Some approximate relations in the photoelasticity of strongly-anisotropic crystals

(Article begins on next page)

Received <day> <Month>, <year>; | Revised <day> <Month>, <year>; | Accepted <day> <Month>, <year>

DOI: xxx/xxxx

RESEARCH ARTICLE

Some approximate relations in the photoelasticity of strongly-anisotropic crystals

Fabrizio Daví

[1]Dipartimento di Ingegneria Civile, Edile ed Architettura and Interdipartimental Research Center for Global Analysis of Crystals, Universitá Politecnica delle Marche, 60131 Ancona, Italy,Italy

Correspondence
Email: davi@univpm.it

Abstract

The knowledge of the principal refractive indeces and their dependence on the applied stress are the starting point for any experimental analysis of transparent crystals. Since for strongly-anisotropic crystals and general state of stress an explicit analytical solution can be difficult to obtain, then we need some approximated solutions. Here we propose a coherent approximation procedure which generalizes results already obtained and give a simple example of application.

KEYWORDS:
Photoelastic crystals; Anisotropic materials

1 | INTRODUCTION

The optical properties of transparent crystals can be completely described by the *optical indicatrix* or Fresnel ellipsoid, *i.e.* the locus of normalized constant dielectric energy [1],[2]:

$$\mathbf{B}\mathbf{x}\cdot\mathbf{x} = 1\,, \tag{1}$$

where **B** is the second-order positive-definite and symmetric *inverse permittivity* tensor, whose principal values B_k are related to the *refractive indeces* $n_k > 1$ by

$$B_k = n_k^{-2}\,, \quad k = 1, 2, 3\,. \tag{2}$$

A well-known property of (1) is that for a given direction of light propagation, each section trough the origin and orthogonal to the light direction is an ellipse whose principal axis $(B_a\,, B_b)$ represents the two different refractive indeces $(n_a\,, n_b)$ associated with such a direction. Accordingly for any direction of propagation we have two differente rays with different velocities $(v_a = c/n_a\,, v_b = c/n_b)$, a phenomena called double refraction and which is measured by the *birifringence* $\Delta\, n = n_a - n_b$.

In the general case, when **B** has three distinct eigenvalues then there exist at most two directions, called the *optic axes*, such that the ellipse degenerates into a circle and there is no birifringence; the angle 2φ between these axis is called the *optic angle* and we say *optically biaxial* a crystal with two optic axis. The plane spanned by these optic axis is called *Optic plane*. When **B** has two equal eigenvalues the crystal is *optically uniaxial* and there is an unique optic axis, whereas when all the eigenvalues are equal there are no optic axis and the material is *optically isotropic*. The optic angle is related to the principal values of **B** by the relation

$$\sin\varphi = \sqrt{\frac{B_1 - B_2}{B_1 - B_3}}\,, \tag{3}$$

or other equivalent relations [1],[2].

The raylights generate within the crystals ”surfaces of equal phase difference” or *Bertin surfaces* (*vid. e.g.* [1] and [7] for a general study) which generate, on a projection plane, families of isochromate interference fringes. On suitable planes, these fringes are fourth-order symmetric closed curves, the so-called *Cassini-like* curves; accordingly it make sense to introduce a measurable parameter, namely the *Ellipticity Ratio* C, first introduced into[3] and further studied in[4] and[5]:

$$C = \frac{a}{b} - 1\,, \tag{4}$$

where $a > b$ are the semi-axes of the Cassini-like curves.

The refractive indeces, the optic angle and the ellipticity ratio can be measured experimentally by the means of optical techniques and characterize the optical properties of a crystal.

A crystal is *linearly photoelastic* when $\mathbf{B}$ is a linear function of the stress tensor $\mathbf{T}$ (a relation which is credited to Maxwell):

$$\mathbf{B}(\mathbf{T}) = \mathbf{B}_o + \mathbb{M}[\mathbf{T}]\,, \tag{5}$$

where $\mathbf{B}_o$ denotes the inverse permittivity of the unstressed material and $\mathbb{M}$ is the fourth-order *piezo-optic* tensor. Accordingly the refractive index, the optic angle and the photoelastic constant are function of the stress which can be expressed as:

$$\begin{aligned} n_j(\mathbf{T}) &= n_j^o + \mathbf{H}^{(j)} \cdot \mathbf{T} + o(\|\mathbf{T}\|^2)\,, \quad j = 1,2,3\,,\\ \varphi(\mathbf{T}) &= \varphi_o + \mathbf{K} \cdot \mathbf{T} + o(\|\mathbf{T}\|^2)\,,\\ C(\mathbf{T}) &= C_o + \mathbf{F} \cdot \mathbf{T} + o(\|\mathbf{T}\|^2)\,; \end{aligned} \tag{6}$$

here $\mathbf{F}$ denotes the tensor of *photoelastic constants*, first introduce in[4], $\mathbf{H}^{(j)}$, $\mathbf{K}$ are four tensors which depend on $\mathbf{B}_o$ and $\mathbb{M}$ and n_j^o, φ_o, C_o are the corresponding values in the unstressed state.

The problem of finding the set $\{\mathbf{F}, \mathbf{K}, \mathbf{H}^{(j)}\}$ for anisotropic materials was addressed and partially solved into[4] and[5] (*vid. also*[7] for a general approach to the problem). However, for strongly anisotropic material and for general state of stress the analytical determination of these quantities is complex[6] and the analysis done there left out crystals of the Triclinic group and was limited for other crystallographic groups to simple state of stress.

Such an approach makes sense (aside for the Triclinic case) when the stress are the result of an applied external load and are therefore know quantities: however, one of the most important concerns in crystals for optical applications is the presence of residual stress generated by the crystal growth process and by the subsequent cut and polishing of the crystal boule. Residual stress are completely unknown and indeed the photoelastic-based experimental techniques are widely used in order to detect and evaluate the presence of residual stress.

In[2], §.20, an approximate technique for the evaluation of the eigenvalues of $\mathbf{B}(\mathbf{T})$ was proposed. In this paper, by using the language and the notation of the tensor algebra, we took the same issue and arrive at a general and coherent approximation scheme in order to obtain the quantities defined by (6) for a general state of stress and for any anisotropic material and under weaker and more general hypothesis than in[2]: we further show how the relations proposed there can be obtained within our approximation. We shall finish by studying a relevant examples for Monoclinic crystals.

1.1 | Notation

We shall denote with $\mathcal{V}$ the vector space isomorphous with $\mathbb{R}^3$, whose elements we denote with small boldface as $\mathbf{a}$; with Lin we denote the space of second-order tensors which map $\mathcal{V}$ into itself and whose typical element we denote with capital boldface as $\mathbf{A}$. With Sym and Skw we respectively denote the subspaces of symmetric and skew-symmetric second order tensors with $\mathrm{Lin} = \mathrm{Sym} \oplus \mathrm{Skw}$. For $\mathbf{A} \in \mathrm{Lin}$ we define its symmetric and skw-symmetric parts by:

$$\mathrm{sym}\,\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)\,, \quad \mathrm{skw}\,\mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)\,. \tag{7}$$

For any $\mathbf{w} \in \mathcal{V}$ we define its unique axial tensor $\mathbf{W} \in \mathrm{Skw}$ as:

$$\mathbf{W}\mathbf{a} = \mathbf{w} \times \mathbf{a}\,, \quad \forall \mathbf{a} \in \mathcal{V}\,. \tag{8}$$

We denote with Sym^+ the subspace of symmetric and positive definite tensor:

$$\mathrm{Sym}^+ \equiv \{\mathbf{A} \in \mathrm{Sym} \mid \mathbf{A}\mathbf{u} \cdot \mathbf{u} > 0\,, \quad \forall \mathbf{u} \in \mathcal{V}/\{\mathbf{0}\}\}\,, \tag{9}$$

