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Analytically Tractable Stochastic Volatility Models in Asset and Option Pricing

Yu Sun

Supervisore
Prof. Maria C. Recchioni

Coordinatore
Prof. Riccardo Lucchetti

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1 Abstract

This dissertation consists of four related essays on stochastic volatility models in asset and option pricing. More precisely, this dissertation focuses on stochastic interest rate and multiscale stochastic volatility models, with applications in various financial products such as European vanilla options, U.S. government bond yield, health insurance policy, implied volatility, and foreign exchange rate.

Volatility study has became one of the most important topics since 2007-2008 financial crisis. Quantitative easing policy (QE), initialized by U.S. Federal Reserve (the Fed) since 2008, has spread world-wide. The expansionary monetary policy significantly escalates global debt level and creates massive bubbles especially in emerging markets. Financial instability leads to severe market volatilities and declares the necessity of investigation on sophisticated stochastic volatility models.

In first essay, a hybrid Heston-CIR (HCIR) model with a stochastic interest rate process is presented. This hybrid model is analytically tractable and inspired by the model illustrated by Grzelak and Oosterlee 2011. Their model is modified to preserve the affine structure and to permit a "direct" correlation between the equity and the interest rate. In this essay, explicit elementary formulas for the moments of the asset price variables as well as efficient formulas to approximate the option prices are deduced. Using European call and put option prices on U.S. S&P 500 index, empirical study shows that the HCIR model outperforms Heston model in interpreting and predicting both call and put option prices.

The second essay is a further extension of the HCIR model with two different applications. The first application is using HCIR model to interpret bond yield term structure. Specially, the model shows ability to capture the relationship between short and long term bond yields and to forecast their upward/downward trend. The second analysis is based on the values of the long-term health endowment policy (i.e. Credit Agricole index linked policy, Azione Piu Capitale Garantito Em.64). The empirical analysis shows that the stochastic interest rate plays a crucial role as a volatility factor and provides a multi-factor model that outperforms the Heston model in predicting health endowment policy price.

The third essay aims to extend the Heston model in order to efficiently solve option pricing problems when negative values of interest rate are observed. A hybrid Heston Hull-White (HHW) model is designed to describe the dynamics of an asset price under stochastic volatility and interest rate that allows negative values. Explicit elementary formulas for the transition probability density function of the asset price variable and closed-form formulas to approximate the option prices are deduced. In first empirical analysis, the HHW model is calibrated by using implied volatility, and the result shows that the stochastic but possible negative values of interest rate plays a significant role in the option pricing. The second empirical analysis focuses on the Eurodollar futures prices and the corresponding European options prices with a generalization of the Heston model in the stochastic interest rate framework. The results are impressive for both approximation and prediction. This confirms the efficiency of HHW model and the necessary to allow for negative values of interest rate.

The fourth essay describes a multiscale hybrid Heston model of the spot FX rate which is an extension of the model De Col, Gnoatto and Grasselli 2013 in order to allow stochastic interest rate. The analytical treatment of the model is described in detail both under physical measure and risk neutral measure. In particular, a formula for the transition probability density function is derived as a one dimensional integral of an elementary integral function which is used to price European Vanilla call and put options.

2 Introduction

2.1 Motivation and Research Background

The collapse of the Bretton Woods system, due to the oversupply of U.S. dollar, has caused severe global monetary instability. Strong exchange rate and interest rate fluctuations have destroyed the conventional financial market disciplines. Sense of uncertainty and unsafety urges people's incentive on risk hedging. Therefore, to meet enormous demand on innovative financial instruments for hedging risk, derivatives market has substantially and rapidly expanded. Besides risk hedging, corporates' pursuit for maximum profits become another fundamental driving force of financial innovation, which further develops financial system, and improves the efficiency of financial intermediation in both primary and secondary markets, so as to better fulfil the economic duties of the financial system, such as liquidity control, risk management, information extraction, acceleration the trade, and so on.

A financial derivative is a contract between two or more parties who are willing to pay based on some agreed-upon terms. Derivatives vary according to different types of contracts. In derivatives market, individual investors could hedge risk by transferring risk to institutional investors who are able and willing to take the risk. The integration and globalisation of financial markets increase market volatility, assert trading volume, and types of derivatives including futures, forward contracts, options, swaps, and hybrid products. Today, the derivatives market has great influence on world economy.

For one example, credit default swap (CDS), which is recognized as the trigger of subprime mortgage crisis. In 2007, the year before the crisis, the global CDS amount was \$62 trillion. About \$14 trillion CDS were held in U.S. top 25 banks at the third quarter of 2007 which was higher than U.S. entire year's GDP, \$13.8 trillion. Obviously, it is not sufficient to cover CDS losses even liquidating the total GDP. Ironically, as a risk hedging instrument, CDS did not reduce the risk, but created horrible financial crisis.

It has been over 7 years since 2008 financial crisis, however, we are still unclear about the trend of the world economy and the stability of financial market. Quantitative easing policy (QE), initialized by U.S. Federal Reserve (the Fed) in 2008, has spread world-wide, not only in developed market such as EU and Japan, but also in emerging market, e.g. China. The effectiveness of expansionary monetary policy (money printing) is always controversial. Furthermore, referring to the Greek sovereign debt default in 2012 and 2015, the economic consequences of EU-U.S. sanctions against Russia over Ukraine crisis since March 2014, the Syria war and European refugee crisis since April 2015, we can not help but ask: do we believe that our economy is recovering? In other words, is the world economy gradually away from the last recession, or on the contrary, heading to the next crisis?

The answer to the above questions is embarrassing. One important reason is our knowledge limitation on risk, since the traditional theories and models of financial markets substantially underestimate the risk. Thus, today's challenge and opportunity for economists and financial engineers is to innovate market adapted financial models to predict and hedge risk. Undoubtedly, financial innovation ability has become the heart of competitiveness. Innovators could get significant competitive advantage and occupy larger share of market by providing various interesting products and services. Under fast economic globalization and technology development, financial markets are providing enormous benefits while creating devastating uncertainty and risk.

This thesis focuses on stochastic interest rate and multidimensional stochastic volatility models, with applications in various financial products such as European vanilla options, U.S. government bond yield, health insurance policy, implied volatility, and foreign exchange rate. Interesting examples will be discussed detailedly in each chapter.

2.2 Mathematical Finance Background

This section presents basic knowledge on probability theory and stochastic calculus. Some definitions and relevant formulas will be frequently used in the following chapters.

2.2.1 Martingale

Probability Space

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space which consists three elementary parts as follows:

- Ω : sample space which is a non-empty set.
- $\mathcal{F}$: a set of events. It is a subset of power set of sample space Ω i.e. $\mathcal{F} \subseteq 2^{\Omega}$. $\mathcal{F}$ is a σ-**algebra** since it satisfies the following three properties:
  - $\Omega \in \mathcal{F}$: the sample space Ω is an element of the events set $\mathcal{F}$.
  - if $A \in \mathcal{F}$, then $\bar{A} \in \mathcal{F}$: $\mathcal{F}$ is closed under complements.
  - if $A_i \in \mathcal{F}$, $i = 1, 2, 3, ...$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$: $\mathcal{F}$ is closed under (countable)unions.
- $\mathbb{P}$: probability measure, $\mathbb{P} : \mathcal{F} \to [0, 1]$, and especially, $\mathbb{P}(\Omega) = 1$, i.e. the measure of the whole space is equal to one.

Filtration

Suppose when $0 \leq t_1 \leq t_2 \leq t_3 \cdots$, a sequence of σ-algebra satisfies $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}_{t_3} \cdots$, with $\mathcal{F}_{t_i} \subseteq \mathcal{F}$, $i = 1, 2, 3, ...$. Then filtration is defined as:

$$\mathbb{F} = (\mathcal{F}_{t_i})_{t_i \in \mathbb{Z}^+} \qquad [2.1]$$

In financial market, $(\mathcal{F}_{t_i})_{t_i \in \mathbb{Z}^+}$ refers to information set including historical stock price at time t_i. Clearly, as time moving i.e. t_i increasing, more data joins in this information set which increases as well. Here $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called filtered probability space or stochastic basis.

Martingale

Suppose $(S_t)_{t \in \mathbb{Z}^+}$ is a $\mathbb{F}$ adapted process in $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ i.e. for any $t \geq 0$, S_t is revealed ($\mathcal{F}_t$ measurable). If the following two properties are satisfied,

- $\mathrm{E}_{\mathbb{P}}(\mid S_t \mid) < \infty$, $t \in \mathbb{Z}^+$ or in other words, $S_t \in \ell^1(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- $\mathrm{E}_{\mathbb{P}}(S_{t+1} \mid \mathcal{F}_t) = S_t$, $t \in \mathbb{Z}^+$

then $(S_t)_{t \in \mathbb{Z}^+}$ is called a **martingale** relative to filtration $\mathbb{F}$ and probability $\mathbb{P}$. Here, it is worth to remark that l^1 refers to an one-dimensional vector space which is a special case of an important Banach space–l^p space ($p < \infty$). In l^1 space, the sequence of series $(S_t)_{t \in \mathbb{Z}^+}$ is absolutely convergent i.e.

$$\|S\|_1 = (|S_1| + |S_2| + \cdots + |S_n|) < \infty$$

where $\| \cdot \|_1$ is 1-dimensional Euclidean norm of vector space.
Clearly if $(S_t)_{t \in \mathbb{Z}^+}$ is a martingale, then $\mathrm{E}_{\mathbb{P}}(\Delta S_t \mid \mathcal{F}_t) = \mathrm{E}_{\mathbb{P}}(S_{t+1} - S_t \mid \mathcal{F}_t) =$

$0\,,\ t \in \mathbb{Z}^+$. This is an important martingale property which indicates that the future movement of the random variable is unpredictable. In addition, **Supermartingale** and **Submartingale** are defined as following:

- if $\mathrm{E}_{\mathbb{P}}(S_{t+1} \mid \mathcal{F}_t) \leq (\geq) S_t\ ,\ t \in \mathbb{Z}^+$, then $(S_t)_{t\in\mathbb{Z}^+}$ is a supermartingale(submartingale).

2.2.2 Brownian Motion and Levi Process

Brownian Motion (BM) is the first stochastic process that has been seriously studied since 827. It was named by British biologist, Robert Brown, who firstly observed the motion of tiny pollen particles suspended in a liquid suspension due to water molecules successive impacts. In 1900, Louis Bachelier tried to describe the stock price movement using Brownian Motion. In 1905, Albert Einstein made a reasonable physical interpretation of this process and obtained the transition probability density function. In 1918, Norbert Wiener firstly serious defined Brownian Motion in mathematics, thus Brownian Motion is also called Wiener process. Afterwards, Levy and other scientists studied the important properties of Brownian Motion. The later sections will show some profound meanings of these properties and important applications. Now Brownian Motion works as a cornerstone in studying stochastic process and random phenomena.

Definition 2.2.1. A stochastic process $(W_t)_{t\in[0,\infty]}$ is a standard **Brownian Motion** (or **Wiener Process**) on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if

1. $W_0 = 0$.
2. W_t is almost surely everywhere continuous.
3. $\Delta W \sim \mathcal{N}(0, \Delta t)$ i.e ΔW_t is normally distributed with mean 0 and variance Δt, where for $0 \leq s \leq t\ ,\ \Delta t = t - s$ and $\Delta W = W_t - W_s$.
4. W_t has independent increments. i.e. $\forall\ \Delta t \neq \Delta s$, ΔW and ΔW_s are independent random variables.

Remark: condition (1) is to determine the initial condition. Condition (2) is sometimes written as: $\Delta W = \phi\sqrt{\Delta t}$, where ϕ follows a standard normal distribution. For a time interval $[0, T]$, with $\Delta t = \dfrac{T}{N}$, $N = 1, 2, 3, \ldots$, the change of $(W_t)_{t\in[0,T]}$ from W_0 to W_T follows as:

$$\Delta W_T := W_T - W_0 = \sum_{i=1}^{N} \phi_i \sqrt{\Delta t} \qquad [2.2]$$

Property (4) implies that ϕ_i follows standard identical and independent distribution (i.i.d), i.e. $\phi_i \overset{i.i.d}{\sim} \mathcal{N}(0,1)$. Thus $\Delta W_T \sim \mathcal{N}(0,T)$. Based on the property (2),(3), (4), Karatzas and Shreve 1991 proved that Wiener process is continuous everywhere but differentiable nowhere. Moreover, the sum of first order difference (i.e. $\sum_{i=1}^{N} \mid \Delta W_t \mid$) is unbounded, and the sum of second order difference is convergence with probability 1, i.e. $\sum_{i=1}^{N} \mid \Delta W_t \mid^2 \xrightarrow{p} 1$, as $N \to \infty$ (or $\Delta t \to 0$). Formally speaking, if $\Delta t \to 0$, we obtain the continuous form of Eq.(2.2).

$$dW = \phi\sqrt{dt} \tag{2.3}$$

It is worth to remark that $(W_t)_{t\in[0,\infty]}$ is actually differentiable nowhere. Thus dW is just a notation indicating the instantaneously small time interval. Furthermore, the following theorem can be proved.

Theorem 2.2.1. *suppose $(W_t)_{t\in[0,\infty]}$ (or in brief, W_t) is a Brownian motion, then*

- *W_t is a martingale.*
- *$W_t^2 - t$ is a martingale.*
- *For any c, $e^{cW_t-\frac{1}{2}c^2t}$ is a martingale.*

Although Wiener Process is a basic mathematical tool in financial modelling, it does not clarify or match various financial phenomena (e.g. price movement) in practice. For instance, sometimes the starting point of stock price and the mathematically expectation of incremental are non-zero. Therefore, the following sections will introduce several generalized Wiener process.

2.2.3 Markov Processes

Definition 2.2.2. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, a $\mathbb{F}$ adapted stochastic process $(X_t)_{t\in[0,T]}$ is called **Markov Process** if $\exists$ a Borel measurable function $g(\cdot)$ such that

$$\mathrm{E}\left[g(X_t) \mid \mathcal{F}_s\right] = \mathrm{E}\left[g(X_t) \mid X_s\right], \qquad s \leq t \tag{2.4}$$

Basically, Markov process is a special stochastic process where its historical value emerged into today's value which is sufficient to predict the future. More precisely, a stochastic process $(S_t)_{t\in[0,T]}$ is a **Markov Process**, if for $n = 1, 2, 3, \dots$, and $0 \le t_0 \le t_1 \le \cdots t_n \le T$, the following holds:

$$\begin{aligned} &P(S_{t_n} = X_n \mid S_{t_{n-1}} = X_{n-1}, S_{t_{n-2}} = X_{n-2}, \dots, S_{t_0} = X_0) \\ = &P(S_{t_n} = X_n \mid S_{t_{n-1}} = X_{n-1}) \end{aligned} \qquad [2.5]$$

It is worth nothing that the historical status has no effect on this process. Moreover, standard Wiener process is a homogeneous Markov process.

2.2.4 Itô Calculus

Itô integral, named in honor of Kiyoshi Itô who firstly introduced in 1944, is the integral used in the stochastic calculus, and takes the following form:

$$\int_C f(s,\xi)dW_s(\xi), \qquad C \subset \mathbb{R}^+ \qquad [2.6]$$

Definition 2.2.3. Suppose $W_t(\xi)$ is a Wiener process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and function $f(t,\xi) : [0,\infty) \times \Omega \to \mathbb{R}$, then **Itô integral** is well defined if the following two properties hold:

1. $f(t,\xi)$ is $\mathcal{F}_t^W$ adapted, and $\mathcal{B} \times \mathcal{F}$ measurable, where $\mathcal{F}_t^W$ means the smallest σ-algebra on Ω and $\mathcal{B}$ are Borel sets on $\mathbb{R}^+$.
2. $\mathrm{E}_{\mathbb{P}}\left(\int_C f^2(t,\xi)dt\right) < \infty$, where $C \subset \mathbb{R}$.

In addition, **Ito's Isometry** implies

$$\mathrm{E}_{\mathbb{P}}\left[\left(\int_C f(t,\xi)dW_t(\xi)\right)^2\right] = \mathrm{E}_{\mathbb{P}}\left(\int_C f^2(t,\xi)dt\right) \qquad [2.7]$$

Lemma 2.2.2. *Suppoe $X_t(\xi) = (X_t^1(\xi), X_t^2(\xi), \dots, X_t^n(\xi))^T$ follows a n-dimensional Itô process, i.e.*

$$dX_t(\xi) = \mu(t,\xi)dt + \sigma(t,\xi)dW_t(\xi) \qquad [2.8]$$

with $\mu(t,\xi) = (\mu_1(t,\xi), \dots, \mu_2(t,\xi))^T$, $dW_t(\xi) = (dW_t^1(\xi), \dots, dW_t^n(\xi))^T$, $\sigma(t,\xi) = \begin{bmatrix} \sigma_1(t,\xi) & 0 & \cdots & 0 \\ 0 & \sigma_2(t,\xi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n(t,\xi) \end{bmatrix}$, *and covariance*

$\mathrm{Cov}\left(dW_t^i\, dW_t^j\right) = \rho_{ij}dt$ *($\rho_{ij} = 1$ if $i = j$), then $f = f(t, X_t(\xi))$ satisfies the following stochastic differential equation (**Ito's Lemma**):*

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i=1}^{n} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 f}{\partial X_i \partial X_j}(t, X_t)dt \quad [2.9] \\
&= \left[\frac{\partial f}{\partial t}(t, X_t) + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial X_i}(t, X_t) + \frac{1}{2}\sum_{i=1}^{n} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 f}{\partial X_i \partial X_j}(t, X_t)\right] dt \\
&\quad + \sum_{i=1}^{n} \sigma_i \frac{\partial f}{\partial X_i}(t, X_t)dW_t^i \quad [2.10]
\end{aligned}
$$

It is worth noting that for one dimensional Itô process $X_t(\xi)$, $f = f(t, X_t(\xi))$ satisfies the stochastic differential equation:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial X}(t, X_t)dX_t + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2}(t, X_t)dt \quad [2.11] \\
&= \left[\frac{\partial f}{\partial t}(t, X_t) + \mu \frac{\partial f}{\partial X}(t, X_t) + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2}(t, X_t)\right] dt + \sigma \frac{\partial f}{\partial X}(t, X_t)dW_t
\end{aligned}
$$

2.2.5 Geometric Brownian Motion

Geometric Brownian Motion (GBM) or Log-normal Brownian Motion, is a generalized Wiener process. It is worth nothing that if $Y \sim \mathcal{N}(\mu\,,\, \sigma^2)$, then $x = e^Y$ follows **log-normal distribution** with density $p(x)$ as follows :

$$
p(x) = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, & x \geq 0 \\ 0 & x < 0 \end{cases} \quad [2.12]
$$

It is not difficult to deduce its expectation $\mathrm{E}(x) = e^{\mu + \frac{1}{2}\sigma^2}$ and variance $\mathrm{Var}(x) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$. In practice, GBM is a simple and efficient Stochastic Differential Equation (SDE) to model the assert price under linear drift and diffusion. The mathematical expression is defined as follows:

$$
dX(t) = \mu X(t)dt + \sigma X(t)dW_t \quad [2.13]
$$

where the coefficients of drift μ and volatility σ are constants. $(W_t)_{t\in[0,\infty)}$ is a standard Wiener process and $dW_t \sim \mathcal{N}(0, dt)$. Thanks to Itô Lemma, the log-price $x(t) = \ln X(t)$ satisfies the following SDE:

$$
dx(t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \quad [2.14]
$$

Furthermore, integrating Eq.(2.14), we obtain:

$$X(t) = X(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t} \qquad [2.15]$$

It is worth to note that many options pricing models assume the assert price following GBM which is a Markov process.

2.2.6 Change of Numeraire and Equivalence Probability Measure

Change of Numeraire

In an arbitrage-free and complete market, the value of any contingent claim can be uniquely determined as the expectation of the payoff normalized by the money market account under a unique equivalent measure. (see, for example, Harrison and Kreps 1979, Harrison and Pliska, 1981). Under this measure, the expected return on all assets is equal to the risk-free rate. Hence this measure is named as the risk-neutral measure, often denoted by Q. The normalizing asset is called the numeraire.

The paper of Geman et al. 1995 has shown that not only the money market account can be used as numeraire, but every strictly positive self-financing portfolio of traded assets can be used as numeraire. Furthermore, the changing numeraire approach demonstrates the way to change numeraire by switching between different probability measures. As a byproduct, every positive non-dividend paying asset divided by its numeraire is a martingale under the measure associated with that numeraire.

Equivalence Probability Measure

Theorem 2.2.3. *Considering two probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\Omega, \mathcal{F}, \mathbb{Q})$ *, the two probability measures* $\mathbb{P}$ *and* $\mathbb{Q}$ *are equivalent if*

- $P(A) > 0 \Rightarrow Q(A) > 0$ *, and for all* $A \in \Omega$ *,*
- $P(A) = 0 \Rightarrow Q(A) = 0$ *, and for all* $A \in \Omega$ *.*

Radon-Nikodym Derivative

Following equivalence measure theorem, **Radon Nikodym derivative** is defined as follows:

$$D(t) = \frac{dQ}{dP}(t) \qquad [2.16]$$

Radon Nikodym derivative is widely used in changing measures. Hence for any random variable X, the following equation holds.

$$\mathrm{E}^P(XM) = \int_\Omega X(\omega)M(t,\omega)dP(\omega) = \int_\Omega X(\omega)dQ(\omega) = \mathrm{E}^Q[X] \qquad [2.17]$$

This interchangeability of the expected values under two different measures confirms the importance of Radon-Nikodym derivative as an intermediate between two measures.

Girsanov Theorem

Girsanov's theorem gives us some concrete instructions to change the measures of stochastic processes. Considering a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ $(T < \infty)$ and a stochastic process $G(t)$

$$G(t) = e^{-\int_0^t \lambda(s)dW_s^P - \frac{1}{2}\int_0^t \lambda^2(s)ds}, \quad t \in [0,T] \qquad [2.18]$$

where W_s^P is a Brownian Motion under probability measure P. $\lambda(t)$ is a $\mathcal{F}_t$ measurable process that satisfies the condition as follows:

$$\mathrm{E}^P\left\{e^{\frac{1}{2}\int_0^t \lambda^2(s)ds} < \infty\right\}, \quad t \in [0,T] \qquad [2.19]$$

in addition, supposing W_t^Q is defined by

$$W_t^Q = W_t^P + \int_0^t \lambda(s)ds\,, \quad t \in [0,T] \qquad [2.20]$$

then we can obtain the following results:

- $G(t)$ defines a Radon-Nikodym derivative,
- W_t^Q is a Brownian motion with respect to $\mathcal{F}_t$ under probability measure Q.

2.2.7 Kolmogorov Equation

In probability theory, Kolmogorov equations, normally refereed to Kolmogorov forward equation and Kolmogorov backward equation, are used to characterize random dynamic processes. Assume we know a statistical description of a stochastic process x(t), then the Kolmogorov equations are used for determining features of a correlated transformation process y(t).

Kolmogorov Forward Equation

Kolmogorov Forward Equation (or **Kokker Planck Equation**) named after Adriaan Fokker and Max Planck who describe the dynamics of the probability density function (pdf) over time.

Assuming that the state variable x_t satisfies the following stochastic differential equation:

$$dx(t) = \mu(x(t), t)\, dt + \sigma(x(t), t)\, dW(t) \qquad [2.21]$$

then the pdf of x_t satisfies **Kokker Planck Equation** as follows

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\sigma^2(x,t)p(x,t)\right] \qquad [2.22]$$

Kolmogorov Backward Equation

Assume that random variable x_t in Eq.(2.21), then the pdf of x_t satisfies **Kolmogorov backward equation** as follows:

$$-\frac{\partial}{\partial t} p(x,t) = \mu(x,t)\frac{\partial}{\partial x}p(x,t) + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}p(x,t) \qquad [2.23]$$

for $t \leq s$, and the final condition $p(x,s) = u_s(x)$.

2.3 Stochastic Volatility model and Financial Derivatives

Option pricing theory represents one of the most significant concepts of modern finance. In the breakthrough work of Black, Scholes and Merton in 1973, an explicitly tractable formula of option pricing is deduced, where option is applied as a risk-free instrument (dynamic hedging). Thanks to the rapid improvement of computer technology and the foundation of Chicago Board Option Exchange (CBOE) in 1973, both theoretical and empirical study on options, futures and other derivatives grow explosively. Derivatives, as major portion of financial innovation, have significantly expanded modern financial market.

The Black-Scholes model (BS) is constructed with the fundamental concept of Brownian Motion. Especially, the underlying stock price is assumed to follow a Geometric Brownian Motion which will be detailedly explained in the following section. The analytical treatment of BS model is well defined and the closed form solution is deduced. Under certain assumptions, the option price from BS model is equal to its replicating portfolio. In addition, it is worth to note that BS option pricing formula is identical and independent from the investor's risk attitude and expectation. Therefore, BS model triggers the growing applications of options and other derivatives.

2.3.1 Options

According to the type of transaction, there are usually two different types of options: call and put Options. A call (or put) option is a financial instrument giving the holder right, but not the obligation to buy (or sell) an underlying asset at or by a certain specified date T at a certain specified price X. Buying a call (or put) option contract hedges upward (or downwards) movement of the price of underlying asset.

The most basic classification of option contracts is made by the expiration time. European options can be expired only at the expiration time T, while American options can be expired at every moment by the expiration time T defined in the contract. The most basic (European and American) options are called plain vanilla options.

European Vanilla Option

The pay-off functions of a purchased European vanilla options are given by the following expressions:

$$C(S,T) = \mathbf{max}(S-K,0) \tag{2.24}$$

$$P(S,T) = \mathbf{max}(K-S,0) \tag{2.25}$$

where C and P stand for call and put options. The strike price K is constant. Another way of writing the pay-off functions is $V(S,T) = \mathbf{max}(\lambda(S-K),0)$, with $\lambda = 1$ for call option and $\lambda = -1$ for put option. The value of a European plain vanilla option can be calculated as a solution of the Black-Scholes partial differential equation. The detail deduction will show in the following section.

American Vanilla Option

The difference between American-style and European-style options is the possibility to claim the contract before the expiration date. Such feature gives an advantage to the holders of an American style options. Consequently, there is inequality relationship between values of these two types of options. Besides these two types of options, the other types of options are usually marked as exotic options, however, their classification is not exact. The following section will discuss the fundamental option pricing theory.

2.3.2 Option Pricing and Black Scholes Model

Black-Scholes Option Pricing Model

In 1973, Black,Scholes and Merton suggested the following stochastic differential equation (SDE) as a dynamic model for approximating stock price.

$$dS_t = \mu S_t + \sigma S_t dW_t \qquad [2.26]$$

where S_t , μ and σ denote the spot price at time t, the return on the stock, and the volatility of the stock. In addition, σ is also defined as the standard deviation of the log-returns. W_t is a standard Brownian Motion. The first term of the right hand side is called the drift term which is the deterministic part of the equation, and drives the process value S_t in a deterministic way. The second part is called the diffusion term and is the stochastic part. Diffusion term contributes to the process S_t with a random noise that is amplified by the volatility parameter σ. The previous section tell us the stock price follows a Geometric Brownian Motion (GBM), i.e.

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \qquad [2.27]$$

It is worth nothing that the stock price in this model follows log-normal distribution. Please recall that option price formula $V(S,t) = \mathbf{max}(\lambda(S - E), 0)$ with $\lambda = 1$ for call option and $\lambda = -1$ for put option, and the strike price E is constant. From Itô's lemma in Eq.(2.9), we obtain:

$$\begin{aligned} dV = & \left[\frac{\partial V}{\partial t}(t, S_t) + \mu S \frac{\partial V}{\partial S}(t, S_t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(t, S_t) \right] dt \\ & + \sigma S \frac{\partial V}{\partial S}(t, S_t) dW_t \end{aligned} \qquad [2.28]$$

Considering an agent who owns a portfolio with value $\Pi(S,t) = V(S,t) - \Delta S$ at time t, this portfolio is invested in option and fixed $(-\Delta)$ shares of stocks. For a time-step jump dt, we have

$$d\Pi = dV - \Delta dS \qquad [2.29]$$

After substituting Eqs.(2.26),(2.28) into (2.29), we obtain

$$d\Pi = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mu \Delta S \right] dt + \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dW_t \quad [2.30]$$

Clearly, $\Pi(t, S)$ follows a random walk process, and the random component disappears if

$$\Delta = \frac{\partial V}{\partial S} \tag{2.31}$$

where Δ (**Delta**) measures the sensitivity of the option value, and is chosen at the beginning of time-step dt. Δ is an important **Greek** in finance, and we will explain this detailedly in the next section.

Without random component, the portfolio satisfies a deterministic increment as follows.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{2.32}$$

It is worth nothing that if an agent invests an amount Π in risk-less money market, the return should be $\frac{d\Pi}{dt} = r\Pi$ for time jump dt. **No arbitrage condition** will guarantee the following equality.

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r\Pi dt \tag{2.33}$$

Substituting $\Pi(S, t) = V(S, t) - \Delta S$ and Eq.(2.31) into (2.33), we obtain **Black-Scholes-Merton Differential Equation**.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.34}$$

It is worth to remark that the Black-Scholes-Merton Differential Equation can be obtained from another point of view. Let us suppose a hedging portfolio $X(t)$ that has the same value and plays the same role as option $V(t, S)$.

$$X(t) = V(t, S) \tag{2.35}$$

The portfolio $X(t)$ is composed by one share of Δ in stock S_t and another share of risk free assert with interest rate r i.e.

$$X(t) = (X(t) - \Delta S) + \Delta S \tag{2.36}$$

For one step time jump dt, we can obtain:

$$dX = r(X - \Delta S)dt + \Delta dS \tag{2.37}$$

Substituting Eq.(2.26) into (2.37), we obtain

$$dX = rXdt + (\mu - r)S\Delta dt + \sigma S\Delta dW \tag{2.38}$$

where $rXdt$ stands for the return of portfolio under risk free interest rate r, and $(\mu - r)$ is the risk premium due to investing in stock. $\sigma S\Delta dW$ means the volatility of portfolio that is proportional to the share of stock investment.

Substituting formulas Eqs.(2.28) and (2.38) into Eq.(2.35), we obtain the following equality:

$$\begin{aligned} & [rV + (\mu - r)S\Delta]dt + \sigma S\Delta dW \\ = & \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S\frac{\partial V}{\partial S}dW \end{aligned} \qquad [2.39]$$

By comparing coefficients of both left hand side and right hand side of the equation, we obtain the following equalities.

$$\Delta = \frac{\partial V}{\partial S} \qquad [2.40]$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad [2.41]$$

Eqs.(2.40) and (2.41) are exactly Eq.(2.31) and Eq.(2.34), i.e. Black-Scholes-Merton Differential Equation. This partial differential equation (PDE) can be solved analytically for different pay-off functions. The following bounder conditions are necessary to guarantee the uniqueness of the Black-Scholes-Merton Differential Equation's solution. The idea is to impose some economically justified constraints on the solution of the PDE. Taking European Call option as an example, the constrains are

1. $V(S,T) = C(S,T) = \max(S - K, 0) \qquad t = T, \quad S \in (0, \infty)$
   i.e. Call option will only be exercised if $S_T > K$, and the gain is exactly $S_T - K$. Hereby, the Brownian motion implies that the process is absorbed by 0 when $S_T \leq E$. This is also called the final condition.

2. $C(0,t) = 0, \qquad t \in [0,T)$.

3. $\lim\limits_{S\to\infty} C(S,t) = S, \qquad t \in [0,T)$.

Eq.(2.34) the Black-Scholes-Merton Differential Equation can be easily solved by transforming the Black-Scholes-Merton PDE into the heat equation (diffusion equation). Here we omit the details of deductions, and directly give the final solution as following:

$$C(t, S_t, K, T, \sigma, r) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \qquad [2.42]$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad [2.43]$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad [2.44]$$

with strike price K and maturity time T. The spot price at time t and the risk-less interest rate are denoted by S_t and r. It is worth to note $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$ represents the cumulative distribution formula for a standard normal random variable.

The corresponding European put option price $P(S, \tau)$ can be deduced similarly by solving Black-Scholes-Merton Differential Equation under certain boundary conditions. But in general, it will be more convenient to use the following **Call-Put Parity** property.

Theorem 2.3.1. *Given the price formulas of a European call option and European put option defined in previous section with the same strike price and maturity date, the following equality holds:*

$$C_t + Ke^{-rT} = P_t + S_t \quad [2.45]$$

Proof. We denote $\mathbf{max}(S-K, 0) = (S-K, 0)_+$, and clearly the following equality holds.

$$S_T - K = (S_T - K)_+ - (K - S^T)_+ \quad [2.46]$$

or

$$(S_T - K)_+ + K = (K - S^T)_+ + S_T \quad [2.47]$$

Multiplying both sides with the discount factor $e^{-r(T-t)}$, $t \leq T$ and taking conditional expectations given stock price at time t S_t under the risk neutral measure Q , we obtain

$$\begin{aligned} & \mathrm{E}^Q[e^{-r(T-t)}(S_T - K)_+ \mid S_t = s] + \mathrm{E}^Q[e^{-r(T-t)}K \mid S_t = s] \\ = \; & \mathrm{E}^Q[e^{-r(T-t)}(K - S_T)_+ \mid S_t = s] + \mathrm{E}^Q[e^{-r(T-t)}S_t \mid S_t = s] \quad [2.48] \end{aligned}$$

Under risk neutral measure, the discounted value of a risky asset is a martingale. Hence, the first expectation on the left (right) side of the Eq.(2.48) is the price of a call (put) option at time t, i.e.

$$\mathrm{E}^Q[e^{-r(T-t)}(S_T - K)_+ \mid S_t = s] = C(S, t) \quad [2.49]$$

$$\mathrm{E}^Q[e^{-r(T-t)}(K - S_T)_+ \mid S_t = s] = P(S, t) \quad [2.50]$$

The second expectation of the left (right) side of the equation is deterministic. So this proves **Call-Put Parity** in Eq.(2.46). □

Given Eqs.(2.42)-(2.44), European put option price $P(S, \tau)$ can be deduced straightforward using Call-Put Parity as follows.

$$P(t, S_t, K, T, \sigma, r) = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1) \qquad [2.51]$$

where the variables and parameters are defined in Eqs.(2.43) and (2.44)

Risk Neutral Valuation

An alternative approach to solve Black-Scholes-Merton Differential Equation is **risk-neutral valuation**. In risk neutral world, the investors do not require risk premium for their investment. In other words, the investors only demand the risk free return of interest rate as average. Hence in a risk-neutral world, no arbitrage price of the call option C at time t equals to the expected value of discounted (by the risk free rate) option price at maturity T. Taking European call option as an example, the following equality holds:

$$\begin{aligned} C(t, S) &= \mathbb{E}^Q(e^{-r(T-t)}C(T, S) \mid_{S=S_T}) && [2.52] \\ &= e^{-r(T-t)}\mathbb{E}^Q((S_T - K)_+ \mid_{S=S_T}) && [2.53] \end{aligned}$$

and for $t = 0$,

$$C(0, S) = e^{-rT}\mathbb{E}^Q((S_T - K)_+ \mid_{S=S_T}) \qquad [2.54]$$

Let us recall Black-Scholes-Merton SDE in Eq.(2.26), where W_t represents a standard Brownian Motion under the physical measure. Here we slightly modify the equation as follows:

$$dS_t = rS_tdt + \sigma S_t \left[\left(\frac{\mu - r}{\sigma} \right) dt + dW_t \right] \qquad [2.55]$$

Now we define

$$\tilde{W}_t = W_t + \left(\frac{\mu - r}{\sigma} \right) t \qquad [2.56]$$

Thanks to **Girsanov's theorem** in previous section, we know $\tilde{W}_t$ represents a Brownian motion under probability measure Q. The market price of risk is defined as $\dfrac{\mu - r}{\sigma}$, and Eq.(2.55) now becomes

$$dS_t = rS_tdt + \sigma S_td\tilde{W}_t \qquad [2.57]$$

Furthermore, we define $\tilde{S}_t = e^{-rt}S_t$ as the discounted stock price. Applying Ito's lemma, we obtain the following:

$$d\tilde{S}_t = \sigma \tilde{S}_td\tilde{W}_t \qquad [2.58]$$

Therefore, we obtain a unique risk neutral measure Q, and the discounted payoff of European call option price $C(t,S) = \mathrm{E}_Q(C(T,S_T)|S_t = s)$ which is a martingale under risk neutral measure Q. Moreover, as described in previous section, the stock price follows a geometric Brownian motion(GBM), i.e Eq.(2.42). Choosing $t = T$, we obtain:

$$\begin{aligned} S_T &= S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \\ &= S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma \tilde{W}_T} \end{aligned} \qquad [2.59]$$

From the log-normal distribution density function in Eq.(2.12), we obtain:

$$S_T = e^Y \qquad [2.60]$$

with

$$\begin{aligned} Y &= \ln(S_0) + (r - \frac{1}{2}\sigma^2)T + \sigma \tilde{W}_T && [2.61] \\ &:= \ln(S_0) + \tilde{\mu} + \tilde{\sigma} && [2.62] \end{aligned}$$

where

$$\begin{aligned} \tilde{\mu} &= (r - \frac{1}{2}\sigma^2)T && [2.63] \\ \tilde{\sigma} &= \sigma \tilde{W}_T && [2.64] \end{aligned}$$

Since $W_T \sim \mathcal{N}(0,T)$, then $Y \sim \mathcal{N}(\ln S_0 + \tilde{\mu}, \sigma^2 T)$. Eq.(2.12) in section of GBM indicates that S_T follows a Log-normal distribution with the density function as follows (here we denote $X := S_T$):

$$p(X) = \begin{cases} \frac{1}{X\sigma\sqrt{2\pi T}} e^{-\frac{(\ln \frac{X}{S_0} - \tilde{\mu})^2}{2\sigma^2 T}}, & X \geq 0 \\ 0 & X < 0 \end{cases} \qquad [2.65]$$

Substituting into Eq.(2.54), we obtain:

$$\begin{aligned} C(0,S) &= e^{-rT}\mathbb{E}^Q((X-K)_+ \,|_{X=x}) \\ &= e^{-rT} \int_K^\infty \frac{1}{X\sigma\sqrt{2\pi T}} (X-K) e^{-\frac{(\ln \frac{X}{S_0} - \tilde{\mu})^2}{2\sigma^2 T}} dX \\ &= e^{-rT} \int_K^\infty \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{(\ln \frac{X}{S_0} - \tilde{\mu})^2}{2\sigma^2 T}} dX - e^{-rT} \int_K^\infty \frac{1}{X\sigma\sqrt{2\pi T}} K e^{-\frac{(\ln \frac{X}{S_0} - \tilde{\mu})^2}{2\sigma^2 T}} dX \end{aligned} \qquad [2.66]$$

We focus on the first item of right hand side of Eq.(2.66).

$$
\begin{aligned}
e^{-rT}\int_K^\infty \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{(\ln\frac{X}{S_0}-\tilde{\mu})^2}{2\sigma^2T}}dX &= S_0e^{-rT}\int_K^\infty \frac{X}{S_0}\frac{1}{X\sigma\sqrt{2\pi T}}e^{-\frac{(\ln\frac{X}{S_0}-\tilde{\mu})^2}{2\sigma^2T}}dX \\
&= S_0\int_K^\infty \frac{1}{\sigma\sqrt{2\pi T}}e^{\ln\frac{X}{S_0}-rT-\frac{(\ln\frac{X}{S_0}-\tilde{\mu})^2}{2\sigma^2T}}d\ln\frac{X}{S_0} \\
&= S_0\int_K^\infty \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{(Z-(r+\sigma^2T))^2}{2\sigma^2T}}dZ \qquad [2.67] \\
&= S_0\left[1-\int_{-\infty}^K \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{(Z-(r+\sigma^2T))^2}{2\sigma^2T}}dZ\right] \\
&= S_0N(d_1) \qquad [2.68]
\end{aligned}
$$

where $Z=\ln\frac{X}{S_0}$ and $N(d_1)$ represents the cumulative distribution function for the standard normal variable d_1, i.e.

$$d_1=\frac{\ln\frac{S_0}{K}+rT+\frac{\sigma^2T}{2}}{\sigma\sqrt{T}} \qquad [2.69]$$

The property of standard normal distribution indicates the following:

$$
\begin{aligned}
1-N\left(\frac{\ln\frac{K}{S_0}-rT-\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}\right) &= N\left(-\frac{\ln\frac{K}{S_0}-rT-\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}\right) \qquad [2.70] \\
&= N\left(\frac{\ln\frac{S_0}{K}+rT+\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}\right) \qquad [2.71]
\end{aligned}
$$

Similarly, we will obtain the second item of right hand side of Eq.(2.66)

$$-e^{-rT}\int_K^\infty \frac{1}{X\sigma\sqrt{2\pi T}}Ke^{-\frac{(\ln\frac{X}{S_0}-\tilde{\mu})^2}{2\sigma^2T}}dX=-Ke^{-rT}N\left(\frac{\ln\frac{S_0}{K}+rT-\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}\right) [2.72]$$

Finally substituting Eqs.(2.68) and (2.72) in formula (2.66), we could obtain the price of European vanilla call option which is identical to Eq.(2.42). Similarly, we could also obtain the same European vanilla put option price formula in Eq.(2.51).

2.3.3 The Greeks

The sensitivity of an option price with respect to underlying variables or parameters is commonly referred to as a **Greek**. For instance, let us recall the fundamental Black-Scholes-Merton Differential Equation in Eq.(2.34), i.e.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad [2.73]$$

where $\frac{\partial V}{\partial t}$, $\frac{\partial^2 V}{\partial S^2}$ and $\frac{\partial V}{\partial S}$ are important 'Greeks' for option pricing. The name 'Greeks' stems from the fact that these quantities are usually denoted by Greek letters. Greeks, also called risk sensitivities, play an important role in risk management. The Greeks can be classified as first-order Greeks, second-order Greeks, and third-order Greeks.

First Order Greeks

Delta, Δ measures the sensitivity of the option price with respect to the current stock price S_t, i.e. $\Delta = \frac{\partial V}{\partial S}$, where V denotes the option price. In Black-Scholes model in Eq.(2.42), the Delta of European call option is

$$\Delta_C = N(d_1) \qquad [2.74]$$

In practice for vanilla options, a long call (or short put) option has $\Delta_C \in [0, 1]$, and a long put (or a short call) option has $\Delta_P \in [-1, 0]$. Because of the relationship of **Call-Put Parity**, the sum of the absolute values of the Deltas with respect to put and call options are equal to 1, i.e. $\Delta_C - \Delta_P = 1$.

Kappa (or **Vega**), κ (or ν):= $\frac{\partial V}{\partial \sigma}$ measures the sensitivity of option price with respect to the volatility of underlying assert i.e σ. **Kappa** can be thought as the gain or lose of option's value when volatility increases or decreases 1 unit. **Kappa** is an important Greek in violate market where the option values are sensitive to the change in volatility.

Theta, Θ measures the sensitivity an option's value on time, i.e $\Theta = \frac{\partial V}{\partial t}$, or sometimes $\Theta = -\frac{\partial V}{\partial \tau}$, where $\tau = T - t$ is defined as time to maturity. **Theta** is important to measure an option's time value. The option's time value decreases when time is close to maturity. In other words, Θ is almost always negative for long call and put options (or positive short call and put options).

Rho, ρ measures sensitivity of option's value to interest rate, i.e $\rho = \dfrac{\partial V}{\partial r}$. If we focus on risk less interest rate, ρ is less sensitive than other Greeks, such as Δ, κ, Θ.

Lambda, $\lambda = \dfrac{\partial V}{\partial S}\dfrac{S}{V}$ represents the elasticity of option's value to stock price.

Second Order Greeks

Gamma, Γ measures the sensitivity of delta with respect to stock price. It is defined as the second order derivative of V with respect to S_t, i.e $\Gamma = \dfrac{\partial^2 C}{\partial S_t^2}$. It represents the convexity of the option's value, and Γ is positive for all long options(negative for all short Option). For an option trader, who seeks an effective Δ-hedge over a wider range of underlying price movements, would like to neutralize the portfolio's Γ. In Black-Scholes model, the gamma equals

$$\Gamma = \frac{p(d_1)}{S_t \Sigma \sqrt{T-t}} \qquad [2.75]$$

where $p(d_1)$ is the density function of standard normal distribution.

Vanna is the second derivative of the option's value to both spot price and volatility, i.e **Vanna** $= \dfrac{\partial^2 V}{\partial S \partial V} = \dfrac{\partial \Delta}{\partial \sigma} = \dfrac{\partial \kappa}{\partial S}$. Thus **Vanna** measures the sensitivity of option's **Delta** with respect to volatility. Meanwhile, **Vanna** measures the sensitivity of **Kappa** with respect to spot price. Therefore, **Vanna** is an important **Greek** for the traders managing the Δ-hedge or κ-hedge portfolios.

Vomma measures the second order sensitivity the option's value to volatility, i.e **Vomma** $= \dfrac{\partial^2 V}{\partial \sigma^2} = \dfrac{\partial \kappa}{\partial \sigma}$. In addition, **Vomma** also measures the sensitivity of **Kappa** with respect to volatility. For instance, positive **Vomma** implies the increase of volatility, and leads to long κ.

Charm (or Δ decay) measures the sensitivity of Δ change over time, and Θ with respect to spot price i.e. **Charm** $= \dfrac{\partial^2 V}{\partial S \partial t} = \dfrac{\partial \Delta}{\partial t} = \dfrac{\partial \Theta}{\partial S}$.

Veta measures both the sensitivity of κ over time, and Θ with respect to volatility of underlying assert, i.e **Venta** $= \dfrac{\partial^2 V}{\partial\sigma\partial t} = \dfrac{\partial\kappa}{\partial t} = \dfrac{\partial\Theta}{\partial\sigma}$.

Vera measures the second order sensitivity of option's value to volatility and interest rate, i.e **Vera** $= \dfrac{\partial^2 V}{\partial\sigma\partial r} = \dfrac{\partial\rho}{\partial\sigma}$. Meanwhile, it also measures the sensitivity of ρ with respect to volatility. Thus, it is important for traders who focus on volatility effect on ρ-hedging.

Third Order Greeks

Speed $= \dfrac{\partial\Gamma}{\partial S} = \dfrac{\partial^3 V}{\partial S^3}$ measures the sensitivity of Γ with respect to spot price. It is an important Greek for Δ-hedging or Γ-hedge portfolio.

Ultima $= \dfrac{\partial\textbf{Vomma}}{\partial\sigma} = \dfrac{\partial^3 V}{\partial\sigma^3}$ measures the sensitivity of **Vomma** with respect to volatility. It is a n important index for κ-hedging or **Vomma**-hedge portfolio.

Color $= \dfrac{\partial\Gamma}{\partial\tau} = \dfrac{\partial^3 V}{\partial S^2\partial\tau}$, $\tau = T - t$ measures the sensitivity of Γ over time. It is an important indicator of Γ-hedge portfolio across time.

Zomma $= \dfrac{\partial\Gamma}{\partial\sigma} = \dfrac{\partial\textbf{Vanna}}{\partial S} = \dfrac{\partial^3 V}{\partial S^2\partial\sigma}$ measures the sensitivity of Γ with respect to volatility. It is an important indicator of Γ-hedge portfolio when volatility fluctuates.

2.3.4 Implied Volatility (IV)

Implied volatility (IV) is a by-product of the option pricing. In Black-Scholes (BS) model, the option price is determined by five variables: current price of underlying assert, maturity (time to expiration), strike price,risk-free interest rate, and implied volatility (IV), where IV is the only unobservable factor and normally refers to annual volatility of stock price (the standard deviation of the short-term returns over one year). Mathematically speaking, we have the following expression of implied volatility:

$$V^{BS}(t, S_t, K, T, \sigma^{IV}) = V^{MKT} \qquad [2.76]$$

where V^{BS} is a monotonically increasing function of volatility, and σ^{IV} represents IV which is implied by the market price V^{MKT}. It is worth to remark that the BS pricing function V^{BS} does not have a closed-form solution for its inverse, however, a unique IV can be deduced numerically using Newton

Raphson method.

Theoretically speaking, IV should refer to the future volatility of the return of underlying assert, however, the future volatility is impossible to foresee. In practice, the future volatility is predicted by historical volatility. Therefore, IV is a statistically weighted approximation.

It is worth to highlight that BS model only tells us the theoretical option price, however, the real option price, which is driven by the market (e.g. S& P 500 Index Options), is not necessarily identical to the modelled option price. From the model construction point of view, the cause of the biases between the theoretical option price and the real price can be attributed to the volatility variable which is the only unobservable factor from the market. In other words, if we assume that all the other four input variables (spot price of underlying assert, maturity, strike price, and risk-free interest rate) are market-based, the true IV will enable the theoretical option price to match the market price.

In conclusion, IV reflects the expectation of future volatility and market's view of the fluctuations. Not surprisingly, IV plays import role in derivative market with various applications. For instance, two fundamental applications are discussed in the next section.

Volatlity Smile

In BS option pricing model, the volatility variable is assumed to be constant over time. This is practically not true, however. Plotting the implied volatilities (IV) as vertical axe against the strike prices as horizon axe, we will obtain a U-shaped curve often referred to as the 'smile' shape. Specifically, for different options having the same expiration date and underlying asset, the more the exercise prices are away from current spot price (in-the-money or out-of-the-money), the larger the IV becomes. Hence, this particular volatility skew pattern looks like a smile shape.

Theoretically, IV smile is attributed to the assumption of BS model, where the yield of assert is assumed to follow a normal distribution (or the underlying assert follows the log-normal distribution). Yet, this assumption is not confirmed by the market. Empirical study found that yield's distribution have strong characteristics of high peak and fat tails. Therefore, the probability of extreme values are significantly underestimated by the assumption of normality. Practically speaking, the values of deep in the money or out of the money options are underestimated.

Of course, there are many other explanations to this skew patter in practice, such as option premiums, asymmetric transaction cost and so on. Due to space limitations, the detail description will not present here. Volatility smile tells us there are greater demand for options that are either deeper in the money or out of the money. The volatility skew pattern is commonly observed in near-term equity options and options in the FX market.

Application of IV

1. IV in Prediction Stock Price Movement
IV tends to move in the opposite direction of underlying assert price in stock market. An intuitive explanation is that people who hold stocks will become panic and pay irrational price for put options in case of stock prices falling for a period. Hence the IV of corresponding options will increase. When the stock prices increase, IV decreases because people become less panic. Moreover, high price stocks normally have lower volatility values than low price stocks. For non-equity assets, the relationship between IV and the corresponding underlying assert is less clear. Nevertheless, it is worth to highlight that IV reflects market sentiment, and its extreme values are important signals of the market rebound or correction.

Given the important role of IV, Chicago Board Options Exchange (CBOE) introduced CBOE Volatility Index (VIX, also called Fear Index) in 1993. VIX represents a key measure of market's expectation of near-term volatility (over the next 30-day period) conveyed by S&P 500 stock index option prices.

2. Volatility Trading Strategies
IV provides us an efficient approach to optimize option strategies. Among various option strategies, there is a class of volatility trading strategy based on options' asymmetrical risk-return structure. In derivatives market, the options' owners have limited risk but unlimited return. Taking call option as an example, the more the stock price increase (above the exercise price), the more the return they will obtain. On the contrary, no matter how stock price falls, the maximum loss is option's premium which is the amount that an option buyer has paid to the seller at the beginning of the contract. With this asymmetry property (given other conditions unchanged), the greater the IV, the higher the option premium.

The principle of volatility trading strategy is to buy 'low IV' and sell 'high

IV' in order to make profit when market fluctuation changes significantly. Specifically, there are two fundamental strategies as follows.

- When the market fluctuation is expected to substantially increase, but the direction (positive or negative) is less clear, we could profit from buying straddles and strangles. At the maturity, if the option's payoff is over its premium due to the IV increase, we are better off with positive gain, no matter fluctuation's direction. If the degree of increase is not as expected, we will get loss.

- When the market fluctuation is expected to substantially decrease, but the direction (positive or negative) is less clear, we could profit from selling straddles and strangles. At the maturity, if the stock price lies between two exercise prices due to IV decrease, we are better off with positive gain regardless of fluctuation's direction. Otherwise, we get loss.

It is worth to remark that buying (or selling) straddle means buying (or selling) both a call option and a put option with the same maturity and strike price, and buying (or selling) strangle means buying (or selling) both a call option and a put option with the same maturity but different strike prices.

2.3.5 Classical Stochastic Volatility Models

The drawback of constant volatility assumption limits BS model's application in option pricing. This section will introduce several important stochastic volatility models which generalize the assumption of constant volatility.

