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Ordering properties of radial ground states and singular ground states of quasilinear elliptic equations

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Abstract. In this paper we discuss the ordering properties of positive radial solutions of the equation

$$\Delta_p u(x) + k|x|^\delta u^{q-1}(x) = 0$$

where $x \in \mathbb{R}^n, n > p > 1, k > 0, \delta > -p, q > p$. We are interested both in regular ground states u (GS), defined and positive in the whole of $\mathbb{R}^n$, and in singular ground states v (SGS), defined and positive in $\mathbb{R}^n \setminus \{0\}$ and such that $\lim_{|x| \to 0} v(x) = +\infty$. A key role in this analysis is played by two bifurcation parameters $p^{JL}(\delta)$ and $p_{jl}(\delta)$, such that $p^{JL}(\delta) > p^*(\delta) > p_{jl}(\delta) > p$: $p^{JL}(\delta)$ generalizes the classical Joseph–Lundgren exponent, and $p_{jl}(\delta)$ its dual. We show that GS are well ordered, i.e. they cannot cross each other if and only if $q \geq p^{JL}(\delta)$; this way we extend to the $p > 1$ case the result proved in Miyamoto (Nonlinear Differ Equ Appl 23(2):24, 2016), Miyamoto and Takahashi (Arch Math Basel 108(1):71–83, 2017) for the $p \geq 2$ case. Analogously we show that SGS are well ordered, if and only if $q \leq p_{jl}(\delta)$; this latter result seems to be known just in the classical $p = 2$ and $\delta = 0$ case, and also the expression of $p_{jl}(\delta)$ has not appeared in literature previously.

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1. Introduction

In this paper we continue the discussion started by Miyamoto [25] and Miyamoto et al [26] concerning the ordering properties of radial solutions for a class of quasilinear elliptic equations, including p-Laplacian and k-Hessian.

Let us start our discussion from the p-Laplace setting, i.e. we consider radial solutions $u(|x|) = u(r)$ for the following equation

$$\Delta_p u(x) + k|x|^\delta u(x)^{q-1} = 0, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\Delta_p(u) = \text{div}(\nabla u |\nabla u|^{p-2})$. In the whole paper we always assume the following relations for the parameters

H $k > 0$, $n > p$, $\delta > -p$ and $q > p$.

Since we are just interested in radial solutions we restrict to consider the following singular ODE:

$$(u'|u'|^{p-2} r^{n-1})' + r^{\delta+n-1} k u^{q-1} = 0, \tag{1.2}$$

where $' = \frac{\partial}{\partial r}$.

Let us introduce some terminology. We say that a definitively positive solution $u(r)$ is *regular* if $u(0) = d > 0$ and that it is singular if $\lim_{r \to 0} u(r) = +\infty$; analogously we say that $u(r)$ has *fast decay* if $\lim_{r \to \infty} u(r) r^{(n-p)/(p-1)} = L > 0$ and that it has slow decay if $\lim_{r \to +\infty} u(r) r^{(n-p)/(p-1)} = +\infty$. We denote by $u(r; d)$ a regular solution to (1.2) such that $u(0; d) = d > 0$ and by $v(r; L)$ a fast decay solution to (1.2) such that $\lim_{r \to +\infty} v(r) r^{(n-p)/(p-1)} = L > 0$.

In fact for any $d > 0$ and any $L > 0$ there is a unique regular solution $u(r; d)$ and a unique fast decay solution $v(r; L)$ to (1.2), see Proposition 1.5 and Theorem 1.6.

A Ground State (GS) and a Singular Ground State (SGS) are respectively a regular and a singular solution of (1.2) which are positive for any $r > 0$; it is easy to check that in both cases $\lim_{r \to +\infty} u(r) = 0$.

It is well known that the structure of positive solutions to (1.1) undergoes several bifurcations as q passes through some critical exponents, in particular

$$p_*(\delta) := \frac{p(n-1) + \delta(p-1)}{n-p} < p^*(\delta) := \frac{p(n+\delta)}{n-p}. \tag{1.3}$$

If $q > p_*(\delta)$ there exists an explicitly known SGS with slow decay $u(r; \infty) := P_x r^{-\alpha}$ where P_x is defined in (2.3). Further, if $q \geq p^*(\delta)$ all the regular solutions are GS. In particular we have the following classical results see e.g. [12, §2].

Theorem A. *Assume **H**. If $q > p^*(\delta)$, $p > 1$, all the regular solutions to (1.2) are GS with slow decay, and there is a unique SGS with slow decay $u(r; \infty) := P_x r^{-\alpha}$, where $\alpha := \frac{p+\delta}{q-p}$ and $P_x > 0$ is an explicitly known constant, see (2.3).*

There are no other positive solutions for any $r > 0$: fast decay solutions exist but they are sign changing.

Theorem B. *Assume* **H**. *If* $p_*(\delta) < q < p^*(\delta)$ *all the fast decay solutions are SGS with fast decay, and there is a unique SGS with slow decay* $u(r; \infty) := P_x r^{-\alpha}$. *There are no other solutions positive for any* $r > 0$: *regular solutions exist but they are sign changing.*

In order to complete the picture we recall that if $q = p^*(\delta)$ then regular solutions are GS with fast decay and in fact they are explicitly known, see e.g. [25, Eq. (1.17)] and there is a two parameter family of SGS with slow decay, see e.g. [12].

We think it is worthwhile to point out that $p_* := p_*(0)$ is related to the continuity of the trace operator in L^q, while $p^* := p^*(0)$ is the Sobolev critical exponent so it is the upper bound for the compactness of the embedding of L^q in $W^{1,p}$.

In the classical $p = 2$ and $\delta = 0$ case the intersection properties of positive solutions depend on two further critical exponents, i.e.

$$
\begin{aligned}
2_{jl} &:= \begin{cases} 2 + \frac{4}{n-4+2\sqrt{n-1}}, & \text{if } n \geq 3 \\ \infty, & \text{if } n = \{1,2\} \end{cases} \\
2^{JL} &:= \begin{cases} 2 + \frac{4}{n-4-2\sqrt{n-1}}, & \text{if } n \geq 11 \\ \infty, & \text{if } n \leq 10. \end{cases}
\end{aligned}
\tag{1.4}
$$

Notice that $2 < 2_*(0) < 2_{jl} < 2^*(0) < 2^{JL}$. The latter, 2^{JL}, is the so called Joseph- Lundgren exponent and it was introduced in [23], while 2_{jl} was introduced in [4]: 2^{JL} and 2_{jl} are used to determine the intersection properties respectively of regular and singular solutions of (1.2).
Namely if $2^* < q < 2^{JL}$ all the GS of Theorem A intersect each other indefinitely, while if $q \geq 2^{JL}$ they are well ordered, i.e. if $d_2 > d_1$ then $u(r; d_2) > u(r; d_1)$, for any $r \geq 0$.

These results are crucial for determining the long time behaviour of positive solutions of the following parabolic problem

$$
\begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + |x|^\delta u^{q-1}(t, x) \\ u(0, x) = U_0(x), \end{cases}
\tag{1.5}
$$

where $U_0(x)$ need not be radial. We emphasise that (1.5) is a simple model for an explosion: u describes the temperature and the non linearity $|x|^\delta u^{q-1}$ represents a spatial-dependent esothermic reaction. The two main expected behaviors are the blowing up in finite time of the solution (the explosion takes place) or the convergence to zero of the solution (the temperature is too low to initiate the explosion).

If $2^* < q < 2^{JL}$, using the intersection property in Theorem A, it is possible to construct sub and super solutions and to show that the radial GS are on the threshold between blowing up and fading solutions, see [2,19,31] for details. Further, if $q \geq 2^{JL}$, the ordering property described in Theorem B is essential to prove that GS enjoy some stability in appropriated L^∞ weighted space, see again [2,19,31]. In fact Theorem B was a key stone for a flourishing of interesting papers, concerning the possible rate of convergence either to the null solution or to the GS, or to determine the speed of the blow up of the

solutions of (1.5), see e.g. [9,21,27] and references therein, see also Corollary 3.6 below.

The generalisation of 2^{JL} to a p-Laplace context firstly appeared in [3] (see also [15] and [25]).

Its complicated expression can be found explicitly in [25, (1.13)] and it is obtained as the largest solution of a quadratic equation, see Sect. 4.1. The exponent 2_{jl}, is the bifurcation parameter for the ordering properties of SGS. This characteristic is again crucial for (1.5): in fact Sato and Yanagida in the interesting paper [28] managed to prove local uniqueness of singular solutions of (1.5) (assuming a priori that the type of singularity is preserved for any $t \geq 0$) and some stability properties of SGS with slow and fast decay when $p = 2$, $\delta = 0$ and $2_*(0) < q < 2_{jl}$, see also [22] and Corollary 3.7 below.

One achievement of this article is to obtain the expression of 2_{jl} in a p-Laplace context: it is denoted by $p_{jl}(\delta)$ and it is the smallest solution of the quadratic equation mentioned above, see Sect. 4.1 and in particular equation (4.4). For any value of the parameters we have:

$$p < p_*(\delta) < p_{jl}(\delta) < p^*(\delta) < p^{JL}(\delta) \leq \infty.$$

One of our main results is the following

Theorem 1.1. *Assume* **H**, *$q \geq \max\{2, p^{JL}(\delta)\}$ and consider* (1.2). *If $0 < d_1 < d_2$, then $0 < u(r; d_1) < u(r; d_2)$ for any $r \geq 0$.*

We emphasise that this result has already been proved in [31] in the case $p = 2$, and in [25,26] when $p > 2$. Here we extend it to the $1 < p < 2$ case too. After the paper was completed we were informed that Theorem 1.1 was recently proved by Guo and Zhou in the $1 < p < 2$ case too, see [20], which, however, is in Chinese.

Theorem 1.1 is in some sense optimal. In fact we have the following known result, see e.g. [15,25].

Theorem C. *Assume* **H**, *$p^*(\delta) < q < p^{JL}(\delta)$ and consider* (1.2). *Then for any $R > 0$ and $0 < d_1 < d_2$ the function $u(r; d_1) - u(r; d_2)$ changes sign infinitely many times when $r \geq R$.*

In the classical $p = 2$ case Theorem C has been proved in [19,31] and it was an important ingredient to prove the weak asymptotic stability of the GS, $u(r; d)$, for (1.5), and several nice results concerning the rate of convergence of the solutions of (1.5), see e.g. [27] and references therein.

We give a second main contribution concerning the dual situation, i.e. the ordering properties of SGS in the subcritical case.

Theorem 1.2. *Assume* **H**, *$p_*(\delta) < q \leq p_{jl}(\delta)$, $q \geq 2$ and consider the SGS with fast decay $v(r; L)$. If $0 < L_1 < L_2$, then $0 < v(r; L_1) < v(r; L_2)$ for any $r > 0$.*

We emphasize that the rather weak condition $q \geq 2$ in Theorems 1.1 and 1.2 seems to be technical. However it is always satisfied if $p \geq 2$.

As far as we are aware the ordering properties of SGS are known just when $p = 2$ and $\delta = 0$ (see [4, Proposition 2.5]).

TABLE 1. Summary of the results for the p-Laplace setting

Range of q	$(p_*, p_{jl}]$	(p_{jl}, p^*)	(p^*, p^{JL})	$[p^{JL}, \infty)$
Solution type	SGS	SGS	GS	GS
Structure	Ordered ($q \geq 2$)	Indefinite	Indefinite	Ordered ($q \geq 2$)
Theorem	Th. 1.2	Th. 1.3	Th. C	Th. 1.1

One of the main purposes of this paper is to generalise this result to the $p > 1$ and $\delta > -p$ case. In fact in the $p = 2$ case Theorem 1.2 can be obtained from Theorem 1.1 by combining Kelvin inversion and Fowler transformation, however as far as we are aware the result has not yet appeared in literature (we provide a short proof in this easier case at the end of Sect. 3). Anyway in the general $p > 1$ case the Kelvin inversion is not available, so we have to argue differently, in fact even the exponent $p_{jl}(\delta)$ was known just in the case $p = 2$ and $\delta = 0$.

Also in this context the result is optimal, i.e. we have the following result which seems to be new, as far as we are aware.

Theorem 1.3. *Assume **H**, $p_{jl}(\delta) < q < p^*(\delta)$; then the SGS $v(r; L)$ of (1.2) intersect each other indefinitely. More precisely for any $R > 0$ and $0 < L_1 < L_2 \leq \infty$ the function $v(r; L_1) - v(r; L_2)$ changes sign infinitely many times when $r \leq R$. Here $v(r; \infty) = P_x r^{-\alpha}$.*

We wish to spend few lines to point out that, if $\delta \leq 0$ and $p = 2$, positive solutions to (1.1) have to be radial, see e.g. [17], while if $\delta > 0$ even in the $p = 2$ case we may have non-radial positive solutions, see e.g. [7] where the authors constructed a positive non-radial singular solution of the Hénon equation in $\mathbb{R}^n \setminus \{0\}$; see also [30] for an example concerning a non-radial regular solution in a ball in the critical case.

In fact there are some symmetry results also in the $p > 1$ case, when $\delta = 0$: e.g. radial symmetry is ensured in the critical case for any $p > 1$, see [29], and by requiring $1 < p \leq 2$ and some a priori estimates on the asymptotic behavior of positive solutions, see [6].

The results concerning existence and ordering properties of GS and SGS are summed up in Table 1.

Let us denote by B the unit ball in $\mathbb{R}^n$. Following [25], as a Corollary of Theorems 1.1 and C we also find the bifurcation diagram for the positive radial solutions of the following problem

$$\Delta_p U(x) + \lambda |x|^\delta (U+1)^{q-1}(x) = 0, \qquad \text{in } B, \qquad (1.6)$$

with Dirichlet boundary conditions. Abusing the notation we denote once again by $U(x) = U(r)$ when $r = |x|$ since U is radially symmetric; in fact (1.6) is reduced to a singular O.D.E. analogous to (1.2). So let $U(r; D)$ denote the unique solution of (1.6) such that $U(0; D) = D > 0$, see Proposition 1.5 just below, and denote by $\rho(D; \lambda)$ the first zero of $U(r; D)$. It is not difficult to

show that $U'(r; D) < 0$ for any $0 < r < \rho(D; \lambda)$; then, following [25], we see that for any D we have a value $\lambda = \lambda(D)$ for which (1.6) admits a solution.

Then we have this result concerning the shape of $\lambda(D)$, which is a generalisation of [25, Theorem B], [26, Theorem B].

Theorem 1.4. *Assume $\boldsymbol{H}$ and $q > p^*(\delta)$; the function $\lambda = \lambda(D)$ satisfies $\lim_{D \searrow 0} \lambda(D) = 0$ and $\lim_{D \to +\infty} \lambda(D) \to \lambda^* := (P_x)^{q-p}$, where $P_x > 0$ is given in (2.3). Further*

(i) *If $q \geq \max\{p^{JL}(\delta), 2\}$ then $\lambda(D)$ is strictly increasing.*

(ii) *If $p^*(\delta) < q < p^{JL}(\delta)$ then $\lambda(D) > 0$ for any $D > 0$ and $\lambda(D)$ oscillates indefinitely around λ^*. Moreover*

$$\lambda(D_2) < \cdots < \lambda(D_{2j}) < \cdots < \lambda^* < \cdots < \lambda(D_{2j+1}) < \cdots < \lambda(D_1)$$

where D_k is the set of the critical points of $\lambda(D)$. In particular D_k is a local minimum point for k even and a local maximum point for k odd.

Proof. The proof is a consequence of Theorem 1.1 and can be obtained by a straightforward repetition of the argument developed in [25, Theorem B] for the case $p \geq 2$. □

To complete the picture we recall that if $p < q \leq p^*(\delta)$ then there is a Λ such that the Dirichlet problem associated to (1.6) admits two solutions if $0 < \lambda < \Lambda$, one solution for $\lambda = \Lambda$ and no solution if $\lambda > \Lambda$, see [25, Theorem B], [8, Theorem 4].