With Rot we denote the group of the proper orthogonal tensors (rotations), *i.e.*:

$$\mathrm{Rot} \equiv \{\mathbf{R} \in \mathrm{Lin} \mid \mathbf{R}^{-1} = \mathbf{R}^T\,, \quad \det \mathbf{R} = 1\,\}\,; \tag{10}$$

for any $\mathbf{R} \in \mathrm{Rot}$, let $(\theta, \boldsymbol{\omega})$ with $\|\boldsymbol{\omega}\| = 1$ be the rotation angle and the axis of rotation, then $\mathbf{R}$ can by expressed by the means of the Rodriguez formula:

$$\mathbf{Q}(\theta, \boldsymbol{\omega}) = \mathbf{I} + \sin\theta \mathbf{W} + (1 - \cos\theta)\mathbf{W}^2 \,; \tag{11}$$

where $\mathbf{W}$ is the axial tensor of $\boldsymbol{\omega}$. For $\mathbf{Q} \in \mathrm{Rot}$ we shall denote the *orthogonal conjugator* $\mathbb{Q}(\mathbf{Q})$ the fourth-order tensor such that:

$$\mathbb{Q}(\mathbf{Q})[\mathbf{A}] = \mathbf{Q}^T \mathbf{A} \mathbf{Q} \,, \quad \forall \mathbf{A} \in \mathrm{Lin} \,. \tag{12}$$

For $\mathbf{e} \in \mathcal{V}$, $\|\mathbf{e}\| = 1$ we define the two projectors

$$\mathbf{P}(\mathbf{e}) = \mathbf{e} \otimes \mathbf{e} \,, \quad \mathbf{P}^{\perp}(\mathbf{e}) = \mathbf{I} - \mathbf{e} \otimes \mathbf{e} \,, \tag{13}$$

which respectively project any $\mathbf{v} \in \mathcal{V}$ on the direction $\mathbf{e}$ and on the plane orthogonal to $\mathbf{e}$.

2 | AN APPROXIMATE RELATION FOR THE EIGENVALUES OF B(T)

2.1 | The eigenvalues of B(T)

We shall make use of three different and related orthonormal frames in $\mathcal{V}$: a frame $\mathcal{S} \equiv \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ which is related to the crystallographic directions; a frame $\Sigma \equiv \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which is the principal frame for the inverse permittivity tensor in the unstressed state $\mathbf{B}_o$ and a third frame $\Sigma_{\mathbf{B}} \equiv \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ which is the principal frame for the inverse permittivity tensor in the stressed state, $\mathbf{B}(\mathbf{T})$. The three frames are related by

$$\mathbf{e}_k = \mathbf{R}\mathbf{u}_k \,, \quad k = 1, 2, 3 \,, \quad \mathbf{R} \in \mathrm{Rot} \,, \tag{14}$$

and

$$\mathbf{w}_k = \mathbf{Q}\mathbf{e}_k \,, \quad k = 1, 2, 3 \,, \quad \mathbf{Q} \in \mathrm{Rot} \,. \tag{15}$$

We start from the Maxwell relation (5)

$$\mathbf{B}(\mathbf{T}) = \mathbf{B}_o + \mathbb{M}[\mathbf{T}] \,, \tag{16}$$

where $\mathbf{T} \in \mathrm{Sym}$ is the Cauchy stress tensor, $\mathbb{M} : \mathrm{Sym} \to \mathrm{Sym}$, is the fourth-order piezo-optic tensor, $\mathbf{B}_o \in \mathrm{Sym}^+$, $\mathbf{B}(\mathbf{T}) \in \mathrm{Sym}^+$; their components in the frame $\mathcal{S}$ read:

$$B_{ij}(\mathbf{T}) = B_{ji}(\mathbf{T}) \,, \quad B^o_{ij} = B^o_{ji} \,, \quad \mathbb{M}_{ijhk} = \mathbb{M}_{jihk} = \mathbb{M}_{ijkh} \,, \quad T_{ij} = T_{ji} \,. \tag{17}$$

The components of (16) in the principal frame Σ, since (14) holds, are given by:

$$\begin{aligned} \mathbf{B}(\mathbf{T}) \cdot \mathbf{e}_i \otimes \mathbf{e}_j &= \mathbf{B}(\mathbf{T}) \cdot \mathbf{R}\mathbf{u}_i \otimes \mathbf{R}\mathbf{u}_j = \mathbb{Q}(\mathbf{R})\mathbf{B}(\mathbf{T}) \cdot \mathbf{u}_i \otimes \mathbf{u}_j \,, \\ &= (\mathbb{Q}(\mathbf{R})\mathbf{B}_o + \mathbb{Q}(\mathbf{R})\mathbb{M}[\mathbf{T}]) \cdot \mathbf{u}_i \otimes \mathbf{u}_j \,; \end{aligned} \tag{18}$$

if we set

$$\hat{\mathbf{B}} = \mathbb{Q}(\mathbf{R})\mathbf{B}_o \,, \quad \hat{\mathbb{M}} = \mathbb{Q}(\mathbf{R})\mathbb{M} \,, \tag{19}$$

whose component in the frame $\mathcal{S}$ are given by

$$\hat{B}_{ij} = \begin{cases} B^o_{hk} R_{hi} R_{kj} = \hat{B}_i \,, i = j \,, \\ B^o_{hk} R_{hi} R_{kj} = 0 \,, i \neq j \,, \end{cases} \qquad \hat{\mathbb{M}}_{ijlm} = \mathbb{M}_{hklm} R_{hi} R_{kj} \,, \tag{20}$$

then $\mathbf{B}_o$ and $\mathbb{M}[\mathbf{T}]$ have representation in the principal frame Σ:

$$\mathbf{B}_o = \sum_{k=1}^{3} \hat{B}_k \mathbf{e}_k \otimes \mathbf{e}_k \,, \quad \mathbb{M}[\mathbf{T}] = \hat{\mathbb{M}}_{ijhk} T_{hk} \mathbf{e}_i \otimes \mathbf{e}_j \,. \tag{21}$$

The components in $\Sigma_{\mathbf{B}}$, which is the principale frame for $\mathbf{B}(\mathbf{T})$, are given by:

$$B_{kh} = \mathbf{B}(\mathbf{T}) \cdot \mathbf{w}_k \otimes \mathbf{w}_h = \mathbb{Q}(\mathbf{Q})\mathbf{B}(\mathbf{T}) \cdot \mathbf{e}_k \otimes \mathbf{e}_h = \begin{cases} B_k \,, & h = k \,, \\ 0 \,, & h \neq k \,; \end{cases} \tag{22}$$

relation (22) allows to evaluate the principal values of $\mathbf{B}(\mathbf{T})$ provided the two rotations $\mathbf{R}$ and $\mathbf{Q}$ are given.

We remark that, for Triclinic crystals, the rotation $\mathbf{R}$ can't be obtained in an explicit form and must be calculated either numerically if we know the values of the components B^o_{ij} or experimentally if we are able to measure the principal refraction index or as in[6] by the means of a point-dipole model.

For Monoclinic crystals, instead, it is customary to assume one of the vectors of $\mathcal{S}$ directed as the monoclinic $b-$axis, typically either $\mathbf{u}_2$ or $\mathbf{u}_3$ [8]. In such a case $\mathbf{R}$ is a rotation about the $b-$axis whose amplitude ψ depends on the non-null components of $\mathbf{B}_o$ and accordingly we can give the explicit expressions for the components $\hat{B}_{ij}$ and $\hat{\mathbb{M}}_{ijlm}$.

For all the other crystallographic classes, the frame $\mathcal{S} \equiv \Sigma$ (*vid.* [8]) and therefore $\mathbf{R} = \mathbf{I}$, $\mathbb{Q}(\mathbf{I}) = \mathbb{I}$ with $\hat{B}_{ij} = B^o_{ij}$ and $\hat{\mathbb{M}}_{ijlm} = \mathbb{M}_{ijlm}$.

2.2 | Small stress and the infinitesimal rotation of $\mathbf{B}_o$

We deal mainly with brittle crystals whose ultimate tensile stress is generally "small" and whose piezo-optic tensor has, in the case were they are measured (*vid.* [9]), components whose values are "small" in such a way the product between stress and piezo-optic coefficients is significatively smaller than the principal values of $\mathbf{B}_o$. Accordingly it makes sense to consider situations in which:[1]

$$\|\mathbf{B}(\mathbf{T}) - \mathbf{B}_o\| << 1\,, \tag{23}$$

an hypothesis which implies tacitly that the rotation tensor $\mathbf{Q}$ which brings Σ into $\Sigma_{\mathbf{B}}$ represents an infinitesimal rotation.

