$*$-exponential of Slice Regular Functions

note finali coverpage

(Article begins on next page)

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∗-EXPONENTIAL OF SLICE-REGULAR FUNCTIONS

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ABSTRACT. According to [?] we define the ∗-exponential of a slice-regular function, which can be seen as a generalization of the complex exponential to quaternions. Explicit expressions for $\exp_*(f)$ are provided, also in terms of suitable sine and cosine functions. We completely classify under which conditions the ∗-exponential of a function is either slice-preserving or $\mathbb{C}_J$-preserving for some $J \in \mathbb{S}$ and show that $\exp_*(f)$ is never-vanishing. Sharp necessary and sufficient conditions are given in order that $\exp_*(f + g) = \exp_*(f) * \exp_*(g)$, finding a neat, unexpected case in which equality holds even if f and g do not commute. We also discuss the existence of a square root of a slice-preserving regular function, characterizing slice-preserving functions (defined on the circularization of simply connected domains) which admit square roots. A number of examples is given throughout the paper.

1. INTRODUCTION

In classical complex analysis the exponential map has a tremendous role in the study of growth of holomorphic functions, differential equations and uniformization theorem. In this paper we investigate the behaviour of a quaternionic analogous of the complex exponential map. In particular, special features of the exponential in the complex case include the fact that it never vanishes, the expression of the exponential of a pure imaginary in terms of sine and cosine, and the formula which gives the exponential of a sum as the product of the exponentials of the two summands. These properties will be the object of our study in the setting of slice-regularity.

We briefly introduce quaternions and quaternionic slice-regular functions. Given an alternating triple i, j, k with $i^2 = j^2 = k^2 = -1$ we denote by $\mathbb{H}$ the real algebra of quaternions

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k \; : \; q_0, q_1, q_2, q_3 \in \mathbb{R}\}.$$

The conjugation on $\mathbb{H}$ is given by $q^c = q_0 - (q_1 i + q_2 j + q_3 k)$ and we sometimes write $\mathrm{Re}q = q_0 = \frac{1}{2}(q + q^c)$ and $\vec{q} = q - \mathrm{Re}q$, so that $q = q_0 + \vec{q}$.

The following subsets of $\mathbb{H}$ have special interest:

$$\mathbb{S} = \{q \in \mathbb{H} \; : \; q^2 = -1\} = \{q_1 i + q_2 j + q_3 k \; : \; q_1^2 + q_2^2 + q_3^2 = 1\}$$
$$\mathrm{Im}\mathbb{H} = \{q \in \mathbb{H} : \mathrm{Re}q = 0\} = \cup_{I \in \mathbb{S}} \mathbb{R}I;$$

the first is isomorphic to the real sphere $S^2 \subset \mathbb{R}^3$ and the second to $\mathbb{R}^3$. For any $I \in \mathbb{S}$ we set $\mathbb{C}_I = \mathrm{Span}_{\mathbb{R}}(1, I)$.

Given any set $D \subset \mathbb{C} = \{\alpha + \imath\beta \,|\, \alpha, \beta \in \mathbb{R}\}$, we define its *circularization* as

$$\Omega_D = \{\alpha + \beta I \,|\, \alpha + \imath\beta \in D, I \in \mathbb{S}\} \subseteq \mathbb{H}.$$

Subsets of $\mathbb{H}$ of this form will be called *circular domains*; in order to simplify the notation we will often drop the subscript D.

The following definitions identify the functions we will work with.

Definition 1.1. Let $D \subset \mathbb{C}$ be closed under conjugation. A *stem* function F on D is a function with values in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ such that $F(\bar{z}) = \overline{F(z)}$ where $\overline{p + \imath q} = p - \imath q$ for any $p + \imath q \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. A *slice* function $f : \Omega_D \to \mathbb{H}$ is a function induced by a suitable stem function $F = F_1 + \imath F_2$ in the following way $f(\alpha + \beta I) = F_1(\alpha + \imath\beta) + I F_2(\alpha + \imath\beta)$. Such a function will also be denoted by $f = \mathcal{I}(F)$.

Definition 1.2. Let $\Omega \subset \mathbb{H}$ be a circular domain. A *slice* function $f = \mathcal{I}(F) : \Omega \to \mathbb{H}$ is *regular* if F is holomorphic with respect to the natural complex structures of $\mathbb{C}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. In particular, the set of slice-regular function $\mathcal{S}(\Omega)$ is a right $\mathbb{H}$-module.

If Ω contains real points, a function is slice-regular if and only if it is regular in the sense of Cullen (see [**?**]). A useful result for slice-regular functions is the following

Proposition 1.3 (Representation formula)**.** *Let $f \in \mathcal{S}(\Omega)$ and let $\alpha + \beta J \in \Omega$. For all $I, J \in \mathbb{S}$ we have*

$$f(\alpha + \beta J) = \frac{1 - JI}{2} f(\alpha + \beta I) + \frac{1 + JI}{2} f(\alpha - \beta I).$$

Therefore if $f_I : \Omega_I = \Omega \cap \mathbb{C}_I \to \mathbb{H}$ is a holomorphic function, then there exists a unique slice-regular function $g : \Omega \to \mathbb{H}$ such that $f_I = g|_{\Omega_I}$; such a function will be called the *regular extension of f_I*, (see [**?**], p 9).

In general the pointwise product of two slice-regular functions is no more slice. Nonetheless, this problem can be overcome by defining the following non-commutative product (see [**?**, **?**]).

Definition 1.4. Let $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ be two slice functions on Ω. We denote by $f * g$ their *∗-product* defined by $f * g = \mathcal{I}(FG)$ where FG is the pointwise product with values in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, *i.e.* $(p + \imath q)(p' + \imath q') = pp' - qq' + \imath(pq' + qp')$.

In $\mathcal{S}(\Omega)$ we can also define the *conjugate function* of f that is $f^c = \mathcal{I}(F_1^c + \imath F_2^c)$ if $f = \mathcal{I}(F_1 + \imath F_2)$ and the functions F_1^c and F_2^c are obtained from F_1 and F_2 by pointwise conjugation in $\mathbb{H}$. Given $f \in \mathcal{S}(\Omega)$ its *symmetrized function* f^s is given by $f^s = f^c * f = f * f^c$.

It is known that if $\alpha + \beta I$ is such that $f(\alpha + \beta I) = 0$, then any point in the set $\mathbb{S}_{\alpha + \beta I} = \{\alpha + \beta J \,|\, J \in \mathbb{S}\}$ is a zero for f^s. In particular if f^s is never-vanishing then also f is. Moreover, the zero set of a slice-regular function not identically equal to zero is closed with empty interior and, if $f^s \not\equiv 0$, it is a union of isolated points and isolated spheres of the form $\mathbb{S}_{\alpha + \beta J}$ for suitable $\alpha, \beta \in \mathbb{R}$ (see [**?**, **?**]).

We now introduce two special classes of slice-regular functions. A function $f = \mathcal{I}(F_1 + \imath F_2) \in \mathcal{S}(\Omega)$ is *slice-preserving* if both F_1 and F_2 are real-valued; the class of such functions will be denoted by $\mathcal{S}_{\mathbb{R}}(\Omega)$ (they are called $\mathbb{H}$-intrinsic in [**?**] and quaternionic-intrinsic in [**?**]). We remark that for any $f \in \mathcal{S}(\Omega)$ its symmetrized function f^s belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$.