Now we briefly observe how our discussion can be used to obtain information on the analogous problems where the p-Laplace operator is replaced by the K-Hessian. Following [25] we denote by $\{\lambda_j\}_{j=1}^N$ the set of the eigenvalues of the Hessian matrix D^2u in $\mathbb{R}^N$. Let $K \in \{1, 2, \ldots, N\}$; the K-Hessian operator is defined by

$$S_K(D^2u) := \sum_{1 \leq j_1 < j_2 < \cdots < j_K \leq N} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_K},$$

so that the classical Laplace operator is $S_1(D^2u) = \Delta u$, while the Monge-Ampere operator is $S_N(D^2u) = \det(D^2(u))$. Let us set

$$C_{N,K} := \frac{(-1)^{K-1}K(N-K)!(K-1)!}{(N-1)!}.$$

Then, following the introduction of [25] (see page 16) the radial solutions of the K- Hessian equation

$$c_{N,K}S_K(D^2u) + u^{q-1} = 0, \qquad \text{in } \mathbb{R}^N,$$

solve an equation of the form (1.2), where $n = N - K + 1$, $p = K + 1$ $\delta = K - 1$, $k = 1$. Hence all the results of this paper concerning the p-Laplace operator are immediately translated for the K-Hessian operator.

Let us conclude the introduction by stating the basic facts concerning regular and fast decay solutions.

Theorem 1.5. [16,25] *Assume $\boldsymbol{H}$; then for any $d > 0$ there exists a unique solution $u(r; d)$ of (1.2) such that $u(0; d) = d$.*

A proof of this result can be found in [16, Appendix] and in [25]. Analogously we have the following.

Theorem 1.6. *Assume **H**, $q > p_*(\delta)$. Then, for any $L > 0$ there exists a fast decay solution $v(r; L)$ of (1.2) and it is unique and C^2 for r large enough.*

The existence part is in fact already known, see, e.g. [12], but the uniqueness is new as far as we are aware, and it will be proved in Sect. 4.2.1

Let us recall some well known facts.

Lemma 1.7. *If $\delta > -1$ then $u'(0; d) = 0$ for any $d > 0$ and regular solutions are in fact smooth also regarded as solutions of (1.1), i.e. $U(x) := u(|x|; d)$ is at least C^1 in the whole of $\mathbb{R}^n$.*

Proof. From a straightforward application of De l'Hospital rule we find

$$\lim_{r \to 0} \frac{u'(r)|u'(r)|^{p-2}}{r^{1+\delta}} = \lim_{r \to 0} \frac{u'(r)|u'(r)|^{p-2}r^{n-1}}{r^{n+\delta}}$$
$$= \lim_{r \to 0} \frac{[u'(r)|u'(r)|^{p-2}r^{n-1}]'}{(n+\delta)r^{n-1+\delta}} = \lim_{r \to 0} \frac{ku(r)^{q-1}r^{n-1+\delta}}{(n+\delta)r^{n-1+\delta}} = \frac{kd^{q-1}}{(n+\delta)}.$$

Then the statement easily follows. $\qquad\square$

Lemma 1.8. *Assume that a solution $u(r)$ of (1.2) is non-negative and decreasing for r large, then it is positive and there exists $L > 0$, possibly $L = +\infty$, such that*

$$\lim_{r \to +\infty} u(r)r^{\frac{n-p}{p-1}} = L, \qquad \lim_{r \to +\infty} u'(r)r^{\frac{n-1}{p-1}} = -\frac{n-p}{p-1}L$$

Proof. From (1.2) we find that

$$\{[-u'(r)]^{p-1}r^{n-1}\}' = ku(r)^{q-1}r^{\delta+n-1} > 0,$$

hence $u'(r)r^{\frac{n-1}{p-1}}$ is negative and decreasing and admits limit as $r \to +\infty$, say $-\frac{n-p}{p-1}L$, where $L > 0$ may be $L = +\infty$. It is easy to check that $\lim_{r \to +\infty} u(r) = 0$; so from De l'Hospital rule we find

$$\lim_{r \to +\infty} \frac{u(r)}{r^{-\frac{n-p}{p-1}}} = \lim_{r \to +\infty} \frac{u'(r)}{-\frac{n-p}{p-1}r^{-\frac{n-1}{p-1}}} = L, \qquad (1.7)$$

and this concludes the proof. $\qquad\square$

Most of the proofs of this article rely on a change of variable known as Fowler transformation, which enables us to use phase plane analysis, and to profit of invariant manifold theory and dynamical systems techniques. The plan of the paper is as follows. In Sect. 2 we introduce the Fowler transformation for p-Laplace equations and system (2.2). In §2.1 we define the critical exponents $p^{JL}(\delta)$ and $p_{jl}(\delta)$ even if the lengthy computation needed for their evaluation is postponed to §4.1; further we explore the dynamics of (2.2) in a neighborhood of the critical point $\boldsymbol{P}$, corresponding to singular and slow decay solutions of (1.2). In §2.2 we introduce the unstable manifold W^u_+ and the stable manifold W^s_+ of the origin of (2.2) corresponding to regular and fast decay solutions of

(1.2). In §2.3 we see how the Pohozaev identity is interpreted in this context and enables us to locate W^u_+ and W^s_+. In §3 we prove Theorems 1.1 and 1.2. In §4 we have some technical lemmas, concerning explicit evaluation of $p^{JL}(\delta)$ and $p_{jl}(\delta)$, in §4.1, and the construction of W^u_+ and W^s_+ in a non-smooth context in Sect. 4.2.

2. Fowler transformation

In this section we introduce a change of variables known as Fowler transformation which allows to pass from the non-autonomous singular ODE (1.2) to a two dimensional autonomous dynamical systems. Hence we set

$$\alpha = \frac{p+\delta}{q-p}, \qquad \beta = (\alpha+1)(p-1), \qquad \gamma = \beta - (n-1),$$
$$x = u(r)r^\alpha, \qquad y = u'(r)|u'(r)|^{p-2}r^\beta, \qquad r = \mathrm{e}^t, \tag{2.1}$$

and we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{f}\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y|y|^{\frac{2-p}{p-1}} \\ -kx^{q-1} \end{pmatrix}. \tag{2.2}$$

Here and later "$\cdot$" stands for $\frac{d}{dt}$, while "$'$" stands for $\frac{d}{dr}$. In the whole paper we denote by $\boldsymbol{\phi}(t;\boldsymbol{Q}) = (x(t;\boldsymbol{Q}), y(t;\boldsymbol{Q}))$ the trajectory of (2.2) passing through $\boldsymbol{Q}$ at $t=0$, omitting the dependence on $\boldsymbol{Q}$ when it is not needed.

This change of variables was developed by Fowler in the 30s and extended to the p-Laplace case by Bidaut-Veron in [1] and independently by Franca in [10].

One of the main advantage in studying (2.2) lies in the fact that the system is autonomous and we can profit of phase plane techniques and of invariant manifold theory.

Remark 2.1. The well known scaling invariance property of (1.2) here is translated in the fact that (2.2) is autonomous. The property that if $u(r)$ is a solution of (1.2) then $u(cr)c^\alpha$ solves (1.2) for any $c > 0$, here becomes the fact that if $\phi(t)$ solves (2.2) then $\phi_\tau(t) = \phi(t+\tau)$ solves (2.2) for any $\tau \in \mathbb{R}$.

However in the $p \neq 2$ case we have the following problem.

Remark 2.2. System (2.2) is C^1 if $1 < p \leq 2 \leq q$ but it is just Holder continuous on the x axis if $p > 2$ and on the y axis if $p < q < 2$.

2.1. Definition of the critical exponents and their dynamical interpretation

As we said in the introduction equation (1.2) and system (2.2) change their characteristics as q crosses some critical values, see (1.3).

In particular $q > p_*(\delta)$ iff $\gamma < 0$; consequently (2.2) admits two further critical points apart from the origin: $\boldsymbol{P} = (P_x, P_y)$, where $P_y < 0 < P_x$ and $-\boldsymbol{P}$, where

$$P_x = [(-\gamma)(\alpha)^{p-1}/k]^{1/(q-p)}, \qquad P_y = -(\alpha P_x)^{p-1}. \tag{2.3}$$

Hence (1.2) admits a SGS with slow decay $u(r,\infty) = P_x r^{-\alpha}$ iff $q > p_*(\delta)$.

Let $\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P})$ be the linearisation of (2.2) on $\boldsymbol{P}$, i.e.

$$\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P}) = \begin{pmatrix} \alpha & \frac{|P_y|^{\frac{2-p}{p-1}}}{p-1} \\ -k(q-1)P_x^{q-2} & \gamma \end{pmatrix}. \tag{2.4}$$

Let T and D be the trace and the determinant of $\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P})$ respectively, we easily see that

$$\begin{aligned} T &= \alpha + \gamma = \alpha p - (n-p), \\ D &= \alpha|\gamma|\frac{q-p}{p-1}, \quad \text{for} \quad q > p \quad \text{and} \quad p > 1, \end{aligned} \tag{2.5}$$

see (4.2) for a detailed computation. Hence, $T < 0$ if and only if $\alpha < \frac{n-p}{p}$ or equivalently $q > p^*(\delta)$.

Thus the critical point $\boldsymbol{P}$ is asymptotically unstable if $p_*(\delta) < q < p^*(\delta)$, a center if $q = p^*(\delta)$, asymptotically stable if $q > p^*(\delta)$: in this case (1.2) is respectively subcritical, critical or supercritical (see Theorems A, B).

Further the eigenvalues $\lambda_1 = \lambda_1(q)$ and $\lambda_2 = \lambda_2(q)$ of $\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P})$ have the form

$$\lambda_1 = \frac{\alpha + \gamma - \sqrt{\Delta}}{2}, \qquad \lambda_2 = \frac{\alpha + \gamma + \sqrt{\Delta}}{2}, \tag{2.6}$$

where $\Delta = \Delta(q)$ will be determined explicitly in Sect. 4.1, see in particular (4.4). From a tedious computation, which is postponed to Sect. 4.1, we see that $\Delta(q)$ is a parabola. Further $\Delta(p^*(\delta)) < 0$ hence the equation $\Delta(q) = 0$ has at most one solution in $(p_*(\delta), p^*(\delta))$, denoted by $p_{jl}(\delta)$, and at most one solution in $(p^*(\delta), +\infty)$, denoted by $p^{JL}(\delta)$. We set $p_{jl}(\delta) = p_*(\delta)$ if there are no solution of $\Delta(q) = 0$ in $(p_*(\delta), p^*(\delta))$ and $p^{JL}(\delta) = +\infty$ if there are no solution of $\Delta(q) = 0$ in $(p^*(\delta), +\infty)$.

Hence the Joseph-Lundgren exponent $p^{JL}(\delta)$ and its dual $p_{jl}(\delta)$ are the values such that $\lambda_i(q)$ has non-zero imaginary part if $p_{jl}(\delta) < q < p^{JL}(\delta)$ and it has zero imaginary part (i.e. it is real) if either $p_*(\delta) < q \le p_{jl}(\delta)$ or $q \ge p^{JL}(\delta)$, for $i = 1, 2$. Summing up we have the following

Lemma 2.3. $\boldsymbol{P}$ and $-\boldsymbol{P}$ *are unstable nodes if* $p_*(\delta) < q \le p_{jl}(\delta)$, *unstable foci if* $p_{jl}(\delta) < q < p^*(\delta)$, *centers if* $q = p^*(\delta)$, *stable foci if* $p^*(\delta) < q < p^{JL}(\delta)$, *stable nodes if* $q \ge p^{JL}(\delta)$.

We emphasise that in the $p = 2$ and $\delta = 0$ case $p^{JL}(\delta)$ reduces to the classical Joseph-Lundgren exponent $2^{JL}(0)$.

From a straightforward computation we see that when λ_1 and λ_2 are real and distinct their eigenvectors are

$$\begin{aligned} \boldsymbol{v_1} &= \left(-1, \frac{(\alpha - \lambda_1)(p-1)}{(\alpha P_x)^{2-p}}\right) := (-1, m_1), \\ \boldsymbol{v_2} &= \left(-1, \frac{(\alpha - \lambda_2)(p-1)}{(\alpha P_x)^{2-p}}\right) := (-1, m_2). \end{aligned} \tag{2.7}$$

Notice that $m_1 \ge m_2 > 0$, and we have the same expression if either $p_*(\delta) < q \le p_{jl}(\delta)$, or $q \ge p^{JL}(\delta)$. If $\lambda_1 = \lambda_2$, i.e. when $q \in \{p_{jl}(\delta); p^{JL}(\delta)\}$, their geometric multiplicity is 1, so the unique eigenvector of $\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P})$ is $\boldsymbol{v_1} = \boldsymbol{v_2}$

and it is again given by (2.7), and $\frac{df}{d\phi}(\boldsymbol{P})$ has a nilpotent part (hence the corresponding linear differential equation is resonant).

2.2. The stable and unstable manifolds W_+^s and W_-^u

Now we turn to consider the stability properties of the origin for system (2.2). In this subsection we assume for simplicity that (2.2) is C^1, cf. Remark 2.2, so that the origin is a saddle (a discussion of the non-smooth case is postponed to Sect. 4.2). So we can define the following sets

$$
\begin{aligned}
W^u &:= \{\boldsymbol{Q} \mid \lim_{t \to -\infty} \phi(t; \boldsymbol{Q}) = (0,0)\}, \\
W^s &:= \{\boldsymbol{Q} \mid \lim_{t \to +\infty} \phi(t; \boldsymbol{Q}) = (0,0)\}.
\end{aligned}
\tag{2.8}
$$

Lemma 2.4. *Assume $q > p_*(\delta)$ and that (2.2) is C^1. Then W^u and W^s are 1 dimensional (immersed) manifolds.*

This Lemma easily follows from the Hartman-Grobman Theorem, see e.g. [18, §1.3].

It is easy to check that W^u (respectively W^s) is split by the origin in two connected components: since we are interested in definitively positive solutions, we consider the one leaving the origin and entering $x \geq 0$, denoted by W_+^u (respectively W_+^s). In fact we have the following Lemma

Lemma 2.5. *Let $\boldsymbol{Q^u} \in W_+^u$ and $\boldsymbol{Q^s} \in W_+^s$, then*

$$
\begin{aligned}
W_+^u &:= \{\phi(t; \boldsymbol{Q^u}) \mid t \in \mathbb{R}\} \cup \{(0,0)\}, \\
W_+^s &:= \{\phi(t; \boldsymbol{Q^s}) \mid t \in \mathbb{R}\} \cup \{(0,0)\}.
\end{aligned}
$$

From the Hartman-Grobman theory we also get the following useful result, see again [18, §1.3].

Lemma 2.6. *Assume $q > p_*(\delta)$ and that (2.2) is C^1. Then W_+^u is tangent in the origin to the x axis, while W_+^s is tangent to the y axis if $1 < p < 2$ and to the line $y = -(n-2)x$ if $p = 2$.*

Then we immediately obtain the following.

Lemma 2.7. *Assume $q > p_*(\delta)$ and that (2.2) is C^1. Then there is a ball Ω centered in the origin such that the sets*

$$
\begin{aligned}
W_{loc}^u &:= \{\boldsymbol{Q} \in (\Omega \cap W_+^u) \mid \phi(t; \boldsymbol{Q}) \in \Omega \text{ for any } t \leq 0\} \subset W_+^u, \\
W_{loc}^s &:= \{\boldsymbol{Q} \in (\Omega \cap W_+^s) \mid \phi(t; \boldsymbol{Q}) \in \Omega \text{ for any } t \geq 0\} \subset W_+^s,
\end{aligned}
$$

are C^1 embedded 1 dimensional manifolds. Further

$$
\begin{aligned}
W_{loc}^u &\subset \{(x,y) \mid \alpha x - |y|^{p-1} > 0, \ y < 0\}, \\
W_{loc}^s &\subset \{(x,y) \mid \alpha x - |y|^{p-1} < 0 < x\}.
\end{aligned}
$$

Using standard tools of dynamical system theory and some integral estimates, we find the following correspondences between trajectories of (2.2), and solutions of the original equation (1.2).

