Special matchings and parabolic Kazhdan–Lusztig polynomials

note finali coverpage

(Article begins on next page)

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SPECIAL MATCHINGS AND PARABOLIC KAZHDAN–LUSZTIG POLYNOMIALS

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ABSTRACT. We prove that the combinatorial concept of a special matching can be used to compute the parabolic Kazhdan–Lusztig polynomials of doubly laced Coxeter groups and of dihedral Coxeter groups. In particular, for this class of groups which includes all Weyl groups, our results generalize to the parabolic setting the main results in [Advances in Math. 202 (2006), 555-601]. As a consequence, the parabolic Kazhdan–Lusztig polynomial indexed by u and v depends only on the poset structure of the Bruhat interval from the identity element to v and on which elements of that interval are minimal coset representatives.

1. INTRODUCTION

Kazhdan–Lusztig polynomials are polynomials $\{P_{u,v}(q)\}_{u,v\in W}$ in one variable q, which are indexed by a pair of elements u, v in a Coxeter group W. These polynomials were introduced by Kazhdan and Lusztig in [10] as a tool for the construction of certain important representations of the Hecke algebra associated with W. Since then, Kazhdan–Lusztig polynomials have been shown to have applications in many contexts and now play a central role in Lie theory and representation theory. In particular, when W is a Weyl group, the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ is the Poincaré polynomial of the local intersection cohomology groups (in even degrees) of the Schubert variety associated with v at any point of the Schubert variety associated with u (see [11]). Thus, for Weyl groups, the coefficients of $P_{u,v}(q)$ are nonnegative, a fact which is not at all evident from the definition of $P_{u,v}(q)$. In fact, Elias and Williamson [7] have recently shown, much more generally, that the coefficients of Kazhdan–Lusztig polynomials are nonnegative for every Coxeter group.

At present, from a combinatorial point of view, the most challenging conjecture about Kazhdan–Lusztig polynomials is arguably the following, made by Lusztig in private and, independently, by Dyer [5].

Conjecture 1.1. *The Kazhdan–Lusztig polynomial $P_{u,v}(q)$ depends only on the combinatorial structure of the Bruhat interval $[u, v]$ (i.e., the isomorphism type of $[u, v]$ as a poset under Bruhat order).*

Conjecture 1.1 is usually referred to as the Combinatorial Invariance Conjecture. It is equivalent to the analogous conjecture on the Kazhdan–Lusztig R-polynomials. These

are also polynomials $\{R_{u,v}(q)\}_{u,v \in W}$ in one variable q, indexed by a pair of elements u, v in a Coxeter group W, and were introduced by Kazhdan–Lusztig in the same work [10]. The knowledge of the entire family $\{R_{u,v}(q)\}_{u,v \in W}$ of the Kazhdan–Lusztig R-polynomials of a Coxeter group W is equivalent to the knowledge of the entire family $\{P_{u,v}(q)\}_{u,v \in W}$ of the Kazhdan–Lusztig polynomials of W.

The Combinatorial Invariance Conjecture asserts that, given two Coxeter groups W and W' and two pairs of elements $u, v \in W$ and $u', v' \in W'$ such that $[u, v] \cong [u', v']$, it holds that $R_{u,v}(q) = R_{u',v'}(q)$ and $P_{u,v}(q) = P_{u',v'}(q)$. This was proved in [2] to hold when u and u' are the identity elements of W and W'. The proof of this result is constructive since it describes an algorithm to compute the Kazhdan–Lusztig R-polynomial $R_{u,v}(q)$ depending only on the lower Bruhat interval $[e, v]$ (where e denotes the identity element). This algorithm is based on combinatorial tools named *special matchings*, which are abstractions of the maps given by the multiplication (on the left or on the right) by a Coxeter generator. A special matching of $[e, v]$ is an involution $M : [e, v] \to [e, v]$ such that

(1) either $u \lhd M(u)$ or $u \rhd M(u)$, for all $u \in [e, v]$,
(2) if $u_1 \lhd u_2$ then $M(u_1) \le M(u_2)$, for all $u_1, u_2 \in [e, v]$ such that $M(u_1) \ne u_2$.

(Here, $\lhd$ denotes the covering relation, i.e., $x \lhd y$ means that $x < y$ and there is no z with $x < z < y$). The concept of special matching is purely poset-theoretic, that is, it depends only on the poset structure of $[e, v]$ and not on other structures (in particular, the algebraic structure of group plays no role). In [2], it is proved that special matchings may be used in place of multiplication maps in the recurrence formula which computes the R-polynomials.

In order to find a method for the computation of the dimensions of the intersection cohomology modules corresponding to Schubert varieties in G/P, where P is a parabolic subgroup of the Kac–Moody group G, Deodhar [4] defined two parabolic analogues of the Kazhdan–Lusztig polynomials, which correspond to the roots $x = q$ and $x = -1$ of the equation $x^2 = q + (q - 1)x$. Also, Deodhar defined two parabolic analogues of the R-polynomials, denoted $\{R_{u,w}^{H,x}(q)\}_{u,w \in W^H}$, whose knowledge is again equivalent to the knowledge of the parabolic Kazhdan–Lusztig polynomials. These polynomials are indexed by pairs of elements in the set W^H of minimal coset representatives with respect to the standard parabolic subgroup W_H generated by a subset $H \subseteq S$. The parabolic Kazhdan–Lusztig and R-polynomials coincide with the ordinary Kazhdan–Lusztig and R-polynomials when $H = \emptyset$.

Since the appearance of [2], the authors have been asked many times whether something analogous could be done in the parabolic setting, i.e., whether a poset-theoretic way to compute the parabolic Kazhdan–Lusztig polynomials could be found. In this work, we generalize the main results in [2] to the parabolic setting, when (W, S) is a doubly laced Coxeter system (and, also, in the case of dihedral Coxeter systems, i.e. Coxeter systems of rank 2, which is much easier).

Let (W, S) be a Coxeter system which is either doubly laced or dihedral. Let $H \subseteq S$ and $w \in W^H$. Let us consider the special matchings M of the lower Bruhat interval $[e, w]$ satisfying the following further property:

$$u \le w, u \in W^H, M(u) \lhd u \implies M(u) \in W^H.$$

We call such matchings H-*special.* Note that the concept of H-special matching depends both on the poset structure of the complete interval $[e, w]$ and on how the parabolic interval $[e, w]^H = \{z \in W^H : e \leq z \leq w\}$ embeds in $[e, w]$. In this work, we show that the H-special matchings may be used in place of left multiplication maps in the recurrence formula for the parabolic Kazhdan–Lusztig R-polynomials: precisely, if M is an H-special matching of $[e, w]$, we have

(1.1)
$$
R_{u,w}^{H,x}(q) = \begin{cases} R_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \lhd u, \\ (q-1)R_{u,M(w)}^{H,x}(q) + qR_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \in W^H, \\ (q-1-x)R_{u,M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \notin W^H. \end{cases}
$$

The parabolic R-polynomials can be computed by means of H-special matchings by iterating (1.1). As a corollary, the parabolic R-polynomial $R_{u,w}^{H,x}(q)$ depends only on the poset structure of the complete interval $[e, w]$ and on which of the elements in this interval are minimal coset representatives. Indeed, we have the following result (Corollary 5.1 in the paper).

Theorem. *Let (W_1, S_1) and (W_2, S_2) be two doubly laced or dihedral Coxeter systems, with identity elements e_1 and e_2, and let $H_1 \subseteq S_1$ and $H_2 \subseteq S_2$. Let $v_1 \in W_1^{H_1}$ and $v_2 \in W_2^{H_2}$ be such that there exists a poset-isomorphism ψ from $[e_1, v_1]$ to $[e_2, v_2]$ which restricts to a poset-isomorphism from $[e_1, v_1]^{H_1}$ to $[e_2, v_2]^{H_2}$. Then, for all $u, w \in [e_1, v_1]^{H_1}$, we have*

$$
R_{u,w}^{H_1,x}(q) = R_{\psi(u),\psi(w)}^{H_2,x}(q) \quad \text{and} \quad P_{u,w}^{H_1,x}(q) = P_{\psi(u),\psi(w)}^{H_2,x}(q).
$$

Since the $\emptyset$-special matchings are exactly the special matchings, the preceding result implies the main result of [2] for the ordinary Kazhdan–Lusztig and R-polynomials.

In the proofs, we use some algebraic properties of the special matchings of a lower Bruhat interval $[e, w]$ which are valid for any arbitrary Coxeter group W (see [12]), while the further hypotheses on W are needed only in few cases. We believe that the main result of this work might be generalized.

In studying the parabolic Kazhdan–Lusztig and R-polynomials $\{P_{u,w}^{H,x}(q)\}_{u,w \in W^H}$ and $\{R_{u,w}^{H,x}(q)\}_{u,w \in W^H}$, attention has been focused on the parabolic intervals $[u, w]^H$. As a consequence, the parabolic analogue of the Combinatorial Invariance Conjecture (Conjecture 1.1) has been considered to be the following.

Conjecture 1.2. *The parabolic Kazhdan–Lusztig polynomial $P_{u,v}^{H,x}(q)$ depends only on the combinatorial structure of the parabolic Bruhat interval $[u, v]^H$.*

Conjecture 1.2 has recently been shown to be false in the case $x = q$ by Mongelli [13], who provides the following counterexample for the Coxeter system (W, S) of type F_4. Let $S = \{s_1, s_2, s_3, s_4\}$ with $(s_1 s_2)^3 = e$, $(s_2 s_3)^4 = e$, $(s_3 s_4)^3 = e$, and $(s_i s_j)^2 = e$ for the other values of i and j. Consider the subset $H = \{s_1, s_2, s_3\}$ of S and the elements $u = s_3 s_1 s_2 s_3 s_4$, $v = s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4$, $x = s_2 s_3 s_4$, and $y = s_4 s_3 s_1 s_2 s_3 s_4$ of W^H. Then the parabolic intervals $[u, v]^H$ and $[x, y]^H$ are isomorphic while $P_{u,v}^{H,q}(q) = q$ and $P_{x,y}^{H,q}(q) = 0$. We point out that $P_{u,v}^{H,-1}(q) = q + 1$ and $P_{x,y}^{H,-1}(q) = 1$, so that Conjecture 1.2 is false also in the case $x = -1$. In particular, there cannot be a

general method to compute the parabolic Kazhdan–Lusztig polynomial $P_{u,w}^{H,x}(q)$ (or R-polynomial $R_{u,w}^{H,x}(q)$) just from the isomorphism type of the parabolic interval $[u,w]^H$.