To make thing more precise, when the angle is infinitesimal (11) reduces to

$$\mathbf{Q}(\theta\,,\boldsymbol{\omega}) = \mathbf{I} + \theta\mathbf{W} + O(\theta^2)\,; \tag{24}$$

in the frame Σ the non-null components of $\mathbf{W}$ can be expressed in terms of the components $\omega_k = \boldsymbol{\omega}\cdot\mathbf{e}_k$ by:

$$W_{21} = -W_{12} = \omega_3\,,\quad W_{13} = -W_{31} = \omega_2\,,\quad W_{32} = -W_{23} = \omega_1\,. \tag{25}$$

and therefore, by (16) and (21), then $(22)_1$ reduces, to within higher-order terms in $o(\theta^2)$, to

$$B_k = (\mathbf{B}(\mathbf{T}) + \theta(\mathbf{B}(\mathbf{T})\mathbf{W} - \mathbf{W}\mathbf{B}(\mathbf{T}))\cdot\mathbf{e}_k\otimes\mathbf{e}_k\,, \tag{26}$$

whereas, to the same degree of approximation, the eigenvectors are given by:

$$\mathbf{w}_k = \mathbf{e}_k + \theta\mathbf{W}\mathbf{e}_k = \mathbf{e}_k + \theta\boldsymbol{\omega}\times\mathbf{e}_k\,. \tag{27}$$

By (16) and (26) we get, for k fixed:

$$\begin{aligned} B_k &= \mathbf{B}_o\cdot\mathbf{e}_k\otimes\mathbf{e}_k + \mathbb{M}[\mathbf{T}]\cdot\mathbf{e}_k\otimes\mathbf{e}_k \\ &+ \theta(\mathbf{B}_o\mathbf{W} - \mathbf{W}\mathbf{B}_o)\cdot\mathbf{e}_k\otimes\mathbf{e}_k + \theta(\mathbb{M}[\mathbf{T}]\mathbf{W} - \mathbf{W}\mathbb{M}[\mathbf{T}])\cdot\mathbf{e}_k\otimes\mathbf{e}_k \\ &= \mathbf{B}_o\cdot\mathbf{e}_k\otimes\mathbf{e}_k + \mathbb{M}[\mathbf{T}]\cdot\mathbf{e}_k\otimes\mathbf{e}_k \\ &+ 2\theta\,\mathrm{sym}(\mathbf{B}_o\mathbf{W})\cdot\mathbf{e}_k\otimes\mathbf{e}_k + 2\theta\,\mathrm{sym}(\mathbb{M}[\mathbf{T}]\mathbf{W})\cdot\mathbf{e}_k\otimes\mathbf{e}_k\,, \end{aligned} \tag{28}$$

which in the components in the frame Σ read:

$$B_k = \hat{B}_k + \left(\hat{\mathbb{M}}_{kkij} + 2\theta W_{hk}\hat{\mathbb{M}}_{hkij}\right)T_{ij}\,. \tag{29}$$

Accordingly we have these approximate values for the principal components of $\mathbf{B}(\mathbf{T})$:

$$\begin{aligned} B_1(\mathbf{T}) &= \hat{B}_1 + \mathcal{M}_{1ij}T_{ij} \\ B_2(\mathbf{T}) &= \hat{B}_2 + \mathcal{M}_{2ij}T_{ij} \\ B_3(\mathbf{T}) &= \hat{B}_3 + \mathcal{M}_{3ij}T_{ij}\,. \end{aligned} \tag{30}$$

provided we define the third-order matrix $\mathcal{M}_{ijk}$ as

$$\begin{aligned} \mathcal{M}_{1ij} &= \hat{\mathbb{M}}_{11ij} + 2\theta\omega_3\hat{\mathbb{M}}_{12ij} + 2\theta\omega_2\hat{\mathbb{M}}_{13ij}\,, \\ \mathcal{M}_{2ij} &= \hat{\mathbb{M}}_{22ij} + 2\theta\omega_1\hat{\mathbb{M}}_{23ij} + 2\theta\omega_3\hat{\mathbb{M}}_{12ij}\,, \\ \mathcal{M}_{3ij} &= \hat{\mathbb{M}}_{33ij} + 2\theta\omega_1\hat{\mathbb{M}}_{23ij} + 2\theta\omega_2\hat{\mathbb{M}}_{13ij}\,. \end{aligned} \tag{31}$$

[1] For instance, the Lead-Tungstate $PbWO_4$ (PWO) has a measured ultimate tensile stress $T_u = 26 \div 32$ MPa [10]; the piezo-optic coefficients measured in [11] are of the order $\mathbb{M} = 26\cdot 10^{-12}$ Pa. Since the minimum eigenvalue of $\mathbf{B}_o$ is, on the wavelenght range $\lambda = 375 \div 700$ nm, $B_{min} = 0.189 \div 0.215$ [12], then $\|\mathbf{B}(\mathbf{T}) - \mathbf{B}_o\| \approx \mathbb{M}T_u = 26\cdot 10^{-6}$ a value which is significatively smaller than B_{min}.

Moreover, since $\Sigma_{\mathbf{B}}$ is a principal frame, then from $(22)_2$ we must have:

$$(\mathbf{B}(\mathbf{T}) + \theta(\mathbf{B}(\mathbf{T})\mathbf{W} - \mathbf{W}\mathbf{B}(\mathbf{T})) \cdot \mathbf{e}_i \otimes \mathbf{e}_j = 0\,, \quad i \neq j\,, \tag{32}$$

which yields

$$\mathbb{M}[\mathbf{T}] \cdot \mathbf{e}_i \otimes \mathbf{e}_j + 2\theta\, \mathrm{sym}(\mathbf{B}_o\mathbf{W} + \mathbb{M}[\mathbf{T}]\mathbf{W}) \cdot \mathbf{e}_i \otimes \mathbf{e}_j = 0\,, \quad i \neq j\,. \tag{33}$$

Condition (33) is equivalent, given the symmetry of $\mathbf{B}(\mathbf{T})$, to the three scalar conditions:

$$A_{ij}v_j = K_i\,, \quad v_j = \theta\omega_j\,, \quad i,j = 1,2,3\,, \tag{34}$$

where:

$$\begin{aligned} A_{11} &= \hat{B}_2 - \hat{B}_3 + (\hat{\mathbb{M}}_{22hk} - \hat{\mathbb{M}}_{33hk})T_{hk}\,, \\ A_{22} &= \hat{B}_3 - \hat{B}_1 + (\hat{\mathbb{M}}_{33hk} - \hat{\mathbb{M}}_{11hk})T_{hk}\,, \\ A_{33} &= \hat{B}_1 - \hat{B}_2 + (\hat{\mathbb{M}}_{11hk} - \hat{\mathbb{M}}_{22hk})T_{hk},, \\ A_{ij} &= \hat{\mathbb{M}}_{ijhk}T_{hk}\,, \quad i \neq j\,, \end{aligned} \tag{35}$$

and

$$K_1 = -A_{23}\,, \quad K_2 = -A_{13}\,, \quad K_3 = A_{12}\,; \tag{36}$$

from the components v_k, $k = 1,2,3$ solution of (35), whose explicit expression in terms of the components of $\mathbb{M}[\mathbf{T}]$ and $\mathbf{B}_o$ is given by equation (85) of the Appendix B, then we obtain

$$\theta = \sqrt{v_1^2 + v_2^2 + v_3^2}\,, \quad \omega_k = \theta^{-1}v_k\,, \quad k = 1,2,3\,. \tag{37}$$