Lemma 2.8. *Assume $q > p_*(\delta)$ and that (2.2) is C^1; let $u(r)$ be a solution of (1.2) and let $\phi(t; \mathbf{Q})$ be the corresponding trajectory of (2.2) via (2.1).*
Then $u(r)$ is a regular solution if and only if $\mathbf{Q} \in W_+^u$, while $u(r)$ has fast decay if and only if $\mathbf{Q} \in W_+^s$.
Moreover, if $q \neq p^*$ then singular and slow decay solutions $u(r)$ of (1.2) correspond to the trajectories $\phi(t)$ converging to $\mathbf{P}$ respectively as $t \to -\infty$ or as $t \to +\infty$; thus $\lim_{r \to 0} u(r) r^\alpha = P_x$ and $\lim_{r \to +\infty} u(r) r^\alpha = P_x$ respectively. No other solutions of (1.2) definitively positive either for r small or for r large exist.

Proof of Lemma 2.8. If $u(r)$ is a regular solution, from (2.1) it follows easily that $\phi(t) \to (0,0)$ as $t \to -\infty$.

Viceversa assume that $\lim_{t \to -\infty} \phi(t) = (0,0)$. Since $\phi(t) \in W_+^u$ we easily see that $y(t) < 0 < x(t)$ when $t \ll 0$, hence $u(r)$ is positive and decreasing for $0 < r \ll 1$. Using standard tools of invariant manifold theory, see e.g. [18, §1.3], we see that the trajectory in W_+^u has the same asymptotic behaviour as a trajectory of the unstable space of the linearisation of system (2.2) in the origin, i.e. $\lim_{t \to -\infty} \|\phi(t)\| e^{-\alpha t} = d(u) > 0$. Further W_+^u is tangent to the x axis in the origin; hence we find

$$0 < d(u) = \lim_{t \to -\infty} x(t) e^{-\alpha t} = \lim_{r \to 0} u(r),$$

and the claim concerning regular solutions is proved.

Now we turn to consider a fast decay solution $v(r; L)$ of (1.2) and the corresponding trajectory $\phi(t)$ of (2.2).

If $v(r; L)$ is a fast decay solution then from Lemma 1.8 it follows that $\lim_{r \to +\infty} v'(r; L) r^{\frac{n-1}{p-1}} = -\frac{n-p}{p-1} L$; so from (2.1) we easily see that $\phi(t) \to (0,0)$ as $t \to +\infty$.

Viceversa assume that $\lim_{t \to +\infty} \phi(t) = (0,0)$. Since $\phi(t) \in W_+^s$ we easily see that $y(t) < 0 < x(t)$ when $t \gg 0$, hence $u(r)$ is positive and decreasing for $r \gg 1$. From Lemma 1.8 we see that $\lim_{r \to +\infty} v(r) r^{\frac{n-p}{p-1}}$ exists and it is either a positive constant or $+\infty$.

Again from standard tools of invariant manifold theory, [18, §1.3], we see that the trajectories in W_+^s satisfy $\lim_{t \to +\infty} \|\phi(t)\| e^{-\gamma t} = \frac{n-p}{p-1} L$, for some $L > 0$. Further W_+^s is tangent to the y axis in the origin hence we find

$$\lim_{t \to +\infty} |y(t)| e^{-\gamma t} = \lim_{r \to +\infty} |v'(r)|^{p-1} r^{n-1} = \left[\frac{n-p}{p-1} L \right]^{p-1} > 0,$$

where we used the fact that $\beta - \gamma = n - 1$. Hence we obtain $\lim_{r \to +\infty} |v'(r)| r^{\frac{n-1}{p-1}} = \frac{n-p}{p-1} L$ and using (1.7) we prove that $v(r)$ has fast decay. The claim concerning singular and slow decay solutions follows from an elementary analysis of the phase portrait of (2.2), see, e.g., [10, Observation 2.14] and Fig. 1. $\square$

When $q = p^*$ the results in Lemma 2.8 need to be modified slightly. In fact in this case (2.2) is Hamiltonian and admits periodic trajectories which correspond to singular solutions $u(r)$ of (1.2) which have a slightly different behaviour. We remand the interested reader, e.g., to [10, Proposition 2.11].
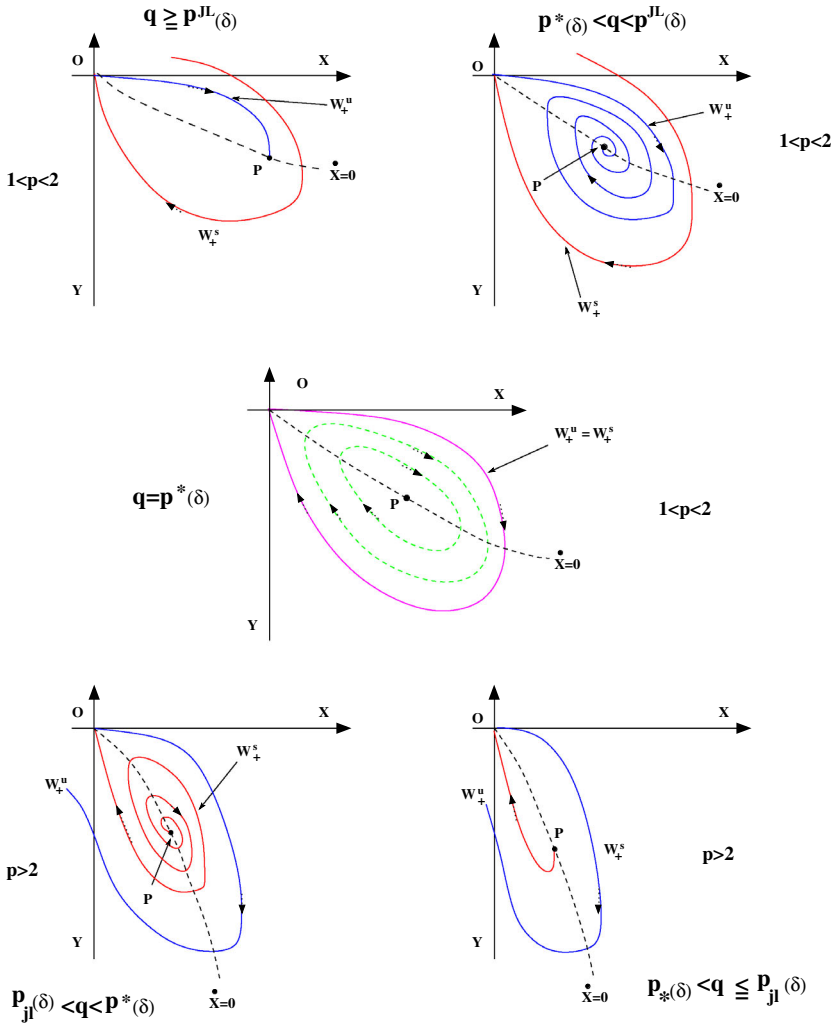
FIGURE 1. The phase portrait as q passes through the critical values. The unstable manifold W_+^u is in blue while the stable manifold W_+^s is in red. When $q = p^*(\delta)$ we have drawn the manifold $W_+^u = W_+^s$ in magenta, and we have drawn in green some periodic trajectories corresponding to singular solutions with slow decay

Lemma 2.9. *Let* (2.2) *be smooth. Then W_+^u may be parametrized by $d \geq 0$. So if we follow W_+^u from the origin towards $x > 0$ we go from $d = 0$ to $d = +\infty$. Analogously W_+^s may be parametrized by $L \geq 0$: following W_+^s from the origin towards $x > 0$ we go from $L = 0$ to $L = +\infty$.*

Proof. Let $x(t; \boldsymbol{Q^u})$, $\boldsymbol{Q^u} \in W_+^u$, be the trajectory of (2.2) corresponding to $u(r; 1)$. From (2.1), Lemma 2.8 and using the invariance for t-translations of (2.2), for any $d > 0$, we can write

$$u(r; d)r^\alpha = x(t; \boldsymbol{Q}) = x(t - \tau; \boldsymbol{Q^u}) = u(re^{-\tau}, 1)r^\alpha e^{-\alpha\tau}, \qquad (2.9)$$

where $r = e^t$ and $\boldsymbol{Q} := x(-\tau; \boldsymbol{Q^u}) \in W_+^u$ (since W_+^u is invariant, see Lemma 2.5). Then passing to the limit as r tends to 0 we find

$$d = \lim_{r \to 0} u(r; d) = \lim_{r \to 0} u(re^{-\tau}, 1)e^{-\alpha\tau},$$

from which

$$d = e^{-\alpha\tau}, \qquad u(r; d) \equiv u\left(r\sqrt[\alpha]{d}; 1\right)d \quad \text{for any } r \geq 0. \qquad (2.10)$$

Similarly let $v(r; 1)$ be a fast decay solution of (1.2) and let $x(t; \boldsymbol{Q^s})$ be the corresponding trajectory of (2.2), so that $\boldsymbol{Q^s} \in W_+^s$.
Reasoning as above we see that $\boldsymbol{\phi}_\tau(t) := \phi(t - \tau; \boldsymbol{Q^s})$ is a trajectory of (2.2) and $\boldsymbol{\phi}_\tau(0) = \boldsymbol{R} \in W_+^s$, see again Lemma 2.5.
So $\boldsymbol{\phi}_\tau(t; \boldsymbol{R})$ corresponds to a fast decay solution $v(r; L)$ of (1.2) such that

$$\begin{aligned} v(r; L)r^\alpha &= x_\tau(t) = x(t - \tau; \boldsymbol{Q^s}) = v(re^{-\tau}; 1)r^\alpha e^{-\alpha\tau} \\ v(r; L)r^{\frac{n-p}{p-1}} &= v(re^{-\tau}; 1)r^{\frac{n-p}{p-1}} e^{-\frac{n-p}{p-1}\tau} e^{(\frac{n-p}{p-1} - \alpha)\tau}, \end{aligned} \qquad (2.11)$$

where we have used the same idea as in (2.9)
Hence passing to the limit as $r \to +\infty$ on the left hand side and as $re^{-\tau} \to +\infty$ on the right hand side we find

$$L = e^{(\frac{n-p}{p-1} - \alpha)\tau}, \qquad v(r; L) \equiv v(rL^{-\frac{p-1}{n-p-\alpha(p-1)}}; 1)L^{-\frac{\alpha(p-1)}{n-p-\alpha(p-1)}} \quad \text{for any } r > 0.$$

$\square$

Proposition 2.10. *Assume* $\boldsymbol{H}$. *Lemmas* 2.4, 2.7, 2.8 *and Lemma* 2.9 *hold also if* (2.2) *is not smooth, i.e. whenever* $q > p_*(\delta)$, $\delta > -p$, $p > 1$ *(even if* $p < q < 2$ *or* $p > 2$*).*

The proof of this result is rather technical, and in fact the part concerning W_+^u and regular solutions can be found in literature, with some effort, even in a non-autonomous context, see [11] and references in Sect. 4.2. We give a new and shorter proof suitable for this simpler autonomous context in Sect. 4.2 for completeness.

2.3. Pohozaev function and heteroclinic connections

Let us introduce the following energy like function

$$H(x, y) = \frac{n-p}{p}xy + \frac{p-1}{p}|y|^{\frac{p}{p-1}} + k\frac{|x|^q}{q} \qquad (2.12)$$

which is closely related to the Pohozaev identity, see e.g. [13, §2], or [10, §2]:
From a straightforward computation it is easy to check that when $q = p^*(\delta)$ then $H(x, y)$ is a first integral, see e.g. [13, §2]. In general we have the following result.

Lemma 2.11. *Let $\boldsymbol{Q^u} \in W^u_+$ and $\boldsymbol{Q^s} \in W^s_+$. If $q > p^*(\delta)$, then $H(\boldsymbol{Q^u}) < 0 < H(\boldsymbol{Q^s})$. If $p_*(\delta) < q < p^*(\delta)$, then $H(\boldsymbol{Q^s}) < 0 < H(\boldsymbol{Q^u})$. If $q = p^*(\delta)$, then $W^u_+ = W^s_+$ and $H(\boldsymbol{Q^u}) = 0$ for any $\boldsymbol{Q^u} \in W^u_+$.*

We provide here the proof of this known result for the convenience of the reader, however see, e.g. [13, §2], for a proof in a non autonomous context.

Proof. For any solution $u(r)$ of (1.2) we can define the following Pohozaev function

$$P(u(r), u'(r), r) = \frac{n-p}{p} r^{n-1} u(r) u'(r) |u'(r)|^{p-2}$$

$$+ r^n \frac{p-1}{p} |u'(r)|^p + k r^{\delta+n+1} \frac{|u(s)|^q}{q}.$$

One of the main tool in the analysis of this equation is the well known Pohozaev identity, see, e.g., [24], that in this context reads as follows:

$$\frac{d}{dr} P(u(r), u'(r), r) = k \frac{(n-p)(p^*(\delta) - q)}{pq} r^{n+\delta} |u(r)|^q. \qquad (2.13)$$

Therefore $P(u(r), u'(r), r)$ is monotone increasing if $p < q < p^*(\delta)$, it is constant if $q = p^*(\delta)$, and it is monotone decreasing if $q > p^*(\delta)$.
Further observe that if $u(r)$ is a regular solution and $v(r)$ is a fast decay solution we have

$$\lim_{r \to 0} P(u(r), u'(r), r) = 0, \qquad \lim_{r \to +\infty} P(v(r), v'(r), r) = 0. \qquad (2.14)$$

Then we go back to system (2.2) and we see that if $\boldsymbol{\phi}(t) = (x(t), y(t))$ is the trajectory of (2.2) corresponding to $u(r)$ we have

$$H(x(t), y(t)) = P(u(e^t), u'(e^t), e^t) e^{(\alpha+\gamma)t}, \qquad (2.15)$$

where H is the function defined in (2.12).

Let $\boldsymbol{\phi^u}(t)$, $\boldsymbol{\phi^v}(t)$ be the trajectories of (2.2) corresponding to the regular and the fast decay solution $u(r)$ and $v(r)$ of (1.2). Assume to fix the ideas that $q > p^*(\delta)$; then from (2.13) and (2.14) we find that $P(u(r), u'(r), r) < 0 < P(v(r), v'(r), r)$ for any $r > 0$. Hence from (2.15) we see that $H(\boldsymbol{\phi^u}(t)) < 0 < H(\boldsymbol{\phi^v}(t))$ for any $t \in \mathbb{R}$, so the Lemma follows. The case $p < q < p^*(\delta)$ is analogous.

With the same argument we easily see that if $q = p^*(\delta)$ then H is a first integral; hence W^u_+ and W^s_+ are obtained as the subset of the 0-level set of H which is contained in $x \geq 0$. $\qquad \square$

Then from an elementary analysis of the phase portrait we obtain the following, cf. Fig. 1.

Proposition 2.12. *If $q > p^*(\delta)$ there is $\boldsymbol{Q^u} = (Q^u_x, Q^u_y)$ such that $Q^u_x > 0$, $H(\boldsymbol{Q^u}) < 0$ and*

$$\lim_{t \to -\infty} \boldsymbol{\phi}(t; \boldsymbol{Q^u}) = (0, 0), \qquad \lim_{t \to +\infty} \boldsymbol{\phi}(t; \boldsymbol{Q^u}) = \boldsymbol{P},$$

$$W^u_+ := \{\boldsymbol{\phi}(t; \boldsymbol{Q^u}) \mid t \in \mathbb{R}\} \cup \{(0, 0)\}.$$

If $p_*(\delta) < q < p^*(\delta)$ there is $\boldsymbol{Q^s} = (Q^s_x, Q^s_y)$ such that $Q^s_x > 0$, $H(\boldsymbol{Q^s}) < 0$ and

$$\lim_{t \to -\infty} \phi(t; \boldsymbol{Q^s}) = \boldsymbol{P}, \qquad \lim_{t \to +\infty} \phi(t; \boldsymbol{Q^s}) = (0,0),$$

$$W^s_+ := \{\phi(t; \boldsymbol{Q^s}) \mid t \in \mathbb{R}\} \cup \{(0,0)\}.$$

To complete the picture we observe that if $p_*(\delta) < q < p^*(\delta)$ then W^u_+ is made up by an unbounded trajectory which converges to the origin as $t \to -\infty$ and rotates clockwise indefinitely as t increases. Similarly if $q > p^*(\delta)$ then W^s_+ is made up by an unbounded trajectory which converges to the origin as $t \to +\infty$ and rotates clockwise indefinitely as t decreases. If $q = p^*(\delta)$ then $W^u_+ = W^s_+$ and it is the graph of an homoclinic trajectory and coincide with the set $\{\boldsymbol{Q} = (Q_x, Q_y) \mid H(\boldsymbol{Q}) = 0, Q_x > 0\}$, see again Fig. 1. We do not give a full fledged proof of this known facts (which will not be used in this article) remanding the interested reader, e.g., to [10].