Given $J \in \mathbb{S}$, a function $f = \mathcal{I}(F_1 + \imath F_2) \in \mathcal{S}(\Omega)$ is said to be $\mathbb{C}_J$-*preserving* if both F_1 and F_2 are $\mathbb{C}_J$-valued; the class of such functions will be denoted by $\mathcal{S}_J(\Omega)$ (see [?]).

For slice-preserving and $\mathbb{C}_J$-preserving functions, the ∗-product has special features:

- for any $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ and $g \in \mathcal{S}(\Omega)$ we have $f * g = g * f = fg$ (that is, the ∗-product $f * g$ coincides with the pointwise product fg);
- choosen $J \in \mathbb{S}$, for any $f, g \in \mathcal{S}_J(\Omega)$ we have $f * g = g * f$ (and $f * g$ coincides with the regular extension of the pointwise product of f_J and g_J);
- if $\rho_1, \rho_2 \in \mathcal{S}_{\mathbb{R}}(\Omega)$ and $a_1, a_2 \in \mathbb{H}$ then $(\rho_1 a_1) * (\rho_2 a_2) = \rho_1 \rho_2 a_1 a_2$.

We have now the main tools to define the ∗-exponential of a slice-regular function. The first definition is the natural extension of the complex exponential to $\mathbb{H}$ and indeed its restriction to any complex line $\mathbb{C}_I$, $I \in \mathbb{S}$, coincides with the complex exponential.

The function $\exp : \mathbb{H} \to \mathbb{H}$ is given by $\exp q = e^q = \sum_{n \in \mathbb{N}} \frac{q^n}{n!}$; trivially $\exp \in \mathcal{S}_{\mathbb{R}}(\mathbb{H})$.

Given $f \in \mathcal{S}(\Omega)$ the composition $\exp \circ f$ is not always slice-regular. In [?], Colombo, Sabadini and Struppa gave the following definition which coincides with $\exp \circ f$ if $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ (see also [?] and [?], where several different regular compositions are introduced).

Definition 1.5. If $f \in \mathcal{S}(\Omega)$ the ∗-exponential of f is defined as $\exp_*(f) = \sum_{n \in \mathbb{N}} \frac{f^{*n}}{n!}$.

This function will be the main object of our study, which will be organized as follows.

Section 2 contains a new interpretation of the ∗-product given by Proposition **??**. This result allows us to give a necessary and sufficient condition for the commutation of two functions with respect to this product. We will exploit Formula (**??**) extensively in Section 4, where it will be a new, useful tool to simplify calculations.

In Section 3 we investigate on the possibility of finding a square root of a slice-preserving regular function in the space $\mathcal{S}_{\mathbb{R}}(\Omega)$, with $\Omega_I = \mathbb{C}_I \cap \Omega$ simply connected, obtaining a complete answer to this question. The result is the following.

Proposition 1.6. *Given* $h \in \mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$ *there exists* $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ *such that* $f^2 = h$ *if and only if*

(i) *the zero set of h consists of isolated zeroes of even multiplicity and isolated spheres with real center with multiplicity multiple of 4;*

(ii) *if* $\Omega \cap \mathbb{R} \neq \emptyset$ *then* $h(\Omega \cap \mathbb{R}) \subseteq [0, +\infty)$.

In particular we apply this result when $h = g^s$ for some $g \in \mathcal{S}(\Omega)$.

Last section is devoted to the study of the ∗-exponential of a slice-regular function. We first give a sufficient condition, namely commutativity of the ∗-product of f and g, for the equality $\exp_*(f + g) = \exp_*(f) * \exp_*(g)$ to hold. Given $f = f_0 + f_v$ (see Definition **??** and subsequent remark for the notation), we then find an explicit expression for $\exp_*(f)$ in terms of f_0 and f_v. In Corollaries **??** and **??** this allows us to rewrite $\exp_*(f)$ in terms of suitable sine and cosine functions. Moreover we completely classify under which conditions the ∗-exponential of a function is either in $\mathcal{S}_{\mathbb{R}}(\Omega)$ or in $\mathcal{S}_J(\Omega)$ for some $J \in \mathbb{S}$.

In Proposition **??** we compute the symmetrized function of the ∗-exponential of f in terms of the real part of f, proving as a consequence that $\exp_*(f)$ is never-vanishing. Finally, in Theorem **??** we give necessary and sufficient conditions on f and g in order that

$$\exp_*(f + g) = \exp_*(f) * \exp_*(g);$$

quite surprisingly we are able to find a large bunch of slice regular functions which do not commute under the $*$-product and for which nonetheless the above equality holds. This last result is followed by four examples which illustrate the sharpness of the required conditions.

2. REGULAR PRODUCT: A FURTHER INTERPRETATION

The following result, which is due to Colombo, Gonzales-Cervantes and Sabadini (see [**?**], Proposition 3.12), gives a way to decompose a given slice-regular function by means of 4 slice-preserving regular functions (see also [**?**]).

Proposition 2.1. *Let $\{1, i, j, k\}$ be a basis of $\mathbb{H}$. Then the map*

$$(\mathcal{S}_\mathbb{R}(\Omega))^4 \ni (f_0, f_1, f_2, f_3) \mapsto f_0 + f_1 i + f_2 j + f_3 k \in \mathcal{S}(\Omega)$$

is bijective. In particular it follows that given any $f \in \mathcal{S}(\Omega)$ there exist and are unique $f_0, f_1, f_2, f_3 \in \mathcal{S}_\mathbb{R}(\Omega)$ such that

$$f = f_0 + f_1 i + f_2 j + f_3 k.$$

Moreover if $i \in \mathbb{S}$ then $f \in \mathcal{S}_i(\Omega)$ iff $f_2 \equiv f_3 \equiv 0$.

Remark 2.2. Notice that (f_0, f_1, f_2, f_3) is the 4-tuple associated to f if and only if $(f_0, -f_1, -f_2, -f_3)$ is the 4-tuple associated to f^c; in particular $f_0 = \frac{f + f^c}{2}$. This implies that the function f_0 does not depend on the chosen real basis of $\mathbb{H}$, provided the first element is equal to 1.

Definition 2.3. Given $f \in \mathcal{S}(\Omega)$, we define the slice-regular function f_v on Ω by $f_v = \frac{f - f^c}{2}$.

Remark 2.4. Trivially $f_0 + f_v = f$ and $f_0 - f_v = f^c$. According to the notation of Proposition **??** we have $f_v = f_1 i + f_2 j + f_3 k$.

The reason we choose the underscore v for the above function is that, in a certain sense, it represents the "vector" part of the quaternionic-valued function f, according to the splitting $q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + \vec{q}$.

The above result and definition allow us to describe the $*$-product of 2 slice-regular functions in terms of intrinsic operators.

Definition 2.5. Given $f, g \in \mathcal{S}(\Omega)$, we denote by $f \mathbb{A} g$ and $\langle f, g \rangle_*$ the slice-regular functions given by

$$(f \mathbb{A} g)(q) = \frac{(f * g)(q) - (g * f)(q)}{2}, \qquad \langle f, g \rangle_*(q) = (f * g^c)_0$$

for any $q \in \Omega$.