2.3 | The Sirotin and Shaskolskaya approximation

In[2], §.20, in order to obtain an approximate formula for the eigencouples of $\mathbf{B}(\mathbf{T})$ two approximations are done, which in our language reads:

$$S_k >> \sup_{i,j=1,2,3} \{\, |\mathbb{M}_{ijlm}T_{lm}|\,\}\,, \quad k = 1,2,3\,, \tag{38}$$

and

$$|S_k - S_h| >> \sup_{i,j=1,2,3} \{\, |\mathbb{M}_{ijlm}T_{lm}|\,\}\,, \quad h,k = 1,2,3\,. \tag{39}$$

the hypothesis (38) is equivalent to (23), whereas (39) is a stronger one. If we set

$$\sup_{i,j=1,2,3} \{\, |\mathbb{M}_{ijlm}T_{lm}|\,\} = O(\varepsilon)\,, \tag{40}$$

where ε is a small parameter, then from (35) we get:

$$\begin{aligned} A_{11} &= \hat{B}_2 - \hat{B}_3 + O(\varepsilon)\,, \\ A_{22} &= \hat{B}_3 - \hat{B}_1 + O(\varepsilon)\,, \\ A_{33} &= \hat{B}_1 - \hat{B}_2 + O(\varepsilon)\,, \\ A_{ij} &= O(\varepsilon)\,, \quad i \neq j\,, \end{aligned} \tag{41}$$

and accordingly from (85) we obtain, to within higher-order terms in $o(\varepsilon^2)$, the simpler relations (*cf.*[2], eqn. (20.9)-Right)

$$\begin{aligned} v_1 &= \frac{\hat{\mathbb{M}}_{23hk}T_{hk}}{\hat{B}_3 - \hat{B}_2}\,, \\ v_2 &= \frac{\hat{\mathbb{M}}_{13hk}T_{hk}}{\hat{B}_3 - \hat{B}_1}\,, \\ v_3 &= \frac{\hat{\mathbb{M}}_{12hk}T_{hk}}{\hat{B}_1 - \hat{B}_2}\,. \end{aligned} \tag{42}$$

and the infinitesimal rotation of order $O(\varepsilon)$:

$$\theta = \sqrt{(\frac{\hat{\mathbb{M}}_{23hk}T_{hk}}{\hat{B}_3 - \hat{B}_2})^2 + (\frac{\hat{\mathbb{M}}_{13hk}T_{hk}}{\hat{B}_3 - \hat{B}_1})^2 + (\frac{\hat{\mathbb{M}}_{12hk}T_{hk}}{\hat{B}_1 - \hat{B}_2})^2}\,, \tag{43}$$

with the components of $\boldsymbol{\omega}$ still given by $(37)_2$. Further, relations (20.9)-Left of[2] can be obtained from (30) provided we set $v_k = 0$, $k = 1, 2, 3$ into (31). The main limitation of these is that (42) hold only for biaxial crystals with three different principal refractive indeces.

Indeed for uniaxial crystal with *e.g.* $\hat{B}_1 = \hat{B}_2$ a different procedure was proposed into[2]. First it was considered the projections of the vectors $\mathbf{w}_k$:

$$\begin{aligned} \bar{\mathbf{w}}_1 &= \mathbf{P}^{\perp}(\mathbf{e}_3)\mathbf{w}_1 = \mathbf{w}_1 + v_2\mathbf{e}_3 = \mathbf{e}_1 - v_3\mathbf{e}_2\,, \\ \bar{\mathbf{w}}_2 &= \mathbf{P}^{\perp}(\mathbf{e}_3)\mathbf{w}_2 = \mathbf{w}_2 - v_1\mathbf{e}_3 = \mathbf{e}_2 + v_3\mathbf{e}_1\,, \\ \bar{\mathbf{w}}_3 &= \mathbf{P}(\mathbf{e}_3)\mathbf{w}_3 = \mathbf{e}_3\,; \end{aligned} \tag{44}$$

then the eigenvalues of $\mathbf{B}(\mathbf{T})$ were represented as

$$B_1(\mathbf{T}) = \hat{B}_1 + \lambda_1\,, \quad B_2(\mathbf{T}) = \hat{B}_1 + \lambda_2\,, \quad B_2(\mathbf{T}) = \hat{B}_3 + \lambda_3\,. \tag{45}$$

By writing the eigenvalue problem

$$(\mathbf{B}_o + \mathbb{M}[\mathbf{T}])\mathbf{w}_k = B_k(\mathbf{T})\mathbf{w}_k\,, \quad k = 1, 2, 3\,, \tag{46}$$

in terms of the new frame $\{\bar{\mathbf{w}}_k\}$ we arrive to:

$$\lambda_k = \mathbb{M}[\mathbf{T}] \cdot \bar{\mathbf{w}}_k \otimes \bar{\mathbf{w}}_k + o(\varepsilon^2)\,, \quad k = 1, 2, 3\,. \tag{47}$$

Relations (47) can be recovered within our present approach provided the identification

$$\lambda_k = \mathcal{M}_{kij}T_{ij}\,, \quad k = 1, 2, 3\,, \tag{48}$$

by setting $v_1 = v_2 = 0$ into (31). In[2] finally the vectors $\bar{\mathbf{w}}_1$ and $\bar{\mathbf{w}}_2$ were evaluated by solving the two-dimensional eigenvalue problem for the projection of $\mathbf{M} = \mathbf{P}^{\perp}(\mathbf{e}_3)\mathbb{M}[\mathbf{T}]\mathbf{P}^{\perp}(\mathbf{e}_3)$ on the plane orthogonal to $\mathbf{e}_3$. Within the present approach however these vectors can be evaluated from their definition (44) provided the components v_k be calculated from (86) and the assumptions (38) and (39). In such a case these components reduce to

$$\begin{aligned} v_1 &= \frac{(\hat{\mathbb{M}}_{22hk}\hat{\mathbb{M}}_{33ij} + \hat{\mathbb{M}}_{13hk}\hat{\mathbb{M}}_{12ij})T_{ij}T_{hk}}{(\hat{B}_3 - \hat{B}_1)\hat{\mathbb{M}}_{33hk}T_{hk}} + o(\varepsilon)\,, \\ v_2 &= \frac{(\hat{\mathbb{M}}_{33hk}\hat{\mathbb{M}}_{13ij} + \hat{\mathbb{M}}_{23hk}\hat{\mathbb{M}}_{12ij})T_{ij}T_{hk}}{(\hat{B}_3 - \hat{B}_1)\hat{\mathbb{M}}_{33hk}T_{hk}} + o(\varepsilon)\,, \\ v_3 &= \frac{\hat{\mathbb{M}}_{12hk}T_{hk}}{\hat{\mathbb{M}}_{33hk}T_{hk}} + o(\varepsilon)\,. \end{aligned} \tag{49}$$

The use of (49) rather than (86) or the formulae obtained into[2] accordingly depends on the role of hypothesis (39) in the desired approach.

2.4 | Refraction indeces, Birifringences and Optic angle

We write the principal refraction indeces $n_k(\mathbf{T}) = \hat{B}_k^{-2}$ $k = 1, 2, 3$, with the aid of (30) and by expanding the result about $\mathbf{T} = \mathbf{0}$ we get

$$n_k(\mathbf{T}) = \frac{1}{\sqrt{B_k(\mathbf{T})}} = \frac{1}{\sqrt{B_k}}\bigg|_{\mathbf{T}=\mathbf{0}} - \frac{1}{2(B_k)^{3/2}}\frac{\partial B_k}{\partial T_{ij}}\bigg|_{\mathbf{T}=\mathbf{0}} T_{ij} + o(\|\mathbf{T}\|^2)\,, \tag{50}$$

which, after some calculation and by the means of (30), to within higher-order terms leads to

$$n_k(\mathbf{T}) = \frac{1}{\sqrt{\hat{B}_k}} - \frac{1}{2(\hat{B}_k)^{3/2}}\mathbb{M}_{kkij}T_{ij}\,, \quad k = 1, 2, 3\,, \tag{51}$$

since $v_k = 0$ for $\mathbf{T} = \mathbf{0}$; such an expression can be put in the form $(6)_1$ provided we set:

$$\hat{n}_k = \frac{1}{\sqrt{\hat{B}_k}}\,, \quad H_{ij}^{(k)} = -\frac{\hat{n}_k^3}{2}\mathbb{M}_{kkij}\,. \tag{52}$$

From $(6)_1$, (51) and (52) the explicit expression for the principal birifringences as a linear function of the stress follows trivially:

$$\begin{aligned} n_1 - n_2 &= \hat{n}_1 - \hat{n}_2 + (\mathbf{H}^{(1)} - \mathbf{H}^{(2)}) \cdot \mathbf{T}\,, \\ n_2 - n_3 &= \hat{n}_2 - \hat{n}_3 + (\mathbf{H}^{(2)} - \mathbf{H}^{(3)}) \cdot \mathbf{T}\,, \\ n_1 - n_3 &= \hat{n}_1 - \hat{n}_3 + (\mathbf{H}^{(1)} - \mathbf{H}^{(3)}) \cdot \mathbf{T}\,. \end{aligned} \tag{53}$$