Hull-White Model

The Hull-White (HW) stochastic volatility model 1897 is composed by two dimensional Hull-White formulas (i.e assert process and volatility process) as follows:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^Q \quad [2.77]$$
$$d\sigma_t^2 = \mu\sigma^2 dt + \gamma\sigma^2 dZ_t^Q \quad [2.78]$$
$$dW^{Q_t} dZ_t^W = 0 \quad [2.79]$$

where W_t^Q, Z_t^Q are two standard Brownian motions under risk neutral probability measure Q. We denote with S_t, σ^2 and r equity price, its volatility and interest rate. Hence, both spot price process S_t , and volatility σ_t^2 follows Geometric Brownian motion. By Ito's lemma, the log-price $x_t = \ln S_t$ satisfies the following

$$dx_t = (r - \frac{1}{2}\sigma^2)dt + \sigma_t dW_t^Q \qquad [2.80]$$

The European call option of underlying stock S_t with strike price K and maturity T takes the form

$$C(t, S_t, K, T) \ = \ \mathrm{E}^Q[e^{-r(T-t)}C(T, S_T, K, T) \mid \mathcal{F}_t] \qquad [2.81]$$

Using the law of iterated expectations i.e $E[E(y \mid x)] = E(y)$, we obtain

$$C(t, S_t, K, T) = \mathrm{E}^Q \left\{ \mathrm{E}^Q[e^{-r(T-t)}C(T, S_T, K, T) \mid \mathcal{F}_t, \sigma_s, s \in [t, T]] \mid \mathcal{F}_t \right\} [2.82]$$

The inner expectation is the call option when volatility is time-dependent but deterministic. Integrating underlying stock price in Eq.(2.80), we obtain:

$$S_T = S_t e^{r(T-t) - \frac{1}{2}\int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_t^Q} \qquad [2.83]$$

where σ_t is a deterministic function over time. The property of the stochastic integrals leads the following:

$$\ln(S_T/S_t) \sim \mathcal{N}\left((r - \frac{1}{2}\bar{\sigma^2})(T-t) \ , \ \bar{\sigma^2}(T-t)\right) \qquad [2.84]$$

where $\bar{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma_s^2 ds$ denotes the mean squired volatility. The distribution in Eq.(2.84) will reduce to the risk neutral price distribution in a Black-Scholes model in case of fixed volatility, i.e $\sigma_{BS}^2 = \bar{\sigma^2}$. The option price formula in HW model is deduced as follows:

$$C(t, S_t, K, T) \ = \ \mathrm{E}^Q[C^{BS}(t, r, S_t, \sqrt{\bar{\sigma^2}}, K, T) \mid \mathcal{F}_t] \qquad [2.85]$$

Like Black-Scholes model, the closed form solution also exists in Hull-White model. It is worth to note that HW model assumes independence relationship between underlying assert and the volatility process i.e $W_t^Q \perp Z_t^Q$. This assumption can be generalized, and we will discuss it in Heston model.

Heston Model

Heston Model 1993 is one of the most popular stochastic volatility models in option pricing. This model has an analytical European option pricing formula where the correlation between spot price and volatility process is considered. The model is defined as follows.

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^Q \quad [2.86]$$
$$dv_t = \kappa(\theta - v_t)dt + \gamma\sqrt{v_t} dZ_t^Q \quad [2.87]$$
$$dW_t^Q dZ_t^Q = \rho dt \quad [2.88]$$

where W_t^Q and Z_t^Q denote two Brownian Motions under risk neutral probability measure Q . Using Ito's lemma, we can obtain the log-price $x_t = \ln S_t$ process which satisfies the following SDE:

$$dx_t = (r - \frac{1}{2}v)dt + \sqrt{v_t} dW_t^Q \quad [2.89]$$

Substituting Eq.(2.89) into the system of Eqs.(2.86)-(2.88), we obtain:

$$\begin{aligned} dx_t &= (r - \frac{1}{2}v)dt + \sqrt{v_t} dW_t^Q \\ dv_t &= \kappa(\theta - v_t)dt + \gamma\sqrt{v_t} dZ_t^Q \\ dW_t^Q dZ_t^Q &= \rho dt \end{aligned} \quad [2.90]$$

Integrating Eq.(2.89) on $[t, T]$, we obtain the moment generation function of x_t as follows:

$$\phi(x_t, v_t, t, z) = \mathrm{E}[e^{zx(T)} \mid \mathcal{F}_t] \quad z \in \mathbf{C} \quad [2.91]$$

It is worth to remark that the characteristic function satisfies the Kolmogrov backward equation as follows.

$$-\frac{\partial\phi}{\partial t} = \kappa(\theta - v)\frac{\partial\phi}{\partial v} + (r - \frac{1}{2}v)\frac{\partial\phi}{\partial x} + \frac{1}{2}v\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}\gamma^2 v\frac{\partial^2\phi}{\partial v^2} + \rho\gamma v\frac{\partial^2\phi}{\partial x\partial v} \quad [2.92]$$

Denoting $\tau = T - t$, we obtain:

$$\frac{\partial\phi}{\partial\tau} = \kappa(\theta - v)\frac{\partial\phi}{\partial v} + (r - \frac{1}{2}v)\frac{\partial\phi}{\partial x} + \frac{1}{2}v\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}\gamma^2 v\frac{\partial^2\phi}{\partial v^2} + \rho\gamma v\frac{\partial^2\phi}{\partial x\partial v} \quad [2.93]$$

together with initial condition when $t = T$ (or $\tau = 0$).

$$\phi(x_T, V, T, \lambda) = \phi_x(\lambda) = e^{\imath\lambda x_T} \quad [2.94]$$

Heston (1993) suggests the solution takes the following form:

$$\phi(x_T, V, T, \lambda) = e^{A(\tau,\lambda) + B(\tau,\lambda)v + \lambda x} \quad [2.95]$$

Substitution Eq.(2.95) into Eq.(2.93), we obtain $A(\tau, \lambda)$ and $B(\tau, \lambda)$ which must satisfy the following ordinary differential equations:

$$\frac{\partial A}{\partial \tau}(\tau, \lambda) = \kappa\theta B(\tau, \lambda) \qquad [2.96]$$

$$\frac{\partial B}{\partial \tau}(\tau, \lambda) = \frac{1}{2}\gamma^2 B^2 + (\rho\gamma\lambda - \kappa)B + \frac{1}{2}(\lambda^2 - \lambda) \qquad [2.97]$$

with initial conditions

$$A(0, \lambda) = 0\,, \quad B(0, \lambda) = 0 \qquad [2.98]$$

Eq.(2.97) is Riccati equation that can be solved analytically. By substituting the solution of $B(\tau, \lambda)$ into Eq.(2.96), and integrating with respect to τ, we obtain A in the following:

$$A(\tau, \lambda) = A(0, \lambda) + \frac{\kappa\theta}{\gamma^2}\left\{(\mu + \zeta)\tau - 2\ln\left[\frac{1 - \eta e^{\zeta\tau}}{1 - \eta}\right]\right\} \qquad [2.99]$$

$$B(\tau, \lambda) = B(0, \lambda) + \frac{(\mu + \zeta - \gamma^2 B(0, \lambda))(1 - e^{\zeta\tau})}{\gamma^2(1 - \eta e^{\zeta\tau})} \qquad [2.100]$$

with

$$\mu = \kappa - \rho\gamma\lambda\,, \quad \zeta = \sqrt{\mu^2 - \gamma^2(\lambda^2 - \lambda)}\,, \quad \eta = \frac{\mu + \eta - \gamma^2 B(0, \lambda)}{\mu - \eta - \gamma^2 B(0, \lambda)} \qquad [2.101]$$

Option Pricing Under the Heston Model

Here we only focus on the deduction of European vanilla call option, and the European put option price formula can be obtained simply using Call-Put Parity.

$$C(t, S_t, K, T) = \mathrm{E}^Q[e^{-r(T-t)}C(T, S_T, K, T) \mid \mathcal{F}_t] \qquad [2.102]$$

$$= \mathrm{E}^Q[e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_t] \qquad [2.103]$$

$$= e^{-r(T-t)}S_t\mathrm{E}^Q\left[(\frac{S_T}{S_0} - \frac{K}{S_0})_+ \mid \mathcal{F}_t\right] \qquad [2.104]$$

$$= e^{-r(T-t)}S_t\left\{\mathrm{E}^Q\left[e^{x_T}\mathbf{1}_{\{S_T>K\}} \mid \mathcal{F}_t\right] - \frac{K}{S_t}\mathrm{E}^Q\left[\mathbf{1}_{\{S_T>K\}} \mid \mathcal{F}_t\right]\right\} \qquad [2.105]$$

Without loss of generality, we choose $t_0 = 0$ to simplify option price formula as follows:

$$C(t_0, S_0, K, T) = \mathrm{E}^Q[e^{-rT}(S_T - K)_+ \mid \mathcal{F}_0] \qquad [2.106]$$

$$= e^{-rT}S_0\mathrm{E}^Q\left[(\frac{S_T}{S_0} - \frac{K}{S_0})_+ \mid \mathcal{F}_0\right] \qquad [2.107]$$

$$= e^{-rT}S_0\left\{\mathrm{E}^Q\left[e^{x_T}\mathbf{1}_{\{S_T>K\}} \mid \mathcal{F}_0\right] - \frac{K}{S_0}\mathrm{E}^Q\left[\mathbf{1}_{\{S_T>K\}} \mid \mathcal{F}_0\right]\right\} \qquad [2.108]$$

Using the moment generating function $\phi(x_t, V_t, t, \lambda) = \mathrm{E}[e^{zx(T)} \mid \mathcal{F}_t]$, we obtain:

$$\mathrm{E}^Q\left[\mathbf{1}_{\{S_T>K\}} \mid \mathcal{F}_0\right] = \frac{\phi(T,0)}{2} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-\imath \ln\frac{K}{S_0}}\phi(T,\imath\xi)\}}{\xi}d\xi \qquad [2.109]$$

$$\mathrm{E}^Q\left[e^{x_T}\mathbf{1}_{\{S_T>K\}} \mid \mathcal{F}_0\right] = \frac{\phi(T,1)}{2} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-\imath \ln\frac{K}{S_0}}\phi(T,1+\imath\xi)\}}{\xi}d\xi \qquad [2.110]$$

Substituting Eqs.(2.109) and (2.110) into Eq.(2.106), we can obtain closed form option price formula. The following is the detail analytical treatment of Heston model.

Analytical Treatment of Heston Model

Integrating Eq.(2.89) on $[t, T]$, we obtain:

$$x_T = x_t + \int_t^T (r - \frac{1}{2}v_s)ds + \int_t^T \sqrt{v_s}dW_s^Q \qquad [2.111]$$

The characteristic function $\phi_{x_t}(\lambda)$ of x_t under Q satisfies the following equation:

$$\begin{aligned}
\phi_{x_t}(\lambda) &= \mathrm{E}^Q\left[e^{\imath\lambda x_t} \mid \mathcal{F}_t\right] && [2.112]\\
&= \mathrm{E}^Q\left[e^{\imath\lambda[x_t+\int_t^T(r-\frac{1}{2}v_s)ds+\int_t^T\sqrt{v_s}dW_s^Q]} \mid \mathcal{F}_t\right] && [2.113]\\
&= e^{\imath\lambda[x_t+r(T-t)]}\mathrm{E}^Q\left[e^{\imath\lambda[-\frac{1}{2}\int_t^T v_s ds+\int_t^T\sqrt{v_s}dW_s^Q]} \mid \mathcal{F}_t\right] && [2.114]
\end{aligned}$$

Decomposing one Brownian Motion W_s^Q into two independent Brownian Motions Z_s^Q and B_t^Q as follows:

$$W_s^Q = \rho Z_t^Q + \sqrt{1-\rho^2}B_t^Q \qquad [2.115]$$

then, we obtain:

$$\begin{aligned}
\phi_{x_t}(\lambda) &= e^{\imath\lambda[x_t+r(T-t)]}\mathrm{E}^Q\left[e^{\imath\lambda[-\frac{1}{2}\int_t^T v_s ds+\rho\int_t^T\sqrt{v_s}dZ_s^Q+\sqrt{1-\rho^2}\int_t^T\sqrt{v_s}dB_s^Q]} \mid \mathcal{F}_t\right]\\
&= e^{\imath\lambda[x_t+r(T-t)]}\mathrm{E}^Q\left\{\mathrm{E}^Q\left[e^{\imath\lambda[-\frac{1}{2}\int_t^T v_s ds+\rho\int_t^T\sqrt{v_s}dZ_s^Q+\sqrt{1-\rho^2}\int_t^T\sqrt{v_s}dB_s^Q]} \mid \mathcal{F}_t, \sqrt{v_s}, s\in[t,T]\right] \mid \mathcal{F}_t\right\}\\
&= e^{\imath\lambda[x_t+r(T-t)]}\mathrm{E}^Q\left[e^{[-\frac{1}{2}\imath\lambda\int_t^T v_s ds+\imath\lambda\rho\int_t^T\sqrt{v_s}dZ_s^Q+\frac{1}{2}(1-\rho^2)(\imath\lambda)^2\int_t^T v_s ds]} \mid \mathcal{F}_t\right] \qquad [2.116]
\end{aligned}$$

It is worth to note that $\int_t^T \sqrt{v_s}dZ_s^2$ remains a function of Z^Q since $\sqrt{v_t}$ is defined in term of Z_t^Q. Integrating the volatility process in Eq.(2.87), we obtain:

$$v_T^2 - v_t^2 = \kappa\theta(T-t) - \kappa\int_t^T v_s ds + \gamma\int_t^T \sqrt{v_s}dZ_s^Q \qquad (2.117)$$

Reallocating terms in both sides of the formula, we obtain:

$$\int_t^T \sqrt{v_s} dZ_s^Q = \frac{1}{\gamma}\left[v_T^2 - v_t^2 - \kappa\theta(T-t) + \kappa\int_t^T v_s ds\right] \tag{2.118}$$

Substituting Eq.(2.118) into Eq.(2.116), we obtain:

$$\begin{aligned}\phi_{x_t}(\lambda) &= e^{\imath\lambda[x_t+r(T-t)]}\mathrm{E}^Q\left[e^{[-\frac{1}{2}\imath\lambda\int_t^T v_s ds+\imath\lambda\frac{\rho}{\gamma}\left[v_T^2-v_t^2-\kappa\theta(T-t)+\kappa\int_t^T v_s ds\right]+\frac{1}{2}(1-\rho^2)(\imath\lambda)^2\int_t^T v_s ds]} \mid \mathcal{F}_t\right] \\ &= e^{\imath\lambda[x_t+r(T-t)-\alpha_2(v_t+\kappa(T-t))]}\mathrm{E}^Q\left[e^{[-\alpha_1\int_t^T v_s ds+\alpha_2 v_T]} \mid \mathcal{F}_t\right] \qquad [2.119]\end{aligned}$$

where

$$\begin{aligned}\alpha_1 &= -\imath\lambda\left[\frac{\rho\kappa}{\gamma} - \frac{1}{2} + \frac{1}{2}\imath\lambda(1-\rho^2)\right] && [2.120] \\ \alpha_2 &= \imath\lambda\frac{\rho}{\gamma} && [2.121]\end{aligned}$$

Furthermore, we need to solve the following:

$$Y(t, v_t) = \mathrm{E}^Q\left[e^{[-\alpha_1\int_t^T v_s ds+\alpha_2 v_T]} \mid \mathcal{F}_t\right] \qquad [2.122]$$

Accounting to **Feynman-Kac Theorem**, $Y(t, v_t)$ fulfils the following partial differential equation(PDE):

$$-\frac{\partial Y}{\partial t} + \frac{1}{2}\gamma^2 v_t\frac{\partial^2 Y}{\partial v^2} + \kappa(\theta - v)\frac{\partial Y}{\partial v} - \alpha_1 v Y = 0 \qquad [2.123]$$

with boundary condition at $t = T$

$$Y(T, v_T) = e^{\alpha_2 v_T} \qquad [2.124]$$

The solution of this PDE satisfies affine form result (Duffie 1990, Zhu,2009). Denoting $\tau = T - t$, we obtain:

$$Y(t, v_t) = e^{A(\tau)+B(\tau)v_t} \qquad [2.125]$$

Like the previous Riccati equations, we obtain analytical solution as follows:

$$\begin{aligned}A(\tau) &= \frac{2\kappa\theta}{\gamma^2}\ln\left[\frac{2\beta_1}{\beta_2}e^{\frac{1}{2}(\kappa-\beta_1)\tau}\right] && [2.126] \\ B(\tau) &= \frac{1}{\beta_2}[\beta_1\alpha_2(1+e^{-\beta_1\tau}) - (1-e^{-\beta_1\tau})(2\alpha_1+\kappa\alpha_2)] && [2.127]\end{aligned}$$

with α_1 and α_2 defined in Eqs.(2.120) and (2.121).

$$\beta_1 = \sqrt{\kappa^2 + 2\gamma^2\alpha_1} \quad [2.128]$$
$$\beta_2 = 2\beta_1 e^{-\beta_1\tau} + (\kappa + \beta_1 - \gamma^2\alpha_2)(1 - e^{-\beta_1\tau}) \quad [2.129]$$

Substituting Eq.(2.125) into (2.116), we obtain the characteristic function $\phi_x(\lambda)$ as follows:

$$\phi_x(\lambda) = e^{\imath\lambda(x_t + r\tau) - \alpha_2(v_t + \kappa\theta\tau) + A(\tau) + B(\tau)v_t} \quad [2.130]$$

Using the characteristic function, we can obtain Heston option pricing formula straightforward. The detailed formulas will not deduced here, however, a more complicated treatment will be explicitly deduced in Hybrid Heston CIR model in Chapter 2.

Schöbel-Zhu Model

Schöbel and Zhu model 1999 is an extension of Stein and Stein model 1991. The model has an analytic solution for option pricing. The Schobel-Zhu model is defined as follows:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^Q \quad [2.131]$$
$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \gamma dZ_t^Q \quad [2.132]$$
$$dW_t^Q dZ_t^Q = \rho dt \quad [2.133]$$

where volatility σ_t follows a Hull-White type stochastic process. W_t^Q , Z_t^Q are two Brownian motions under risk neutral probability measure Q. Using Ito's lemma, we could obtain log-price process $x_t = \ln S_t$, which satisfies the following equation.

$$dx_t = (r - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t^Q \quad [2.134]$$

Substituting Eq.(2.90) in to system of Eqs.(2.131)-(2.133), we obtain:

$$\begin{aligned} dx_t &= (r - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t^Q \quad [2.135] \\ d\sigma_t &= \kappa(\theta - \sigma_t)dt + \gamma dZ_t^Q \\ dW_t^Q dZ_t^Q &= \rho dt \end{aligned}$$

It is worth nothing that the standard deviation σ_t follows an **Ornstein-Uhlenbeck** Process. By Ito's lemma, $v_t = \sigma_t^2$ satisfies the following equation:

$$dv_t = [\gamma^2 + 2\kappa\sigma_t(\theta - \sigma_t)]dt + 2\gamma\sigma_t dZ_t^Q \quad [2.136]$$

It is worth to note that replacing σ_t with $\sqrt{v_t}$ is not straightforward since σ_t could take negative values (i.e $\sigma = -\sqrt{v_t}$). Similar to the analytical treatment of Heston model, Schöbel-Zhu Model can also be solved analytically. Omitting the detailed deduction, we focus on the characteristic function $\phi_x(\lambda)$ of x_t under risk neutral probability Q.

$$\begin{aligned}
\phi_{x_t}(\lambda) &= \mathrm{E}^Q\left[e^{\imath\lambda x_t} \mid \mathcal{F}_t\right] && [2.137]\\
&= e^{\imath\lambda[x_t + r(T-t) - \alpha_2\sigma^2 - \frac{1}{2}\imath\phi\rho\gamma(T-t)]} \cdot \\
&\quad \mathrm{E}^Q\left[e^{[-\alpha_1\int_t^T \sigma_s^2 ds - \alpha_2\int_t^T \sigma_s ds + \alpha_3\sigma_T^2 + \alpha_4\sigma_T]} \mid \mathcal{F}_t\right] && [2.138]\\
&= e^{[\imath\lambda(x_t + r\tau) + A(\tau) + B(\tau)\sigma_t + \frac{1}{2}C(\tau)\sigma_t^2]} && [2.139]
\end{aligned}$$

with $\tau = T - t$ and

$$\alpha_1 = -\frac{1}{2}\imath\lambda\left[\imath\lambda(1-\rho^2) - 1 + \frac{2\rho\kappa}{\gamma}\right], \quad \alpha_2 = \imath\lambda\frac{\rho\kappa\theta}{\gamma}, \quad \alpha_3 = \imath\lambda\frac{\rho}{2\gamma}, \quad \alpha_4 = 0 \qquad [2.140]$$

where

$$\begin{aligned}
A(\tau) &= -\frac{1}{2}\ln\beta_4 + \frac{[(\kappa\theta\beta_1 - \beta_2\beta_3)^2 - \beta_3^2(1-\beta_2^2)]\sinh(\beta_1\tau)}{2\gamma^2\beta_1^3\beta_4}\\
&\quad + \frac{(\kappa\theta\beta_1 - \beta_2\beta_3)\beta_3(\beta_4 - 1)}{\gamma^2\beta_1^3\beta_4} + \tau\left[\frac{\kappa\beta_1^2(\gamma^2 - \kappa\theta^2) + \beta_3^2}{2\gamma^2\beta_1^2}\right]\\
&\quad + \frac{\alpha_4}{\beta_1^2\beta_4}\left[\beta_3(\beta_4 - 1) + \left(\kappa\theta\beta_1 + \frac{1}{2}\gamma^2\beta_1\alpha_4 + \beta_2\beta_3\right)\sinh(\beta_1\tau)\right]\\
&\quad - \alpha_3\gamma_2\tau && [2.141]\\
B(\tau) &= \frac{(\kappa\theta\beta_1 - \beta_2\beta_3)(1 - \cosh(\beta_1\tau)) - (\kappa\theta\beta_1\beta_2 - \beta_3)\sinh(\beta_1\tau)}{\gamma^2\beta_1\beta_4}\\
&\quad + \frac{\alpha^4}{\beta_4} && [2.142]\\
C(\tau) &= \frac{\kappa}{\gamma^2} - \frac{\beta_1}{\gamma^2}\frac{\sinh(\beta_1\tau) + \cosh(\beta_1\tau)}{\beta_4} - 2\alpha_3 && [2.143]
\end{aligned}$$

with

$$\begin{aligned}
&\beta_1 = \sqrt{2\gamma^2\alpha_1 + \kappa^2}, \quad \beta_2 = \frac{\kappa - 2\gamma^2\alpha_3}{\beta_1},\\
&\beta_3 = \kappa^2\theta - \alpha_2\gamma^2 \quad \beta_4 = \cosh(\beta_1\tau) + \beta_2\sinh(\beta_1\tau)\\
&\cosh(\beta_1\tau) = \frac{e^{\beta_1\tau} + e^{-\beta_1\tau}}{2}, \quad \sinh(\beta_1\tau) = \frac{e^{\beta_1\tau} - e^{-\beta_1\tau}}{2} && [2.144]
\end{aligned}$$

Using the characteristic function $\phi_x(\lambda)$, we can obtain option pricing formula straightforward. The detail formulas will not deduced here, however, a general treatment will be explicitly deduced in Chapter 2.

A Comparison Of Schöbel-Zhu Model With Heston Model

The major difference between the Heston and Schöbel-Zhu model is the volatility process. In Heston model 1993, the stochastic volatility process refers to the variance process, however, in the Schöbel and Zhu model 1999, the volatility σ_t process referred to the square-root of variance process in Eq.(2.136). Therefore, the variance process σ_t^2 satisfies the follow formula:

$$d\sigma_t^2 = 2\kappa\left[\frac{\gamma^2}{2\kappa} + \sigma_t\theta - \sigma_t^2\right]dt + 2\gamma\sigma_t dZ_t^Q \qquad [2.145]$$

$$= \tilde{\kappa}\left(\tilde{\theta} + \sigma_t\theta - \sigma_t^2\right)dt + \tilde{\gamma}\sigma_t dZ_t^Q \qquad [2.146]$$

where $\tilde{\kappa} = 2\kappa$, $\tilde{\theta} = \dfrac{\gamma^2}{2\kappa}$, $\tilde{\gamma} = 2\gamma$. Comparing Eq.(2.87) with (2.146), we can easily observe that the difference lies in the drift term. Specifically, Heston and Schöbel-Zhu models are equivalent if the long-term mean of volatility process in Eq.(2.132) is zero i.e $\theta = 0$. Not surprisingly, we can establish a relationship between the two models' characteristic functions (Lord and Kahl 2007). Moreover, the characteristic function of Schöbel-Zhu model can be viewed as a modified Heston model with an additional simple multiplier.

3 An Explicitly Solvable Hybrid Heston CIR Model With Stochastic Interest Rate

3.1 Introduction

3.1.1 Motivation and Research Background

Since 2008, central banks' frequent interventions on interest rate have caused severe instability of derivatives market. Intensive market fluctuations motivate us to introduce stochastic interest rate models to price options and other derivatives. It is worth to trace back to a series of quantitative easing (QE) monetary policies that are initialized by Federal Reserve (Fed) after 2008 financial crisis, and then spreads worldwide. The most influenced one is Fed's third round of quantitative easing (QE3, or nick name QE-infinity) with the aim to improve the performance of the U.S. job market. QE3 was launched in September 2012 and ended in October 2014. Compared with QE1 in 2008 and QE2 in 2010, QE3 has the largest scale that launched overall \$85 billion per month for purchasing mortgage-backed securities (MBS) and long term treasury securities. Consequently, only one year later (in 2013), monetary base has increased by \$1 trillion. Till November 2015, the Fed's balance sheet has reached \$4.5 trillion which is 4.5 times larger than the level before the 2008 crisis.

Nevertheless, the effectiveness of QE is always controversial. Many central banks and economists hold optimistic opinions, especially Ben Bernanke (chairman of the former Federal Reserve, 2006-2014) and Janet Yellen (current chairman of the Federal Reserve). They believe QE not only saved the world economy and financial system from ultimate recession since 2008 crisis, but led foreseeable economic recovery. However, several other influential economists and economic organizations, including International Monetary Fund (IMF), The Organisation for Economic Co-operation and Development (OECD), and Bank for International Settlements (BIS) express concern and worry. According to William White, chairman of economic development and review committee of OECD's, and former chief economist of BIS, "All the previous imbalances are still there. Total public and private debt levels are 30% higher as a share of GDP in the advanced economies than they were then, and we have added a whole new problem with bubbles in emerging markets that are ending in a boom-bust cycle." Said, William White in 2013 "This looks like to me like 2007 all over again, but even worse." Mr.White's concern indeed makes sense since interest rate swap(IRS) are trigging potential avalanche on financial market. If we believe that credit default swap(CDS) has initiated 2008 subprime securities crisis and seriously crippled the world economy, then the IRS market may pose a potential death sentence since its scale is over 7 times bigger than CDS.

Acting like an insurance contract on interest rate market, IRS permits big banks to hold U.S. local governments and agencies as hostage, and the ransom accounts for hundreds of millions of dollars. In 2010 and 2011, New York state and its local governments have to pay 1.4 billion due to the IRS contract. Especially, New York's Metropolitan Transit Authority (MTA) has to pay hundreds of millions. New York public finance problems arouse people's strong dissatisfaction and thus lead to famous 'Occupy Wall Street (OWS)' protest movement in 2011. IRS caused similar problem in Pennsylvania in 2012. In July 2013, Detroit declared bankruptcy which accounted the largest city bankrupt in U.S. history. The main reason of Detroit bankruptcy is government cash flow crisis resulted from the IRS contract on 1.4 billion debt signed with two big banks, Bank of America and Union Bank of Switzerland (UBS).

Now the world focuses on the Fed's decision whether or not to raise interest rate in December 2015. Considering the potential swap crisis, uncontrollable growing risk of inflation, and belief on economic recovery, Fed has declared to raise interest rate. Even many economics anticipated the raise will take place in September 2015, however, concerning the unsatisfactory economic indicators, Fed decides to postpone the raise in December 2015 or beginning of 2016. Obviously, this will be the first U.S. interest rate rise since 2006. Despite the significant economic influence, it will certainly lead to severe fluctuation on global financial markets.

3.1.2 Literature Review and Chapter Outline

The pricing of derivatives is probably one of the most challenging topics in modern financial theory. The modern financial market includes not only bonds but also derivative securities sensitive to interest rates. In this chapter, a modification of the hybrid model illustrated in Grzelak and Oosterlee

2011 is considered. This model is described by a system of stochastic differential equations (SDEs) which combines different models for equity, interest rate and volatility in order to efficiently price European vanilla call and put options. Specifically, this chapter focuses on the model which combines the Heston model 1993 for equity and its volatility and the Cox-Ingersoll-Ross 1985 (CIR) model for the interest rate.

The use of the hybrid SDE models is motivated by the empirical evidence that the asset volatility and the interest rate are not constant over time. The relaxation of the constant volatility assumption first appears in well known time continuous stochastic volatility models, such as Hull-White 1988, Stein-Stein 1991, Heston 1993, Ball and Roma 1994, and Schöbel and Zhu 1999. The Heston model is one of the most celebrated models because it allows for closed-form formulas for option pricing. In fact, this model accurately describes the asset price behaviour when the assumption of constant interest rate is realistic. Recently, some modified versions of the Heston model are found, like the model of Wong and Lo 2009, Fatone et al 2009, 2013 and Date and Islyaev 2015.

Furthermore, the relaxation of the constant interest rate assumption can be found in the last decade. Although far from being exhaustive, the chapter cites the papers of Chiarella, Kwon 2003, Trolle, Schwartz 2009, Andersen, Benzoni 2010, and Christensen et al. 2011, which show that stochastic interest rates should be used in order to capture the bond yield behaviour.

In line with the attempt to deal with stochastic interest rates, several hybrid SDE models have been introduced since 2000. In fact, Zhu 2000 introduces a model capable of generating a skew pattern for the equity, using a stochastic interest rate that is not correlated to the equity. Later, Andreasen 2007 generalizes the Zhu model by using the Heston stochastic volatility model, and some indirect correlation between the equity and the interest rate process.

Grzelak, Oosterlee and Weeren 2012 propose the so called Schöbel-Zhu-Hull-White hybrid model. This is an affine model whose analytical treatment is done by Grzelak, Oosterlee and Weeren 2012 following the approach proposed by Duffie, Pan and Singleton 2000. However, as highlighted in Grzelak and Oosterlee 2011, the model allows for negative volatility and interest rates. In order to overcome this problem, they propose the use of a Cox-Ingersoll-Ross (CIR) process to describe the variance process (see, Grzelak and Oosterlee 2011). Local volatility models have also been extended to deal with stochastic interest rates. For example, Deelstra and Rayee 2012 propose a three factor pricing model with local volatility and domestic and foreign interest rates modelled by the Hull and White (HW) model 1993. In line with the latter, Benhamou, Gobet and Miri 2012 provide analytical formulas for European option prices when the underlying asset is described by a local volatility model with stochastic rates. As previously mentioned, this chapter focuses on a modification of the hybrid Heston-CIR model illustrated by Grzelak and Oosterlee 2011. The contribution is threefold.

Firstly, the hybrid SDE model of Grzelak and Oosterlee 2011 is modified in order to preserve the affine structure and to permit a "direct" correlation between the equity and the interest rate. As highlighted by Grzelak and Oosterlee 2011 and confirmed by our empirical analysis, this correlation plays a fundamental role to get a good match between the observed and the theoretical option prices.

Secondly, the analytical treatment of the model is described. Specifically, an integral representation formula of the probability density function of the stochastic process is derived by solving the backward Kolmogorov equation using some ideas illustrated in Fatone et al.2009, 2013. In the solution of the backward Kolmogorov equation, this chapter uses a suitable change of the dependent variable which allows us to express the prices of European call and put options as one dimensional integrals, and to get elementary formulas for the moments of the asset price. These elementary formulas do not involve integrals and are used to prove that the existence of bounded moments that depends on the value of the correlation coefficients. This finding is similar to the results obtained by Lions and Musiela 2007, Andersen and Piterbarg, 2007 on the explosion of moments of some well known probability distributions. As a by-product of this analytical treatment, an efficient approximation of the stochastic integral appearing in the discount factor is obtained. This approximation is suggested by the explicit formula of the zero-coupon bond in the CIR model (see Eq. (3.40)) and permits us to approximate the option prices as one dimensional integrals of elementary functions. A further consequence of this is the obtainment of an explicit formula to approximate the zero-coupon bond. This chapter measures the quality of this approximation by comparing it with the true value given in Eq.(3.40). This permits us to measure the accuracy of the formulas used to price the options.

Thirdly, a well defined model is calibrated in order to measure its performance in interpreting real data and forecasting European call and put option prices. The calibration procedure is based on the solution of a non-linear constrained optimization problem whose objective function measures the relative squared difference between the observed and theoretical put and call option prices. Numerical simulations show that the approximate formulas for the discount factor and option prices work satisfactorily for any maturities (see Section 3.4). Moreover, the empirical study conducted using real data (i.e. call and put options on U.S. S&P 500 index) shows that the model is capable of fitting and predicting satisfactorily call and put option prices using only one set of model parameters obtained from the calibration procedure.

This chapter is organized as follows. In Section 3.2, the hybrid SDE model is described to illustrate the main relevant formulas. In Appendix A, the formula for the probability density function and explicit formulas of the moments of the asset price are derived. In Section 3.3, analytical formulas are proposed to approximate the European vanilla call and put option prices as one-dimensional integrals of explicitly known functions. In Section 3.4, some experiments involving the moments of the price variable and the zero coupon bond formula are illustrated.

Furthermore, the model parameters implied by the option prices solving a constrained optimization problem are estimated. The data includes the prices of some European vanilla call and put options on the U.S. S&P 500 index in the year 2012. Finally, conclusions are drawn in Section3.5.

3.2 The Hybrid Heston CIR Model

Hereafter, S_t, v_t and r_t are denoted the equity price, its volatility and the interest rate at time $t > 0$ respectively. The hybrid model studied by Grzelak and Oosterlee 2011 describes the process (S_t, v_t, r_t) via the following system of stochastic differential equations:

$$\begin{aligned} dS_t &= S_t\, r_t dt + S_t\sqrt{v_t} dW_t^x & t > 0, && [3.1] \\ dv_t &= \chi(v^* - v_t)dt + \gamma\sqrt{v_t} dW_t^v, & t > 0, && [3.2] \\ dr_t &= \lambda(\theta - r_t)dt + \eta\,\sqrt{r}_t dW_t^r, & t > 0, \quad . && [3.3] \end{aligned}$$

where W_t^x, W_t^v, W_t^r are Wiener processes with $W_0^x = W_0^v = W_0^r = 0$, and dW_t^x, dW_t^v, dW_t^r, $t > 0$, are their stochastic differentials satisfying the following conditions:

$$\begin{aligned} E(dW_t^x\, dW_t^v) &= \rho_{x,v} dt,\ t > 0, && [3.4] \\ E(dW_t^x\, dW_t^r) &= \rho_{x,r} dt,\ t > 0, && [3.5] \\ E(dW_t^v\, dW_t^r) &= \rho_{v,r} dt,\ t > 0, && [3.6] \end{aligned}$$

where $E(\cdot)$ denotes the expected value of $\cdot$, and $\rho_{x,v}$, $\rho_{x,r}$, $\rho_{v,r}$ are constant quantities known as correlation coefficients, $\rho_{x,v}$, $\rho_{x,r} \in (-1,1)$. Roughly speaking, this model generalizes the Heston model within a framework of stochastic interest rates. This stochastic rate is described by the Cox Ingersoll Ross (HCIR for short) model. The stochastic model (3.1)–(6.3) is written with respect the risk neutral world measure. This is motivated by the fact that the model calibration proposed in Section 3.4 is carried out against derivative data.

The system of equations (3.1)-(6.3) are equipped with the following initial conditions:

$$S_0 = S_0^*, \qquad [3.7]$$

$$v_0 = v_0^*, \qquad [3.8]$$

$$r_0 = r_0^*, \qquad [3.9]$$

where S_0^* and v_0^*, r_0^* are random variables concentrated in a point with probability one. For simplicity, these random variables are identified with the points where they are concentrated and S_0^*, $v_0^* > 0$ are chosen. The quantities χ, v^*, γ, λ, θ, η are positive constants. More precisely, the quantity χ is the speed of mean reversion, v^* is the long term mean and γ is the so called volatility of volatility (vol of vol for short).

It is worthy of note that the variance v_t remains positive for any $t > 0$ with probability one given that $2\chi\, v^*/\gamma^2 > 1$ and $v_0 = v_0^* > 0$ (see Heston 1991). As a consequence, the equity price S_t remains positive for any $t > 0$ with probability one given that $S_0^* > 0$ with probability one. The interest rate, r_t, described by the HCIR model remains positive with probability one for any $t > 0$ given that $2\lambda\theta/\eta^2 > 1$ and $r_0 = r_0^* > 0$ with probability one (see Heston 1991).

The HCIR model (3.1)-(6.3) with the correlation structure (3.4)–(3.6) is not an affine model so that its analytical treatment is a challenging problem. To overcome this difficulty, this chapter follows the approach of Grzelak and Oosterlee 2011. That is, HCIR model (3.1)–(6.3) is modified in order to get an analytically tractable model and to allow for a direct correlation between the equity and interest rate processes. More precisely, the proposed model describes the dynamics of the process (S_t, v_t, r_t), $t > 0$, trough the following system of stochastic differential equations:

$$dS_t = S_t\, r_t dt + S_t\sqrt{v_t}dW_t^{p,v} + S_t\Delta\sqrt{v_t}dW_t^v + S_t\,\Omega_t\,\sqrt{r_t}dW_t^{p,r},\ \ t > 0, \quad [3.10]$$
$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \qquad t > 0, \quad [3.11]$$
$$dr_t = \lambda(\theta - r_t)dt + \eta\,\sqrt{r_t}dW_t^r, \qquad t > 0 \quad [3.12]$$

where Δ is a positive constant, Ω_t is a positive function and the constants χ, v^*, γ, λ, θ, η are the same appearing in the model (3.1), (6.2), (6.3) and $W_t^{p,v}$, $W_t^{p,r}$, W_t^v, W_t^r are standard Wiener processes. The assumption of correlation structure is the following :

$$E(dW_t^{p,v}dW_t^v) = \rho_{p,v}\,dt,\ \ t > 0, \quad [3.13]$$
$$E(dW_t^{p,v}dW_t^r) = 0,\ \ t > 0, \quad [3.14]$$
$$[3.15]$$

and

$$E(dW_t^{p,r}dW_t^r) = \rho_{p,r}\,dt,\ \ t > 0, \quad [3.16]$$
$$E(dW_t^{p,r}dW_t^v) = 0,\ \ t > 0, \quad [3.17]$$
$$E(dW_t^r dW_t^v) = 0,\ \ t > 0, \quad [3.18]$$

where the quantities $\rho_{p,v}$, $\rho_{p,r} \in (-1, 1)$ are constant correlation coefficients. As explained in Grzelak and Oosterlee 2011, the good performance of this model in interpreting real data is due to the terms $S_t\Delta\sqrt{v_t}dW_t^v$ and $S_t\Omega_t\sqrt{r_t}dW_t^{p,r}$ that correspond to the terms $S_t\Delta\sqrt{v_t}dW_t^v$ and $S_t\Omega_t\sqrt{r_t}dW_t^r$ in the Grzelak and Oosterlee model. Hence, this chapter's model differs from the latter in the term $S_t\Omega_t\sqrt{r_t}dW_t^{p,r}$ that replaces $S_t\Omega_t\sqrt{r_t}dW_t^r$. This term is modified to allow a direct correlation between equity and interest rate. In the Grzelak and Oosterlee model the term $S_t\Omega_t\sqrt{r_t}dW_t^r$ implies a covariance between equity and rate equal to $S_t^2\Omega_t^2\,r_t\,dt$ while in my proposed method, the term $S_t\Omega_t\sqrt{r_t}dW_t^{p,r}$ implies a covariance equal to $S_t^2\Omega_t^2\,\rho_{p,r}\,r_t\,dt$. As shown in the empirical study of Section 3.4), the new correlation structure plays a crucial role to get accurate forecast prices of the European call and put options. For simplicity, in order to illustrate the analytical treatment, $\Omega_t = \Omega$ is assumed, where Ω is a positive constant. More general choices can be considered while preserving the analytical treatment of the model.

Here is the main formulas derived in this chapter and used for the option pricing. To this aim, the model (3.10)is re-written in term of the log-price, $x_t = \ln(S_t/S_0)$, $t > 0$. Using Ito's lemma and Eq. (3.10), the process

(x_t, v_t, r_t) satisfies the following dynamics:

$$dx_t = \left[r_t - \frac{1}{2}\left(\tilde{\psi}v_t + \Omega^2 r_t\right)\right]dt + \sqrt{v_t}dW_t^{p,v} + \Delta\sqrt{v_t}dW_t^v + \Omega\sqrt{r_t}dW_t^{p,r} \quad [3.19]$$

$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \qquad t > 0, \quad [3.20]$$

$$dr_t = \lambda(\theta - r_t)dt + \eta\sqrt{r_t}dW_t^r, \qquad t > 0, \quad [3.21]$$

where $\tilde{\psi}$ is the quantity defined by:

$$\tilde{\psi} := 1 + \Delta^2 + 2\Delta\rho_{p,v}. \quad [3.22]$$

The system of stochastic differential equations (3.19)–(3.21) are equipped with the initial conditions:

$$x_0 = x_0^* = 0, \quad v_0 = v_0^*, \quad r_0 = r_0^*. \quad [3.23]$$

In Eq.(3.23), as already specified, x_0^* is a random variable assumed to be concentrated in a point with probability one.
Now, let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of the positive real numbers, and $\mathbb{R}^n$ the n-dimensional Euclidean vector space. Let $\underline{\Theta}_v$ and $\underline{\Theta}_r$ denote the vectors $\underline{\Theta}_v = (\gamma, \chi, v^*, \rho_{p,v}, \Delta) \in \mathbb{R}^5$ and $\underline{\Theta}_r = (\eta, \lambda, \theta, \rho_{p,r}, \Omega) \in \mathbb{R}^5$ containing the parameters of the volatility and interest rate processes respectively. Let $p_f(x, v, r, t, x', v', r', t')$, $(x, v, r), (x', v', r') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$, $t, t' \geq 0$, $t - t' > 0$, be the transition probability density function associated with the stochastic differential system (3.19), (3.20), (3.21), that is, the probability density function of having $x_{t'} = x'$, $v_{t'} = v'$, $r_{t'} = r'$ given that $x_t = x$, $v_t = v$, $r_t = r$, when $t' - t > 0$. This transition probability density function $p_f(x, v, r, t, x', v', r', t')$ as a function of the "past" variables (x, v, r, t) satisfies the following backward Kolmogorov equation:

$$\begin{aligned}
-\frac{\partial p_f}{\partial t} &= \frac{1}{2}[(1 + \Delta^2 + 2\Delta\rho_{p,v})v + \Omega^2 r]\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\gamma^2 v\frac{\partial^2 p_f}{\partial v^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 p_f}{\partial r^2} \\
&+\gamma(\rho_{p,v} + \Delta)v\frac{\partial^2 p_f}{\partial x\partial v} + \eta\rho_{p,r}\Omega\, r\frac{\partial^2 p_f}{\partial x\partial r} + \chi(v^* - v)\frac{\partial p_f}{\partial v} \\
&+\lambda(\theta - r)\frac{\partial p_f}{\partial r} + \left(r - \frac{1}{2}\left[(1 + 2\rho_{p,v}\Delta + \Delta^2)v + \Omega^2 r\right]\right)\frac{\partial p_f}{\partial x}, \\
&\qquad (x, v, r) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \leq t < t',
\end{aligned} \quad [3.24]$$

with final condition:

$$p_f(x, v, r, t, x', v', r', t') = \delta(x' - x)\delta(v' - v)\delta(r' - r),$$
$$(x, v, r), (x', v', r') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t \geq 0,$$

and appropriate boundary conditions.

In Appendix A, using a suitable parametrization of the transition probability density function, the following formula is proved to be hold:

$$\begin{aligned}
&p_f(x, v, r, t, x', v', r', t') \\
&= e^{q(x-x')} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} L_{v,q}(t'-t, v, v', k;\, \underline{\Theta}_v)\, L_{r,q}(t'-t, r, r', k;\, \underline{\Theta}_r), \\
&(x, v, r),\, (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t, t' \geq 0,\ \ q \in \mathbb{R},\ t'-t > 0, \qquad [3.25]
\end{aligned}$$

where $\imath$ is the imaginary unit,the functions $L_{v,q}$ and $L_{r,q}$ are explicitly known functions given in Eqs.(7.32)-(7.34), which depend on the modified Bessel functions, $I_{(2\chi v^*/\gamma^2)-1}$ and $I_{(\lambda\theta/\eta^2)-1}$ (see, for example, Abramowitz and Stegun, 1970), where $(2\chi v^*/\gamma^2)-1$ and $(\lambda\theta/\eta^2)-1$ are real indices. The indices of these Bessel functions are positive under the conditions $2\chi v^*/\gamma^2 > 1$ and $2\lambda\theta/\eta^2 > 1$. Positive indices imply that the modified Bessel functions are bounded at zero and this guarantees that the function p_f given in (3.25) is a probability density function with respect to the future variables. On the other hand, these conditions are identical to the ones, as already mentioned, that guarantee positive values of the variance and interest rate processes for any time (with probability one) given that the positive initial stochastic conditions v_0, r_0(with probability one).

Moreover, it is worth noting that formula (3.25) can be interpreted as the inverse Fourier transform of the convolution of the probability density functions associated with the stochastic processes described by Eqs. (3.19)- (3.21) when one of the two factors, v_t, r_t, is dropped. This specific form of the transition probability density function is a consequence of the correlation structure (6.4)-(6.4).

A further good feature of the functions $L_{v,q}$ and $L_{r,q}$ is that the integrals of $L_{v,q}$ and $L_{r,q}$ with respect the future variables v' and r' are given by elementary functions, $W^0_{v,q}$ and $W^0_{r,q}$ respectively (see Eqs.(7.37)–(7.38)), as well as their products for integer powers of the future variables v', r' (see Eqs.(7.39)–(7.40)). That is, $W^m_{v,q}$ and $W^m_{r,q}$, $m = 0, 1, \ldots$ are defined as follows:

$$W^m_{v,q}(t'-t, v, k; \underline{\Theta}_v) = \int_0^{+\infty} dv'\, (v')^m L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v), \qquad [3.26]$$

$$W^m_{r,q}(t'-t, r, k; \underline{\Theta}_r) = \int_0^{+\infty} dr'\, (r')^m L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r), \qquad [3.27]$$

These functions together with the function $L_{r,q}$ can be used to get an integral representation formula for the marginal probability density function $D_{v,q}(x,v,r,t,x',r',t')$ of the future variables (x',r'):

$$D_{v,q}(x,v,r,t,x',r',t') = \int_0^{+\infty} dv'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v) L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r), \quad [3.28]$$

and for the marginal probability density function, $D_{v,r,q}(x,v,r,t,x',t')$, of the price variable x':

$$D_{v,r,q}(x,v,r,t,x',t') = \int_0^{+\infty} dr' \int_0^{+\infty} dv'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v) W^0_{r,q}(t'-t,r,k;\underline{\Theta}_r)\,. \quad [3.29]$$

In formulas (3.28) and (3.29), it's worth to highlight that the variables x, v, r are the initial values of the log-price, the stochastic variance and the stochastic interest rate respectively. These last two variables are not observable in the financial market and should be estimated using an appropriate calibration procedure. The estimation of the initial stochastic volatility is a common practice as suggested by Bühler, 2002.

In addition, formulas (3.28) and (3.29) are useful in estimating model parameters using a maximum likelihood approach and deriving the explicit formulas for the moments of the price variable and the mixed moments. In fact, the formula for the $m-th$ moment of the price $S_{t'} = S_0 e^{x'}$ conditioned to the observation at time $t=0$ is obtained computing the following integral:

$$\mathcal{M}_m = E(S^m_{t'}) = S^m_0 \int_{-\infty}^{+\infty} dx' e^{mx'} D_{v,r,q}(0,v_0,r_0,0,x',t') =$$
$$S^m_0 \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \left(\int_{-\infty}^{+\infty} dx' e^{mx'} e^{-qx'} e^{\imath k\, x'}\right) W^0_{v,q}(t',v_0,k;\underline{\Theta}_v)\, W^0_{r,q}(t',r_0,k;\underline{\Theta}_r)$$
$$(S_0,v_0,r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ q \in \mathbb{R},\ t' > 0. \quad [3.30]$$

Choosing $m = q$, the integral in the bracket leads to a delta Dirac's function of the conjugate variable k which allows the following explicit formula for the moments:

$$\mathcal{M}_m = E(S^m_{t'}) = S^m_0 W^0_{v,m}(t',v_0,0;\underline{\Theta}_v)\, W^0_{r,m}(t',r_0,0;\underline{\Theta}_r),$$
$$(S_0,v_0,r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0. \quad [3.31]$$

Similarly, using the explicit formulas for the integrals (3.26) and (3.27), the following explicit formulas for the mixed moments are obtained:

$$E(S_{t'}^{m_1} r_{t'}^{m_2}) = S_0^{m_1} W_{v,m_1}^0(t', v_0, 0; \underline{\Theta}_v)\, W_{r,m_1}^{m_2}(t', r_0, 0; \underline{\Theta}_r),$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0, \qquad [3.32]$$

and

$$E(S_{t'}^{m_1} v_{t'}^{m_2}) = S_0^{m_1} W_{v,m_1}^{m_2}(t', v_0, 0; \underline{\Theta}_v)\, W_{r,m_1}^0(t', r_0, 0; \underline{\Theta}_r),$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0, \qquad [3.33]$$

where the functions $W_{v,q}^0$, $W_{r,q}^0$, $W_{v,q}^m$, $W_{r,q}^m$, $m = 1, 2, \ldots$ are elementary functions given by (7.37), (7.38), (7.42) and (7.43) respectively. Moreover, using this approach, the following expressions for the moments of the log-return variable used in Section 3.4 are obtained:

$$E(x_{t'}^m) = \imath^m \frac{d^m}{dk^m} \left[W_{v,0}^0(t', v_0, k; \underline{\Theta}_v)\, W_{r,0}^0(t', r_0, k; \underline{\Theta}_r) \right]_{k=0},$$
$$(v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0, \qquad [3.34]$$

when $m = 1$, it gives:

$$E(x_{t'}) = \left(1 - \frac{\Omega^2}{2}\right) \left(\theta\, t' + (r_0 - \theta) \frac{1 - e^{-\lambda t'}}{\lambda}\right)$$
$$-\frac{1}{2}(1 + \Delta^2 + 2\rho_{p,v}\Delta) \left(v^*\, t' + (v_0 - v^*) \frac{1 - e^{-\chi t'}}{\chi}\right). \qquad [3.35]$$

It is worth to highlight that the formulas (3.31), (3.33), (3.32), (3.34), (3.35) are elementary formulas that do not involve integrals. The derivation of these formulas is possible thanks to Eq.(3.25). This equation reduces the computation of the transition probability density function, which depends on a "regularization" parameter, q, to a one dimensional integral whose integrand function is the product of smooth functions of the future variables. This smoothness is a result of the specific way in which the formula is deduced. This together with an appropriate choice for q allows to derive elementary formulas for the marginal probability density functions and the moments illustrated above. These formulas are used to estimate the model parameters with great savings of computational time.

In fact, the marginal probability density function, $D_{v,q}$, can be used to price European call and put options with payoff functions independent of the variance process in the framework of stochastic interest rates. Thus, we use $D_{v,q}$

to deduce formulas for European call and put vanilla options. As stressed in Christoffersen et al. 2009, a multi-factor model is more flexible for conditional kurtosis and skewness. Thanks to formula (3.31), these indicators can be easily computed. Finally, a useful byproduct of these formulas are the moments and the mixed moments associated with the Heston model.

3.3 Integral Formulas For European Vanilla Call and Put Options

In the framework of the model (3.19)-(3.21), integral formulas can be derived to approximate the prices of European call and put vanilla options with strike price E and maturity time T. This is done using the no arbitrage pricing theory. As illustrated in Grzelak and Oosterlee in 2011, the option price is computed as the expected value of the discounted payoff with respect to an equivalent martingale measure known as a risk-neutral measure (see, for example, Duffie, 2001; Schoutens, 2003, Wong, 2006). That is, let S_0 be the spot price at time zero, the prices of European call and put options with strike price E and maturity time T can be commutated as follows:

$$C(S_0, T, E, r_0, v_0) = E^Q\left(\frac{(S_0 e^{x_T} - E)_+}{e^{\int_0^T r_t\, dt}}\right), \quad S_0,\ T,\ r_0,\ v_0 > 0, \qquad [3.36]$$

$$P(S_0, T, E, r_0, v_0) = E^Q\left(\frac{(E - S_0 e^{x_T})_+}{e^{\int_0^T r_t\, dt}}\right), \quad S_0,\ T,\ E,\ r_0,\ v_0 > 0, \qquad [3.37]$$

where $(\,\cdot\,)_+ = max\{\,\cdot\,,\, 0\}$, and the expectation is taken under the risk-neutral measure Q. In the numerical experiments, only the option price is used to calibrate the model, that is we use only the risk-neutral measure and not the physical one. As a consequence, it is not necessary introduce the risk premium parameters.