Remark 2.6. In terms of the notation of Proposition **??** we can rewrite the above intrinsic expressions in the following form:

$$f \mathbb{A} g = (f_2 g_3 - f_3 g_2)i + (f_3 g_1 - f_1 g_3)j + (f_1 g_2 - f_2 g_1)k$$
$$\langle f, g \rangle_* = f_0 g_0 + f_1 g_1 + f_2 g_2 + f_3 g_3.$$

which in particular show that

$$f \mathbb{A} g = f_v \mathbb{A} g_v = -g_v \mathbb{A} f_v$$
$$\langle f, g \rangle_* = \langle g, f \rangle_* = \langle f^c, g^c \rangle_* = f_0(q)g_0(q) - (f_v * g_v)_0(q).$$

These operators can be used to write the ∗-product of two slice-functions in a more explicit form which recalls the formula for the product of two quaternions in the scalar-vector form. This different expression of the ∗-product will turn out to be a useful tool in the study of slice regular function, simplifying and speeding formerly long and complicated computations.

Proposition 2.7. *The ∗ product of $f = f_0 + f_v$ and $g = g_0 + g_v$ is given by*

$$(2.1) \qquad f * g = f_0 g_0 - \langle f_v, g_v \rangle_* + f_0 g_v + g_0 f_v + f_v \wedge g_v$$

Proof. Choosing an orthonormal basis $\{1, i, j, k\}$ of $\mathbb{H}$ on $\mathbb{R}$ and writing $f = f_0 + f_1 i + f_2 j + f_3 k$ and $g = g_0 + g_1 i + g_2 j + g_3 k$ we have

$$\begin{aligned}
f * g &= (f_0 + f_1 i + f_2 j + f_3 k) * (g_0 + g_1 i + g_2 j + g_3 k) \\
&= f_0 g_0 - f_1 g_1 - f_2 g_2 - f_3 g_3 + f_0(g_1 i + g_2 j + g_3 k) + g_0(f_1 i + f_2 j + f_3 k) \\
&\quad + (f_2 g_3 - f_3 g_2)i + (f_3 g_1 - f_1 g_3)j + (f_1 g_2 - f_2 g_1)k \\
&= f_0 g_0 - \langle f_v, g_v \rangle_* + f_0 g_v + g_0 f_v + f_v \wedge g_v
\end{aligned}$$

where the second equality is a consequence of the definition of the ∗-product and of the fact that $f_0, \ldots, f_3, g_0, \ldots, g_3$ belong to $\mathcal{S}_{\mathbb{R}}(\Omega)$. □

Remark 2.8. Given $f \in \mathcal{S}(\Omega)$, we can recognize in $\langle f, f \rangle_*$ its symmetrized function f^s. Indeed chosen any orthonormal basis $\{1, i, j, k\}$ of $\mathbb{H}$ on $\mathbb{R}$ and written f as $f_0 + f_1 i + f_2 j + f_3 k$, since $f^c = f_0 - f_v$ we have

$$\langle f, f \rangle_* = f_0^2 + f_1^2 + f_2^2 + f_3^2 = f_0 g_0 - \langle f_v, -f_v \rangle_* + f_0(-f_v) + f_0 f_v + f_v \wedge (-f_v) = f * f^c.$$

Remark 2.9. Notice that if Ω contains real points, then $f_v^s \equiv 0$ if and only if $f_v \equiv 0$. If Ω does not contain any real point, there exist examples in which $f_v^s \equiv 0$ and $f_v \not\equiv 0$. Indeed if Ω contains real points and $f_v^s \equiv 0$ then $f_1^2 + f_2^2 + f_3^2 \equiv 0$. On real points f_1, f_2 and f_3 take real values and therefore $f_1^2 + f_2^2 + f_3^2 \equiv 0$ implies $f_1 = f_2 = f_3 = 0$ on the intersection between Ω and $\mathbb{R}$, which entails $f_1 \equiv f_2 \equiv f_3 \equiv 0$ by the identity principle (see [**?**], Theorem 1.12). If the domain Ω does not contain any real point there exist $f \in \mathcal{S}(\Omega)$ such that $f_v \not\equiv 0$ and $f_v^s \equiv 0$ (see [**?**], Example 2).

In a certain sense, Formula **??** allows us to untangle the ∗-product on $\mathcal{S}(\Omega)$, confining the skewness to $\mathbb{H}$ and thus decidedly simplifying its computaion with respect to Definition **??** or Theorem 3.4 in [**?**]. Next result characterizes slice-regular functions whose $\wedge$-product vanishes identically on Ω and shows the effectiveness of Proposition **??**. Notice that by definition the vanishing of the $\wedge$-product is equivalent to the fact that f and g commute with respect to the ∗-product.

Proposition 2.10. *Given $f, g \in \mathcal{S}(\Omega)$, the function $f \wedge g$ vanishes identically if and only if f_v and g_v are linearly dependent on $\mathcal{S}_{\mathbb{R}}(\Omega)$.*

Proof. If $f_v \equiv 0$ the assertion is trivial by setting $1 \cdot f_v + 0 \cdot g_v \equiv 0$. Otherwise choose an orthonormal basis $\{1, i, j, k\}$ of $\mathbb{H}$ on $\mathbb{R}$, up to a rearrangement of the basis we can suppose $f_1 \not\equiv 0$. The identity $f \wedge g \equiv 0$ is equivalent to $(f_2 g_3 - f_3 g_2)i + (f_3 g_1 - f_1 g_3)j + (f_1 g_2 - f_2 g_1)k \equiv 0$.

Thanks to Proposition **??**, this gives

$$\begin{cases} f_2 g_3 - f_3 g_2 \equiv 0, \\ f_3 g_1 - f_1 g_3 \equiv 0, \\ f_1 g_2 - f_2 g_1 \equiv 0. \end{cases}$$

Last two equations of the previous system give $f_1 g_v - g_1 f_v \equiv 0$; as $f_1 \not\equiv 0$, the functions f_v and g_v are linearly dependent on $\mathcal{S}_{\mathbb{R}}(\Omega)$.

Vice versa, if f_v and g_v are linearly dependent on $\mathcal{S}_{\mathbb{R}}(\Omega)$, we can suppose, up to a rearrangment, that there exist $\alpha, \beta \in \mathcal{S}_{\mathbb{R}}(\Omega)$, with $\alpha \not\equiv 0$, such that $\alpha f_v + \beta g_v \equiv 0$. By taking the $\mathbb{A}$-product with g_v we obtain

$$0 \equiv \alpha f_v \mathbb{A} \ g_v + \beta g_v \mathbb{A} \ g_v = \alpha f_v \mathbb{A} \ g_v.$$

Since $\alpha \in \mathcal{S}_{\mathbb{R}}(\Omega)$ is not identically zero, then its zero set is a disjoint union of isolated real zeros and isolated spheres with real center. Therefore the equality $\alpha f_v \mathbb{A} \ g_v \equiv 0$ implies that $f_v \mathbb{A} \ g_v \equiv 0$ since the product of α and $f_v \mathbb{A} \ g_v$ is the pointwise product. $\qquad\square$

3. Square roots of slice-preserving regular functions

Since the symmetrized of any slice-regular function always belongs to $\mathcal{S}_{\mathbb{R}}(\Omega)$ and the symmetrized of a slice-preserving function coincides with its square, it is natural to ask when an element in $\mathcal{S}_{\mathbb{R}}(\Omega)$ has a square root in $\mathcal{S}_{\mathbb{R}}(\Omega)$ and, more generally, when the symmetrized function of a slice-regular function has a square root in $\mathcal{S}_{\mathbb{R}}(\Omega)$. The following results completely solve the question, giving necessary and sufficient conditions on a non-zero slice-preserving function in order to be the square of a slice-preserving function; in particular they apply when we look for the square root of the symmetrized of a given function.