Proof of Theorems C and 1.3

Theorems C and 1.3 simply follow putting together Lemma 2.8, Proposition 2.12 and Lemma 2.3. $\qquad \square$

3. Proof of the main results

In this section we show that if $q \geq p^{JL}(\delta)$ then W^u_+ is a graph on $\pi := \{(x,0) \mid 0 \leq x < P_x\}$ while if $p_*(\delta) < q \leq p_{jl}(\delta)$ then W^s_+ is a graph on π. Whence Theorems 1.1 and 1.2 easily follow.

The idea is inspired by the work by Miyamoto [25] and it is obtained by constructing suitable positively and negatively invariant sets.

Proposition 3.1. *Let r_1 be the semi-line through $\boldsymbol{P}$ with the direction of $\boldsymbol{v_1}$ and let r_2 be the semi-line through $\boldsymbol{P}$ with the direction of $\boldsymbol{v_2}$ and with equation*

$$r_1: \quad y = -m_1(x - P_x) + P_y, \qquad x < P_x,$$
$$r_2: \quad y = -m_2(x - P_x) + P_y, \qquad x < P_x.$$

Let $q > p$. If $q \geq p^{JL}(\delta)$, $q \geq 2$ and $p > 1$ then both the lines r_1 and r_2 intersect the x positive semi-axis respectively in the points $S^i_x = (X^i, 0)$, for $i = 1, 2$, with $X^1 \geq X^2 > 0$.
If $p_*(\delta) < q \leq p_{jl}(\delta)$ and $p \in (1, 2]$ then the semi-line r_1 and r_2 intersect the y positive semi-axis in the points $S^i_y = (0, -Y^i)$ with $Y^2 \geq Y^1 > 0$.

Proof. The proof is based on a geometric argument. We first observe that on the positive x-semi-axis we have $\dot{x} > 0$ and $\dot{y} < 0$. We restrict to consider the 4th quadrant so the x-nullcline is

$$\alpha x - |y|^{\frac{1}{p-1}} = 0, \tag{3.1}$$

and it is represented in Fig. 2 for $p \in (1, 2)$ and for $p \geq 2$.
The y-nullcline is (see Fig. 3)

$$\gamma y - k x^{q-1} = 0, \tag{3.2}$$

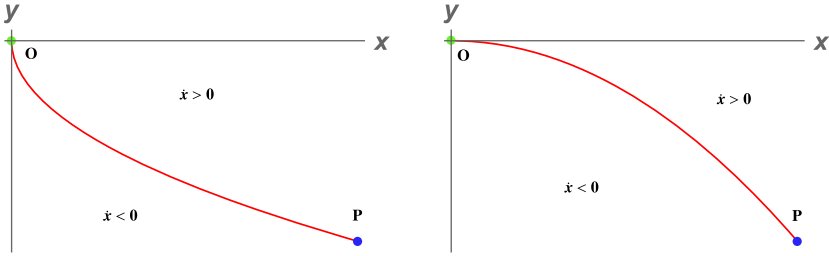so $\dot{y} > 0$ if $y < -\frac{k}{|\gamma|} x^{q-1}$.

FIGURE 2. The x-nullcline in the case $p \in (1,2)$ and $p \geq 2$ respectively
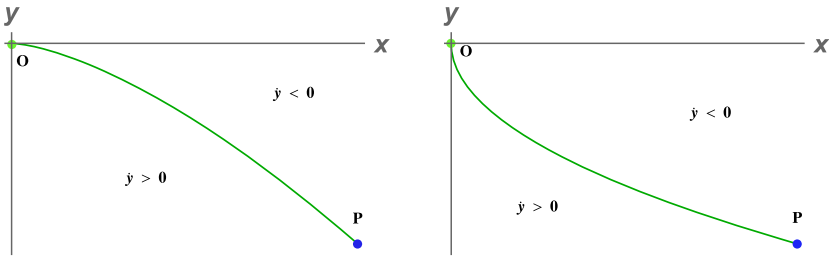

FIGURE 3. The y-nullcline when $q > 2$ on the left and when $q \in (1,2)$ on the right

Since $q > p$, the x-nullcline is below the y-nullcline when $x \in (0, P_x)$ that is

$$-\frac{k}{|\gamma|}x^{q-1} > -(\alpha x)^{p-1}, \qquad 0 < x < P_x,$$

while we have the opposite situation for $x > P_x$.

The two nullclines, the vertical line $x = P_x$ and the horizontal line $y = P_y$ define 8 regions in the fourth quadrant (see Fig. 4). In order to conclude the proof we consider two cases.

Case 1: $q > p > 1, q \geq 2$ and $q \geq p^{JL}(\delta)$.

In this case the fixed point $\boldsymbol{P}$ is a stable node; so there are solutions of the linearised system which converge to $\boldsymbol{P}$ along the semi-lines r_1 and r_2; correspondingly there are solutions of the nonlinear system which converge to $\boldsymbol{P}$ tangentially to r_1 and r_2. Then the semi-lines r_i, spanned by the eigenvectors $\boldsymbol{v}_i$, must lie inside the regions of the space which allow convergence to $\boldsymbol{P}$. By the analysis of the vector field we conclude that there are only two regions in which we can find solutions converging to $\boldsymbol{P}$, that is region 3 and 7 (see Fig. 4). Whence both r_1 and r_2 are in region 3 and, since the y-nullcline is concave, the lines r_1 and r_2 both intersect the x-axis at a certain $X^i > 0$. We note that this is not ensured when $q < 2$ since the y-nullcline is convex (a priori r_i may intersect the x-axis for a negative value of x).

Case 2: $p \in (1,2]$, $p < p_*(\delta) < q \leq p_{jl}(\delta)$.

In this case the fixed point $\boldsymbol{P}$ is an unstable node; reasoning as above we see that the semi-lines r_i must lie inside the regions of the space which allow
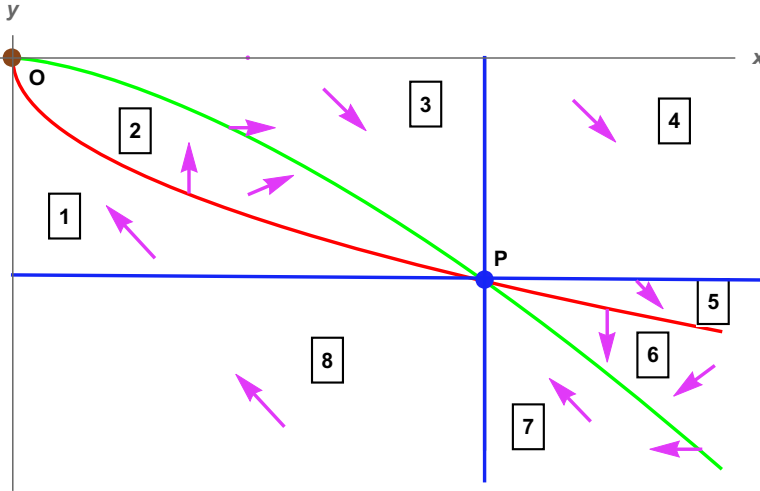
FIGURE 4. The eight regions of the fourth quadrant defined by the x-nullcline (in red), the y-nullcline (in green) the vertical line $x = P_x$ and the horizontal line $y = P_y$ (in blue) (color figure online)

convergence to P in the past, i.e. regions 1 and 5. Then, r_1 and r_2 lie in region 1, below the convex x-nullcline; hence r_1 and r_2 both intersect the y-negative semi-axis. We note that this argument does not work for $p > 2$ since the x-nullcline is concave. $\qquad\square$

Let us denote by S_x the intersection between the semi-line r_2 and the x axis, and by S_y the intersection between the semi-line r_1 and the y axis, see Fig. 5.

Proposition 3.2. (i) *Let* $q \geq p^{JL}(\delta)$, $q \geq 2 \geq p > 1$. *Then the region* A^+ *(see Fig. 5) delimitated by the x-nullcline, the segments $\overline{OS_x}$ and $\overline{PS_x}$ is positively invariant.*

(ii) *Let* $p \in (1, 2]$, *and* $p_*(\delta) < q \leq p_{jl}(\delta)$, $q \geq 2$. *Then the region* A^-, *delimitated by the the x-nullcline, the segments $\overline{S_y O}$ and $\overline{PS_y}$, is negatively invariant.*

Proof. We divide the proof into two parts.

(i). On the x-nullcline and on the segment $\overline{OS_x}$ the vector field points inside A^+. It remains to check the vector field on the segment $\overline{PS_x}$ which lies on the line r_2.

We rewrite (2.2) in the following form

$$\dot{\phi} = f(\phi) = \frac{\partial f}{\partial \phi}(P)(\phi - P) + R(\phi), \qquad (3.3)$$
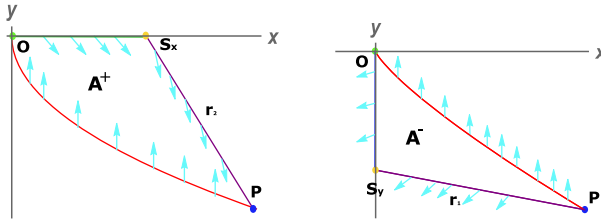
FIGURE 5. The sets A^+ on the left and A^- on the right: they are respectively positively and negatively invariant for (2.2). We have represented the x-nullcline in red, the lines r_2 and r_1 in purple and the segments $\overline{OS_x}$ and $\overline{S_yO}$ in green and in blue respectively

where

$$\boldsymbol{R}(\phi) := \begin{pmatrix} R_1(y) \\ R_2(x) \end{pmatrix} = \begin{pmatrix} y|y|^{\frac{2-p}{p-1}} - P_y|P_y|^{\frac{2-p}{p-1}} - \frac{|P_y|^{\frac{2-p}{p-1}}}{p-1}(y - P_y) \\ -k[x^{q-1} - P_x^{q-1} - (q-1)P_x^{q-2}(x - P_x)] \end{pmatrix},$$

and $\phi = (x, y)$. We observe that $\hat{\psi}(t) = \boldsymbol{P} + \boldsymbol{v_1}\mathrm{e}^{\lambda_1 t}$ is a solution of the linear system

$$\dot{\psi} = \boldsymbol{f}(\psi) = \frac{\partial \boldsymbol{f}}{\partial \phi}(\boldsymbol{P})(\psi - \boldsymbol{P}), \tag{3.4}$$

and its graph is r_2. Further

$$R_1(y) = g(y) - [g(P_y) + g'(P_y)(y - P_y)],$$

where $g(y) = -|y|^{\frac{1}{p-1}}$.
Since $R_1(P_y) = R_1'(P_y) = 0$ and for any $y \neq 0$

$$R_1''(y) = g''(y) = -\frac{2-p}{(p-1)^2}|y|^{\frac{3-2p}{p-1}} \leq 0,$$

we have that $R_1(y) < 0$ for all $y \in (P_y, 0)$. Moreover

$$R_2(x) = h(x) - [h(P_x) + h'(P_x)(x - P_x)],$$

where

$$h(x) = -kx^{q-1}, \quad \text{and} \quad h''(x) = -k(q-1)(q-2)x^{q-3}.$$

Since $R_2(P_x) = R_2'(P_x) = 0$ and $R_2''(x) = h''(x) \leq 0$, we see that $R_2(x) < 0$ when $0 < x < P_x$. Let $\phi(t; \boldsymbol{Q}) = (x(t; \boldsymbol{Q}), y(t; \boldsymbol{Q}))$ and $\psi(t; \boldsymbol{Q}) = (\bar{x}(t; \boldsymbol{Q}), \bar{y}(t; \boldsymbol{Q}))$ be solutions of (3.3) and (3.4) respectively, departing from $\boldsymbol{Q} = (Q_x, Q_y) \in r_2$. We find

$$\dot{y}(0; \boldsymbol{Q}) - \dot{\bar{y}}(0; \boldsymbol{Q}) + m_1[\dot{x}(0; \boldsymbol{Q}) - \dot{\bar{x}}(0; \boldsymbol{Q})] = R_2(Q_x) + m_1 R_1(Q_y) < 0. \tag{3.5}$$

Then observing that r_2 is invariant for the flow of (3.4) and using (3.5) we conclude that A^+ is positive invariant on r_2 too, so claim (i) is proved.

(ii) We observe that on the negative y-axis the vector field points outside A^-. The rest of the proof is identical to case (i).

$\square$

Proof of Theorems 1.1 *and* 1.2 *in the* $1 < p \le 2$ *case*

Let $q \ge p^{JL}(\delta)$: we claim that W_+^u *is a graph on* $\pi := \{(x,0) \mid 0 \le x < P_x\}$.

From Lemma 2.7 we know that there is a neighborhood Ω of the origin such that $W_{loc}^u \subset A^+$, and if $\boldsymbol{Q} \in W_{loc}^u$ then $\phi(t; \boldsymbol{Q}) \in W_{loc}^u \subset A^+$ for any $t \le 0$. From Proposition 3.2 we see that $\phi(t; \boldsymbol{Q}) \in A^+$ for any $t \ge 0$. Hence from Lemma 2.5 we see that $\phi(t; \boldsymbol{Q}) \in A^+$ for any $t \in \mathbb{R}$ and $W_+^u \subset A^+$. Thus $\dot{x}(t; \boldsymbol{Q}) > 0$ for any $t \in \mathbb{R}$ and the claim easily follows from Lemma 2.5.

Now let $0 < d_1 < d_2 < +\infty$; let $\phi(t; \boldsymbol{Q}(d_i))$ be the trajectory of (2.2) corresponding to the regular solution $u(r, d_i)$ of (1.2), for $i = 1, 2$, so that $\boldsymbol{Q}(d_i) = (Q_x(d_i), Q_y(d_i)) \in W_+^u$. From Lemma 2.9 we see that $0 < Q_x(d_1) < Q_x(d_2)$ and $0 < x(t; Q(d_1)) < x(t; Q(d_2)) < P_x$ for any $t \in \mathbb{R}$; then Theorem 1.1 immediately follows.

Analogously let $p_*(\delta) < q \le p_{jl}(\delta)$: from Lemma 2.7 we see that $W_{loc}^s \subset A^-$, for a suitable neighborhood Ω of the origin. Using again Lemma 2.7 and Proposition 3.2 and reasoning as above we see that if $\boldsymbol{Q} \in W_+^s$ then $\phi(t; \boldsymbol{Q}) = (x(t; \boldsymbol{Q}), y(t; \boldsymbol{Q}))$ is such that $\dot{x}(t; \boldsymbol{Q}) < 0$ for any $t \in \mathbb{R}$, and W_+^s is a graph on π.

Now let $0 < L_1 < L_2 < +\infty$, and let $\phi(t; \boldsymbol{Q}(L_i))$ be the trajectory of (2.2) corresponding to the fast decay solution $v(r, L_i)$ of (1.2), for $i = 1, 2$, so that $\boldsymbol{Q}(L_i) = (Q_x(L_i), Q_y(L_i)) \in W_+^s$. From Lemma 2.9 we see that $0 < Q_x(L_1) < Q_x(L_2)$ and that $0 < x(t; Q(L_1)) < x(t; Q(L_2)) < P_x$ for any $t \in \mathbb{R}$; then Theorem 1.2 immediately follows. $\square$

Now we turn to consider the $p > 2$ case: in this setting we use the argument developed by Miyamoto in [25]. So, with a slight adaption of [25] we rewrite (2.2) as follows:

$$\begin{cases} \dot{x} = z, \\ \dot{z} = -\frac{\alpha\gamma}{p-1}x + [\alpha + \frac{\gamma}{p-1}]z - \frac{k}{p-1}\tilde{g}(x, z), \end{cases} \tag{3.6}$$

where

$$\tilde{g}(x, z) = \frac{x^{q-1}}{|\alpha x - z|^{p-2}}.$$

We denote by $\tilde{\phi}(t; \boldsymbol{Q}) = (\tilde{x}(t; \boldsymbol{Q}), \tilde{z}(t; \boldsymbol{Q}))$ a solution of (3.6) leaving from $\boldsymbol{Q}$ at $t = 0$. Moreover, we rewrite (3.6) as $\dot{\tilde{\phi}}(t) = \tilde{\boldsymbol{f}}(\tilde{\phi})$, i.e. $\tilde{\boldsymbol{f}}(\tilde{\phi})$ is the right hand side of (3.6).