Roughly speaking, the basic idea of the present work is that, for a poset-theoretic approach, also the elements that are not minimal coset representatives carry some information, and what should be considered is not just the parabolic interval $[u,w]^H$ but the complete interval $[u,w]$, together with the notion of how $[u,w]^H$ embeds in $[u,w]$. This is reflected, for instance, in the fact that the matchings for which (1.1) holds are the H-special matchings, which are special matchings of the complete interval with a good behaviour with respect to the elements in the parabolic interval. Within this perspective, the right approach to the generalization of the Combinatorial Invariance Conjecture to the parabolic setting would be studying to what extent the following conjecture is true.

Conjecture 1.3. *Let (W_1, S_1) and (W_2, S_2) be two Coxeter systems, $H_1 \subseteq S_1$ and $H_2 \subseteq S_2$. Let $u_1, v_1 \in W_1^{H_1}$ and $u_2, v_2 \in W_2^{H_2}$ be such that there exists a poset-isomorphism from $[u_1, v_1]$ to $[u_2, v_2]$ which restricts to a poset-isomorphism from $[u_1, v_1]^{H_1}$ to $[u_2, v_2]^{H_2}$. Then $P_{u_1,v_1}^{H_1,x}(q) = P_{u_2,v_2}^{H_2,x}(q)$ (equivalently, $R_{u_1,v_1}^{H_1,x}(q) = R_{u_2,v_2}^{H_2,x}(q)$).*

Evidently, Conjecture 1.3 reduces to Conjecture 1.1 for $H_1 = H_2 = \emptyset$. Mongelli's is not a counterexample to Conjecture 1.3 since the two Bruhat intervals $[u, v]$ and $[x, y]$ of the counterexample are not isomorphic. The results of this work imply that, for doubly laced and dihedral Coxeter groups, Conjecture 1.3 holds when u_1 and u_2 are the identity elements.

2. Notation, definitions and preliminaries

This section reviews the background material that is needed in the rest of this work. We follow [1] and [14, Chapter 3] for undefined notation and terminology concerning, respectively, Coxeter groups and partially ordered sets.

2.1. **Coxeter groups.** Given a Coxeter system (W, S), we denote the entries of its Coxeter matrix M by $m(s, s')$, for all $(s, s') \in S \times S$. As usual, we say that a Coxeter system - or a Coxeter group, by abuse of language - is simply laced (respectively, doubly laced) if $m(s, s') \leq 3$ (respectively, $m(s, s') \leq 4$), for all $(s, s') \in S \times S$. We denote by e the identity of W, and we let $T = \{wsw^{-1} : w \in W, \ s \in S\}$ be the set of *reflections* of W.

Given $w \in W$, we denote by $\ell(w)$ the length of w with respect to S, and we let

$$\begin{aligned} D_R(w) &= \{s \in S : \ell(ws) < \ell(w)\}, \\ D_L(w) &= \{s \in S : \ell(sw) < \ell(w)\}. \end{aligned}$$

We call the elements of $D_R(w)$ and $D_L(w)$, respectively, the *right descents* and the *left descents* of w.

We now recall a result due to Tits (see [15] or [1, Theorem 3.3.1]). Given $s, s' \in S$ such that $m(s, s') < \infty$, let $\alpha_{s,s'}$ denote the alternating word $ss'ss' \ldots$ of length $m(s, s')$. Two expressions are said to be linked by a braid-move (respectively, a nil-move) if it is possible to obtain one from the other by replacing a factor $\alpha_{s,s'}$ by a factor $\alpha_{s',s}$ (respectively, by deleting a factor ss).

Theorem 2.1 (Word Property). *Let $u \in W$. Then:*

- *any two reduced expressions of u are linked by a finite sequence of braid-moves;*
- *any expression of u (not necessarily reduced) is linked to any reduced expression of u by a finite sequence of braid-moves and nil-moves.*

The *Bruhat graph* of W (see [6], or, e.g., [1, §2.1] or [9, §8.6]) is the directed graph having W as vertex set and having a directed edge from u to v if and only if $u^{-1}v \in T$ and $\ell(u) < \ell(v)$. The transitive closure of the Bruhat graph of W is a partial order on W that is usually called the *Bruhat order* (see, e.g., [1, §2.1] or [9, §5.9]) and that we denote by $\leq$. Throughout this work, we always assume that W, and its subsets, are partially ordered by $\leq$. There is a well known characterization of Bruhat order on a Coxeter group (usually referred to as the *Subword Property*) that we will use repeatedly in this work, often without explicit mention. We recall it here for the reader's convenience (a proof of it can be found, e.g., in [1, §2.2] or [9, §5.10]). By a *subword* of a word $s_1 s_2 \cdots s_q$ we mean a word of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq q$.

Theorem 2.2 (Subword Property). *Let $u, w \in W$. Then the following are equivalent:*

- *$u \leq w$ in the Bruhat order,*
- *every reduced expression for w has a subword that is a reduced expression for u,*
- *there exists a reduced expression for w having a subword that is a reduced expression for u.*

The Coxeter group W, partially ordered by Bruhat order, is a graded poset having ℓ as its rank function.

For each subset $J \subseteq S$, we denote by W_J the parabolic subgroup of W generated by J, and by W^J the set of minimal coset representatives:

$$W^J = \{w \in W \ : \ D_R(w) \subseteq S \setminus J\}.$$

The following result is well known and a proof of it can be found, e.g., in [1, §2.4] or [9, §1.10].

Proposition 2.3. *Let $J \subseteq S$. Then:*

- (i) *every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ with $w^J \in W^J$ and $w_J \in W_J$;*
- (ii) *for this factorization, $\ell(w) = \ell(w^J) + \ell(w_J)$.*

There are, of course, left versions of the above definition and result. Namely, if we let

(2.1) $$^J W = \{w \in W \ : \ D_L(w) \subseteq S \setminus J\} = (W^J)^{-1},$$

then every $w \in W$ can be uniquely factorized $w = {}_J w \cdot {}^J w$, where ${}_J w \in W_J$, ${}^J w \in {}^J W$, and $\ell(w) = \ell({}_J w) + \ell({}^J w)$.

The following is a well known result (see, e.g., [8, Lemma 7]).

Proposition 2.4. *Let $J \subseteq S$ and $w \in W$. The set $W_J \cap [e, w]$ has a unique maximal element $w_0(J)$, so that $W_J \cap [e, w]$ is the interval $[e, w_0(J)]$.*

2.2. **Special matchings.** Given x, y in a partially ordered set P, we say that y *covers* x and we write $x \lhd y$ if the interval $[x, y]$ coincides with $\{x, y\}$. An element $z \in [x, y]$ is said to be an *atom* (respectively, a *coatom*) of $[x, y]$ if $x \lhd z$ (respectively, $z \lhd y$). We say that a poset P is *graded* if P has a minimum and there is a function $\rho : P \to \mathbb{N}$ (the *rank function* of P) such that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ for all $x, y \in P$ with $x \lhd y$. (This definition is slightly different from the one given in [14], but is more convenient for our purposes.) The *Hasse diagram* of P is the graph having P as vertex set and $\{\{x, y\} \in \binom{P}{2} : \text{ either } x \lhd y \text{ or } y \lhd x\}$ as edge set.

A *matching* of a poset P is an involution $M : P \to P$ such that $\{v, M(v)\}$ is an edge in the Hasse diagram of P, for all $v \in V$. A matching M of P is *special* if

$$u \lhd v \implies M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$.

The two simple results in the following lemma will be often used without explicit mention (see [2, Lemmas 2.1 and 4.1]). Given a poset P, two matchings M and N of P, and $u \in P$, we denote by $\langle M, N \rangle(u)$ the orbit of u under the action of the subgroup of the symmetric group on P generated by M and N. We call an interval $[u, v]$ in a poset P *dihedral* if it is isomorphic to a finite Coxeter system of rank 2 ordered by Bruhat order.

Lemma 2.5. *Let P be a graded poset.*
  (1) *Let M be a special matching of P, and $u, v \in P$ be such that $M(v) \lhd v$ and $M(u) \rhd u$. Then M restricts to a special matching of the interval $[u, v]$.*
  (2) *Let M and N be two special matchings of P. Then, for all $u \in P$, the orbit $\langle M, N \rangle(u)$ is a dihedral interval.*

2.3. **Special matchings in Coxeter groups.** Let (W, S) be a Coxeter system and recall that the Bruhat order is a partial order on W. For $w \in W$, we say that M is a matching of w if M is a matching of the lower Bruhat interval $[e, w]$. If $s \in D_R(w)$ (respectively, $s \in D_L(w)$) we define a matching ρ_s (respectively, λ_s) of w by $\rho_s(u) = us$ (respectively, $\lambda_s(u) = su$) for all $u \leq w$. From the "Lifting Property" (see, e.g., [3, Theorem 1.1], [1, Proposition 2.2.7] or [9, Proposition 5.9]), it easily follows that ρ_s (respectively, λ_s) is a special matching of w. We call a matching M of w a *left multiplication matching* if there exists $s \in S$ such that $M = \lambda_s$ on $[e, w]$, and we call it a *right multiplication matching* if there exists $s \in S$ such that $M = \rho_s$ on $[e, w]$.

We recall the following result, which will be needed in the proof of the main result of this work (see [2, Lemma 4.3] for a proof).