As it will be seen in Corollary **??**, the results contained in this section can be applied to obtain a more explicit form for the $*$-exponential of a slice-regular function.

Throught the whole section we assume that $\Omega_I = \Omega \cap \mathbb{C}_I$ is simply connected, that is $\pi_1(\Omega_I)$ is trivial for some $I \in \mathbb{S}$ (and so for all $I \in \mathbb{S}$, being Ω a circular domain).

Proposition 3.1. *Given $h \in \mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$ there exists $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that $f^2 = h$ if and only if*

(i) *the zero set of h consists of real isolated zeroes of even multiplicity and isolated spheres with real center with multiplicity multiple of 4;*

(ii) *if $\Omega \cap \mathbb{R} \neq \emptyset$ then $h(\Omega \cap \mathbb{R}) \subseteq [0, +\infty)$.*

Proof. The necessity of the first condition is due to the fact that functions belonging to $\mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$ only have real isolated zeroes and isolated spherical zeroes with real center (see [**?**]). Squaring a slice-preserving function therefore produces real isolated zeroes with even multiplicity and isolated spherical zeroes with real center with multiplicity multiple of 4, as the spherical multiplicity is always an even number (see [**?**], Definition 3.37). The necessity of the second condition is straightforward.

We first perform the proof of the sufficiency of conditions (i) and (ii) in the case when Ω intersects the real axis. As $h \not\equiv 0$, we can choose a point $q_0 \in \Omega \cap \mathbb{R}$ with $h(q_0) \neq 0$. Now fix $I \in \mathbb{S}$ and choose a family $\{\Omega_n\}$ of nested domains with compact closure in $\Omega_I = \Omega \cap \mathbb{C}_I$ such that each Ω_n is simply connected and contains q_0 (such a family certainly exists thanks to Riemann mapping theorem).

The restriction h_I of h to Ω_I is a holomorphic function from Ω_I to $\mathbb{C}_I$ which does not vanish at q_0 and, by condition (i), has double zeroes both at real points and at conjugate points outside $\mathbb{R}$. Now denote by Z_n the intersection of the zero set of h_I with Ω_n: since $h_I(q_0) \neq 0$ then Z_n is finite and we can find a polynomial $P_n(z)$ with real coefficients and a holomorphic function $g_n : \Omega_I \to \mathbb{C}_I$ such that g_n does not vanish on $\overline{\Omega_n}$ and $h_I(z) = P_n^2(z)g_n(z)$ for any $z \in \Omega_I$. As g_n does not vanish on Ω_n, we can find a holomorphic square root γ_n of g_n on Ω_n such that $P_n(q_0)\gamma_n(q_0)$ is real and positive. Since $P_n(z)$ has real coefficients, condition (ii) implies that $\gamma_n(\Omega_n \cap \mathbb{R}) \subset \mathbb{R}$. Then the function $\varphi_n = P_n \cdot \gamma_n$ defined on Ω_n has the following properties

- is holomorphic on Ω_n,
- is a square root of h_I on Ω_n, that is $\varphi_n^2(z) = h_I(z)$ for any $z \in \Omega_n$,
- maps $\Omega_n \cap \mathbb{R}$ into $\mathbb{R}$,
- takes positive value at q_0.

It is easily seen that the above properties entail the uniqueness of φ_n and therefore $\varphi_n \equiv \varphi_{n+1}$ on Ω_n; hence setting $\varphi(z) = \varphi_n(z)$ for any $z \in \Omega_n$ defines a holomorphic function on Ω_I which is a square root of h_I. Now denote by f the regular extension of φ to Ω; it is easily seen that such a function is a square root of h. Indeed $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ because f coincides with φ on $\Omega \cap \mathbb{R}$, moreover $f^2 = h$ on $\Omega \cap \mathbb{R}$ which ensures $f^2 \equiv h$ on Ω.

Now suppose that $\Omega \cap \mathbb{R} = \emptyset$. Fix $I \in \mathbb{S}$ and consider $\Omega_I^+ = \Omega \cap \{z \in \mathbb{C}_I \,|\, \mathrm{Im}\, z > 0\}$. As Ω_I is simply connected, also Ω_I^+ is simply connected; since the set of zeroes of h_I in Ω_I consists of conjugate double points, by reasoning as above, we can find a square root φ^+ of h_I on Ω_I^+. As the function $h \in \mathcal{S}_{\mathbb{R}}(\Omega)$, then h_I coincides with its Schwarz reflection on $\Omega_I^- = \Omega_I \setminus \Omega_I^+$, so we can define a holomorphic map φ on Ω_I by extending φ^+ by Schwarz reflection on Ω_I^-. In this way we obtain a holomorphic square root φ of h_I on Ω_I; now denote by f the regular extension of φ to Ω; as above it is easily seen that $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ is a square root of h. $\qquad \square$

The above result can be applied to infer the existence of the square root of the symmetrized of a regular function with suitable zeroes. As previously stated, we assume that $\pi_1(\Omega_I)$ is trivial.

Corollary 3.2. *Given $g \in \mathcal{S}(\Omega)$ there exists $f \in \mathcal{S}_{\mathbb{R}}(\Omega)$ such that $f^2 = g^s$ if and only if the zero set of g does not contain non real isolated zeroes of odd multiplicity.*

Proof. If $g^s \equiv 0$, we can take $f \equiv 0$. Then we are left to study the case when $g^s \not\equiv 0$. In [?] the zero set of such g's is explicitely described as the union of isolated points and isolated spheres with real center. As g^s maps $\Omega \cap \mathbb{R}$ in $[0, +\infty)$, thanks to Proposition ?? it is enough to check that the zeroes of the symmetrized function g^s are only real zeroes of even multiplicity and spherical zeroes of multiplicity multiple of 4. This request exactly means that the zero set of g cannot contain non real isolated zeroes of odd multiplicity. $\qquad \square$

Next example gives an explicit application of the above corollary.

Example 3.3. For $i \in \mathbb{S}$ set $g(q) = q + i$. Then $g^s(q) = q^2 + 1$ has a spherical zero given by $\mathbb{S}$ with multiplicity 2 and so it is not the square of any slice-preserving regular function.

4. The ∗-exponential of a slice-regular function

We now enter in the discussion of the ∗-exponential. In [**?**], Colombo, Sabadini e Struppa introduced the following definitions, motivated by the natural challenge of functional calculus in the non-commutative setting, see also [**?**].

Definition 4.1. Given $f \in \mathcal{S}(\Omega)$ we set

$$\exp_*(f) = \sum_{n \in \mathbb{N}} \frac{f^{*n}}{n!}; \qquad \cos_*(f) = \sum_{n \in \mathbb{N}} \frac{(-1)^n f^{*(2n)}}{(2n)!}; \qquad \sin_*(f) = \sum_{n \in \mathbb{N}} \frac{(-1)^n f^{*(2n+1)}}{(2n+1)!}.$$

Natural estimates on compact subsets of Ω show that the series converge uniformly and therefore $\exp_*(f)$, $\cos_*(f)$, $\sin_*(f)$ are well defined and belong to $\mathcal{S}(\Omega)$.