Note that v_0 and r_0 are not observable in the market, so that they are considered as model parameters that must be estimated (see Section 3.4).
The numerical evaluation of formula (3.36) is very time consuming. Thus a formula to evaluate these prices approximating the stochastic integral defines the discount factor as follows:

$$e^{-\int_0^T r_t dt} \approx e^{-r_0 \frac{T}{(1+e^{\lambda T})} - r_T \frac{T e^{\lambda T}}{(1+e^{\lambda T})}}. \qquad [3.38]$$

Roughly speaking, formula (5.37) has been obtained approximating r_t as a suitable weighted sum of short rate r_t at $t = 0$ and $t = T$. The choice of these weights are inspired by the analytical expression of zero-coupon bond as following:

$$E^Q\left(e^{\int_0^T r_t\,dt}\right) \approx E^Q\left(e^{-r_0\frac{T}{(1+e^{\lambda T})}-r_T\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right). \tag{3.39}$$

The expected values appearing in formula (3.39) are given by explicit formulas. Specifically, the expected value on the left hand side of Eq.(3.39) is the exact formula to price zero-coupon bonds in the CIR model (see Cox, Ingersoll and Ross 1985) which is given by:

$$\begin{aligned} &B(r_0,T) \\ &= \left(\frac{2\,he^{(\lambda-h)T/2}}{2h+(\lambda-h)(1-e^{-hT})}\right)^{\frac{2\lambda\theta}{\eta^2}} e^{-\left(\frac{2\,he^{(\lambda-h)T/2}}{2h+(\lambda-h)(1-e^{-hT})}\right)r_0 e^{-(h+\lambda)T/2}(e^{hT}-1)/h}, \\ &\qquad T>0,\ r_0>0\,, \end{aligned} \qquad [3.40]$$

where $h=\sqrt{\lambda^2+2\eta^2}$. The expected value on the right hand side of Eq. (3.39) is computed using formula (3.28) and the following formula (see Erdely et al. 1954, Vol I, p. 197 formula (18)):

$$\begin{aligned} &\int_0^{+\infty} dr'\,(r')^{\nu_r/2}\,e^{-(M_{q,r}+b)r'}I_{\nu_r}(2M_{q,r}(r_q r')^{1/2}) \\ &= [(M_{q,r})^2\tilde{r}_q]^{\frac{\nu_r}{2}}\,(M_{q,r}+b)^{-\nu_r-1}e^{\frac{(M_{q,r})^2\tilde{r}_q}{(M_{q,r}+b)}}\,, \end{aligned} \qquad [3.41]$$

where $M_{q,r}$ is given in (7.35). That is we have:

$$\begin{aligned} &B_A(r_0,T) \\ &= e^{-\frac{r_0 T}{(1+e^{\lambda T})}}\left(\frac{2\lambda}{2\lambda+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\eta^2(1-e^{-\lambda T})}\right)^{\frac{2\lambda\theta}{\eta^2}} e^{-\left(\frac{2\lambda}{2\lambda+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\eta^2(1-e^{-\lambda T})}\right)\left(\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\right)r_0\,e^{-\lambda T}}, \\ &\qquad T>0,\quad r_0>0\,. \end{aligned} \qquad [3.42]$$

Note that in formulas (3.40) and (3.42), this chapter has denoted the expected values appearing in the left and right hand sides of Eq.(3.39) with $B(r_0,T)$ and $B_A(r_0,T)$ respectively to highlight the dependence of these formulas on r_0 and T. It is worthy to note that formula (3.42) differs from that presented in Choi and Wirjanto 2007 in that the latter is based on a suitable approximation of the stochastic differential equation that defines the interest rate while our formula is obtained using the integral representation formula of the transition probability density and a specific approximation of the stochastic interest rate process.

The simulation study, illustrated in Section 3.4, shows that formula (3.42) satisfactorily approximates zero-coupon bonds with maturities up to twenty years. In fact, comparing the zero-coupon bond values obtained using Eq.(3.42) and Eq.(3.40), it shows that the approximation (3.42) guarantees at least five correct significant digits for maturity up to 20 years (see Table 3.1). This good performance of formula (3.42) guarantees a good approximation when used to price options.

As shown in Section 3.4, this approximation works well also for long maturity. Furthermore, the use of formula (5.37) allows reducing the computation of the option prices to the evaluation of a one dimensional integral. In fact, let $p_f(x,v,t,x',v',t')$, $(x,v,t),(x',v',t') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$, $t,\ t' \geq 0$, $\tau = t'-t > 0$ be the transition probability density function of the stochastic process described by Eqs.(3.19)–(3.21). Using formula (3.25) for p_f and (5.37), we obtain:

$$C_A(S_0,T,E,r_0,v_0) = e^{-r_0\frac{T}{(1+e^{\lambda T})}} \int_{\ln(E/S_0)}^{+\infty} dx'\, e^{-qx'}(S_0 e^{x'} - E) \int_0^{+\infty} dr' e^{-r'\frac{Te^{\lambda T}}{(1+e^{\lambda T})}} D_{v,q}(0,v_0,r_0,0,x',r',T), \quad [3.43]$$
$$S_0,\ T,\ E,\ r_0,\ \ v_0 > 0,\ \ q > 1,$$

$$P_A(S_0,T,E,r_0,v_0) = e^{-r_0\frac{T}{(1+e^{\lambda T})}} \int_{-\infty}^{\ln(E/S_0)} dx'\, e^{-qx'}(E - S_0 e^{x'}) \int_0^{+\infty} dr' e^{-r'\frac{Te^{\lambda T}}{(1+e^{\lambda T})}} D_{v,q}(0,v_0,r_0,0,x',r',T), \quad [3.44]$$
$$S_0,\ T,\ E,\ r_0,\ \ v_0 > 0,\ \ q < -1,$$

where $D_{v,q}$ is given in (3.28). Using formulas (3.28) and (6.153) with $q=2$ in (3.43), the following approximation of the call option price C is obtained:

$$C_A(S_0,T,E,r_0,v_0) = e^{-r_0\frac{T}{(1+e^{\lambda T})}} \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} dk\, \frac{\left(\dfrac{S_0}{E}\right)^{(1-\imath k)}}{-k^2 - 3\imath k + 2} \cdot$$
$$W^0_{v,q}(T,v_0,k;\underline{\Theta}_v) W^0_{r,q}(T,r_0,k;\underline{\Theta}_r) \left(\frac{M_{q,r}}{M_{q,r} + \frac{T\,e^{\lambda T}}{(1+e^{\lambda T})}}\right)^{\nu_r+1} e^{-\left(\frac{T\,e^{\lambda T}}{(1+e^{\lambda T})}\right)\left(\frac{M_{q,r}\tilde{r}_q}{M_{q,r}+\frac{T\,e^{\lambda T}}{(1+e^{\lambda T})}}\right)},$$
$$S_0,\ T,\ E,\ r_0,\ v_0,\ \ q = 2, \quad [3.45]$$

Proceeding in a similar way, the following approximation, P_A, of the put option price P is obtained:

$$P_A(S_0,T,E,r_0,v_0) = e^{-r_0 \frac{T}{(1+e^{\lambda T})}} \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} dk \, \frac{\left(\frac{S_0}{E}\right)^{-(3+\imath k)}}{-k^2+5\imath k+6} \cdot$$

$$W^0_{v,q}(T,v_0,k;\underline{\Theta}_v) W^0_{r,q}(T,r_0,k;\underline{\Theta}_r) \left(\frac{M_{q,r}}{M_{q,r}+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}} \right)^{\nu_r+1} e^{-\left(\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\right)\left(\frac{M_{q,r}\tilde{r}_q}{M_{q,r}+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right)},$$

$$S_0,\ T,\ E,\ r_0,\ v_0,\ \ q=-2. \qquad [3.46]$$

Taking the limit $\Omega \to 0^+$, $\lambda \to 0^+$, $\eta \to 0^+$ in Eqs.(3.45) and (3.46), we derive the following exact formulas for the price of the European call and put options under the Heston model:

$$\begin{aligned} &C_H(S_0,T,E,r_0,v_0) \\ &= e^{-r_0T} S_0 e^{2r_0T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)} e^{-\imath k r_0 T}}{-k^2-3\imath k+2} W^0_{v,q}(T,v_0,k;\underline{\Theta}_v), \\ &\qquad S_0,\ T,\ E,\ r_0,\ v_0,\ \ q=2, \end{aligned} \qquad [3.47]$$

$$\begin{aligned} &P_H(S_0,T,E,r_0,v_0) \\ &= e^{-r_0T} S_0 e^{-2r_0T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{-(3+\imath k)} e^{-\imath k r_0 T}}{-k^2+5\imath k+6} W^0_{v,q}(T,v_0,k;\underline{\Theta}_v), \\ &\qquad S_0,\ T,\ E,\ r_0,\ v_0,\ \ q=-2. \end{aligned} \qquad [3.48]$$

These formulas are used in the empirical analysis to compare the performance of the Heston model and the stochastic model proposed here in interpreting real data. It is worth noting that the integrand functions appearing in formulas (3.45), (3.46), (3.47), (3.48) are smooth functions whose integration does not require a specific care. This regularity is due to the specific approach used to derive them.

3.4 Simulation And Empirical Studies

In this section, some experiments are illustrated on simulated data as well as an empirical analysis relative to the U.S. S&P 500 index and its European options. This analysis uses a suitable calibration of the model parameters.

3.4.1 Simulation Study

Zero Coupon Bond Approximation

The first experiment analyses the performance of formula (3.42) in approximating the bond price. Two sets of parameter values are considered here. The first one, *Set A*, is as follows: $\theta = 0.02$, $\eta = 0.01$, $\lambda = 0.01$, $\rho_{p,r} = -0.23$, $\Omega = 1$, $r_0 = 0.02$, $\Delta = 0.01$, $\rho_{p,v} = -0.3$, $\chi = 0.3$, $\gamma = 0.6$, $v^* = 0.05$, $v_0 = 0.05$, $S_0 = 100$.The parameters of the volatility process are those employed by Grzelak and Oosterlee 2011 in Table 1.

The second one, *Set B* is as follows: $\theta = 0.00044$, $\eta = 0.0098$, $\lambda = 3.62$, $\rho_{p,r} = -0.81$, $\Omega = 2.51$, $r_0 = 0.00022$, $\Delta = 1.98$, $\rho_{p,v} = -0.97$, $\chi = 0.65$, $\gamma = 0.018$, $v^* = 0.0345$, $v_0 = 0.089$, $S_0 = 12.456$. *Set B* are the parameter values estimated in the empirical analysis illustrated in the following of this Section.

Table 3.1 shows, from left to right, the time to maturity τ, the true bond value computed via formula (3.40), the approximate formula (3.42) and the relative error given by the ratio of the absolute error (i.e. the absolute value of the difference between the true and approximate values) to the true value. The notation x.xxx e-n is equivalent to $x.xxx \cdot 10^{-n}$. The relative errors in Table 3.1 show that the approximate formula guarantees at least five correct significant digits for maturity up to 20 years. This is a satisfactory result that supports the approximation of the discount factor given in formula (5.37).

Monte Carlo Simulation On Moments

The aim of second experiment is to compare the first two theoretical moments given in Eq. (3.31) (i.e. $m = 1$, $m = 2$) with those attained using the Monte Carlo method. The latter is implemented integrating numerically the stochastic differential equations with the explicit Euler method with variable step-size. The largest value of the Euler step-size is 10^{-5} and the number of the Monte Carlo simulations is 10000.

The upper panels of Figure 3.1 show the first and second moments of the asset price variable as a function of time computed using Eq.(3.31) (solid line) and the Monte Carlo method (dotted line). The lower panel shows the corresponding relative errors. The values of the model parameters are those of *Set A*. The moments obtained using the Monte Carlo method approximate quite well those obtained with the theoretical formulas. However, the theoretical formulas lead to significant savings in computing time. In fact,

TABLE 3.1 – *Performance of formula (3.42) to approximate the zero-coupon bond value*

Set A: $\theta = 0.02$, $\eta = 0.01$, $\lambda = 0.01$, $\rho_{p,r} = -0.23$, $\Omega = 1$, $r_0 = 0.02$
$\Delta = 0.01$, $\rho_{p,v} = -0.3$, $\chi = 0.3$, $\gamma = 0.6$, $v^* = 0.05$, $v_0 = 0.05$, $S_0 = 100$

τ	True bond value	Approximate bond value	Relative Error
0.25	9.950125e-1	9.950125e-1	1.292331e-9
0.50	9.900499e-1	9.900499e-1	1.026083e-8
0.75	9.851121e-1	9.851120e-1	3.436840e-8
1.00	9.801990e-1	9.801989e-1	8.084669e-8
1.50	9.704466e-1	9.704464e-1	2.686953e-7
2.00	9.607920e-1	9.607914e-1	6.270951e-7
3.00	9.417728e-1	9.417709e-1	2.050759e-6
5.00	9.048737e-1	9.048657e-1	8.896235e-6
10.0	8.189837e-1	8.189348e-1	5.969022e-5
20.0	6.718534e-1	6.716448e-1	3.105461e-4

Set B: $\theta = 0.00044$, $\eta = 0.0098$, $\lambda = 3.62$, $\rho_{p,r} = -0.81$, $\Omega = 2.51$, $r_0 = 0.00022$
$\Delta = 1.98$, $\rho_{p,v} = -0.97$, $\chi = 0.65$, $\gamma = 0.018$, $v^* = 0.0345$, $v_0 = 0.089$, $S_0 = 12.456$

τ	True bond value	Approximate bond value	Relative Error
0.25	9.999262e-1	9.999250e-1	1.206098e-6
0.50	9.998308e-1	9.998245e-1	6.328031e-6
0.75	9.997268e-1	9.997125e-1	1.429458e-5
1.00	9.996192e-1	9.995961e-1	2.314066e-5
1.50	9.994007e-1	9.993621e-1	3.865490e-5
2.00	9.991811e-1	9.991322e-1	4.893588e-5
3.00	9.987416e-1	9.986838e-1	5.785582e-5
5.00	9.978631e-1	9.978026e-1	6.057676e-5
10.0	9.956702e-1	9.956100e-1	6.049504e-5
20.0	9.912989e-1	9.912398e-1	5.963593e-5

the Monte Carlo method requires about ten minutes while formula (3.31) requires only a few milliseconds when their Matlab code is run on a laptop with Intel core i3 processor and 8GB of RAM.

3.4.2 An Empirical Analysis Of Index Options

Finally, an empirical analysis is used to calibrate the model parameters against real data. The calibration procedure is based on the solution of an appropriate non-linear constrained least squares problem, whose set of

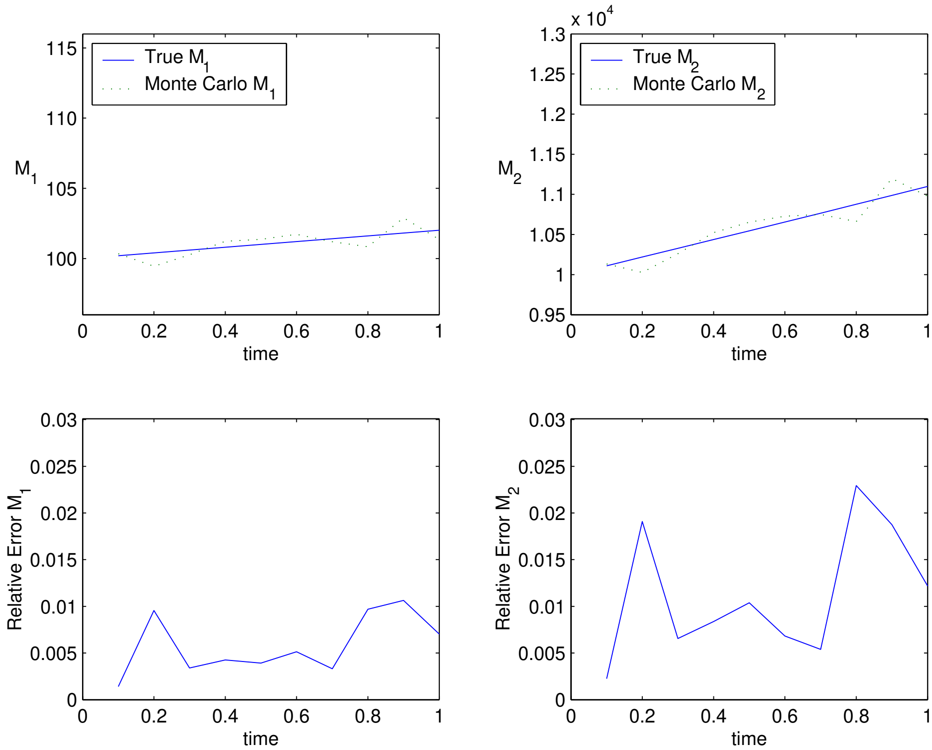

Figure 3.1 – Graphs of $\mathcal{M}_1$ (upper left panel) and $\mathcal{M}_2$ (upper right panel) versus time computed using formula (3.31) (solid line) and Monte Carlo method (dotted line). The lower panels show the corresponding relative errors. The values of the model parameters are those of Set A.

feasible vectors, $\mathcal{V}$, is as follows:

$$\mathcal{V}= \Big\{\underline{\Theta}= (\Delta, \gamma, v^*, \chi, \rho_{p,v}, v_0, \eta, \lambda, \theta, \rho_{p,r}, r_0, \Omega) \in \mathbb{R}^{12} \,|\, \Delta, \gamma, v^*, \chi, v_0, \eta, \lambda, \theta > 0, \quad \frac{2\chi v^*}{\gamma^2} > 1, \quad -1 < \rho_{p,v}, \rho_{p,r} < 1\Big\}, \qquad [3.49]$$

where $\mathbb{R}^{12}$ denotes the 12-dimensional Euclidean real space.

Please note that the set $\mathcal{V}$ also includes the initial stochastic volatility and interest rate, v_0, r_0, of the stochastic model (3.19)–(3.21). That is, the initial values v_0, r_0 are considered parameters to be estimated via the calibration procedure. This choice is motivated by the fact that v_0 cannot be observed in the market while r_0 refers to a risk free interest rate associated with the risk neutral measure and, subsequently, its value is not clearly identified by the financial market. The estimation of the initial stochastic volatility is a common practice since 2002 (see, for example, Bühler 2002), while the interest rate has been estimated only recently (see, for example, Fatone et al. 2009, 2013 and Grzelak and Oosterlee 2011).

In order to formulate the least square problem, the objective function is defined as following. Let n_D be a positive integer, $\tilde{t} \geq 0$ be the observation time and $\tilde{S}_{\tilde{t}}$ be the asset price observed at time $t = \tilde{t}$. In addition, let $C^{\tilde{t}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$, $C^{\tilde{t},\underline{\Theta}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$, $i = 1, 2, \dots, n_D$ be the observed price and the model price (3.45) at time $t = \tilde{t}$ of the European call option having maturity times T_i and strike prices E_i, $i = 1, 2, \dots, n_D$. Similarly, let us denote with $P^{\tilde{t}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$ and $P^{\tilde{t},\underline{\Theta}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$ the observed price and the model price (3.46) at time $t = \tilde{t}$ of the European put option having maturity times T_i and strike prices E_i where $i = 1, 2, \dots, n_D$.

Let n_T be a positive integer, the objective function, F_{n_T}, of our constrained optimization problem is as follows:

$$F_{n_T}(\underline{\Theta}) = \frac{1}{n_T} \sum_{j=1}^{n_T} \frac{1}{n_D} \sum_{i=1}^{n_D} \left[\frac{C^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i) - C^{\tilde{t}_j,\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)}{C^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)} \right]^2 +$$
$$\frac{1}{n_T} \sum_{j=1}^{n_T} \frac{1}{n_D} \sum_{i=1}^{n_D} \left[\frac{P^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i) - P^{\tilde{t}_j,\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)}{P^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)} \right]^2, \quad \underline{\Theta} \in \mathcal{V}, \quad [3.50]$$

where $\tilde{t}_j$, $j = 1, 2, \dots, n_T$, are the observation times. The optimization problem used to estimate the model parameters can be stated as follows:

$$\min_{\underline{\Theta} \in \mathcal{V}} F_{n_T}(\underline{\Theta}). \tag{3.51}$$

Heston model (Heston 1993) is also calibrated here by solving problem (3.51) with the appropriate adjustments. That is, removing the parameters of the stochastic interest rate model with the exception of the parameter r_0.

Formulas (3.45), (3.46) and (3.47), (3.48) are used to evaluate option prices. The one-dimensional integrals appearing in these formulas are computed using the midpoint quadrature rule with 2^{14} nodes. This quadrature rule gives satisfactory approximations since the integrand functions appearing in Eqs.(3.45), (3.46), (3.47), (3.48) are smooth functions whose numerical integration does not require special care.

Moreover, problem (3.51) is solved using a steepest descent algorithm with variable metric (see, for example, Recchioni and Scoccia 2002, Fatone et al 2013). The real data analyzed are the daily closing values of the U.S. S&P

500 index and the daily closing prices of the European call and put options on this index. These options have an expiry date of March 16th, 2013 with strike prices $E_i = 1075 + 25(i-1)$, $i = 1, 2, \dots, 4$, $E_5 = 1170$.

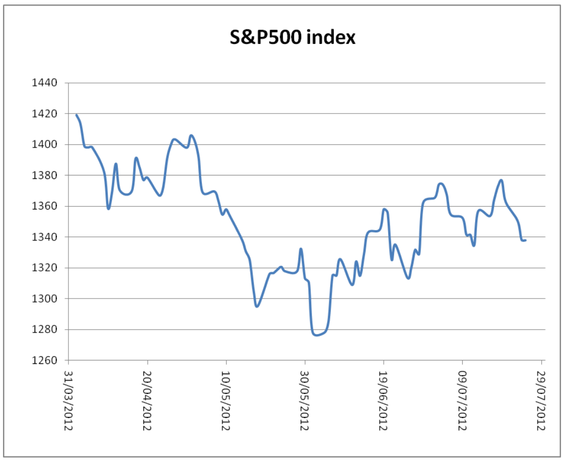

Figure 3.2 – The U.S.A. S&P S500 index versus time.

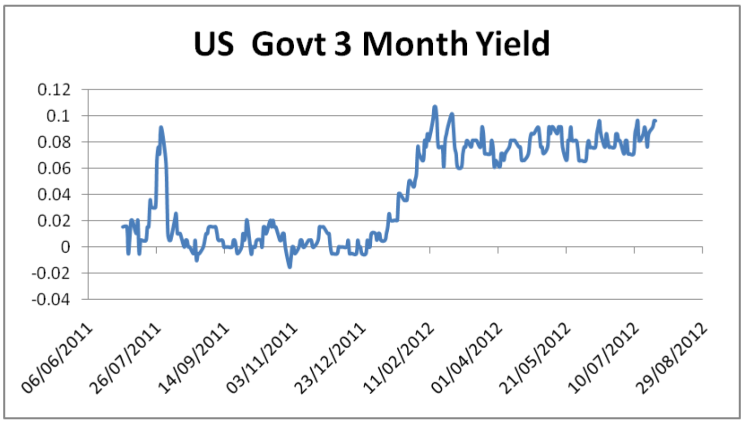

Figure 3.3 – U.S. three-month government yield versus time.

Figure 3.2 shows the U.S. S&P 500 index while Figures 3.4 and 3.5 show the corresponding call and put option prices as a function of time (April 2nd, 2012, July 27th, 2012). Figure 3.3 shows the U.S. three month government yields (in percent) as a function of time. Usually, the short-dated government bonds are used as proxy of the risk free interest rate and we expect that the initial stochastic interest rate r_0 and the long-term mean θ have values similar to those in Figure 3.3 from April 2nd to July 27th, 2012.

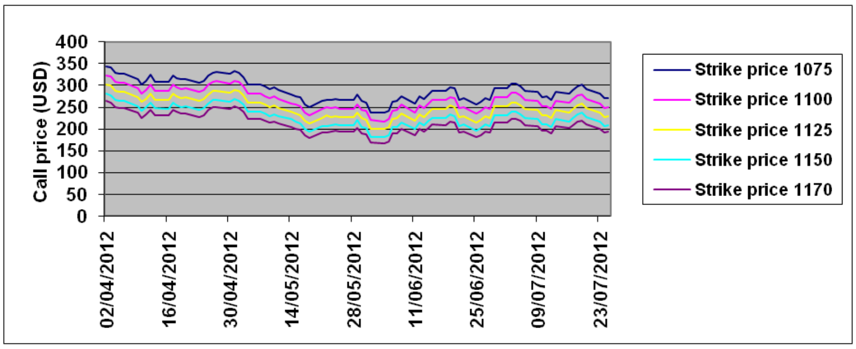

Figure 3.4 – Prices of the call options on the U.S.A. S&P 500 index with strike prices $E_i = 1075 + 25(i-1)$, $i = 1, 2, \ldots, 4$ *and* $E_5 = 1170$, *and with expiry date* T= *March 16th, 2013 versus time.*

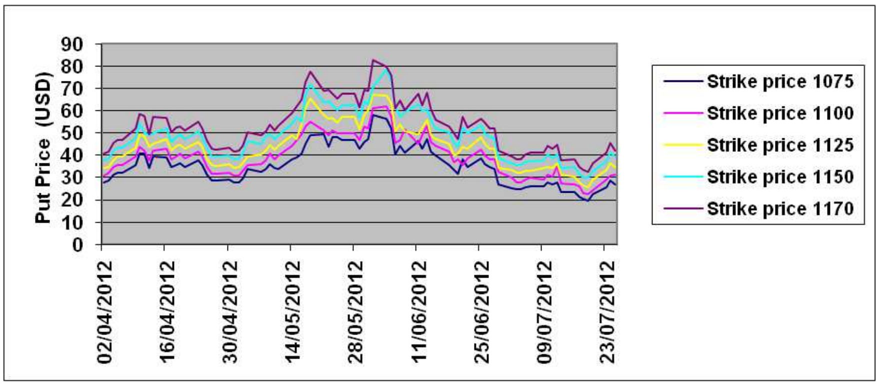

Figure 3.5 – Prices of the put options on the U.S.A. S&P 500 index with strike prices $E_i = 1075 + 25(i-1)$, $i = 1, 2, \ldots, 4$ *and* $E_5 = 1170$, *and with expiry date* T= *March 16th, 2013 versus time.*

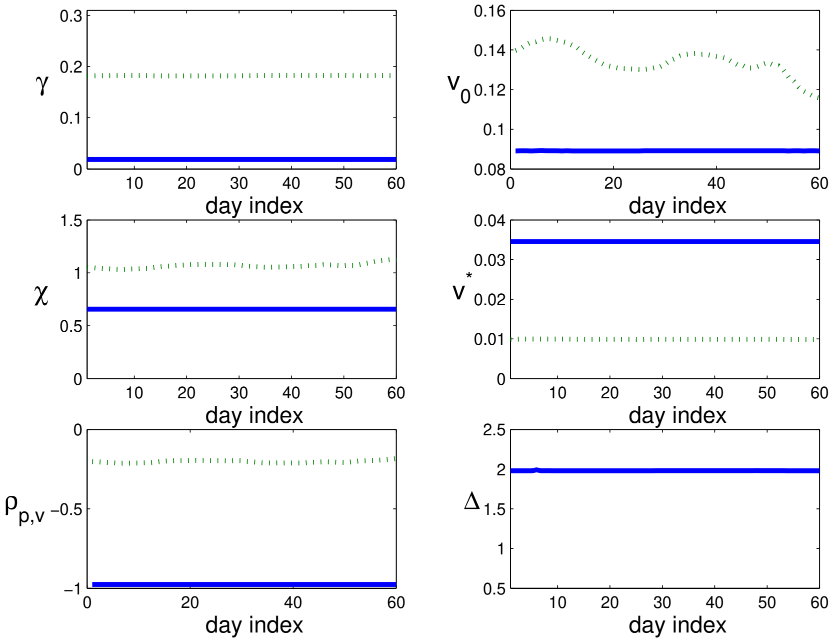

Figure 3.6 – Estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$ and Δ versus window index (six-day window) resulting from the calibration of the hybrid Heston model (solid line) and the Heston model (dotted line)

In the empirical analysis, a rolling window of six consecutive trading day data (i.e. $n_T = 6$) is considered. This window is moved by one day along the historical series. The time window covers the period April 2nd to July 2nd, 2012. In this way, $66 - n_T$ calibration problems are solved(3.51). That is, one problem for each six-day window, j, where $j = 1, 2, \ldots, 66 - n_T$. As a consequence, in each window sixty option values are used to calibrate the twelve parameters of the model (i.e. $n_D = 5$ put option price and $n_D = 5$ call option prices for $n_T = 6$ days).

Shifting the rolling window by one day along the time series and the choice of $n_T = 6$ have a twofold effect. First, there is a sufficient number of data to validate the model (sixty option values). Second, a "daily" time series of the estimated parameters are obtained. The values of the parameters obtained in the j-th window are representative of the last day of the j-th window.

It is worth noting that when the time series of the parameter values are constant, the model (3.19)–(3.21) will correctly interpret the asset price dynamics. In fact, when the values of the estimated parameters are constant in time, the model is able to reproduce the asset price dynamics in the analyzed

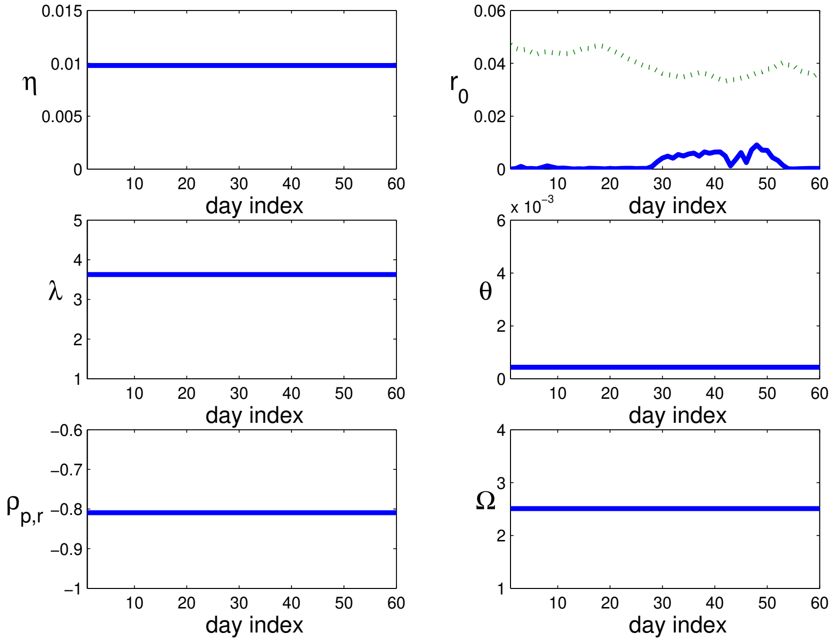

Figure 3.7 – Estimated parameters η, r_0, λ, θ, $\rho_{p,r}$ and Ω versus window index (six-day window) resulting from the calibration of the hybrid Heston model (solid line). Note that the dotted line in the upper right panel shows the estimated values of the parameter r_0 resulting from the calibration of the Heston model.

period by using only one set of model parameters. Figures 5.13 and 5.14 show the parameter values as a function of the index j, $j = 1, 2, \ldots, 66 - n_T$.
It is observed from our empirical analysis that these values are relatively constant as a function of time.

Moreover, there is a significant difference between the initial interest rate r_0 of the hybrid Heston CIR model (i.e. $r_0 \approx 0.00022$) and the Heston model (i.e. $r_0 \approx 0.04$). The two models also differ in the value of the vol of vol γ and the correlation coefficient $\rho_{p,v}$. In fact, as already stressed by Grzelak and Oosterlee 2011, the hybrid Heston model shows lower values of the parameter γ and a more negative correlation coefficient $\rho_{p,v}$ with respect to the Heston model. The lower value of γ may be due to the additional volatility coming from the interest rate process, while the more negative correlation may be due to an increase of the leverage effect caused by the previously mentioned additional volatility.

The values of the initial stochastic interest rate r_0 and of θ are about 0.02% and 0.04%, and those are values comparable with those shown in Figure 3.3. Moreover, Figure (3.3) shows an abrupt change of the yield trend in February. This abrupt change may explain the fluctuations of the initial stochastic rate shown in Figure 5.14. Figures 3.8 and 3.9 show the in-sample values of the European call and put option prices obtained using the Heston (dashed line) and the hybrid Heston CIR(dotted line) models with the parameters estimated in the period April 2nd, 2012 to July 2nd, 2012.

These figures show that the theoretical option prices of the hybrid Heston CIR model provide satisfactory approximations of observed put prices for all values of the strike prices and time to maturity. These values outperform those obtained with the Heston model. Both models overestimate the observed values of the call options, but their relative errors are reduced by over one half using the hybrid Heston model. In fact, the average relative errors of the call and put options are 9.6% and 6.9% for the hybrid Heston model while 21.2% and 11.2% for the Heston model.

Figures 3.6, 3.7, 3.8 and 3.9 show that the hybrid Heston CIR model is capable of matching with sufficient accuracy both call and put option prices for several strike prices and expiry dates using only one set of parameters. This results from the use of a stochastic interest rate.

Furthermore, the value of the model parameters estimated in the last window, June 25th, 2012 - July 2nd, 2012, are used to evaluate the out-of-sample

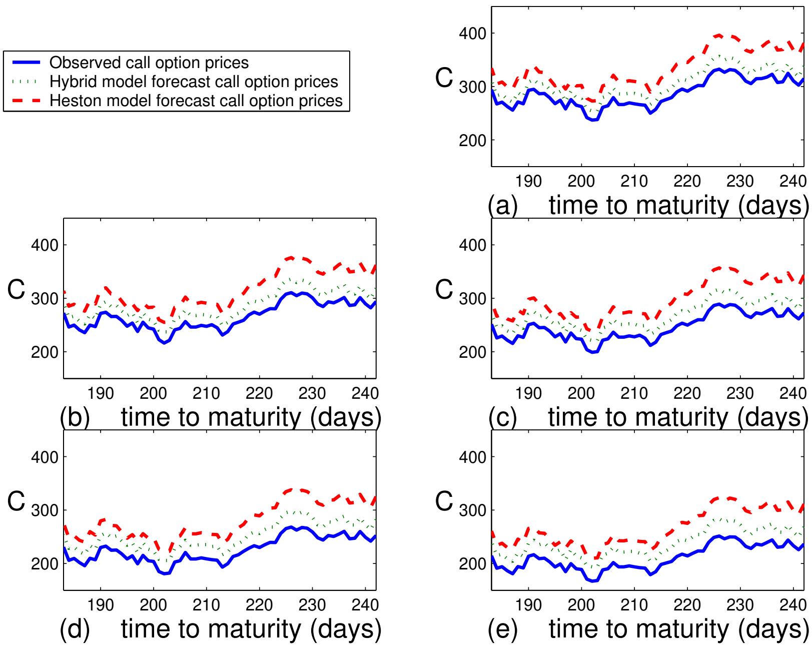

Figure 3.8 – Observed (solid line) and in-sample call option prices (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

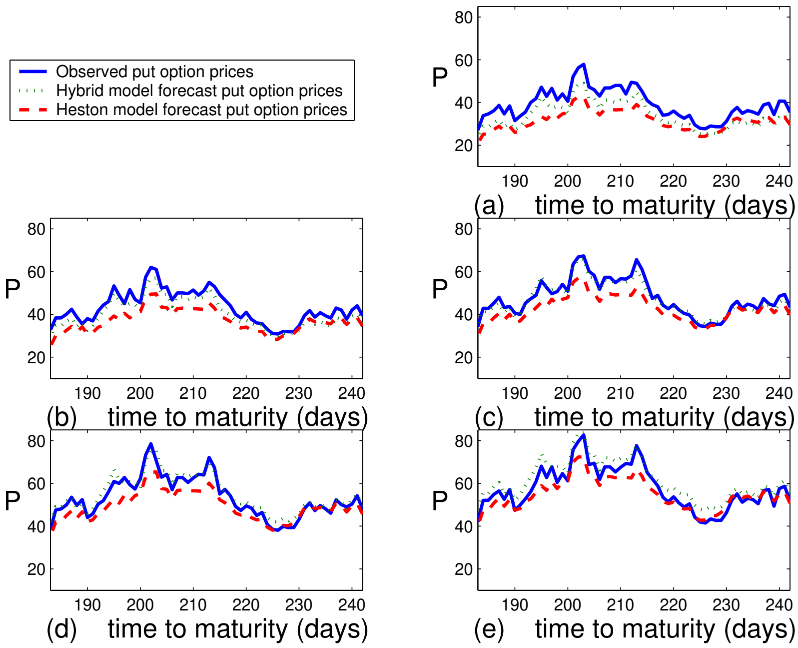

Figure 3.9 – Observed (solid line) and in-sample put option price (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

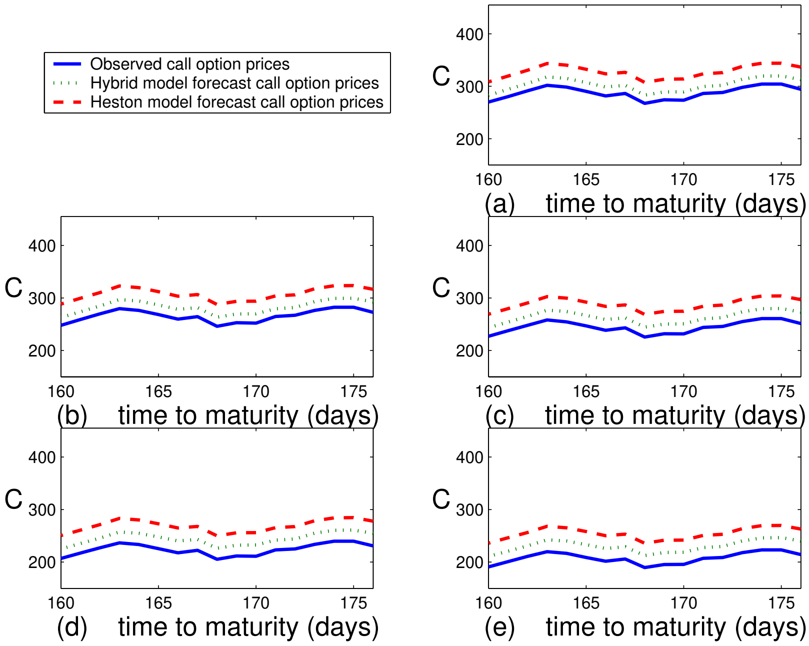

Figure 3.10 – Observed (solid line) and out-of-sample call option price forecast (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

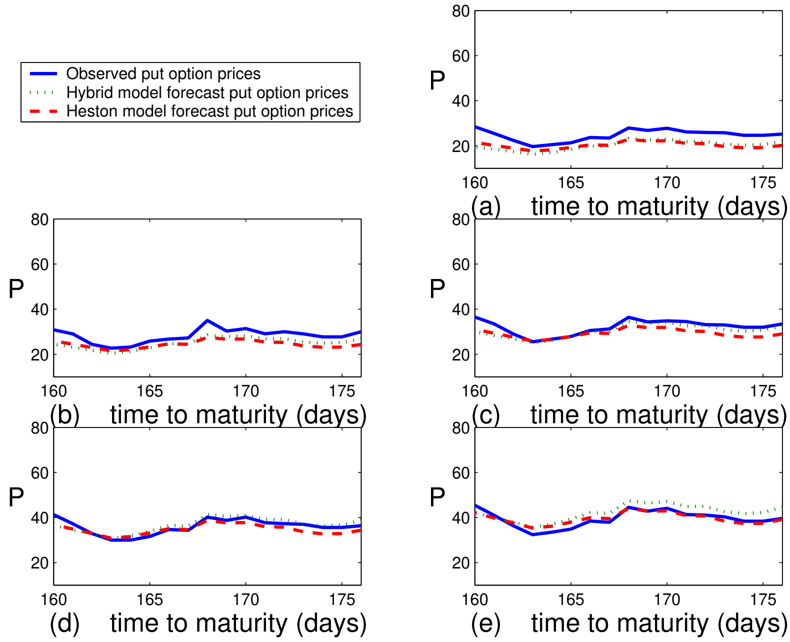

Figure 3.11 – Observed (solid line) and out-of-sample put option price forecast (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

European call and put option prices. The out-of-sample period is July 3rd to July 27th, 2012. The time to maturity for this period is 176 to 160 days. The performance of the stochastic model proposed and its parameter estimation procedure are measured with an “a posteriori” validation. That is, to compare the observed out-of-sample option prices with those obtained using formulas (3.47), (3.48), (3.45), (3.46) which use estimated parameters and observed spot prices.

Figures 3.10 and 3.11 show the out-of-sample option prices for the hybrid model (dotted line) and for the Heston model (dashed line). The out-of-sample put option prices of the Heston model are very accurate while the call option prices are not. The hybrid Heston CIR model provides accurate approximations of put option prices and outperforms the Heston model in approximating the call options. In fact, the average relative errors on the put and call options obtained using the hybrid Heston CIR model are 9.5% and 7.8% and using the Heston model are 9.6% and 17.9%.

In conclusion, the empirical analysis shows that the hybrid model interprets satisfactorily the real data considered in the period April 2nd to July 27th 2012 using only one set of model parameters. Moreover, the values of the initial stochastic rate r_0 could be considered a proxy of the short-dated government bond yield.

3.4.3 Two Stage Calibration

The results illustrated here are consistent with those obtained using the multi-factor stochastic volatility model of Christoffersen et al. 2009. Indeed, in the latter the authors use two stochastic factors but they are not specified. The results shown in this subsection suggest that the stochastic interest rate is one of these volatility factors.

To provide further evidence on the role of the stochastic interest rate, this subsection is concluded by repeating the previous analysis with an alternative calibration procedure which includes the U.S. three month government bond yields as data in the parameter estimation. Specifically, a two-stage calibration procedure is used to estimate the model parameters. Several two stage procedures applied to multi-factor stochastic volatility models can be found in the literature such as those illustrated in Christoffersen et al. 2009 and Islyaev and Date 2015.

The two stage calibration used here consists of a first stage where the

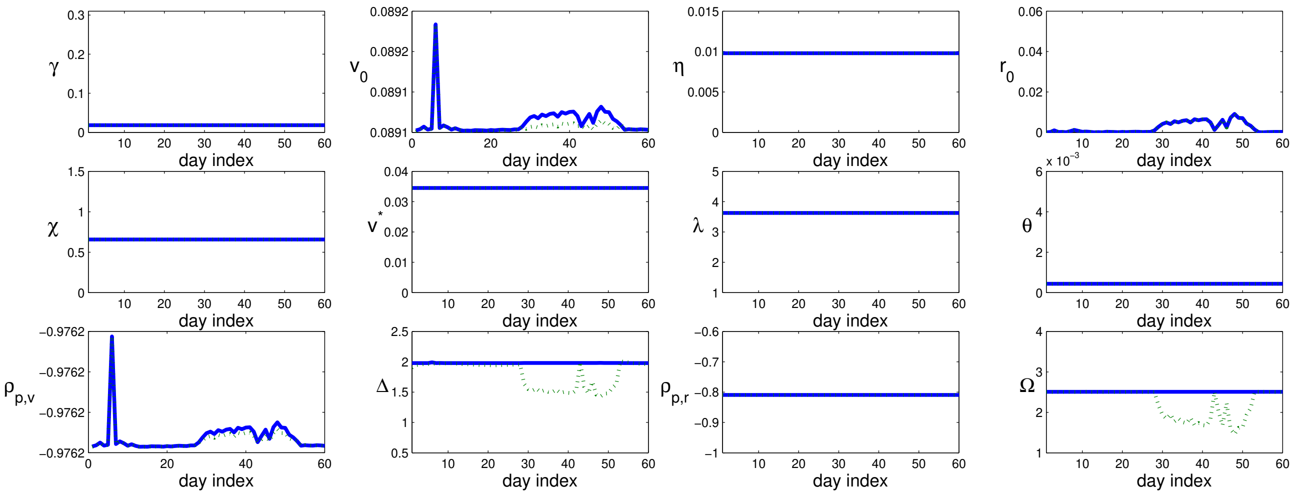

Figure 3.12 – Estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$, Δ (left panel) and η, r_0, λ, θ, $\rho_{p,r}$ and Ω (right panel) versus window index (six-day window) resulting from the calibration of the hybrid Heston model using only option prices (solid line) and the two stage procedure (dotted line).

parameters λ, θ, η and r_0 of the interest rate process are estimated using the U.S. three month government bond yields. This is done by minimizing the squared residuals of the observed and theoretical bond values employing the same rolling window of the experiment described immediately above. Then, in the second stage the remaining parameters are calibrated using option prices. As in the previous experiment, both in-sample and out-of-sample observed option prices are compared with the option prices provided by the hybrid Heston CIR model calibrated with this two stage procedure. The sample mean of the in-sample relative errors of the call and put options are 8.3% and 9.4% for the hybrid Heston model calibrated with the two stage procedure while, as already mentioned, they are 21.2% and 11.2% for the Heston model. The sample mean of the out-of-sample relative errors of the call and put options are 8.3% and 8.8% for the hybrid Heston model calibrated with the two stage procedure while they are 17.9% and 9.6% for the Heston model.

Figure 3.12 shows the parameters estimated with the two stage approach (dotted line) and those estimated using option prices (solid line) which have already been shown in Figures (3.6), (3.7). As can be seen, the parameter values estimated by using the two step procedure are basically the same obtained without using the two stage calibration except for Δ and Ω in the time windows indexed by 30 to 55, which corresponds to the period May 23, 2012 to June 25, 2012. Note that in this period the U.S. S&P 500 index shows abrupt oscillations (see Figure 5.10(a)). Please note that the y-scale of the graphs relative to the parameters $\rho_{p,v}$, v_0 of Figure 3.12 differs from the y-scale of Figure 3.6 in order to show the slight difference in the estimated values of these parameters.

The results obtained using the two stage calibration provides empirical evidence that the stochastic interest rate is a crucial volatility factor. In fact, the hybrid Heston CIR model significantly outperforms the Heston one even when we use the two-stage calibration where the interest rate process parameters are estimated without using option prices.

3.5 Conclusions

A hybrid Heston model with a stochastic interest rate is presented here. This hybrid model is analytically tractable and is a modified version of the model illustrated by Grzelak and Oosterlee 2011. The analytical treatment is based on a simple "trick" which allows expressing the transition probability density function as a one-dimensional integral of a smooth integrand function that depends on a real parameter q. Thanks to this formula and suitable choices for the parameter q, explicit elementary formulas for the moments of the asset price variable as well as efficient formulas to approximate the option prices are deduced. The proposed calibration procedure is used to conduct an empirical analysis of European call and put options on the U.S. S&P 500 index in 2012.

This analysis shows that the hybrid Heston CIR model outperforms the Heston model in interpreting both call and put option prices. Moreover, the values of the parameters of the stochastic interest rate model could provide useful insights into the relationship between "risk neutral" and "physical" measures. The empirical analysis of European call and put options on the U.S. S&P 500 index shows that the hybrid Heston model outperforms the Heston model in interpreting both call and put option prices. In fact, the use of a two stage calibration provides empirical evidence that the stochastic interest rate plays a significant role as a volatility factor in the option pricing.

In conclusion, the hybrid model seems to interpret satisfactorily call and put option prices, and consequently, the volatility implied by these prices. Using this model to forecast implied volatility and using the theoretical moments derived here to calibrate the model following the approach of Date and Islyaev 2015 will be objects of future research.

4 Is Consideration Of Stochastic Interest Rate Model Necessary For Long Term Product? Evidence From Yield Curve And Health Insurance Policy

4.1 Introduction

4.1.1 Motivation and Research Background

Predicting long-term bond yields is an interesting and challenging topic. The central bank's intervention clearly increases the volatility in government bond market. Since 2011, Fed has used QE and 'Operation Twist' program to control the interest rate and twist the yield curve. Through open market operations (OMO), the Fed has successfully lowered interest rate at target level. Figure 4.1 shows the Fed has kept short-term bond yields near zero since 2008 crisis.

Theoretically, it is the market, not central bank, should determine the long-term money supply and demand. In other words, the fact that central banks have controlled interest rates violates the most fundamental and widely accepted principles of price theory. Nevertheless, long-time intervention in interest rate market is clearly a risk-taking behaviour which is not consistent with central bank's role as risk manager and supervisor.

Empirical evidences tell us the super low interest rate dose not boost the real economy recovery, but increase the risk of default. For example, the debt to GDP ratio has gradually increased since 2008 crisis. Hence, the risk of default is indeed increasing. But the investors are still enlarging their investment in U.S. government bond. Because they are confident that the Fed is continuously repurchasing bonds via repurchase agreement (repo). Hence, even the yield is decreasing, the price of the bond is increasing. Investors look forward to the price return from the market instead of yield return at maturity.

Therefore, we are looking for robust approach to efficiently predict the trend of long-term bond yield. Empirical analysis in this chapter confirms that our stochastic interest approach is efficient. Last but not least, an application of Heston CIR model on Health endowment policy is presented.

4.1.2 Literature Review and Chapter Outline

In this chapter, a procedure to estimate the hybrid Heston CIR model parameters is proposed and validated using two different data sets: the U.S. three-month, two and ten year government bond yields, and the values of the Credit Agricole index linked policy, Azione Più Capitale Garantito Em.64.

The first analysis is some preliminary results regarding the application of the hybrid Heston CIR model from previous chapter in interpreting bond yield term structure are illustrated. Specifically, the model shows ability to capture the relationship between short and long term bond yields and to forecast their upward/downward trend. This application is possible because the hybrid model proposed here is closely related to the multi-factor stochastic volatility model of Trolle and Schwartz 2009.

In fact, roughly speaking, the hybrid model is similar to the Trolle and Schwartz model when only two stochastic factors are chosen: one is the short term yield, and the other is the long-term yield volatility. Recently, Cieslak and Povala 2014 show that the use of the short term rate as a volatility factor provides a good fitting of the bond yield term structure. Specifically, the generic U.S. two(ten)-year government bond yield is described using Eq. (3.19) where one factor is the volatility of the two(ten)-year bond yield and another factor is the generic U.S. three-month government bond yield. Figure 4.1 shows the yield values in the period December 31, 2002 to April 3, 2014.

The empirical analysis on long-term endowment policy shows that the stochastic interest rate plays a crucial role as a volatility factor and provides a multi-factor model that outperforms the Heston model in predicting health endowment policy price. This chapter is composed by two sections. One is an empirical analysis of U.S. government bond yields, and the other is an empirical analysis of a long-term endowment policy.

4.2 An Empirical Analysis of U.S. Government Bond Yields

In this experiment, the model parameters are estimated by maximizing the likelihood function using n_T (i.e. n_T =22,44,66,132) daily data for each bond (i.e. the daily data observed in one month $n_T = 22$, two months $n_T = 44$, three months $n_T = 66$ and six months $n_T = 132$). After having estimated the model parameters, we move this window along the time series discarding the n_T observations already used and inserting the new n_T observations. That is, the following problem are solved:

Figure 4.1 – Daily observations of U.S. three-month (solid line), two-year (dotted line) and ten-year (dashed line) government bond yields.

$$\max_{\underline{\Theta}\in\mathcal{V}} L_{n_T}(\underline{\Theta}). \tag{4.1}$$

where $\mathcal{V}$ is given in (3.49), n_T is the number of daily observations used and L_{n_T} is the objective function given by:

$$L_{n_t} = \sum_{j=1}^{n_T} D_{v,r,q}(x_j, v_0, r_{t_1}, t_j, x_{j+1}, t_{j+1}). \tag{4.2}$$

In Eq.(4.2) the function $D_{v,r,q}$ is given by (3.29) where q is equal to zero, x_j is the observation of the two(ten)-year bond yield at $t = t_j$, $j = 1, 2, \ldots, n_T$, and r_{t_1} is the observed value of the three month government bond on the first day of the time window (i.e. $t = t_1$). Thus, calibration problems needs to be solved is 128 ($n_T = 22$), 64 ($n_T = 44$), 42 ($n_T = 66$) and 21 ($n_T = 132$) . To each parameter value obtained solving problem (4.1), this chapter associates the last date $t = t_{n_T}$ of the time window used in the estimation procedure. In this way, there is a time series of monthly observations for $n_T = 22$, bi-monthly for $n_T = 44$ and so on. As mentioned above, in this experiment, the initial stochastic interest rate, r_0 is chosen, to be the value of the three-month bond yield on the first date of the time window used in the calibration.

Figure 4.2 shows the model parameters estimated with daily data of the three-month and two-year government bond yields using a time window of one month, $n_T = 22$ (dotted line) and a time window of three months, $n_T = 66$ (solid line). The observation indicates that the estimates obtained behave similarly except for the parameters v_0, Δ and Ω. In fact, the estimates of these parameters obtained using one-month time windows show oscillations which are more pronounced than those obtained using a three month window. This is especially true when the index value is in the interval $[1900, 2200]$ which corresponds approximately to the period July 2010 to September 2011. These oscillations seem to precede the U.S. debt-ceiling crisis of 2011.

Then formula (3.35) is used to forecast the log-return at $t = t_{n_T} + \delta t$. That is, we forecast $r_{t_{n_T}+\delta t} = x_{t_{n_T}+\delta t} - x_{t_{n_T}}$ where x_t is the logarithm of the two(ten)-year bond yield at time t. The forecasts are computed using different δt: $\delta t = 1\,month$, $\delta t = 2\,months$, $\delta t = 3\,months$ and $\delta t = 6\,months$.