Proposition 2.6. *Given a Coxeter system (W, S) and an element $w \in W$, let M and N be two special matchings of w such that $M(e) = s$ and $N(e) = t$, with $s \neq t$. Let $u \leq w$. Then the lower dihedral interval $[e, w] \cap W_{\{s,t\}}$ contains an orbit of $\langle M, N \rangle$ having the same cardinality as the orbit $\langle M, N \rangle(u)$ of u.*

Let (W, S) be a Coxeter system and $w \in W$. By Proposition 2.4, the intersection of the lower Bruhat interval $[e, w]$ with the dihedral parabolic subgroup $W_{\{s,t\}}$ generated by any two given generators $s, t \in S$ has a maximal element; for short, we denote it by $w_0(s, t)$ instead of $w_0(\{s, t\})$.

We give the following symmetric definitions.

Definition 2.7. *A right system for w is a quadruple (J, s, t, M_{st}) such that:*

R1. $J \subseteq S$, $s \in J$, $t \in S \setminus J$, *and* M_{st} *is a special matching of* $w_0(s, t)$ *such that* $M_{st}(e) = s$ *and* $M_{st}(t) = ts$;

R2. $(u^J)^{\{s,t\}} \cdot M_{st}\left((u^J)_{\{s,t\}} \cdot {}_{\{s\}}(u_J) \right) \cdot {}^{\{s\}}(u_J) \leq w$, *for all* $u \leq w$;

R3. *if* $r \in J$ *and* $r \leq w^J$, *then* r *and* s *commute;*

R4. (a) *if* $s \leq (w^J)^{\{s,t\}}$ *and* $t \leq (w^J)^{\{s,t\}}$, *then* $M_{st} = \rho_s$,

    (b) *if* $s \leq (w^J)^{\{s,t\}}$ *and* $t \not\leq (w^J)^{\{s,t\}}$, *then* M_{st} *commutes with* λ_s,

    (c) *if* $s \not\leq (w^J)^{\{s,t\}}$ *and* $t \leq (w^J)^{\{s,t\}}$, *then* M_{st} *commutes with* λ_t;

R5. *if* $v \leq w$ *and* $s \leq {}^{\{s\}}(v_J)$, *then* M_{st} *commutes with* ρ_s *on* $[e, v] \cap [e, w_0(s, t)] = [e, v_0(s, t)]$.

Definition 2.8. *A left system for w is a quadruple (J, s, t, M_{st}) such that:*

L1. $J \subseteq S$, $s \in J$, $t \in S \setminus J$, *and* M_{st} *is a special matching of* $w_0(s, t)$ *such that* $M_{st}(e) = s$ *and* $M_{st}(t) = st$;

L2. $({}_J u)^{\{s\}} \cdot M_{st}\left(({}_J u)_{\{s\}} \cdot {}_{\{s,t\}}({}^J u) \right) \cdot {}^{\{s,t\}}({}^J u) \leq w$, *for all* $u \leq w$;

L3. *if* $r \in J$ *and* $r \leq {}^J w$, *then* r *and* s *commute;*

L4. (a) *if* $s \leq {}^{\{s,t\}}({}^J w)$ *and* $t \leq {}^{\{s,t\}}({}^J w)$, *then* $M_{st} = \lambda_s$,

    (b) *if* $s \leq {}^{\{s,t\}}({}^J w)$ *and* $t \not\leq {}^{\{s,t\}}({}^J w)$, *then* M_{st} *commutes with* ρ_s,

    (c) *if* $s \not\leq {}^{\{s,t\}}({}^J w)$ *and* $t \leq {}^{\{s,t\}}({}^J w)$, *then* M_{st} *commutes with* ρ_t;

L5. *if* $v \leq w$ *and* $s \leq ({}_J v)^{\{s\}}$, *then* M_{st} *commutes with* λ_s *on* $[e, v] \cap [e, w_0(s, t)] = [e, v_0(s, t)]$.

Given a right system for w, the matching M associated with it is the matching of w acting in the following way: for all $u \leq w$,

$$M(u) = (u^J)^{\{s,t\}} \cdot M_{st}\left((u^J)_{\{s,t\}} \cdot {}_{\{s\}}(u_J) \right) \cdot {}^{\{s\}}(u_J).$$

Note that M acts as λ_s on $[e, w_0(s, r)]$ for all $r \in J$, and as ρ_s on $[e, w_0(s, r)]$ for all $r \in S \setminus (J \cup \{t\})$; moreover, if $s \in D_R(w)$, for the trivial choises $J = \{s\}$ and $M_{st} = \rho_s$, we obtain right multiplication matchings ($M = \rho_s$ on the entire interval $[e, w]$).

Symmetrically, given a left system for w, the matching M associated with it is the matching of w acting in the following way: for all $u \leq w$,

$$M(u) = ({}_J u)^{\{s\}} \cdot M_{st}\left(({}_J u)_{\{s\}} \cdot {}_{\{s,t\}}({}^J u) \right) \cdot {}^{\{s,t\}}({}^J u).$$

We obtain left multiplication matchings as special cases.

We comment that distinct systems for w might give rise to the same matching of w.

The following result is needed in the proofs of the main results of this work (see [12]).

Theorem 2.9. *Let w be any element of any arbitrary Coxeter group W and M be a special matching of w. Then M is associated with a right or a left system of w.*

2.4. **Kazhdan–Lusztig polynomials.** In introducing the (ordinary and parabolic) R-polynomials and Kazhdan–Lusztig polynomials, among all the equivalent definitions, we choose the combinatorial ones, since they suit our purposes best.

Given a Coxeter system (W, S) and $H \subseteq S$, we consider the set of minimal coset representatives W^H as a poset with the partial ordering induced by the Bruhat order on W. Given $u, v \in W^H$, $u \leq v$, we let

$$[u, v]^H = \{z \in W^H : u \leq z \leq v\},$$

be the (parabolic) interval in W^H with bottom element u and top element v.

The following two results are due to Deodhar, and we refer to [4, §§2-3] for their proofs.

Theorem 2.10. *Let (W, S) be a Coxeter system, and $H \subseteq S$. Then, for each $x \in \{-1, q\}$, there is a unique family of polynomials $\{R_{u,v}^{H,x}(q)\}_{u,v \in W^H} \subseteq \mathbf{Z}[q]$ such that, for all $u, v \in W^H$:*

(1) $R_{u,v}^{H,x}(q) = 0$ *if $u \not\leq v$;*
(2) $R_{u,u}^{H,x}(q) = 1$;
(3) *if $u < v$ and $s \in D_L(v)$, then*

$$R_{u,v}^{H,x}(q) = \begin{cases} R_{su,sv}^{H,x}(q), & \text{if } s \in D_L(u), \\ (q-1)R_{u,sv}^{H,x}(q) + qR_{su,sv}^{H,x}(q), & \text{if } s \notin D_L(u) \text{ and } su \in W^H, \\ (q-1-x)R_{u,sv}^{H,x}(q), & \text{if } s \notin D_L(u) \text{ and } su \notin W^H. \end{cases}$$

In the sequel, we will often use the inductive formula of Theorem 2.10 without explicit mention.

Theorem 2.11. *Let (W, S) be a Coxeter system, and $H \subseteq S$. Then, for each $x \in \{-1, q\}$, there is a unique family of polynomials $\{P_{u,v}^{H,x}(q)\}_{u,v \in W^H} \subseteq \mathbf{Z}[q]$, such that, for all $u, v \in W^H$:*

(1) $P_{u,v}^{H,x}(q) = 0$ *if $u \not\leq v$;*
(2) $P_{u,u}^{H,x}(q) = 1$;
(3) $deg(P_{u,v}^{H,x}(q)) \leq \frac{1}{2}(\ell(v) - \ell(u) - 1)$, *if $u < v$;*
(4) $q^{\ell(v)-\ell(u)} P_{u,v}^{H,x}\left(\frac{1}{q}\right) = \sum_{z \in [u,v]_H} R_{u,z}^{H,x}(q) P_{z,v}^{H,x}(q)$.

The polynomials $R_{u,v}^{H,x}(q)$ and $P_{u,v}^{H,x}(q)$ are called the *parabolic R-polynomials* and *parabolic Kazhdan–Lusztig polynomials* of W^H of type x. For $H = \emptyset$, $R_{u,v}^{\emptyset,-1}(q)$ $(= R_{u,v}^{\emptyset,q}(q))$ and $P_{u,v}^{\emptyset,-1}(q)$ $(= P_{u,v}^{\emptyset,q}(q))$ are the ordinary R-polynomials $R_{u,v}(q)$ and Kazhdan–Lusztig polynomials $P_{u,v}(q)$ of W. Another relationship between the parabolic Kazhdan–Lusztig polynomials and their ordinary counterparts is established by the following result (see [4, Proposition 3.4, and Remark 3.8]).

Proposition 2.12. *Let (W, S) be a Coxeter system, $H \subseteq S$, and $u, v \in W^H$. Then we have that*

$$P_{u,v}^{H,q}(q) = \sum_{w \in W_H} (-1)^{l(w)} P_{uw,v}(q).$$

Furthermore, if W_H is finite, then

$$P_{u,v}^{H,-1}(q) = P_{uw_0^H, vw_0^H}(q),$$

where w_0^H is the longest element of W_H.

We refer to [1], [4], and [9] for more details concerning general Coxeter group theory and parabolic Kazhdan–Lusztig polynomials.

3. Commuting special matchings

In this section, we prove existence results for special matchings commuting with a given one. These results are needed in Section 4.

We will make repeated use of the following easy result.

Lemma 3.1. *Let (W, S) be any arbitrary Coxeter system. Two special matchings M and N of $w \in W$ commute if and only if they commute on the lower dihedral intervals containing $M(e)$ and $N(e)$ (in particular, on $[e, w_0(s, t)]$ if $s = M(e), t = N(e), s \neq t$).*

Proof. The result follows directly from Proposition 2.6. $\square$

We recall that a special matching M of w stabilizes the intersection of $[e, w]$ with any parabolic subgroup containing $M(e)$ (see [2, Proposition 5.3]). We will use this fact without explicit mention.

