. There exists $d > 2$ such that f^{*d} belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ if and only if the zero set of f_v does not contain non real isolated zeroes of odd multiplicities and there exists $\xi \in \Sigma_d$ such that f_0 is a square root of $\xi^2 f_v^s$.*

Proof. By Proposition 7.4 there exists $\xi \in \Sigma_d$ such that $f_0^2 \equiv \xi^2 f_v^s$ which is equivalent to $f_v^s = \frac{f_0^2}{\xi^2}$. Then the functions f_v^s has a square root $\frac{f_0}{\xi} \in \mathcal{S}_{\mathbb{R}}(\Omega)$; the hypothesis on the first fundamental group of Ω_I together with Corollary 3.2 in [3], entail that this is equivalent to the fact that the zero set of f_v does not contain non real isolated zeroes of odd multiplicities. □

The following example contains the explicit expressions of Q_d and Σ_d for $d = 3, \dots, 10$.

Example 7.6.

d	$Q_d(x,y)$	Σ_d
3	$3x^2y - y^3$	$\left\{\pm\frac{\sqrt{3}}{3}\right\}$
4	$4x^3y - 4xy^3$	$\{\pm 1\}$
5	$5x^4y - 10x^2y^3 + y^5$	$\left\{\pm\frac{\sqrt{25\pm 10\sqrt{5}}}{5}\right\}$
6	$6x^5y - 20x^3y^3 + 6xy^5$	$\left\{\pm\frac{\sqrt{3}}{3}, \pm\sqrt{3}\right\}$
7	$7x^6y - 35x^4y^3 + 21x^2y^5 - y^7$	$\left\{\pm\left(\frac{1}{3}\left(5+8\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{3}}{13}\right)\right)\right)\right)^{\frac{1}{2}}, \pm\left(\frac{1}{3}\left(5\pm 4\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{3}}{13}\right)\right)+ -4\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{3}}{13}\right)\right)\right)\right)^{\frac{1}{2}}\right\}$
8	$8x^7y - 56x^5y^3 + 56x^3y^5 - 8xy^7$	$\left\{\pm 1, \pm\sqrt{3\pm 2\sqrt{2}}\right\}$
9	$9x^8y - 84x^6y^3 + 126x^4y^5 - 36x^2y^7 + y^9$	$\left\{\pm\frac{\sqrt{3}}{3}, \pm\left(3+\frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}}\right)^{\frac{1}{2}}, \pm\left(3\pm 4\sin\left(\frac{\pi}{18}\right)-\frac{4\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}}\right)^{\frac{1}{2}}\right\}$
10	$10x^9y - 120x^7y^3 + 252x^5y^5 - 120x^3y^7 + 10xy^9$	$\left\{\pm\sqrt{1\pm\frac{2}{\sqrt{5}}}, \pm\sqrt{5\pm 2\sqrt{5}}\right\}$

Example 7.7. If $d = 4$ and $\pi_1(\Omega_{I_0}) = 0$, a function $f = f_0 + f_v \in \mathcal{S}(\Omega)$ which preserves no slice and has non-zero real part is such that f^{*4} is slice preserving if and only if f_v does not have non-real isolated zeroes of odd multiplicity and $f_0 = \pm\sqrt{f_v^s}$.

References

1. A. Altavilla, Some properties for quaternionic slice-regular functions on domains without real points. Complex Var. Elliptic Equ. 60, n. 1 (2015), 59–77.
2. A. Altavilla, Twistor interpretation of slice regular functions, J. Geometry Phys. 123C (2018) pp. 184-208, https://doi.org/10.1016/j.geomphys.2017.09.007.
3. A. Altavilla, C. de Fabritiis, $*$-exponential of slice-regular functions, to appear.
4. F. Colombo, G. Gentili, I. Sabadini, D. C. Struppa, Extension results for slice regular functions of a quaternionic variable. Adv. Math. 222(5), (2009), 1793–1808.
5. F. Colombo, J. Oscar Gonzalez-Cervantes, I. Sabadini, The C-property for slice regular functions and applications to the Bergman space, Compl. Var. Ell. Eq., 58, n. 10 (2013), 1355–1372.
6. A. Causa, R. Re, On the maximum rank of a real binary form, Ann. Mat. Pura, Appl. vol. 190, (2010) p. 55–59.
7. F. Colombo, I. Sabadini, D. C. Struppa, Noncommutative Functional Calculus, Progress In Mathematics, Birkhauser, 2011.
8. F. Colombo, I. Sabadini, D. C. Struppa, Entire Slice Regular Functions, SpringerBriefs in Mathematics, Springer, 2016.
9. C. de Fabritiis, G. Gentili, G. Sarfatti, Quaternionic Hardy Spaces, to appear in Ann. SNS., DOI: 10.2422/2036-2145.201502_001
10. S. G. Gal, I. Sabadini, Approximation by polynomials on quaternionic compact sets, Math. Meth. Appl. Sci., 38 (2015) 3063–3074.
11. G. Gentili, S. Salamon, C. Stoppato, Twistor transforms of quaternionic functions and orthogonal complex structures, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 11, 2323–2353.
12. G.Gentili, D. C. Struppa, A new theory of regular functions of a quaternionic variable, Adv. Math. 216 (2007), no. 1, 279–301.
13. G. Gentili, C. Stoppato, Zeros of regular functions and polynomials of a quaternionic variable. Mich. Math. J. 56(3) (2008), 655–667.
14. G. Gentili, C. Stoppato, D. C. Struppa, Regular Functions of a Quaternionic Variable, Springer Monographs in Matehmatics, Springer, 2013.
15. G. Gentili, I. Vignozzi, The Weierstrass factorization theorem for slice regular functions over the quaternions. Ann. Global Anal. Geom. 40(4), (2011), 435–466.
16. R. Ghiloni, V. Moretti, A. Perotti, Continuous Slice Functional Calculus in Quaternionic Hilbert Spaces, Rev. Math. Phys. 25 (2013), 1350006-1-1350006-83.
17. R. Ghiloni, A. Perotti, Power and spherical series over real alternative *-algebras, Indiana University Mathematics Journal, Vol. 63, No. 2, Pages 495-532 (2014)
18. R. Ghiloni, A. Perotti, Slice regular functions on real alternative algebras, Adv. in Math., v. 226, n. 2 (2011), 1662-1691.
19. M. Maccioni, Tensor rank and eigenvectors, Ph. D. thesis, available at http://web.math.unifi.it/users/ottavian/tesi/tesimaccioni.pdf
20. G. Ren, X. Wang, Slice Regular Composition Operators, Compl. Var. Ell. Equ., 61-5 (2015) 682–711.
21. W.Rudin, Real and Complex Analysis, Third Edition, McGraw–Hill, 2011
22. C. Stoppato, Regular Moebius transformations of the space of quaternions, Ann. Glob. Anal. Geom. 39 (4), (2011) 387–401.
23. F. Vlacci, Regular Composition for Slice–Regular Functions of Quaternionic Variable, Advances in Hypercomplex Analysis, Springer INDAM Series 1, Springer-Verlag pp. 141–148, 2013.

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