Likewise, from (3) and (51) we obtain the explicit expression for the optic angle (here under the hypothesis that $B_1(\mathbf{T}) > B_2(\mathbf{T}) > B_3(\mathbf{T})$):

$$\varphi(\mathbf{T}) = \sin^{-1}\sqrt{\frac{B_1 - B_2}{B_1 - B_3}} = \sin^{-1}\sqrt{\frac{\hat{B}_1 - \hat{B}_2 + (\mathbb{M}_{11ij} - \mathbb{M}_{22ij})T_{ij}}{\hat{B}_1 - \hat{B}_3 + (\mathbb{M}_{11ij} - \mathbb{M}_{33ij})T_{ij}}}\,. \tag{54}$$

By the linearization of (54) about $\mathbf{T} = \mathbf{0}$ we arrive at the linear representation $(6)_2$, provided we denote φ_o the optic angle in the unstressed state and set:

$$K_{ij} = \frac{(\hat{B}_2 - \hat{B}_3)\mathbb{M}_{11ij} + (\hat{B}_3 - \hat{B}_1)\mathbb{M}_{22ij} + (\hat{B}_1 - \hat{B}_2)\mathbb{M}_{33ij}}{2(\hat{B}_1 - \hat{B}_3)\sqrt{(\hat{B}_1 - \hat{B}_2)(\hat{B}_2 - \hat{B}_3)}}\,. \tag{55}$$

Relation (55) clearly holds for biaxial crystal when the three eigenvalues of $\mathbf{B}_o$ are different; for Uniaxial crystals, were we have two equal eigenvalues, say $\hat{B}_1 = \hat{B}_2$, relation (54) becomes

$$\varphi(\mathbf{T}) = \sin^{-1}\sqrt{\frac{(\mathbb{M}_{11ij} - \mathbb{M}_{22ij})T_{ij}}{\hat{B}_1 - \hat{B}_3 + (\mathbb{M}_{11ij} - \mathbb{M}_{33ij})T_{ij}}}\,; \tag{56}$$

it is easy to show that

$$\lim_{T_{ij}\to 0}\left|\frac{\partial\varphi}{\partial T_{ij}}\right| \to +\infty\,, \tag{57}$$

and accordingly we cannot write (54) in the linearized form $(6)_2$, relation (56) being an estimate within our approximation procedure.

2.5 | The Photoelastic constants tensor

Besides the Fresnel ellipsoid (1), the other surface which describes the crystal optical properties is the surface of equal phase difference or *Bertin surface* which can be obtained from the Fresnel equation[1]. The Bertin surfaces are fourth-order surfaces whose mathematical aspects were dealt with into[7] and whose equation in the frame Σ is

$$\begin{aligned} z_1^4\cos^4\varphi + z_2^4 + z_3^4\sin^4\varphi \;+\; 2z_1^2z_2^2\cos^2\varphi + 2z_2^2z_3^2\sin^2\varphi \\ -\; 2z_1^2z_3^2\sin^2\varphi\cos^2\varphi - H^2\|\mathbf{x}\|^2 = 0\,; \end{aligned} \tag{58}$$

here $z_k = \mathbf{x}\cdot\mathbf{e}_k$ and the parameter H, which has the dimension of a length, is defined as

$$H = \frac{N\lambda}{n_{max} - n_{min}}\,, \tag{59}$$

where N is the fringe order and λ is the wave length of the light source.

The section of (58) with the crystal surface or a projection plane gives the isochromate interference fringes, fourth-order plane curves which can be either closed or open and two-folded. The equation for a generic plane are given into[7]: here we shall limit our analysis to planes which are orthogonal to the bisector of the optic axes in biaxial crystals or to the optic axis in uniaxial and hence with $z_3 = z^o$. In both cases we have closed fourth-order curves $f = f(z_1, z_2, z^o)$ parameterized on z^o that we call *Cassini-like* for their similarity with the Cassini's curves. Let $\{\pm a, \pm b\}$ with $a > b$ be the solutions of the equations $f = f(z_1, 0, z^o) = 0$ and $f = f(0, z_2, z^o) = 0$, then we define an *ellipticity ratio*[3]

$$C = \frac{a}{b} - 1 > 0\,, \tag{60}$$

in such a way that $C = 0$ in uniaxial crystal. By (58), (60) and the definition of a, b, we obtain an explicit expression of C in terms of $u = \sin^2\varphi \in [0, 1]$:

$$C(u) = \frac{1}{1-u}\sqrt{\frac{1 + 2K^2(1-u)u + \sqrt{1 + 4K^2(1-u)}}{1 - 2K^2u + \sqrt{1 + 4K^2(1-u)}}} - 1\,, \quad K = \frac{z^o}{H}\,. \tag{61}$$

By (16), then (61) depends on the stress $\mathbf{T}$ and therefore, for small stress we can write the expansion about $\mathbf{T} = \mathbf{0}$:

$$C(\mathbf{T}) = C(\mathbf{0}) + \frac{\partial C}{\partial \mathbf{T}}\Big|_{\mathbf{T}=\mathbf{0}} \cdot \mathbf{T} + o(\|\mathbf{T}\|^2)\,, \tag{62}$$

which leads, to within higher-order terms, to the linear expression $(6)_3$ provided the identification

$$C_o = C(\mathbf{0})\,, \quad \mathbf{F} = \frac{\partial C}{\partial \mathbf{T}}\Big|_{\mathbf{T}=\mathbf{0}}\,. \tag{63}$$

In order to evaluate the components F_{ij} of the photoelastic constants tensor $\mathbf{F}$ we first notice that, from (61)

$$\frac{\partial C}{\partial \mathbf{T}} = \frac{dC}{du}\frac{\partial u}{\partial \mathbf{T}}\Big|_{\mathbf{T}=\mathbf{0}}\,, \tag{64}$$

and then by simple calculations we get

$$\begin{aligned}\frac{dC}{du}\Big|_{\mathbf{T}=\mathbf{0}} = A(K\,,\varphi_o) \;=\; & \frac{1}{(1-u_o)^2}\sqrt{\frac{N(u_o)}{D(u_o)}} \\ & + \frac{1}{1-u_o}\frac{N_{,u}(u_o)D(u_o) - N(u_o)D_{,u}(u_o)}{2D(u_o)\sqrt{N(u_o)D(u_o)}}\,,\end{aligned} \tag{65}$$

where $u_o = \sin^2\varphi_o$ and

$$\begin{aligned} N(u) \;&=\; 1 + 2K^2(1-u)u + \sqrt{1+4K^2(1-u)}\,, \\ D(u) \;&=\; 1 - 2K^2u + \sqrt{1+4K^2(1-u)}\,; \end{aligned}$$

we notice that (65) is a term which is independent on both the stress $\mathbf{T}$ and the piezo-optic tensor $\mathbb{M}$ and depends only on φ_o and on the adimensional ratio $K = z^o/H$.[2]

From (3) we have:

$$\frac{\partial u}{\partial T_{ij}} = \frac{\partial}{\partial T_{ij}}\frac{\hat{B}_1 - \hat{B}_2}{\hat{B}_1 - \hat{B}_3} = \frac{(\hat{B}_1 - \hat{B}_2)_{,ij}(\hat{B}_1 - \hat{B}_3) - (\hat{B}_1 - \hat{B}_3)_{,ij}(\hat{B}_1 - \hat{B}_2)}{(\hat{B}_1 - \hat{B}_3)^2}\,, \tag{66}$$

and since for $\mathbf{T} = \mathbf{0}$ it is $\hat{B}_k = \hat{n}_k^{-2}$, $k = 1, 2, 3$, then (66), evaluated for $\mathbf{T} = \mathbf{0}$, reduces to:

$$\frac{\partial u}{\partial T_{ij}}\Big|_{\mathbf{T}=\mathbf{0}} = \frac{1}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}\Big(\frac{\partial(\hat{B}_1 - \hat{B}_3)}{\partial T_{ij}} - \frac{\hat{n}_1^{-2} - \hat{n}_2^{-2}}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}\frac{\partial(\hat{B}_1 - \hat{B}_3)}{\partial T_{ij}}\Big)\Big|_{\mathbf{T}=\mathbf{0}}\,. \tag{67}$$