3.1. **Right (resp. left) systems and left (resp. right) multiplication matchings.**

Proposition 3.2. *Let W be a doubly laced Coxeter group, $w \in W$ and M a special matching of w associated with a right system (J, s, t, M_{st}), M not a left multiplication matching. Then there exists a left multiplication matching λ of w commuting with M such that $\lambda(w) \neq M(w)$, unless $m(s, t) = 4$ and we are in one of the following cases:*
First case:

(1) $w_0(s, t) = tst$,
(2) $(w^J)^{\{s,t\}} = e$,
(3) $w = tst \cdot {}^{\{s\}}(w_J)$,

Second case:

(1) $w_0(s, t) = tsts = stst$,
(2) $(w^J)^{\{s,t\}} = e$,
(3) *either*
   - $w = tsts \cdot {}^{\{s\}}(w_J)$, *or*
   - $w = tst \cdot {}^{\{s\}}(w_J)$ *with* $s \leq {}^{\{s\}}(w_J)$,
(4) $M(tsts) = sts$ *(and then M_{st} must be the matching mapping e to s, t to ts, st to tst, and sts to $tsts$).*

Proof. If $(w^J)^{\{s,t\}} \neq e$, there exists $l \in D_L((w^J)^{\{s,t\}})$. Then $l \in D_L(w)$ and λ_l is a special matching of w which satisfies $M(w) \neq \lambda_l(w)$ since

$$M(w) = (w^J)^{\{s,t\}} \cdot M_{st}\left((w^J)_{\{s,t\}} \cdot {}_{\{s\}}(w_J)\right) \cdot {}^{\{s\}}(w_J)$$

while

$$\lambda_l(w) = l \cdot (w^J)^{\{s,t\}} \cdot \left((w^J)_{\{s,t\}} \cdot {}_{\{s\}}(w_J)\right) \cdot {}^{\{s\}}(w_J).$$

We have to show that M and λ_l commute. We distinguish the following cases, in which we apply Lemma 3.1.

(a) $l \notin \{s,t\}$

By Property R3 of the definition of a right system, either $l \notin J$ or l commutes with s. In the first case, M acts as ρ_s on $[e, w_0(s,l)]$ and hence commutes with λ_l. In the second case, M and λ_l clearly commutes on $[e, w_0(s,l)]$ because $[e, w_0(s,l)]$ has just 4 elements.

(b) $l = t$

By Property R4, M commutes with λ_t on the lower interval $[e, w_0(s,t)]$ (we are either in case (a) or in case (c) of Property R4).

(c) $l = s$

By Lemma 3.1, we need to show that M and λ_s commute on every lower dihedral intervals $[e, w_0(s,r)]$, with $r \in S \setminus \{s\}$. For $r = t$, it follows from Property R4 (we are either in case (a) or in case (b) of Property R4). For $r \neq t$, M acts on $[e, w_0(s,r)]$ as ρ_s or λ_s, and in both cases M commutes with λ_s on $[e, w_0(s,r)]$.

We now suppose $(w^J)^{\{s,t\}} = e$. By the definition of a right system, M acts as λ_s on every lower dihedral interval $[e, w_0(s,r)]$, $r \in S \setminus \{s,t\}$. On $[e, w_0(s,t)]$, M does not act as λ_s as otherwise M would coincide with λ_s everywhere, but M is not a left multiplication matching by hypothesis. In particular, $M(t) = ts \neq st$. If $w_0(s,t) = ts$, then either $w = ts \cdot {}^{\{s\}}(w_J)$, or $w = t \cdot {}^{\{s\}}(w_J)$ with $s \leq {}^{\{s\}}(w_J)$. Indeed, the second case cannot occur since $M(w)$ would be $M(t) \cdot {}^{\{s\}}(w_J) = ts \cdot {}^{\{s\}}(w_J)$ and we would have $M(w) \rhd w$, which is impossible. So $w = ts \cdot {}^{\{s\}}(w_J)$, $t \in D_L(w)$, λ_t is a special matching of w, and M commutes with λ_t by Lemma 3.1. Moreover,

$$M(w) = M(ts) \cdot {}^{\{s\}}(w_J) = t \cdot {}^{\{s\}}(w_J),$$

while

$$\lambda_t(w) = s \cdot {}^{\{s\}}(w_J),$$

so $M(w) \neq \lambda_s(w)$.

So we may assume that the lower dihedral interval $[e, w_0(s,t)]$ has at least 6 elements (hence it has 6 or 8 elements, since W is doubly laced). Suppose that $[e, w_0(s,t)]$ has 6 elements. Necessarily, $M(e) = s$, $M(t) = ts$, and $M(st) = w_0(s,t) \in \{tst, sts\}$. If $w_0(s,t) = tst \neq sts$, then, since $tst \leq w$ and $t \not\leq {}^{\{s\}}(w_J)$, we have $w = tst \cdot {}^{\{s\}}(w_J)$ and we are in the first case of the statement of the proposition. If $w_0(s,t) = sts$, then, since $sts \leq w$ and $t \not\leq {}^{\{s\}}(w_J)$, we have only two possibilities: either $w = sts \cdot {}^{\{s\}}(w_J)$, or $w = st \cdot {}^{\{s\}}(w_J)$ with $s \leq {}^{\{s\}}(w_J)$. Indeed, the second one cannot occur since, in that case, $M(w) = M(st) \cdot {}^{\{s\}}(w_J) = sts \cdot {}^{\{s\}}(w_J)$ and we would have $M(w) \rhd w$, which is impossible. So $w = sts \cdot {}^{\{s\}}(w_J)$; thus $s \in D_L(w)$, λ_s is a special matching of w, and M commutes with λ_s by Lemma 3.1. Moreover,

$$M(w) = M(sts) \cdot {}^{\{s\}}(w_J) = st \cdot {}^{\{s\}}(w_J),$$

while

$$\lambda_s(w) = ts \cdot {}^{\{s\}}(w_J),$$

so $M(w) \neq \lambda_s(w)$.

Suppose that $[e, w_0(s,t)]$ has 8 elements, i.e., $w_0(s,t) = stst = tsts$ since W is doubly laced. Then, since $tsts \leq w$ and $t \not\leq {}^{\{s\}}(w_J)$, we have either $w = tsts\,{}^{\{s\}}(w_J)$, or $w = tst\,{}^{\{s\}}(w_J)$ with $s \leq {}^{\{s\}}(w_J)$. In both cases $t \in D_L(w)$, hence λ_t is a special matching

of w. The matching λ_t always commutes with M since $M(t) = ts$ by the definition of a right system; moreover, evidently, $\lambda_t(w) = M(w)$ if and only if $M(tsts) = sts$.

The proof is complete. □

Note that the conditions in the cases of the statement of Proposition 3.2 (and also of the forthcoming Propositions 3.3, 3.5, and 3.6) are redundant but we prefer to emphasize them since they are needed later.

The symmetric version of Proposition 3.2 is the following.

Proposition 3.3. *Let W be a doubly laced Coxeter group, $w \in W$ and M a special matching of w associated with a left system (J, s, t, M_{st}), M not a right multiplication matching. Then there exists a right multiplication matching ρ of w commuting with M such that $\rho(w) \neq M(w)$, unless $m(s, t) = 4$ and we are in one of the following cases:*

First case:

(1) $w_0(s, t) = tst$,
(2) $^{\{s,t\}}(^J w) = e$,
(3) $w = (_J w)^{\{s\}} \cdot tst$,

Second case:

(1) $w_0(s, t) = stst = tsts$,
(2) $^{\{s,t\}}(^J w) = e$,
(3) *either*
  - $w = (_J w)^{\{s\}} \cdot stst$, *or*
  - $w = (_J w)^{\{s\}} \cdot tst$ *with* $s \leq (_J w)^{\{s\}}$,
(4) $M(stst) = sts$ *(and then M_{st} must be the matching mapping e to s, t to st, ts to tst, and sts to $stst$).*

Remark 3.4. *We make the following observations.*

(1) *In the proof of Proposition 3.2, we use the hypothesis that W be doubly laced only in the case $(w^J)^{\{s,t\}} = e$ and we actually use only the fact that $m(s, t) \leq 4$.*
(2) *Proposition 3.2 does not hold in the special cases we excluded. A trivial example can be found in the dihedral Coxeter system $(W, \{s, t\})$ with $m(s, t) = 4$. Since usually dihedral Coxeter systems can be treated separately, we give also the following less trivial counterexample which explains better what obstructions may occur. Let $(W, \{s, t, r\})$ be the Coxeter system with Coxeter matrix satisfying $m(s, t) = 4$, $m(s, r) = m(t, r) = 3$. Consider the element $w = tstrs \in W$ and the matching associated with the right system $(J = \{s, r\}, s, t, M_{st})$ with $M_{st}(st) = tst$ and $M_{st}(sts) = tsts$. Then M commutes with the unique left multiplication special matching λ_t but it coincides with it on w.*

These observations similarly hold true, mutatis mutandis, also for Proposition 3.3.

3.2. **Right (resp. left) systems and right (resp. left) multiplication matchings.**

Proposition 3.5. *Let W be a doubly laced Coxeter group, $w \in W$ and M a special matching of w associated with a right system (J, s, t, M_{st}), M not a right multiplication matching. Then there exists a right multiplication matching ρ of w commuting with M such that $\rho(w) \neq M(w)$, unless $m(s, t) = 4$ and we are in the following case:*

(1) $w_0(s,t) = tst$,
(2) $w_J = e$,
(3) $w = (w^J)^{\{s,t\}} \cdot tst$.

Proof. If $^{\{s\}}(w_J) \neq e$, there exists $r \in D_R(^{\{s\}}(w_J))$. Then $r \in D_R(w)$ and ρ_r is a special matching of w which satisfies $M(w) \neq \rho_r(w)$ since

$$M(w) = (w^J)^{\{s,t\}} \cdot M_{st}\Big((w^J)_{\{s,t\}} \cdot {}_{\{s\}}(w_J)\Big) \cdot {}^{\{s\}}(w_J)$$

while

$$\rho_r(w) = (w^J)^{\{s,t\}} \cdot \Big((w^J)_{\{s,t\}} \cdot {}_{\{s\}}(w_J)\Big) \cdot {}^{\{s\}}(w_J) \cdot r.$$

In order to show that M and ρ_r commute, we apply Lemma 3.1. If $r \neq s$, M acts as λ_s on the lower dihedral interval $[e, w_0(s,r)]$ and hence it commutes with ρ_r. If $r = s$, we need to show that M and ρ_s commute on every lower dihedral interval $[e, w_0(s,l)]$, with $l \in S \setminus \{s\}$. For $l = t$, it follows from Property R5. For $l \neq t$, M acts on $[e, w_0(s,l)]$ either as λ_s or as ρ_s: in both cases it commutes with ρ_s on $[e, w_0(s,l)]$.