The point $\tilde{\boldsymbol{P}} = (P_x, 0)$ (where P_x is given in (2.3)) is the critical point of system (3.6) corresponding to the critical point $\boldsymbol{P}$ of (2.2). Notice that from a lengthy but straightforward computation we find

$$\frac{d\tilde{\boldsymbol{f}}}{d\tilde{\phi}}(\tilde{\boldsymbol{P}}) = \begin{pmatrix} 0 & 1 \\ -\frac{\alpha\gamma + k\frac{\partial\tilde{g}}{\partial x}(\tilde{\boldsymbol{P}})}{p-1} & \alpha + \frac{\gamma - k\frac{\partial\tilde{g}}{\partial z}(\tilde{\boldsymbol{P}})}{p-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\alpha\gamma(q-p)}{p-1} & \alpha + \gamma \end{pmatrix}. \tag{3.7}$$

We emphasize that $\frac{d\tilde{\boldsymbol{f}}}{d\tilde{\phi}}$ has the same trace T and determinant D as $\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P})$, cf. (2.5). Hence, as it has to be expected, $\tilde{\boldsymbol{P}}$ has the same stability properties as $\boldsymbol{P}$, i.e. we have the following.
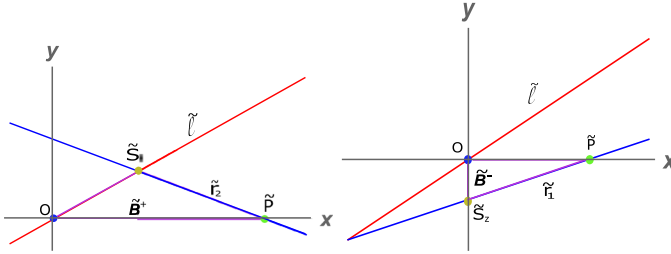
FIGURE 6. The sets $\tilde{B}^+$ on the left and $\tilde{B}^-$ on the right: they are respectively positively and negatively invariant for (3.6)

Lemma 3.3. *The critical point $\tilde{P}$ is an unstable node if $p_*(\delta) < q \le p_{jl}(\delta)$, an unstable focus if $p_{jl}(\delta) < q < p^*(\delta)$, a center if $q = p^*(\delta)$, a stable focus if $p^*(\delta) < q < p^{JL}(\delta)$, a stable node if $q \ge p^{JL}(\delta)$.*
Further its eigenvalues are the λ_i given in (2.6), and when $\tilde{P}$ is a node the eigenvectors are given by $\tilde{v}_i = (1, \lambda_i)$.

Passing from (2.2) to (3.6) the 1 dimensional manifolds W^u_+ and W^s_+ are driven into the 1 dimensional manifolds $\tilde{W}^u_+$ and $\tilde{W}^s_+$ by a global diffeomorphism (which brings the nullcline $\dot{x} = 0$ into the x axis, and it is linear when $p = 2$). Obviously the trajectories in $\tilde{W}^u_+$ and $\tilde{W}^s_+$ correspond respectively to regular and fast decay solutions of (1.2) and all the results in §2 hold for $\tilde{W}^u_+$ and $\tilde{W}^s_+$ too, apart from the ones regarding the tangent in the origin.
We omit the computation which is quite similar to the one carried on for system (2.2), see also the analogous computation performed in [25, Lemma 2.5].
Let us set

$$\tilde{r}_i := \{(x, z) : \quad z = \lambda_i(x - P_x), \quad x < P_x\}, \quad \text{ for } i = 1, 2$$

and

$$\tilde{\ell} := \{(x, z) : \quad z = \alpha x > 0\}.$$

Notice that the semi-line $\tilde{\ell}$ of (3.6) corresponds to the x-positive semi-axis of (2.2), so to solutions $u(r)$ of (1.2) such that $u'(r) = 0$. We stress that if $p_*(\delta) < q \le p_{jl}(\delta)$ then $\lambda_2 > \lambda_1 > 0$; hence $\tilde{r}_1$ lies in the semiplane $z < 0$ and it intersects the z negative semi-axis in a point $\tilde{S}_z = (0, Z_-)$, with $Z_- < 0$, while if $q \ge p^{JL}(\delta)$ then $\lambda_1 < \lambda_2 < 0$; hence $\tilde{r}_2$ lies in the $z > 0$ semiplane and it intersects $\tilde{\ell}$ in the point $\tilde{S}_\ell$.
If $p_*(\delta) < q \le p_{jl}(\delta)$ we denote by $\tilde{B}^-$ the compact set enclosed by the segments $\overline{\tilde{S}_z O}$, $\overline{\tilde{S}_z \tilde{P}}$ and $\overline{O\tilde{P}}$.
Similarly if $q \ge p^{JL}(\delta)$ we denote by $\tilde{B}^+$ the compact set enclosed by the segments $\overline{O\tilde{P}}$, $\overline{O\tilde{S}_\ell}$ and $\overline{\tilde{S}_\ell \tilde{P}}$ (see Fig. 6).

Proposition 3.4. *Assume $p \geq 2$. If $p_*(\delta) < q \leq p_{jl}(\delta)$, then the set $\tilde{B}^-$ is negatively invariant for (3.6), while if $q \geq p^{JL}(\delta)$ then $\tilde{B}^+$ is positively invariant for (3.6).*

Proof. The flow of (3.6) on the segment of the x positive semi-axis between the origin and $\tilde{P}$ points upwards, this can be checked directly; similarly we see that on the z-negative semi-axis the flow points towards $x < 0$.
We recall that the line $\tilde{\ell}$ of (3.6) corresponds to the x positive semi-axis of (2.2): hence the vector field points towards the interior of $\tilde{B}^+$. It remains to check the vector field on $\tilde{r}_i$.
Observe first that $\tilde{\psi}_i(t) = \tilde{P} + \boldsymbol{v}_i e^{\lambda_i} = (P_x + \mathrm{e}^{\lambda_i t}, \lambda_i \mathrm{e}^{\lambda_i t})$, for $i = 1, 2$ is a solution of the autonomous linear equation

$$\dot{\tilde{\psi}}_i = \frac{d\tilde{\boldsymbol{f}}}{d\tilde{\phi}}(\tilde{\boldsymbol{P}})[\tilde{\psi}_i - \tilde{\boldsymbol{P}}], \tag{3.8}$$

and that $\tilde{r}_i := \{\tilde{\psi}_i(t) \mid t \in \mathbb{R}\}$.
Assume $p_*(\delta) < q \leq p_{jl}(\delta)$ (respectively $q \geq p^{JL}(\delta)$) we claim *that if* $\boldsymbol{Q} \in \overline{\tilde{S}_z \tilde{P}}$ (respectively $\boldsymbol{Q} \in \overline{\tilde{S}_\ell \tilde{P}}$), then $\dot{\tilde{\phi}}(t; \boldsymbol{Q}) - \dot{\tilde{\psi}}(\boldsymbol{Q}) = (0, -c_2(\boldsymbol{Q}))$ where $c_2(\boldsymbol{Q}) > 0$.
Now we prove the claim.
Let $s \in (0, P_x]$; we define the following function:

$$\tilde{\boldsymbol{c}}(s) = \begin{pmatrix} \tilde{c}_1(s) \\ \tilde{c}_2(s) \end{pmatrix} = \tilde{\boldsymbol{f}}(s; \lambda_1(s - P_x)) - \frac{d\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{P}})}{d\tilde{\phi}} \begin{pmatrix} s \\ \lambda_1(s - P_x) \end{pmatrix}. \tag{3.9}$$

By construction $\tilde{\boldsymbol{c}}(s)$ evaluates $\dot{\tilde{\phi}}(t; \boldsymbol{Q}) - \dot{\tilde{\psi}}_1(\boldsymbol{Q})$ for $\boldsymbol{Q} = (s; \lambda_1(s - P_x)) \in \tilde{r}$.
Since the non-linear part of $\tilde{\boldsymbol{c}}(s)$ just depend on the presence of $\tilde{g}$ in (3.6) we have $\tilde{c}_2(0) = \tilde{c}_2'(0) = 0$; further $\tilde{c}_1(s) \equiv s - s = 0$. To prove the claim it is sufficient to show that $\tilde{c}_2''(s) < 0$ for any $s \in (0, P_x]$.
Once again, since all the linear terms cancel out, it is enough to differentiate

$$\tilde{\mathfrak{G}}(s) := -\frac{k}{p-1}\tilde{g}(s, \lambda_1(s - P_x)) = -\frac{k}{p-1}\frac{s^{q-1}}{A(s)^{p-2}},$$

where we have set for simplicity $A(s) := \alpha s - \lambda_1(s - P_x)$.
Differentiating the previous expression we find

$$\frac{d}{ds}\tilde{\mathfrak{G}}(s) = -\frac{k}{p-1}\left\{(q-1)\frac{s^{q-2}}{A(s)^{p-2}} - (p-2)\frac{(\alpha - \lambda_1)s^{q-1}}{A(s)^{p-1}}\right\},$$

$$\frac{d^2}{ds^2}\tilde{\mathfrak{G}}(s) = -\frac{k}{p-1}\left\{(q-1)(q-2)\frac{s^{q-3}}{A(s)^{p-2}}\right.$$
$$\left. - 2(q-1)(p-2)\frac{(\alpha - \lambda_1)s^{q-2}}{A(s)^{p-1}} + (p-1)(p-2)\frac{(\alpha - \lambda_1)^2 s^{q-1}}{A(s)^p}\right\}.$$

Hence we find

$$\frac{d^2}{ds^2}\tilde{\mathfrak{G}}(s) = -\frac{ks^{q-3}}{(p-1)A(s)^p}\left\{(q-1)(q-2)\left[A(s) - \frac{p-2}{q-2}(\alpha - \lambda_1)s\right]^2\right.$$
$$\left. + \frac{p-2}{q-2}(q-p)(\alpha - \lambda_1)^2 s^2\right\} < 0 \tag{3.10}$$

for any $s \neq 0$. Hence $\frac{d^2\tilde{c}_2}{ds^2}(s) = \frac{d^2\tilde{\mathfrak{G}}}{ds^2}(s) < 0$ for any $0 < s < P_x$ and the claim is proved. Now, using the claim, we easily conclude in both the cases: $q \geq p^{JL}(\delta)$ and $p_*(\delta) < q \leq p_{jl}(\delta)$ $\qquad \square$

Remark 3.5. The second part of the Lemma can be obtained by [25, Lemma 2.6] and in fact our proof is a slight simplification and a geometrical interpretation of the one by Miyamoto. The first part is obtained using Miyamoto's idea in the subcritical context.

Proof of Theorems 1.1 *and* 1.2 *in the $p > 2$ case*

We develop the proof just for Theorem 1.2, Theorem 1.1 is analogous. Let $p_*(\delta) < q \leq p_{jl}(\delta)$: we claim that $\tilde{W}_+^s$ *is a graph on* $\pi := \{(x,0) \mid 0 \leq x < P_x\}$. Let $\tilde{W}_{loc}^s := \{\boldsymbol{Q} \in \tilde{W}_+^s \mid \tilde{\boldsymbol{\phi}}(t;\boldsymbol{Q}) \in \Omega$ for any $t \geq 0\}$, for a suitable neighbourhood Ω of the origin. In fact applying Lemma 2.7 and Proposition 2.10 to W_+^s and passing to system (3.6), we can choose Ω so that if $\boldsymbol{Q} \in \tilde{W}_{loc}^s$, then $\tilde{\boldsymbol{\phi}}(t;\boldsymbol{Q}) = (\tilde{x}(t;\boldsymbol{Q}), \tilde{y}(t;\boldsymbol{Q})) \in \tilde{B}^-$ for any $t \geq 0$. Further from Proposition 3.4 we see that $\tilde{\boldsymbol{\phi}}(t;\boldsymbol{Q}) \in \tilde{B}^-$ for any $t \leq 0$. Hence $\dot{\tilde{x}}(t;\boldsymbol{Q}) < 0$ for any $t \in \mathbb{R}$, and $\tilde{W}_+^s$ is a graph on π.

Then we easily conclude the proof repeating the argument of the proof of the $1 < p \leq 2$ case. $\qquad \square$

We observe that in the $p = 2$ and $\delta \neq 0$ case Theorem 1.2 can be obtained in a simpler way combining Kelvin inversion and Fowler transformation.

Proof of Theorem 1.2 *in the $p = 2$, $\delta > -2$ case.*

Let us recall first the Kelvin inversion. If $u(r)$ is a solution of (1.2) then $\tilde{u}(s) = u(s^{-1})s^{2-n}$ solves

$$\frac{d}{ds}\left[s^{n-1}\frac{d\tilde{u}}{ds}(s)\right] + ks^{\tilde{\delta}}\tilde{u}^{q-1} = 0, \tag{3.11}$$

where $\tilde{\delta} = (n-2)(q-2^*) - \delta$. Notice that regular solutions $u(r;d)$ of (1.2) become fast decay solutions $\tilde{v}(r;d)$ of (3.11) and fast decay solutions $v(r;L)$ of (1.2) become regular solutions $\tilde{u}(r;L)$ of (3.11), and viceversa.

If we apply (2.1) to (3.11) with the new parameters $\tilde{\alpha} = -\gamma$, $\tilde{\gamma} = -\alpha$ ($\tilde{\beta} = \tilde{\alpha} + 1$) we obtain again an autonomous system of the form (2.2). More precisely if we denote by $x(t)$, $y(t)$, $z(t) = \dot{x}(t)$ the Fowler variables obtained from the original (1.2) and by $\tilde{x}(t)$, $\tilde{y}(t)$, $\tilde{z}(t) = \dot{\tilde{x}}(t)$ the Fowler variables obtained from (3.11) we pass from

$$\begin{cases} \dot{x} = z, \\ \dot{z} = -\alpha\gamma x + (\alpha + \gamma)z - kx^{q-1}, \end{cases} \tag{3.12}$$

to

$$\begin{cases} \frac{d\tilde{x}}{ds} = \tilde{z}, \\ \frac{d\tilde{z}}{ds} = -\alpha\gamma\tilde{x} - (\alpha + \gamma)\tilde{z} - k\tilde{x}^{q-1}. \end{cases} \tag{3.13}$$

We remand the interested reader to [14, pag. 521] for the tedious computation. Further if $q > 2^*(\delta)$, $q \geq 2^{JL}(\delta)$, $2_*(\delta) < q < 2_{jl}(\delta)$ in the original system then $q < 2^*(\tilde{\delta})$, $q \leq 2_{jl}(\tilde{\delta})$, $q > 2^{JL}(\tilde{\delta})$ for the system obtained after Kelvin inversion, and viceversa

We emphasise that if $(x(t), z(t))$ solves (3.12) then $(x(-t), -z(-t))$ solves (3.13) and viceversa, therefore the stable manifold W^s_+ of (3.13) where $2_*(\delta) < q \le 2_{jl}(\delta)$ is obtained from the unstable manifold $\tilde{W}^u_+$ of (3.12) where $q > 2^{JL}(\tilde{\delta})$ simply by a reflection with respect to the $z = 0$ axis. From Theorem 1.1 we know that $\tilde{W}^u_+$ is a graph on $\{(x, 0) \mid 0 < x < P_x\}$ and that it is contained in the $\tilde{z} > 0$ semi-plane, hence W^s_+ is a graph on $\{(x, 0) \mid 0 < x < P_x\}$ and that it is contained in the $\tilde{z} > 0$ semi-plane. So the proof of Theorem 1.2 easily follows reasoning as above. $\square$

From Theorems 1.1 and 1.2 , we easily deduce the following results which are useful in a parabolic context.

Let $O(r^a)$ as $r \to +\infty$ (respectively as $r \to 0$) denote a function such that $O(r^a)r^{-a}$ has a finite limit, possibly null as $r \to +\infty$ (respectively as $r \to 0$).