From (63), (64), (65) and (67) finally:

$$\frac{\partial C}{\partial T_{ij}} = \frac{A(K\,,\sin\varphi_o)}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}\Big(\frac{\partial(\hat{B}_1 - \hat{B}_2)}{\partial T_{ij}} - \frac{\hat{n}_1^{-2} - \hat{n}_2^{-2}}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}\frac{\partial(\hat{B}_1 - \hat{B}_3)}{\partial T_{ij}}\Big)\Big|_{\mathbf{T}=\mathbf{0}}\,, \tag{68}$$

and accordingly, by (30) we obtain the expression of the six components of the photoelastic constants tensor:

$$F_{ij} = \frac{A(K\,,\sin\varphi_o)}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}\left(\mathbb{M}_{11ij} - \mathbb{M}_{22ij} - (\mathbb{M}_{11ij} - \mathbb{M}_{33ij})\frac{\hat{n}_1^{-2} - \hat{n}_2^{-2}}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}\right); \tag{69}$$

for uniaxial crystals with $\hat{B}_1 = \hat{B}_2$ and $\varphi_o = 0$, relation (69) reduces to

$$F_{ij} = \frac{A(K\,,0)}{\hat{n}_1^{-2} - \hat{n}_3^{-2}}(\mathbb{M}_{11ij} - \mathbb{M}_{22ij})\,. \tag{70}$$

3 | AN EXAMPLE

In[7] we presented a detailed analysis of the changes induced on the refractive indeces, optic axes and plane induced by a generic stress applied to an anisotropic crystal: further such an analysis was made explicit in term of analytical solution for tetragonal crystals into[4] and for anisotropic crystals of other groups in[5], leaving out only optically isotropic crystals.

[2]To arrive at these results we assumed $H =$ const. *vid. e.g.* the discussion in[7].

However, for some groups these analysis lead to explicit solutions only for particular cases of stress, since for some state of stress and for some crystallographic groups the analytical expression of the eigenvalues of $\mathbf{B}(\mathbf{T})$ required the solution of a full three-dimensional eigenvalue problem. Accordingly, in these cases we gave only a qualitative estimate of the solutions.

Here, by using the approximate expression (30) for the eigenvalues of $\mathbf{B}(\mathbf{T})$, with either (42) or (85) as an estimate of the couple $(\theta\,,\boldsymbol{\omega})$, we give the explicit expressions of the quantities (6) in a case we left out in the aforementioned papers, namely Monoclinic crystal acted by shear stress on planes parallel to the monoclinic $b-$ axis.

We assume that the monoclinic $b-$ axis is parallel to the vector $\mathbf{u}_2$ of $\mathcal{S}$. Accordingly the matrix of $\mathbf{B}_o$ in the frame $\mathcal{S}$ is

$$\mathbf{B}_o \equiv \begin{bmatrix} B_{11} & 0 & B_{13} \\ \cdot & B_{22} & 0 \\ \cdot & \cdot & B_{33} \end{bmatrix}, \tag{71}$$

with the principal components of $\mathbf{B}_o$:

$$\hat{B}_{1,3} = \frac{B_{11}+B_{33}}{2} \pm \sqrt{(\frac{B_{11}-B_{33}}{2})^2 + B_{13}^2}\,, \quad \hat{B}_2 = B_{22}\,, \tag{72}$$

and the angle ψ of the rotation $\mathbf{R}$ about $\mathbf{u}_2$ is given by

$$\tan\psi = \frac{B_{13}}{\hat{B}_1 - B_{11}}\,. \tag{73}$$

In the frame $\mathcal{S}$ the matrix of $\mathbb{M}$ is, for all classes[8]:

$$[\mathbb{M}] \equiv \begin{bmatrix} \mathbb{M}_{1111} & \mathbb{M}_{1122} & \mathbb{M}_{1133} & 0 & \mathbb{M}_{1113} & 0 \\ \mathbb{M}_{2211} & \mathbb{M}_{2222} & \mathbb{M}_{2233} & 0 & \mathbb{M}_{2213} & 0 \\ \mathbb{M}_{3311} & \mathbb{M}_{3322} & \mathbb{M}_{3333} & 0 & \mathbb{M}_{3313} & 0 \\ 0 & 0 & 0 & \mathbb{M}_{2323} & 0 & \mathbb{M}_{2312} \\ \mathbb{M}_{1311} & \mathbb{M}_{1322} & \mathbb{M}_{1333} & 0 & \mathbb{M}_{1313} & 0 \\ 0 & 0 & 0 & \mathbb{M}_{1223} & 0 & \mathbb{M}_{1212} \end{bmatrix}, \tag{74}$$

the corresponding components $\hat{\mathbb{M}}_{ijhk}$ being given into the Appendix.

Let π a plane parallel to monoclinic $b-$ axis $\mathbf{e}_2$ whose unit normal is $\mathbf{m} = \cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_3$: the shear stress acting on π is accordingly given by:

$$T_{m2} = \mathbf{Tm}\cdot\mathbf{e}_2 = \cos\alpha\, T_{12} + \sin\alpha\, T_{23}\,. \tag{75}$$

If we consider for instance only the the shear stress $T_{12} = \tau$ then we have

$$\mathbf{B}(\mathbf{T}) = \begin{bmatrix} \hat{B}_1 & \hat{\mathbb{M}}_{1212}\tau & 0 \\ \cdot & \hat{B}_2 & \hat{\mathbb{M}}_{2312}\tau \\ \cdot & \cdot & \hat{B}_3 \end{bmatrix}, \tag{76}$$

and we cannot write an analytical expression for the eigencouples.

In our approximation then, we have from (30):

$$\begin{aligned} B_1(\mathbf{T}) &= \hat{B}_1 + \mathcal{M}_{112}\tau\,, \\ B_2(\mathbf{T}) &= \hat{B}_2 + \mathcal{M}_{212}\tau\,, \\ B_3(\mathbf{T}) &= \hat{B}_3 + \mathcal{M}_{312}\tau\,; \end{aligned} \tag{77}$$

were the components $\mathcal{M}_{k12}$, $k = 1,2,3$ evaluated with (31) are

$$\begin{aligned} \mathcal{M}_{112} &= 2v_3\hat{\mathbb{M}}_{1212}\,, \\ \mathcal{M}_{212} &= 2v_1\hat{\mathbb{M}}_{2312} + 2v_3\hat{\mathbb{M}}_{1212}\,, \\ \mathcal{M}_{312} &= 2v_1\hat{\mathbb{M}}_{2312}\,, \end{aligned} \tag{78}$$

wheras the components v_k, $k = 1,2,3$, given either by (85) or (42), depending on the desired approximation. If for instance we choose (42) then

$$v_1 = \frac{\hat{\mathbb{M}}_{2312}}{\hat{B}_3 - \hat{B}_2}\tau\,, \quad v_2 = \frac{\hat{\mathbb{M}}_{1312}}{\hat{B}_3 - \hat{B}_1}\tau\,, \quad v_3 = \frac{\hat{\mathbb{M}}_{1212}}{\hat{B}_1 - \hat{B}_2}\tau\,, \tag{79}$$

with the infinitesimal rotation angle:

$$\theta = \tau\sqrt{(\frac{\hat{\mathbb{M}}_{2312}}{\hat{B}_3-\hat{B}_2})^2+(\frac{\hat{\mathbb{M}}_{1312}}{\hat{B}_3-\hat{B}_1})^2+(\frac{\hat{\mathbb{M}}_{1212}}{\hat{B}_1-\hat{B}_2})^2}\,, \tag{80}$$

As it was pointed out in[7], the optic plane rotates about a direction $\boldsymbol{\omega}$ which, by (79) and (80) is independent on the magnitude τ of the shear stress.

The principal refraction indeces, the change in the optic angle and in the photoelastic constant are given, by (6), (51), (55), (69) and (74):

$$\begin{aligned} n_k(\tau) &= \hat{n}_k + O(\tau^2)\,, \quad k=1,2,3\,,\\ \varphi(\tau) &= \varphi_o + O(\tau^2)\,,\\ C(\tau) &= C_o + O(\tau^2)\,. \end{aligned} \tag{81}$$

Accordingly we are able to evaluate, with the proposed approximation, the infinitesimal rotation of the optic plane, the other relevant quantities remaining unchanged to within higher-order term in the shear stress. A similar result can be obtained for the shear stress T_{23}.