We now suppose $^{\{s\}}(w_J) = e$. By the definition of a right system, M acts as ρ_s on every lower dihedral interval $[e, w_0(s,r)]$, $r \in S \setminus \{s,t\}$. On $[e, w_0(s,t)]$, M does not act as ρ_s as otherwise M would coincide with ρ_s everywhere, but M is not a right multiplication matching by hypothesis. Since $M(e) = \rho_s(e) = s$ and $M(t) = \rho_s(t) = ts$, $w_0(s,t)$ cannot be sts, which implies that $m(s,t) = 4$ (since W is a doubly laced Coxeter group). If $w_0(s,t) = tst$, then also $_{\{s\}}(w_J) = e$, as otherwise $tsts$ would be $\leq w$ by the Subword Property. Hence $w_J = e$ and $w = (w^J)^{\{s,t\}} \cdot (w^J)_{\{s,t\}}$. Since $M(w) \lhd w$ and $M(w) = (w^J)^{\{s,t\}} \cdot M((w^J)_{\{s,t\}})$ by definition, we have $M((w^J)_{\{s,t\}}) \lhd (w^J)_{\{s,t\}}$ and hence $(w^J)_{\{s,t\}} \notin \{e,t,st\}$. Moreover, since $s \in J$, clearly $s \notin D_R(w^J)$, which implies $s \notin D_R((w^J)_{\{s,t\}})$, and thus $(w^J)_{\{s,t\}} \notin \{s,ts\}$. The only possibility left is $(w^J)_{\{s,t\}} = tst$. Hence $w = (w^J)^{\{s,t\}} \cdot tst$ and we are in the case we excluded in the statement of the proposition.

Now suppose that $w_0(s,t) = stst = tsts$; since $M \neq \rho_s$, necessarily $M(st) = tst$ and $M(sts) = stst$. By Property R4, (a), it is not possible that both s and t are $\leq (w^J)^{\{s,t\}}$. Since M does not commute with λ_s on $[e, w_0(s,t)]$, we have $s \not\leq (w^J)^{\{s,t\}}$ by Property R4, (b). Since $stst \leq w$, we have that $(w^J)_{\{s,t\}} \cdot {}_{\{s\}}(w_J)$ can be either $stst$ or sts. The second case is impossible since $M(w) = (w^J)^{\{s,t\}} \cdot M(sts) = (w^J)^{\{s,t\}} \cdot stst$ would be greater than w. Hence

$$w = (w^J)^{\{s,t\}} \cdot stst,$$

$s \in D_R(w)$, and ρ_s is a special matching of w. Since M and ρ_s commute on every lower dihedral interval, they commute everywhere by Lemma 3.1. Moreover $M(w) \neq \rho_s(w)$ since

$$M(w) = M\Big((w^J)^{\{s,t\}} \cdot stst\Big) = (w^J)^{\{s,t\}} \cdot M(stst) = (w^J)^{\{s,t\}} \cdot sts$$

while

$$\rho_s(w) = \rho_s\Big((w^J)^{\{s,t\}} \cdot stst\Big) = (w^J)^{\{s,t\}} \cdot tst.$$

The proof is complete. $\square$

The symmetric version of Proposition 3.5 is the following.

Proposition 3.6. *Let W be a doubly laced Coxeter group, $w \in W$ and M a special matching of w associated with a left system (J, s, t, M_{st}), M not a left multiplication matching. Then there exists a left multiplication matching λ of w commuting with M such that $\lambda(w) \neq M(w)$, unless $m(s, t) = 4$ and we are in the following case:*

(1) $w_0(s, t) = tst$,
(2) $_J w = e$,
(3) $w = tst \cdot {}^{\{s,t\}}({}^J w)$.

.

Remark 3.7. *We make the following observations.*

(1) *In the proof of Proposition 3.5, we use the hypothesis that W be doubly laced only in the case ${}^{\{s\}}(w_J) = e$ and we actually use only the fact that $m(s, t) \leq 4$.*
(2) *Proposition 3.5 does not hold if $m(s, t) = 5$. A trivial counterexample can be found in the dihedral Coxeter system $(W, \{s, t\})$ with $m(s, t) = 5$. Since usually dihedral Coxeter systems can be treated separately, we give also the following less trivial counterexample which explains better what obstruction may occur. Consider the Coxeter system $(W, \{s, t, r\})$ with $m(s, t) = 5, m(s, r) = m(t, r) = 3$, the element $w = trstst \in W$ and the matching associated with the right system $(J = \{s, r\}, s, t, M_{st})$ with $M_{st}(st) = tst$, $M_{st}(sts) = stst$, and $M_{st}(tsts) = tstst$. Then M does not commute with the unique right multiplication special matching ρ_t.*

These observations similarly hold true, mutatis mutandis, also for Proposition 3.6.

As an immediate consequence, we have the following result.

Corollary 3.8. *Let W be a simply laced Coxeter group, $w \in W$, and M a special matching of w. Then there exist a right multiplication matching and a left multiplication matching that commute with M and such that do not agree with M on w.*

Proof. The assertion follows from Theorem 2.9 and Propositions 3.2, 3.3, 3.5, 3.6. $\square$

4. Parabolic R-polynomials and H-special matchings

In this section, for all $H \subseteq S$ and $w \in W^H$, we give a method for computing the parabolic Kazhan–Lusztig R-polynomials $\{R_{u,w}^{H,x}(q)\}_{u \in W^H}$ from the only knowledge of:

- the isomorphism type (as a poset) of the interval $[e, w]$,
- which elements of the interval $[e, w]$ are in W^H.

We prove this result in the case W is either a doubly laced Coxeter group (i.e., $m(r, r') \leq 4$, for all $r, r' \in S$), or a dihedral Coxeter group (i.e., a Coxeter group of rank 2). In particular, the result holds for all Weyl groups. It is worth noting that this is also a result on parabolic Kazhdan–Lusztig polynomials since these are equivalent to the parabolic R-polynomials.

4.1. **H-special matchings and calculating special matchings.** We now define the tools that compute the parabolic Kazhan–Lusztig polynomials.

Let (W, S) be an arbitrary Coxeter system, $H \subseteq S$, and $w \in W^H$. An *H-special matching of w* is a special matching of w such that

$$u \leq w, u \in W^H, M(u) \lhd u \Rightarrow M(u) \in W^H.$$

Note that the $\emptyset$-special matchings are exactly the special matchings and that a left multiplication matching is H-special for all $H \subseteq S$.

For convenience' sake, we say that an H-special matching M of w *calculates* the parabolic Kazhdan–Lusztig R-polynomials (or is *calculating*, for short) if, for all $u \in W^H$, $u \leq w$, we have

(4.1)
$$R_{u,w}^{H,x}(q) = \begin{cases} R_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \lhd u, \\ (q-1)R_{u,M(w)}^{H,x}(q) + qR_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \in W^H, \\ (q-1-x)R_{u,M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \notin W^H. \end{cases}$$

For this definition, it is essential that the special matching be H-special. Note that all left multiplication matchings are calculating.

We want to show that all H-special matchings are calculating for all doubly laced Coxeter groups and all dihedral groups (indeed, we prove it for a larger class of situations).

4.2. Commuting matchings and calculating matchings. We need the following easy lemma.

Lemma 4.1. *Let (W, S) be an arbitrary Coxeter system, $H \subset S$, and $v \notin W^H$. Then at most one of the coatoms of v belongs to W^H.*

Proof. Since $v \notin W^H$, the set $H \cap D_R$ is non-empty. Let $s \in H \cap D_R$. Then there exists $y \in W$ such that $v = y \cdot s$ and $\ell(v) = \ell(y) + 1$. By the Deletion Property, s is a right descent of all coatoms of v except y. $\qquad\square$

In the proof of the following theorem and in the sequel, we use the inductive formula of Theorem 2.10 without explicit mention.

Theorem 4.2. *Given a Coxeter system (W, S) and $H \subseteq S$, let $w \in W^H$ and M be an H-special matching of w. Suppose that*

- *every H-special matching of v is calculating, for all $v \in W^H$, $v < w$,*
- *there exists a calculating special matching N of w commuting with M and such that $M(w) \neq N(w)$.*

Then M is calculating.

Proof. We prove the claim by induction on $\ell(w)$, the result being clearly true if $\ell(w) \leq 2$. So assume $\ell(w) \geq 3$, and let $u \in [e, w]$, $u \in W^H$. The orbit of w under the action of the group generated by M and N is $\langle N, M \rangle(w) = \{w, N(w), M(w), MN(w) = NM(w)\}$ and is contained in W^H since both N and M are H-special. On the other hand, the orbit $\langle N, M \rangle(u)$ can have either cardinality 4 or cardinality 2, and can be contained in W^H or not. We may assume that it is not contained in W^H as otherwise we could prove that M is calculating by the same arguments as in [2, Theorem 7.8].

If the cardinality of $\langle N, M \rangle(u)$ is 2, then $u \lhd N(u) = M(u) \notin W^H$, since otherwise the orbit would be contained in W^H since N and M are H-special. In this case, by our induction hypothesis

$$
\begin{aligned}
R_{u,w}^{H,x}(q) &= (q-1-x)R_{u,N(w)}^{H,x} = (q-1-x)^2 R_{u,MN(w)}^{H,x} \\
&= (q-1-x)^2 R_{u,NM(w)}^{H,x} = (q-1-x)R_{u,M(w)}^{H,x},
\end{aligned}
$$

as desired. (Here and in the sequel, we use the fact that M restricts to a special matching of $N(w)$ and hence we can use the induction hypothesis since $\ell(N(w)) < \ell(w)$.)

Now assume that the cardinality of $\langle N, M \rangle(u)$ is 4. By Lemma 4.1 and the fact that N and M are H-special, there are 5 cases to be considered.