These forecasts (hereafter, model forecasts) are compared with those ob-

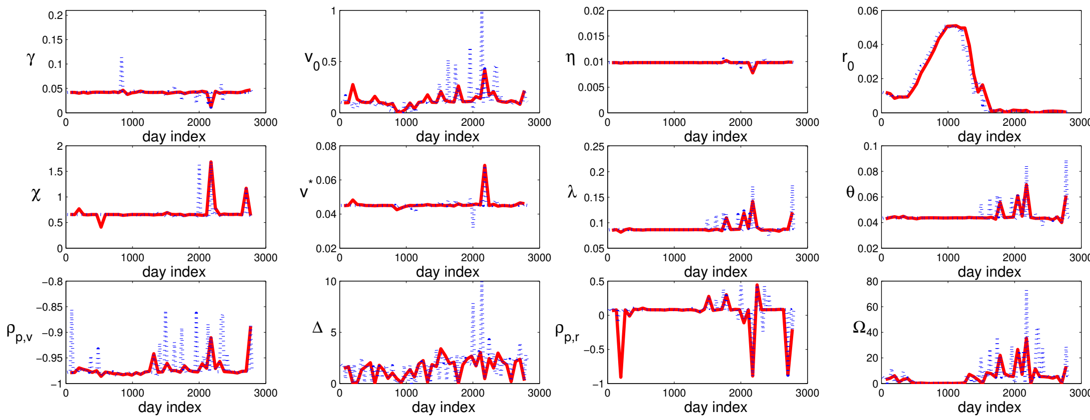

Figure 4.2 – Parameter estimated by using a time window of one month (dotted line) and a time window of three months (solid line). The data are daily data of the three-month and two-year government bond yields.

tained using the following naive formula, $\hat{r}_{t_{n_T}+\delta t} = x_{t_{n_T}} - x_{t_{n_T}-\delta t}$. Figure 4.3 shows the results obtained using the three-month bond yield as a volatility factor to describe the two year government bond yield. Specifically, Figure 4.3 displays the true log-returns (solid line and squares), the forecast log-

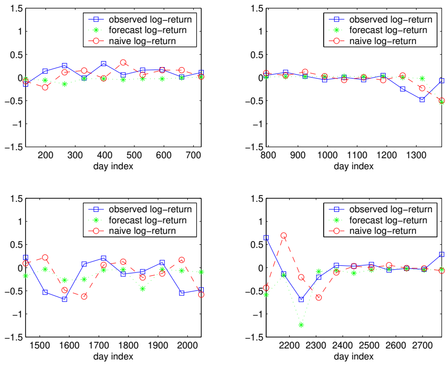

Figure 4.3 – Three month ahead 2-year bond yield forecasts. True log-returns (solid line), model log-return forecasts (dotted line) and naive log-return forecasts (dashed line). Upper left panel: January 2003 - March 2006. Upper right panel: April 2006 - July 2008. Lower left panel August 2008 - April 2011. Lower right panel: May 2011 - April 2014

returns (dotted line and stars) and the naive forecast log-returns (dashed line and circles). It is necessary to be observed that the forecast values obtained using the hybrid model are able to capture the trend of the log-return better than the naive approach in periods where the yields experienced strong fluctuations such as the crisis that followed the Lehman bankruptcy in the middle of September 2008 (lower left panel of Figure 4.3) and the United States debt-ceiling crisis of 2011 (lower right panel of Figure 4.3). To better

TABLE 4.1 – *Percentage of correct trend forecasts in one, two, three and six month ahead bond yield forecasts*

2-year bond yield trend forecasts			10-year bond yield trend forecasts		
Forecast horizon	Model Forecast	Naive Forecast	Forecast horizon	Model Forecast	Naive Forecast
one-month	47.24%	39.37%	one-month	58.51%	63.41%
two-month	69.35%	37.10%	two-month	52.38%	46.03%
three-month	57.50%	32.50%	three-month	70.0%	32.50%
six-month	52.63%	31.58%	six-month	52.63%	36.84%

analyze the quality of the forecasts in predicting the trend, it is necessary to compute how many times the forecast values match the upward/downward trend of the observed values. Note that the upward/downward trend is given by the sign of the difference between the value at $t = \tau$ and at $t = \tau + \delta t$. A negative (positive) sign signifies downward (upward) trend in the yield.

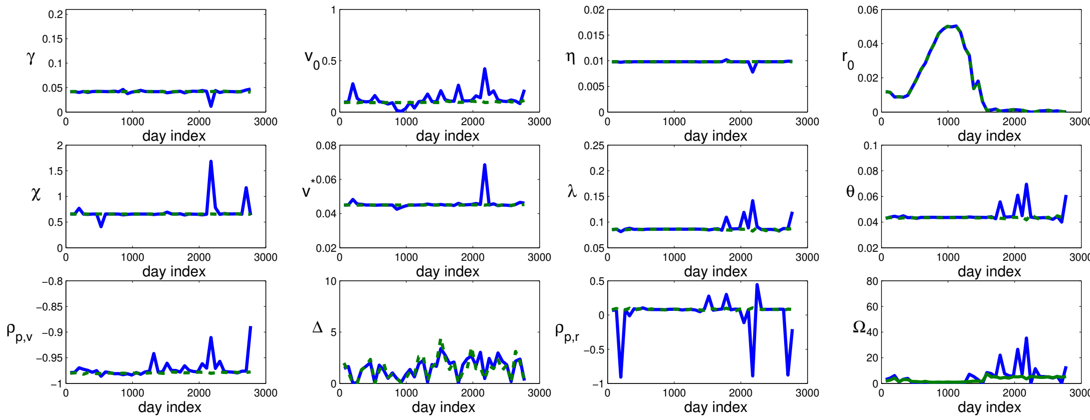

Figure 4.4 – Parameter estimated with a three month rolling time window using 3-month and 2-year bond yields (solid line) and using 3-month and 10-year bond yields (dashed line).

Table 4.1 illustrates the results of the trend forecasts obtained by first using the three-month and two-year government bond yields and then using the three-month and ten-year ones. Table 4.1 shows that the hybrid model forecasts outperform the naive ones in all cases considered except for the one-month ahead forecasts of the 10-year bond yield. This may be due to the fact that the maturity of the short-term bond yield used in the calibration is three months which may be too long for one-month ahead forecasts.

Finally, Figure 4.4 compares the estimates of the parameters obtained calibrating the hybrid model with a rolling time window of three months. The solid line indicates the values of the model parameters obtained by three-month and two year bond yields while the dashed line those obtained by three-month and ten-year yields. It is worth noting that the estimated values of the parameters obtained using the ten-year bond yields display less fluctuations than those obtained using two-year yields. This is particularly true for the parameters γ, χ, v_0, v^* and η, λ, θ which are related to the volatility of the stochastic processes which describe the long-term yield. This finding is coherent with the fact that bond yields with long-maturity usually are less volatile than those with short maturity. As previously mentioned, this is only a preliminary analysis on the use of this hybrid model to study the term structure of yields. However, the results seem to be encouraging.

4.3 An Empirical Analysis Of A Long-Term Endowment Policy

In this section, the hybrid model is shown to have the ability to price long-run products. The model parameters are estimated by using the time series of a pure endowment policy of the Crédit Agricole insurance company. Specifically, the weekly data of the index linked policy, Azione Più Capitale Garantito Em.64,are used here. The data is freely available at the website *http://www.previdoc.it/d/Ana/CREM64/_credit-agricole-vita-azione-piu-em64-01082017*.

This policy is a single-premium index linked life insurance policy whose benefits are directly linked to the performance of the Dow Jones Eurostoxx 50 Index. The duration of the Azione Più Capitale Garantito Em.64 policy covers the period from October 29, 2010 (date of issue) to August 1, 2017 (expiration date). On the maturity date, Action Capital Piú Guaranteed Em.64 policy guarantees the insured, should he be living, the payment of the premium plus a variable bonus obtained by multiplying the premium by 35 percent of the relative difference $(S_T - S_r)/S_r$ of the Dow Jones Eurostoxx 50 values between October 29, 2010 (i.e. $S_r = 2844.99$) and July 22, 2017, in the case of a positive difference. In the case of a negative difference, the variable bonus will be equal to zero. The payoff of this policy is given by:

$$\begin{aligned} & P_1\left(\frac{S_T}{S_r}, T\right) \\ & = N + 0.35N \max\left[\frac{S_T - S_r}{S_r}; 0\right] = N + 0.35\frac{N}{S_r} \max\left[S_T - S_r; 0\right] \end{aligned} \quad [4.3]$$

where N is the nominal capital paid at the start of the contract. For this contract $N = 100$.

The risky asset specified in the contract is modelled under the risk-neutral measure (using the hybrid Heston CIR model) and the mortality risk under the physical measure (using the mean reverting Gompertz model, see Milevsky and Promislow 2001) with the assumption that these two measures are independent. As a consequence, the pricing of this policy consists in the evaluation of the following product:

$$C_P(S_0, T) = E\left(e^{-\int_0^T r_\tau\, d\tau} P_1(S_T)\right) E\left(e^{-\int_0^T h_u du}\right), \quad [4.4]$$

where P_1 is the payoff function associated with the policy, r is the stochastic interest rate and h_t is the mortality rate. The first expected value in (4.4) is the value of a European option in the hybrid Heston model and the second expected value is the survival probability. We model the mortality rate h_t, $t > 0$, using the stochastic model:

$$dh_t = (g + \frac{1}{2}(\sigma^*)^2 + b\ln(\hat{h}_0) + b\,g\,t + b\,\ln(h_t))h_t\,dt + \sigma^*\,e^{at}\,h_t dQ_t,\ t > 0, \quad [4.5]$$

where the quantities g, b, h_0, a, σ^* are real constants, and Q_t, $t > 0$, is a standard Wiener process. The survival probability is evaluated using the following approximation (for further details see Recchioni and Screpante 2014):

$$\begin{aligned}
&E(e^{-\int_t^T h_\tau d\tau}\,|\,h_0 = \hat{h}_0) \approx e^{-\hat{h}_0 e^{gt}(e^{g(T-t)}-1)/g}\cdot\{1-\\
&\frac{\hat{h}_0(\sigma^*)^2}{4(a+b)}\left[\left(\frac{e^{(g+2a)t}(e^{(g+2a)(T-t)}-1)}{g+2a} - \frac{e^{(g-2b)t}(e^{(g-2b)(T-t)}-1)}{g-2b}\right)\right]\\
&+\frac{\hat{h}_0^2(\sigma^*)^2}{2}\left[-\frac{1}{2(a+b)}\left(\frac{e^{(g-b)t}(e^{(g-b)(T-t)}-1)}{g-b}\right)^2 + \frac{1}{(a+b)}\,\frac{1}{(g+b+2a)}\cdot\right.\\
&\left.\left[\left(\frac{e^{2(g+a)t}(e^{2(g+a)(T-t)}-1)}{2(g+a)}\right) - \left(\frac{e^{(g-b)(T+t)+2(a+b)t} - e^{2(g+a)t}}{g-b}\right)\right]\right]\Bigg\},\\
&\qquad 0 \le t < T. \qquad [4.6]
\end{aligned}$$

The parameters appearing in formula (4.6) are chosen as shown in Table 4.2. These values are motivated by the analysis proposed in Recchioni and Screpante 2014 on a similar Credit Agricole pure endowment policy. The

Table 4.2 – *Parameter values of the demographic component of the Credit Agricole index linked policies.*

cohort	h_0	a	b	g	σ^*
1977	0.0001175	0.0005	0.6315	0.0722	0.0311

model parameters are estimated by solving the calibration problem (3.51) after appropriate adjustments (i.e. removing the put option prices and replacing the call option prices with policy prices). We use formula (3.45) to evaluate the first expected value appearing in Eq.(4.4). As in Section 3.4, the integrals are computed by using the midpoint quadrature rule with 2^{14} nodes.

Weekly prices are used here covering the period April 4, 2012 to April 13, 2015 corresponding to 152 observations. In formula (3.50), $n_T = 60$ (i.e. sixty weekly observations) and $n_D = 1$ (i.e. one strike price S_r) are chosen. A rolling window of size n_T is considered and this window is moved along the

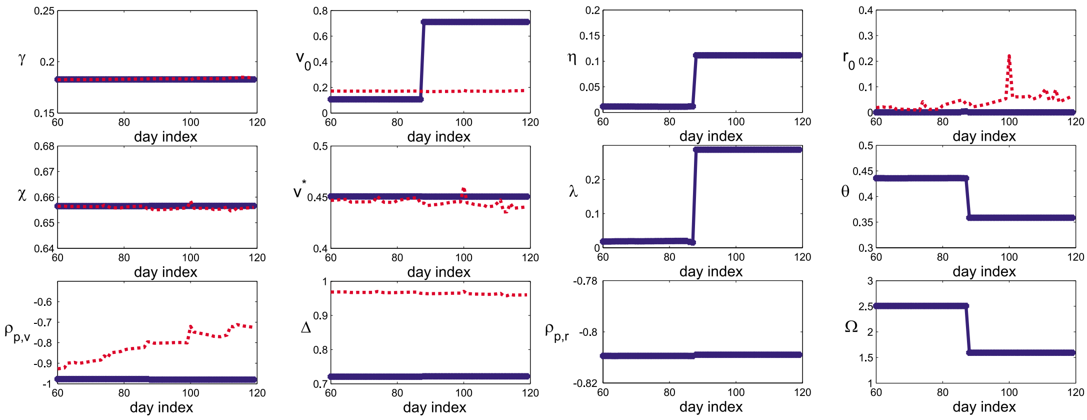

Figure 4.5 – Left panel: estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$ and Δ versus window index resulting from the calibration of the hybrid Heston model (solid line) and the Heston model (dotted line). Right Panel: estimated parameters η, r_0, λ, θ, $\rho_{p,r}$ and Ω versus window index resulting from the calibration of the hybrid Heston model (solid line). Note that the dotted line in the upper right sub-panel shows the estimated values of the parameter r_0 resulting from the calibration of the Heston model. The data used in the calibration are the Credit Agricole pure endowment policy values.

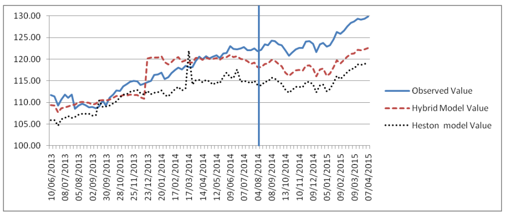

Figure 4.6 – Credit Agricole pure endowment policy. In-sample (on the left of the vertical bar) and out-of-sample (on the right of the vertical bar) approximations obtained using the hybrid model (dashed line) and the Heston model (dotted line) and the observed policy values (solid line).

time series discarding the oldest observed price of the window and inserting the new observed one. In this way, sixty problems covering the period April 4, 2012 to August 4, 2014 are solved, and this section obtains a time series of each parameter made of weekly observations. The parameter value estimated in a given time window is associated with the last date of the window.

Figure 4.5 shows the estimated values as a function of the window index. It can be observed that the time series are approximately constant except for the time series of r_0 which shows some oscillations in the year 2014. The model parameters estimated in the last window are used to evaluate the policy out-sample that is in the period August 11, 2014 to April 13, 2015. Figure 4.6 shows the in-sample and out-sample observed policy values (solid line), the approximations obtained with the hybrid model (dashed-line) and those obtained with the Heston model (dotted line). The observed policy values and their approximations before the vertical bar are in-sample while those after the vertical bar are out-of sample. It can be observed that the hybrid Heston CIR model outperforms the Heston one in both the in-sample and out-of-sample. More specifically, the average relative error of the hybrid model approximations is 2.5% and the maximum is 5.7%, while those of the Heston approximations is 4.9% and 8.4% respectively.

4.4 Conclusions

In this chapter, we extend our previous hybrid Heston CIR model to conduct an empirical analysis of two different data-sets. In chapter 2, it is observed that the hybrid Heston CIR model outperforms the Heston model in interpreting both call and put option prices. Thus, it provides empirical evidence that the stochastic interest rate plays a significant role as a volatility factor in the option pricing. Moreover, in this chapter, the hybrid Heston CIR model outperforms the Heston one even when the Credit Agricole policy values are analyzed. This result is not surprising since the assumption of constant interest rate is not realistic in the pricing of long-term products.

Finally, the preliminary results of the empirical analysis on U.S. government bond yields confirm the validity of the Heston-like multi-factor stochastic volatility models to interpret bond yield term structure. The use of short-term yields as a volatility factor is a recent approach of Cieslak and Povala 2014 that deserves further investigation.

5 Can Negative Interest Rates Really Affect Option Pricing? Empirical Evidence From An Explicitly Solvable Stochastic Volatility Model

5.1 Introduction

5.1.1 Motivation And Research Background

Intuitively speaking, it is believed that banks as lender of assets prefer high interest rate in order to maximize the profit of interest return. This is not necessarily the case, however. Low interest rate and inflation are bank's 'gospel'. Because low interest will raise the price of all the assets, which will create more benefit to banks. In other words, as the primary beneficiaries of assets depreciation, banks prefer super low interest rate, or if possible, permanent low interest rate.

After 2008 financial crisis, Fed and other major central banks believe that the trend of interest rate should oscillating decline in order to guarantee an increasing trend of market price. Thus, the value of assets will raise in order to avoid great economic recession and generate the economic recovery. During a series of QEs after 2008 crisis, i.e. QE1 in 2008, QE2 in 2010 and QE3 in 2012, Fed contentiously intervenes in the interest rate market for a long time. Specifically since late 2011, Fed has used QE and 'Operation Twist' program to successfully control the short-term interest rate around zero. Besides Fed, many other central banks are also strongly in favor of QE and super low interest rate. For instance in 2014, the European Central Bank (ECB) instituted a negative interest rate which is only applied to bank deposits with the aim of preventing the Euro zone from falling into deflationary spiral.

Since 'expanded asset purchase programme' announced in February 2015 by Mario Draghi, president of the European Central Bank, ECB decides to continuously stimulate the Euro Zone economy till September 2016. This will lead to a total QE of at least €1.1 trillion in Euro zone. Therefore, we believe low interest rate will continue for quite a long period of time. Moreover, referring to the Syria war and European refugee crisis since April 2015, it is indeed necessary to consider negative values of interest rate with non-neglected probability in the future. How negative values of interest rate will challenge the traditional approach of approximating bond yields and derivatives pricing? This is what we will explore in this chapter.

5.1.2 Literature Review and Chapter Outline

The profound 2008 financial crisis and 2011 European sovereign debt crisis have caused negative government bond yields both in U.S. and EU area. This chapter aims at the understanding the following questions.

1. Whether and how can negative values of short-term government bond yields affect the prices of Foreign EXchange rate (FX) and the associated index options?

2. Will the approach allowing for negative values of interest rates improve option pricing model and implied volatility forecasts?

Empirical evidence shows the importance of including stochastic volatility in derivative pricing approach, such as Hull and White 1988, Stein-Stein 1991, Heston 1993 and Ball and Roma 1994. Heston 1993 has shown a closed-form solution for options with stochastic volatility, while the interest rate process is assumed constant. Indeed, the Heston model is not hybrid SDE model with stochastic interest rate. However, with one dimensional stochastic volatility process, it is able to describe the smile shape of implied volatility with respect to the spot price.

The relaxation of the constant interest rate assumption is described in Amin and Ng 1993, Scott 1997, Bakshi et al. 2000, Zhu 2000, Giese 2006, and Andreason 2007. Approaches that including stochastic interest rate as an additional stochastic factor to approximate the derivatives price can be found in Bakshi et al 2007, Antonove et al 2008, Haastrech 2010 and Grzelak et al 2011, 2012. Nevertheless, the analytical treatment of full scaled hybrid SDE model is not straightforward. Thus these models use numerical approximation instead of analytical solution to formulate derivative pricing.

In line with the attempt to provide exact closed-form solution of multi-scale stochastic volatility models, Duffie, Pan, and Singleton 2000 studied the assert pricing model with the assumption of affine diffusion process (AD). Fouque and han 2004, Christoffersen et al. 2009, Fatone et al. 2009, 2013, Wong and Lo 2009, Date and Islyaev 2015, Islyaev and Date 2015, Pun et al. 2015 studied a multi-scale stochastic volatility model. These models extend the Heston model into a multi-scale volatility case where price volatility dynamics are better captured, and option pricing models are improved. In addition, several hybrid models cover both stochastic interest rates and stochastic volatilities. Zhu 2000 developed a model using a stochastic interest rate which is not correlated to the equity. Grzelak, Oosterlee and Weeren 2012 proposed the so called Schobel-Zhu-Hull-White hybrid model with closed-form solutions. This model follows the affine model approach proposed by Due, Pan and Singleton 2000. An analytical treatment model for the spot Foreign EXchange (FX) rate with stochastic volatility and stochastic domestic as well as foreign rates was studied by Ahlip.

In this capter, a generalization of the Hybrid Hull and White model with affine diffusion processes is considered. For accurately modelling the derivatives, this model allows negative values of interest rate, and the correlation between the diffusion process of equity and interest rate, i.e $\rho_{p,r}$ is generalized in a range of $[-1, 1]$ (not necessarily equal to 1, in Grzelak et.al 2011). Moreover, in stead of numerical approximation, an analytical closed-form solution for probability density function is studied here.

The aim of the Heston-Hull-White (HHW) model, which describes the dynamics of an asset price under stochastic volatility and interest rate, is to extend the Heston model in order to efficiently solve option pricing problems. A formula for the transition probability density function is derived as a one dimensional integral of an elementary integral function which is used to price European Vanilla call and put options based on HHW model.

To this end, an empirical analysis of the prices of call and put options on the U.S. S&P 500 index is carried out. Apart from this, this chapter uses implied volatility to calibration the model. More precisely, the calibration procedure is based on a nonlinear constrained optimization problem whose objective function measures the relative squared difference between the observed and theoretical implied volatilities associated with call and put options. Similar study can be found in Pacati et al. 2015. Furthermore, this chapter studies another empirical analysis of the Eurodollar futures prices and the corresponding European options prices with a generalization of the Heston model in the stochastic interest rate framework. Specifically, the dynamics of the underlying asset is described by two factors: a stochastic variance and a stochastic interest rate. The volatility is not allowed to be negative while the interest rate is allowed for negative values. Explicit formulas for the transition probability density function and moments are derived. These formulas are used to efficiently estimate the model parameters. The illustrated model belongs to the class of full-scale Heston-Hull-White model recently illustrated in Grzelak et.al 2011, 2012, Guo et.al 2013, and Recchioni et. al 2015. The empirical analysis shows that the use of models which allows for negative values of interest rates can reproduce implied volatility and smile effect.

This chapter is organized as follows. In Section 5.1, the hybrid Heston-Hull-White model and some relevant formulas are explicitly illustrated. In Section 5.2 some formulas are proposed to approximate the European vanilla call and put options price as one-dimensional integrals of explicitly known functions. In Section 5.3, some experiments are proposed on simulated data to test the accuracy of the approximation proposed. Furthermore, in Section 5.4.3 an empirical analysis is illustrated. Specifically,the model parameters are estimated by solving constrained optimization problems whose objective functions involves implied volatility. There are two proposed empirical analysis. The first one deals with U.S. S&P 500 index from April 2, 2012 to July 2, 2012 and the prices of the corresponding European call and put options with expiry on March 16th, 2013. The second experiment analyzes the futures price of the EUR/USD currency's exchange rate that having maturity on September 16th, 2011, and the daily prices of the corresponding European call and put options with expiry date on September 9th, 2011, observed in the time period from September 27th, 2010, to July 19th, 2011. In Section 5.5 are conclusions. in the Appendix B, the formula for the probability density function and explicit formulas of the moments are derived.

5.2 Generalized Hybrid Heston Hull-White Model

This section focuses on a generalization of the Heston model in the stochastic interest rate framework. This model is illustrated by Grzelak et al. 2011, 2012, and Guo et al. 2013. Roughly speaking, it can also be interpreted as a multi-factor Heston model where one of the stochastic factor is the interest rate. As already mentioned, multi-factor Heston stochastic volatility models are proposed in Christoffersen et al. 2009 and Fatone et al. 2009. However, the introduction of the stochastic interest rate as a factor which drives the asset makes the analytical treatment of the model different from those illustrated in Christoffersen et al.2009 and Fatone et.al. 2009. Moreover, the use of the one-factor Vasicek model (Vasicek 1977) to describe the stochastic interest rate makes the proposed multi-factor model rather different from the multifactor models previously mentioned where the factors are driven by an one factor CIR model.

In fact, the Vasicek model allows for negative values of the interest rate and this has been considered a weakness of this model. However, the negative values of the U.S. short term government bond yields observed in the last years (see Figure 5.3) makes this model appealing. Hereafter, S_t, v_t and r_t are denoted the equity price, its volatility and the interest rate at time $t > 0$ respectively.

$$\begin{aligned} dS_t &= S_t\, r_t dt + S_t\sqrt{v_t}dW_t^x && t > 0, && [5.1] \\ dv_t &= \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dW_t^v, && t > 0, && [5.2] \\ dr_t &= \lambda(\theta - r_t)dt + \eta\, dW_t^r, && t > 0 && [5.3] \end{aligned}$$

where χ, v^*, γ, λ, η are positive constants, and $W_t^{p,v}$, $W_t^{p,r}$, W_t^v, W_t^r are standard Wiener processes. We assume the following correlation structure:

$$\begin{aligned} E(dW_t^x dW_t^v) &= \rho_{x,v}dt,\ t > 0, && [5.4] \\ E(dW_t^x dW_t^r) &= \rho_{x,r}dt,\ t > 0, && [5.5] \\ E(dW_t^v dW_t^r) &= 0,\ t > 0, && [5.6] \end{aligned}$$

where $E(\cdot)$ denotes the expected value of $\cdot$ and $\rho_{x,v}$, $\rho_{x,r}$, $\in [-1, 1]$ are constants known as correlation coefficients.

Roughly speaking, this model generalizes the Heston model within a framework of stochastic interest rates. When the stochastic rate is described by a Vasicek model, or a more general Hull and White model. It is worth to highlight that the stochastic model (5.1)–(5.3) is written with respect the risk neutral world measure. This is motivated by the fact that, in Section 5.4.3, the model parameters are estimated by using derivative data. The system of equations (5.1)-(5.3) are equipped with the following initial conditions:

$$\begin{aligned} S_0 &= S_0^*, && [5.7] \\ v_0 &= v_0^*, && [5.8] \\ r_0 &= r_0^*, && [5.9] \end{aligned}$$

where S_0^*, v_0^*, and r_0^* are random variables concentrated in a point with probability one, and for the sake of simplicity, these random variables are denoted with the points where they are concentrated.

The quantities χ, v^*, γ, λ, θ, η are positive constants. More precisely, The quantity χ is the speed of mean reversion, v^* is the long term mean and γ is the so called volatility of volatility (vol of vol for short). It is worth noting that the variance v_t remains positive for any $t > 0$ with probability one given that $2\chi\, v^*/\gamma^2 > 1$ and $v_0 = v_0^* > 0$ (see Heston 1991). As a consequence, the equity price S_t remains positive for any $t > 0$ with probability one given that $S_0^* > 0$ with probability one.

The hybrid Heston-Hull-White (HHW) model allows for negative values of the interest rate r_t. Nowadays, this is a nice feature since negative short-term bond yields have been experienced in the both U.S. and EURO zone. The HHW model (5.1)-(5.3) with the correlation structure (5.4)–(5.16) is not an affine model. Following the approach illustrated in Chatper 2, the HHW model (5.1)–(5.3) is modified in order to get an analytically tractable model, and to allow for a direct correlation between the equity and interest rate processes. That is, the dynamics of the process (S_t, v_t, r_t), $t > 0$ is described as follows:

$$
\begin{aligned}
dS_t &= S_t\, r_t dt + S_t\sqrt{v_t}dW_t^{p,v} + S_t\Delta\sqrt{v_t}dW_t^v + S_t\,\Omega\, dW_t^{p,r}, \quad t > 0, && [5.10]\\
dv_t &= \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \qquad t > 0, && [5.11]\\
dr_t &= \lambda(\theta - r_t)dt + \eta\, dW_t^r, \qquad t > 0 && [5.12]
\end{aligned}
$$

where Δ and Ω are positive functions, and the constants χ, v^*, γ, λ, θ, η are the same appearing in the model (5.1), (5.2), (5.3), while $W_t^{p,v}$, $W_t^{p,r}$, W_t^v, W_t^r are standard Wiener processes. The correlation structure is assumed as follows:

$$
\begin{aligned}
E(dW_t^{p,v}dW_t^v) &= \rho_{p,v}\, dt,\ t > 0, && [5.13]\\
E(dW_t^{p,v}dW_t^r) &= 0,\ t > 0, && [5.14]\\
E(dW_t^{p,r}dW_t^r) &= \rho_{p,r}\, dt,\ t > 0, && [5.15]\\
E(dW_t^{p,r}dW_t^v) &= 0,\ t > 0, && [5.16]\\
E(dW_t^r dW_t^v) &= 0,\ t > 0, && [5.17]
\end{aligned}
$$

where the quantities $\rho_{p,v}$, $\rho_{p,r} \in (-1, 1)$ are constant correlation coefficients. Using Ito's lemma and Eq.(5.10), the process of the log-price, $x_t = \ln(S_t/S_0)$, $t > 0$ satisfies the following dynamics:

$$
\begin{aligned}
dx_t &= \left[r_t - \frac{1}{2}\left(\tilde{\psi} v_t + \Omega^2 r_t\right)\right] dt + \sqrt{v_t}dW_t^{p,v} + \Delta\sqrt{v_t}dW_t^v + \Omega dW_t^{p,r}, && [5.18]\\
dv_t &= \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \qquad t > 0, && [5.19]\\
dr_t &= \lambda(\theta - r_t)dt + \eta\, dW_t^r, \qquad t > 0, && [5.20]
\end{aligned}
$$

where $\tilde{\psi}$ is the quantity defined by:

$$\tilde{\psi} := 1 + \Delta^2 + 2\rho_{p,v}\,\Delta. \tag{5.21}$$

The process (x_t, v_t, r_t) satisfies the following initial conditions:

$$x_0 = x_0^* = 0, \quad v_0 = v_0^*, \quad r_0 = r_0^*, \qquad [5.22]$$

where x_0^*, v_0^*, and r_0^* are random variables that are assumed to be concentrated in a point with probability one. As explained in Chapter 2, and Grzelak and Oosterlee 2011, the good performance of the hybrid models in interpreting real data is due to the correlations between equity and interest rate. This finding is confirmed by the empirical study of Section 5.5, which shows the crucial role of the stochastic rate and its correlation with the equity to get accurate forecast of the European call and put option prices.

Let me now illustrate the main formulas derived in this section and used in the simulation and empirical analysis. To this end, it is convenience to introduce some notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of the positive real numbers, and $\mathbb{R}^n$ the n-dimensional Euclidean vector space. Let $\underline{\Theta}_v$ and $\underline{\Theta}_r$ denote the vectors $\underline{\Theta}_v = (\gamma, \chi, v^*, \rho_{p,v}, \Delta) \in \mathbb{R}^5$ and $\underline{\Theta}_r = (\eta, \lambda, \theta, \rho_{p,r}, \Omega) \in \mathbb{R}^5$, containing the parameters of the volatility and interest rate processes respectively. Let $p_f(x, v, r, t, x', v', r', t')$, $(x, v, r), (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, t, $t' \geq 0$, $t - t' > 0$ be the transition probability density function associated with the stochastic differential system (5.18), (5.19), (5.20). The function p_f as a function of the variables (x, v, r, t) satisfies the following backward Kolmogorov equation:

$$\begin{aligned}
-\frac{\partial p_f}{\partial t} &= \frac{1}{2}(\tilde{\psi} v + \Omega^2)\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\gamma^2 v \frac{\partial^2 p_f}{\partial v^2} + \frac{1}{2}\eta^2 \frac{\partial^2 p_f}{\partial r^2} + \gamma(\rho_{p,v} + \Delta) v \frac{\partial^2 p_f}{\partial x \partial v} \\
&+ \eta \rho_{p,r} \Omega \frac{\partial^2 p_f}{\partial x \partial r} + \chi(v^* - v)\frac{\partial p_f}{\partial v} + \lambda(\theta - r)\frac{\partial p_f}{\partial r} + \left(r - \frac{1}{2}(\tilde{\psi} v + \Omega^2)\right)\frac{\partial p_f}{\partial x}, \\
&(x, v, r) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \ 0 \leq t < t', \qquad [5.23]
\end{aligned}$$

$$\text{where} \quad \tilde{\psi} := 1 + \Delta^2 + 2\Delta\rho_{p,v}. \qquad [5.24]$$

with final condition:

$$\begin{gathered}
p_f(x, v, r, t, x', v', r', t') = \delta(x' - x)\delta(v' - v)\delta(r' - r), \\
(x, v, r), (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \ t \geq 0,
\end{gathered}$$

and appropriate boundary conditions. In Appendix B, it's proved that the following formula holds:

$$p_f(x,v,r,t,x',v',r',t') = e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} L_{v,q}(t'-t,v,v',k;\,\underline{\Theta}_v)\cdot$$
$$L_{r,q}(t'-t,r,r',k;\,\underline{\Theta}_r),$$
$$(x,v,r),\,(x',v',r') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ q\in\mathbb{R},\ t'-t>0, \quad [5.25]$$

where $\imath$ is the imaginary unit, $L_{v,q}$ and $L_{r,q}$ are explicitly known functions given in Eqs.(7.110) and (7.106). The function $L_{v,q}$ has already been introduced in Chapter 2, and it depends on the modified Bessel function, $I_{(2\chi v^*/\gamma^2)-1}$ (see, for example, Abramowitz and Stegun, 1970), where $(2\chi v^*/\gamma^2)-1$ is a positive real index under the condition $2\chi v^*/\gamma^2 > 1$. The last condition guarantees that the modified Bessel function is bounded at zero, and this guarantees that the function p_f given in (3.25) is a probability density function with respect to the future variables.

Moreover, it is worth noting that formula (5.25) can be interpreted as the inverse Fourier transform of the convolution of the probability density functions associated with the stochastic processes described by Eqs. (5.18)- (5.20) when one of the two factors, v_t, r_t, is dropped. This specific form of the transition probability density function is a consequence of the correlation structure (5.13)-(5.17). A further good feature of the functions $L_{v,q}$ and $L_{r,q}$ is that the integrals of $L_{v,q}$ and $L_{r,q}$ with respect the future variables v' and r' are given by elementary functions, $W^0_{v,q}$ and $W^0_{r,q}$ respectively (see formulas (7.116) and (7.113)) as well as their products for integer powers of the future variables v', r' (see formulas (7.117) and (7.112)). That is, $W^m_{v,q}$ and $W^m_{r,q}$, $m=0,1,\ldots$ are defined as follows:

$$W^m_{v,q}(t'-t,v,k;\underline{\Theta}_v) = \int_0^{+\infty} dv'\,(v')^m L_{v,q}(t'-t,v,v',k;\underline{\Theta}_v), \qquad [5.26]$$

$$W^m_{r,q}(t'-t,r,k;\underline{\Theta}_r) = \int_0^{+\infty} dr'\,(r')^m L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r), \qquad [5.27]$$

These functions can be used together to get an integral representation formula for the marginal probability density function, $D_{v,q}(x,v,r,t,x',r',t')$, of the future variables (x',r').

$$D_{v,q}(x,v,r,t,x',r',t') = \int_0^{+\infty} dv'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v) L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r), \quad [5.28]$$

and for the marginal probability density function, $D_{v,r,q}(x,v,r,t,x',t')$, of the price variable x':

$$D_{v,r,q}(x,v,r,t,x',t') = \int_0^{+\infty} dr' \int_0^{+\infty} dv'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v) W^0_{r,q}(t'-t,r,k;\underline{\Theta}_r)\,. \quad [5.29]$$

Please note that in formulas (5.28) and (5.29), the variables x, v, r are the initial values of the log-price, the stochastic variance and the stochastic interest rate respectively. These two last variables are not observable in the financial market and should be estimated using an appropriate calibration procedure. The estimation of the initial stochastic volatility is a common practice as suggested by Bühler, 2002.

In addition, formulas (5.28) and (5.29) are useful in estimating model parameters using a maximum likelihood approach, and deriving the explicit formulas for the moments of the price variable and the mixed moments. In fact, the formula for the $m-th$ moment of the price $S_{t'} = S_0 e^{x'}$ conditioned to the observation at time $t=0$ is obtained by computing the following integral:

$$\mathcal{M}_m = E(S^m_{t'}) = S^m_0 \int_{-\infty}^{+\infty} dx' e^{mx'} D_{v,r,q}(0,v_0,r_0,0,x',t') =$$
$$S^m_0 \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \left(\int_{-\infty}^{+\infty} dx' e^{mx'} e^{-qx'} e^{\imath k\, x'}\right) W^0_{v,q}(t',v_0,k;\underline{\Theta}_v)\, W^0_{r,q}(t',r_0,k;\underline{\Theta}_r)$$
$$(S_0,v_0,r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ q \in \mathbb{R},\ t'>0. \quad [5.30]$$

When $m=q$ is chosen, the integral in the bracket shows a delta Dirac's function of the conjugate variable k, which allows to having the following explicit formula for the moments:

$$\mathcal{M}_m = E(S^m_{t'}) = S^m_0 W^0_{v,m}(t',v_0,0;\underline{\Theta}_v)\, W^0_{r,m}(t',r_0,0;\underline{\Theta}_r),$$
$$(S_0,v_0,r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t'>0. \quad [5.31]$$

Similarly, using the explicit formulas for the integrals (5.26) and (5.27), the following explicit formulas for the mixed moments is obtained:

$$E(S^{m_1}_{t'} r^{m_2}_{t'}) = S^{m_1}_0 W^0_{v,m_1}(t',v_0,0;\underline{\Theta}_v)\, W^{m_2}_{r,m_1}(t',r_0,0;\underline{\Theta}_r),$$
$$(S_0,v_0,r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t'>0, \quad [5.32]$$

and

$$E(S_{t'}^{m_1} v_{t'}^{m_2}) = S_0^{m_1} W_{v,m_1}^{m_2}(t', v_0, 0; \underline{\Theta}_v)\, W_{r,m_1}^{0}(t', r_0, 0; \underline{\Theta}_r),$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0, \qquad [5.33]$$

where the functions $W_{v,q}^0$, $W_{r,q}^0$, $W_{v,q}^m$, $W_{r,q}^m$, $m = 1, 2, \ldots$ are elementary functions given by (5.26), (7.111), (5.26) and (5.27) respectively.

The highlight here is that formulas (7.129), (7.131), (7.130) are elementary formulas that do not involve integrals. The derivation of these formulas is possible, thanks to Eq.(5.25). This equation reduces the computation of the transition probability density function, which depends on a "regularization" parameter, q, to a one dimensional integral whose integrand function is the product of smooth functions of the future variables. This smoothness is a result of the specific way in which the formula is deduced. This together with an appropriate choice for q permits deriving elementary formulas for the marginal probability density functions and the moments illustrated above. These formulas are used to estimate the model parameters with great savings of computational time.

In fact, the marginal probability density function, $D_{v,q}$, can be used to price European call and put options with payoff functions independent of the variance process in the framework of stochastic interest rates. Thus, we use $D_{v,q}$ to deduce formulas for European call and put vanilla options.

5.3 Integral Formulae To Price European vanilla Call and Put Options Under HW Interest Rate Model

In the framework of the model (5.10), (5.11), (5.12), this section derived the integral formulas to price European vanilla call and put options with strike price $K > 0$ and maturity time T. The prices of European vanilla call and put options are derived using the no arbitrage pricing theory. The option price is computed as the expected value of a discounted payoff with respect to an equivalent martingale measure known as a risk-neutral measure (see, for example, Duffie, 2001; Schoutens, 2003) given by (5.10), (5.11), and (5.12), with the initial conditions (5.7), (5.8), and (5.9). Let $p_f(x, v, t, x', v', t')$, $(x, v, t), (x', v', t') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t,\ t' \geq 0$, $\tau = t' - t > 0$ be the transition probability density function given in (7.121), associated with the stochastic differential equations (5.18),(5.11), and (5.12). The prices of European vanilla call and put option can be computed with strike price K and maturity time T as the expected value of the discounted payoff, that is:

$$C(S_t,\tau,E,r_t,v_t) = E^Q\left(\frac{(S_t e^{x'}-E)_+}{e^{\int_0^T r_t\,dt}}\right),\ \ S_t,\ \tau,\ E\ ,\ v_t > 0, \qquad [5.34]$$

$$P(S_t,\tau,E,r_t,v_t) = E^Q\left(\frac{(E-S_t e^{x'})_+}{e^{\int_0^T r_t\,dt}}\right),\ \ S_t,\ \tau,\ E,\ v_t, > 0, \qquad [5.35]$$

where $(\,\cdot\,)_+ = max\{\,\cdot\,,\,0\}$, and the expectation is taken under the risk-neutral measure Q. In the numerical experiments, the option price is only used to calibrate the model, that is only the risk-neutral measure is used and not the physical one. As a consequence, it is not necessary introduce the risk premium parameters.

$$\begin{aligned}
&C_A(S_t,\tau,E,r,v)\\
&= \int_{-\infty}^{+\infty} dr' \int_{-\infty}^{+\infty} dx' \int_0^{+\infty} dv' \frac{\mathbf{Payoff}(x',\tau)}{\mathbb{B}(r',\tau)} \cdot p_f(\tau,r,x,v,r',x',v'),\\
&= \int_{-\infty}^{+\infty} dr' \int_{-\infty}^{+\infty} dx' \int_0^{+\infty} dv' \left\{\frac{(S_t e^{x'}-E)_+}{e^{\int_t^{t'} r(t)dt}}\right\} \cdot p_f(t'-t,r,x,v,r',x',v'),\\
&\qquad\qquad \tau > 0, S_t,\ v, > 0, \qquad [5.36]
\end{aligned}$$

Moreover in formula (5.36), there is a three dimensional integral whose explicit computation is important to make (5.36) for practical use. Indeed, the integral appearing in (5.36) cannot be computed explicitly; however, it is possible to reduce the computation of the option price from the numerical evaluation of the three dimensional integral in (5.36) to the numerical evaluation of a one-dimensional integral. Recalling that, since $\tau = t'-t > 0$, the variables (x',v',r',t') are those in the "future" and the variables (x,v,r,t) are those in the"past". Inspired by Trapezoidal Rule, a better approximate the stochastic integral of discounting rate $e^{-\int_t^{t'} r(t)dt}$ expresses as follows.

$$e^{-\int_t^{t'} r(t)dt} \doteq e^{-r\,(1-\omega)(t'-t)} \cdot e^{-r'\,\omega(t'-t)} \qquad [5.37]$$

where $\omega = \dfrac{1-\Psi_1/(t'-t)}{\lambda\,\Psi_1}$ based on our Zero Coupon bond approximation. For further details, please see Section 4.4 Bond Price. In addition, it is worth to remark that in Trapeziodal Rule, $\omega = 1/2$. Here $\tau = T$ is the time to maturity at time $t = 0$. S_0, v_0, v' are chosen, and r_0 have been defined in the introduction. Thus Eq.(5.36) becomes

$$
\begin{aligned}
&C_A(S_0,T,E,r_0,v_0)\\
=&\int_{-\infty}^{+\infty} dr' \int_{-\infty}^{+\infty} dx' \int_{0}^{+\infty} dv' (S_0 e^{x'} - E)_+ \left\{ e^{-\int_0^T r(t)dt} \right\} p_f(T,r_0,0,v_0,r',x',v'),\\
=&e^{-r_0\,(1-\omega)\,T} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dx' (S_0 e^{x'} - \frac{E}{S_0})_+ \int_{-\infty}^{+\infty} dr' e^{-r'\,\omega\,T} \cdot\\
&D_{v,q}(0,v_0,r_0,0,x',r',T), \qquad T>0, S_0, E,\ v_0>0,\ q>1, \qquad [5.38]
\end{aligned}
$$

Moreover, let us re-order the components in Eq.(5.38) as follows:

$$
\begin{aligned}
&C_A(S_0,T,E,r_0,v_0)\\
=&\ e^{-r_0\,(1-\omega)\,T} S_0 \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dx' (e^{x'} - \frac{E}{S_0})_+ \int_{-\infty}^{+\infty} dr' e^{-r'\,\omega\,T} \cdot\\
&D_{v,q}(0,v_0,r_0,0,x',r',T),\\
=&e^{-r_0\,(T-Q(T))} \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} dk\, PT1(S_0,E,k,q)\, W^0_{v,q}(T,v_0,k;\underline{\Theta}_v) \int_{-\infty}^{+\infty} dr' e^{-r'\,Q(T)},\\
&L_{r,q}(\tau,r_0,r',k;\underline{\Theta}_r), \qquad T>0, S_0, E,\ v_0>0,\ q=2, \qquad [5.39]
\end{aligned}
$$

where $Q(T) := \omega\, T$, and

$$
\begin{aligned}
PT1(S_0,E,k,q) =& \int_{-\infty}^{+\infty} dx'\, e^{(-q+\imath k)x'} \left(e^{x'} - \frac{E}{S_0} \right)_+ \cdot\\
=& \int_{\ln\left(\frac{E}{S_0}\right)}^{+\infty} dx'\, e^{(-q+\imath k)x'} \left(e^{x'} - \frac{E}{S_0} \right)\\
=& \frac{\left(\frac{S_0}{E}\right)^{q-1-\imath k}}{-k^2 - (2q-1)\imath k + q(q-1)} \qquad [5.40]
\end{aligned}
$$

Without loss of generality, let us choose $q = 2$, then we obtain:

$$
\begin{aligned}
PT1(S_0,E,k,2) =& \int_{-\infty}^{+\infty} dx'\, e^{(-2+\imath k)x'} \left(e^{x'} - \frac{E}{S_0} \right)_+ \cdot\\
=& \int_{\ln\left(\frac{E}{S_0}\right)}^{+\infty} dx'\, e^{(-2+\imath k)x'} \left(e^{x'} - \frac{E}{S_0} \right)\\
=& \frac{\left(\frac{S_0}{E}\right)^{1-\imath k}}{-k^2 - 3\imath k + 2} \qquad [5.41]
\end{aligned}
$$

Then let us focus on the last integral of Eq.(5.39) as follows:

$$\int_{-\infty}^{+\infty} dr' e^{-r'\,Q(T)}\, L_{r,q}(T, r_0, r', k; \underline{\Theta}_r)$$

$$= \int_{-\infty}^{+\infty} dr'\, e^{Q_0(T,k,r_0)} \frac{1}{2\sqrt{\pi\, Q_2(T)}}\, e^{-r'\,Q(T) - \frac{1}{2}\left(\frac{r' - \tilde{Q}_1(T,k,r)}{\sqrt{2Q_2(T)}}\right)^2}$$

$$= W^0_{r,q}(T, r_0, k; \underline{\Theta}_r)\, e^{Q^2 Q_2 - Q\tilde{Q}_1}\, \frac{1}{2\sqrt{\pi\, Q_2}} \int_{-\infty}^{\infty} dr'\, e^{-\frac{1}{2}\left(\frac{r' + (2Q\,Q_2 - \tilde{Q}_1)}{\sqrt{2Q_2}}\right)^2} \qquad [5.42]$$

Obviously, the integral part in Eq.(5.42) is Gaussian Integral. Thus, we have

$$\frac{1}{2\sqrt{\pi\, Q_2}} \int_{-\infty}^{\infty} dr'\, e^{-\frac{1}{2}\left(\frac{r' + (2Q\,Q_2 - \tilde{Q}_1)}{\sqrt{2Q_2}}\right)^2} = 1 \qquad [5.43]$$

Substituting Eqs.(5.43), (5.42) and (5.41) into Eq.(5.39), we will obtain

$$\begin{aligned} C_A(S_0, T, E, r_0, v_0) =& e^{-r_0\,(T - Q(T))} \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)}}{-k^2 - 3\imath k + 2} \cdot \\ W^0_{v,q}(T, v_0, k; \underline{\Theta}_v) \cdot W^0_{r,q}&(T, r_0, k; \underline{\Theta}_r) \cdot\ e^{Q^2(T) Q_2(T) - Q(T)\tilde{Q}_1(T,k,r_0)} \\ & T > 0, S_0,\ v_0 > 0,\ q = 2 \end{aligned} \qquad [5.44]$$

where

$$W^0_{r,q}(T, r_0, k; \underline{\Theta}_r) = e^{Q_0(T,k,r_0)} \qquad [5.45]$$

and

$$Q(T) = \omega\, T = \frac{T - \Psi_1}{\lambda \Psi_1} = \frac{-1 + (1/\Psi_1)\; T}{\lambda} \qquad [5.46]$$

$$T - Q(T) = (1 - \omega)\, T = \frac{1 + (\lambda - 1/\Psi_1)\; T}{\lambda} \qquad [5.47]$$

$$[5.48]$$

$Q_2(T)$, and $\tilde{Q}_1(T,k,r)$ are defined in Eqs.(7.99) and (7.98). Moreover, Formula(5.44) can be written in the following brief form:

$$C_A(S_0,T,E,r_0,v_0) = e^{-r_0\,(T-Q(T))}\frac{S_0}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)}}{-k^2-3\imath k+2}\cdot$$
$$W^0_{v,q}(T,v_0,k;\underline{\Theta}_v)\cdot W^0_{r,q}(T,r_0,k;\underline{\Theta}_r)\cdot\ e^{\hat{M}(T,k)+\hat{N}(T,k)\,r_0}$$

$$T>0, S_0,\ v_0>0,\ q=2 \qquad [5.49]$$

where

$$\hat{M}(T,k) = Q^2(T)Q_2(T) - Q(T)\hat{Q}_1(T,k) \qquad [5.50]$$
$$\hat{N}(T) = -Q(T)e^{-\lambda T} \qquad [5.51]$$

Following the same approach, the approximation for European put option $P_A(S_0,T,E,r_0,v_0)$ can be obtained similarly.

$$P_A(S_0,T,E,r_0,v_0)$$
$$= e^{-r_0\,(T-Q(T))}\int_{-\infty}^{\ln(E/S_0)} dx'\, e^{-qx'}(E-S_0e^{x'})\int_0^{+\infty} dr' e^{-r'\,Q(T)}D_v(0,v_0,r_0,0,x',r',T),$$
$$S_0,\ T,\ E\,r_0,\ \ v_0>0,\ \ q<-1, \qquad [5.52]$$

where D_v is given in (5.28). Choosing $q<-1$, we obtain $PT1$ as follows:

$$\int_{-\infty}^{\ln(E/S_0)} dx'\, e^{-q\,x'}(E-S_0e^{x'})e^{\imath\,k\,x'} = \frac{S_0\left(\frac{S_0}{E}\right)^{q-1-\imath\,k}}{-k^2-(2q-1)\imath\,k+q(q-1)}. \qquad [5.53]$$

Without loss of generality, we can set $q=-2$. Then, we have

$$PT1(S_0,E,k,-2) = \frac{\left(\frac{S_0}{E}\right)^{-3-\imath k}}{-k^2+5\imath k+6} \qquad [5.54]$$

Proceeding in a similar way, the following approximation, P_A, of the put option price P is obtained:

$$P_A(S_0,T,E,r_0,v_0) = e^{-r_0\,(T-Q(T))}\frac{S_0}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{(-3+\imath k)}}{-k^2+5\imath k+6}\cdot$$
$$W^0_{v,q}(T,v_0,k;\underline{\Theta}_v)\cdot W^0_{r,q}(T,r_0,k;\underline{\Theta}_r)\cdot\ e^{\hat{M}(T,k)+\hat{N}(T,k)\,r_0}$$

$$S_0,\ T,\ E,\ r_0,\ v_0,\ \ q=-2. \qquad [5.55]$$

Taking the limit $\Omega \to 0^+$, $\lambda \to 0^+$, $\eta \to 0^+0$ in Eqs.(5.49) and (5.55), we can derive the following exact formulas for the price of the European call and put options under the Heston model:

$$
\begin{aligned}
& C_H(S_0, T, E, r_0, v_0) \\
&= e^{-r_0 T} S_0 e^{2r_0 T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)} e^{-\imath k r_0 T}}{-k^2 - 3\imath k + 2} W^0_{v,q}(T, v_0, k; \underline{\Theta}_v) \\
& \qquad S_0,\ T,\ E,\ r_0,\ v_0,\ \ q = 2,
\end{aligned} \qquad [5.56]
$$

$$
\begin{aligned}
& P_H(S_0, T, E, r_0, v_0) \\
&= e^{-r_0 T} S_0 e^{-2 r_0 T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{-(3+\imath k)} e^{-\imath k r_0 T}}{-k^2 + 5\imath k + 6} W^0_{v,q}(T, v_0, k; \underline{\Theta}_v) \\
& \qquad S_0,\ T,\ E,\ r_0,\ v_0,\ \ q = -2.
\end{aligned} \qquad [5.57]
$$

These formulas are used in the simulation study to compare the performance of the Heston model and hybrid HHW model in interpreting real data. It is worthy to note that the integrand functions appearing in formulas (5.49), (5.55), (5.103), (5.108) are smooth functions whose integration does not require a specific care. This regularity is due to the specific approach used to derive them. Next section will introduce the analytical approach to evaluate the bond price using hybrid Heston HW model.