In some special cases, $\exp_*$, $\cos_*$, $\sin_*$ take the usual form of the complex case and we sometimes denote them also dropping the underscore $*$.

Remark 4.2. If $f \in \mathcal{S}_\mathbb{R}(\Omega)$ then $\exp_*(f)$ can be written as $\sum_{n \in \mathbb{N}} \frac{f^n}{n!}$ and belongs to $\mathcal{S}_\mathbb{R}(\Omega)$; the same holds for $\cos_*(f)$ and $\sin_*(f)$. If there exists $J \in \mathbb{S}$ such that $f \in \mathcal{S}_J(\Omega)$ then $\exp_*(f)$ can be obtained as the regular extension to Ω of the complex function e^{f_J}, where f_J denotes the restriction of f to $\Omega_J = \Omega \cap \mathbb{C}_J$; in this case $\exp_*(f)$ belongs to $\mathcal{S}_J(\Omega)$. Again, the same holds for $\cos_*(f)$ and $\sin_*(f)$, too.

In the complex case, one of the most peculiar features of the exponental is its behaviour with respect to the sum; in the quaternionic case this happens under special conditions. The most natural hypothesis is the commutation of f and g which, due to Proposition **??**, is equivalent to the fact that f_v and g_v are linearly dependent on $\mathcal{S}_\mathbb{R}(\Omega)$. At the end of the paper we will prove a more refined result, see Theorem **??**, containing an unexpected couple of conditions on the functions f and g which are equivalent to the equality $\exp_*(f + g) = \exp_*(f) * \exp_*(g)$.

Proposition 4.3. If f_v and g_v are linearly dependent on $\mathcal{S}_\mathbb{R}(\Omega)$, then

$$\exp_*(f + g) = \exp_*(f) * \exp_*(g).$$

In particular the above equality holds if either f or g belong to $\mathcal{S}_\mathbb{R}(\Omega)$.

Proof.

$$\exp_*(f+g) = \sum_{n \in \mathbb{N}} \frac{(f+g)^{*n}}{n!} = \sum_{n \in \mathbb{N}} \frac{1}{n!} \sum_{m \leq n} \binom{n}{m} f^{*m} * g^{*(n-m)} = \sum_{n \in \mathbb{N}} \sum_{m \leq n} \frac{1}{m!(n-m)!} f^{*m} * g^{*(n-m)}$$

$$= \sum_{m \in \mathbb{N}} \sum_{n \geq m} \frac{f^{*m}}{m!} * \frac{g^{*(n-m)}}{(n-m)!} = \sum_{m \in \mathbb{N}} \frac{f^{*m}}{m!} * \sum_{n \geq m} \frac{g^{*(n-m)}}{(n-m)!} = \left(\sum_{m \in \mathbb{N}} \frac{f^{*m}}{m!} \right) * \left(\sum_{\nu \in \mathbb{N}} \frac{g^{*\nu}}{\nu!} \right)$$

$$= \exp_*(f) * \exp_*(g),$$

where in the second equality we took into account the fact that f and g commute because of Proposition **??**. $\square$

As an immediate consequence we obtain the following

Corollary 4.4. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$, then*

$$\exp_*(f) = \exp_*(f_0) * \exp_*(f_v) = \exp(f_0) \exp_*(f_v).$$

In particular if $f_v(q_0) = 0$ then $\exp_*(f)(q_0) = \exp(f_0)(q_0)$.

The above corollary allows us to interpretate the ∗-exponential in a cosine-sine form.

Proposition 4.5. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$, then*

$$(4.1) \qquad \exp_*(f) = \exp_*(f_0) \left(\sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m)!} + \sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m+1)!} f_v \right)$$

Proof. Thanks to Corollary **??** we can perform the computation in the case when $f_0 \equiv 0$. Notice that Proposition **??** and Remark **??** imply $(f_v)^{*2} = f_v * f_v = -\langle f_v, f_v \rangle_* = -f_v^s$. Now

$$\exp_*(f_v) = \sum_{n \in \mathbb{N}} \frac{f_v^{*n}}{n!} = \sum_{n \text{ even}} \frac{f_v^{*n}}{n!} + \sum_{n \text{ odd}} \frac{f_v^{*n}}{n!} = \sum_{m \in \mathbb{N}} \frac{f_v^{*(2m)}}{(2m)!} + \sum_{m \in \mathbb{N}} \frac{f_v^{*(2m+1)}}{(2m+1)!}$$

$$= \sum_{m \in \mathbb{N}} \frac{(f_v^{*2})^{*m}}{(2m)!} + \sum_{m \in \mathbb{N}} \frac{(f_v^{*2})^{*m}}{(2m+1)!} * f_v = \sum_{m \in \mathbb{N}} \frac{(-f_v^s)^{*m}}{(2m)!} + \sum_{m \in \mathbb{N}} \frac{(-f_v^s)^{*m}}{(2m+1)!} * f_v$$

$$= \sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m)!} + \sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m+1)!} f_v$$

because f_v^s is slice-preserving and the ∗-product becomes the pointwise product. □

This last expression allows us to look to the ∗-exponential in terms of a local cosine-sine analogous of the complex exponential.

Corollary 4.6. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$. If*

(1) $f_v^s(q_0) = 0$, then

$$\exp_*(f)(q_0) = \exp_*(f_0)(q_0) \left(1 + f_v(q_0) \right);$$

(2) $f_v^s(q_0)$ is a positive real, denote by x_0 one of the square roots of $f_v^s(q_0)$, then

$$\exp_*(f)(q_0) = \exp_*(f_0)(q_0) \left(\cos x_0 + \frac{\sin x_0}{x_0} f_v(q_0) \right);$$

(3) $f_v^s(q_0)$ is a negative real, denote by x_0 one of the square roots of $-f_v^s(q_0)$, then

$$\exp_*(f)(q_0) = \exp_*(f_0)(q_0) \left(\cosh x_0 + \frac{\sinh x_0}{x_0} f_v(q_0) \right);$$

(4) $f_v^s(q_0) = \alpha_0 + \beta_0 J$ with $\beta_0 \neq 0$ denote by $a_0 + b_0 J$ one of the square roots of $f_v^s(q_0)$, then

$$\exp_*(f)(q_0) = \exp_*(f_0)(q_0) \left(\cos(a_0 + b_0 J) + \frac{\sin(a_0 + b_0 J)}{a_0 + b_0 J} f_v(q_0) \right).$$

Proof. The proof is a trivial application of Formula (**??**) since in (1) the two power series sum both up to 1; in (2) we have $f_v^s(q_0) = x_0^2$; in (3) we have $f_v^s(q_0) = -x_0^2$ and in (4) we have $f_v^s(q_0) = (a_0 + b_0 J)^2$. Notice that the expression $\frac{\sin(a_0 + b_0 J)}{a_0 + b_0 J} f_v(q_0)$ in (4) is well defined because all factors lie in the same $\mathbb{C}_J$. □

If the zero set of f_v does not contain non real isolated zeroes of odd multiplicity, the above result can be made more precise. Indeed in this case Corollary **??** ensures we can find a regular function in $\mathcal{S}_{\mathbb{R}}(\Omega)$ whose square is f_v^s and which therefore gives a global determination of the square root of f_v^s; in order to stress the analogy with the complex case we denote it by $\sqrt{f_v^s}$.