Acknowledgments

This work is within the scope of CERN R&D Experiment 18 "Crystal Clear Collaboration" and the PANDA Collaboration at GSI-Darmstadt.

APPENDIX

A: The components $\hat{\mathbb{M}}_{ijhk}$ for monoclinic crystals

For monoclinic crystals an explicit expression of the components $\hat{\mathbb{M}}_{ijhk}$ can be done in terms of the components $\mathbb{M}_{ijhk}$ and the rotation angle ψ. Provided (71)-(74) hold and since in this case the non-null components of the rotation $\mathbf{R}$ of an angle ψ about $\mathbf{u}_2$ are $R_{11}=R_{33}=\cos\psi$, $R_{13}=-R_{31}=\sin\psi$ and $R_{22}=1$, then from relation $(20)_2$ we get:

$$\begin{aligned}
\hat{\mathbb{M}}_{1111} &= \cos^4\psi\,\mathbb{M}_{1111}+\sin^4\psi\,\mathbb{M}_{3333}+\sin^2 2\psi\,\mathbb{M}_{1313}+\frac{1}{4}\sin^2 2\psi\frac{\mathbb{M}_{3311}+\mathbb{M}_{1133}}{4}\\
&+\sin 2\psi(\cos^2\psi\,(\mathbb{M}_{1113}+\mathbb{M}_{1311})+\sin^2\psi\,(\mathbb{M}_{3313}+\mathbb{M}_{1333}))\,,\\
\hat{\mathbb{M}}_{2222} &= \mathbb{M}_{2222}\,,\\
\hat{\mathbb{M}}_{3333} &= \sin^4\psi\,\mathbb{M}_{1111}+\cos^4\psi\,\mathbb{M}_{3333}+\sin^2 2\psi\,\mathbb{M}_{1313}+\frac{1}{4}\sin^2 2\psi\frac{\mathbb{M}_{3311}+\mathbb{M}_{1133}}{4}\\
&+\sin 2\psi(\sin^2\psi(\mathbb{M}_{1113}+\mathbb{M}_{1311})+\cos^2\psi(\mathbb{M}_{3313}+\mathbb{M}_{1333}))\,,\\
\hat{\mathbb{M}}_{2323} &= \cos^2\psi\,\mathbb{M}_{2323}+\sin^2\psi\,\mathbb{M}_{1212}-\sin 2\psi\frac{\mathbb{M}_{2312}+\mathbb{M}_{1223}}{2}\\
\hat{\mathbb{M}}_{1313} &= \sin^2 2\psi\frac{\mathbb{M}_{1111}+\mathbb{M}_{3333}-\mathbb{M}_{1133}-\mathbb{M}_{3311}}{4}+\cos^2 2\psi\mathbb{M}_{1313}\,,\\
&-\sin 2\psi\cos 2\psi\frac{\mathbb{M}_{1113}+\mathbb{M}_{1311}+\mathbb{M}_{3313}+\mathbb{M}_{1333}}{2}\,,\\
\hat{\mathbb{M}}_{1212} &= \sin^2\psi\,\mathbb{M}_{2323}+\cos^2\psi\,\mathbb{M}_{1212}+\sin 2\psi\frac{\mathbb{M}_{2312}+\mathbb{M}_{1223}}{2}\,,\\
\hat{\mathbb{M}}_{1122} &= \cos^2\psi\,\mathbb{M}_{1122}+\sin^2\psi\,\mathbb{M}_{3322}+\sin 2\psi\,\mathbb{M}_{1322}\,,
\end{aligned}$$

$$
\begin{aligned}
\hat{\mathbb{M}}_{1133} &= \cos^4\psi\, \mathbb{M}_{1133} + \sin^4\psi\, \mathbb{M}_{3311} + \sin^2 2\psi \frac{\mathbb{M}_{1111} + \mathbb{M}_{3333} + 4\mathbb{M}_{1313}}{4} \\
&+ \sin 2\psi(\cos^2\psi\, (\mathbb{M}_{1113} + \mathbb{M}_{1333}) + \sin^2\psi\, (\mathbb{M}_{1311} + \mathbb{M}_{3313}))\,, \\
\hat{\mathbb{M}}_{2233} &= \cos^2\psi\, \mathbb{M}_{2233} + \sin^2\psi\, \mathbb{M}_{2211} + \sin 2\psi\, \mathbb{M}_{2213}\,, \\
\hat{\mathbb{M}}_{2211} &= \sin^2\psi\, \mathbb{M}_{2233} + \cos^2\psi\, \mathbb{M}_{2211} + \sin 2\psi\, \mathbb{M}_{2213}\,, \\
\hat{\mathbb{M}}_{3311} &= \cos^4\psi\, \mathbb{M}_{3311} + \sin^4\psi\, \mathbb{M}_{1133} + \sin^2 2\psi \frac{\mathbb{M}_{1111} + \mathbb{M}_{3333} + 4\mathbb{M}_{1313}}{4} \\
&+ \sin 2\psi(\cos^2\psi\, (\mathbb{M}_{3313} + \mathbb{M}_{1311}) + \sin^2\psi\, (\mathbb{M}_{1113} + \mathbb{M}_{1333}))\,, \\
\hat{\mathbb{M}}_{3322} &= \sin^2\psi\, \mathbb{M}_{1122} + \cos^2\psi\, \mathbb{M}_{3322} + \sin 2\psi\, \mathbb{M}_{1322}\,, \\
\hat{\mathbb{M}}_{1113} &= \sin 2\psi(\cos^2\psi \frac{\mathbb{M}_{1133} - \mathbb{M}_{1111}}{2} + \sin^2\psi \frac{\mathbb{M}_{3333} - \mathbb{M}_{3311}}{2}) \\
&+ \cos 2\psi(\cos^2\psi\, \mathbb{M}_{1113} + \sin^2\psi\, \mathbb{M}_{3313} + \sin 2\psi\, \mathbb{M}_{1313}) \\
&- \sin^2 2\psi \frac{\mathbb{M}_{1333} + \mathbb{M}_{1311}}{2}\,, \\
\hat{\mathbb{M}}_{2213} &= \sin 2\psi \frac{\mathbb{M}_{2233} - \mathbb{M}_{2211}}{2} + \cos 2\psi\, \mathbb{M}_{2213}\,, \\
\hat{\mathbb{M}}_{3313} &= \sin 2\psi(\sin^2\psi \frac{\mathbb{M}_{3333} - \mathbb{M}_{1133}}{2} - \cos^2\psi \frac{\mathbb{M}_{3311} + \mathbb{M}_{1111}}{2}) \\
&+ \cos 2\psi(\cos^2\psi\, \mathbb{M}_{3313} + \sin^2\psi\, \mathbb{M}_{1113} + \sin 2\psi\, \mathbb{M}_{1313}) \\
&- \sin^2 2\psi \frac{\mathbb{M}_{1311} + \mathbb{M}_{1333}}{2}\,, \\
\hat{\mathbb{M}}_{1311} &= \sin 2\psi(\cos^2\psi \frac{\mathbb{M}_{1133} - \mathbb{M}_{1111}}{2} + \sin^2\psi \frac{\mathbb{M}_{3333} - \mathbb{M}_{3311}}{2}) \\
&+ \cos 2\psi(\cos^2\psi\, \mathbb{M}_{1311} + \sin^2\psi\, \mathbb{M}_{1333} + \sin 2\psi\, \mathbb{M}_{1313}) \\
&- \sin^2 2\psi \frac{\mathbb{M}_{1333} + \mathbb{M}_{1311}}{2}\,, \\
\hat{\mathbb{M}}_{1322} &= \sin 2\psi \frac{\mathbb{M}_{3322} - \mathbb{M}_{1122}}{2} + \cos 2\psi\, \mathbb{M}_{1322}\,, \\
\hat{\mathbb{M}}_{1333} &= \sin 2\psi(\sin^2\psi \frac{\mathbb{M}_{3311} - \mathbb{M}_{1111}}{2} - \cos^2\psi \frac{\mathbb{M}_{3333} + \mathbb{M}_{1133}}{2}) \\
&+ \cos 2\psi(\cos^2\psi\, \mathbb{M}_{1333} + \sin^2\psi\, \mathbb{M}_{1311} - \sin 2\psi\, \mathbb{M}_{1313}) \\
&- \sin^2 2\psi \frac{\mathbb{M}_{1113} + \mathbb{M}_{3313}}{2}\,, \\
\hat{\mathbb{M}}_{2312} &= \sin^2\psi\, \mathbb{M}_{1223} - \cos^2\psi\, \mathbb{M}_{2312} + \sin 2\psi \frac{\mathbb{M}_{2323} - \mathbb{M}_{1212}}{2}\,, \\
\hat{\mathbb{M}}_{1223} &= \sin^2\psi\, \mathbb{M}_{2312} - \cos^2\psi\, \mathbb{M}_{1223} + \sin 2\psi \frac{\mathbb{M}_{2323} - \mathbb{M}_{1212}}{2}\,.
\end{aligned}
$$