5.4 Analytical Treatment for Pricing Zero Coupon Bond

5.4.1 Pricing Zero Coupon Bond Under Hull-White (HW) Model

Let us consider the following Hull and White interest rate model:

$$
dr_t = \lambda(\theta - r_t)dt + \eta\, dW_t^r \qquad [5.58]
$$

Let us define $B(r,t)$, zero-coupon bond at time t , should satisfy the Kolmogorov Backward equation. Moreover $B(r,t)$ yields 1 at maturity T, i.e. $B(r,T) = 1$.

$$
\frac{\partial B(r,t)}{\partial t} + \frac{\eta^2}{2}\frac{\partial^2 B(r,t)}{\partial r^2} + \lambda(\theta - r)\frac{\partial B(r,t)}{\partial r} - r\, B(r,t) = 0 \qquad [5.59]
$$

with

$$
r \in \mathbb{R}\,, \qquad t \in [0,T]\,, \quad B(r,T) = 1\,. \qquad [5.60]
$$

Now let us define $\tau = T - t$, and thus $B(r,\tau)$ satisfies $B(r,0) = 1$ as well as the following differential equation:

$$\frac{\partial B(r,r)}{\partial t} = -\frac{\partial B(r,\tau)}{\partial \tau} \qquad [5.61]$$

Moreover, substituting $\tau = T - t$ in Eq.(5.59) , we will obtain the following equation:

$$-\frac{\partial B(r,\tau)}{\partial \tau} + \frac{\eta^2}{2}\frac{\partial^2 B(r,\tau)}{\partial r^2} + \lambda(\theta - r)\frac{\partial B(r,\tau)}{\partial r} - r\,B(r,\tau) = 0 \qquad [5.62]$$

Assuming zero coupon takes the following form

$$B(r,\tau) = e^{m(\tau) - n(\tau)\,r} \qquad [5.63]$$

and substituting Eq.(5.63) into Eq.(5.62), we will obtain:

$$-\dot{m}(\tau) + \frac{\eta^2}{2}n^2(\tau) - \lambda\theta\,n(\tau) = 0 \qquad [5.64]$$

$$\dot{n}(\tau) + \lambda\theta\,n(\tau) - 1 = 0 \qquad [5.65]$$

Eq.(5.65) is a first-order difference equation which can be easily solved as follows:

$$n(\tau) = \frac{1 - e^{-\lambda\tau}}{\lambda} := \Psi_1(\lambda,\tau) \qquad [5.66]$$

Substituting Eq.(5.66) into Eq.(5.64), and integrating both sides of the equations with respect to time τ, we will obtain:

$$\begin{aligned} m(\tau) &= \frac{\eta^2}{2}\int_0^\tau n^2(\tau)d\tau \; - \lambda\theta\int_0^\tau n(\tau)d\tau \\ &= \frac{\eta^2}{2\lambda^2}\left[\tau - 2\frac{1 - e^{-\lambda\tau}}{\lambda} + \frac{1 - e^{-2\lambda\tau}}{2\lambda}\right] - \theta\left[\tau - \frac{1 - e^{-\lambda\tau}}{\lambda}\right] \\ &= \frac{\eta^2}{2\lambda^2}\left[\tau - 2\Psi_1(\lambda,\tau) + \Psi_2(\lambda,\tau)\right] - \theta\left[\tau - \Psi_1(\lambda,\tau)\right] \\ &= \frac{\eta^2}{2\lambda^2}\Psi_2(\lambda,\tau) - \frac{\eta^2}{\lambda^2}\Psi_1(\lambda,\tau) + \frac{\eta^2}{2\lambda^2}\tau - \theta\left[\tau - \Psi_1(\lambda,\tau)\right]\,. \end{aligned} \qquad [5.67]$$

Substituting Eqs.(5.66) and (5.67) into Eq.(5.63), we will obtain the Zero Coupon price formula as follows:

$$B(r,\tau) = e^{-r\,\Psi_1(\lambda,\tau)}\,e^{\frac{\eta^2}{2\lambda^2}\Psi_2(\lambda,\tau) - \frac{\eta^2}{\lambda^2}\Psi_1(\lambda,\tau) + \frac{\eta^2}{2\lambda^2}\tau - \theta[\tau - \Psi_1(\lambda,\tau)]} \qquad [5.68]$$

with $B(r,0) = 1$. Especially, setting $t = 0$ (thus $\tau = T$), we will obtain:

$$B(r_0,T) = e^{-r\,\Psi_1(\lambda,T)}\,e^{\frac{\eta^2}{2\lambda^2}\Psi_2(\lambda,T) - \frac{\eta^2}{\lambda^2}\Psi_1(\lambda,T) + \frac{\eta^2}{2\lambda^2}T - \theta[T - \Psi_1(\lambda,T)]} \qquad [5.69]$$

5.4.2 Pricing Zero Coupon Bond Under Hybrid Heston-Hull-White (HHW) Model

From the call option price formula in Eq.(5.44), we can directly obtain Zero Coupon price formula by setting

$$x' = \ln \frac{E}{S^0}\,,\ k = 0\,,\ \text{and}\, q = 0 \qquad [5.70]$$

$$\begin{aligned}
&W^0_{v,q}(T, v_0, 0; \underline{\Theta}_v) = e^0 = 1 && [5.71]\\
&W^0_{r,q}(T, r_0, 0; \underline{\Theta}_r) = e^0 = 1 && [5.72]\\
&Q_0(T, 0, r_0) = 0 && [5.73]\\
&\imath Q_1(T, 0, r_0) = \lambda\theta\Psi_1(T) + e^{-\lambda T}\, r_0 && [5.74]\\
&Q_2(T) = \frac{\eta^2}{2}\Psi_2(T) && [5.75]\\
&\hat{Q}_0(T, 0) = 0 && [5.76]\\
&\hat{Q}_1(T, 0) = \lambda\theta\Psi_1(T)\,. && [5.77]
\end{aligned}$$

Moreover, the general form of $Q(T)$ is defined as follows:

$$Q(T) = \omega\, T \qquad [5.78]$$

Therefore, the Zero Coupon price formula is obtained as follows:

$$\begin{aligned}
B_A(r_0, T) &= e^{-r_0(1-\omega)T}\, e^{\omega^2 T^2 Q_2(T) - \omega T\left[\lambda\theta\Psi_1(T) + e^{-\lambda T} r_0\right]}\\
&= e^{-r_0\left[1-\omega\left(1-e^{-\lambda T}\right)\right]T}\, e^{\frac{\eta^2}{2}(\omega\, T)^2\Psi_2(T) - (\omega\, T)\lambda\theta\Psi_1(T)} && [5.79]
\end{aligned}$$

5.4.3 Comparison Between HW and HHW Model: Choosing Best Approximation Indicator ω

Let us compare Hull-White Zero Coupon Bond Price $B(r_0, T)$ in Eq.(5.69) with the Heston-Hull-White Zero Coupon Bond Price in Eq.(5.79). The task is to determine the parameter ω in order to better approximate the interval of interest rate. Let us firstly look at the conclusion, and then the proof. Through comparison between HW and HHW models, ω satisfies the following equation:

$$\Psi_1(T) = \left[1 - \omega\left(1 - e^{-\lambda T}\right)\right] T \qquad [5.80]$$

Thus,

$$\omega = \frac{1 - \Psi_1/T}{\lambda\,\Psi_1}\,, \qquad T > 0 \qquad [5.81]$$

Moreover, $\omega = 0$ (when $T = 0$) is also a solution of Eq.(5.80). Now let us prove that ω is a probability parameter.

Lemma 5.4.1. *When ω is defined in Eq.(5.81), i.e. $\omega = \dfrac{1-\Psi_1/T}{\lambda\,\Psi_1}$, $T > 0$ with $\Psi_1(T) = \dfrac{1-e^{-\lambda T}}{\lambda}$, or $\omega = 0$, $T = 0$, then ω is bounded between 0 and 1 i.e.*

$$0 \le \omega \le 1\,. \qquad [5.82]$$

Proof. (i) '$\omega = 0$'. Clearly when $T = 0$, then $\omega = 0$ satisfies Eq.(5.80).
(ii) '$\omega > 0$'. For $T > 0$, denominator of Eq.(5.81) (i.e. $\lambda\Psi_1$), is clearly greater than zero. Then let us focus on numerator as follows:

$$\begin{aligned} 1-\Psi_1/T \quad &= 1-\frac{1-e^{-\lambda T}}{\lambda T} \\ &= \frac{e^{-\lambda T}}{\lambda T}\left[(\lambda T-1)e^{\lambda T}+1\right] \\ &:= \frac{e^{-\varrho}}{\varrho}\,\phi(\varrho)\,, \qquad [5.83] \\ \text{with} \quad \varrho = \lambda T \ge 0\,, &\quad \phi(\varrho) = (\varrho-1)e^{\varrho}+1 \qquad [5.84] \end{aligned}$$

Focusing on Eq(5.84), we have

$$\frac{\partial\phi(\varrho)}{\partial\varrho} = \varrho\,e^{\varrho} \ge 0 \qquad [5.85]$$

Thus

$$\min_{\varrho\ge 0}\phi(\varrho) = \phi(\varrho=0) = 0 \qquad [5.86]$$

$$\text{and} \quad \phi(\varrho) \ge \phi(0) = 0 \qquad [5.87]$$

Back to Eq.(5.83), and since $\dfrac{e^{-\varrho}}{\varrho} \ge 0$, we can see $1-\Psi_1/T \ge 0$. Therefore, both denominator and numerator are strictly larger than zero, thus we have $\omega > 0$.
(iii) '$\omega \le 1$'. Form (ii), we know both Ψ_1/T and $e^{-\lambda T}$ are strictly larger than zero, thus

$$\begin{aligned} \omega \le 1 \qquad &\Leftrightarrow \quad \Psi_1/T \ge\; e^{-\lambda T} \\ &\Leftrightarrow \quad \frac{1-e^{-\lambda T}}{\lambda T} \ge e^{-\lambda T} \\ &\Leftrightarrow \quad \frac{e^{\lambda T}-1}{\lambda T} \ge 1 \\ &\Leftrightarrow \quad e^{\lambda T} \ge 1+\lambda T \end{aligned}$$

This clearly holds from the Taylor expansion in Eq.(5.88), i.e.

$$\begin{aligned} e^{\lambda T} &= 1 + \lambda T + \frac{\lambda^2}{2} T^2 + \cdots \\ &> 1 + \lambda T \end{aligned} \qquad [5.88]$$

Combining (i),(ii),(iii), the proof of $\omega \in [0, 1]$ is done. □

From Eq.(5.81), we obtain

$$\omega T = \frac{T - \Psi_1}{\lambda \Psi_1} \qquad [5.89]$$

Substituting Eq.(5.89) into approximate Bond price formula (5.79), we obtain

$$\begin{aligned} B_A(r_0, T) &= e^{-r_0 \Psi_1} e^{\frac{\eta^2}{2\lambda^2}\left(1 - \frac{T}{\Psi_1}\right)^2 \Psi_2 - \theta(T - \Psi_1)} \\ &= e^{-r_0 \Psi_1} e^{\frac{\eta^2}{2\lambda^2}\Psi_2 - \frac{\eta^2}{\lambda^2}\frac{T}{\Psi_1}\Psi_2 + \frac{\eta^2}{2\lambda^2}\left(\frac{T}{\Psi_1}\right)^2 \Psi_2 - \theta(T - \Psi_1)} \end{aligned} \qquad [5.90]$$

Through comparison, it is easy to observe that Eq.(5.90) and Eq.(5.69) are exactly the same if the following two equations hold.

$$\Psi_1 = \frac{T}{\Psi_1} \Psi_2 \qquad [5.91]$$

$$T = \left(\frac{T}{\Psi_1}\right)^2 \Psi_2 \qquad [5.92]$$

Actually, Eq.(5.91) and (5.92) are identical. There, the next step is to prove the left-hand side and right-hand side of Eq.(5.91) are approximately equal.

Lemma 5.4.2. *For short term and middle term maturity, i.e. for small $T \geq 0$.*

$$\Psi_1 \doteq \frac{T}{\Psi_1} \Psi_2 \qquad [5.93]$$

Proof. Firstly

$$\begin{aligned} \Psi_2 &= \frac{1 - e^{-2\lambda T}}{2\lambda} \\ &= \frac{(1 - e^{-\lambda T})(1 + e^{-\lambda T})}{2\lambda} \\ &= \Psi_1 \frac{1 + e^{-\lambda T}}{2} \end{aligned} \qquad [5.94]$$

Thus, substituting Eq.(5.94) into (5.93), we can obtain

$$
\begin{aligned}
\Psi_1 \doteq \frac{T}{\Psi_1}\Psi_2 \qquad & \Leftrightarrow \quad \Psi_1 \doteq T\,\frac{1+e^{-\lambda T}}{2} \\
& \Leftrightarrow \quad \Psi_1 \doteq T\left(1-\frac{\lambda}{2}\Psi_1\right) \\
& \Leftrightarrow \quad \underbrace{\left(1+\frac{\lambda}{2}T\right)}_{\doteq e^{\frac{\lambda}{2}T}}\Psi_1 \doteq T \\
& \Leftrightarrow \quad \frac{e^{\frac{\lambda}{2}T}-e^{-\frac{\lambda}{2}T}}{2} \doteq T
\end{aligned}
\qquad [5.95]
$$

Referring to the Taylor expansion which is similar to Eq.(5.88), we can obtain the following approximation:

$$
e^{\pm\frac{\lambda}{2}T} \doteq 1 \pm \frac{\lambda}{2}T \qquad [5.96]
$$

Therefore, substituting Eq.(5.96) into (5.95), the approximation in (5.95) clearly satisfies. □

In summary, based on **Lemma 4.4.1** and **Lemma 4.4.2**, HHW Zero Coupon Bond $B_A(r_0, T)$ is indeed a perfect approximation of HW Zero Coupon model $B(r_0, T)$ when choosing $\omega = \dfrac{1-\Psi_1/T}{\lambda\,\Psi_1}$ (for short term maturity $T > 0$).

5.5 An Empirical Analysis

In this section, an empirical analysis is proposed for the Eurodollar futures prices, the U.S. S&P 500 index and the corresponding European options prices. Specifically, this section conducts an empirical analysis calibrating the model parameters against these real data. The three month U.S. government bond index has been used as proxy of the initial stochastic interest rate.

In the numerical experience illustrated in this section, the option prices have been computed evaluating with numerical quadratures the integrals contained in formulas (5.49), and (5.55) which are compared with the option prices actually observed. The numerical quadratures are performed using the composite midpoint quadrature rule with 2^{14} nodes. These choices guarantee approximately six significant digits correct in the option prices.

In this section, $C^{\tilde{t}}(\tilde{S}_{\tilde{t}}, T, E)$, and $C^{\tilde{t},\underline{\Theta}}(\tilde{S}_{\tilde{t}}, T, E)$ are denoted as the observed price and the model price in Eq.(5.49) at time $t = \tilde{t}$ of the European call option having maturity time T and strike price E. Similarly, $P^{\tilde{t}}(\tilde{S}_{\tilde{t}}, T, E)$ and $P^{\tilde{t},\underline{\Theta}}(\tilde{S}_{\tilde{t}}, T, E)$ are denoted as the observed price and the model price in Eq.(5.55) at time $t = \tilde{t}$ of the European put option having maturity time T and strike prices E.

The model parameters are estimated by solving an appropriate nonlinear constrained least squares problem. Going into the details, let $\mathbb{R}^{12}$ denote the 12-dimensional Euclidean real space, the set of the constraints, $\mathcal{V}$ is defined as follows:

$$\mathcal{V}= \Big\{\underline{\Theta}= (\Delta, \gamma, v^*, \chi, \rho_{p,v}, v_0, \eta, \lambda, \theta, \rho_{p,r}, r_0, \Omega) \in \mathbb{R}^{12} \,|\, \Delta, \gamma, v^*, \chi, v_0, \eta, \lambda, \theta > 0, \\ \frac{2\chi v^*}{\gamma^2} > 1, \quad -1 < \rho_{p,v}, \rho_{p,r} \; < 1\Big\}. \qquad [5.97]$$

It is worth noting that the initial values v_0, r_0 are considered parameters to be estimated via the calibration procedure. This is motivated by the fact that v_0 and r_0 are latent variables not really observable in the market. In fact, both v_0 and r_0 refer to a risk neutral measure and, subsequently, their values have not been yet clearly identified by the financial market. The estimation of these parameters is not new, in fact it can be found, for example, in Bühler 2002, Fatone et al. 2009, 2013, Grzelak and Oosterlee 2011, and Chapter 2 of this thesis.

Let us now formulate the calibration problem. To this end, some notations are introduced here. Let n_D and n_T be two nonnegative integers, we denote with $C^{HB}_{\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)$, $P^{HB}_{\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)$ and $C^o(\tilde{S}_{\tilde{t}_j}, T_i, E_i)$, $P^o(\tilde{S}_{\tilde{t}_j}, T_i, E_i)$,$i = 1, 2, \ldots, n_D$, the prices of the European call and put option in the hybrid Hull and White Heston model and the observed option prices, at time $t = \tilde{t}_j$, $j = 1, 2, \ldots, n_T$. The notations here $\Sigma^{o,C}(\tilde{S}_{\tilde{t}_j}, r, T_i, E_i)$ and $\Sigma^C_{\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)$, $\Sigma^{o,P}(\tilde{S}_{\tilde{t}_j}, r, T_i, E_i)$, $\Sigma^P_{\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)$, $j = 1, 2, \ldots, n_T$, $i = 1, 2, \ldots, n_D$, are, respectively, the observed and the theoretical implied volatilities associated to the call and put option prices. For the sake of the clarity, the implied volatility $\Sigma^{o,C}(\tilde{S}, r, T, E)$ is defined as the quantity such that the following equality holds:

$$C_{BS}(T - \tilde{t}, E, \tilde{S}_{\tilde{t}}, \Sigma^{o,C}) = C^o(T, E, \tilde{S}_{\tilde{t}}), \;\; , \qquad [5.98]$$

where C^o is the observed call option price while C_{BS} is the Black Scholes price at time $t = \tilde{t}$ of the European call option with strike price $E > 0$, and maturity time $T > 0$, that is:

$$C_{BS}(T-\tilde{t}, E, \tilde{S}_{\tilde{t}}, \Sigma^{0,C}) = \tilde{S}_{\tilde{t}}\, N(d_1) - E\, e^{-r_F\, T-\tilde{t}} N(d_2), \tag{5.99}$$

where $N(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-y^2/2} dy$, and d_1, d_2 are:

$$\begin{aligned} d_1 &= \frac{\ln(\tilde{S}_0/E) - (r + \frac{1}{2}\Sigma^2)\, T}{\Sigma\, \sqrt{T}}\,, \\ d_2 &= d_1 - \frac{1}{2}\Sigma\, \sqrt{T},\ T > 0, E > 0, \tilde{S}_0 > 0,\ \Sigma > 0, \end{aligned} \tag{5.100}$$

and r_F is the risk free interest rate that are choose to be the U.S. three month government bond yield.

The model parameters are estimated using the implied volatilities motivated by the following two facts. First, implied volatilities play a crucial role in hedging portfolios, so that a model may be considered efficient/performing whether it is able to fit and forecast them. Second, the minimization of the implied volatility allows avoiding biased approximations caused by price very different in magnitude (few dollars against one hundred dollars).

Following this approach, the model parameters are estimated by solving the following nonlinear constrained optimization problem:

$$\min_{\underline{\Theta} \in \mathcal{V}} F(\underline{\Theta}). \tag{5.101}$$

where the objective function, F_{n_T}, is as follows:

$$\begin{aligned} F(\underline{\Theta}) = &\frac{1}{n_C\, n_T} \sum_{j=1}^{n_T} \sum_{i=1}^{n_C} \left[\frac{\Sigma^{o,C}(\tilde{S}_{\tilde{t}_j}, r, T_{C,i}, E_{C,i}) - \Sigma^{C}_{\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_{C,i}, E_{C,i})}{\Sigma^{o,C}(\tilde{S}_{\tilde{t}_j}, r, T_{C,i}, E_{C,i})} \right]^2 \\ &+ \frac{1}{n_P\, n_T} \sum_{j=1}^{n_T} \sum_{i=1}^{n_P} \left[\frac{\Sigma^{o,P}(\tilde{S}_{\tilde{t}_j}, r, T_{P,i}, E_{P,i}) - \Sigma^{P}_{\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_{P,i}, E_{P_i})}{\Sigma^{o,P}(\tilde{S}_{\tilde{t}_j}, r, T_{P,i}, E_{P,i})} \right]^2, \end{aligned} \tag{5.102}$$

In order to show empirical evidence that the use of stochastic interest rates is crucial, the proposed model is compared with the Heston model (Heston 1993). The latter is calibrated by solving problem (5.101) in a feasible set which does not contain the parameters of the stochastic interest rate model except for the parameter r_0 which is allowed for negative values. This is

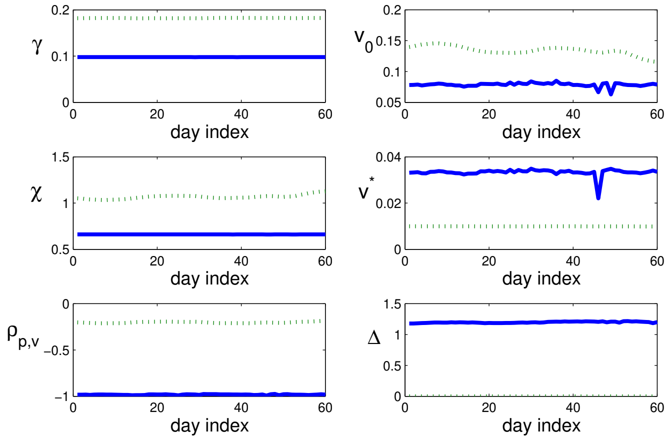

Figure 5.1 – Estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$ and Δ versus window index (six-day window) resulting from the calibration of the hybrid Heston model (solid line) and the Heston model (dotted line)

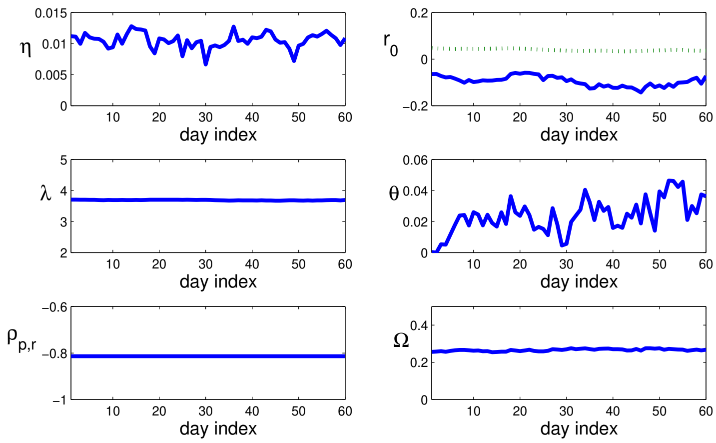

Figure 5.2 – Estimated parameters η, r_0, λ, θ, $\rho_{p,r}$ and Ω versus window index (six-day window) resulting from the calibration of the hybrid Heston model (solid line). Note that the dotted line in the upper right panel shows the estimated values of the parameter r_0 resulting from the calibration of the Heston model.

motivated by the fact of stretching the limit of negative interest rate. The following formulas are used to price European call and put options in the

Heston model (see, Fatone et al. 2009)

$$C_H(\tau, E, \tilde{S}_0, \tilde{v}_0) = \frac{\tilde{S}_0^2}{E}\frac{1}{2\pi}e^{r\tau}\int_{-\infty}^{+\infty} dk \frac{e^{-\imath\, k(\ln(\tilde{S}_0/E)+r\tau)}}{-k^2-3\imath\, k+2}\cdot$$
$$\left(e^{-2\chi\theta((\nu^c+\zeta^c)\tau+\ln(s_b^c/(2\zeta^c)))/\varepsilon^2}e^{-2\tilde{v}_0((\zeta^c)^2-(\nu^c)^2)s_g^c/(\varepsilon^2 s_b^c)}\right),$$
$$\tau = T,\ E,\ \tilde{S}_0,\ \tilde{v}_0 > 0, \qquad [5.103]$$

where the quantities ν^c, ζ^c, s_g^c, s_b^c are given by:

$$\nu^c = -\frac{1}{2}\left(\chi + \imath\, k\, \varepsilon\rho - 2\rho\varepsilon\right),\ k \in \mathbb{R}, \qquad [5.104]$$

$$\zeta^c = \frac{1}{2}\left(4(\nu^c)^2 + \varepsilon^2(k^2 + 3\imath\, k - 2)\right)^{1/2},\ k \in \mathbb{R}, \qquad [5.105]$$

$$s_g^c = 1 - e^{-2\zeta^c\tau},\ \ k \in \mathbb{R},\ \tau = T > 0, \qquad [5.106]$$

$$s_b^c = \zeta^c - \nu^c + (\zeta^c + \nu^c)e^{-2\zeta^c\tau},\ \ k \in \mathbb{R},\ \tau = T > 0, \qquad [5.107]$$

and that:

$$P_H(\tau, E, \tilde{S}_0, \tilde{v}_0) = \frac{E^2}{\tilde{S}_0}\frac{1}{2\pi}e^{-2r\tau}\int_{-\infty}^{+\infty} dk \frac{e^{-\imath k(\ln(\tilde{S}_0/E)+r\tau)}}{-k^2+3\imath\, k+2}$$
$$\left(e^{-2\chi\theta((\nu^p+\zeta^p)\tau+\ln(s_b^p/(2\zeta^p)))/\varepsilon^2}e^{-2\tilde{v}_0((\zeta^p)^2-(\nu^p)^2)s_g^p/(\varepsilon^2 s_b^p)}\right),$$
$$\tau = T, E, \tilde{S}_0,\ \tilde{v}_0 > 0, \qquad [5.108]$$

where the quantities ν^p, ζ^p, s_g^p, s_b^p are given by:

$$\nu^p = -\frac{1}{2}\left(\chi + \imath\, k\, \varepsilon\rho + \rho\varepsilon\right),\ k \in \mathbb{R}, \qquad [5.109]$$

$$\zeta^p = \frac{1}{2}\left(4(\nu^p)^2 + \varepsilon^2(k^2 - 3\imath\, k - 2)\right)^{1/2},\ k \in \mathbb{R}, \qquad [5.110]$$

$$s_g^p = 1 - e^{-2\zeta^p\tau},\quad k \in \mathbb{R},\quad \tau = T > 0, \qquad [5.111]$$

$$s_b^p = \zeta^p - \nu^p + (\zeta^p + \nu^p)e^{-2\zeta^p\tau},\quad k \in \mathbb{R}, \tau = T > 0, . \qquad [5.112]$$

The formulas (5.49), (5.55) and (5.103), (5.108) are used to evaluate option prices. The one-dimensional integrals appearing in these formulas are computed using the midpoint quadrature rule with 2^{14} nodes. This quadrature rule gives satisfactory approximations since the integrand functions appearing in Eqs. (5.49), (5.55), (5.103), (5.108) are smooth functions whose numerical integration does not require special care. Moreover, problem (5.101) is solved using a steepest descent algorithm with variable metric (see, for example, Recchioni and Scoccia 2002, and Chapter 2).

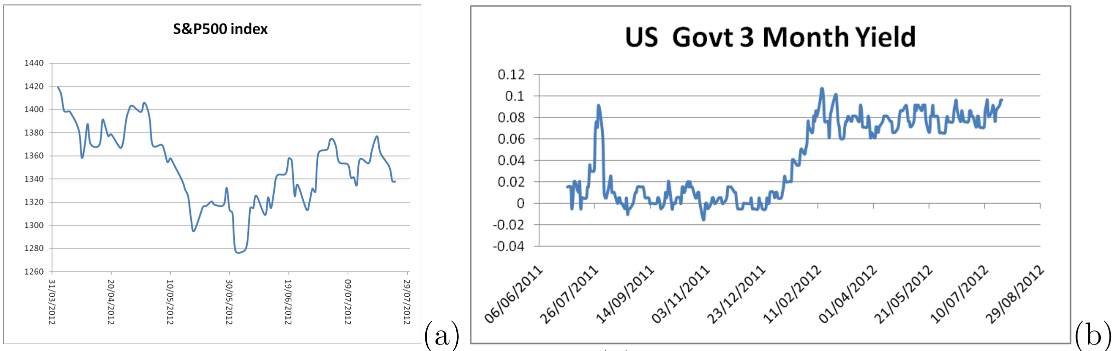

Figure 5.3 – The U.S.A. S&P S500 index (a) and U.S. three-month government yield (b) versus time.

5.5.1 U.S. S&P 500 index options

The empirical analysis presented in this subsection concerns the daily closing values of the U.S. S&P 500 index and the daily closing prices of the European call and put options on this index. The expiry date of these options is March 16th, 2013 and their strike prices are $E_i = 1075 + 25(i-1)$, $i = 1, 2, \dots, 4$, $E_5 = 1170$.

Figure 5.3 (a) shows the U.S. S&P 500 index while Figures 5.4 (a) and 5.4 (b) show the corresponding call and put option prices as a function of time (April 2nd, 2012, July 27th, 2012). Figure 5.3 (b) shows the U.S. three month government yields (in percent) as a function of time. These short-term bond yields are used as values of the risk free interest rate. The option data is analysed by using a rolling window of six consecutive trading day data (i.e. $n_T = 6$). This window is moved by one day along the historical series. The time window covers the period April 2nd to July 2nd, 2012 and the calibration problems (5.101) solved are $66 - n_T$. As a consequence, in each window sixty option values are used to calibrate the twelve parameters of the model (i.e. $n_D = 5$ put option price and $n_D = 5$ call option prices for $n_T = 6$ days) so that we get an historical series of daily observation for each parameter. The values of the parameters obtained in the j-th window are representative of the last day of the j-th window.

We underline that when the values of the estimated parameters are constant in time, the model is able to reproduce the asset price dynamics in the analyzed period by using only one set of model parameters. Figures 5.5 and 5.6 show the in-sample values of the European call and put option prices obtained using the Heston (dashed line) and the hybrid Heston (dotted line) models with the parameters estimated in the period April 2nd, 2012 to July

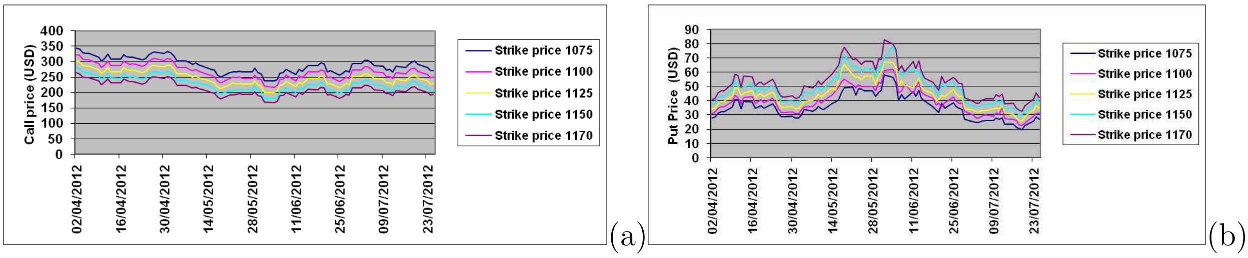

Figure 5.4 – Prices of the call (a) and put (b) options on the U.S. S&P 500 index with strike prices $E_i = 1075 + 25(i-1)$, $i = 1, 2, \ldots, 4$ and $E_5 = 1170$, and with expiry date $T=$ March 16th, 2013 versus time.

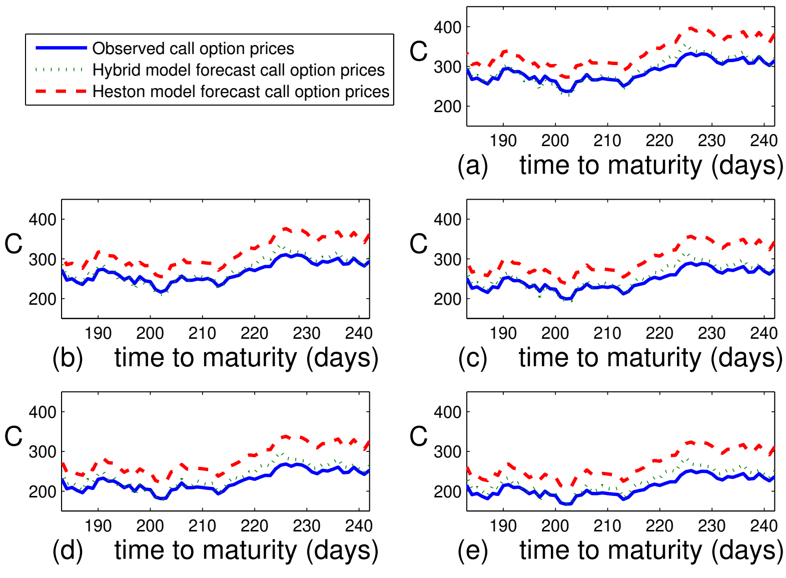

Figure 5.5 – Observed (solid line) and in-sample call option prices (in USD) obtained using the hybrid HW-Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

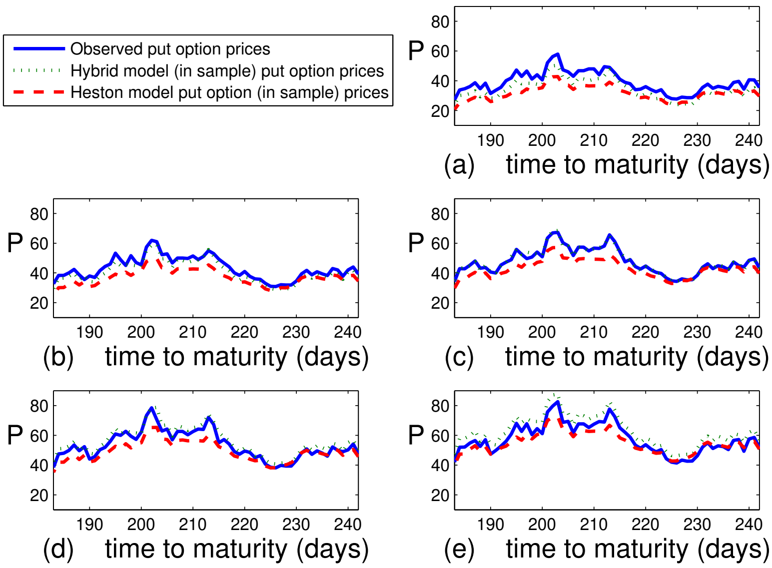

Figure 5.6 – Observed (solid line) and in-sample put option price (in USD) obtained using the hybrid HW-Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

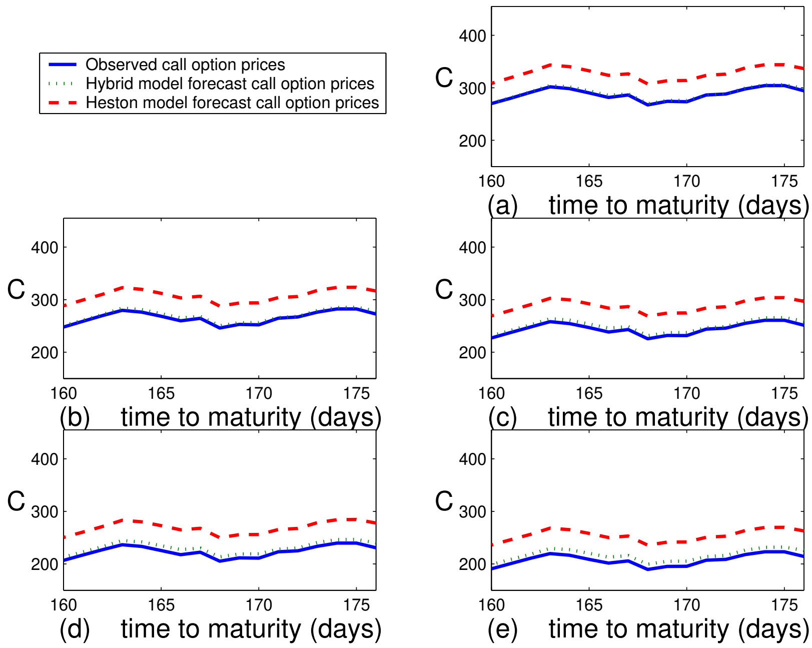

Figure 5.7 – Observed (solid line) and out-of-sample call option price forecast (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

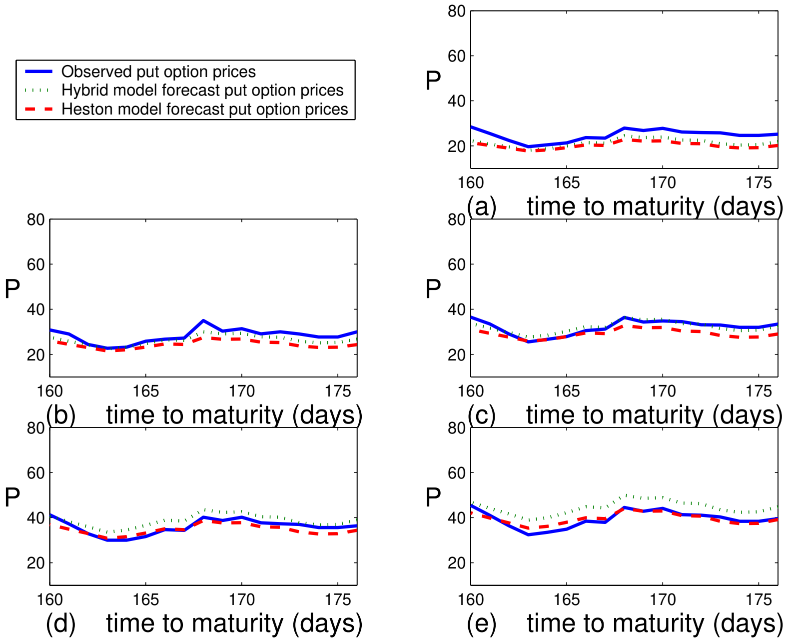

Figure 5.8 – Observed (solid line) and out-of-sample put option price forecast (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$*, (b)* $E_2 = 1100$*, (c)* $E_3 = 1125$*, (d)* $E_4 = 1150$*, (e)* $E_5 = 1170$*) versus time to maturity expressed in days.*

2nd, 2012.

These figures show that the theoretical option prices of the hybrid HHW model provide satisfactory approximations of observed put prices for all values of the strike prices and time to maturity. These values outperform those obtained with the Heston model. The results obtained using the hybrid HHW model outperform the Heston model. In fact, the sample mean of the relative errors of the call and put options are 4.1% and 6.7% for the hybrid HHW model while 21.2% and 11.2% for the Heston model.

Figures 5.1, 5.2, 5.5 and 5.6 show that the hybrid Heston model is capable of matching with sufficient accuracy both call and put option prices for several strike prices and expiry dates using only one set of parameters. This good performance is achieved from the use of a stochastic interest rate which is allowed for negative values. We use the value of the model parameters estimated in the last window, June 25th, 2012 - July 2nd, 2012, to evaluate the out-of-sample European call and put option prices. The out-of-sample period is July 3rd to July 27th, 2012. The time to maturity for this period is 176 to 160 days. We measure the performance of the stochastic model proposed and its parameter estimation procedure with an "a posteriori" validation. That is, we compare the observed out-of-sample option prices with those obtained using formulas (5.103), (5.108), (5.49), (5.55) which use estimated parameters and observed spot prices.

Figures 5.1 and 5.2 show the parameter values as a function of the index j, $j = 1, 2, \ldots, 66 - n_T$. We can observe that these values are relatively constant as a function of time.

Figures 5.7 and 5.8 show the out-of-sample option prices for the hybrid model (dotted line) and for the Heston model (dashed line). The out-of-sample put option prices of the Heston model are very accurate while the call option prices are not. The hybrid Heston model provides accurate approximations of put option prices and outperforms the Heston model in approximating the call options. In fact, the sample mean of the relative errors on the put and call options obtained using the hybrid Heston model are 2.3% and 9.1% and using the Heston model are 9.6% and 17.9%.

In conclusion, the empirical analysis shows that the hybrid model interprets satisfactorily the real data considered in the period April 2nd to July 27th 2012 using only one set of model parameters. Moreover, the values of the initial stochastic rate, r_0, could be considered a proxy of the short-dated government bond yield. The results illustrated here are consistent with those obtained using the multi-factor stochastic volatility model of Christoffersen et al. 2009. Indeed, in the latter the authors use two stochastic factors but they are not specified. The results shown in this subsection suggest that the stochastic interest rate is one of these volatility factors.

5.5.2 FX options

In the second experiment, this section considers the daily values of the futures price on the EUR/USD currency's exchange rate having maturity September 16th, 2011 (the third Friday of September 2011), and the daily prices of the corresponding European call and put options with expiry date September 9th, 2011 and strike prices $E_i = 1.375 + 0.005 * (i - 1)$, $i = 1, 2, \ldots, 18$. The strike prices E_i, $i = 1, 2, \ldots, 18$, are expressed in USD. These prices are observed in the time period that goes from September 27th, 2010, to July 19th, 2011. The observations are made daily and the prices considered are the closing prices of the day. Recall that a year is made of about 250-260 trading days and a month is made of about 21 trading days. Figure 5.9

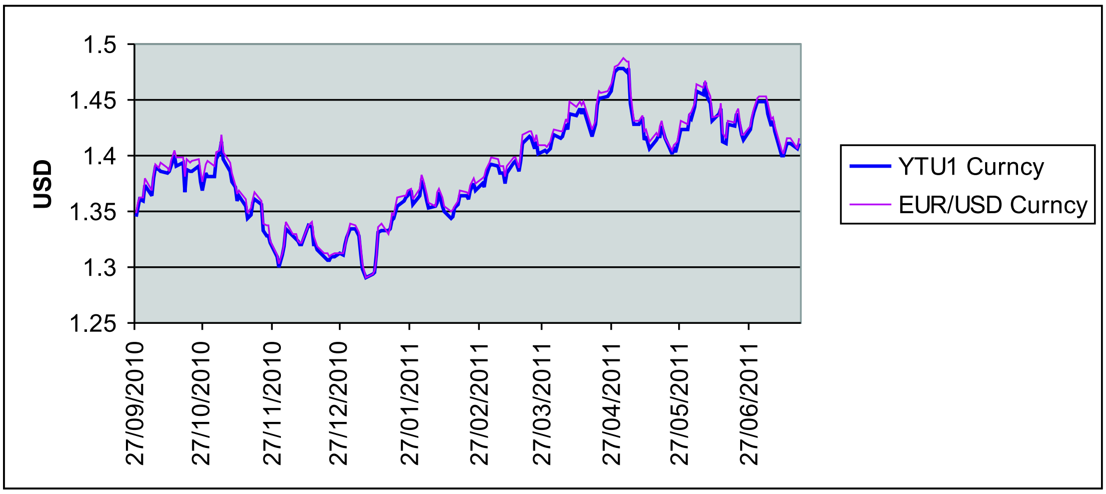

Figure 5.9 – YTU1 (blue line) and EUR/USD currency's exchange rate (pink line) versus time.

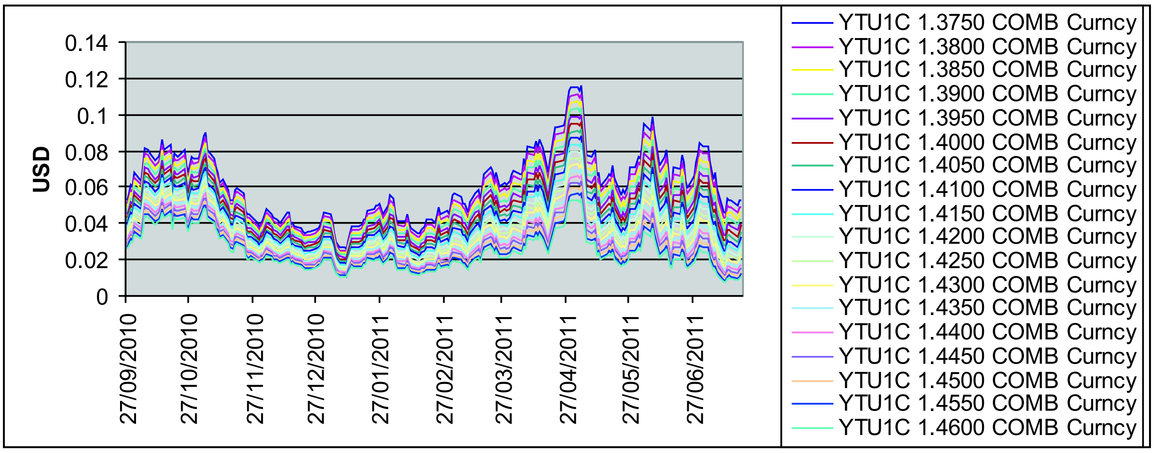

Figure 5.10 – Call option prices on YTU1 with strike price $E_i = 1.375 + 0.005 * (i-1)$, $i = 1, 2, \ldots, 18$, and expiry date T= September 9th, 2011 versus time.

shows the futures price EUR/USD (ticker YTU1 Curncy) (blue line) and the EUR/USD currency's exchange rate (pink line) as a function of time. Figures 5.10 and 5.11 show respectively the prices (in USD) of the corresponding call and put options with maturity time September 9th, 2011 and strike price E_i, $i = 1, 2, \ldots, 18$, as a function of time.

Let us define the moneyness of an option a given day as the ratio between the strike price of the option and the futures price on the EUR/USD exchange rate of that day. As in the previous subsection we consider a rolling containing the priced of one trading day (i.e. $n_T = 1$). This window is moved by one day along the historical series. The time window covers the period September 27th to December 17th, 2010 and the calibration problems (3.51) solved are $61 - n_T$. As a consequence, in each window thirty-six option values are used to calibrate the twelve parameters of the model (i.e. $n_D = 18$ put option price and $n_D = 18$ call option prices). In this way we generate an historical series of daily observation for each parameter. The date associated to the values of the parameters obtained in the j-th window are is the date corresponding to the observed call and put option prices of the j-th window.

Figures 5.12 and 5.13 show the in-sample and the out-of-sample option prices as a function of time to maturity. We can observe that the quality of the out-of-sample call option prices slightly outperforms that of the put option prices while the in-sample put option prices are more accurate than the in-sample call option prices. In fact, the sample mean of the relative errors of the in-sample call and put options for the HW-Heston model are are 3.21% and 1.49% while the sample mean of the relative errors of the out-of-sample call and put options are are 5.01% and 5.76%

The left panels of Figure 5.14 show the time average out-of-sample implied volatility of call (upper panel) and put (lower panel) options as a function of the strike price. The right panels show the strike average out-of-sample implied volatility of call (upper panel) and put (lower panel) options as a function of time to maturity. Specifically 183 days to maturity corresponds to December 20, 2010 while 163 days to maturity corresponds to January 14, 2011. We observe that the forecast values of the implied volatility are satisfactory up to ten days from the last calibration carried out on December 17, 2010.

Figure 5.15 shows the model parameters estimated using the FX data. We observe that r_0 is negative while the long term mean θ is positive and increasing with respect to time. Looking at the times series of the parameters

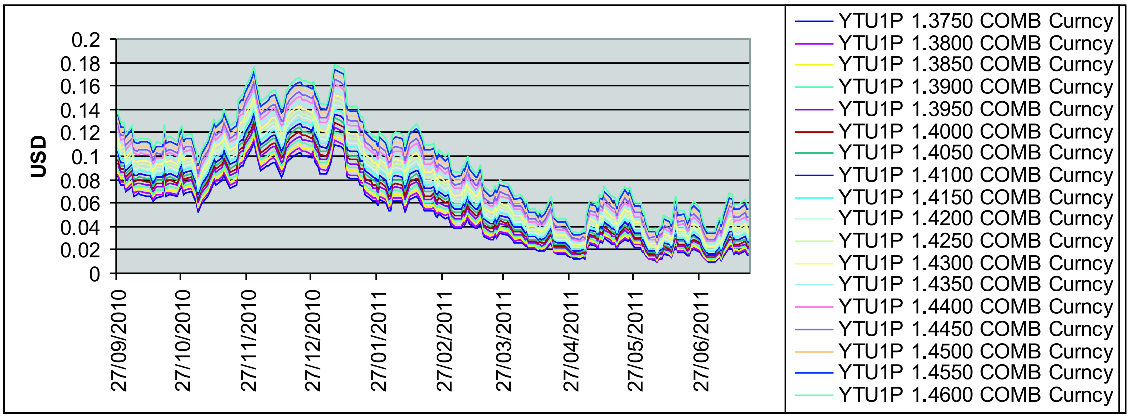

Figure 5.11 – Put option prices on YTU1 with strike price $E_i = 1.375 + 0.005 * (i-1)$, $i = 1, 2, \ldots, 18$, *and expiry date T= September 9th, 2011 versus time.*

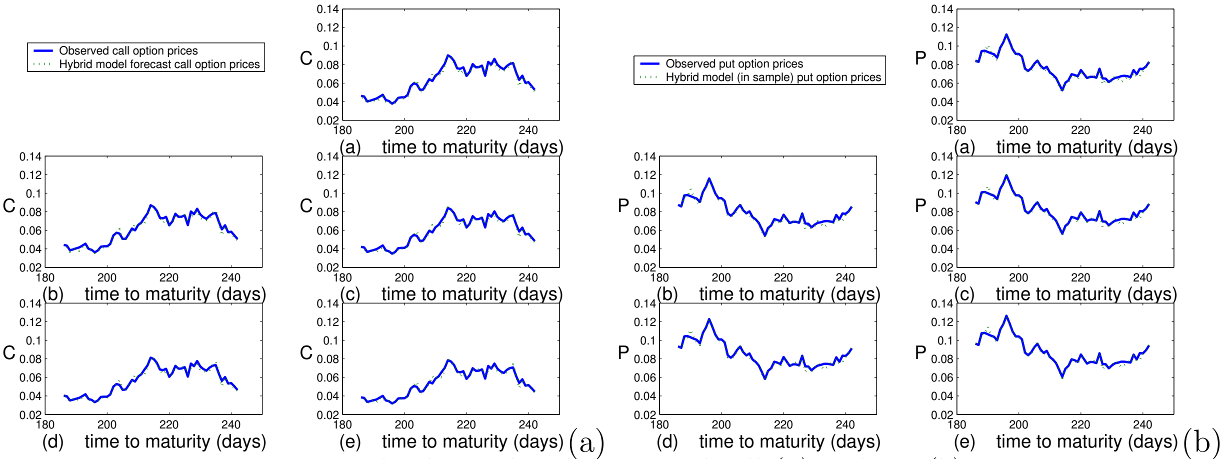

Figure 5.12 – In-sample observed and theoretical call (a) and put (b) option prices on YTU1 with strike price $E_i = 1.375{+}0.005(i{-}1)$, $i = 1, 2, \ldots, 18$, *and expiry date T= September 9th, 2011 versus time to maturity (September 27, 2010 - December 17, 2010).*

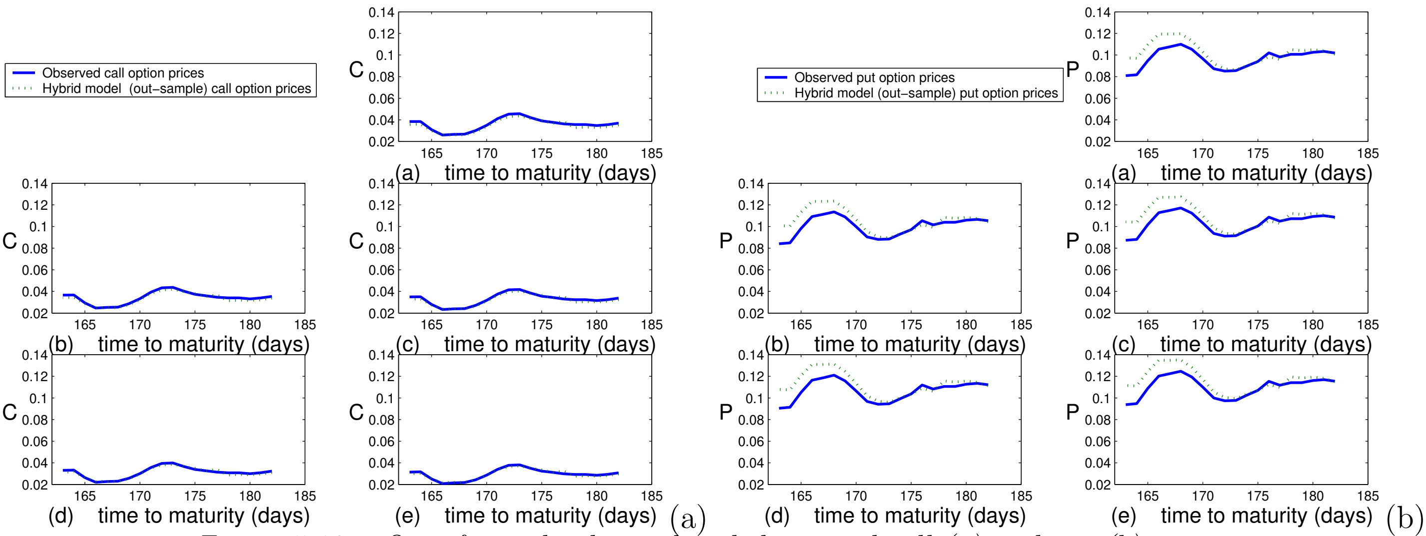

Figure 5.13 – Out-of-sample observed and theoretical call (a) and put (b) option prices on YTU1 with strike price $E_i = 1.375 + 0.005(i-1)$, $i = 1, 2, \ldots, 18$, and expiry date T= September 9th, 2011 versus time to maturity (December 20, 2010 - January 14, 2011).