Corollary 4.7. *Let $f = f_0 + f_v \in \mathcal{S}(\Omega)$. If $\pi_1(\Omega_I)$ is trivial, f_v^s is not identically zero and the zero set of f_v does not contain non real isolated zeroes of odd multiplicity then*

$$(4.2) \qquad \exp_*(f) = \exp_*(f_0) \left(\cos_* \left(\sqrt{f_v^s} \right) + \frac{\sin_* \left(\sqrt{f_v^s} \right)}{\sqrt{f_v^s}} f_v \right)$$

Proof. As

$$\sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m)!} = \sum_{m \in \mathbb{N}} \frac{(-1)^m (\sqrt{f_v^s})^{2m}}{(2m)!} = \cos_* \left(\sqrt{f_v^s} \right)$$

and

$$\sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m+1)!} = \sum_{m \in \mathbb{N}} \frac{(-1)^m (\sqrt{f_v^s})^{2m}}{(2m+1)!} = \frac{1}{\sqrt{f_v^s}} \sum_{m \in \mathbb{N}} \frac{(-1)^m (\sqrt{f_v^s})^{2m+1}}{(2m+1)!} = \frac{\sin_* \left(\sqrt{f_v^s} \right)}{\sqrt{f_v^s}},$$

the assertion follows from (**??**).   □

Remark 4.8. In the hypothesis of Corollary **??**, at points where f_v^s does not vanish, we can write Formula (**??**) in the form

$$\exp_*(f) = \exp_*(f_0) \left(\cos_* \left(\sqrt{f_v^s} \right) + \sin_* \left(\sqrt{f_v^s} \right) \frac{f_v}{\sqrt{f_v^s}} \right).$$

We underline that $\left(\frac{f_v}{\sqrt{f_v^s}} \right)^s \equiv 1$ outside the zero set of f_v^s and therefore the function $\frac{f_v}{\sqrt{f_v^s}}$ imitates the behaviour of the imaginary unit in $\mathbb{C}$.

As a further consequence of Proposition **??** we can describe the set of all functions whose $*$-exponential is either slice-preserving or $\mathbb{C}_J$-preserving for a suitable $J \in \mathbb{S}$.

To simplify computations we set

$$\mu(f) = \sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m)!}, \qquad \nu(f) = \sum_{m \in \mathbb{N}} \frac{(-1)^m (f_v^s)^m}{(2m+1)!}$$

which are slice-preserving regular functions. Then (**??**) can be reformulated as

$$(4.3) \qquad \exp_*(f) = \exp_*(f_0) \left(\mu(f) + \nu(f) f_v \right).$$

Notice that, thanks to Corollary **??**, the equality $\nu(f)(q_0) = 0$ holds if and only if $f_v^s(q_0)$ belongs to $\mathcal{Z} = \{ n^2 \pi^2 \,|\, n \in \mathbb{Z} \setminus \{0\} \}$, that is there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $f_v^s(q_0) = n^2 \pi^2$. Notice that at such points $\mu(f)(q_0)$ is equal to $(-1)^n$.

Formula (**??**) enables us to characterize when the $*$-exponential of a regular function preserves one or all slices. Next example shows that things are not as simple as one could imagine.

Example 4.9. If $f(q) = \pi \cos(q)i + \pi \sin(q)j$, then $f_1(q) = \pi \cos(q)$, $f_2(q) = \pi \sin(q)$ and therefore $f_v^s(q) = f_1^2(q) + f_2^2(q) = \pi^2 \cos^2(q) + \pi^2 \sin^2(q) = \pi^2$ for all $q \in \mathbb{H}$. By Corollary **??** this implies that $\exp_*(f) = \cos(\pi) = -1$. It is easy to check that the function f does not preserve any slice but its $*$-exponential trivially preserves all slices.

Proposition 4.10. *Let $f \in \mathcal{S}(\Omega)$. Then*

(i) $\exp_*(f) \in \mathcal{S}_\mathbb{R}(\Omega)$ *if and only if either $f_v \equiv 0$ or f_v^s is a constant which belongs to $\mathcal{Z}$,*

(ii) *there exists $J \in \mathbb{S}$ such that $\exp_*(f) \in \mathcal{S}_J(\Omega) \setminus \mathcal{S}_\mathbb{R}(\Omega)$ if and only if $f_v \in \mathcal{S}_J(\Omega)$ and it is not a constant which belongs to $\mathbb{Z}\pi J$.*

Proof. (i) If $f_v \equiv 0$ then $f \in \mathcal{S}_\mathbb{R}(\Omega)$ and therefore $\exp_*(f) \in \mathcal{S}_\mathbb{R}(\Omega)$. If $f_v \not\equiv 0$ and f_v^s is a constant which belongs to $\mathcal{Z}$, let us denote by $n \neq 0$ an integer such that $f_v^s \equiv n^2\pi^2$; then Corollary **??** gives $\exp_*(f) = \exp_*(f_0)\left(\cos_*(n\pi) + \frac{\sin_*(n\pi)}{n\pi}f_v\right) = \exp_*(f_0)\cos(n\pi) = (-1)^n\exp_*(f_0) \in \mathcal{S}_\mathbb{R}(\Omega)$.

Now if $\exp_*(f) \in \mathcal{S}_\mathbb{R}(\Omega)$, then the function $\exp_*(f_0)\left(\mu(f) + \nu(f)f_v\right)$ does; since $\exp_*(f_0)$ is never-vanishing, this entails that $\mu(f) + \nu(f)f_v$ belongs to $\mathcal{S}_\mathbb{R}(\Omega)$. This fact implies that $\nu(f)f_v$ is identically zero. If $\nu(f) \equiv 0$, then $\sum_{m\in\mathbb{N}} \frac{(-1)^m(f_v^s)^m}{(2m+1)!} \equiv 0$ and therefore f_v^s belongs to one of the zeroes of the power series which are given by $\mathcal{Z} = \{n^2\pi^2 \,|\, n \in \mathbb{Z} \setminus \{0\}\}$ and hence f_v^s is a constant which belongs to $\mathcal{Z}$. If $\nu(f) \not\equiv 0$, we have that $f_v = 0$ outside the zero set of $\nu(f)$ which is closed and has empty interior, therefore by continuity $f_v \equiv 0$, that is $f \in \mathcal{S}_\mathbb{R}(\Omega)$.

(ii) If $f_v \in \mathcal{S}_J(\Omega)$ and it is not a constant which belongs to $\mathbb{Z}\pi J$, then we can write $f = f_0 + f_1 J$ where $f_1 \not\equiv n\pi$ for some $n \in \mathbb{Z}$. As $f_v^s = f_1^2$, by Formula (**??**) we obtain that $\exp_*(f) = \exp_*(f_0)\left(\cos_*(f_1) + \sin_*(f_1)J\right)$. As $\exp_*(f_0)$ is never-vanishing and f_1 is not a constant which belongs to $\mathbb{Z}\pi$, then $\exp_*(f)$ belongs to $\mathcal{S}_J(\Omega) \setminus \mathcal{S}_\mathbb{R}(\Omega)$.