Corollary 3.6. *Assume **H** and $q > p^{JL}(\delta)$, $q \ge 2$, then for any $\varpi > 0$ the GS have the following expansion as $r \to +\infty$:*

$$u(r; d) = P_x r^{-\alpha} + c(d)r^{-\alpha - |\lambda_2|} + O(r^{-\alpha - |\lambda_1|} + r^{-\alpha - 2|\lambda_2| + \varpi}) \qquad (3.14)$$

where $c(d) < 0$, $c'(d) > 0$. Further if we set $w(r; d_0) = \frac{\partial}{\partial d}u(r; d)\rfloor_{d=d_0}$ we have $w(r; d_0) \ge 0$ for any $r \ge 0$ and $d_0 \ge 0$.

In the $p = 2$, $\delta = 0$ case Corollary 3.6 was an essential ingredient to construct sub-super solutions for (1.5); then these sub-super solutions allowed to prove interesting results concerning the rate of convergence of the solutions of (1.5) to the stationary GS, see e.g. [9, 19, 21] and references therein.

Proof. Since $u(r; d)$ is monotone increasing in d for any $r \ge 0$, we immediately find that $w(r; d_0) \ge 0$.

We claim that, as $r \to +\infty$, we find

$$u(r; 1) = P_x r^{-\alpha} + c(1)r^{-\alpha - |\lambda_2|} + O(r^{-\alpha - |\lambda_1|} + r^{-\alpha - 2|\lambda_2| + \varpi}) \qquad (3.15)$$

where $c(1) < 0$ and $\varpi > 0$ is an arbitrarily small positive constant.
Then from (2.10) we see that (3.14) holds and $c(d) = c(1)d^{-|\lambda_2|/\alpha}$, hence $c(d) < 0$, $c'(d) > 0$, and we prove the Corollary.

Now we prove the claim, using a new geometrical idea. Let $\phi(t) = (x(t), y(t))$ be the trajectory corresponding to $u(r; 1)$. From standard facts of invariant manifold theory, see e.g. [5, §13.4], we know that any trajectory converging to **P** as $t \to +\infty$ satisfies

$$\phi(t) = \boldsymbol{P} + b(1)\boldsymbol{v_2}e^{\lambda_2 t} + a(1)\boldsymbol{v_1}e^{\lambda_1 t} + \boldsymbol{R}(t) \qquad (3.16)$$

where $\boldsymbol{R}(t) = O(e^{(2\lambda_2 + \varpi)t})$, and $(a(1), b(1)) \in \mathbb{R}^2$. So we immediately see that $u(r; 1)$ can be expanded as in (3.15), but we have to show that $c(1) < 0$.
Generically, i.e. if $b(1) \ne 0$, a trajectory converging to **P** is tangent to the line $y = -m_2(x - P_x) + P_y$, cf. (2.7), but if $b(1) = 0$ then it is tangent to $y = -m_1(x - P_x) + P_y$. However from an inspection of the proof of Theorem 1.1 and of the positively invariant regions for $1 < p \le 2$ and for $p > 2$, we see that just the former possibility takes place, whence $b(1) \ne 0$. Further, since $x(t) - P_x < 0$ for any $t \in \mathbb{R}$ we see that $c(1) < 0$ and the claim is proved. $\square$

Corollary 3.7. *Assume* **H**, $p_*(\delta) < q < p_{jl}(\delta)$, *and* $q \geq 2$; *then for any* $\varpi > 0$, *the SGS have the following expansion as* $r \to 0$:

$$v(r; L) = P_x r^{-\alpha} + C(L)r^{-\alpha + |\lambda_1|} + O(r^{-\alpha + |\lambda_2|} + r^{-\alpha + 2|\lambda_1| - \varpi}) \qquad (3.17)$$

where $C(L) < 0$ *and* $C'(L) > 0$. *Further if we set* $W(r; L_0) = \frac{\partial}{\partial L} v(r; L)\lfloor_{L = L_0}$ *we have* $W(r; L_0) \geq 0$ *for any* $r \geq 0$ *and* $L_0 \geq 0$.

Proof. The proof is analogous to the one of Corollary 3.6. The fact that $W(r; L_0) \geq 0$ follows from the monotonicity in L of the SGS $v(r; L)$.

Let $\phi(t) = (x(t), y(t))$ be the trajectory corresponding to $v(r; 1)$. From standard facts of invariant manifold theory, see again [5, §13.4], any trajectory converging to P as $t \to -\infty$ satisfies (3.16) where $R(t) = O(e^{(2\lambda_1 - \varpi)t})$ as $t \to -\infty$, and $(a(1), b(1)) \in \mathbb{R}^2$. So we immediately see that $v(r; 1)$ can be expanded as in (3.17), but we have to show that $C(1) < 0$.
Again, from an inspection of the proof of Theorem 1.2 and of the positively invariant regions for $1 < p \leq 2$ and for $p > 2$, we see that $\phi(t)$ is tangent to the line $y = -m_1(x - P_x) + P_y$ (in fact to the semi-line r_1), whence $a(1) \neq 0$. Further, since $x(t) - P_x < 0$ for any $t \in \mathbb{R}$ we see that $C(1) < 0$ and the claim is proved. $\qquad \square$

Corollary 3.7 is again important in the parabolic context: when $p = 2$, $\delta = 0$, it is crucial to get local uniqueness results for the Cauchy problem with singular data, and to prove some stability properties of SGS, see the nice papers [22,28] and references therein.

4. Technical results

In this section we provide the detailed proof of several results needed for the main theorems.

4.1. Evaluation of the critical exponents $p_{jl}(\delta)$, $p^{JL}(\delta)$

In the following lines we perform the detailed computation needed to define the critical exponents $p_{jl}(\delta)$ and $p^{JL}(\delta)$. We recall that they are obtained as the positive numbers such that the eigenvalues $\lambda_1(q)$ and $\lambda_2(q)$ of $\frac{df}{d\phi}(P)$ defined in (2.6), have non-zero imaginary part for any $q \in (p_{jl}(\delta); p^{JL}(\delta))$.
Linearising the right hand side f of (2.2) in P, we see that the Jacobian $\frac{df}{d\phi}(P)$ satisfies

$$\frac{df}{d\phi}(P) = \begin{pmatrix} \alpha & \frac{|P_y|^{\frac{2-p}{p-1}}}{p-1} \\ -k(q-1)P_x^{q-2} & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \frac{(\alpha P_x)^{2-p}}{p-1} \\ -k(q-1)P_x^{q-2} & \gamma \end{pmatrix}. \quad (4.1)$$

The trace T of $\frac{df}{d\phi}(P)$ satisfies $T = \alpha + \gamma$, see (2.5), and the determinant D can be written as follows

$$
\begin{aligned}
D &= \alpha\gamma + \frac{k(q-1)}{p-1}\alpha^{2-p}P_x^{q-p} = \alpha\gamma + \frac{q-1}{p-1}\alpha|\gamma| = \alpha|\gamma|\frac{q-p}{p-1} \\
&= |\gamma|\frac{p+\delta}{p-1} = \left[\frac{n-p}{p-1} - \alpha\right](p+\delta).
\end{aligned}
\qquad (4.2)
$$

Hence $\frac{d\boldsymbol{f}}{d\phi}(\boldsymbol{P})$ has real eigenvalues if and only if $\Delta := T^2 - 4D \geq 0$ where Δ is given by the following expression:

$$\Delta = [\alpha p - (n - p)]^2 + 4(p + \delta)\left[\alpha - \frac{n - p}{p - 1}\right]. \tag{4.3}$$

Thus the critical values $p_{jl}(\delta) < p^{JL}(\delta)$ are obtained as the smallest and the largest root of the following second order equation in α:

$$\Delta(\alpha) = \alpha^2 p^2 - 2\alpha\left[p(n - p) - 2(p + \delta)\right] + (n - p)\left[(n - p) - 4\frac{p + \delta}{p - 1}\right] = 0. \tag{4.4}$$

From a lengthy but straightforward computation the discriminant Ψ of (4.4) is always positive, in fact we have

$$\Psi := 16(p + \delta)\left[\frac{p(n - p)}{p - 1} + (p + \delta)\right] > 0.$$

So the equation $\Delta(\alpha) = 0$ has always two solutions, say $\alpha_* < \alpha^*$. Further from (2.1) we see that $\alpha = \alpha(q)$ is monotone decreasing, and $\alpha(q) : (p_*(\delta); +\infty) \to (0, \frac{n-p}{p-1})$ is a bijection whose inverse is $Q(\alpha)$. Hence $p_{jl}(\delta) = Q(\alpha^*)$, while $p^{JL}(\delta) = Q(\alpha_*)$, however these values are defined only if $\alpha^* < \frac{n-p}{p-1}$ and $\alpha_* > 0$ respectively. The condition $\alpha_* > 0$, which guarantees the existence of $p^{JL}(\delta)$, can be written as follows (simply requiring the last term in (4.4) to be positive):

$$n > 4\frac{p + \delta}{p - 1} + p, \tag{4.5}$$

which gives back the known condition $n > 10$ for the classical Joseph-Lundgren exponent.

   The condition $\alpha^* < \frac{n-p}{p-1}$ which guarantees the existence of $p_{jl}(\delta)$, is less restrictive but more difficult to handle; it is certainly satisfied if

$$\alpha_* + \alpha^* = 2\frac{n - p}{p} - 4\frac{p + \delta}{p^2} < \frac{n - p}{p - 1},$$

which is equivalent to

$$(n - p)\frac{p - 2}{p - 1} < 4\frac{p + \delta}{p},$$

and it is always satisfied if $n > p$ and $1 < p \leq 2$.

4.2. Construction of W_+^u and W_+^s in a non smooth setting

In this section we prove again Lemmas 2.4, 2.7, 2.8 and Lemma 2.9 removing the simplifying assumption that (2.2) is C^1, i.e. we prove Proposition 2.10.

4.2.1. Existence and uniqueness of fast decay solutions in a non-smooth setting. Here, working directly on (1.2) we prove the existence and the uniqueness of fast decay solutions. In fact the existence has already been proved, e.g. in [11], but the uniqueness of fast decay solutions is new up to our knowledge.

   We recall that when (2.2) is C^1 these results may be obtained using invariant manifold tools by combining Lemma 2.8 and Lemma 2.9.

Theorem 4.1. *Assume $\boldsymbol{H}$ and $q > p_*(\delta)$. Then, for any $L > 0$ there exists a unique fast decay solution $v(r; L)$ of (1.2) and it is C^2 for r large enough.*

Proof. Let $\nu := (n-p)/(p-1)$. Fix $L > 0$ and set $\rho := (L\nu)^{p-1}$; let $r_0 > 0$ to be chosen later, we consider the following space:

$$\mathcal{A} = \{u \in \mathcal{C}([r_0, +\infty)) \mid u(r) \text{ is decreasing}, \qquad \lim_{r \to +\infty} u(r)r^\nu = L\}$$

endowed with the norm $\|v\|_\mathcal{A} = \sup\{\|v(r)r^\nu\| \mid r \geq r_0\}$. Then we consider the ball in $\mathcal{A}$ of radius $c > 0$ centered in $Lr^{-\nu}$, i.e.

$$\mathcal{B} = \{u \in \mathcal{A} \mid \|u(r) - Lr^{-\nu}\|_\mathcal{A} \leq L/2\}.$$

We introduce the integral operator $\mathcal{F} : \mathcal{B} \to \mathcal{A}$ defined as follows:

$$\mathcal{F}(u) := \int_r^{+\infty} \frac{1}{t^{\nu+1}} \left\{ \rho - k \int_t^{+\infty} s^{\delta+n-1} u^{q-1}(s)ds \right\}^{\frac{1}{p-1}} dt.$$

We claim that $v(r; L)$ is a fast decay solution of (1.2) if and only if it is a fixed point of $\mathcal{F}$ for a suitable $r_0 > 0$.

Let $v(r; L)$ be a fast decay solution; from Lemma 1.8 we see that $\lim_{r \to +\infty} |v'(r)|^{p-1} r^{n-1} = \rho := (L\nu)^{p-1}$. Hence from (1.2) we find

$$|v'(r)|^{p-1} r^{n-1} = \rho - k \int_r^{+\infty} s^{\delta+n-1} v^{q-1}(s)ds. \tag{4.6}$$

Since $\lim_{r \to +\infty} v'(r)r^{\nu+1} = -L\nu < 0$ we can find $r_0 > 0$ such that $v'(r) < 0$ for $r > r_0$. Hence integrating (4.6), for any $r > r_0$ we find $v(r) = \mathcal{F}(v)$.

The viceversa can be obtained from a straightforward computation. Then the claim is proved.

We will show that $\mathcal{F}$ maps $\mathcal{B}$ into itself and it is a contraction, then we obtain the existence and uniqueness of the fixed point of $\mathcal{F}$, and as a consequence of the above discussion Theorem 1.6 follows.

We begin by showing that $\mathcal{F}$ maps $\mathcal{B}$ into itself. Observe that

$$\left| \mathcal{F}(u) - Lr^{-\nu} \right| \leq$$

$$\leq \left| \int_r^{+\infty} \frac{1}{t^{\nu+1}} \left\{ L\nu - \left[\rho - k \int_t^{+\infty} s^{\delta+n-1} u^{q-1}(s)ds \right]^{\frac{1}{p-1}} \right\} dt \right| \tag{4.7}$$

$$\leq \int_r^{+\infty} \frac{1}{t^{\nu+1}} \left| \rho^{\frac{1}{p-1}} - \left[\rho - k \int_t^{+\infty} s^{\delta+n-1} u^{q-1}(s)ds \right]^{\frac{1}{p-1}} \right| dt.$$

Before to proceed we need to prove the following estimate: let $c_1 := \frac{k(2L)^{q-1}}{\nu[q-p_*(\delta)]}$ and set $r_1 := \left[\frac{2c_1}{\rho} \right]^{\frac{1}{\nu[q-p_*(\delta)]}}$ then, if $u \in \mathcal{B}$, for any $t > r_1$ we have

$$I_1(t) := k \int_t^{+\infty} s^{\delta+n-1} u^{q-1}(s)ds \leq c_1 t^{-\nu} \leq \frac{\rho}{2}. \tag{4.8}$$

In order to prove the previous estimate we first note that

$$\nu(p_*(\delta) - 1) = n + \delta > 0. \tag{4.9}$$

Further, if $u \in \mathcal{B}$ we have $|u(t)t^\nu| \leq L + L/2 \leq 2L$ for any $t > r_1$. Hence

$$I_1(t) \leq k(2L)^{q-1} \int_t^{+\infty} s^{\delta+n-1-\nu(q-1)} ds \leq k(2L)^{q-1} \int_t^{+\infty} s^{-\nu[q-p_*(\delta)]-1} ds$$

$$\leq \frac{k(2L)^{q-1}}{\nu[q-p_*(\delta)]} t^{-\nu[q-p_*(\delta)]} = c_1 t^{-\nu[q-p_*(\delta)]} < c_1 r_1^{-\nu[q-p_*(\delta)]} = \frac{\rho}{2}.$$

So the estimate (4.8) is proved.