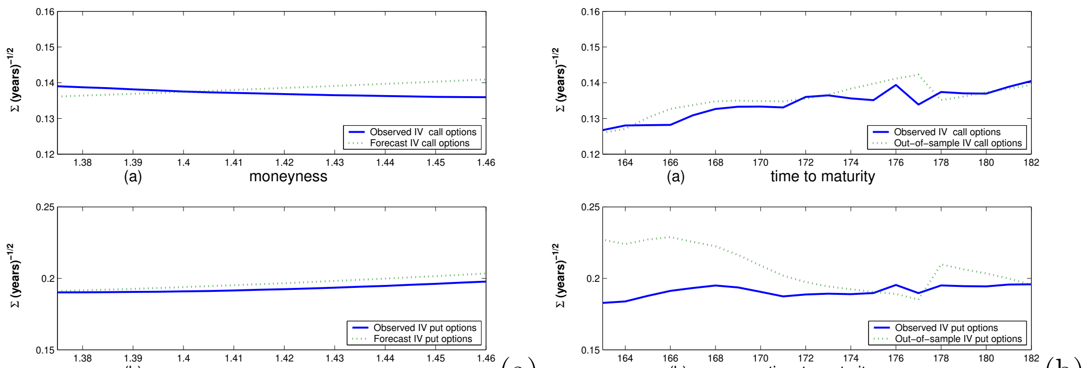

Figure 5.14 – Left panel: time average out-of-sample implied volatility associated with observed and theoretical call (left upper panel) and put (left lower panel) option prices on YTU1 versus strike price $E_i = 1.375 + 0.005(i-1)$, $i = 1, 2, \ldots, 18$ (expiry date T= September 9th, 2011). Right panel: strike average out-of-sample implied volatility associated with observed and theoretical call (right upper panel) and put (right lower panel) versus time to maturity (December 20, 2010 (184-days) - January 14, 2011 (163-days)).

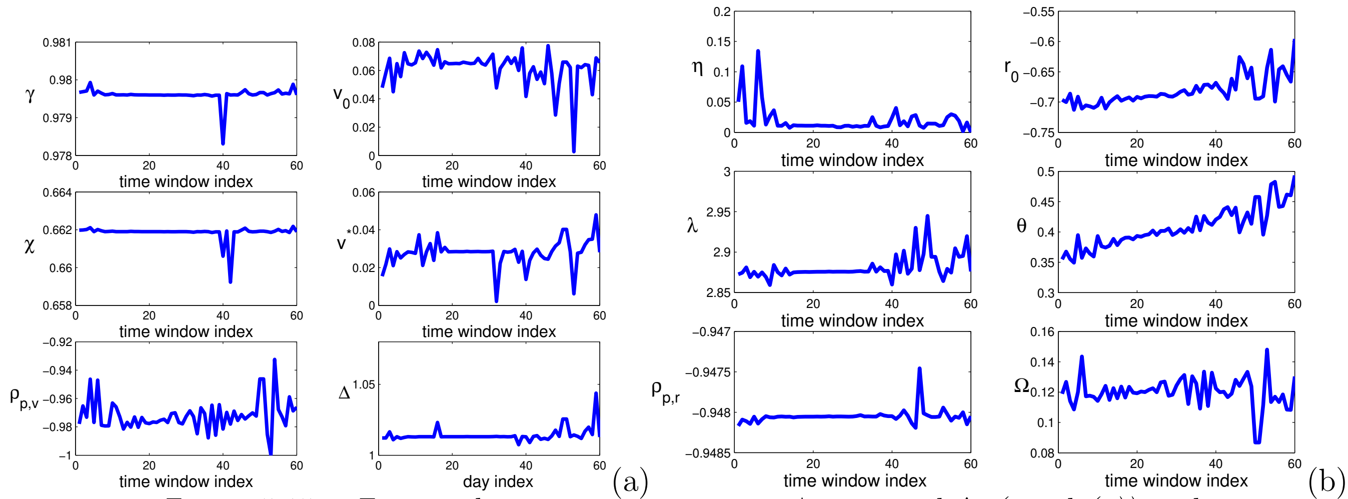

Figure 5.15 – Estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$ and Δ (panel (a)) and Estimated parameters η, r_0, λ, θ, $\rho_{p,r}$ and Ω (panel (b)) versus window index resulting from the calibration of the hybrid Heston model to FX data.

relative to the stochastic interest rate we can observe that in the last perion corresponding to November-December 2010 we observe a slight turbulence in the time series of r_0 and θ.

5.6 Conclusions

This chapter designs a hybrid Heston Hull-White model which describes the dynamics of an asset price under stochastic volatility and interest rate that allows negative values. The aim is to extend the Heston model in order to efficiently solve option pricing problems when negative values of interest rate are observed. A formula for the transition probability density function is derived as a one dimensional integral of an elementary integral function which is used to price European Vanilla call option based on HHW model.

An empirical analysis shows that the hybrid HHW model outperforms the Heston model in interpreting both call and put option prices. Thus, it provides empirical evidence that the stochastic but possible negative values of interest rate plays a significant role as a volatility factor in the option pricing. Apart from this, this chapter uses implied volatility to calibration the model.

Furthermore, this chapter studies an empirical analysis of the Eurodollar futures prices and the corresponding European options prices with a generalization of the Heston model in the stochastic interest rate framework. The results are impressive for both approximation and prediction. This confirms the efficiency of HHW model and the necessary to allow for negative values of interest rate.

6 An Analytical Tractable Mutiscale Hybrid Heston SDE Model On FX Market

6.1 Introduction

6.1.1 Motivation and Research Background

After the collapse of the Bretton Woods system in 1971, U.S. dollar was no longer pegged to gold. Meanwhile, most of the world's currencies exited from the pegged exchange rate with respect to to U.S. dollar. The world economy has to face the risk of floating exchange rate, which can result in the possible investment loss in foreign exchange (FX) reserves and the associated derivatives.

Foreign exchange derivatives have been thought as efficient weapons against the FX rate risk by some scholars and policy makers. A growing number of FX derivatives have been generated in the derivatives market, such as Barrier option which adds an obstacle term to the European call or put option. Specifically, if the FX rate exceeds the barrier price before the option's maturity, the holder is unable to exercise.

Moreover, in the emerging financial Markets, for example China, FX derivatives are playing important roles in maintaining domestic economic stability. Since joining the World Trade Organisation (WTO) in 2001, the volume of Chinese international trade has explosively increased. Till the end of June 2015, FX reserves of China has amounted to 3.69 trillion U.S. dollar. Facing with the huge amount of FX reserves, and global financial instability since 2008 crisis, China needs efficient tools to hedge the risk of FX rate fluctuation. In addition, the explosive growth of China's emerging middle class stimulates domestic financial innovation in FX derivatives, e.g. FX options. Therefore, Bank of China has firstly launched FX options to individual investors since 2002, and the trading volume is continuously growing afterwards.

World economic integration increases the complexity of FX options, investment companies and banks take FX options as fundamental tools to hedge FX risk and stabilize returns of portfolios management. Therefore, from risk management point of view, how to accurately model the price of FX derivatives becomes particularly important today. Needless to say, precisely predicting the FX moment is a continuing demand by various financial sectors.

6.1.2 Literature Review and Chapter Outline

A model of a single FX spot underlying problem can be found in Wystep 2006, while Lipton 2001 and Clark 2011 study a multiple FX rates. These models can be seen as an extension of Black-Scholes model where the volatility is assumed to be constant. Volatility smile effect is studied by Carr and Wu 2007, where empirical evidence rejects the hypothesis of normal distribution of FX returns. Shiraya and Takahashi 2012 use SABR model to study the stochastic volatility effect, and obtain approximation formulas for pricing FX options. It is worth to note that FX rates has inversion and triangulation symmetries, e.g. EUR/USD can be derived from EUR/GBP and GBP/USD. SABR type model does not satisfy this symmetry property, since the inversion's volatility process has an additional drift term. Nevertheless, Heston-type model satisfy this symmetry property (see, Del Bano Rollin 2008).

Ahlip 2008 studies a model of the spot FX rate with stochastic volatility and stochastic domestic as well as foreign rates. Specifically, these rates are modelled by Ornstein-Uhlenbbeck processes, and the volatility by a mean-reverting Ornstein-Uhlenbeck process correlated with the spot FX rate. Ahlip also derives an analytical formula for the price of European call options on the spot FX rate. Despite the complexity and intractability, multiscale FX volatility models usually outperform the single dimensional volatility models with the advantage of well presenting real market data. For instance, Alvise De Col, Gnoatto and Grasselli 2013 propose a multi-factor Heston stochastic volatility model which is an (semi)-affine model whose analytical treatment is deduced using the approach proposed by Duffie, Pan and Singleton 2000. However, the model does not allow for stochastic interest rates. In order to overcome this problem, this chapter proposes the use of a stochastic process to describe the interest rate process (see, Section 2.2.1).

This chapter describes a Multiscale hybrid Heston model which is an extension of De Col, Gnoatto and Grasselli 2013. The contributions are twofold.

Firstly, the Multiscale Heston SDE model of De Col, Gnoatto and Grasselli 2013 is modified here in order to allow a stochastic interest rate. As highlighted in Chapter 2, 3 and 4, stochastic interest rate plays a fundamental role in better matching the market values of the options with the theoretical option prices.

Secondly, the analytical treatment of the model is described both under physical measure and risk neutral measure. In particular, closed-form formulas for FX rate approximation (under physical measure) and option pricing (under risk neutral measure) are obtained. In addition, an integral representation formula of the probability density function (pdf) of the stochastic process is derived by solving the backward Kolmogorov equation using some ideas illustrated in Fatone et al. 2009, 2013. This pdf has practical applications in calculating moment generating functions, model calibration, and empirical analysis which deserve further investigation.

This chapter is organized as follows. In Section 6.2.1, the Multiscale hybrid Heston SDE model is described to illustrate the main relevant formulas under physical measure. In Section 6.2.1, the analytical treatment and the corresponding formulas for the probability density function are deduced. Section 6.2.2 describes the approach of measurement change on Multiscale Hybrid Heston model. In Section 6.2.3, the model treatment under risk neutral measure is briefly described, and the corresponding formulas are deduced detailedly in Appendix C. In Section 6.2.4, analytical formulas are proposed to approximate the FX European vanilla call and put option prices as one-dimensional integrals of explicitly known functions.

6.2 The Multiscale Hybrid Heston SDE Model

This chapter considers the FX market where various currencies are mutually traded. Particularly, this chapter considers FX spot trading model under physical measure and the corresponding European vanilla options pricing model under risk neutral measure. This section presents the analytical treatment of Multiscale Heston hybrid model and some important results (i.e. transition probability density function, moment functions, etc) which are used to deduce option pricing formulas in the Section 6.2.4.

The proposed multiscale hybrid Heston model is inspired by De Col, Gnoatto and Grasselli 2013 model where the artificial currency is used as a universal numéraire, and each values of currencies are considered in units of the artificial currency. Hereafter, this chapter denotes with $S_t^{i,j}$, $\mathbf{V}_t$ and $\mathbf{R}_t$ the exchange rate between currency i and j, d-dimensional independent volatility matrix and 2-dimensional independent interest rate matrix at time $t > 0$. Moreover, let us denote index $j = 0$ for artificial currency, however, the exact specification of the universal numéraire is not necessary here. It is worth to highlight that, for formulating stochastic interest rate process, this chapter considers two fundamental SDE models, i.e. Cox Ingersoll Ross (CIR) model, and Hull-White (HW) model which allows negative values of interest rate. The dynamic of the process is described as follows:

$$\frac{dS_t^{0,i}}{S_t^{0,i}} = (r^0 - r^i)dt - (\mathbf{a}^i)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{W}_t^{p,v} - (\mathbf{b}^{0,i})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{0,i}(t))\right]^{\alpha} d\mathbf{W}_t^{p,r},$$

$$t > 0,\ i = 1, \ldots, N\,,\ \alpha = 0, \frac{1}{2}; \qquad [6.1]$$

$$dv_n(t) = \chi_n(v_n^* - v_n(t))dt + \gamma_n\sqrt{v_n(t)}dW_{n,t}^v, \qquad n = 1, \ldots, d\,,\ t > 0; \quad [6.2]$$

$$dr_m(t) = \lambda_m(\theta_m - r_m(t))dt + \eta_m\, r_m^{\alpha}(t)dW_{m,t}^r, \qquad t > 0 \quad m = \{0, i\}\,; \quad [6.3]$$

where χ_n, v_n^*, γ_n, λ_m, η_m are positive constants. $\mathbf{a}^i = (a_1^i, \ldots, a_d^i)^{\mathrm{T}}$ and $\mathbf{a}^i \in \mathbb{R}^d$, $i = 1, \ldots, N$. $\sqrt{\mathrm{Diag}(\mathbf{V}(t))}$ (or $\sqrt{\mathrm{Diag}(\mathbf{V})}$) is a $d \times d$ dimensional diagonal matrix denoting the square root of the principle diagonal in the elements of the matrix $\mathbf{V}$, i.e. $\sqrt{\mathrm{Diag}(\mathbf{V})} = \begin{bmatrix} \sqrt{v_1} & 0 & \cdots & 0 \\ 0 & \sqrt{v_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{v_d} \end{bmatrix}$. $\mathbf{W}_t^{p,v}$ is a d-dimensional standard Wiener process with $\mathbf{W}_t^{p,v} = (W_{1,t}^{p,v}, \ldots, W_{d,t}^{p,v})^{\mathrm{T}}$, and $W_{n,t}^v$ is standard Wiener processes. Furthermore, $\mathbf{b}^{0,i} = (b_0, b_i)^{\mathrm{T}} \in \mathbb{R}^2$, and $[\mathrm{Diag}(\mathbf{R}_{0,i}(t))]^{\alpha}$ (or $[\mathrm{Diag}(\mathbf{R}_{0,i})]^{\alpha}$) is a 2×2 dimensional diagonal matrix which denotes the α-power of the principle diagonal in the elements of the vector $\mathbf{R}_{0,i}$, and $[\mathrm{Diag}(\mathbf{R}_{0,i})]^{\alpha} = \begin{bmatrix} r_0^{\alpha} & 0 \\ 0 & r_i^{\alpha} \end{bmatrix}$ where r_0 is the artificial currency rate in our universal numéraire. $\mathbf{W}_t^{p,r}$ is a 2-dimensional standard Wiener process with $\mathbf{W}_t^{p,r} = (W_{0,t}^{p,r}, W_{i,t}^{p,r})^{\mathrm{T}}$, and $W_{n,t}^v$ is standard Wiener processes. In contrary to the Del Col, Gnoatto and Grasselli 2013 model assuming deterministic interest rates, our proposed model assumes stochastic interest in Eq.(6.3) following CIR process($\alpha = \frac{1}{2}$) or HW process ($\alpha = 0$). Moreover, the following correlation structure are assumed:

$$
\begin{aligned}
E(dW^{p,r}_{m,t}dW^{p,v}_{n,t}) &= 0,\ m,n,t>0, & [6.4]\\
E(dW^{r}_{m,t}dW^{v}_{n,t}) &= 0,\ m,n,t>0, & [6.5]\\
E(dW^{p,v}_{n,t}dW^{v}_{n,t}) &= \rho_{n,v}dt,\ t>0, & [6.6]\\
E(dW^{p,v}_{n,t}dW^{v}_{l,t}) &= 0,\ n\neq l, t>0, & [6.7]\\
E(dW^{p,v}_{n,t}dW^{r}_{m,t}) &= 0,\ n,m,t>0, & [6.8]\\
E(dW^{p,r}_{m,t}dW^{r}_{m,t}) &= \rho_{m,r}dt,\ t>0, & [6.9]\\
E(dW^{p,r}_{m,t}dW^{r}_{m',t}) &= 0,\ m\neq m', t>0, & [6.10]\\
E(dW^{p,r}_{m,t}dW^{v}_{n,t}) &= 0,\ m,n,t>0, & [6.11]
\end{aligned}
$$

where $E(\cdot)$ denotes the expected value, and $\rho_{n,v}$, $\rho_{m,r}$ $\in[-1,1]$ are constants known as correlation coefficients. Moreover, it is worth nothing that Feller condition i.e. $\forall n=1,2\ldots,d$, $\frac{2\chi_n v_n^*}{\gamma_n^2}>1$ should be satisfied in order to guarantee positive variance $v_n(t)$ with probability one for any $t>0$ given that $v_n^*>0$.

Before giving the main formulas derived in this chapter, let us rewrite the formula Eq.(6.1) in term of the log-price. Using Ito's lemma, we will obtain that the log-price $x_t^{0,i}=\ln S_t^{0,i}$, $t>0$ satisfies the following dynamics:

$$
\begin{aligned}
dx^{0,i} &= \left[(r^0-r^i)-\frac{1}{2}(\mathbf{a}^i)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^i)-\frac{1}{2}(\mathbf{b}^{0,i})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{0,i})\right]^{2\alpha}(\mathbf{b}^{0,i})\right]dt\\
&\quad -(\mathbf{a}^i)^{\mathrm{T}}\sqrt{\mathrm{Diag}\mathbf{V}}d\mathbf{W}_t^{p,v}-(\mathbf{b}^{0,i})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{0,i})\right]^{\alpha}d\mathbf{W}_t^{p,r},\\
&\qquad t>0,\ \alpha=0,\frac{1}{2}, \qquad [6.12]
\end{aligned}
$$

Substituting index i with j, we have the following FX stochastic process of $x_t^{0,j}=\ln S_t^{0,j}$.

$$
\begin{aligned}
dx^{0,j} &= \left[(r^0-r^j)-\frac{1}{2}(\mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^j)-\frac{1}{2}(\mathbf{b}^{0,j})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{r}_{0,j})\right]^{2\alpha}(\mathbf{b}^{0,j})\right]dt\\
&\quad -(\mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})d\mathbf{W}_t^{p,v}-(\mathbf{b}^{0,j})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{0,j})\right]^{2\alpha}d\mathbf{W}_t^{p,r},\\
&\qquad t>0,\ \alpha=0,\frac{1}{2}, \qquad [6.13]
\end{aligned}
$$

where $\mathbf{b}^{0,j}=(b_0,b_j)^{\mathrm{T}}\in\mathbb{R}^2$. Since $S^{i,j}$ denotes the exchange rate between different currencies i and j, and $S^{i,j}=\dfrac{S^{0,j}}{S^{0,i}}$ by definition, therefore we have the following triangular symmetry relationship:

$$
x^{i,j}=\ln S^{i,j}=\ln S^{0,j}-\ln S^{0,i}=x^{0,j}-x^{0,i} \qquad [6.14]
$$

as well as

$$dx^{i,j} = dx^{0,j} - dx^{0,i} \qquad [6.15]$$

Substituting Eq.(6.12) into (6.13), we obtain:

$$\begin{aligned} & dx^{i,j} \\ = & \Big\{ (r_i - r_j) + \frac{1}{2} \left[(\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}} (\mathrm{Diag}\mathbf{V})(\mathbf{a}^i + \mathbf{a}^j) + (b_i, -b_j) \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{2\alpha} (b_i, b_j)^{\mathrm{T}} \right] \\ & \Big\} dt + (a^i - a^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}\mathbf{V}} d\mathbf{W}_t^{p,v} + (b_i, -b_j) \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r}, \qquad [6.16] \\ & \qquad t > 0,\ \alpha = 0, \frac{1}{2} \end{aligned}$$

Lemma 6.2.1. *Assuming the log-price $x_t^{i,j} = \ln S_t^{i,j}$, $t > 0$ satisfies the SDE in Eq.(6.16), then spot price $S_t^{i,j}$ satisfies the following dynamic:*

$$\begin{aligned} \frac{dS_t^{i,j}}{S_t^{i,j}} &= \left[r_i - r_j + (\boldsymbol{a}^i - \boldsymbol{a}^j)^{\mathrm{T}} (\mathrm{Diag}\, \boldsymbol{V}) \boldsymbol{a}^i + \boldsymbol{b}_i^2 r_i^{2\alpha} \right] dt \\ & + (\boldsymbol{a}^i - \boldsymbol{a}^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}(\boldsymbol{V})} d\, \boldsymbol{W}_t^{p,v} + (b_i, -b_j) \left[\mathrm{Diag}(\boldsymbol{R}_{i,j})\right]^{\alpha} d\, \boldsymbol{W}_t^{p,r}, \\ & \qquad t > 0,\ i, j = 1, \ldots, N\,,\ \alpha = 0, \frac{1}{2} \qquad [6.17] \end{aligned}$$

Proof. By definition in Eq.(6.1), the dynamic of underlying of FX rate $S_t^{i,j}$ satisfies a general geometric Brownian motion with drift coefficient $\mu^{i,j}$ as follows:

$$\begin{aligned} \frac{dS_t^{i,j}}{S_t^{i,j}} &= \mu^{i,j} dt + (\mathbf{a}^{i,j})^{\mathrm{T}} \sqrt{\mathrm{Diag}(\mathbf{V}(t))} d\mathbf{W}_t^{p,v} + (\mathbf{b}^{i,j})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{i,j}(t))\right]^{\alpha} d\mathbf{W}_t^{p,r}, \\ & \qquad t > 0,\ i, j = 1, \ldots, N\,,\ \alpha = 0, \frac{1}{2} \qquad [6.18] \end{aligned}$$

where $\mathbf{b}^{i,j} = (b_i, -b_j)^{\mathrm{T}}$. Using Ito's Lemma in Eq.(6.18), we will obtain $x_t^{i,j} = \ln S_t^{i,j}$ as follows:

$$\begin{aligned} dx^{i,j} &= \left[\mu^{i,j} - \frac{1}{2} (\mathbf{a}^{i,j})^{\mathrm{T}} (\mathrm{Diag}\mathbf{V})(\mathbf{a}^{i,j}) - \frac{1}{2} (\mathbf{b}^{i,j})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{2\alpha} (\mathbf{b}^{i,j}) \right] dt \\ & + (\mathbf{a}^{i,j})^{\mathrm{T}} \sqrt{\mathrm{Diag}\mathbf{V}} d\mathbf{W}_t^{p,v} + (\mathbf{b}^{i,j})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r}, \\ & \qquad t > 0,\ \alpha = 0, \frac{1}{2} \qquad [6.19] \end{aligned}$$

It is worth nothing that Eqs.(6.16) and (6.19) should be identical. Thus, we have the following equalities:

$$\mathbf{a}^{i,j} = \mathbf{a}^i - \mathbf{a}^j \in \mathbb{R}^d \,,\text{ or } \mathbf{a}^{i,j} = (\mathbf{a}_1^{i,j}, \ldots, \mathbf{a}_d^{i,j})^{\mathrm{T}} \,, \text{with } \mathbf{a}_k^{i,j} = \mathbf{a}_k^i - \mathbf{a}_k^j \,, \quad k = 1, 2, \ldots d \qquad [6.20]$$

$$\mathbf{b}^{i,j} = (b_i, -b_j)^{\mathrm{T}} \in \mathbb{R}^2 \,, \qquad [6.21]$$

$$\mu^{i,j} = r_i - r_j + (\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{DiagV})\mathbf{a}^i + \underbrace{(b_i, -b_j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^{2\alpha}\,(b_i, 0)^{\mathrm{T}}}_{=b_i^2 r_i^{2\alpha}} \qquad [6.22]$$

Substituting Eqs.(6.20)-(6.22) into Eq.(6.16), we obtain

$$\begin{aligned} \frac{dS_t^{i,j}}{S_t^{i,j}} &= \left[r_i - r_j + (\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})\mathbf{a}^i + \mathbf{b}_i^2 r_i^{2\alpha}\right] dt \\ &\quad + (\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}(\mathbf{V})} d\mathbf{W}_t^{p,v} + (b_i, -b_j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^{\alpha}\, d\mathbf{W}_t^{p,r}, \\ &\qquad t > 0,\ i, j = 1, \ldots, N\,,\ \alpha = 0, \frac{1}{2} \end{aligned}$$

Without using matrix components, we could re-write Eq.(6.23) in the following linear equations:

$$\frac{dS_t^{i,j}}{S_t^{i,j}} = \left[r_i - r_j + \sum_{k=1}^{d}(a_k^i - a_k^j)a_k^i v_k + b_i^2 r_i^{2\alpha}\right] dt \qquad [6.23]$$

$$+\sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k} dW_k^{p,v} + b_i r_i^{\alpha} dW_i^{p,r} - b_j r_j^{\alpha} dW_j^{p,r} \,, \qquad [6.24]$$

$$t > 0,\ i, j = 1, \ldots, N\,,\ \alpha = 0, \frac{1}{2}$$

□

In order to simplify the following computations, we can re-write the matrix components of Eq.(6.16) with expression of linear components as follows:

$$\begin{aligned} & (\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^i + \mathbf{a}^j) \\ = \ & [a_1^i - a_1^j, a_2^i - a_2^j, \ldots, a_d^i - a_d^j] \cdot \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_d \end{bmatrix} \cdot \begin{bmatrix} a_1^i + a_1^j \\ a_2^i + a_2^j \\ \vdots \\ a_d^i + a_d^j \end{bmatrix} \\ = \ & \sum_{k=1}^{d} [(a_k^i)^2 - (a_k^j)^2] \cdot v_k \end{aligned} \qquad [6.25]$$

and

$$
\begin{aligned}
& (b_i, -b_j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^{2\alpha}\,(b_i, b_j)^{\mathrm{T}} \\
= & \begin{bmatrix} b_i, -b_j \end{bmatrix} \begin{bmatrix} r_i^{2\alpha} & 0 \\ 0 & r_j^{2\alpha} \end{bmatrix} \begin{bmatrix} b_i \\ b_j \end{bmatrix} \\
= & b_i^2 \cdot r_i^{2\alpha} - b_j^2 \cdot r_j^{2\alpha} \qquad [6.26]
\end{aligned}
$$

Thus, Eq.(6.19) can be re-written in the following form:

$$
\begin{aligned}
dx^{i,j} &= \left[r_i - r_j + \frac{1}{2}\sum_{k=1}^{d}[(a_k^i)^2 - (a_k^j)^2]v_k + \frac{1}{2}(b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}) \right] dt \\
&\quad + \sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v} + b_i r_i^{\alpha} dW_i^{p,r} - b_j r_j^{\alpha} dW_j^{p,r}\,, \qquad [6.27] \\
&\qquad t > 0,\ \alpha = 0, \frac{1}{2},
\end{aligned}
$$

Combining Eqs.(6.2), (6.3) and (6.27) into a system of equations, we obtain:

$$
\begin{aligned}
dx^{i,j} &= \left[r_i - r_j + \frac{1}{2}\sum_{k=1}^{d}[(a_k^i)^2 - (a_k^j)^2]v_k + \frac{1}{2}(b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}) \right] dt \\
&\quad + \sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v} + b_i r_i^{\alpha} dW_i^{p,r} - b_j r_j^{\alpha} dW_j^{p,r}\,, \qquad [6.28] \\
&\qquad t > 0,\ \alpha = 0, \frac{1}{2}, \\
dv_k &= \chi_k(v_k^* - v_k)dt + \gamma_k\sqrt{v_k}dW_{k,t}^{v}, \qquad k = 1, \ldots, d\,,\ t > 0; \qquad [6.29] \\
dr_i &= \lambda_i(\theta_i - r_i)dt + \eta_i\, r_i^{\alpha} dW_i^{r}, \qquad [6.30] \\
dr_j &= \lambda_j(\theta_j - r_j)dt + \eta_j\, r_j^{\alpha} dW_j^{r}. \qquad [6.31]
\end{aligned}
$$

6.2.1 The Model Treatment Under Physical Measure

First of all, let us equip Eqs.(6.28), (6.29), (6.30) and (6.31) with the initial conditions as follows:

$$
\begin{aligned}
x^{i,j}(0) &= \tilde{x}_0^{i,j}\,, \qquad [6.32] \\
v_k(0) &= \tilde{v}_{k,0}\,, \quad k = 1, \ldots, d \qquad [6.33] \\
r_m(0) &= \tilde{r}_{m,0}\,, \quad m = i, j \qquad [6.34]
\end{aligned}
$$

where $\tilde{x}_0^{i,j}$, $\tilde{v}_{k,0}$, $\tilde{r}_{m,0}$ are random variables that are assumed to be concentrated in a point with probability one. For simplicity, we identify the random variables $\tilde{x}_0^{i,j}$, $\tilde{v}_{k,0}$, $\tilde{r}_{m,0}$ with the points where they are concentrated. This chapter assumes $\tilde{v}_{k,0}$, $\tilde{r}_{m,0}$, χ_k, λ_m, γ_k, η_m, v_k^*, θ_m to be positive constant. In addition, this chapter assumes $\frac{2\chi_k v_k^*}{\gamma_k^2} > 1$ and $\frac{2\lambda_m \theta_m}{\eta_m^2} > 1$ (Feller condition). Hence, we mainly consider CIR type (i.e. $\alpha = 1/2$) volatility and interest rate processes where positive values of volatility and interest rate are guaranteed under the above assumptions. Nevertheless, in case of negative values of interest rate, the model can be explicitly extended by choosing a HW type interest rate process (i.e. $\alpha = 0$), and relaxing the assumptions of the second Feller condition for interest rate. This deserves further investigation.

Let $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$, $(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2$, t, $t' \geq 0$, $t' - t > 0$ be the transition probability density function associated with the stochastic differential system (6.28), (6.29), (6.30), (6.31). That is, the probability density function having $x' = x_{t'}^{i,j}$, $\underline{v}' = (v_1', \ldots, v_d')^{\mathrm{T}}$, $\underline{r}' = (r_i', r_j')^{\mathrm{T}}$ given that $x = x_t^{i,j}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, when $t' - t > 0$. In analogy with Lipton (2001)(pages 602–605), this transition probability density function $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$ as a function of the "past" variables $(x, \underline{v}, \underline{r}, t)$ satisfies the following backward Kolmogorov equation as follows:

$$
\begin{aligned}
-\frac{\partial p_f}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_f}{\partial v_n^2} \\
&+\frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_f}{\partial r_i^2} + \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_f}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_f}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_i^{2\alpha}} \\
&-\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_j} + \sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_f}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_f}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_f}{\partial r_j} \\
&+\left((r_i - r_j) + \frac{1}{2}\sum_{n=1}^{d}\left[(a_n^i)^2 - (a_n^j)^2\right]v_n + \frac{1}{2}[b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}]\right)\frac{\partial p_f}{\partial x} \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \leq t < t', \qquad [6.35]
\end{aligned}
$$

with final condition:

$$
\begin{aligned}
p_f(t, x, \underline{v}, \underline{r}, t', x', \underline{v}', \underline{r}' | t' = t) &= \delta(x' - x)\prod_{m \in \{i,j\}}\delta(r_m' - r_m) \cdot \prod_{n=1}^{d}\delta(v_n' - v_n) \\
&= \delta(x' - x) \cdot \delta(r_i' - r_i)\delta(r_j' - r_j) \cdot \prod_{n=1}^{d}\delta(v_n' - v_n), \qquad [6.36] \\
\forall n \in \{1, \ldots, d\} \text{ and } m \in \{i, j\}, &\quad (x, v_n, r_m), (x', v_n', r_m') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t \geq 0,
\end{aligned}
$$

and the appropriate boundary conditions. Defining $\tau = t' - t$, the function p_b is defined as follows:

$$p_b(\tau, x, \underline{v}, r, x', \underline{v}', r') = p_f(t, x, \underline{v}, r, t', x', \underline{v}', r'),$$
$$(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2,\ t' = t + \tau,\ \tau > 0\,. \qquad [6.37]$$

The representation (6.37) holds since the coefficients of the Kolmogorov backward equation and condition (6.36) are invariant by time translation. Using the change of the time variable $\tau = t - t'$ and equation (6.35), it is easy to see that p_b is the solution of the following problem:

$$\begin{aligned}
\frac{\partial p_b}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_b}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_b}{\partial v_n^2} \\
&+\frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_b}{\partial r_i^2} + \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_b}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_b}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_b}{\partial x \partial r_i^{2\alpha}} \\
&-\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_b}{\partial x \partial r_j} + \sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_b}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_b}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_b}{\partial r_j} \\
&+\left((r_i - r_j) + \frac{1}{2}\sum_{n=1}^{d}\left[(a_n^i)^2 - (a_n^j)^2\right]v_n + \frac{1}{2}[b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}]\right)\frac{\partial p_b}{\partial x} \\
&\quad (x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \le t < t', \qquad [6.38]
\end{aligned}$$

with the initial condition:

$$\begin{aligned}
p_b(0, x, \underline{v}, r, x', \underline{v}', r') &= \delta(x' - x)\prod_{m=i}^{j}\delta(r_m' - r_m)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n) \\
&= \delta(x' - x)\cdot\delta(r_i' - r_i)\delta(r_j' - r_j)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n)\,, \qquad [6.39] \\
&\quad (x, v_n, r_m), (x', v_n', r_m') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t \ge 0,
\end{aligned}$$

and with the appropriate boundary conditions. For the convenience of later computation, let us consider the following change of dependent variable with a 'regularization' parameter q, which enables us to derive elementary formulas for the marginal probability density function and the FX option prices in Section 6.2.4.

$$p_b(\tau, x, \underline{v}, r, x', \underline{v}', r') = e^{q(x-x')}p_q(\tau, x, \underline{v}, r, x', \underline{v}', r') \qquad [6.40]$$
$$(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2,\ t' = t + \tau,\ \tau > 0\,.$$

Substituting Eq.(6.40) into (6.38) and (6.39), it is easy to get that p_q is the solution of the following problem:

$$
\begin{aligned}
&\frac{\partial p_q}{\partial \tau} = \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_q}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_q}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_q}{\partial r_i^2} \\
&+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_q}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_q}{\partial x\partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_q}{\partial x \partial r_i} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_q}{\partial x\partial r_j} \\
&+\sum_{n=1}^{d}\left[\chi_n(\overline{v}_n - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial p_q}{\partial v_n} + \left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial p_q}{\partial r_i} \\
&+\left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial p_q}{\partial r_j} + \left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left(2q + \frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right. \\
&+\left.\left(r_i + \frac{r_i^{2\alpha}}{2}b_i^2(2q+1)\right) + \left(-r_j + \frac{r_j^{2\alpha}}{2}b_j^2(2q-1)\right)\right]\frac{\partial p_q}{\partial x} + \left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left(q^2 + q\frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right. \\
&+\left.\left(qr_i + \frac{r_i^{2\alpha}}{2}b_i^2(q^2+q)\right) + \left(q(-r_j) + \frac{r_j^{2\alpha}}{2}b_j^2(q^2-q)\right)\right]p_q \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t', \qquad [6.41]
\end{aligned}
$$

with the initial condition:

$$
\begin{aligned}
p_q(0, x, \underline{v}, r, x', \underline{v}', r') &= e^{q(x-x')}\delta(x'-x)\prod_{m=i}^{j}\delta(r_m' - r_m)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n) \\
&= e^{q(x-x')}\delta(x'-x)\cdot\delta(r_i' - r_i)\delta(r_j' - r_j)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n), \qquad [6.42] \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ \tau = 0,
\end{aligned}
$$

Now let us consider the following representation formula for p_q with a Fourier transform:

$$
\begin{aligned}
&p_q(\tau, x, \underline{v}, r, x', \underline{v}', r') = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\, f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k), \\
&(\underline{v}, \underline{r}) \in (\mathbb{R}^+)^d, (\underline{v}', \underline{r}') \times (\mathbb{R}^+)^2,\ k \in \mathbb{R},\ \tau > 0. \qquad [6.43]
\end{aligned}
$$

This is possible since the coefficients (6.38) and the initial condition (6.39) are independent of translation in the log-price variable. Substituting Eq.(6.43) into (6.38), we obtain that the function f is the solution of the following problem:

$$
\begin{aligned}
\frac{\partial f}{\partial \tau} &= \frac{-k^2}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right] f + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 f}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 f}{\partial r_i^2} \\
&+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 f}{\partial r_j^2} + \sum_{n=1}^{d}(-\imath k)\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial f}{\partial v_n} + (-\imath k)\rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial f}{\partial r_i} \\
&-(-\imath k)\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial f}{\partial r_j} + \sum_{n=1}^{d}\left[\chi_n(v_n^* - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial f}{\partial v_n} \\
&+\left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial f}{\partial r_i} + \left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial f}{\partial r_j} \\
&+\Bigg\{\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left[\left(q^2 + q\frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right) - \imath k\left(2q + \frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right] \\
&\quad + \left(r_i(q - \imath k) + \frac{r_i^{2\alpha}}{2}b_i^2\left[(q^2 + q) - \imath k(2q + 1)\right]\right) \\
&\quad + \left(r_j(-q + \imath k) + \frac{r_j^{2\alpha}}{2}b_j^2\left[(q^2 - q) - \imath k(2q - 1)\right]\right)\Bigg\} f \qquad [6.44] \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t'.
\end{aligned}
$$

with the initial condition:

$$
\begin{gathered}
f(0, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \delta(r_i' - r_i)\delta(r_j' - r_j)\prod_{n=1}^{k}\delta(v_i' - v_i)\,, \qquad [6.45] \\
(v_i, r_i, r_j), (v_i', r_i', r_j') \in \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ k \in \mathbb{R}.
\end{gathered}
$$

Now let us represent f as the inverse Fourier transform of the future variables $(\underline{v}', \underline{r}')$ whose conjugate variables are denoted by $(\underline{l}, \underline{\xi})$, that is:

$$
\begin{gathered}
f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \left(\frac{1}{2\pi}\right)^{d+2}\prod_{n=1}^{d}\int_{\mathbb{R}} dl_n\, e^{\imath l_n v_n'} \cdot \prod_{m\in\{i,j\}}\int_{\mathbb{R}} d\xi_m\, e^{\imath\,\xi_m r_m'}\, g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}), \\
m = i, j\,,\ (v_n, r_m) \in \mathbb{R}^+\times\mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau > 0. \qquad [6.46]
\end{gathered}
$$

It is easy to see that the function g satisfies Eq.(6.44) with the following initial condition:

$$g(0,\underline{v},\underline{r},k,\underline{l},\underline{\xi}) = \prod_{m\in\{i,j\}} e^{-\imath\,\xi_m\,r_m} \prod_{n=1}^{k} e^{-\imath\,l_n\,v_n}, \qquad [6.47]$$

$$= e^{-\imath\,\xi_i\,r_i} e^{-\imath\,\xi_j\,r_j} \prod_{n=1}^{k} e^{-\imath\,l_n\,v_n}, \qquad [6.48]$$

$$m \in \{i,j\}\ ,\ (v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau > 0.$$

It is worth to note that g is the Fourier transform with respect to the future variables $(\underline{v}', \underline{r}')$ of the function obtained by extending f, as a function of the variables $(\underline{v}, \underline{r})$, with zero when $v_n \notin \mathbb{R}^+$ and/or $r_m \notin \mathbb{R}^+$. The coefficients of the partial differential operator appearing on the right hand side of Eq.(6.38) are first degree polynomials in $\underline{v}$ and $\underline{r}$. Thus, we seek a solution of problem (6.44), (6.45) in the form (see Lipton, 2001):

$$g(\tau,\underline{v},\underline{r},k,\underline{l},\underline{\xi}) = e^{A(\tau,k,\underline{l},\underline{\xi})} \prod_{m\in\{i,j\}} e^{-r_m\,B_{r_m}(\tau,k,\xi_m)} \prod_{n=1}^{d} e^{-v_n\,B_{v_n}(\tau,k,l_n)},$$

$$= e^{A(\tau,k,\underline{l},\underline{\xi})} e^{-r_i\,B_{r_i}(\tau,k,\xi_i)} e^{-r_j\,B_{r_j}(\tau,k,\xi_j)} e^{-\sum_{n=1}^{d} v_n\,B_{v_n}(\tau,k,l_n)}, \qquad [6.49]$$

$$(v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, \underline{l}, \underline{\xi}) \in \mathbb{R}\times(\mathbb{R})^d\times(\mathbb{R})^2,\ \tau > 0.$$

From now on, let us only focus on the CIR type interest rate process, i.e choosing $\alpha = \dfrac{1}{2}$. In this case, non-negative values of interest rate is guaranteed with probability one. Substituting Eq.(6.49) into Eq.(6.44) , the formulas $A(\tau,k,\underline{l},\underline{\xi})$, $B_{v_n}(\tau,k,l_n)$, $B_{r_i}(\tau,k,\xi_i)$, and $B_{r_j}(\tau,k,\xi_j)$ must satisfy the following ordinary differential equations:

$$\frac{dA}{d\tau}(\tau,k,\underline{l},\underline{\xi}) = -\lambda_i\,\theta_i\,B_{r_i}(\tau,k,\xi_i) - \lambda_j\,\theta_j\,B_{r_j}(\tau,k,\xi_j) - \sum_{n=1}^{d} \chi_n v_n^* B_{v_n}(\tau,k,l_n)$$

$$(k, l_n, \xi_i, \xi_j) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau > 0, \qquad [6.50]$$

$$
\begin{aligned}
&\frac{dB_{v_n}}{d\tau}(\tau,k,l_n)\\
&= \frac{k^2}{2}(a_n^i - a_n^j)^2 - \frac{(a_n^i - a_n^j)^2}{2}\left[\left(q^2 + q\frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right) - \imath k\left(2q + \frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right]\\
&\quad - \left[\chi_n + (\imath k - q)\gamma_n\rho_{n,v}(a_n^i - a_n^j)\right] B_{v_n} - \frac{\gamma_n^2}{2}B_{v_n}^2 \qquad [6.51]\\
&= \varphi_q^{v_n}(k)(a_n^{i,j})^2 - (\chi_n + (\imath k - q)\gamma_n\tilde{\rho}_{n,v})B_{v_n}(\tau,k,l_n) - \frac{\gamma_n^2}{2}B_{v_n}^2(\tau,k,l_n), \qquad [6.52]\\
&\qquad (k,l_n)\in\mathbb{R}\times\mathbb{R}, \tau > 0
\end{aligned}
$$

where $\varphi_q^{v_n}(k) = \frac{k^2}{2} - \frac{1}{2}\left[\left(q^2 + q\frac{\tilde{a}_n^{i,j}}{a_n^{i,j}}\right) - \imath k\left(2q + \frac{\tilde{a}_n^{i,j}}{a_n^{i,j}}\right)\right]$, $a_n^{i,j} = a_n^i - a_n^j$, $\tilde{a}_n^{i,j} = a_n^i + a_n^j$, and $\tilde{\rho}_{n,v} = \rho_{n,v}(a_n^i - a_n^j)$.

$$
\begin{aligned}
&\frac{dB_{r_i}}{d\tau}(\tau,k,\xi_i)\\
&= \frac{k^2}{2}b_i^2 + (\imath k - q) - \frac{b_i^2}{2}\left[(q^2+q) - \imath k(2q+1)\right] - \left[\lambda_i + (\imath k - q)\eta_i\rho_{i,r}b_i\right]B_{r_i}(\tau,k,\xi_i)\\
&\quad -\frac{\eta_i^2}{2}B_{r_i}^2(\tau,k,\xi_i)\\
&= \varphi_q^{r_i}(k)b_i^2 + (\imath k - q) - \left[\lambda_i + (\imath k - q)\eta_i\rho_{i,r}b_i\right]B_{r_i}(\tau,k,\xi_i) - \frac{\eta_i^2}{2}B_{r_i}^2(\tau,k,\xi_i) \qquad [6.53]\\
&\qquad (k,\xi_i)\in\mathbb{R}\times\mathbb{R}, \tau > 0,
\end{aligned}
$$

where $\varphi_q^{r_i}(k) = \frac{k^2}{2} - \frac{1}{2}\left[(q^2+q) - \imath k(2q+1)\right]$.

$$
\begin{aligned}
&\frac{dB_{r_j}}{d\tau}(\tau,k,\xi_i)\\
&= \frac{k^2}{2}b_j^2 + (q - \imath k) - \frac{b_j^2}{2}\left[(q^2-q) - \imath k(2q-1)\right] - \left[\lambda_j + (\imath k - q)\eta_j\rho_{j,r}(-b_j)\right]B_{r_j}\\
&\quad -\frac{\eta_j^2}{2}B_{r_j}^2\\
&:= \varphi_q^{r_j}(k)b_j^2 + (q - \imath k) - \left[\lambda_j + (q - \imath k)\eta_j\rho_{i,r}b_j\right]B_{r_j}(\tau,k,\xi_j) - \frac{\eta_j^2}{2}B_{r_j}^2(\tau,k,\xi_j)\\
&\qquad (k,\xi_j)\in\mathbb{R}\times\mathbb{R}, \tau > 0, \qquad [6.54]
\end{aligned}
$$

with $\varphi_q^{r_j}(k) = \frac{k^2}{2} - \frac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right]$, and initial conditions:

$$
\begin{aligned}
&A(0,k,\underline{l},\underline{\xi}) = 0, \quad B_{v_n}(0,k,l_n) = \imath\, l_n, \quad B_{r_i}(0,k,\xi_i) = \imath\,\xi_i, \quad B_{r_j}(0,k,\xi_j) = \imath\,\xi_j,\\
&\qquad \text{with } (k,\underline{l},\underline{\xi})\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^2. \qquad [6.55]
\end{aligned}
$$

Eqs.(6.52)-(6.54) are Riccati equations that can be solved analytically. Then substituting their solutions into (6.50) and integrating with respect to τ, $A(\tau, k, \underline{l}, \underline{\xi})$ can be obtained straightforward.

Let us firstly solve Eq.(6.52), and the solutions of (6.53) and (6.54) can be obtained analogously. Let us seek for the solution of Eq.(6.52) in the following form:

$$B_{v_n}(\tau, k, l_n) = \frac{2}{\gamma_n^2} \frac{\frac{d}{d\tau} C_{v_n}}{C_{v_n}}. \tag{6.56}$$

Replacing Eq.(6.65) into (6.52), we can obtain that C_v^i must satisfy the following problem:

$$\frac{d^2 C_{v_n}}{d\tau^2} + (\chi_n + (\imath k - q)\gamma_n \tilde{\rho}_{n,v}) \frac{dC_{v_n}}{d\tau} - \frac{\gamma_i^2}{2} \varphi_q^{v_n}(k)(a_n^{i,j})^2 C_{v_n} = 0, \tag{6.57}$$

$$C_{v_n}(0, k, l_n) = 1, \quad \frac{dC_{v_n}}{d\tau}(0, k, l_n) = \imath\, l_i \frac{\gamma_n^2}{2}, \ (k, l_n) \in \mathbb{R} \times \mathbb{R}. \tag{6.58}$$

Please note that Eq.(6.57) is a second order ordinary differential equation with constant coefficients. Therefore, the solution is given by:

$$C_{v_n}(\tau, k, l_n) = C_{v_n}(\tau, k, l_n) = e^{(\mu_{q,v_n} + \zeta_{q,v_n})\tau} \left[\frac{s_{q,v_n,b} + \imath\, l_n \frac{\gamma_n^2}{2} s_{q,v_n,g}}{2\zeta_{q,v_n}} \right], \quad [6.59]$$

where

$$\mu_{q,v_n} = -\frac{1}{2}(\chi_n + (\imath k - q)\gamma_n \tilde{\rho}_{n,v}), \quad \zeta_{q,v_n} = \frac{1}{2}\left[4\mu_{v_n}^2 + 2\gamma_n^2 \varphi_q^{v_n}(k)(a_n^{i,j})^2\right]^{1/2}, \quad [6.60]$$

$$s_{q,v_n,g} = 1 - e^{-2\zeta_{q,v_n}\tau}, \quad [6.61]$$
$$s_{q,v_n,b} = (\zeta_{q,v_n} + \mu_{q,v_n})e^{-2\zeta_{q,v_n}\tau} + (\zeta_{q,v_n} - \mu_{q,v_n}). \quad [6.62]$$

Furthermore, we can obtain:

$$\frac{dC_{v_n}}{d\tau}(\tau, k, l_n) = e^{(\mu_{q,v_n} + \zeta_{q,v_n})\tau} \cdot$$
$$\left[\frac{(\mu_{q,v_n} - \zeta_{q,v_n})(\mu_{q,v_n} + \zeta_{q,v_n} - \frac{\gamma_n^2}{2}\imath\, l_n)e^{-2\zeta_{q,v_n}\tau} + (\mu_{q,v_n} + \zeta_{q,v_n})(\zeta_{q,v_n} - \mu_{q,v_n} + \frac{\gamma_n^2}{2}\imath\, l_n)}{2\zeta_{q,v_n}} \right]$$
$$= \frac{e^{(\mu_{q,v_n} + \zeta_{q,v_n})\tau}}{2\zeta_{q,v_n}} \left[[\zeta_{q,v_n}^2 - \mu_{q,v_n}^2] s_{q,v_n,g} + \frac{\gamma_n^2}{2}\imath\, l_n\, s_{q,v_n,d} \right], \quad [6.63]$$

where

$$s_{q,v_n,d} = (\zeta_{q,v_n} - \mu_{q,v_n})e^{-2\zeta_{q,v_n}\tau} + (\zeta_{q,v_n} + \mu_{q,v_n}). \quad [6.64]$$

Substituting Eqs.(6.59) and (6.63) into (6.65), we obtain:

$$B_{v_n}(\tau,k,l_n) = \frac{2}{\gamma_n^2}\frac{\left((\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}+\frac{\gamma_n^2}{2}\imath\, l_n\, s_{q,v_n,d}\right)}{s_{q,v_n,b}+\imath\, l_i\frac{\gamma_n^2}{2}s_{q,v_n,g}}. \tag{6.65}$$

Then, the solutions of Eqs.(6.53) and (6.54) are obtained analogously:

$$B_{r_m}(\tau,k,\xi_m) = \frac{2}{\eta_m^2}\frac{\left((\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}+\frac{\eta_m^2}{2}\imath\,\xi_m\, s_{q,r_m,d}\right)}{s_{q,r_m,b}+\imath\,\xi_m\frac{\eta_m^2}{2}s_{q,r_m,g}}, \quad m=i,j \tag{6.66}$$

where

$$s_{q,r_m,g} = 1-e^{-2\zeta_{q,r_m}\tau}, \qquad [6.67]$$
$$s_{q,r_m,b} = (\zeta_{q,r_m}+\mu_{q,r_m})e^{-2\zeta_{q,r_m}\tau}+(\zeta_{q,r_m}-\mu_{q,r_m}), \qquad [6.68]$$
$$s_{q,r_m,d} = (\zeta_{q,r_m}-\mu_{q,r_m})e^{-2\zeta_{q,r_m}\tau}+(\zeta_{q,r_m}+\mu_{q,r_m}). \qquad [6.69]$$

and for $m=i$

$$\mu_{q,r_i} = -\frac{1}{2}\left(\lambda_i+(\imath k-q)\eta_i\rho_{i,r}b_i\right), \qquad [6.70]$$
$$\zeta_{q,r_i} = \frac{1}{2}\left[4\mu_{q,r_i}^2+2\eta_i^2(\varphi_q^{r_i}(k)b_i^2-q+\imath\, k_i)\right]^{1/2}, \qquad [6.71]$$

for $m=j$

$$\mu_{q,r_j} = -\frac{1}{2}\left(\lambda_j+(q-\imath k)\eta_i\rho_{i,r}b_i\right), \qquad [6.72]$$
$$\zeta_{q,r_i} = \frac{1}{2}\left[4\mu_{q,r_i}^2+2\eta_i^2(\varphi_q^{r_i}(k)b_i^2+q-\imath\, k_i)\right]^{1/2}, \qquad [6.73]$$

where $\varphi_q^{v_n}(k)\,,\varphi_q^{r_i}(k)\,,\varphi_q^{r_j}(k)$ and $a_n^{i,j}\,,b_i\,,b_j$ are defined in Eqs.(6.52), (6.53) and (6.54).