Now, if $\exp_*(f) \in \mathcal{S}_J(\Omega) \setminus \mathcal{S}_\mathbb{R}(\Omega)$, then the function $\exp_*(f_0)\left(\mu(f) + \nu(f)f_v\right)$ does; since $\exp_*(f_0)$ is never-vanishing, this entails that $\mu(f) + \nu(f)f_v$ belongs to $\mathcal{S}_J(\Omega) \setminus \mathcal{S}_\mathbb{R}(\Omega)$. This fact implies that $\nu(f)f_v$ is equal to $g_1 J$ for some $g_1 \in \mathcal{S}_\mathbb{R}(\Omega) \setminus \{0\}$. Since $\nu(f) \in \mathcal{S}_\mathbb{R}(\Omega) \setminus \{0\}$ we obtain that $f_v \in \mathcal{S}_J(\Omega) \setminus \{0\}$. Would $f_v \equiv n\pi J$ for some $n \in \mathbb{Z} \setminus \{0\}$, then $f_v^s \equiv n^2\pi^2$ and therefore $\nu(f) \equiv 0$, which is a contradiction. $\square$

Formula (**??**) can also be used to obtain an explicit expression for $(\exp_*(f)) * (\exp_*(g))$.

Proposition 4.11. *Let $f, g \in \mathcal{S}(\Omega)$. Then $\exp_*(f) * \exp_*(g)$ is equal to*

$$\exp_*(f_0)\exp_*(g_0)\left(\mu(f)\mu(g) - \nu(f)\nu(g)\langle f_v, g_v\rangle_* + \nu(f)\nu(g)f_v \wedge g_v + \mu(f)\nu(g)g_v + \mu(g)\nu(f)f_v\right).$$

Proof. The assumption is a direct offspring of Formulas (**??**) and (**??**). $\square$

Since $f^c = f_0 - f_v$, Formula (**??**) also gives the expression of $\exp_*(f^c)$ in terms of $\exp_*(f)$. Indeed we have

Remark 4.12. Let $f \in \mathcal{S}(\Omega)$. Then $\exp_*(f^c) = (\exp_*(f))^c$.

When applied to f and f^c the above Proposition shows that, as it is natural to expect, the $*$-exponential of a regular function never vanishes. Indeed we have the following

Proposition 4.13. *Let $f \in \mathcal{S}(\Omega)$. Then*

(4.4)
$$(\exp_*(f))^s = \exp_*(2f_0)$$

(4.5)
$$\frac{\exp_*(f) + \exp_*(f^c)}{2} = \exp_*(f_0)\mu(f),$$

(4.6)
$$\frac{\exp_*(f) - \exp_*(f^c)}{2} = \exp_*(f_0)\nu(f)f_v,$$

(4.7)
$$\exp_*(f) * \exp_*(-f) \equiv 1.$$

In particular $\exp_*(f)$ is a never-vanishing function.

Proof. Indeed we have

$$(\exp_*(f))^s = (\exp_*(f)) * (\exp_*(f))^c = (\exp_*(f)) * (\exp_*(f^c))$$
$$= \exp_*(f_0)\exp_*(f_0)\left(\mu(f)\mu(f) + \nu(f)\nu(f)\langle f_v, f_v \rangle_* - \mu(f)\nu(f)f_v + \mu(f)\nu(f)f_v\right)$$
$$= \exp_*(2f_0)\left(\mu(f)\mu(f) + \nu(f)\nu(f)f_v^s\right) = \exp_*(2f_0)\left(\mu(f)^2 + \nu(f)^2 f_v^s\right)$$

Now using the same notation as in Corollary **??** we have that $\mu(f)^2 + \nu(f)^2 f_v^s \equiv 1$ and therefore we obtain Equality (**??**). As $\exp_*(2f_0) = \exp(2f_0)$ is never-vanishing, then also $\exp_*(f)$ has no zeroes on Ω. Equalities (**??**) and (**??**) follow immediately from (**??**), while (**??**) can be obtained from Proposition **??** following the same kind of computations as above. $\square$

Proposition **??** enables us to deepen our understanding of the comparison between the $*$-product of $\exp_*(f)$ and $\exp_*(g)$ and $\exp_*(f + g)$, thus giving, when Ω contains real points, a necessary and sufficient condition for them to be equal. We underline that, while Condition (i) appears quite natural, Condition (ii) is in some sense surprising: in particular it holds for a class of functions larger than one could initially presume (see Example **??**).

Theorem 4.14. *Let Ω contain real points. Take $f, g \in \mathcal{S}(\Omega)$. If*

(4.8)
$$\exp_*(f + g) = \exp_*(f) * \exp_*(g)$$

then either

(i) *f_v and g_v are linearly dependent over $\mathcal{S}_\mathbb{R}(\Omega)$ or*
(ii) *there exist $n, m, p \in \mathbb{Z} \setminus \{0\}$ such that $f_v^s \equiv n^2\pi^2$, $g_v^s \equiv m^2\pi^2$, $2\langle f_v, g_v \rangle_* = (p^2 - n^2 - m^2)\pi^2$ and $n + m \equiv p \mod 2$.*

*Vice versa, if either (i) or (ii) are satisfied, then (**??**) holds.*

Proof. By Formula (**??**) and Proposition **??** we can suppose that f_0 and g_0 vanish identically. Then we have

$$\exp_*(f + g) = \mu(f + g) + \nu(f + g)(f + g)_v$$
$$\exp_*(f) * \exp_*(g) = \mu(f)\mu(g) - \nu(f)\nu(g)\langle f_v, g_v \rangle_*$$
$$+ \nu(f)\nu(g)f_v \wedge g_v + \mu(f)\nu(g)g_v + \mu(g)\nu(f)f_v.$$

First we study the case when neither $\nu(f)$ nor $\nu(g)$ are identically zero. Since $(f+g)_v = f_v + g_v$ belongs to $\mathrm{Span}_{\mathcal{S}_\mathbb{R}(\Omega)}(f_v, g_v)$, if $\exp_*(f + g) = \exp_*(f) * \exp_*(g)$ then $\nu(f)\nu(g)f_v \wedge g_v$ belongs to $\mathrm{Span}_{\mathcal{S}_\mathbb{R}(\Omega)}(f_v, g_v)$ too. Then there exist $\alpha, \beta \in \mathcal{S}_\mathbb{R}(\Omega)$ such that $\nu(f)\nu(g)f_v \wedge g_v = \alpha f_v + \beta g_v$. If α

and β are both identically zero, then $\nu(f)\nu(g)f_v \mathbb{A} g_v \equiv 0$. Thus $f_v \mathbb{A} g_v \equiv 0$ and Proposition **??** ends the proof. If α and β are not both identically zero, up to rearrangement, we can suppose $\alpha \not\equiv 0$. Now $0 \equiv \langle \nu(f)\nu(g)f_v \mathbb{A} g_v, f_v \rangle_* = \alpha f_v^s + \beta \langle f_v, g_v \rangle_*$ and $0 \equiv \langle \nu(f)\nu(g)f_v \mathbb{A} g_v, g_v \rangle_* = \alpha \langle f_v, g_v \rangle_* + \beta g_v^s$ that is

$$\begin{cases} \alpha f_v^s + \beta \langle f_v, g_v \rangle_* \equiv 0, \\ \alpha \langle f_v, g_v \rangle_* + \beta g_v^s \equiv 0. \end{cases}$$

Taking a suitable combination of these equations we obtain $\alpha \left(f_v^s g_v^s - \langle f_v, g_v \rangle_*^2 \right) \equiv 0$. As $\alpha \in \mathcal{S}_{\mathbb{R}}(\Omega) \setminus \{0\}$, we obtain that $f_v^s g_v^s - \langle f_v, g_v \rangle_*^2 = (f_v \mathbb{A} g_v)^s$ is identically zero. Since Ω contains real points, then $\mathcal{S}(\Omega)$ is an integral domain, and therefore $f_v \mathbb{A} g_v$ is identically zero. Again, Proposition **??** entails that f_v and g_v are linearly dependent over $\mathcal{S}_{\mathbb{R}}(\Omega)$.