Then, using the mean value theorem, for any $0 < \xi < \eta$ we find

$$
\begin{cases}
\left| \xi^{\frac{1}{p-1}} - \eta^{\frac{1}{p-1}} \right| \leq \frac{\xi^{\frac{2-p}{p-1}}}{p-1} |\xi - \eta|, & \text{if } p \geq 2, \\[2mm]
\left| \xi^{\frac{1}{p-1}} - \eta^{\frac{1}{p-1}} \right| \leq \frac{\eta^{\frac{2-p}{p-1}}}{p-1} |\xi - \eta|, & \text{if } 1 < p < 2.
\end{cases}
\tag{4.10}
$$

So, if $p \geq 2$, plugging (4.10) into (4.7), and using (4.8) we find

$$|\mathcal{F}(u) - Lr^{-\nu}| \leq \frac{\left(\frac{\rho}{2}\right)^{\frac{2-p}{p-1}}}{p-1} \int_r^{+\infty} \frac{k\,dt}{t^{\nu+1}} \int_t^{+\infty} s^{\delta+n-1} u^{q-1}(s)\,ds$$

$$\leq \frac{\left(\frac{\rho}{2}\right)^{\frac{2-p}{p-1}}}{p-1} \int_r^{+\infty} \frac{c_1}{t^{1+\nu[q-p_*(\delta)+1]}}\,dt \leq \frac{c_2 2^{\frac{p-2}{p-1}}}{r^{\nu[q-p_*(\delta)+1]}},$$

(4.11)

where $c_2 := \frac{c_1 \rho^{\frac{2-p}{p-1}}}{\nu[q-p_*(\delta)+1](p-1)}$. Similarly if $1 < p < 2$, plugging (4.10) into (4.7), and using (4.8) we find

$$|\mathcal{F}(u) - Lr^{-\nu}| \leq \frac{(\rho)^{\frac{2-p}{p-1}}}{p-1} \int_r^{+\infty} \frac{k\,dt}{t^{\nu+1}} \int_t^{+\infty} s^{\delta+n-1} u^{q-1}(s)\,ds$$

$$\leq \frac{(\rho)^{\frac{2-p}{p-1}}}{p-1} \int_r^{+\infty} \frac{c_1}{t^{1+\nu[q-p_*(\delta)+1]}}\,dt \leq \frac{c_2}{r^{\nu[q-p_*(\delta)+1]}}.$$

(4.12)

Therefore if we set $\bar{c} := \max\{2^{(p-2)/(p-1)}; 1\}$, $r_2 := \left[\frac{2\bar{c}c_2}{L}\right]^{\frac{1}{\nu[q-p_*(\delta)]}}$ and $r_a := \max\{r_1; r_2\}$, from (4.11) and (4.12) we find

$$\|\mathcal{F}(u) - Lr^{-\nu}\|_{\mathcal{A}} \leq \sup_{r \geq r_a} \{\bar{c}c_2 r^{-\nu[q-p_*(\delta)]}\} \leq \bar{c}c_2 r_a^{-\nu[q-p_*(\delta)]} \leq \frac{L}{2}. \tag{4.13}$$

Now we pass to prove that $\mathcal{F}$ is a contraction. Observe that

$$|\mathcal{F}(u_1) - \mathcal{F}(u_2)| \leq \int_r^{+\infty} \frac{I_2(t)}{t^{\nu+1}},$$

where

$$I_2(t) := \left| \left[\rho - k \int_t^{+\infty} s^{\delta+n-1} u_1^{q-1}(s)\,ds \right]^{\frac{1}{p-1}} - \left[\rho - k \int_t^{+\infty} s^{\delta+n-1} u_2^{q-1}(s)\,ds \right]^{\frac{1}{p-1}} \right|.$$

Then, using (4.10) and (4.6) and reasoning as in (4.11) and (4.12) we find

$$I_2(t) \leq \frac{k\bar{c}\rho^{\frac{2-p}{p-1}}}{p-1} \int_t^{+\infty} s^{\delta+n-1} \left| u_1^{q-1}(s) - u_2^{q-1}(s) \right| ds. \tag{4.14}$$

Since $u_1, u_2 \in \mathcal{B}$, using the mean value theorem we find $\tilde{u}(s)$ between $u_1(s)$ and $u_2(s)$ such that

$$
\begin{aligned}
\left| u_1^{q-1}(s) - u_2^{q-1}(s) \right| &\leq (q-1)[\tilde{u}(s)]^{q-2}|u_1(s) - u_2(s)| \\
&\leq (q-1)(2L)^{q-2}s^{-\nu(q-1)} \left\| u_1(s) - u_2(s) \right\|_{\mathcal{A}} .
\end{aligned}
\tag{4.15}
$$

Hence plugging (4.15) into (4.14), setting $c_4 := (q-1)(2L)^{q-2}\frac{k\bar{c}\rho^{\frac{2-p}{p-1}}}{p-1}$ and using (4.9) we find

$$
|\mathcal{F}(u_1) - \mathcal{F}(u_2)| \leq c_4 \int_r^{+\infty} \frac{1}{t^{\nu+1}} \left(\int_t^{+\infty} s^{\delta+n-1-\nu(q-1)} \left\| u_1 - u_2 \right\|_{\mathcal{A}} ds \right) dt
$$

$$
= c_4 \left\| u_1 - u_2 \right\|_{\mathcal{A}} \int_r^{+\infty} \frac{1}{t^{\nu+1}} \int_t^{+\infty} \frac{ds\,dt}{s^{\nu[q-p_*(\delta)]+1}} \leq c_5 \frac{\left\| u_1 - u_2 \right\|}{r^{\nu[q-p_*(\delta)+1]}},
$$

where $c_5 := \frac{c_4}{\nu^2[q-p_*(\delta)][q-p_*(\delta)+1]}$. Then, setting $r_b = (2c_5)^{\frac{1}{\nu[q-p_*(\delta)]}}$, for any $r > r_b$ we find

$$
\left\| \mathcal{F}(u_1) - \mathcal{F}(u_2) \right\|_{\mathcal{A}} \leq \left(\frac{r_b}{r} \right)^{\nu[q-p_*(\delta)]} \frac{\left\| u_1 - u_2 \right\|_{\mathcal{A}}}{2} < \frac{\left\| u_1 - u_2 \right\|_{\mathcal{A}}}{2}.
\tag{4.16}
$$

Hence for any $r > r_0 := \max\{r_a; r_b\}$ we find that both (4.13) and (4.16) hold, so $\mathcal{F}$ admits a unique fixed point, thanks to the Banach Theorem. $\qquad\square$

4.2.2. Construction of W_+^u and W_+^s via Wazewski's principle.
In this section we prove again Lemmas 2.4, 2.8 and Lemma 2.9 removing the simplifying assumption that (2.2) is C^1.

We begin by the analogous of Lemma 2.8, then we use an idea inspired by Wazewski's principle to construct compact and connected stable and unstable sets. Finally we combine these results with the ones of Sect. 4.2.1 to show that W_+^u and W_+^s are 1 dimensional manifolds.

Proposition 4.2. *Lemma* 2.8 *holds for any* $q > p_*(\delta)$, *whenever* $q > p > 1$.

The proof of these results can be found in [11, Lemmas 5.4, 5.5] in a non-autonomous context. The argument in [11] is based on an iterative application of Gronwall inequality. Here we give a new proof, which is much simpler and in fact can be adapted to the non-autonomous setting too.

Proof. If $u(r)$ is either a regular or a fast decay solution from (2.1) we immediately see that $\phi(t; \boldsymbol{Q}) \to (0,0)$ respectively as $t \to -\infty$ or as $t \to +\infty$.

So let us prove the viceversa: assume that $\boldsymbol{Q} \in W_+^u$ so that $\lim_{t \to -\infty} \phi(t; \boldsymbol{Q}) = (0,0)$. Then by an elementary phase-plane analysis it follows that there is $T \in \mathbb{R}$ such that $\dot{y}(t) < 0 < \dot{x}(t)$ for any $t \leq T$, where $\phi(t; \boldsymbol{Q}) = (x(t), y(t))$. Since $\dot{y}(t) < 0$ for any $t \leq T$, we find

$$
|y(t)| < \frac{k}{|\gamma|}x(t)^{q-1}.
\tag{4.17}
$$

Hence, plugging (4.17) in the expression of $\dot{x}$ we find

$$
\alpha x(t) - \left[\frac{k}{|\gamma|} \right]^{\frac{1}{p-1}} x(t)^{\frac{q-1}{p-1}} < \dot{x}(t) < \alpha x(t).
\tag{4.18}
$$

Let us denote by $\overline{x}(t)$ and by $\underline{x}(t)$ the solutions of the following scalar smooth ODEs:

$$\begin{cases} \dot{\overline{x}}(t) = \alpha\overline{x}(t) - \left[\frac{k}{|\gamma|}\right]^{\frac{1}{p-1}} \overline{x}(t)^{\frac{q-1}{p-1}}, \\ \overline{x}(T) = x(T), \end{cases} \qquad \begin{cases} \dot{\underline{x}}(t) = \alpha\underline{x}(t), \\ \underline{x}(T) = x(T). \end{cases}$$

We claim that $\underline{x}(t) < x(t) < \overline{x}(t)$ for any $t \leq T$.

In fact $\underline{x}(T) = x(T) = \overline{x}(T)$ by construction. Further from (4.18) it follows that $\dot{\underline{x}}(T) > \dot{x}(T) > \dot{\overline{x}}(T)$ so the inequality holds in a left neighborhood of $t = T$.

Assume by contradiction that there is $T_0 < T$ such that $\underline{x}(T_0) = x(T_0) = \overline{x}(T)$ and $\underline{x}(t) < x(t) < \overline{x}(t)$ for $T_0 < t < T$. Then from (4.18) it follows that $\dot{\underline{x}}(T_0) > \dot{x}(T_0) > \dot{\overline{x}}(T_0)$, hence $\underline{x}(t) > x(t) > \overline{x}(t)$ in a right neighbourhood of $t = T_0$; but this is a contradiction and the claim is proved.

Now from standard fact in ODE theory we see that both $\underline{x}(t)\mathrm{e}^{-\alpha t}$ and $\overline{x}(t)\mathrm{e}^{-\alpha t}$ have positive finite limit, say $0 < \underline{d} \leq \overline{d}$.

Hence there is $\rho > 0$ such that $\frac{\underline{d}}{2} < u(r) < 2\overline{d}$ for any $0 < r \leq \rho$. Since $u(r)$ is positive and decreasing we see that $\lim_{r\to 0} u(r) = d > 0$, where $\underline{d} \leq d \leq \overline{d}$ and the part of the Proposition concerning regular solutions is proved.

So let us assume now that $\mathbf{Q} \in W_+^s$ i.e. $\lim_{t\to+\infty}\boldsymbol{\phi}(t;\mathbf{Q}) = (0,0)$; then by an elementary phase-plane analysis we find $T \in \mathbb{R}$ such that $\dot{x}(t) < 0$ and $\dot{y}(t) > 0$ for any $t \geq T$, where $\boldsymbol{\phi}(t) = (x(t), y(t))$. Since $\dot{x}(t) < 0$ for any $t \geq T$, we find

$$x(t) < \frac{|y(t)|^{1/(p-1)}}{\alpha}. \tag{4.19}$$

Plugging (4.19) in $\dot{y}$, see (2.1), we find

$$\gamma y(t) - k\frac{|y(t)|^{\frac{q-1}{p-1}}}{\alpha^{q-1}} < \dot{y}(t) < \gamma y(t). \tag{4.20}$$

Let us denote by $\underline{y}(t)$ and by $\overline{y}(t)$ the solutions of the following scalar smooth ODEs:

$$\begin{cases} \dot{\underline{y}}(t) = \gamma\underline{y}(t) - k\frac{|y(t)|^{\frac{q-1}{p-1}}}{\alpha^{q-1}}, \\ \underline{y}(T) = y(T), \end{cases} \qquad \begin{cases} \dot{\overline{y}}(t) = \gamma\overline{y}(t), \\ \overline{y}(T) = y(T). \end{cases}$$

Reasoning as above we see that $\underline{y}(t) < y(t) < \overline{y}(t)$ for any $t \geq T$.

Then from standard fact in ODE theory we see that both $\underline{y}(t)\mathrm{e}^{-\gamma t}$ and $\overline{y}(t)\mathrm{e}^{-\gamma t}$ have negative finite limit, say $-\underline{\ell} \leq -\overline{\ell} < 0$.

Hence there is $\tau > T$ such that $-2\underline{\ell} < y(t)\mathrm{e}^{-\gamma t} < -\frac{\overline{\ell}}{2}$ for any $t \geq \tau$. Recalling that $y(\ln(r))r^{-\gamma} = -|v'(r)|^{p-1}r^{n-1}$ and that $\lim_{r\to+\infty}|v'(r)|^{p-1}r^{n-1}$ exists, see Lemma 1.8, we show that $v(r)$ is a fast decay solution and the Proposition is proved.

The proof concerning singular and slow decay solution proceeds as in the smooth case. $\qquad\square$

Now we show that W_+^u and W_+^s are non-empty compact sets. The argument can already be found in [13] in a more general setting: we repeat here a simplified version of the proof for completeness.
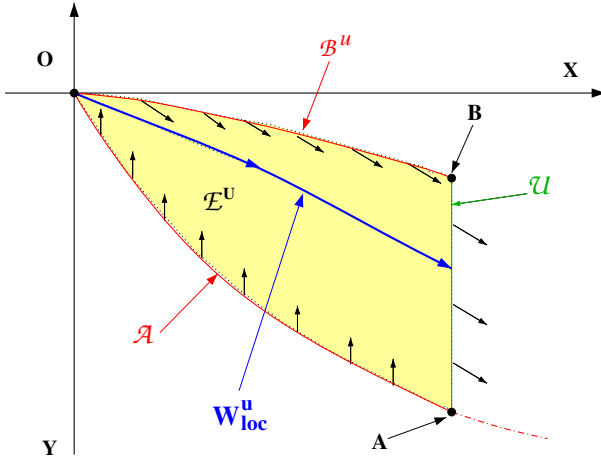
FIGURE 7. Construction of W_{loc}^u and illustration of Lemma 4.3

To construct W_+^u we look for a positively invariant triangular like set $\mathcal{E}^u$, then we conclude with a topological argument based on Wazewski's principle.

Let $A_x = \frac{P_x}{2}$ and $A_y = -\left(\frac{\alpha P_x}{2}\right)^{p-1}$ so that $\boldsymbol{A} = (A_x, A_y)$ is a point in the nullcine $\dot{x} = 0$. Then we set

$$\mathcal{A} := \{(x,y) \mid \alpha x - |y|^{\frac{1}{p-1}} = 0, \quad 0 < x \leq A_x, \quad y < 0\}. \tag{4.21}$$

We now consider the curve $y = -\varepsilon x^{q-1}$ where $\varepsilon > 0$ is a small constant to be fixed below. Let us set $B_y^u = -\varepsilon(A_x)^{q-1}$ and $\boldsymbol{B^u} = (A_x, B_y^u)$; we define the following sets:

$$\mathcal{B}^u := \{(x, -\varepsilon x^{q-1}), \quad 0 < x \leq A_x\}, \qquad \mathcal{U} := \{(A_x, y), \quad A_y \leq y \leq B_y^u\};$$
$$\mathcal{E}^u := \{(x,y) \mid -(\alpha x)^{p-1} \leq y \leq -kx^{q-1}, \quad 0 < x \leq A_x\}. \tag{4.22}$$

Then $\mathcal{E}^u \cup \{(0,0)\}$ is the compact set enclosed by $\mathcal{A}$, $\mathcal{B}^u$ and $\mathcal{U}$, see Fig. 7; notice that $\partial\mathcal{E}^u = \mathcal{A} \cup \mathcal{B}^u \cup \mathcal{U} \cup \{(0,0)\}$. Observe that (2.2) is C^1 in $\mathcal{E}^u$; further we have the following.

Lemma 4.3. *Let $0 < \varepsilon < \varepsilon_u := \frac{k}{|\gamma| + \alpha(q-1)}$; then the flow of (2.2) on $(\mathcal{A} \cup \mathcal{B}^u) \setminus \{\boldsymbol{A}, \boldsymbol{B}\}$ aims towards the interior of $\mathcal{E}^u$, while on $\mathcal{U}$ it aims towards the exterior of $\mathcal{E}^u$.*

Proof. If $\boldsymbol{Q} \in \mathcal{U}$ the proof is obtained simply by observing that, by construction, we are in the set where $\dot{x} > 0$. If $\boldsymbol{Q} \in \mathcal{A}$ it follows from a straightforward computation, or from the fact that we are on the branch of the nullcine $\dot{x} = 0$ between the origin and $\boldsymbol{A}$ (so on the left of $\boldsymbol{P}$). If $\boldsymbol{Q} = (x,y) \in \mathcal{B}^u$ we have to show that

$$D := \frac{d}{dt}\left\{y(t; \boldsymbol{Q}) + \varepsilon[x(t; \boldsymbol{Q})]^{q-1}\right\}\big\lfloor_{t=0} < 0. \tag{4.23}$$

In fact, using (2.2) and the fact that $y = -\varepsilon x^{q-1}$, we find

$$
\begin{aligned}
D = {} & \gamma y - kx^{q-1} + \varepsilon(q-1)x^{q-2}(\alpha x - |y|^{1/(p-1)}) \\
= {} & -\left(k - \varepsilon|\gamma| - \varepsilon\alpha(q-1)\right)x^{q-1} - \varepsilon(q-1)x^{q-2}|y|^{1/(p-1)} \\
< {} & -\left(k - \varepsilon|\gamma| - \varepsilon\alpha(q-1)\right)x^{q-1}.
\end{aligned}
$$

Hence $D < 0$ if $\varepsilon < \varepsilon_u$, so (4.23) follows and the Lemma is proved. □

Let us set

$$
W^u_{\mathrm{loc}} := \{\boldsymbol{Q} \mid \boldsymbol{\phi}(t; \boldsymbol{Q}) \in \mathcal{E}^u \text{ for any } t \le 0\}.
$$

A priori a trajectory may converge to the origin in finite time since local uniqueness on the coordinate axes is not ensured. However, from an inspection of the proof of Proposition 4.2, we find the following.