Transition Probability Density Function Of Multiscale Heston CIR Model

Let us derive the joint transition probability density function p_f in the case of $\alpha=1/2$, that is when the CIR interest rate model is considered. Considering integration on Eq.(6.50) for $A(\tau,k,\underline{l},\underline{\xi})$, we obtain:

$$\begin{aligned} &A(\tau,k,\underline{l},\underline{\xi}) \\ &= -\sum_{n=1}^{d}\frac{2\chi_n v_n^*}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n) - \sum_{m=i}^{j}\frac{2\lambda_m\theta_m}{\eta_m^2}\ln C_{r_m}(\tau,k,\xi_m), \\ &= -\sum_{n=1}^{d}\frac{2\chi_n v_n^*}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n) - \frac{2\lambda_i\theta_i}{\eta_i^2}\ln C_{r_i}(\tau,k,\xi_i) - \frac{2\lambda_j\theta_j}{\eta_j^2}\ln C_{r_j}(\tau,k,\xi_j) \end{aligned} \qquad [6.74]$$

Hence the function $g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi})$ in Eq.(7.6) is given by:

$$g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) = \prod_{n=1}^{d} \left(e^{-\frac{2\chi_n v_n^*}{\gamma_n^2} \ln C_{v_n}(\tau,k,l_n)} \, e^{-\frac{2v_n}{\gamma_n^2} \frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \right) \prod_{m\in\{i,j\}} \Big(e^{-\frac{2\lambda_m\theta_m}{\eta_m^2} \ln C_{r_m}(\tau,k,\xi_m)} e^{-\frac{2r_m}{\eta_m^2} \frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \Big), \quad [6.75]$$

$$(v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau > 0.$$

In order to get an explicit expression for $f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k)$ in Eq.(6.46), that is the inverse Fourier transform of $g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi})$ with respect to the variables $\underline{v}'$ and $\underline{r}'$, we have to compute the following integrals:

$$L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl_n e^{\imath l_n v_n'} \, e^{-\frac{2\chi_n \overline{v}_n}{\gamma_n^2} \ln C_{v_n}(\tau,k,l_n)} \, e^{-\frac{2v_n}{\gamma_n^2} \frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \quad [6.76]$$

$$L_{r_m}(\tau, r_m, r_m', k \mid \underline{\Theta}_{r_m}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi_m \, e^{\imath \xi_m r_m'} e^{-\frac{2\lambda_m\theta_m}{\eta_m^2} \ln C_{r_m}(\tau,k,\xi_m)} e^{-\frac{2r_m}{\eta_m^2} \frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \quad [6.77]$$

Let us show how to compute the integral appearing in (6.76) and (6.77) analytically by using Eqs.(6.59) and (6.63) with the change of variable $l_n' = -l_n \frac{(\gamma_n)^2}{2} \frac{s_{q,v_n,g}}{s_{q,v_n,b}}$, and the following equality:

$$s_{q,v_n,d} s_{q,v_n,b} = \left(\zeta_{q,v_n}^2 - \mu_{v_n}^2\right) s_{q,v_n,g}^2 - \imath\, l_n' \left(\frac{8\zeta_{v_n}^2 e^{-2\zeta_{v_n}\tau}}{1 - 2\imath\, l_n'} \right), \quad [6.78]$$

We can re-write $L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n})$ as follows:

$$L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) = \frac{1}{2\pi} M_{q,v_n} e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n}) + (\mu_{q,v_n}+\zeta_{q,v_n})\tau]} \cdot$$
$$e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2 - \mu_{q,v_n}^2) s_{q,v_n,g}/s_{q,v_n,b}} \int_{-\infty}^{+\infty} dl_n' \, e^{-(\imath\, l_n') M_{q,v_n}} e^{-(2\chi_n v_i^*/\gamma_n^2)\ln(1-\imath\, l_n')} e^{\frac{(M_{q,v_n}\tilde{v}_{q,n})\imath\, l_n'}{1-\imath\, l_n'}}, \quad [6.79]$$

where

$$M_{q,v_n} = \frac{2}{\gamma_n^2}\frac{s_{q,v_n,b}}{s_{q,v_n,g}}, \quad \tilde{v}_{q,n} = \frac{4(\zeta_{q,v_n})^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,b}^2}, \ M_{q,v_n}\tilde{v}_{q,n} = \frac{8}{\gamma_n^2}\frac{\zeta_{q,v_n}^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,g} s_{q,v_n,b}}. \quad [6.80]$$

Now using formula n.34 on p.156 in Oberhettinger 1973, we obtain:

$$L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) = e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n}) + (\mu_{q,v_n}+\zeta_{q,v_n})\tau]} \, e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2 - \mu_{q,v_n}^2) s_{q,v_n,g}/s_{q,v_n,b}} M_{q,v_n}$$
$$(M_{q,v_n}\tilde{v}_{q,n})^{-\nu_{q,v_n}/2} (M_{q,v_n} v_n')^{\nu_{q,v_n}/2} e^{-M_{q,v_n}\tilde{v}_{q,n}} e^{-M_{q,v_n} v_n'} I_{\nu_{q,v_n}}(2\tilde{M}_{q,v_n}(\tilde{v}_{q,n} v_n')^{1/2}),$$
$$\text{, with },\ v_n, v_n' > 0,\ k \in \mathbb{R}. \quad [6.81]$$

where $\nu_{q,v_n} = 2\chi_n v_n^*/\gamma_n^2 - 1$, and $I_{\nu_{q,v_n}}$ is the modified Bessel function of order ν_{q,v_n} (see, for example, Abramowitz and Stegun, 1970). Analogously, we obtain:

$$\begin{aligned}
& L_{r_m}(\tau, r_m, r'_m, k \mid \underline{\Theta}_{r_m}) \\
&= e^{-(2\lambda_m\theta_m/\eta_m^2)[\ln(s_{q,r_m,b}/2\zeta_{q,r_m})+(\mu_{q,r_m}+\zeta_{q,r_m})\tau]}\, e^{-(2r_m/\eta_m^2)(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/s_{q,r_m,b}} M_{q,r_m} \\
& (M_{q,r_m}\tilde{r}_{q,n})^{-\nu_{q,r_m}/2}\,(M_{q,r_m} r'_m)^{\nu_{q,r_m}/2}\, e^{-M_{q,r_m}\tilde{r}} e^{-M_{q,r_m} r'_m} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,m} r'_m)^{1/2}), \\
& \quad \text{with}\,,\ r_m, r'_m > 0,\ k \in \mathbb{R}, \qquad [6.82]
\end{aligned}$$

where

$$M_{q,r_m} = \frac{2}{\eta_m^2}\frac{s_{q,r_m,b}}{s_{q,r_m,g}}, \quad \tilde{r}_{q,m} = \frac{4(\zeta_{q,r_m})^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,b}^2}\,, M_{q,r_m}\tilde{r}_{q,m} = \frac{8}{\eta_m^2}\frac{\zeta_{q,r_m}^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,g} s_{q,r_n,b}}\,,$$

$$\text{and } \nu_{q,r_m} = 2\lambda_m\theta_m/\eta_m^2 - 1\,, \quad m = i, j \qquad [6.83]$$

[6.84]

From Eq.(6.45), we can obtain:

$$\begin{aligned}
& f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) \\
= & \prod_{n=2}^{d} L_{v_n}(\tau, v_n, v'_n, k) \cdot L_{r_i}(\tau, r_i, r'_i, k) L_{r_j}(\tau, r_j, r'_j, k) \\
= & e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})\tau/\gamma_n^2} \\
& \cdot e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\, e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})\tau/\eta_m^2} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\, e^{-\sum\limits_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v'_n)} \cdot \\
& \quad \prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v'_n}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2} \cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\, v'_n)^{1/2}\right)\right]\Bigg\} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum\limits_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r'_m)} \cdot \\
& \quad \prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r'_m}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\, r'_m)^{1/2}\right)\right]\Bigg\}, \qquad [6.85] \\
& (x, v_n, r_m),\ (x', v'_n, r'_m) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t, t' \geq 0,\ t' - t > 0.
\end{aligned}$$

Using Eq.(6.43), we obtain the probability density function $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v'}, \underline{r'}, t')$ as follows:

$$
\begin{aligned}
&p_f(x, \underline{v}, \underline{r}, t, x', \underline{v'}, \underline{r'}, t') = e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)} \cdot \Bigg\{ \\
&\cdot e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2} \\
&\cdot e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\, e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2} \\
&\cdot\Bigg\{ e^{-2\sum\limits_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\, e^{-\sum\limits_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v_n')} \cdot \\
&\quad \prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v_n'}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2} \cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\,v_n')^{1/2}\right)\right]\Bigg\} \\
&\cdot\Bigg\{ e^{-2\sum\limits_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum\limits_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r_m')} \cdot \\
&\quad \prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r_m'}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\,r_m')^{1/2}\right)\right]\Bigg\}\Bigg\}, \qquad [6.86] \\
&(x, v_n, r_m),\, (x', v_n', r_m') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t, t' \geq 0,\ t'-t > 0.
\end{aligned}
$$

The following results are remarkable (see Abramowitz and Stegun 1970 pp. 375 and 486):

$$
\begin{aligned}
&P_{p,v_n}(\tau, v_n, k) \\
&= \int_0^{+\infty} dv_n'(v_n')^{\nu_{q,v_n}/2} I_{\nu_{q,v_n}}(2M_{q,v_n}(\tilde{v}_{q,v_n}v_n')^{1/2})e^{-M_{q,v_n}\,v_n'} = \frac{(\tilde{v}_{q,n})^{\nu_{q,v_n}/2}}{M_{q,v_n}}e^{M_{q,v_n}\tilde{v}_{q,n}}, \\
&\qquad v_n > 0,\ k \in \mathbb{R}, \qquad [6.87]
\end{aligned}
$$

$$
\begin{aligned}
&P_{p,r_m}(\tau, r_m, k) \\
&= \int_0^{+\infty} dr_m'(r_m')^{\nu_{q,r_m}/2} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,r_m}r_m')^{1/2})e^{-M_{q,r_m}\,r_m'} = \frac{(\tilde{r}_{q,m})^{\nu_{q,r_m}/2}}{M_{q,r_m}}e^{M_{q,r_m}\tilde{r}_{q,m}}, \\
&\qquad r_m > 0,\ k \in \mathbb{R}, \qquad [6.88]
\end{aligned}
$$

Using Eq.(6.87), we integrate the joint probability density function over the future variance $\underline{v}'$ to find the marginal density for $(x', \underline{r}')$ as follows:

$$D_v(x,\underline{v},\underline{r},t,\underline{x}',\underline{r}',t') = \prod_{n=1}^{d}\left(\int_0^{+\infty} dv'_n\, p_f(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')\right)$$
$$= e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\cdot\Bigg\{$$
$$\cdot e^{-2\sum_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2}$$
$$\cdot e^{-2\sum_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\, e^{-2\sum_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2}$$
$$\cdot\Bigg\{ e^{-2\sum_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\Bigg\}$$
$$\cdot\Bigg\{ e^{-2\sum_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r'_m)}\,\cdot$$
$$\prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r'_m}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\, r'_m)^{1/2}\right)\right]\Bigg\}\Bigg\}, \qquad [6.89]$$
$$(x,v_n,r_m),\ (x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ t'-t>0.$$

6.2.2 Numéraire Invariance

In derivatives market, it is necessary to consider the risk neutral world and no-arbitrage condition. Thus, it is necessary to study SDE model under risk neutral measure.

Considering a Brownian motion vector $\mathbf{W}_t^{p,v(Q^i)}$ and $\mathbf{W}_t^{p,r(Q^i)}$ under measure Q^i by imposing the Q^i-martingale property and Girsanov Theorem (in Section 1.2.6), we obtain:

$$d\mathbf{W}_t^{p,v(Q^i)} = d\mathbf{W}_t^{p,v} + \sqrt{\mathrm{Diag}\mathbf{V}}\mathbf{a}^i dt \qquad [6.90]$$
$$\begin{aligned} d\mathbf{W}_t^{p,r(Q^i)} &= d\mathbf{W}_t^{p,r} + [\mathrm{Diag}(\mathbf{R}_{i,j})]^\alpha (b_i,0)^{\mathrm{T}}dt \\ &= d\mathbf{W}_t^{p,r} + (r_i^\alpha b^i,0)^{\mathrm{T}}dt \end{aligned} \qquad [6.91]$$

Since $d\mathbf{W}_t^{p,r(Q^i)} = \left(dW_i^{p,r(Q^i)}(t)\,,\ dW_j^{p,r(Q^i)}(t)\right)^{\mathrm{T}}$, Eq.(6.91) can be separated as follows:

$$dW_i^{p,r(Q^i)}(t) = dW_i^{p,r} + r_i^\alpha b_i dt\,, \qquad [6.92]$$
$$dW_j^{p,r(Q^i)}(t) = dW_j^{p,r} \qquad [6.93]$$

Substituting Eqs.(6.90) and (6.92) into Eq.(6.17), a SDE under risk measure Q^i is obtained as follows:

$$\frac{dS_t^{i,j}}{S_t^{i,j}} = \left(r^i - r^j\right) dt + (a^i - a^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}(\mathbf{V})} d\mathbf{W}_t^{p,v(Q^i)} + (b_i, -b_j) \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r(Q^i)}$$
$$= \left(r^i - r^j\right) dt + (a^i - a^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}(\mathbf{V})} d\mathbf{W}_t^{p,v(Q^i)} + b_i r_i^{\alpha} dW_i^{p,r(Q^i)} - b_j r_j^{\alpha} dW_j^{p,r(Q^i)}\,, \quad [6.94]$$
$$t > 0,\ i, j = 1, \ldots, N\,,\ \alpha = 0,\ \frac{1}{2}$$

It is worth to remark that Q^i-measure is clearly the risk neutral measure. Because in each monetary index i, the money market accounts accrues interest based on interest rate r^i . Thus, cash bond price $B^i(r^i, t)$ for currency i is defined as follows:

$$\frac{dB^i}{B^i}(t) = r_t^i\, dt \qquad [6.95]$$

hence

$$B^i(t) = e^{\int_0^t r_i(s) ds} \qquad [6.96]$$

Similarly, we could define cash bond pricing $B^j(t) = e^{\int_0^t r_j(s) ds}$ for currency j. Furthermore, let us define inter-currency cash bond price as follows:

$$\begin{aligned} B^{i,j}(t) &= \frac{B^i(t)}{B^i(t)} \\ &= e^{\int_0^t [r_i(s) - r_j(s)] ds} \end{aligned} \qquad [6.97]$$

Symmetrically, the following holds:

$$\begin{aligned} B^{j,i}(t) &= [B^{i,j}(t)]^{-1} \\ &= e^{\int_0^t -[r_i(s) - r_j(s)] ds} \end{aligned} \qquad [6.98]$$

hence

$$\frac{dB^{j,i}}{B^{j,i}} = -[r_i(t) - r_j(t)] dt \qquad [6.99]$$

Lemma 6.2.2. *Under the risk neutral measure Q^i, $S^{i,j}(t)/B^{i,j}(t)$ or $S^{i,j}(t) \cdot B^{j,i}(t)$ is a martingale process.*

Proof. Here is the simple proof that $S^{i,j}(t) \cdot B^{j,i}(t)$ is a martingale process, while the proof of $S^{i,j}(t)/B^{i,j}(t)$ can be deduced analogously.

$$\begin{aligned} \frac{d(S^{i,j}B^{j,i})}{S^{i,j}B^{j,i}} &= \frac{dS^{i,j}B^{j,i} + S^{i,j}dB^{j,i}}{S^{i,j}B^{j,i}} \\ &= \frac{dS^{i,j}}{S^{i,j}} + \frac{dB^{j,i}}{B^{j,i}} \\ &= \frac{dS^{i,j}}{S^{i,j}} - [r_i(t) - r_j(t)]dt \end{aligned} \qquad [6.100]$$

Substituting Eq.(6.94) into (6.100), we obtain

$$\begin{aligned} d(S^{i,j}B^{j,i}) = S^{i,j}(a^i - a^j)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V})}d\mathbf{W}_t^{p,v(Q^i)} + S^{i,j}(b_i, -b_j)\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} dW_t^{p,r(Q^i)}, \\ t > 0,\ i,j = 1,\ldots,N\,,\ \alpha = 0,\ \frac{1}{2} \end{aligned} \qquad [6.101]$$

Since $d\mathbf{W}_t^{p,v(Q^i)}$ and $d\mathbf{W}_t^{p,r(Q^i)}$ are new Brownian Motion vectors under Q^i measure, $S^{i,j}(t) \cdot B^{j,i}(t)$ is a Q^i-martingale process by imposing the Q^i-local martingale property. □

The measurement change on Brownian Motion vectors also affect variance processes via the correlation coefficients ρ_n^v, $n = 1,\ldots,d$ as follows:

$$\begin{aligned} dW_{n,t}^{v(Q^i)} &= dW_{n,t}^v + \rho_n^v(\mathbf{e}^n)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V(t)})}a^i dt \\ &= dW_{n,t}^v + \rho_n^v\sqrt{v_{n,t}}a_n^i dt \end{aligned} \qquad [6.102]$$

where $\mathbf{e}^n$ is unit vector with $n-th$ element is 1. Moreover, $\{\mathbf{e}^n\,,\ n = 1,\ldots,d\}$ forms a set of d-dimensional mutually orthogonal unit vectors(or standard basis). Therefore, we obtain the following volatility process under Q^i measure:

$$\begin{aligned} dv_n(t) = \chi_n^{Q^i}(v_n^{*(Q^i)} - v_n(t))dt + \gamma_n\sqrt{v_n(t)}dW_{n,t}^{v(Q^i)}, \\ n = 1,\ldots,d\,,\ t > 0 \end{aligned} \qquad [6.103]$$

with coefficients defined as follows:

$$\rho_n^{v(Q^i)} = \rho_n^v \qquad [6.104]$$

$$\chi_n^{Q^i} = \chi_n + \gamma_n\rho_n^v a_n^i \qquad [6.105]$$

$$v_n^{*(Q^i)} = v_n^*\frac{\chi^{Q^i}}{\chi_n^{Q^i}} \qquad [6.106]$$

$$[6.107]$$

Similarly, interest rate process $dr_m\,, r = i, j$ under risk neutral measure Q^i satisfies the following dynamics:

$$dW_{m,t}^{r(Q^i)} \quad = \quad dW_{m,t}^{r} + \rho_m^r (\mathbf{e}^m)^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} (b_i, 0)^{\mathrm{T}} dt \qquad [6.108]$$

or

$$dW_{m,t}^{r(Q^i)} = \begin{cases} dW_{i,t}^{r} + \rho_i^r r_i^{\alpha} b_i dt & \text{if } m = i \\ dW_{j,t}^{r} & \text{if } m = j \end{cases}$$

where $\mathbf{e}^m\,, m = i, j$ is the unit vector, and $\rho_m^r\;, m = i, j$ is the correlation coefficient defined before. Therefore, the following interest rate process under Q^i measure are obtained:

$$\begin{aligned} dr_m(t) = \lambda_m^{Q^i}(\theta_m^{Q^i} - r_m(t))dt + \eta_m\, r_m^{\alpha}(t) dW_{m,t}^{r(Q^i)}, \\ t > 0 \quad m \in \{i, j\} \end{aligned} \qquad [6.109]$$

with coefficients as follows:

- when $m = i$

$$\rho_i^{r(Q^i)} = \rho_i^r \qquad [6.110]$$

$$\lambda_i^{Q^i} = \lambda_i + \eta_i \rho_i^r b_i \qquad [6.111]$$

$$\theta_i^{Q^i} = \theta_i \frac{\lambda_i}{\lambda_i^{Q^i}} \qquad [6.112]$$

- when $m = j$

$$\rho_j^{r(Q^i)} = \rho_j^r \qquad [6.113]$$

$$\lambda_j^{Q^i} = \lambda_j \qquad [6.114]$$

$$\theta_j^{Q^i} = \theta_j \qquad [6.115]$$

From now on, let us write Q instead of Q^i for abbreviation. Moreover, the log-price of FX rate $x_t^{i,j} = \ln S_t^{i,j}$, $t > 0$ under Q measure satisfies the following SDE:

$$\begin{aligned} dx^{i,j} = & \left[(r_i - r_j) - \frac{1}{2}(a^i - a^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(a^i - a^j) - \frac{1}{2}(b_i, -b_j)\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{2\alpha}(b_i, -b_j)^{\mathrm{T}}\right] dt \\ & + (a^i - a^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}\mathbf{V}}\, d\mathbf{W}_t^{p,v(Q)} + (b^i, -b^j)\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r(Q)}, \\ & t > 0,\ \alpha = 0, \frac{1}{2}, \end{aligned} \qquad [6.116]$$

Obviously, Eq.(6.116) can be written in the following form:

$$
\begin{aligned}
dx^{i,j} &= \left[r_i - r_j - \frac{1}{2}\sum_{n=1}^{d}[a_n^i - a_n^j]^2 v_k - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}) \right] dt \\
&\quad + \sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v(Q)} + b_i r_i^{\alpha} dW_i^{p,r(Q)} - b_j r_j^{\alpha} dW_j^{p,r(Q)}\,, \quad [6.117] \\
&\qquad t > 0,\ \alpha = 0, \frac{1}{2},
\end{aligned}
$$

Therefore, a system of equations defined on risk-neutral measure Q^i is obtained as follows:

$$
\begin{aligned}
dx^{i,j} &= \left[r_i - r_j - \frac{1}{2}\sum_{n=1}^{d}[a_n^i - a_n^j]^2 v_k - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}) \right] dt \\
&\quad + \sum_{n=1}^{d}(a_n^i - a_n^j)\sqrt{v_n}dW_k^{p,v(Q)} + b_i r_i^{\alpha} dW_i^{p,r(Q)} - b_j r_j^{\alpha} dW_j^{p,r(Q)}, \\
&\qquad t > 0,\ n = 1, \ldots, d\,,\ \alpha = 0, \frac{1}{2}, \quad [6.118] \\
dv_n(t) &= \chi_n^Q (v_n^{*(Q)} - v_n(t))dt + \gamma_n \sqrt{v_n(t)}dW_{n,t}^{v(Q)}, \qquad n = 1, \ldots, d\,,\ t > 0; \quad [6.119] \\
dr_m(t) &= \lambda_m^Q(\theta_m^Q - r_m(t))dt + \eta_m\, r_m^{\alpha}(t) dW_{m,t}^{r(Q)}, \qquad t > 0 \quad m = \{i, j\}\,; \quad [6.120]
\end{aligned}
$$

with the following correlation structure:

$$
\begin{aligned}
E(dW_{n,t}^{p,v(Q)} dW_{m,t}^{p,r(Q)}) &= 0,\ \ n = 1, \ldots d\,, m = i, j,\ t > 0, \quad [6.121] \\
E(dW_{n,t}^{p,v(Q)} dW_{l,t}^{v(Q)}) &= 0,\ \ n \neq l,\ n, l = 1, \ldots, d,\ t > 0, \quad [6.122] \\
E(dW_{n,t}^{p,v(Q)} dW_{n,t}^{v(Q)}) &= \rho_{n,v}dt,\ \ n = 1, \ldots, d\,,\ t > 0, \quad [6.123] \\
E(dW_{n,t}^{p,v(Q)} dW_{m,t}^{r(Q)}) &= 0,\ \ m = i, j;\ n = 1, \ldots, d,\ t > 0, \quad [6.124] \\
E(dW_{m,t}^{p,r(Q)} dW_{m,t}^{r(Q)}) &= \rho_{m,r}dt,\ m = i, j\, t > 0, \quad [6.125] \\
E(dW_{m,t}^{p,r(Q)} dW_{m',t}^{r}) &= 0,\ \ m \neq m',\ m, m' = i, j\ ,\ t > 0, \quad [6.126] \\
E(dW_{m,t}^{p,r(Q)} dW_{n,t}^{v(Q)}) &= 0,\ \ \ n = 1, \ldots d\,, m = i, j,\ t > 0, \quad [6.127] \\
E(dW_{m,t}^{r(Q)} dW_{n,t}^{v(Q)}) &= 0,\ \ n = 1, \ldots d\,, m = i, j,\ t > 0, \quad [6.128]
\end{aligned}
$$

6.2.3 The Model Treatment Under Risk-Neutral Measure

In this section, the model are treated under risk neutral measure. First of all, let us equipped Eqs.(6.118)-(6.120) with the initial condition:

$$\begin{aligned} x^{i,j}(0) &= \tilde{x}_0^{i,j}, & [6.129]\\ v_n(0) &= \tilde{v}_{n,0}, & [6.130]\\ r_m(0) &= \tilde{r}_{m,0}, & [6.131] \end{aligned}$$

where $\tilde{x}_0^{i,j}$, $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$ are random variables that are assumed to be concentrated in a point with probability one. For simplicity, we identify the random variables $\tilde{x}_0^{i,j}$, $\tilde{v}_{m,0}$, $\tilde{r}_{m,0}$ with the points where they are concentrated. $\tilde{v}_{m,0}$, $\tilde{r}_{m,0}$, χ_k^Q, λ_m^Q, γ_n, η_m, $v_n^{*(Q)}$, θ_m^Q are assumed to be positive constants. In addition, Feller condition is considered here, i.e. $\frac{2\chi_k v_n^*}{\gamma_k^2} > 1$ and $\frac{2\lambda_m \theta_m}{\eta_m^2} > 1$.

Let $p_f^Q(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$, $(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2$, t, $t' \geq 0$, $t' - t > 0$ be the transition probability density function under risk neutral measure Q associated with the stochastic differential system (6.118), (6.119), (6.120), and (6.31). That is, the probability density function having $x' = x_{t'}^{i,j}$, $\underline{v}' = (v_1', \ldots, v_d')^{\mathrm{T}}$, $\underline{r}' = (r_i', r_j')^{\mathrm{T}}$, given that $x = x_t^{i,j}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, when $t' - t > 0$.

In analogy with Lipton (2001) (pages 602–605), this transition probability density function $p_f^Q(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$ as a function of the "past" variables $(x, \underline{v}, \underline{r}, t)$ satisfies the following backward Kolmogorov equation:

$$\begin{aligned} -\frac{\partial p_f^Q}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_f^Q}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_f^Q}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_f^Q}{\partial r_i^2} \\ &+ \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_f^Q}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_f^Q}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_f^Q}{\partial x \partial r_i^{2\alpha}} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_f^Q}{\partial x \partial r_j} \\ &+ \sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_f^Q}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_f^Q}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_f^Q}{\partial r_j} \\ &+ \left((r_i - r_j) - \frac{1}{2}\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha})\right)\frac{\partial p_f^Q}{\partial x} \qquad [6.132] \\ &\quad (x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \ 0 \leq t < t', \end{aligned}$$

with final condition:

$$
\begin{aligned}
p_f^Q(t,x,\underline{v},\underline{r},t',x',\underline{v'},\underline{r'}|t'=t) &= \delta(x'-x)\prod_{m\in\{i,j\}}\delta(r'_m-r_m)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n) \\
&= \delta(x'-x)\cdot\delta(r'_i-r_i)\delta(r'_j-r_j)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n)\,, \qquad [6.133]
\end{aligned}
$$

$\forall n\in\{1,\ldots,d\}\,\text{and}\, m\in\{i,j\}\,,\quad (x,v_n,r_m),(x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\, t\geq 0,$

and the appropriate boundary conditions. The analytical treatment is similar to Section 6.2.1, and the detail deduction is shown in Appedix C, where the following transition probability density function is obtained using a suitable parametrization.

$$
\begin{aligned}
&p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v'},\underline{r'},t') \\
&= e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}}dk\,e^{\imath k(x'-x)}\underline{L_v^Q}(t'-t,\underline{v},\underline{v'},k\mid\underline{\Theta_v})\cdot\underline{L_r^Q}(t'-t,\underline{r},\underline{r'},k\mid\underline{\Theta_r})\,, \\
&(x,v_n,r_m),\,(x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\,t,t'\geq 0,\,t'-t>0. \qquad [6.134]
\end{aligned}
$$

where $\imath$ stands for the imaginary unit. $\underline{L_v^Q}(t'-t,\underline{v},\underline{v'},k\mid\underline{\Theta_v})$ and $\underline{L_r^Q}(t'-t,\underline{r},\underline{r'},k\mid\underline{\Theta_r})$ are explicitly known functions given in Eqs.(6.135) and (6.136) which both depend on the product of the modified Bessel functions, $\prod_n^d I_{\nu^Q_{q,v_n}}$ and $\prod_{m\in\{i,j\}} I_{\nu^Q_{q,r_m}}$ (see, for example, Abramowitz and Stegun, 1970). $\nu^Q_{q,v_n}=(2\chi v_n^{*(Q)}/\gamma_n^2)-1$ and $\nu_{q,r_m}=(\lambda\theta_m^Q/\eta_m^2)-1$ are positive real indices under the conditions $2\chi v_n^{*(Q)}/\gamma_n^2>1$ and $2\lambda\theta_m^Q/\eta_m^2>1$.

As mentioned in Chapter 2 and Section 6.2.1, positive indices imply non negative modified Bessel functions which guarantee that the function p_f^Q given in Eq.(6.134) is a probability density function with respect to the future variables. Moreover, these conditions guarantee positive values of the variance and interest rate processes $\forall t>0$ (with probability one) given that the initial stochastic conditions $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$ are positive (with probability one).

$$\underline{L_{v}^{Q}}(t'-t,\underline{v},\underline{v'},k\mid\underline{\Theta_{v}})=\prod_{n=1}^{d}L_{v_n}^{Q}(t'-t,v_n,v'_n,k\mid\underline{\Theta}_{v_n})$$
$$=e^{-2\sum_{n=1}^{d}\chi_n v_n^{*}\ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}e^{-2\sum_{n=1}^{d}\chi_n v_n^{*}(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2}$$
$$\cdot\left\{e^{-2\sum_{n=1}^{d}v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}e^{-\sum_{n=1}^{d}M_{q,v_n}(\tilde{v}_{q,n}+v'_n)}\cdot\right.$$
$$\left.\prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v'_n}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2}\cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\,v'_n)^{1/2}\right)\right]\right\}\quad[6.135]$$

$$\underline{L_{r}^{Q}}(t'-t,\underline{r},\underline{r'},k\mid\underline{\Theta_{r}})=\prod_{m\in\{i,j\}}L_{r_m}^{Q}(t'-t,r_m,r'_m,k\mid\underline{\Theta}_{r_m})$$
$$=e^{-2\sum_{m\in\{i,j\}}\lambda_m\theta_m\ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}e^{-2\sum_{m\in\{i,j\}}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2}$$
$$\cdot\left\{e^{-2\sum_{m\in\{i,j\}}r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}e^{-\sum_{m\in\{i,j\}}M_{q,r_m}(\tilde{r}_{q,m}+r'_m)}\cdot\right.$$
$$\left.\prod_{m\in\{i,j\}}\left[M_{q,r_m}\left(\frac{r'_m}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\,r'_m)^{1/2}\right)\right]\right\},\quad[6.136]$$

Furthermore, let us remark that formula (6.134) can be interpreted as the inverse Fourier transform of the convolution of the probability density functions associated with the stochastic processes described by Eqs.(6.116)- (6.120). It is worth noting that the integrals of $\underline{L_v^Q}$ and $\underline{L_r^Q}$ with respect the future variables $\underline{v'}$ and $\underline{r'}$ are given by elementary functions $\underline{W_{v,q}^{z(Q)}}$ and $\underline{W_{r,q}^{z(Q)}}$ when $z=0$.

$$\underline{W_{v,q}^{z(Q)}}(t'-t,\underline{v},k;\underline{\Theta_v})=\prod_{n=1}^{d}\int_0^{+\infty}dv'_n\,(v'_n)^z L_{v,q}(t'-t,v_n,v'_n,k;\underline{\Theta}_v),\quad(6.137)$$

$$\underline{W_{r,q}^{z(Q)}}(t'-t,r,k;\underline{\Theta_r})=\prod_{m\in\{i,j\}}\int_0^{+\infty}dr'_m\,(r'_m)^z L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r),\quad(6.138)$$

Especially, when $z=0$, we obtain:

$$\underline{W_{v,q}^{0(Q)}}(t'-t,\underline{v},k;\underline{\Theta_v})=e^{-\sum_{n=1}^{d}(2\chi_n^Q v_n^{*(Q)}/\gamma_n^2)\ln(s_{q,v_n,b}/2\zeta_{q,v_n})}e^{-\sum_{n=1}^{d}(2\chi_n^Q v_n^{*(Q)}/\gamma_n^2)(\mu_{q,v_n}+\zeta_{q,v_n})T}\cdot$$
$$e^{-\sum_{n=1}^{d}(2v_n(0)/\gamma_n^2)(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/s_{q,v_n,b}},\quad[6.139]$$

$$\underline{W^{0(Q)}_{r,q}}(t'-t,\underline{r},k;\underline{\Theta_r}) = e^{-\sum\limits_{m\in\{i,j\}}(2\lambda^Q_m\theta^Q_m/\eta^2_m)\ln(s_{q,r_m,b}/2\zeta_{q,r_m})}\, e^{-\sum\limits_{m\in\{i,j\}}(2\lambda^Q_m\theta^Q_m/\eta^2_m)(\mu_{q,r_m}+\zeta_{q,r_m})T}\,\cdot$$
$$e^{-\sum\limits_{m\in\{i,j\}}(2r_m(0)/\eta^2_m)(\zeta^2_{q,r_m}-\mu^2_{q,r_m})s_{q,r_m,g}/s_{q,r_m,b}}, \quad [6.140]$$

Let us highlight that formulas (6.139) and (6.140) are elementary formulas that do not involve integrals. These functions and the function $L_{r,q}$ can be used to get an integral representation formula for the marginal probability density function, $D^Q_{v,q}(x,\underline{v},\underline{r},t,x',\underline{r'},t')$, of the future variables $(x',\underline{r'})$.

$$D^Q_{v,q}(x,\underline{v},\underline{r},t,x',\underline{r'},t') = \prod_{n=0}^{d}\int_0^{+\infty} dv'_n\, p^Q_f(x,\underline{v},\underline{r},t,x',\underline{v'},\underline{r'},t')$$
$$= e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)}\underline{W^{z(Q)}_{v,q}}(t'-t,\underline{v},k;\underline{\Theta_v})\underline{L^Q_r}(t'-t,\underline{r},\underline{r'},k\mid\underline{\Theta_r}), \quad [6.141]$$

and for the marginal probability density function, $D^Q_{v,r,q}(x,\underline{v},\underline{r},t,x',,t')$, of the price variable x':

$$D^Q_{v,r,q}(x,\underline{v},\underline{r},t,x',t') = \prod_{m\in\{i,j\}}\int_0^{+\infty} dr'_m \prod_{n=0}^{d}\int_0^{+\infty} dv'_n\, p^Q_f(x,\underline{v},\underline{r},t,x',\underline{v'},\underline{r'},t')$$
$$= e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)}\underline{W^{z(Q)}_{v,q}}(t'-t,\underline{v},k;\underline{\Theta_v})\underline{W^{z(Q)}_{r,q}}(t'-t,r,k;\underline{\Theta_r})\,. \quad [6.142]$$

It is worth to highlight that in formulas (6.141) and (6.142) the variables x, v, r are the initial values of the log-price, the stochastic variance and the stochastic interest rate respectively. The marginal probability density function, $D_{v,q}$, can be used to price European call and put options with payoff functions independent of the variance process in the framework of stochastic interest rates. Thus, $D^Q_{v,q}$ is used to deduce formulas for European call and put vanilla options.

6.2.4 Integral formulas for Vanilla Foreign Exchange Call and Put Options

Foreign exchange (FX) future (or currency future) is an important futures contract in derivatives market. The holders of FX future are obliged to exchange one currency for another at a fixed price (exchange rate) on a specified date. Thus, the corresponding call and put options on FX future are popular risk hedging tools in fluctuated FX market. In the framework of the model (6.117)-(6.128), integral formulas are derived to approximate the prices of the corresponding European call and put options with strike price

$E > 0$ and maturity time T. This is done using the no arbitrage pricing theory associated with one of the currencies involved, for instance currency i. As illustrated in De Col,Gnoatto and Grasselli 2013, the option price is computed as the expected value of a discounted payoff with respect to an equivalent martingale measure known as a risk-neutral measure (see, for example, Duffie, 2001; Schoutens, 2003, Wong, 2006, Grzelak, 2011). That is, let S_0 be the spot price (future) at time zero we can compute the prices of European call and put options with strike price E and maturity time T as follows:

$$C(S_0^{i,j}, T, E, \underline{r}(0), \underline{v}(0)) = E^Q \left(\frac{(S_0^{i,j} e^{x_T} - E)_+}{e^{\int_0^T r_i(t)\, dt}} \right), \qquad [6.143]$$

$$T > 0, S_0^{i,j} > 0,\ v_n(0)\, , n = 1, \ldots, d\, , r_m(0) > 0, m = i, j\, , \qquad [6.144]$$

$$P(S_0^{i,j}, T, E, \underline{r}(0), \underline{v}(0)) = E^Q \left(\frac{(E - S_0^{i,j} e^{x_T})_+}{e^{\int_0^T r_i(t)\, dt}} \right), \qquad [6.145]$$

$$T > 0, S_0^{i,j} > 0,\ v_n(0)\, , n = 1, \ldots, d\, , r_m(0) > 0, m = i, j \qquad [6.146]$$

where $(\,\cdot\,)_+ = max\{\,\cdot\,,\,0\}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$ and the expectation is taken under the risk-neutral measure Q. In contrast to physical measure, another advantage of using risk-neutral measure lies in the fact that it is not necessary introduce the risk premium parameters.

Please note that $v_n(0)\, , n = 1, \ldots d$ and $r_m(0)\, , m = i, j$ are not observable in the market so that we consider them as model parameters that must be estimated. The numerical evaluation of formula (6.143) is very time consuming. We get a formula to evaluate these prices approximating the stochastic integral that defines the discount factor as follows:

$$e^{-\int_0^T r_i(t) dt} \approx e^{-r_i(t) \frac{T}{(1+e^{\lambda_i T})} - r_T \frac{T e^{\lambda_i T}}{(1+e^{\lambda_i T})}}. \qquad [6.147]$$

Roughly speaking, formula (6.147) has been obtained approximating r_t as a suitable weighted sum of short rate r_t at $t = 0$ and $t = T$. The choice of these weights is inspired by the analytical expression of zero-coupon bond given in Eq.(3.40). As shown in Section 3.4, this approximation works well also for long maturity. Furthermore, the use of formula (6.147) allows us to reduce the computation of the option prices to the evaluation of a one dimensional integral.

In fact, let $p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v}'\underline{r}',t')$, $(x,\underline{v},\underline{r},t),(x',\underline{v}',\underline{r}',t')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$, t, $t'\geq 0$, $\tau=t'-t>0$, be the transition probability density function of the stochastic process described by Eqs.(6.117)–(6.128). Using formula (6.134) for p_f^Q and (6.147), we obtain:

$$C_A(S_0^{i,j},T,E,\underline{r_0},\underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}}\int_{\ln(E/S_0^{i,j})}^{+\infty} dx'\, e^{-qx'}(S_0^{i,j}e^{x'}-E)$$
$$\int_0^{+\infty} dr_i' e^{-r_i'\frac{Te^{\lambda_i T}}{(1+e^{\lambda_i T})}}\int_0^{+\infty} dr_j' D_v(0,\underline{v_0},\underline{r_0},0,\underline{x}',\underline{r}',T), \quad [6.148]$$

$S_0^{i,j}$, T, $E>0$, $\quad \underline{r_0}=(r_i(0),r_j(0))^{\mathrm{T}}\in{\mathbb{R}^+}^2$, $\underline{v_0}=(v_1(0),\ldots,v_d(0))^{\mathrm{T}}\in{\mathbb{R}^+}^d$, $q>1$,

$$P_A(S_0^{i,j},T,E,\underline{r_0},\underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}}\int_{-\infty}^{\ln(E/S_0^{i,j})} dx'\, e^{-qx'}(E-S_0^{i,j}e^{x'})$$
$$\int_0^{+\infty} dr_i' e^{-r'\frac{Te^{\lambda_i T}}{(1+e^{\lambda_i T})}}\int_0^{+\infty} dr_j' D_v(0,\underline{v_0},\underline{r_0},0,x',r',T), \quad [6.149]$$

$S_0^{i,j}$, T, $E>0$, $\underline{r_0}=(r_i(0),r_j(0))^{\mathrm{T}}\in{\mathbb{R}^+}^2$, $\underline{v_0}=(v_1(0),\ldots,v_d(0))^{\mathrm{T}}\in{\mathbb{R}^+}^d$, $q<-1$,

where D_v is given in (6.141). Choosing $q>1$, we obtain:

$$\int_{\ln(E/S_0^{i,j})}^{+\infty} dx'\, e^{-q\,x'}(S_0^{i,j}e^{x'}-E)e^{\imath\,k\,x'} = \frac{S_0^{i,j}\left(\frac{S_0^{i,j}}{E}\right)^{q-1-\imath\,k}}{-k^2-(2q-1)\imath\,k+q(q-1)}, [6.150]$$

and choosing $q<-1$ we obtain:

$$\int_{-\infty}^{\ln(E/S_0^{i,j})} dx'\, e^{-q\,x'}(E-S_0^{i,j}e^{x'})e^{\imath\,k\,x'} = \frac{S_0^{i,j}\left(\frac{S_0^{i,j}}{E}\right)^{q-1-\imath\,k}}{-k^2-(2q-1)\imath\,k+q(q-1)}. [6.151]$$

In addition, the following formula holds (see Erdely et al. Vol I, 1954, p. 197 formula (18)).

$$\int_0^{\infty} t^{\frac{\nu}{2}} I_\nu(2\alpha^{\frac{1}{2}}t^{\frac{1}{2}})e^{-p\cdot t}dt = \alpha^{\frac{\nu}{2}}p^{-\nu-1}e^{\frac{\alpha}{p}}, \quad [6.152]$$

and this implies the following equality:

$$\int_0^{+\infty} dr_i'\,(r')^{\nu_{r_i}/2}\,e^{-(M_{q,r_i}+b)r'}I_{\nu_{r_i}}(2M_{q,r_i}(\tilde{r}_{q,i}r_i')^{1/2})$$
$$= \quad [(M_{q,r_i})^2\tilde{r}_{q,i}]^{\frac{\nu_{r_i}}{2}}(M_{q,r_i}+b)^{-\nu_{r_i}-1}e^{\frac{(M_{q,r_i})^2\tilde{r}_{q,i}}{(M_{q,r_i}+b)}}, \quad b\in\mathbb{R}. \quad [6.153]$$

Now using the expression of D_v given in Eq.(6.141), and Eq.(6.153), (6.150) with $q = 2$, we obtain the approximation C_A of the call option price C given in (6.143), that is:

$$C_A(S_0^{i,j}, T, E, \underline{r_0}, \underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}} \frac{S_0^{i,j}}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\dfrac{S_0^{i,j}}{E}\right)^{(1-\imath k)}}{-k^2 - 3\imath k + 2} \underline{W_{v,q}^{0(Q)}}(t'-t, \underline{v}, k; \underline{\Theta_r})$$

$$\underline{W_{r,q}^{0(Q)}}(t'-t, r, k; \underline{\Theta_r}) \left(\frac{M_{q,r_i}}{M_{q,r_i} + \frac{T e^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)^{\nu_{r_i}+1} e^{-\left(\frac{T e^{\lambda_i T}}{1+e^{\lambda_i T}}\right)\left(\frac{M_{q,r_i}\tilde{r}_{q,i}}{M_{q,r_i} + \frac{T e^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)}, \qquad [6.154]$$

$$n = 1, \ldots, d\,, m = i, j\,,\ (x, v_n, r_m),\ (x', v_n', r_m') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t, t' \geq 0,\ t' - t > 0\,, q = 2,$$

Proceeding in a similar way, we obtain the following approximation, P_A, of the put option price P:

$$P_A(S_0^{i,j}, T, E, \underline{r_0}, \underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}} \frac{S_0^{i,j}}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\dfrac{S_0^{i,j}}{E}\right)^{-(3+\imath k)}}{-k^2 + 5\imath k + 6} \underline{W_{v,q}^{0(Q)}}(t'-t, \underline{v}, k; \underline{\Theta_r})$$

$$\underline{W_{r,q}^{0(Q)}}(t'-t, r, k; \underline{\Theta_r}) \left(\frac{M_{q,r_i}}{M_{q,r_i} + \frac{T e^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)^{\nu_{r_i}+1} e^{-\left(\frac{T e^{\lambda_i T}}{1+e^{\lambda_i T}}\right)\left(\frac{M_{q,r_i}\tilde{r}_{q,i}}{M_{q,r_i} + \frac{T e^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)}, \qquad [6.155]$$

$$n = 1, \ldots, d\,, m = i, j\,,\ (x, v_n, r_m),\ (x', v_n', r_m') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t, t' \geq 0,\ t' - t > 0\,, q = -2,$$

Taking the limit $a^{i,j} \to 0^+$, $\lambda_m \to 0^+$, $\eta_n \to 0^+$ in Eqs.(6.154) and (6.155), we derive the following exact formulas for the price of the European call and put options of the Heston model:

$$C_H(S_0^{i,j}, T, E, \underline{r_0}, \underline{v_0})$$

$$= e^{-r_i(0)T}\, e^{2r_i(0)T}\, \frac{S_0^{i,j}}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\dfrac{S_0^{i,j}}{E}\right)^{(1-\imath k)} e^{-\imath k r_i(0)T}}{-k^2 - 3\imath k + 2} \underline{W_{v,q}^{0(Q)}}(t'-t, \underline{v}, k; \underline{\Theta_r})\,,$$

$$n = 1, \ldots, d\,, m = i, j\,,\ (x, v_n, r_m),\ (x', v_n', r_m') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t, t' \geq 0,\ t' - t > 0\,, q = \pm 2, \qquad [6.156]$$

$$
\begin{aligned}
&P_H(S_0^{i,j}, T, E, \underline{r_0}, \underline{v_0})\\
&= e^{-r_i(0)T}\, e^{-2r_i(0)T}\, \frac{S_0^{i,j}}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\dfrac{S_0^{i,j}}{E}\right)^{-(3+\imath k)} e^{-\imath k r_i(0)T}}{-k^2+5\imath k+6} \underline{W_{v,q}^{0(Q)}}(t'-t, \underline{v}, k; \underline{\Theta_r}) \,, \quad [6.157]\\
&\quad n = 1, \ldots, d\,, m = i, j\,,\ (x, v_n, r_m),\ (x', v_n', r_m') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\\
&\quad t, t' \geq 0,\ t'-t > 0\,, q = -2,
\end{aligned}
$$

It is worthy of note that the integrand functions appearing in formulas (6.154)–(6.157) are smooth functions whose integration do not require a specific care. This regularity is due to the specific approach used to derive them. The future study will focus on the application of this theoretical model. This includes the simulation study to compare the performance of different models in interpreting real data.

7 Appendix

7.1 Appendix A: The Analytical Treatment Of The HCIR model

In Appendix A, Let us derive an integral representation formula for the transition probability density function of the process described by Eqs.(3.19)-(3.21) and initial conditions (3.23). In addition, we deduce the moments of the price variable and the mixed moments. As mentioned in the Chapter 2, we assume that v_0^*, χ, λ, γ, η, v^*, θ are positive constants and that $\frac{2\chi v^*}{\gamma^2} > 1$ and $\frac{2\lambda\theta}{\eta^2} > 1$.