(a) $u \lhd N(u) \notin W^H$, $u \rhd M(u) \in W^H$, $NM(u) = MN(u) \notin W^H$.
By our induction hypothesis,

$$
\begin{aligned}
R_{u,w}^{H,x}(q) &= (q-1-x)R_{u,N(w)}^{H,x} = (q-1-x)R_{M(u),MN(w)}^{H,x} \\
&= (q-1-x)R_{M(u),NM(w)}^{H,x} = R_{M(u),M(w)}^{H,x}.
\end{aligned}
$$

(b) $u \rhd N(u) \in W^H$, $u \lhd M(u) \notin W^H$, $NM(u) = MN(u) \notin W^H$.
By our induction hypothesis,

$$
\begin{aligned}
R_{u,w}^{H,x}(q) &= R_{N(u),N(w)}^{H,x} = (q-1-x)R_{N(u),MN(w)}^{H,x} \\
&= (q-1-x)R_{N(u),NM(w)}^{H,x} = (q-1-x)R_{u,M(w)}^{H,x}.
\end{aligned}
$$

(c) $u \lhd N(u) \in W^H$, $u \lhd M(u) \notin W^H$, $NM(u) = MN(u) \notin W^H$.
By our induction hypothesis,

$$
\begin{aligned}
R_{u,w}^{H,x}(q) &= (q-1)R_{u,N(w)}^{H,x} + qR_{N(u),N(w)}^{H,x} \\
&= (q-1)(q-1-x)R_{u,MN(w)}^{H,x} + q(q-1-x)R_{N(u),MN(w)}^{H,x} \\
&= (q-1-x)((q-1)R_{u,NM(w)}^{H,x} + qR_{N(u),NM(w)}^{H,x}) \\
&= (q-1-x)R_{u,M(w)}^{H,x}.
\end{aligned}
$$

(d) $u \lhd N(u) \notin W^H$, $u \lhd M(u) \in W^H$, $NM(u) = MN(u) \notin W^H$.
By our induction hypothesis,

$$
\begin{aligned}
R_{u,w}^{H,x}(q) &= (q-1-x)R_{u,N(w)}^{H,x} \\
&= (q-1-x)((q-1)R_{u,MN(w)}^{H,x} + qR_{M(u),MN(w)}^{H,x}) \\
&= (q-1-x)((q-1)R_{u,NM(w)}^{H,x} + qR_{M(u),NM(w)}^{H,x}) \\
&= (q-1)R_{u,M(w)}^{H,x} + qR_{M(u),M(w)}^{H,x}.
\end{aligned}
$$

(e) $u \lhd N(u) \notin W^H$, $u \lhd M(u) \notin W^H$, $NM(u) = MN(u) \notin W^H$.
By our induction hypothesis,

$$
\begin{aligned}
R_{u,w}^{H,x}(q) &= (q-1-x)R_{u,N(w)}^{H,x} = (q-1-x)^2 R_{u,MN(w)}^{H,x} \\
&= (q-1-x)^2 R_{u,NM(w)}^{H,x} = (q-1-x)R_{u,M(w)}^{H,x}.
\end{aligned}
$$

The proof is complete.   □

The following result is a special case of Theorem 4.5. Since it concerns an important class of Coxeter groups, including the symmetric group, and its proof is much simpler than the proof of Theorem 4.5, we give it here explicitly.

Corollary 4.3. *Let (W,S) be a simply laced Coxeter system, H be any arbitrary subset of S and w be any arbitrary element of W^H. Then every H-special matching M of w calculates the $R^{H,x}$-polynomials.*

Proof. The assertion follows immediately by induction using Corollary 3.8 and Theorem 4.2, since left multiplication matchings are calculating by definition.   □

4.3. **Doubly laced Coxeter groups.** We now prove that every H-special matching calculates the $R^{H,x}$-polynomials of doubly laced Coxeter groups. The following easy lemma is needed in the proof.

Lemma 4.4. *Let (W,S) be a Coxeter system, $s,t \in S$, $m(s,t) \geq 4$, $H \subseteq S$. Let $w \in W$ be such that $t \not\leq w$ and $stw \in W^H$. Let $v \leq w$ be such that $s \notin D_L(v)$ and $\{v, sv, tv, stv\} \subseteq W^H$. Then*

$$R^{H,x}_{sv,stw} = R^{H,x}_{tv,stw}.$$

Proof. By hypothesis, $\ell(tsv) = \ell(stv) = \ell(sv) + 1 = \ell(tv) + 1 = \ell(v) + 2$. We have

$$R^{H,x}_{sv,stw} = R^{H,x}_{v,tw} = (q-1)R^{H,x}_{v,w} + qR^{H,x}_{tv,w} = (q-1)R^{H,x}_{v,w}$$

(where the last equality holds since $t \not\leq w$ and hence $tv \not\leq w$), and

$$R^{H,x}_{tv,stw} = (q-1)R^{H,x}_{tv,tw} + qR^{H,x}_{stv,tw} = (q-1)R^{H,x}_{tv,tw} = (q-1)R^{H,x}_{v,w}$$

(where the second equality holds since $t \not\leq w$ and hence $stv \not\leq tw$).   □

Theorem 4.5. *Let (W,S) be a doubly laced Coxeter system, H be any arbitrary subset of S and w be any arbitrary element of W^H. Then every H-special matching of w calculates the $R^{H,x}$-polynomials.*

Proof. We proceed by induction on $\ell(w)$, the case $\ell(w) \leq 1$ being trivial.

Let M be a H-special matching of w. We assume that M is not a left multipication matching, since left multiplication matchings are calculating by definition.

If there exists a left multiplication matching λ commuting with M and such that $\lambda(w) \neq M(w)$, then we can conclude by Theorem 4.2. We assume that such a left multiplication matching does not exist. By Theorem 2.9 and Propositions 3.2 and 3.6, necessarily $m(s,t) = 4$ and we are in one of the following cases:
Case 1: M is associated with a right system (J, s, t, M_{st}) and

(1) $w_0(s,t) = tst$,
(2) $(w^J)^{\{s,t\}} = e$,
(3) $w = tst \cdot {}^{\{s\}}(w_J)$,

Case 2: M is associated with a right system (J, s, t, M_{st}) and

(1) $w_0(s,t) = tsts = stst$,
(2) $(w^J)^{\{s,t\}} = e$,

(3) either
- $w = tsts \cdot {}^{\{s\}}(w_J)$, or
- $w = tst \cdot {}^{\{s\}}(w_J)$ with $s \leq {}^{\{s\}}(w_J)$,

(4) $M(tsts) = sts$ (and then M_{st} must be the matching mapping e to s, t to ts, st to tst, and sts to $tsts$).

Case 3: M is associated with a left system (J, s, t, M_{st}) and

(1) $w_0(s, t) = tst$,
(2) $_J w = e$,
(3) $w = tst \cdot {}^{\{s,t\}}({}^J w)$.

.

Case 1. Notice that, in this case, $s \not\leq {}^{\{s\}}(w_J)$ (as otherwise $tsts$ would be $\leq w$ by the Subword Property), $w_J = {}^{\{s\}}(w_J)$, $w = tst \cdot w_J$ and $M(w) = st \cdot w_J$. Then $t \in D_L(w)$ and λ_t is a special matching of w. M and λ_t commute but they agree on w (i.e., $|\langle M, \lambda_t \rangle(w)| = 2$). The lower dihedral interval $[e, w_0(s, t)]$ contains two orbits of $\langle M, \lambda_t \rangle$: one orbit with 2 elements and the other with 4 elements; hence every orbit of $\langle M, \lambda_t \rangle$ has either 2 or 4 elements, by Proposition 2.6.

Let $u \leq w$, $u \in W^H$. We need to show that

(4.2)
$$R_{u,w}^{H,x}(q) = \begin{cases} R_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \lhd u, \\ (q-1)R_{u,M(w)}^{H,x}(q) + qR_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \in W^H, \\ (q-1-x)R_{u,M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \notin W^H. \end{cases}$$

This is trivial if also $|\langle M, \lambda_t \rangle(u)| = 2$ so assume $|\langle M, \lambda_t \rangle(u)| = 4$. Since M agrees with λ_t on st and tst, necessarily $\langle M, \lambda_t \rangle(u) = \{y_J, sy_J, ty_J, tsy_J\}$, for a certain $y_J \in W_J$ with $s \notin D_L(y_J)$. Note that $M(y_J) = sy_J$, $M(ty_J) = tsy_J$, and $t \not\leq y_J$ since $t \notin J$.

Assume that $\langle M, \lambda_t \rangle(u) = \{y_J, sy_J, ty_J, tsy_J\}$ is contained in W^H and note that, by the Word Property (Theorem 2.1), also sty_J is in W^H since $t \not\leq y_J$ and no braid move could involve the leftmost letters s, t (so we cannot obtain a reduced expression ending with a letter in H). We distinguish the following 4 cases according to which of the elements in the orbit is u.

(a) $u = tsy_J$. Then

$$R_{u,w}^{H,x} = R_{tsy_J, tstw_J}^{H,x} = R_{sy_J, stw_J}^{H,x} = R_{ty_J, stw_J}^{H,x} = R_{M(u),M(w)}^{H,x},$$

where the third equality follows from Lemma 4.4.

(b) $u = sy_J$. Then

$$R_{u,w}^{H,x} = R_{sy_J, tstw_J}^{H,x} = (q-1)R_{sy_J, stw_J}^{H,x} + qR_{tsy_J, stw_J}^{H,x} = (q-1)R_{sy_J, stw_J}^{H,x} = (q-1)R_{y_J, tw_J}^{H,x}$$

(where the third equality holds since $s \not\leq w_J$ implies $tsy_J \not\leq stw_J$) and

$$R_{M(u),M(w)}^{H,x} = R_{y_J, stw_J}^{H,x} = (q-1)R_{y_J, tw_J}^{H,x} + qR_{sy_J, tw_J}^{H,x} = (q-1)R_{y_J, tw_J}^{H,x}$$

(where the third equality holds since $s \not\leq w_J$ implies $sy_J \not\leq tw_J$).