Lemma 4.4. *If $\boldsymbol{Q} \in W^u_{loc}$ then $\lim_{t \to -\infty} \boldsymbol{\phi}(t; \boldsymbol{Q}) = (0,0)$ and $x(t; \boldsymbol{Q}) > 0$ for any $t < 0$.*

From an elementary analysis of the phase portrait of (2.2) we easily obtain the viceversa.

Remark 4.5. *If $\lim_{t \to -\infty} \boldsymbol{\phi}(t; \boldsymbol{Q}) = (0,0)$ and $x(t; \boldsymbol{Q}) \ge 0$ for $t \ll 0$, there is $\tau \in \mathbb{R}$ such that $\boldsymbol{\phi}(t; \boldsymbol{Q}) \in W^u_{\mathrm{loc}}$ for any $t \le \tau$.*

Lemma 4.6. *Assume $q > p_*(\delta)$, then W^u_{loc} is a closed connected non-empty set.*

Proof. For any $\boldsymbol{Q} \in \tilde{\mathcal{U}} := \mathcal{U} \setminus W^u_{\mathrm{loc}}$ we define the functions

$$
\begin{aligned}
\tau(\boldsymbol{Q}) &:= \inf\{T \mid \boldsymbol{\phi}(t; \boldsymbol{Q}) \in \mathcal{E}^u \text{ for any } T < t \le 0\}, \\
\boldsymbol{\Gamma}(\boldsymbol{Q}) &:= \boldsymbol{\phi}(\tau(\boldsymbol{Q}); \boldsymbol{Q}).
\end{aligned}
$$

From Lemma 4.3 we find that if $\boldsymbol{Q} \in \tilde{\mathcal{U}}$ then either $\boldsymbol{\Gamma}(\boldsymbol{Q}) \in \mathcal{A}$ or $\boldsymbol{\Gamma}(\boldsymbol{Q}) \in \mathcal{B}^u$. Assume by contradiction that $W^u_{\mathrm{loc}} = \emptyset$ so that $\tilde{\mathcal{U}} = \mathcal{U}$. Notice that $\boldsymbol{\Gamma}^{-1}(\mathcal{A}) := \{\boldsymbol{Q} \in \mathcal{U} \mid \boldsymbol{\Gamma}(\boldsymbol{Q}) \in \mathcal{A}\}$ and $\boldsymbol{\Gamma}^{-1}(\mathcal{B}^u) := \{\boldsymbol{Q} \in \mathcal{U} \mid \boldsymbol{\Gamma}(\boldsymbol{Q}) \in \mathcal{B}^u\}$ are relatively open in $\mathcal{U}$, since the flow of (2.2) on $\mathcal{E}^u$ is C^1. Further $\tau(\boldsymbol{A}) = \tau(\boldsymbol{B}^u) = 0$ and $\boldsymbol{\Gamma}(\boldsymbol{A}) = \boldsymbol{A}$, $\boldsymbol{\Gamma}(\boldsymbol{B}) = \boldsymbol{B}$. Whence $\boldsymbol{\Gamma}^{-1}(\mathcal{A})$ and $\boldsymbol{\Gamma}^{-1}(\mathcal{B}^u)$ are both relatively open and nonempty (they contain respectively $\boldsymbol{A}$ and $\boldsymbol{B}^u$): since $\mathcal{U}$ is connected we have found a contradiction. Hence W^u_{loc} is a closed non-empty set.

Let $\boldsymbol{Q}^u \in (W^u_{\mathrm{loc}} \cap \mathcal{U})$, then by construction $w^u_{\mathrm{loc}} := \{\boldsymbol{\phi}(t; \boldsymbol{Q}^u) \mid t \le 0\}$ is a 1 dimensional manifold and $w^u_{\mathrm{loc}} \subset W^u_{\mathrm{loc}}$. Further, from elementary consideration on the phase portrait it is easy to check that W^u_{loc} is connected. □

Lemma 4.7. *Assume $q > p_*(\delta)$, $p > 1$, then W^u_+ is a 1 dimensional immersed manifold.*

Proof. We claim that $W^u_{\mathrm{loc}} \cap \mathcal{U}$ is a singleton say $\{\boldsymbol{Q}^u\}$. In fact assume by contradiction that there is $\boldsymbol{P}^u \in (W^u_{\mathrm{loc}} \cap \mathcal{U})$, $\boldsymbol{P}^u \ne \boldsymbol{Q}^u$. Let us denote by $w^u_P := \{\boldsymbol{\phi}(t; \boldsymbol{P}^u) \mid t \le 0\}$, and by $w^u_Q := \{\boldsymbol{\phi}(t; \boldsymbol{Q}^u) \mid t \le 0\}$; then by construction $w^u_P \subset W^u_{\mathrm{loc}}$ and $w^u_Q \subset W^u_{\mathrm{loc}}$. Further $w^u_Q \cap w^u_P = \emptyset$, due to local uniqueness of the solutions of (2.2). Finally if $\boldsymbol{R} \in (w^u_P \cup w^u_Q)$ then the trajectory $\boldsymbol{\phi}(t; \boldsymbol{R})$ of (2.2) corresponds to a regular solution $u(r; d(\boldsymbol{R}))$ of (1.2), where $d(\boldsymbol{R}) > 0$,

see Proposition 4.2. From Lemma 1.5 we see that $d(\boldsymbol{R_1}) \neq d(\boldsymbol{R_2})$ if $\boldsymbol{R_1} \neq \boldsymbol{R_2}$, i.e. $d(\cdot) : W_+^u \to [0, +\infty)$ is injective, so in particular $d(\boldsymbol{Q^u}) \neq d(\boldsymbol{P^u})$.

Assume to fix the ideas $d(\boldsymbol{Q^u}) > d(\boldsymbol{P^u})$. Using invariance for t-translations of (2.2) as in (2.9), we find that there are $\tilde{\boldsymbol{R}} \in w_Q^u$ and $\tilde{\tau} = \frac{1}{\alpha} \ln \left(\frac{d(\boldsymbol{Q^u})}{d(\boldsymbol{P^u})} \right) > 0$ such that

$$u(r; d(\tilde{\boldsymbol{R}}))r^\alpha = x(t; \tilde{\boldsymbol{R}}) = x(t - \tilde{\tau}; \boldsymbol{Q^u}) = u(re^{-\tilde{\tau}}; d(\boldsymbol{Q^u}))r^\alpha e^{-\alpha\tilde{\tau}}.$$

Hence

$$d(\tilde{\boldsymbol{R}}) = d(\boldsymbol{Q^u})e^{-\alpha\tilde{\tau}} = d(\boldsymbol{P^u}). \tag{4.24}$$

But this is a contradiction since $\tilde{\boldsymbol{R}} \in w_Q^u$ while $\boldsymbol{P^u} \in w_P^u$ and $w_Q^u \cap w_P^u = \emptyset$ and $d(\cdot)$ is injective; so the claim is proved and $W_{\text{loc}}^u \cap \mathcal{U}$ is a singleton.

Then we easily see that $W_+^u = w_Q^u = \{\phi(t; \boldsymbol{Q^u}) \mid t \in \mathbb{R}\}$, hence W_+^u is a 1 dimensional immersed manifold. $\qquad\square$

The construction of the stable set is completely analogous.

We consider the curve $x = \varepsilon|y|^{1/(p-1)}$ where $\varepsilon > 0$ is a small constant to be fixed below. Let us set $B_x^s = \varepsilon|A_y|^{1/(p-1)}$ and $\boldsymbol{B^s} = (B_x^s, A_y)$. Then we define the following sets.

$$\mathcal{B}^s := \{(\varepsilon|y|^{1/(p-1)}, y) \mid A_y \leq y < 0\}, \qquad \mathcal{S} := \{(x, A_y), \quad B_y^s \leq x \leq A_x\};$$
$$\mathcal{E}^s := \{(x, y) \mid \alpha x \leq |y|^{\frac{1}{p-1}} \leq \frac{x}{\varepsilon}, \quad A_y \leq y < 0\}. \tag{4.25}$$

Notice that $\mathcal{E}^s \cup \{(0,0)\}$ is the compact set enclosed by $\mathcal{A}$, $\mathcal{B}^s$, $\mathcal{S}$ and $\partial\mathcal{E}^s = \mathcal{A} \cup \mathcal{B}^s \cup \mathcal{S} \cup \{(0,0)\}$, see Fig. 8. Again (2.2) is C^1 in $\mathcal{E}^s$; further we have the following.

Lemma 4.8. *Let* $0 < \varepsilon < \varepsilon_s := \frac{p-1}{|\gamma|+\alpha(p-1)}$; *then the flow of* (2.2) *on* $(\mathcal{A} \cup \mathcal{B}^s) \setminus \{\boldsymbol{A}, \boldsymbol{B}\}$ *aims towards the exterior of* $\mathcal{E}^s$, *while on* $\mathcal{S}$ *it aims towards the interior of* $\mathcal{E}^s$.

Proof. Observe that if $0 < x < P_x$ the nullcline $\dot{x} = 0$ lies below the nullcline $\dot{y} = 0$, since $q > p$, cf Fig. 4; hence if $\boldsymbol{Q} \in \mathcal{S}$ the proof is obtained simply by observing that we are in the set where $\dot{y} > 0$ by construction. If $\boldsymbol{Q} \in \mathcal{A}$ it follows from the fact that we are on the nullcline $\dot{x} = 0$ between the origin and $\boldsymbol{A}$ (so on the left of $\boldsymbol{P}$).

If $\boldsymbol{Q} = (x, y) \in \mathcal{B}^s$ we have to show that

$$D^s := \frac{d}{dt}\left\{\varepsilon^{p-1}y(t; \boldsymbol{Q}) + x(t; \boldsymbol{Q})^{p-1}\right\}\lfloor_{t=0} < 0. \tag{4.26}$$

In fact, using (2.2) and the fact that $|y|^{1/(p-1)} = x/\varepsilon$, we find

$$D^s = \varepsilon^{p-1}(\gamma y - kx^{q-1}) + (p-1)x^{p-2}(\alpha x - |y|^{1/(p-1)})$$
$$= \left(|\gamma| + (p-1)\alpha - \frac{p-1}{\varepsilon}\right)x^{p-1} - \varepsilon^{p-1}kx^{q-1}$$
$$< \left(|\gamma| + (p-1)\alpha - \frac{p-1}{\varepsilon}\right)x^{p-1}.$$

Hence $D^s < 0$ if $0 < \varepsilon < \varepsilon_s$, so (4.26) follows and the Lemma is proved. $\qquad\square$
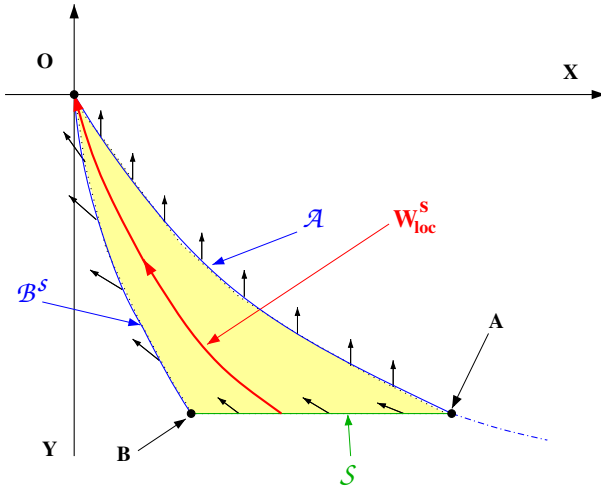
FIGURE 8. Construction of W_{loc}^s and illustration of Lemma 4.8

Once again priori a trajectory may converge to the origin in finite time since local uniqueness on the coordinate axes is not ensured. However, from an inspection of the proof of Proposition 4.2, we find the following.

Lemma 4.9. *If $Q \in W_{loc}^s$ then $\lim_{t \to +\infty} \phi(t; Q) = (0,0)$ and $x(t; Q) > 0$ for any $t > 0$. Conversely if $\lim_{t \to +\infty} \phi(t; Q) = (0,0)$ and $x(t; Q) \geq 0$ for $t \gg 0$ then there is $\tau \in \mathbb{R}$ such that $\phi(\tau; Q) \in W_{loc}^s$.*

Now using Lemma 4.8 and arguing as in Lemma 4.6 we obtain the following.

Lemma 4.10. *Assume $q > p_*(\delta)$, then W_{loc}^s is a compact connected non-empty set.*

Lemma 4.11. *Assume $q > p_*(\delta)$, then W_+^s is a 1 dimensional immersed manifold.*

Proof. We just need to show that $W_{loc}^s \cap \mathcal{S}$ is a singleton, say $\{Q^s\}$; then we conclude the proof arguing as in Lemma 4.7.

Assume by contradiction that there is $P^s \in (W_{loc}^s \cap \mathcal{S})$, $P^s \neq Q^s$. Denote by $w_P^s := \{\phi(t; P^s) \mid t \geq 0\} \subset \mathcal{E}^s$, and by $w_Q^s := \{\phi(t; Q^s) \mid t \geq 0\} \subset \mathcal{E}^s$: by construction $w_P^s \subset W_+^s$ and $w_Q^s \subset W_+^s$, and $w_P^s \cap w_Q^s = \emptyset$, due to local uniqueness of the solutions of (2.2) outside the coordinate axes. If $R \in (w_P^s \cup w_Q^s)$ then the trajectory $\phi(t; R)$ of (2.2) corresponds to a fast decay solution $v(r; L(R))$ of (1.2), where $L(R) > 0$. From Theorem 1.6 we see that $L(\cdot) : W_+^s \to [0, +\infty)$ is injective, so in particular $L(Q^s) \neq L(P^s)$. Assume to fix the ideas $L(Q^s) > L(P^s)$; using invariance for t-translations of (2.2) as in (2.11), we find that there are $\tilde{R} \in w_Q^s$ and $\tilde{\tau} = \frac{p-1}{n-p-\alpha(p-1)} \ln\left(\frac{L(P^s)}{L(Q^s)}\right)$ such

that

$$v(r; L(\tilde{\boldsymbol{R}}))r^{\alpha} = x(t; \tilde{\boldsymbol{R}}) = x(t - \tilde{\tau}; \boldsymbol{Q^s}) = v(r\mathrm{e}^{-\tilde{\tau}}; L(\boldsymbol{Q^s}))r^{\alpha}\mathrm{e}^{-\alpha\tilde{\tau}}.$$

Hence, cf. (2.11), we find

$$L(\tilde{\boldsymbol{R}}) = L(\boldsymbol{Q^s})\mathrm{e}^{\frac{n-p-\alpha(p-1)}{p-1}\tilde{\tau}} = L(\boldsymbol{P^s}). \tag{4.27}$$

But this is a contradiction since $\tilde{\boldsymbol{R}} \in w_Q^s$ while $\boldsymbol{P^s} \in w_P^s$, $w_Q^s \cap w_P^s = \emptyset$, and L is injective so the Lemma is proved. $\qquad\square$

Remark 4.12. Lemma 2.4 in the general $p > 1$ case now immediately follows from Lemmas 4.7 and 4.11 , while Lemma 2.7 is easily obtained observing that by construction $W_{\mathrm{loc}}^u \subset \mathcal{E}^u$ and $W_{\mathrm{loc}}^s \subset \mathcal{E}^s$.

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