If $\nu(f) \equiv 0$ (the case $\nu(g) \equiv 0$ is treated analogously), then f_v^s is identically equal to $n^2\pi^2$ for a suitable $n \in \mathbb{Z} \setminus \{0\}$. In this case $\exp_*(f + g) = \exp_*(f) * \exp_*(g)$ gives

$$\mu(f + g) + \nu(f + g)(f + g)_v = \mu(f)\mu(g) + \mu(f)\nu(g)g_v$$

which, taking the equality of the vector parts, implies

$$\nu(f + g)(f_v + g_v) = \mu(f)\nu(g)g_v,$$

that can be written as

$$\left(\mu(f)\nu(g) - \nu(f + g) \right) g_v = \nu(f + g)f_v.$$

This gives the linear dependency of f_v and g_v over $\mathcal{S}_{\mathbb{R}}(\Omega)$ unless $\mu(f)\nu(g) - \nu(f+g) \equiv \nu(f+g) \equiv 0$. In this last case, we obtain $\mu(f)\nu(g) \equiv \nu(f + g) \equiv 0$. A simple computation gives $\mu(f) = (-1)^n$ and therefore $\nu(g) \equiv \nu(f + g) \equiv 0$. Then there exist $m, p \in \mathbb{Z} \setminus \{0\}$ such that $g_v^s \equiv m^2\pi^2$ and $(f_v + g_v)^s \equiv p^2\pi^2$ and therefore $\mu(f + g) = (-1)^p$.

Since $(f_v + g_v)^s = -(f_v + g_v) * (f_v + g_v) = n^2\pi^2 + m^2\pi^2 + 2\langle f_v, g_v \rangle_*$ we obtain $2\langle f_v, g_v \rangle_* = (p^2 - m^2 - n^2)\pi^2$.

Under these assumptions we have

$$\exp_*(f+g) = \mu(f+g) \equiv (-1)^p \quad \text{and} \quad \exp_*(f)*\exp_*(g) = \mu(f)\mu(g) \equiv (-1)^n(-1)^m = (-1)^{n+m}$$

which holds if and only if $n + m$ and p have the same parity.

If (i) is satisfied, then (**??**) holds because of Proposition **??**. If (ii) is satisfied, then $\nu(f) = \nu(g) = \nu(f + g) \equiv 0$ and $\mu(f) = (-1)^n$, $\mu(g) = (-1)^m$, $\mu(f + g) = (-1)^p$ so that $\exp_*(f + g) = \mu(f + g) = (-1)^p$ and $\exp_*(f) * \exp_*(g) = (-1)^n(-1)^m = (-1)^{n+m} = (-1)^p$ because $n + m$ and p have the same parity. $\qquad \square$

The following two examples show cases where (**??**) holds while f_v and g_v are not linearly dependent on $\mathcal{S}_{\mathbb{R}}(\mathbb{H})$. In particular the second one gives a large class of non constant functions which do not commute but satisfy (**??**).

Example 4.15. If $I, J \in \mathbb{S}$ are orthogonal, $f_v \equiv 3\pi I$ and $g_v \equiv 4\pi J$ then

$$\exp_*(f + g) = -\exp_*(f_0 + g_0) = \exp_*(f) * \exp_*(g).$$

Example 4.16. Let n, p, m be a Pythagorean triple with $m^2 = n^2 + p^2$. Now choose $\alpha, \beta \in \mathcal{S}_{\mathbb{R}}(\Omega)$ and set

$$f(q) = -n\pi \left(\cos(\alpha(q)) \cos(\beta(q))i + \cos(\alpha(q)) \sin(\beta(q))j + \sin(\alpha(q))k \right),$$

$$g(q) = \pi((n\cos(\alpha(q))\cos(\beta(q)) + p\sin(\alpha(q))\cos(\beta(q)))i+$$
$$+ (n\cos(\alpha(q))\sin(\beta(q)) + p\sin(\alpha(q))\sin(\beta(q)))j$$
$$+ (n\sin(\alpha(q)) - p\cos(\alpha(q)))k).$$

As α, β are slice-preserving functions then $f, g \in \mathcal{S}(\Omega)$, moreover $f_v^s = n^2\pi^2$, $g_v^s = m^2\pi^2$ and $\langle f_v, g_v\rangle_* = -n^2\pi^2$. Since n, p, m is a Pythagorean triple then $2\langle f_v, g_v\rangle_* = -2n^2\pi^2 = \pi^2(-n^2 - p^2 - n^2 + p^2) = \pi^2(p^2 - n^2 - m^2)$ and hence the given functions satisfy the requirements of the statement because the parity condition is satisfied for any Pythagorean triple.

Next example gives an explicit couple of functions f and g where the parity relation between n, m and p does not hold.

Example 4.17. If $I, J \in \mathbb{S}$ satisfy $IJ+JI = -\frac{13}{20}$, $f_v \equiv 2\pi I$ and $g_v \equiv 5\pi J$ then $n = 2$, $m = 5$ and $p = 4$ since $((f+g)_v)^s = (2^2+5^2+2\cdot2\cdot5(IJ+JI))\pi^2 = 4^2\pi^2$. Then we have $\mu(f) = \mu(f+g) \equiv 1$, $\mu(g) \equiv -1$ and $\nu(f) = \nu(g) = \nu(f+g) \equiv 0$ which entail $\exp_*(f+g) = \exp_*(f_0 + g_0)$ while $\exp_*(f) * \exp_*(g) = -\exp_*(f_0 + g_0)$.

The example below illustrates a case where Ω does not contain real points.

Example 4.18. Denote by $\tau : \mathbb{H} \setminus \mathbb{R} \to \mathbb{S}$ be given by $\tau(\alpha + \beta I) = \begin{cases} I & \text{if } \beta > 0, \\ -I & \text{if } \beta < 0. \end{cases}$ Choose $i, j, k \in \mathbb{S}$ an orthonormal basis of $\mathrm{Im}\mathbb{H}$ and consider $f, g : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined as

$$f(\alpha + \beta I) = 2\pi i - 2\pi(1 + \tau(\alpha + \beta I)i)j = 2\pi(i - j - \tau(\alpha + \beta I)k)$$
$$g(\alpha + \beta I) = \tau(\alpha + \beta I)i + \pi j + k.$$

Now we have $(f+g)(\alpha+\beta I) = (2\pi+\tau(\alpha+\beta I))i - \pi j + (-2\pi\tau(\alpha+\beta I)+1)k$, $f_v^s = 4\pi^2$, $g_v^s = \pi^2$ and $((f+g)_v)^s = (2\pi + \tau(\alpha + \beta I))^2 + \pi^2 + (-2\pi\tau(\alpha + \beta I) + 1)^2 = \pi^2$.

This entails $\mu(f) \equiv 1$, $\mu(g) \equiv -1$, $\mu(f+g) \equiv -1$, $\nu(f) \equiv \nu(g) \equiv \nu(f+g) \equiv 0$ and therefore $\exp_*(f+g) \equiv -1 \equiv \exp_*(f) * \exp_*(g)$ holds.

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