(c) $u = ty_J$. Then

$$
\begin{aligned}
R^{H,x}_{u,w} &= R^{H,x}_{ty_J,tstw_J} = R^{H,x}_{y_J,stw_J} = (q-1)R^{H,x}_{y_J,tw_J} + qR^{H,x}_{sy_J,tw_J} = (q-1)R^{H,x}_{y_J,tw_J} \\
&= (q-1)\left((q-1)R^{H,x}_{y_J,w_J} + qR^{H,x}_{ty_J,w_J}\right) = (q-1)^2 R^{H,x}_{y_J,w_J},
\end{aligned}
$$

where the fourth equality holds since $s \not\leq w_J$ implies $sy_J \not\leq tw_J$, and the last equality follows since $t \not\leq w_J$. On the other hand

$$
\begin{aligned}
(q-1)R^{H,x}_{u,M(w)} + qR^{H,x}_{M(u),M(w)} &= (q-1)R^{H,x}_{ty_J,stw_J} + qR^{H,x}_{tsy_J,stw_J} \\
&= (q-1)R^{H,x}_{ty_J,stw_J} \\
&= (q-1)\left((q-1)R^{H,x}_{ty_J,tw_J} + qR^{H,x}_{sty_J,tw_J}\right) \\
&= (q-1)^2 R^{H,x}_{ty_J,tw_J} \\
&= (q-1)^2 R^{H,x}_{y_J,w_J},
\end{aligned}
$$

where the second and the fourth equalities hold since $s \not\leq w_J$ implies both $tsy_J \not\leq stw_J$ and $sty_J \not\leq tw_J$.

(d) $u = y_J$. Then

$$
\begin{aligned}
R^{H,x}_{u,w} &= R^{H,x}_{y_J,tstw_J} = (q-1)R^{H,x}_{y_J,stw_J} + qR^{H,x}_{ty_J,stw_J} = (q-1)R^{H,x}_{y_J,stw_J} + qR^{H,x}_{sy_J,stw_J} \\
&= (q-1)R^{H,x}_{u,M(w)} + qR^{H,x}_{M(u),M(w)},
\end{aligned}
$$

where the third equality follows from Lemma 4.4.

Assume now that $\langle M, \lambda_t \rangle(u) = \{y_J, sy_J, ty_J, tsy_J\}$ is not contained in W^H. The case $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J, sy_J, ty_J\}$ is impossible by Lemma 4.1.

Also the case $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J, sy_J\}$ is impossible. Indeed, in this case sy_J would be the unique coatom of tsy_J in W^H, i.e. $tsy_J = sy_J h$ for a certain $h \in H$. Since $t \not\leq sy_J$, $h = t$: being $m(s,t) > 2$, this is in contradiction with the Word Property (Theorem 2.1).

If $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J\}$, then clearly $u = y_J$ and

$$
R^{H,x}_{u,w} = R^{H,x}_{y_J,tstw_J} = (q-1-x)R^{H,x}_{y_J,stw_J} = (q-1-x)R^{H,x}_{u,M(w)},
$$

and the assertion follows.

It remains to treat the case $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J, ty_J\}$, in which we have two possibilities for u.

(1) $u = ty_J$. Then

$$
\begin{aligned}
R^{H,x}_{u,w} &= R^{H,x}_{ty_J,tstw_J} = R^{H,x}_{y_J,stw_J} = (q-1-x)R^{H,x}_{y_J,tw_J} \\
&= (q-1-x)\left((q-1)R^{H,x}_{y_J,w_J} + qR^{H,x}_{ty_J,w_J}\right) = (q-1-x)(q-1)R^{H,x}_{y_J,w_J},
\end{aligned}
$$

where the last equality follows since $t \not\leq w_J$. On the other hand

$$
\begin{aligned}
(q-1-x)R^{H,x}_{u,M(w)} &= (q-1-x)R^{H,x}_{ty_J,stw_J} = (q-1-x)\left((q-1)R^{H,x}_{ty_J,tw_J} + qR^{H,x}_{sty_J,tw_J}\right) \\
&= (q-1-x)(q-1)R^{H,x}_{ty_J,tw_J} = (q-1-x)(q-1)R^{H,x}_{y_J,w_J},
\end{aligned}
$$

where

- the second equality follows by the fact that $ty_J \in W^H$ implies $sty_J \in W^H$ since all reduced expressions of sty_J start with s by the Word Property (Theorem 2.1),
- the third equality holds since $s \not\leq w_J$.

(2) $u = y_J$. Then

$$
\begin{aligned}
R_{u,w}^{H,x} &= R_{y_J,tstw_J}^{H,x} = (q-1)R_{y_J,stw_J}^{H,x} + qR_{ty_J,stw_J}^{H,x} \\
&= (q-1)(q-1-x)R_{y_J,tw_J}^{H,x} + q\Big((q-1)R_{ty_J,tw_J}^{H,x} + qR_{sty_J,tw_J}^{H,x}\Big) \\
&= (q-1)(q-1-x)\Big((q-1)R_{y_J,w_J}^{H,x} + qR_{ty_J,w_J}^{H,x}\Big) + q(q-1)R_{y_J,w_J}^{H,x} \\
&= (q-1)[(q-1-x)(q-1)+q]R_{y_J,w_J}^{H,x},
\end{aligned}
$$

where the third equality follows (as before) by the fact that $ty_J \in W^H$ implies $sty_J \in W^H$, the fourth and the fifth equalities follow since $s, t \not\leq w_J$. On the other hand, we have

$$
\begin{aligned}
(q-1-x)R_{u,M(w)}^{H,x} &= (q-1-x)R_{y_J,stw_J}^{H,x} = (q-1-x)^2 R_{y_J,tw_J}^{H,x} \\
&= (q-1-x)^2\Big((q-1)R_{y_J,w_J}^{H,x} + qR_{ty_J,w_J}^{H,x}\Big) \\
&= (q-1-x)^2(q-1)R_{y_J,w_J}^{H,x},
\end{aligned}
$$

where the last equality follows since $t \not\leq w_J$. The two expressions coincide since $(q-1-x)(q-1)+q = (q-1-x)^2$ is the equation defining x (changing x with $q-1-x$).

Case 2. Notice that, as in the previous case, also now we have $w = tst \cdot w_J$ and $M(w) = st \cdot w_J$. Then the assertion is proved by an argument which is very similar to the argument for the proof in Case 1. However there are some differences and we prefer to write explicitly all subcases where the differences occur.

Since $t \in D_L(w)$, λ_t is a special matching of w. M and λ_t commute but they agree on w (i.e., $|\langle M, \lambda_t\rangle(w)| = 2$). The lower dihedral interval $[e, w_0(s,t)]$ contains 3 orbits of $\langle M, \lambda_t\rangle$: 2 orbits with 2 elements and the other with 4 elements; hence every orbit of $\langle M, \lambda_t\rangle$ must have either 2 or 4 elements, by Proposition 2.6.

Let $u \leq w$, $u \in W^H$. We need to show that

(4.3)
$$
R_{u,w}^{H,x}(q) = \begin{cases}
R_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \lhd u, \\
(q-1)R_{u,M(w)}^{H,x}(q) + qR_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \in W^H, \\
(q-1-x)R_{u,M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \notin W^H.
\end{cases}
$$

This is trivial if also $|\langle M, \lambda_t\rangle(u)| = 2$ so assume $|\langle M, \lambda_t\rangle(u)| = 4$. Since M agrees with λ_t on st, tst, sts, and $tsts$, necessarily $\langle M, \lambda_t\rangle(u) = \{y_J, sy_J, ty_J, tsy_J\}$, for a certain $y_J \in W_J$ with $s \notin D_L(y_J)$. Note that $M(y_J) = sy_J$, $M(ty_J) = tsy_J$, and $t \not\leq y_J$ since $t \notin J$.

Assume that $\langle M, \lambda_t\rangle(u) = \{y_J, sy_J, ty_J, tsy_J\}$ is contained in W^H and note that, by the Word Property (Theorem 2.1), also sty_J and $stsy_J$ are in W^H since $t \not\leq y_J$ and no braid move could involve the leftmost letters s, t (so we cannot obtain a reduced

expression ending with a letter in H). We distinguish the following 4 cases according to which element in the orbit is u.

(a) $u = tsy_J$. Then
$$R_{u,w}^{H,x} = R_{tsy_J,tstw_J}^{H,x} = R_{sy_J,stw_J}^{H,x} = R_{ty_J,stw_J}^{H,x} = R_{M(u),M(w)}^{H,x},$$
where the third equality follows from Lemma 4.4.

(b) $u = sy_J$. Then
$$\begin{aligned}
R_{u,w}^{H,x} &= R_{sy_J,tstw_J}^{H,x} = (q-1)R_{sy_J,stw_J}^{H,x} + qR_{tsy_J,stw_J}^{H,x} \\
&= (q-1)R_{y_J,tw_J}^{H,x} + q\Big((q-1)R_{tsy_J,tw_J}^{H,x} + qR_{stsy_J,tw_J}^{H,x}\Big) \\
&= (q-1)R_{y_J,tw_J}^{H,x} + q(q-1)R_{sy_J,w_J}^{H,x} = (q-1)R_{y_J,tw_J}^{H,x} + qR_{sy_J,tw_J}^{H,x} \\
&= R_{y_J,stw_J}^{H,x} = R_{M(u),M(w)}^{H,x},
\end{aligned}$$
where the fourth equality holds since $t \not\leq w_J$ and the fifth holds since
$$R_{sy_J,tw_J}^{H,x} = (q-1)R_{sy_J,w_J}^{H,x} + qR_{tsy_J,w_J}^{H,x} = (q-1)R_{sy_J,w_J}^{H,x}.$$

(c) $u = ty_J$. Then
$$\begin{aligned}
R_{u,w}^{H,x} &= R_{ty_J,tstw_J}^{H,x} = R_{y_J,stw_J}^{H,x} = (q-1)R_{y_J,tw_J}^{H,x} + qR_{sy_J,tw_J}^{H,x} \\
&= (q-1)\Big((q-1)R_{y_J,w_J}^{H,x} + qR_{ty_J,w_J}^{H,x}\Big) + q\Big((q-1)R_{sy_J,w_J}^{H,x} + qR_{tsy_J,w_J}^{H,x}\Big) \\
&= (q-1)^2 R_{y_J,w_J}^{H,x} + q(q-1)R_{sy_J,w_J}^{H,x},
\end{aligned}$$
where the fifth equality follows since $t \not\leq w_J$. On the other hand
$$\begin{aligned}
(q-1)R_{u,M(w)}^{H,x} + qR_{M(u),M(w)}^{H,x} &= (q-1)R_{ty_J,stw_J}^{H,x} + qR_{tsy_J,stw_J}^{H,x} \\
&= (q-1)\Big((q-1)R_{ty_J,tw_J}^{H,x} + qR_{sty_J,tw_J}^{H,x}\Big) + \\
&\quad q\Big((q-1)R_{tsy_J,tw_J}^{H,x} + qR_{stsy_J,tw_J}^{H,x}\Big) \\
&= (q-1)^2 R_{ty_J,tw_J}^{H,x} + q(q-1)R_{tsy_J,tw_J}^{H,x} \\
&= (q-1)^2 R_{y_J,w_J}^{H,x} + q(q-1)R_{sy_J,w_J}^{H,x},
\end{aligned}$$
where the third equality follows since $t \not\leq w_J$.