B: The components v_k, $k = 1, 2, 3$

We solve (34) with the Cramer's rule; if we write $[A_{ij}] = [\Delta_{ij}] + [S_{ij}]$ with

$$
\Delta_{11} = \hat{B}_2 - \hat{B}_3\,, \quad \Delta_{22} = \hat{B}_3 - \hat{B}_1\,, \quad \Delta_{33} = \hat{B}_1 - \hat{B}_2\,, \quad S_{ij} = \hat{\mathbb{M}}_{ijhk} T_{hk}\,, \tag{82}
$$

then

$$
\det[A_{ij}] = \Delta_{11}\Delta_{22}\Delta_{33} + \Delta^*_{ij} S_{ij} + \Delta_{ij} S^*_{ij} + \det[S_{ij}]\,, \tag{83}
$$

where for any given invertible and symmetric matrix $[B_{ij}]$ we define its *cofactor* as:

$$
[B^*_{ij}] = (\det[B_{ij}])[B^{-1}_{ij}]\,. \tag{84}
$$

Accordingly, the solution of (34) can be written as:

$$\begin{aligned}
v_1 &= (\det[A_{ij}])^{-1}\big(- S_{23}\Delta_{22}\Delta_{33} - \Delta_{22}(S_{23}S_{33} + S_{12}S_{13}) - \Delta_{33}(S_{22}S_{23} + S_{13}S_{12}) \\
&+ S_{23}(S_{23}^2 - S_{22}S_{33} - S_{12}^2 - S_{13}^2) - S_{12}S_{13}(S_{11} + S_{22})\big)\,, \\
v_2 &= (\det[A_{ij}])^{-1}\big(- S_{13}\Delta_{11}\Delta_{33} - \Delta_{11}(S_{33}S_{13} + S_{12}S_{23}) - \Delta_{33}(S_{11}S_{13} - S_{23}S_{12}) \\
&+ S_{13}(S_{13}^2 - S_{11}S_{33} - S_{23}^2 + S_{12}^2) + S_{12}S_{23}(S_{33} + S_{11})\big)\,, \\
v_3 &= (\det[A_{ij}])^{-1}\big(S_{12}\Delta_{11}\Delta_{22} + \Delta_{11}(S_{12}S_{22} + S_{13}S_{23}) + \Delta_{22}(S_{11}S_{12} + S_{22}S_{13}) \\
&- S_{12}(S_{13}^2 + S_{23}^2 - S_{12}^2 + S_{11}S_{22}) - S_{23}S_{13}(S_{11} + S_{22})\big)\,.
\end{aligned} \tag{85}$$

For uniaxial crystals, where for instance $\Delta_{33} = 0$ and $\Delta_{11} = \Delta_{22} = \Delta$, relations (85) reduce to:

$$\begin{aligned}
v_1 &= \frac{-\Delta(S_{22}S_{33} + S_{13}S_{12})}{\Delta^2 S_{33} + \Delta(S_{22}S_{33} + S_{11}S_{33} - S_{13}^2 - S_{23}^2) + \det[S_{ij}]} \\
&+ \frac{S_{23}(S_{23}^2 - S_{22}S_{33} - S_{12}^2 - S_{13}^2) - S_{12}S_{13}(S_{11} + S_{22})}{\Delta^2 S_{33} + \Delta(S_{22}S_{33} + S_{11}S_{33} - S_{13}^2 - S_{23}^2) + \det[S_{ij}]}\,, \\
v_2 &= \frac{-\Delta(S_{33}S_{13} + S_{12}S_{23})}{\Delta^2 S_{33} + \Delta(S_{22}S_{33} + S_{11}S_{33} - S_{13}^2 - S_{23}^2) + \det[S_{ij}]} \\
&+ \frac{S_{13}(S_{13}^2 - S_{11}S_{33} - S_{23}^2 + S_{12}^2) + S_{12}S_{23}(S_{33} + S_{11})}{\Delta^2 S_{33} + \Delta(S_{22}S_{33} + S_{11}S_{33} - S_{13}^2 - S_{23}^2) + \det[S_{ij}]}\,, \\
v_3 &= \frac{S_{12}\Delta^2 + \Delta(S_{12}S_{22} + S_{13}S_{23} + S_{11}S_{12} + S_{22}S_{13})}{\Delta^2 S_{33} + \Delta(S_{22}S_{33} + S_{11}S_{33} - S_{13}^2 - S_{23}^2) + \det[S_{ij}]} \\
&+ \frac{S_{12}(S_{13}^2 + S_{23}^2 - S_{12}^2 + S_{11}S_{22}) - S_{23}S_{13}(S_{11} + S_{22})}{\Delta^2 S_{33} + \Delta(S_{22}S_{33} + S_{11}S_{33} - S_{13}^2 - S_{23}^2) + \det[S_{ij}]}\,.
\end{aligned} \tag{86}$$

References

1. Wahlstrom E. E., *Optical Crystallography*, Wiley, New York, 1960.

2. Sirotin Y. I. and Shaskolskaya M. P., *Fundamentals of crystal physics*, Mir-Moscow, 1982.

3. Cocozzella N., Lebeau M., Majni G., Paone N. and Rinaldi D., *Quality inspection of anisotropic scintillating lead tungstate ($PbWO_4$) crystals through measurement of interferometric fringe pattern parameters.*Nucl. Instr. and Meth. A**469**, 331–339, 2001.

4. Daví F., Rinaldi D. and Montalto L., *On the photoelastic constants and the Brewster law for stressed tetragonal crystals*, Mathematical Models in Applied Sciences, vol **41** 8, 3103–3116, 2018.

5. Daví F., Rinaldi D. and Montalto L.,*On the photoelastic constants for stressed anisotropic crystals*, Nuclear Inst. and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment, **947**, 162782, 2019.

6. Abbott, Jr. R N. *Calculation of the orientation of the optical indicatrix in monoclinic and triclinic crystals: The point-dipole model*, American Mineralogist **78**, 952–956, 1993.

7. Daví F., *On the Bertin Surfaces for Photoelastic Crystals*, J. of the Optical Society of America A, **32**(12), 2323-2337, Dec. 2015.

8. Authier A. (Ed.), *International Tables for Crystallography. Volume D: Physical Properties of Crystals*, Kluwer Academic Publ. Dordrecht, 2003.

9. Weber M. J., *Handbook of optical materials*, CRC Laser and Optical Science and Technology Series, CRC Press, 2003.

10. Ishii M., Harada K., Kobayashi M., Usuki Y. and Yazawa T., *Mechanical properties of PWO*, Nuclear Instruments and Methods in Physics Research A, **376**, 203–207, 1996.

11. Natali P. P., Montalto L., Daví F., Mengucci P., Ciriaco A., Paone N., Rinaldi D., *Theoretical and experimental evaluation of piezo-optic parameters and photoelastic constants in Tetragonal PWO*. Applied Optics, vol **57** 4, 730–737, 2018.

12. Baccaro S., Barone L. M. , Borgia B., Castelli F., Cavallari F., Dafinei I., de Notaristefani F., Diemoz M., Festinesi A. , Leonardi E., Longo E., Montecchi M. , Organtini G. , *Ordinary and extraordinary complex refractive index of the lead tungstate ($PbWO_4$) crystal*, Nuclear Instruments and Methods in Physics Research A, **385**, 09–214, 1997.

How to cite this article: Daví, (2020), Some approximate relations in the photoelasticity of strongly-anisotropic crystals, *Math. Meth. Appl. Sc.*, *2020;00:1–6*.