(d) $u = y_J$. Then
$$\begin{aligned}
R_{u,w}^{H,x} &= R_{y_J,tstw_J}^{H,x} = (q-1)R_{y_J,stw_J}^{H,x} + qR_{ty_J,stw_J}^{H,x} \\
&= (q-1)R_{y_J,stw_J}^{H,x} + qR_{sy_J,stw_J}^{H,x} = (q-1)R_{u,M(w)}^{H,x} + qR_{M(u),M(w)}^{H,x},
\end{aligned}$$
where the third equality follows from Lemma 4.4.

Assume now that $\langle M, \lambda_t \rangle(u) = \{y_J, sy_J, ty_J, tsy_J\}$ is not contained in W^H. The case $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J, sy_J, ty_J\}$ is impossible by Lemma 4.1.

Also the case $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J, sy_J\}$ is impossible. Indeed, in this case sy_J would be the unique coatom of tsy_J in W^H, i.e. $tsy_J = sy_J h$ for a certain $h \in H$. Since $t \not\leq sy_J$, $h = t$: being $m(s,t) > 2$, this is in contradiction with the Word Property (Theorem 2.1).

If $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J\}$, then clearly $u = y_J$ and

$$R_{u,w}^{H,x} = R_{y_J,tstw_J}^{H,x} = (q - 1 - x)R_{y_J,stw_J}^{H,x} = (q - 1 - x)R_{u,M(w)}^{H,x},$$

and the assertion follows.

The subcase $\{y_J, sy_J, ty_J, tsy_J\} \cap W^H = \{y_J, ty_J\}$ can be treated as the corresponding subcase in Case 1 and therefore we omit the proof.

Case 3. Notice that, in this case, $s \not\leq {}^{\{s,t\}}({}^J w)$ (as otherwise $tsts$ would be $\leq w$ by the Subword Property) and $M(w) = st \cdot {}^{\{s,t\}}({}^J w)$. Then this last case is completely analogous to Case 1 and we omit the proof. $\square$

Remark 4.6. *Note that, in the last subcase of Case 1 and then also in the last subcase of Case 2 and Case 3 (whose proofs are omitted) of the proof of Theorem 4.5, we use the fact that x satisfies $x^2 = q + (q - 1)x$.*

4.4. **Dihedral Coxeter groups.** We now prove that the H-special matchings of w are calculating also in the case $[e, w]$ is a dihedral interval (so, in particular, in the case W is a dihedral group, i.e. a Coxeter group of rank 2).

Proposition 4.7. *Given a Coxeter system (W, S) and a subset $H \subset S$, let $w \in W^H$ be such that $[e, w]$ is a dihedral interval and $[e, w] \cap W^H$ is a chain (i.e. a totally ordered set). Then*

$$R_{u,w}^{H,x}(q) = (q - 1)(q - 1 - x)^{\ell(w)-\ell(u)-1}$$

for all $u < w$, $u \in W^H$.

Proof. We proceed by induction on $\ell(w)$, the case $\ell(w) = 1$ being clear.

Fix a left multiplication matching λ of w. Then

$$R_{u,w}^{H,x} = \begin{cases} R_{\lambda(u),\lambda(w)}^{H,x}, & \text{if } u \rhd \lambda(u), \\ (q - 1)R_{u,\lambda(w)}^{H,x} + qR_{\lambda(u),\lambda(w)}^{H,x}, & \text{if } u \lhd \lambda(u) \in W^H, \\ (q - 1 - x)R_{u,\lambda(w)}^{H,x}, & \text{if } u \lhd \lambda(u) \notin W^H, \end{cases}$$

In all three cases, the assertion follows easily by induction. $\square$

Theorem 4.8. *Given a Coxeter system (W, S) and a subset $H \subset S$, let $w \in W^H$ be such that $[e, w]$ is a dihedral interval. Then all H-special matchings of w calculate the $R^{H,x}$-polynomials.*

Proof. The intersection $[e, w] \cap W^H$ is either trivially equal to $[e, w]$ or is a chain. In the first case, the parabolic R-polynomials coincide with the ordinary R-polynomials and the result is known (and easy to proof).

Assume we are in the second case and proceed by induction on $\ell(w)$. Let M be an H-special matching of w and $u \in W^H$, $u \leq w$. We need to show that

$$R_{u,w}^{H,x}(q) = \begin{cases} R_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \lhd u, \\ (q - 1)R_{u,M(w)}^{H,x}(q) + qR_{M(u),M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \in W^H, \\ (q - 1 - x)R_{u,M(w)}^{H,x}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \notin W^H. \end{cases}$$

Note that

- if $u \rhd M(u)$, then $M(u) \in W^H$ since M is H-special,

- if $M(u) \rhd u$ and $M(u) \in W^H$, then $M(u) \le M(w)$ unless $M(w) = u$.

Then the assertion follows by induction using Proposition 4.7. $\square$

5. Combinatorial Invariance

In this brief final section, we show the consequences of Theorems 4.5 and 4.8 on the problem of the Combinatorial Invariance of the parabolic Kazhdan–Lusztig and R-polynomials.

Corollary 5.1. *Let (W_1, S_1) and (W_2, S_2) be two doubly laced or dihedral Coxeter systems, with identity elements e_1 and e_2, and let $H_1 \subseteq S_1$ and $H_2 \subseteq S_2$. Let $v_1 \in W_1^{H_1}$ and $v_2 \in W_2^{H_2}$ be such that there exists a poset-isomorphism ψ from $[e_1, v_1]$ to $[e_2, v_2]$ which restricts to a poset-isomorphism from $[e_1, v_1]^{H_1}$ to $[e_2, v_2]^{H_2}$. Then, for all $u, w \in [e_1, v_1]^{H_1}$, we have*

$$P_{u,w}^{H_1,x}(q) = P_{\psi(u),\psi(w)}^{H_2,x}(q)$$

(equivalently, $R_{u,w}^{H_1,x}(q) = R_{\psi(u),\psi(w)}^{H_2,x}(q)$).

Proof. By Theorem 2.11, (4), the knowledge of all parabolic Kazhdan–Lusztig polynomials indexed by elements in a fixed parabolic interval is equivalent to the knowledge of all parabolic R-polynomials indexed by elements in that interval. Consequently, the statement for the parabolic Kazhdan–Lusztig polynomials is equivalent to the statement for the parabolic R-polynomials. We prove the latter.

By Theorems 4.5 and 4.8, the H-special matchings compute the R-polynomials of both W_1 and W_2. Since $[e_1, v_1]$ and $[e_2, v_2]$ are isomorphic, they have the same special matchings, and, since such isomorphism ψ restricts to an isomorphism from $[e_1, v_1]^{H_1}$ to $[e_2, v_2]^{H_2}$, a special matching is H_1-special if and only if the corresponding matching is H_2-special. The assertion follows from these considerations. $\square$

As an immediate result of Corollary 5.1, we obtain the following theorem which establishes Conjecture 1.3 for lower intervals within the class of doubly laced and dihedral Coxeter groups.

Theorem 5.2. *If (W_1, S_1) and (W_2, S_2) are either doubly laced or dihedral Coxeter systems, then Conjecture 1.3 holds when u_1 and u_2 are the identity elements.*

Proof. Straightforward by Corollary 5.1. $\square$

References

[1] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics **231**, Springer-Verlag, New York, 2005.

[2] F. Brenti, F. Caselli, M. Marietti, *Special Matchings and Kazhdan–Lusztig polynomials*, Advances in Math. **202** (2006), 555-601.

[3] V. V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math., **39** (1977), 187-198.

[4] V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials*, J. Algebra, **111** (1987), 483-506.

[5] M. J. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph. D. Thesis, University of Sydney, 1987.

[6] M. Dyer, *On the Bruhat graph of a Coxeter system*, Compositio Math., **78** (1991), 185-191.

[7] B. Elias, G. Williamson, *The Hodge theory of Soergel bimodules*, Ann. Math, **180** (2014), 1089-1136.

[8] A. van den Hombergh, *About the automorphisms of the Bruhat-ordering in a Coxeter group*, Indag. Math., **36** (1974), 125-131.

[9] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, no.29, Cambridge Univ. Press, Cambridge, 1990.

[10] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.

[11] D. Kazhdan, G. Lusztig, *Schubert varieties and Poincaré duality*, Proc. Symp. Pure Math., **36**, A. M. S., (1980), 185-203.

[12] M. Marietti, *Special matchings in Coxeter groups*, preprint.

[13] P. Mongelli, *Coxeter groups: statistics and Kazhdan–Lusztig polynomials*, Ph. D. Thesis, Sapienza - University of Rome, 2012.

[14] R. P. Stanley, *Enumerative Combinatorics* , vol.1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[15] J. Tits, *Le problème des mots dans les groupes de Coxeter*, in Symposia Mathematica, Vol 1, Academic Press, London, 1969, 175-185.

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