Note that since the coefficients of the Kolmogorov backward equation (3.24) and condition (3.25) are invariant by time translation and log-price translation, the function p_f can be expressed as follows:

$$p_f(x, v, r, t, x', v', r', t') = e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\, f(\tau, v, r, v', r', k),$$
$$(v, r), (v', r') \in \mathbb{R}^+ \times \mathbb{R}^+,\ k \in \mathbb{R},\ \tau > 0, q \in \mathbb{R}. \qquad [7.1]$$

The representation formula (7.1) for the transition probability density function p_f depends on a 'regularization' parameter, q, which plays a crucial role in deriving explicit formulas for option pricing and conditional mixed moments. Substituting Eq.(7.1) into (3.24) , we obtain that the function f is the solution of the following problem:

$$\frac{\partial f}{\partial \tau} = -\frac{k^2}{2}(\tilde{\psi} v + \Omega^2 r) f + \frac{1}{2}\gamma^2 v \frac{\partial^2 f}{\partial v^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 f}{\partial r^2} + [(-\imath\, k)\gamma\tilde{\rho}_{p,v} v + q\,\gamma\tilde{\rho}_{p,v} v]\,\frac{\partial f}{\partial v}$$
$$+ [(-\imath\, k)\eta\rho_{p,r}\Omega\, r + q\,\eta\,\rho_{p,r}\Omega\, r]\,\frac{\partial f}{\partial r} + \chi(v^* - v)\frac{\partial f}{\partial v} + \lambda(\theta - r)\frac{\partial f}{\partial r}$$
$$+ \left[\frac{v}{2}\left((q^2 - q)\tilde{\psi} - \imath\, k(2q-1)\tilde{\psi}\right) + \frac{r}{2}\left((q^2 - q)\Omega^2 + 2q - \imath\, k((2q-1)\Omega^2 + 2)\right)\right] f$$
$$(k, v, r) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ \tau > 0,\ q \in \mathbb{R} \qquad [7.2]$$

with the initial condition:

$$f(0, v, r, v', r', k) = \delta(v' - v)\delta(r' - r),\ (v, r), (v', r') \in \mathbb{R}^+ \times \mathbb{R}^+,\ k \in \mathbb{R}, \ [7.3]$$

where $\tilde{\psi}$ is given by Eq.(3.22) and $\tilde{\rho}_{p,v}$ is given as follows:

$$\tilde{\rho}_{p,v} = \rho_{p,v} + \Delta \ . \tag{7.4}$$

Now let us represent f as the inverse Fourier transform of the future variables (v', r') whose conjugate variables are denoted by (l, ξ), that is:

$$f(\tau, v, r, v', r', k) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dl\, e^{\imath l\, v'} \int_{-\infty}^{+\infty} d\xi\, e^{\imath\, \xi\, r'}\, g(\tau, v, r, k, l, \xi),$$
$$(v, r) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0. \qquad [7.5]$$

It is easy to see that the function g satisfies Eq.(7.2) with the following initial condition:

$$g(0, v, r, k, l, \xi) = e^{-\imath\, \xi\, r} e^{-\imath\, l\, v},$$
$$(v, r) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \qquad [7.6]$$

The coefficients of the partial differential operator appearing on the right hand side of Eq.(7.2) are first degree polynomials in v and r so that we seek a solution of problem (7.2), (7.3) in the following form (see Lipton 2001):

$$g(\tau, v, r, k, l, \xi) = e^{A(\tau,k,l,\xi)} e^{-v\, B_v(\tau,k,l)} e^{-r\, B_r(\tau,k,\xi)},$$
$$(v, r) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0. \qquad [7.7]$$

Substituting formula (7.7) into Eq.(7.2), we obtain that the functions A, B_v and B_r must satisfy the following ordinary differential equations for $(k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\tau > 0$:

$$\frac{dA}{d\tau}(\tau, k, l, \xi) = -\lambda\, \theta\, B_r(\tau, k, \xi) - \chi v^* B_v(\tau, k, l), \qquad [7.8]$$
$$\frac{dB_v}{d\tau}(\tau, k, l) = \varphi_q(k)\tilde{\psi} - (\chi + (\imath\, k - q)\, \gamma\, \tilde{\rho}_{p,v}) B_v(\tau, k, l) - \frac{\gamma^2}{2} B_v^2(\tau, k, l), \qquad [7.9]$$
$$\frac{dB_r}{d\tau}(\tau, k, \xi) = \varphi_q(k)\Omega^2 + \imath\, k - q - \frac{\eta^2}{2} B_r^2(\tau, k, \xi) - (\lambda + (\imath\, k - q)\Omega\, \rho_{p,r}\eta)\, B_r(\tau, k, \xi), \qquad [7.10]$$

with initial conditions:

$$A(0, k, l, \xi) = 0, \quad B_v(0, k, l) = \imath\, l, \quad B_r(0, k, \xi) = \imath\, \xi, \tag{7.11}$$

where φ_q is the quantity given by:

$$\varphi_q(k) = \frac{k^2}{2} + \imath \frac{k}{2}(2q - 1) - \frac{1}{2}(q^2 - q),\ \ k \in \mathbb{R}, \tag{7.12}$$

and the quantities $\tilde{\psi}$, $\tilde{\rho}_{p,v}$ are given by (3.22) and (7.4) respectively. Eqs.(7.9)–(7.10) are Riccati equations that can be easily solved by substituting their solutions into Eq.(7.8) and integrating with respect to τ in order to obtain A.
Let us solve Eq.(7.9), and the solution of Eq.(7.10) can be obtained analogously. We seek for the solution of Eq.(7.9) in the following form:

$$B_v(\tau, k, l) = \frac{2}{\gamma^2} \frac{\frac{d}{d\tau} C_v}{C_v}. \tag{7.13}$$

Replacing (7.13) into Eq.(7.9), we obtain C_v:

$$C_v(\tau, k, l) = e^{(\mu_{q,v}+\zeta_{q,v})\tau} \left[\frac{s_{q,v,b} + \imath\, l \frac{\gamma^2}{2} s_{q,v,g}}{2\zeta_{q,v}} \right], \qquad [7.14]$$

where $\mu_{q,v}$, $\zeta_{q,v}$, $s_{q,v,g}$, $s_{q,v,b}$ are given by:

$$\mu_{q,v} = -\frac{1}{2}(\chi + (\imath\, k - q)\, \gamma\, \tilde{\rho}_{p,v}), \qquad [7.15]$$

$$\zeta_{q,v} = \frac{1}{2}\left[4\mu_{q,v}^2 + 2\gamma^2 \varphi_q(k)\tilde{\psi}\right]^{1/2}, \qquad [7.16]$$

$$s_{q,v,g} = 1 - e^{-2\zeta_{q,v}\tau}, \qquad [7.17]$$

$$s_{q,v,b} = (\zeta_{q,v} + \mu_{q,v})e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} - \mu_{q,v}), \qquad [7.18]$$

and the quantities φ_q, $\tilde{\rho}_{p,v}$ in (7.15) are given by (7.12) and (7.4) respectively. Substituting Eq.(7.14) into (7.13), we obtain:

$$B_v(\tau, k, l) = \frac{2}{\gamma^2} \frac{\left((\zeta_{q,v}^2 - \mu_{q,v}^2) s_{q,v,g} + \frac{\gamma^2}{2} \imath\, l\, s_{q,v,d}\right)}{s_{q,v,b} + \imath\, l \frac{\gamma^2}{2} s_{q,v,g}}, \tag{7.19}$$

where $s_{q,v,d}$ are given by:

$$s_{q,v,d} = (\zeta_{q,v} - \mu_{q,v})e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} + \mu_{q,v}). \qquad [7.20]$$

The solution of Eq.(7.10) is obtained analogously:

$$B_r(\tau, k, \xi) = \frac{2}{\eta^2} \frac{\frac{d}{d\tau} C_r}{C_r} = \frac{2}{\eta^2} \frac{\left((\zeta_{q,r}^2 - \mu_{q,r}^2) s_{q,r,g} + \frac{\eta^2}{2} \imath\, \xi\, s_{q,r,d}\right)}{s_{q,r,b} + \imath\, \xi \frac{\eta^2}{2} s_{q,r,g}}, \tag{7.21}$$

where C_r is given by:

$$C_r(\tau, k, l) = e^{(\mu_{q,r}+\zeta_{q,r})\tau} \left[\frac{s_{q,r,b} + \imath\, \xi \frac{\eta^2}{2} s_{q,r,g}}{2\zeta_{q,r}} \right]. \qquad [7.22]$$

and the quantities $\mu_{q,r}$, $\zeta_{q,r}$, $s_{q,r,g}$, $s_{q,r,b}$ are given by:

$$\mu_{q,r} = -\frac{1}{2}(\lambda + (\imath\, k - q)\, \eta\, \Omega\, \rho_{p,r}), \qquad [7.23]$$

$$\zeta_{q,r} = \frac{1}{2}\left[4\mu_{q,r}^2 + 2\eta^2\left(\varphi_q(k)\Omega^2 - q + \imath\, k)\right)\right]^{1/2}, \qquad [7.24]$$

$$s_{q,r,g} = 1 - e^{-2\zeta_{q,r}\tau}, \qquad [7.25]$$

$$s_{q,r,b} = (\zeta_{q,r} + \mu_{q,r})e^{-2\zeta_{q,r}\tau} + (\zeta_{q,r} - \mu_{q,r}), \qquad [7.26]$$

where the quantity φ_q is given in Eqs. (7.12) while $s_{q,r,d}$ is given by:

$$s_{q,r,d} = (\zeta_{q,r} - \mu_{q,r})e^{-2\zeta_{q,r}\tau} + (\zeta_{q,r} + \mu_{q,r}). \qquad [7.27]$$

Let us carry out the final steps of the computation. From Eq.(7.8), we obtain:

$$A(\tau, k, l, \xi) = -\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau, k, l)\; - \;\frac{2\lambda\theta}{\eta^2}\ln C_r(\tau, k, \xi), \qquad (7.28)$$

so that we obtain the following expression for the function g (see Eq. (7.7)):

$$\begin{aligned} &g(\tau, v, r, k, l, \xi) \\ &= e^{-\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau,k,l)}\, e^{-\frac{2v}{\gamma^2}\frac{dC_v}{d\tau}(\tau,k,l)/C_v}\, e^{-\frac{2\lambda\theta}{\eta^2}\ln C_r(\tau,k,\xi)} e^{-\frac{2r}{\eta^2}\frac{dC_r}{d\tau}(\tau,k,\xi)/C_r}, \\ &\quad (v, r) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0. \qquad [7.29] \end{aligned}$$

Finally, in order to get an explicit expression for f (see Eq. (7.5)) we have to compute the following integrals:

$$L_{v,q}(\tau, v, v', k; \underline{\Theta}_v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl\, e^{\imath\, l\, v'} e^{-\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau,k,l)}\, e^{-\frac{2v}{\gamma^2}\frac{dC_v}{d\tau}(\tau,k,l)/C_v} \qquad [7.30]$$

$$L_{r,q}(\tau, r, r', k; \underline{\Theta}_r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi\, e^{\imath\,\xi\, r'} e^{-\frac{2\lambda\theta}{\eta^2}\ln C_r(\tau,k,\xi)} e^{-\frac{2r}{\eta^2}\frac{dC_r}{d\tau}(\tau,k,\xi)/C_r}. \qquad [7.31]$$

Let us show how to compute the integral in Eq.(7.107). The integral in Eq.(7.31) can be obtained in a similar fashion as Eq.(7.107). Using the explicit expression of C_v of Eq.(7.14)) in Eq.(7.107) and the change of variable $l' = -s_{q,v,g}(\gamma^2/2)l/s_{q,v,b}$ and formula n. 34 on p. 156 in Oberhettinger 1973 we obtain the final expression of $L_{v,q}$:

$$\begin{aligned} &L_{v,q}(\tau, v, v', k; \underline{\Theta}_v) \\ &= e^{-(2\chi v^*/\gamma^2)[\ln(s_{q,v,b}/(2\zeta_v)) + (\mu_{q,v}+\zeta_{q,v})\tau]}\; e^{-(2v/\gamma^2)(\zeta_{q,v}^2 - \mu_{q,v}^2)s_{q,v,g}/s_{q,v,b}}\, \cdot \\ &M_{q,v}\left(\frac{v'}{\tilde{v}_q}\right)^{\nu_v/2} e^{-M_{q,v}\tilde{v}} e^{-M_{q,v}v'} I_{\nu_v}(2M_{q,v}(\tilde{v}_q v')^{1/2}),\ v, v' > 0,\ k \in \mathbb{R}, \qquad [7.32] \end{aligned}$$

where I_ν is the modified Bessel function of order ν (see, for example, Abramowitz and Stegun, 1970), $\nu_v = 2\chi v^*/\gamma^2 - 1$, and $M_{q,v}$, $\tilde{v}_q$ are given by:

$$M_{q,v} = \frac{2}{\gamma^2}\frac{s_{q,v,b}}{s_{q,v,g}}, \quad \tilde{v}_q = \frac{4\zeta_{q,v}^2 v e^{-2\zeta_{q,v}\tau}}{s_{q,v,b}^2}, \quad M_{q,v}\tilde{v}_q = \frac{8}{\gamma^2}\frac{\zeta_{q,v}^2 v e^{-2\zeta_{q,v}\tau}}{s_{q,v,g}s_{q,v,b}}. \quad [7.33]$$

Arguing similarly, from Eq.(7.31) we obtain:

$$\begin{aligned} &L_{r,q}(\tau, r, r', k; \underline{\Theta}_r) \\ &= e^{-(2\lambda\theta/\eta^2)[\ln(s_{q,r,b}/(2\zeta_{q,r}))+(\mu_{q,r}+\zeta_{q,r})\tau]}\, e^{-(2r/\eta^2)(\zeta_{q,r}^2-\mu_{q,r}^2)s_{q,r,g}/s_{q,r,b}} \cdot \\ &M_{q,r}\left(\frac{r'}{\tilde{r}_q}\right)^{\nu_r/2} e^{-M_{q,r}\tilde{r}_q} e^{-M_{q,r}r'} I_{\nu_r}(2M_{q,r}(\tilde{r}_q r')^{1/2}), \; r, r' > 0, \; k \in \mathbb{R}, \quad [7.34] \end{aligned}$$

where $\nu_r = 2\lambda\theta/\eta^2 - 1$ and the quantities $M_{q,r}$, $\tilde{r}_q$ are given by:

$$M_{q,r} = \frac{2}{\eta^2}\frac{s_{q,r,b}}{s_{q,r,g}}, \quad \tilde{r}_q = \frac{4\zeta_{q,r}^2 r e^{-2\zeta_{q,r}\tau}}{s_{q,r,b}^2}, \quad M_{q,r}\tilde{r}_q = \frac{8}{\eta^2}\frac{\zeta_{q,r}^2 r e^{-2\zeta_{q,r}\tau}}{s_{q,r,g}s_{q,r,b}}. \quad [7.35]$$

Finally, using (7.32) and (7.34) in formula (7.5) and then in (7.1) we obtain formula (3.25), that is:

$$\begin{aligned} &p_f(x, v, r, t, x', v', r', t') \\ &= \frac{e^{q(x-x')}}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v) L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r), \\ &(x, v, r), \, (x', v', r') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+, \; t, t' \geq 0, \; q \in \mathbb{R}, \; t'-t > 0. \quad [7.36] \end{aligned}$$

As explained in Section 3.2, the parameter q is a 'regularization' parameter whose choice allows us to explicitly compute some integrals appearing in the formulas of the moments/mixed moments and option prices. In fact, using formula n. 18 in Erdely et al. Vol I, 1954, p. 197, we obtain:

$$\begin{aligned} &W_{v,q}^0(t'-t, v, k; \underline{\Theta}_v) = \int_0^{+\infty} dv'\, L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v) = \\ &e^{-2\chi v^* \ln(s_{q,v,b}/(2\zeta_{q,v}))/\gamma^2} e^{-2\chi v^*(\zeta_{q,v}+\mu_{q,v})(t'-t)/\gamma^2} e^{-(2v/\gamma^2)(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/s_{q,v,b}}, \quad [7.37] \end{aligned}$$

and $W_{r,q}^0$ is the function given by:

$$\begin{aligned} &W_{r,q}^0(t'-t, r, k; \underline{\Theta}_r) = \int_0^{+\infty} dr'\, L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r) = \\ &e^{-2\lambda\theta \ln(s_{q,r,b}/(2\zeta_{q,r}))/\eta^2} e^{-2\lambda\theta(\zeta_{q,r}+\mu_{q,r})(t'-t)/\eta^2} e^{-(2r/\eta^2)(\zeta_{q,r}^2-\mu_{q,r}^2)s_{q,r,g}/s_{q,r,b}}. \quad [7.38] \end{aligned}$$

Furthermore, using Abramowitz and Stegun 1970 p. 375 formula n. 9.6.3, p. 486 formula n.11.4.28, p. 505 13.1.27, p. 509 formula 13.6.9 we obtain:

$$W^m_{v,q}(t'-t,v,k;\underline{\Theta}_v) = \int_0^{+\infty} dv'\,(v')^m L_{v,q}(t'-t,v,v',k;\underline{\Theta}_v) =$$
$$W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v)\frac{1}{(M_{v,q})^m}\frac{\Gamma(m+1+\nu_v)}{\Gamma(\nu_v+1)}\frac{m!}{(\nu_v+1)_m}L_m^{(\nu_v)}\left(-M_{v,q}\tilde{v}_q\right)\ ,[7.39]$$

$$W^m_{r,q}(t'-t,r,k;\underline{\Theta}_r) = \int_0^{+\infty} dr'\,(r')^m L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r) =$$
$$W^0_{r,q}(t'-t,r,k;\underline{\Theta}_r)\frac{1}{(M_{r,q})^m}\frac{\Gamma(m+1+\nu_r)}{\Gamma(\nu_r+1)}\frac{m!}{(\nu_r+1)_m}L_m^{(\nu_r)}\left(-M_{r,q}\tilde{r}_q\right)\ ,[7.40]$$

where $L_m^{(\nu)}$ is the generalized Laguerre polynomials (see Abramowitz and Stegun 1970 p. 775), $m!$ is the factorial of the integer m (i.e $m! = \prod_{j=1}^m j$) and $(\nu)_m = \prod_{j=1}^m(\nu+j-1)$. The explicit expression of $L_m^{(\nu)}$ is given by (see Abramowitz and Stegun 1970 p. 775):

$$L_m^{(\nu)}(y) = \sum_{j=0}^m(-1)^j\binom{m+\nu}{m-j}\frac{1}{j!}\,y^j,\ \ y\geq 0. \qquad (7.41)$$

Substituting Eq.(7.41) into Eqs.(7.39) and (7.40), we deduce the following elementary formulas:

$$W^m_{v,q}(t'-t,v,k;\underline{\Theta}_v)$$
$$= W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v)\frac{\Gamma(m+1+\nu_v)}{\Gamma(\nu_v+1)}\frac{m!}{(\nu_v+1)_m}\sum_{j=1}^m\binom{m+\nu_v}{m-j}\frac{\tilde{v}_q^j}{j!}\left(M_{v,q}\right)^{(j-m)}\ , \qquad [7.42]$$

$$W^m_{r,q}(t'-t,r,k;\underline{\Theta}_r)$$
$$= W^0_{r,q}(t'-t,r,k;\underline{\Theta}_r)\frac{\Gamma(m+1+\nu_r)}{\Gamma(\nu_r+1)}\frac{m!}{(\nu_r+1)_m}\sum_{j=1}^m\binom{m+\nu_r}{m-j}\frac{\tilde{r}_q^j}{j!}\left(M_{r,q}\right)^{(j-m)}\ . \qquad [7.43]$$

Using Eqs.(7.37)–(7.40) and the definitions of the marginal distribution and of the moments with an easy computation we obtain formulas (3.28), (3.29), (3.31), (3.33), and (3.32).

We conclude Appendix A with some comments on the existence of the moments. Formula (3.31) shows that the existence of the moments is guaranteed when $\zeta_{m,v}(0)$, $\zeta_{m,r}(0)$ (see formulas (7.15)-(7.16) and (7.23)-(7.24) are real numbers. It is easy to see that sufficient conditions for the existence of $E(S_{t'}^m)$, $m = 2, 3, \ldots$, are:

$$\Delta \leq 1, \quad \rho_{p,v} \leq \frac{\chi - \gamma\Delta}{m\gamma} - \frac{1}{m\gamma}\sqrt{(m-1)\gamma(2\chi\Delta - \Delta^2\gamma + m\gamma)}, \quad [7.44]$$

$$\lambda \geq (2-\Omega)\eta, \quad \rho_{p,r} \leq \frac{\lambda}{m\eta\Omega} - \frac{1}{m\eta\Omega}\sqrt{m\,\gamma^2((m-1)\Omega^2+2)}. \quad [7.45]$$

Finally, formula (3.31) provides the moments of the price variable of the Heston model in the limit $\Omega \to 0^+$, $\eta \to 0^+$ and $\lambda \to 0^+$, that is:

$$\mathcal{M}_m^H(S,v,r,t,t') = S^m\, e^{m\,r(t'-t)} W_{v,m}^0(t'-t, v, 0; \underline{\Theta}_v),$$
$$(x,v) \in \mathbb{R} \times \mathbb{R}^+,\ t,t' \geq 0,\ t'-t > 0. \quad [7.46]$$

7.2 Appendix B: The Analytical Treatment Of The HHW model

In Appendix B, let us derive an integral representation formula for the transition probability density function of the process described by Eq.(5.18)-(5.20) and first of all we equipped equations (5.18)–(5.20) with the initial condition:

$$x_0 = x_0^*, \quad [7.47]$$
$$v_0 = v_0^*, \quad [7.48]$$
$$r_0 = r_0^*, \quad [7.49]$$

where x_0^*, v_0^*, r_0^* are random variables that we assume to be concentrated in a point with probability one. For simplicity, we identify the random variables x_0^*, v_0^*, r_0^* with the points where they are concentrated. We assume v_0^*, χ, λ, γ, η, v^*, θ to be positive constant. Moreover we assume $\frac{2\chi\overline{v}}{\gamma^2} > 1$.

Let $p_f(x,v,r,t,x',v',r',t')$, $(x,v,r),(x',v',r') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$, t, $t' \geq 0$, $t-t' > 0$ be the transition probability density function associated with the stochastic differential system (5.18)–(5.20), that is, the probability density function of having $x_{t'} = x'$, $v_{t'} = v'$, $r_{t'} = r'$ given that $x_t = x$, $v_t = v$, $r_t = r$, when $t'-t > 0$. This transition probability density function $p_f(x,v,r,t,x',v',r',t')$ as a function of the "past" variables (x,v,r,t) satisfies the following backward Kolmogorov equation:

$$-\frac{\partial p_f}{\partial t} = \frac{1}{2}(\tilde{\psi}v+\Omega^2)\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\gamma^2 v\frac{\partial^2 p_f}{\partial v^2} + \frac{1}{2}\eta^2\frac{\partial^2 p_f}{\partial r^2} + \gamma(\rho_{p,v}+\Delta)v\frac{\partial^2 p_f}{\partial x\partial v}$$
$$+\eta\rho_{p,r}\Omega\,\frac{\partial^2 p_f}{\partial x\partial r} + \chi(v^*-v)\frac{\partial p_f}{\partial v} + \lambda(\theta-r)\frac{\partial p_f}{\partial r} + \left(r - \frac{1}{2}(\tilde{\psi}v+\Omega^2)\right)\frac{\partial p_f}{\partial x},$$
$$(x,v,r)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ 0\leq t<t', \quad [7.50]$$

$$\text{where} \quad \tilde{\psi} := 1+\Delta^2+2\Delta\rho_{p,v}. \quad [7.51]$$

with final condition:

$$p_f(x,v,r,t,x',v',r',t') = \delta(x'-x)\delta(v'-v)\delta(r'-r),$$
$$(x,v,r),(x',v',r')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t\geq 0, \quad (7.52)$$

and the appropriate boundary conditions. Letting $\tau = t'-t$, we can introduce the function p_b defined as follows:

$$p_b(\tau,x,v,r,x',v',r') = p_f(x,v,r,t,x',v',r',t'),$$
$$(x,v,r),(x',v',r')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t'=t+\tau,\ \tau>0\,. \quad (7.53)$$

The representation (7.53) holds since the coefficients of the Kolmogorov backward equation and condition (7.52) are invariant by time translation. Using the change of the time variable $\tau = t - t'$ and equation (7.50), it is easy to see that p_b is the solution of the following problem:

$$\frac{\partial p_b}{\partial \tau} = \frac{1}{2}(\tilde{\psi} v + \Omega^2)\frac{\partial^2 p_b}{\partial x^2} + \frac{1}{2}\gamma^2 v \frac{\partial^2 p_b}{\partial v^2} + \frac{1}{2}\eta^2 \frac{\partial^2 p_b}{\partial r^2} + \gamma \tilde{\rho}_{p,v} v \frac{\partial^2 p_b}{\partial x \partial v} +$$
$$\eta \rho_{p,r} \Omega \frac{\partial^2 p_b}{\partial x \partial r} + \chi(v^* - v)\frac{\partial p_b}{\partial v} + \lambda(\theta - r)\frac{\partial p_b}{\partial r} + \left(r - \frac{1}{2}(\tilde{\psi} v + \Omega^2)\right)\frac{\partial p_b}{\partial x},$$
$$(x, v, r) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \; \tau > 0, \qquad [7.54]$$

with the initial condition:

$$p_b(0, x, v, r, x', v', r') = \delta(x' - x)\delta(v' - v)\delta(r' - r),$$
$$(x, v, r), (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \qquad [7.55]$$

and with the appropriate boundary conditions. For later convenience, we consider the following change of dependent variable:

$$p_b(\tau, x, v, r, x', v', r') = e^{q(x - x')} p_q(\tau, x, v, r, x', v', r'),$$
$$(v, r), (v', r') \in \mathbb{R}^+ \times \mathbb{R}^+, \; k \in \mathbb{R}, \; q \in \mathbb{R}, \; \tau > 0. \quad [7.56]$$

Substituting (7.56) into (7.54) and (7.55), it is easy to see that p_q is the solution of the following problem:

$$\frac{\partial p_q}{\partial \tau} = \frac{1}{2}\left(\tilde{\psi} v + \Omega^2\right)\frac{\partial^2 p_q}{\partial x^2} + \frac{1}{2}\gamma^2 v \frac{\partial^2 p_q}{\partial v^2} + \frac{1}{2}\eta^2 \frac{\partial^2 p_q}{\partial r^2} + \gamma \tilde{\rho}_{p,v} v \frac{\partial^2 p_q}{\partial x \partial v} +$$
$$\eta \rho_{p,r} \Omega \frac{\partial^2 p_q}{\partial x \partial r} + [\chi(v^* - v) + v\, q\, \gamma\, \tilde{\rho}_{p,v}]\frac{\partial p_q}{\partial v} + [\lambda(\theta - r) + \; q \eta \rho_{p,r} \Omega]\frac{\partial p_q}{\partial r}$$
$$+ \left[r + \frac{v}{2}(2q - 1)\tilde{\psi} + \frac{1}{2}(2q - 1)\Omega^2\right]\frac{\partial p_q}{\partial x} + \left[qr + \frac{v}{2}\left(q^2 - q\right)\tilde{\psi} + \frac{1}{2}(q^2 - q)\Omega^2\right] p_q,$$
$$(x, v, r) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \; \tau > 0, \qquad [7.57]$$

with initial condition:

$$p_q(0, x, v, r, x', v', r') = e^{-q(x - x')}\,\delta(x' - x)\delta(v' - v)\delta(r' - r),$$
$$(x, v, r), (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \qquad [7.58]$$

Now we consider the following representation formula for p_q:

$$p_q(\tau, x, v, r, x', v', r') = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x' - x)} f(\tau, v, r, v', r', k),$$
$$(v, r), (v', r') \in \mathbb{R}^+ \times \mathbb{R}^+, \; k \in \mathbb{R}, \; \tau > 0. \qquad [7.59]$$

This is possible since the coefficients (7.54) and the initial condition (7.55) are independent of translation in the log-price variable. Substituting (7.59) into (7.54), we obtain that the function f is the solution of the following problem:

$$\frac{\partial f}{\partial \tau} = -\frac{k^2}{2}(\tilde{\psi} v + \Omega^2) f + \frac{1}{2}\gamma^2 v \frac{\partial^2 f}{\partial v^2} + \frac{1}{2}\eta^2 \frac{\partial^2 f}{\partial r^2} + [(-\imath\, k)\gamma\tilde{\rho}_{p,v} v + q\, \gamma\tilde{\rho}_{p,v} v]\, \frac{\partial\, f}{\partial v}$$
$$+ [(-\imath\, k)\eta\rho_{p,r}\Omega\ + q\,\eta\,\rho_{p,r}\Omega\,]\, \frac{\partial\, f}{\partial r} + \chi(v^* - v)\frac{\partial f}{\partial v} + \lambda(\theta - r)\frac{\partial f}{\partial r}$$
$$+ \left[\frac{v}{2}\left((q^2 - q)\tilde{\psi} - \imath\, k(2q-1)\tilde{\psi}\right) + (q - \imath k)r + \frac{1}{2}\left((q^2 - q)\Omega^2 - \imath\, k(2q-1)\Omega^2\right)\right] f$$
$$(k, v, r) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ \ \tau > 0,\, q \in \mathbb{R} \qquad [7.60]$$

with the initial condition:

$$f(0, v, r, v', r', k) = \delta(v' - v)\delta(r' - r),$$
$$(v, r), (v', r') \in \mathbb{R}^+ \times \mathbb{R},\, k \in \mathbb{R}. \qquad [7.61]$$

Now let us represent f as the inverse Fourier transform of the future variables (v', r') whose conjugate variables are denoted by (l, ξ), that is:

$$f(\tau, v, r, v', r', k) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} dl\, e^{\imath l\, v'} \int_{\mathbb{R}} d\xi\, e^{\imath\,\xi\, r'}\, g(\tau, v, r, k, l, \xi),$$
$$(v, r) \in \mathbb{R}^+ \times \mathbb{R},\, (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\, \tau > 0. \qquad [7.62]$$

It is easy to see that the function g satisfies Eq.(7.60) with the following initial condition:

$$g(0, v, r, k, l, \xi) = e^{-\imath\,\xi\, r} e^{-\imath\, l\, v},$$
$$(v, r) \in \mathbb{R}^+ \times \mathbb{R},\, (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\, \tau > 0. \qquad (7.63)$$

Please note that g is the Fourier transform with respect to the future variables (v', r') of the function obtained by extending f, as a function of the variables (v', r'), with zero when $v' \notin \mathbb{R}^+$. The coefficients of the partial differential operator appearing on the right hand side of (7.54) are first degree polynomials in v and r so that we seek a solution of problem (7.60), (7.61) in the form (see Lipton, 2001):

$$g(\tau, v, r, k, l, \xi) = e^{A(\tau,k,l,\xi)} e^{-v\, B_v(\tau,k,l)} e^{-r\, B_r(\tau,k,\xi)},$$
$$(v, r) \in \mathbb{R}^+ \times \mathbb{R}^+,\, (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\, \tau > 0. \qquad [7.64]$$

Substituting formula (7.64) into equation (7.60), we obtain that the functions A and B_v, B_r must satisfy the following ordinary differential equations:

$$\frac{dA}{d\tau}(\tau,k,l,\xi) = -\lambda\,\theta\,B_r(\tau,k,\xi) - \chi v^* B_v(\tau,k,l) + \Big(-\varphi_q(k)\Omega^2 \\ +(\imath\,k-q)\Omega\,\rho_{p,r}\eta B_r(\tau,k,\xi) + \frac{\eta^2}{2}B_r^2(\tau,k,\xi)\Big) \\ = -\varphi_q(k)\Omega^2 - [\lambda\theta - (\imath\,k-q)\Omega\,\rho_{p,r}\eta]B_r(\tau,k,\xi) + \frac{\eta^2}{2}B_r^2(\tau,k,\xi) - \chi v^* B_v(\tau,k,l) \\ (k,\xi)\in\mathbb{R}\times\mathbb{R},\ \tau>0, (k,l,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau>0, \qquad [7.65]$$

$$\frac{dB_v}{d\tau}(\tau,k,l) = \varphi_q(k)\tilde{\psi} - (\chi + (\imath\,k-q)\,\gamma\,\tilde{\rho}_{p,v})B_v(\tau,k,l) - \frac{\gamma^2}{2}B_v^2(\tau,k,l), \\ (k,l)\in\mathbb{R}\times\mathbb{R},\quad \tau>0, \qquad [7.66]$$

$$\frac{dB_r}{d\tau}(\tau,k,\xi) = (\imath k - q) - \lambda B_r(\tau,k,\xi) \\ (k,\xi)\in\mathbb{R}\times\mathbb{R}, \tau>0, \qquad [7.67]$$

with initial condition:

$$A(0,k,l,\xi) = 0,\quad B_v(0,k,l) = \imath\,l,\quad B_r(0,k,\xi) = \imath\,\xi, \\ (k,l,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}. \qquad [7.68]$$

where φ_q is the quantity given in (7.12) and $\tilde{\psi}$, $\tilde{\rho}_{p,v}$ are given by (5.21) and (7.4) respectively. Firstly, equations (7.67) can be solved:

$$B_r(\tau,k,\xi) = \left(\imath\xi - \frac{\imath k - q}{\lambda}\right)e^{-\lambda\tau} + \frac{\imath k - q}{\lambda} \qquad [7.69]$$

Secondly, Eq.(7.66) is Riccati equations that can be solved elementarily by substituting their solutions into (7.65) and integrating with respect to τ to obtain A. We seek for the solution of Eq.(7.66) in the following form:

$$B_v(\tau,k,l) = \frac{2}{\gamma^2}\frac{\frac{d}{d\tau}C_v}{C_v}. \qquad (7.70)$$

Replacing Eq.(7.70) into Eq.(7.66), we obtain that C_v must satisfy the following problem:

$$\frac{d^2C_v}{d\tau^2} + (\chi + (\imath\,k-q)\,\gamma\,\tilde{\rho}_{p,v})\frac{dC_v}{d\tau} - \frac{\gamma^2}{2}\varphi_q(k)\tilde{\psi}C_v = 0, \qquad (7.71)$$

with initial conditions

$$C_v(0,k,l) = 1, \qquad \frac{dC_v}{d\tau}(0,k,l) = \imath\, l\, \frac{\gamma^2}{2}, \;\; (k,l) \in \mathbb{R} \times \mathbb{R}. \tag{7.72}$$

Please note that Eq.(7.71) is a second order ordinary differential equation with coefficients depends on τ so that is solution is given by:

$$C_v(\tau,k,l) = e^{(\mu_{q,v}+\zeta_{q,v})\tau} \left[\frac{s_{q,v,b} + \imath\, l \frac{\gamma^2}{2} s_{q,v,g}}{2\zeta_{q,v}} \right], \qquad [7.73]$$

where

$$\mu_{q,v} = -\frac{1}{2}(\chi + (\imath\, k - q)\, \gamma\, \tilde{\rho}_{p,v}) \qquad [7.74]$$

$$\zeta_{q,v} = \frac{1}{2} \left[4\mu_v^2 + 2\gamma^2 \varphi_q(k) \tilde{\psi} \right]^{1/2}, \qquad [7.75]$$

$$s_{q,v,g} = 1 - e^{-2\zeta_{q,v}\tau}, \qquad [7.76]$$

$$s_{q,v,b} = (\zeta_{q,v} + \mu_{q,v}) e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} - \mu_{q,v}). \qquad [7.77]$$

where $\mu_{q,v}$, $\zeta_{q,v}$, $s_{q,v,g}$, $s_{q,v,b}$ are given in Eqs. (7.74)–(7.77). We have:

$$\frac{dC_v}{d\tau}(\tau,k,l) = e^{(\mu_{q,v}+\zeta_{q,v})\tau} \cdot$$
$$\left[\frac{(\mu_{q,v} - \zeta_{q,v})(\mu_{q,v} + \zeta_{q,v} - \frac{\gamma^2}{2}\imath\, l) e^{-2\zeta_{q,v}\tau} + (\mu_{q,v} + \zeta_{q,v})(\zeta_{q,v} - \mu_{q,v} + \frac{\gamma^2}{2}\imath\, l)}{2\zeta_{q,v}} \right]$$
$$= \frac{e^{(\mu_{q,v}+\zeta_{q,v})\tau}}{2\zeta_{q,v}} \left[(\zeta_{q,v}^2 - \mu_{q,v}^2) s_{q,v,g} + \frac{\gamma^2}{2} \imath\, l\, s_{q,v,d} \right], \qquad [7.78]$$

where $s_{q,v,d}$ are given by:

$$s_{q,v,d} = (\zeta_{q,v} - \mu_{q,v}) e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} + \mu_{q,v}). \qquad [7.79]$$

Substituting Eqs.(7.73)–(7.78) into (7.70), we obtain:

$$B_v(\tau,k,l) = \frac{2}{\gamma^2} \frac{\left((\zeta_{q,v}^2 - \mu_{q,v}^2) s_{q,v,g} + \frac{\gamma^2}{2} \imath\, l\, s_{q,v,d} \right)}{s_{q,v,b} + \imath\, l \frac{\gamma^2}{2} s_{q,v,g}}. \qquad [7.80]$$

Let us carry out the final steps of the computation. From Eq.(7.65), we obtain:

$$\frac{dA}{d\tau}(\tau,k,l,\xi) = -\varphi_q(k)\Omega^2 - [\lambda\theta - (\imath\, k - q)\Omega\, \rho_{p,r}\eta] B_r(\tau,k,\xi) + \frac{\eta^2}{2} B_r^2(\tau,k,\xi)$$
$$-2\frac{\chi v^*}{\gamma^2} \underbrace{\frac{\dot{C}_v(\tau,k,l)}{C_v(\tau,k,l)}}_{\ln \dot{C}_v(\tau,k,l)} \qquad [7.81]$$

By integration both sides of the equation, we obtain

$$A(\tau,k,l,\xi) = -\varphi_q(k)\Omega^2\tau - [\lambda\theta - (\imath\, k - q)\Omega\,\rho_{p,r}\eta]\int_0^\tau B_r(\tau,k,\xi)d\tau$$
$$+\frac{1}{2}\eta^2\int_0^\tau B_r^2(\tau,k,\xi)d\tau - 2\frac{\chi v^*}{\gamma^2}\ln C_v(\tau,k,l) + \text{CSTA} \qquad [7.82]$$

where

$$\begin{aligned}\int_0^\tau B_r(\tau,k,\xi)d\tau &= \int_0^\tau\left[(\imath\xi - \frac{\imath k - q}{\lambda})e^{-\lambda\tau} + \frac{\imath k - q}{\lambda}\right]d\tau \\ &= (\imath\xi - \frac{\imath k - q}{\lambda})\frac{-e^{-\lambda\tau}}{\lambda} + \frac{\imath k - q}{\lambda}\tau + \text{CST1} \qquad [7.83]\end{aligned}$$

$$\begin{aligned}\int_0^\tau B_r^2(\tau,k,\xi)d\tau &= \int_0^\tau \imath^2\left[(\xi - \frac{k+\imath q}{\lambda})e^{-\lambda\tau} + \frac{k+\imath q}{\lambda}\right]^2 d\tau \\ &= -\left[(\xi - \frac{k+\imath q}{\lambda})^2\frac{-e^{-2\lambda\tau}}{2\lambda} + 2\frac{k+\imath q}{\lambda}(\xi - \frac{k+\imath q}{\lambda})\frac{-e^{-\lambda\tau}}{\lambda}\right. \\ &\quad \left. + \frac{(k+\imath q)^2}{\lambda^2}\tau\right] + \text{CST2} \qquad [7.84]\end{aligned}$$

and CSTA, CST1 and CST2 are constants. Substituting Eqs.(7.83) and (7.84) into (7.82), we can obtain:

$$\begin{aligned}A(\tau,k,l,\xi) = &-\varphi_q(k)\Omega^2\tau - [\lambda\theta - (\imath\, k - q)\Omega\,\rho_{p,r}\eta]\cdot\imath\left[(\xi - \frac{k+\imath q}{\lambda})\frac{-e^{-\lambda\tau}}{\lambda} + \frac{k+\imath q}{\lambda}\tau\right] \\ &-\frac{1}{2}\eta^2\left[(\xi - \frac{k+\imath q}{\lambda})^2\frac{-e^{-2\lambda\tau}}{2\lambda} + 2\,\frac{k+\imath q}{\lambda}(\xi - \frac{k+\imath q}{\lambda})\frac{-e^{-\lambda\tau}}{\lambda} + \frac{(k+\imath q)^2}{\lambda^2}\tau\right] \\ &-2\frac{\chi v^*}{\gamma^2}\ln C_v(\tau,k,\xi) + \text{CSTA2} \qquad [7.85]\end{aligned}$$

where CSTA2 is also a constant with combination of CSTA, CST1 and CST2 multiplied by parameters $\underline{\Theta}_r := (\lambda,\eta,\theta,\Omega,\rho_{p,r})$. Considering initial condition Eqs.(7.68) and (7.72), we could obtain CSTA2 analytically. Moreover, we could reallocate the components of CSTA2 into polynomials in $\tilde{A}(\tau,k,\xi)$ defined as follows:

$$A(\tau,k,l,\xi) = \tilde{A}(\tau,k,\xi) - 2\frac{\chi v^*}{\gamma^2}\ln C_v(\tau,k,l)$$

where

$$\begin{aligned}\tilde{A}(\tau,k,l) = &-\varphi_q(k)\Omega^2\tau - [\lambda\theta - (\imath\, k - q)\Omega\,\rho_{p,r}\eta]\cdot\imath\left[(\xi - \frac{k+\imath q}{\lambda})\frac{1-e^{-\lambda\tau}}{\lambda} + \frac{k+\imath q}{\lambda}\tau\right] \\ &-\frac{1}{2}\eta^2\left[(\xi - \frac{k+\imath q}{\lambda})^2\frac{1-e^{-2\lambda\tau}}{2\lambda} + 2\,\frac{k+\imath q}{\lambda}(\xi - \frac{k+\imath q}{\lambda})\frac{1-e^{-\lambda\tau}}{\lambda} + \frac{(k+\imath q)^2}{\lambda^2}\tau\right] \qquad [7.86]\end{aligned}$$

Substituting Eq.(7.85), B_r and B_v into (7.64), we obtain:

$$\begin{aligned} g(\tau, v, r, k, l, \xi) &= e^{A(\tau,k,l,\xi)} e^{-v\, B_v(\tau,k,l)} e^{-r\, B_r(\tau,k,\xi)}, \\ &:= e^{\Psi_v(\tau,k,l,v)} e^{\Psi_r(\tau,k,\xi,r)}, \\ (v, r) &\in \mathbb{R}^+ \times \mathbb{R},\ (k, l, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0. \end{aligned} \quad [7.87]$$

where

$$\Psi_v(\tau, k, l, v) = -\frac{2\chi v^*}{\gamma^2} \ln C_v(\tau, k, l) - \frac{2v}{\gamma^2} \frac{dC_v}{d\tau}(\tau, k, l)/C_v\,, \quad [7.88]$$

$$\Psi_r(\tau, k, \xi, r) = \tilde{A}(\tau, k, \xi) - r\, B_r(\tau, k, \xi) \quad [7.89]$$

Firstly, we concentrate on the function $\Psi_r(\tau, k, \xi)$ given by:

$$\Psi_r(\tau, k, \xi, r) = Q_0(\tau, k, r) + Q_1(\tau, k, r) \cdot \xi - Q_2(\tau) \cdot \xi^2$$

where

$$\begin{aligned} Q_0(\tau, k, r) &= -\varphi_q(k)\Omega^2\tau - [\lambda\theta - (\imath\, k - q)\Omega\, \rho_{p,r}\eta] \cdot \imath \left[-\frac{k + \imath q}{\lambda} \frac{1 - e^{-\lambda\tau}}{\lambda} + \frac{k + \imath q}{\lambda}\tau \right] \\ &\quad -\frac{1}{2}\eta^2 \left[\left(\frac{k + \imath q}{\lambda}\right)^2 \frac{1 - e^{-2\lambda\tau}}{2\lambda} - 2\left(\frac{k + \imath q}{\lambda}\right)^2 \frac{1 - e^{-\lambda\tau}}{\lambda} + \left(\frac{k + \imath q}{\lambda}\right)^2 \tau \right] \\ &\quad -\imath \left(\frac{k + \imath q}{\lambda}\right)(1 - e^{-\lambda\tau}) \cdot r \\ &= -\varphi_q(k)\Omega^2\tau - \mu_{q,r}\left(\frac{\imath k - q}{\lambda}\right)(\tau - \Psi_1(\tau)) + \frac{\eta^2}{2}\left(\frac{\imath k - q}{\lambda}\right)^2 (\tau - 2\Psi_1(\tau) + \Psi_2) \\ &\quad -(\imath k - q)\Psi_1(\tau)\, r \end{aligned} \quad [7.90]$$

with

$$\Psi_m(\tau) = \frac{1 - e^{-m\lambda\tau}}{m\lambda}\ , \quad m = 1, 2, \ldots \quad [7.91]$$

$$\mu_{q,r} = \lambda\theta - (\imath k - q)\Omega\rho_{p,r}\eta \quad [7.92]$$

Let us define $\hat{Q}_0(\tau, k)$ as follows:

$$\begin{aligned} &\hat{Q}_0(\tau, k) \quad [7.93] \\ &= -\varphi_q(k)\Omega^2\tau - \mu_{q,r}\left(\frac{\imath k - q}{\lambda}\right)(\tau - \Psi_1(\tau)) + \frac{\eta^2}{2}\left(\frac{\imath k - q}{\lambda}\right)^2 (\tau - 2\Psi_1(\tau) + \Psi_2) \end{aligned}$$

Thus, Eq.(7.90) can be written as

$$Q_0(\tau, k, r) = \hat{Q}_0(\tau, k) - (\imath k - q)\Psi_1(\tau)\, r \quad [7.94]$$

$$Q_1(\tau,k,r) = -\imath[\lambda\theta - (\imath\, k - q)\Omega\, \rho_{p,r}\eta]\left(\frac{1-e^{-\lambda\tau}}{\lambda}\right)$$
$$+\eta^2\left(\frac{k+\imath q}{\lambda}\right)\left[\frac{1-e^{-2\lambda\tau}}{2\lambda} - \frac{1-e^{-\lambda\tau}}{\lambda}\right] - \imath\, e^{-\lambda\tau}\, r \qquad [7.95]$$

Moreover, let us define $\tilde{Q}_1(\tau,k,r) := \imath Q_1(\tau,k,r)$

$$\tilde{Q}_1(\tau,k,r) = [\lambda\theta - (\imath\, k - q)\Omega\, \rho_{p,r}\eta]\left(\frac{1-e^{-\lambda\tau}}{\lambda}\right)$$
$$+\eta^2\left(\frac{\imath k - q}{\lambda}\right)\left[\frac{1-e^{-2\lambda\tau}}{2\lambda} - \frac{1-e^{-\lambda\tau}}{\lambda}\right] + e^{-\lambda\tau}\, r$$
$$= \mu_{q,r}\Psi_1(\tau) + \eta^2\left(\frac{\imath k - q}{\lambda}\right)\Big(\Psi_2(\tau) - \Psi_1(\tau)\Big) + e^{-\lambda\tau}\, r \qquad [7.96]$$

Let us define $\hat{Q}_1(T,k)$ as follows

$$\hat{Q}_1(\tau,k) = \mu_{q,r}\Psi_1(\tau) + \eta^2\left(\frac{\imath k - q}{\lambda}\right)(\Psi_2(\tau) - \Psi_1(\tau)) \qquad [7.97]$$

Hence, we obtain

$$\tilde{Q}_1(\tau,k,r) = \hat{Q}_1(\tau,k) + r\, e^{-\lambda\,\tau} \qquad [7.98]$$

$$Q_2(\tau) = \frac{\eta^2}{2}\frac{1-e^{-2\lambda\tau}}{2\lambda} = \frac{\eta^2}{2}\Psi_2(\tau) \qquad [7.99]$$

In order to get an explicit expression for f, that is the inverse Fourier transform of g with respect to the variable v' and r', we have to compute the following integrals.

$$L_{v,q}(\tau,v,v',k;\underline{\Theta}_v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl\, e^{\imath l v'}\, e^{\Psi_v(\tau,k,l,v)} \qquad [7.100]$$
$$L_{r,q}(\tau,r,r',k;\underline{\Theta}_r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi\, e^{\imath\,\xi\, r'} e^{\Psi_r(\tau,k,\xi,r)} \qquad [7.101]$$

Let us focus on Eq.(7.101)

$$L_{r,q}(\tau,r,r',k;\underline{\Theta}_r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi\, e^{\imath\,\xi\, r'} e^{\Psi_r(\tau,k,\xi,r)}$$
$$= \frac{1}{2\pi}\, e^{Q_0(\tau,k,r)}\int_{-\infty}^{+\infty} d\xi\, e^{Q_4(\tau,k,r,r')\,\xi - Q_2(\tau)\,\xi^2} \qquad [7.102]$$
$$= \frac{1}{2\pi}\, e^{Q_0(\tau,k,r)+\frac{Q_4^2(\tau,k,r,r')}{4Q_2(\tau)}}\int_{-\infty}^{+\infty} d\xi\, e^{-Q_2(\tau)\left[\xi - \frac{Q_4(\tau,k,r,r')}{2\,Q_2(\tau)}\right]^2}, \qquad [7.103]$$

where

$$Q_4(\tau,k,r,r') = Q_1(\tau,k,r) + \imath\, r' \qquad [7.104]$$

Thanks to Gaussian Integral (with complex offset), we can obtain the integration part analytically.

$$\int_{-\infty}^{+\infty} d\xi\, e^{-Q_2(\tau)\left[\xi - \frac{Q_4(\tau,k,r,r')}{2\,Q_2(\tau)}\right]^2} = \sqrt{\frac{\pi}{Q_2(\tau)}} \qquad [7.105]$$

where $Q_2(\tau) > 0$, and substituting Eq.(7.105) into (7.102), we obtain:

$$\begin{aligned} L_{r,q}(\tau,r,r',k;\underline{\Theta}_r) &= \frac{1}{2\sqrt{\pi Q_2(\tau)}}\, e^{Q_0(\tau,k,r) + \dfrac{(\imath r' + Q_1(\tau,k,r))^2}{4Q_2(\tau)}} \\ &= \frac{1}{2\sqrt{\pi Q_2(\tau)}}\, e^{Q_0(\tau,k,r) - \dfrac{1}{2}\left(\dfrac{r' - \tilde{Q}_1(\tau,k,r)}{\sqrt{2Q_2(\tau)}}\right)^2} \end{aligned} \qquad [7.106]$$

Furthermore, in order to get an explicit expression for f (see Eq. (7.5)), the following integrals should be computed analytically.

$$L_{v,q}(\tau,v,v',k;\underline{\Theta}_v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl\, e^{\imath\, l\, v'} e^{-\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau,k,l)}\, e^{-\frac{2v}{\gamma^2}\frac{dC_v}{d\tau}(\tau,k,l)/C_v} \qquad [7.107]$$

The integral in Eq.(7.31) can be similarly computed. Substituting Eqs.(7.14) and (7.78) into Eq.(7.107), and using the change of variable $l' = -s_{q,v,g}(\gamma^2/2)l/s_{q,v,b}$, i.e.

$$\begin{aligned} &\frac{2}{\gamma^2}\frac{(\zeta_{q,v}^2 - \mu_{q,v}^2)s_{q,v,g} + \imath\, l\, s_{q,v,d}\gamma^2/2}{s_{q,v,b} + \imath\, l\, s_{q,v,g}\gamma^2/2} \\ &= \frac{2}{\gamma^2}\frac{s_{q,v,g}}{s_{q,v,b}}(\zeta_{q,v}^2 - \mu_{q,v}^2) - \imath\, l'\left(\frac{8\zeta_{q,v}^2 e^{-2\zeta_{q,v}\tau}}{1 - 2\imath\, l'}\right), \end{aligned} \qquad [7.108]$$

we obtain $L_{v,q}$ as follows:

$$\begin{aligned} L_{v,q}(\tau,v,v',k;\underline{\Theta}_v) &= \frac{1}{2\pi}\tilde{M}_{q,v}\; e^{-(2\chi v^*/\gamma^2)[\ln(s_{q,v,b}/2\zeta_{q,v}) + (\mu_{q,v}+\zeta_{q,v})\tau]}\;\cdot \\ e^{-(2v/\gamma^2)(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/s_{q,v,b}} &\int_{-\infty}^{+\infty} dl'\, e^{-(\imath\, l')\tilde{M}_{q,v}} e^{-(2\chi v^*/\gamma^2)\ln(1-\imath\, l')} e^{\frac{(\tilde{M}_{q,v}\tilde{v}_q)\imath\, l'}{1-\imath\, l'}}, \end{aligned} \qquad [7.109]$$

where $\tilde{M}_{q,v}$, $\tilde{v}_q$ are given in Eq.(7.33). Now, using formula n.34 on p.156 in Oberhettinger 1973, we obtain the final expression of $L_{v,q}$:

$$L_{v,q}(\tau, v, v', k; \underline{\Theta}_v) = e^{-(2\chi v^*/\gamma^2)[\ln(s_{q,v,b}/2\zeta_v)+(\mu_{q,v}+\zeta_{q,v})\tau]}\, e^{-(2v/\gamma^2)(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/s_{q,v,b}} \cdot$$
$$\tilde{M}_{q,v}\left(\tilde{M}_{q,v}\tilde{v}_q\right)^{-\nu_v/2}\left(\tilde{M}_{q,v}v'\right)^{\nu_v/2} e^{-\tilde{M}_{q,v}\tilde{v}} e^{-\tilde{M}_{q,v}v'} I_{\nu_v}(2\tilde{M}_{q,v}(\tilde{v}_q v')^{1/2}),$$
$$v, v' > 0,\ k \in \mathbb{R}, \qquad [7.110]$$

where $\nu_v = 2\chi v^*/\gamma^2 - 1$. Let us focus on $W_{r,q}^m(t'-t, r, k; \underline{\Theta}_r)$ as follows:

$$W_{r,q}^m(t'-t, r, k; \underline{\Theta}_r) = \int_{-\infty}^{+\infty} dr'\,(r^m)\, L_{r,q}(\tau, r, r', k; \underline{\Theta}_r)$$
$$= e^{Q_0(\tau,k,r)} \frac{1}{2\sqrt{\pi\, Q_2(\tau)}} \int_{-\infty}^{+\infty} dr'\,(r^m)\, e^{-\frac{1}{2}\left(\frac{r'-\tilde{Q}_1(\tau,k,r)}{\sqrt{2Q_2(\tau)}}\right)^2}, \qquad [7.111]$$

where

$$\frac{1}{2\sqrt{\pi\, Q_2(\tau)}} \int_{-\infty}^{+\infty} dr'\,(r^m)\, e^{-\frac{1}{2}\left(\frac{r'-\tilde{Q}_1(\tau,k,r)}{\sqrt{2Q_2(\tau)}}\right)^2}$$

is the moment of normal distribution $\mathcal{N}(\tilde{Q}_1, 2Q_2)$. Therefore

$$W_{r,q}^m(t'-t, r, k; \underline{\Theta}_r) = e^{Q_0(\tau,k,r)}\,(\sigma^m)\,(-\imath\sqrt{2})^m\, U(-\frac{1}{2}m, \frac{1}{2}, \frac{Q_1(\tau,k,r)^2}{4Q_2(\tau)}) \qquad [7.112]$$

where $U(-\frac{1}{2}m, \frac{1}{2}, \frac{Q_1(\tau,k,r)^2}{4Q_2(\tau)})$ stands for confluent hyper-geometric function. Moreover, when $m = 0$, we can obtain:

$$W_{r,q}^0(t'-t, r, k; \underline{\Theta}_r) = e^{Q_0(t'-t,k,r)} \qquad [7.113]$$

The following result is remarkable when we consider $k = 0$ and $m = 0$.

$$W_{r,q}^0(\tau, r, 0; \underline{\Theta}_r) = \frac{1}{2}(q^2-q)\Omega^2\tau - \frac{\lambda\theta + q\Omega\,\rho_{p,r}\eta}{\lambda} \cdot m\left[\tau - \frac{1-e^{-\lambda\tau}}{\lambda}\right]$$
$$+\frac{\eta^2}{2}\frac{q^2}{\lambda^2}\left[\tau + \frac{1-e^{-2\lambda\tau}}{2\lambda} - 2\frac{1-e^{-\lambda\tau}}{\lambda}\right] + q\left(\frac{1-e^{-\lambda\tau}}{\lambda}\right) r \qquad [7.114]$$

Moreover, from Abramowitz and Stegun 1970 P.509 13.6.9, we obtain

$$W_{r,q}^m(t'-t, r, k; \underline{\Theta}_r) = W_{r,q}^0(t'-t, r, k; \underline{\Theta}_r) \cdot \Xi \qquad [7.115]$$

where

$$\Xi = \begin{cases} (2Q_2)^m \, 2^{\frac{m}{2}} \, \dfrac{\Gamma(\frac{m+1}{2})}{\sqrt{\pi}} \, \dfrac{(\frac{m}{2})!}{(\frac{1}{2})_{\frac{m}{2}}} \, L^{-\frac{1}{2}}_{(\frac{m}{2})} \left(-\frac{\tilde{Q}_1^2}{4\,Q_2} \right) & : m \in \text{even} \\ \tilde{Q}_1 (2Q_2)^{m-1} \, 2^{\frac{m+1}{2}} \, \dfrac{\Gamma(\frac{m+1}{2})}{\sqrt{\pi}} \, \dfrac{(\frac{m-1}{2})!}{(\frac{3}{2})_{\frac{m-1}{2}}} \, L^{\frac{1}{2}}_{(\frac{m-1}{2})} \left(-\frac{\tilde{Q}_1^2}{4\,Q_2} \right) & : m \in \text{odd} \end{cases}$$

As described in Section 5.2, the parameter q is a 'regularization' parameter whose choice allows us to explicitly compute some integrals appearing in the formulas of the moments/mixed moments and option prices. In fact, using formula n. 18 in Erdely et al. Vol I, 1954, p.197, we obtain:

$$W^0_{v,q}(t'-t, v, k; \underline{\Theta}_v) = \int_0^{+\infty} dv' \, L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v) =$$
$$e^{-2\chi v^* \ln(s_{q,v,b}/(2\zeta_{q,v}))/\gamma^2} e^{-2\chi v^* (\zeta_{q,v}+\mu_{q,v})(t'-t)/\gamma^2} e^{-(2v/\gamma^2)(\zeta^2_{q,v}-\mu^2_{q,v}) s_{q,v,g}/s_{q,v,b}}, \quad [7.116]$$

$$W^m_{v,q}(t'-t, v, k; \underline{\Theta}_v) = \int_0^{+\infty} dv' \, (v')^m L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v) =$$
$$W^0_{v,q}(t'-t, v, k; \underline{\Theta}_v) \frac{1}{(M_{v,q})^m} \frac{\Gamma(m+1+\nu_v)}{\Gamma(\nu_v+1)} \frac{m!}{(\nu_v+1)_m} L^{(\nu_v)}_m (-M_{v,q}\tilde{v}_q) \ ,[7.117]$$

where $L^{(\nu)}_m$ are the generalized Laguerre polynomials (see Abramowitz and Stegun 1970 p. 775), $m!$ is the factorial of the integer m (i.e $m! = \prod_{j=1}^m j$) and $(\nu)_m = \prod_{j=1}^m (\nu + j - 1)$. The explicit expression of $L^{(\nu)}_m$ is given by (see Abramowitz and Stegun 1970 p. 775):

$$L^{(\nu)}_m(y) = \sum_{j=0}^m (-1)^j \binom{m+\nu}{m-j} \frac{1}{j!} y^j, \ \ y \geq 0. \tag{7.118}$$

Substituting Eq.(7.118) into Eq.(7.117), we deduce the following elementary formulae

$$W^m_{v,q}(t'-t, v, k; \underline{\Theta}_v)$$
$$= W^0_{v,q}(t'-t, v, k; \underline{\Theta}_v) \frac{\Gamma(m+1+\nu_v)}{\Gamma(\nu_v+1)} \frac{m!}{(\nu_v+1)_m} \sum_{j=1}^m \binom{m+\nu_v}{m-j} \frac{\tilde{v}_q^j}{j!} (M_{v,q})^{(j-m)} \ , \quad [7.119]$$

Using Eqs.(7.116)–(7.115) and the definitions of the marginal distribution and of the moments with an easy computation, we obtain formulas (5.28), (5.29), and (7.129), (7.131), (7.130). Moreover, Eqs.(7.110), (7.34) and (7.61) imply:

$$f(\tau, v, r, v', r', k) = L_{v,q}(\tau, v, v', k; \underline{\Theta}_v) \cdot L_{r,q}(\tau, r, r', k; \underline{\Theta}_r) =$$
$$e^{-2\chi v^* \ln(s_{q,v,b}/2\zeta_{q,v})/\gamma^2} e^{-2\chi v^* (\zeta_{q,v}+\mu_{q,v})\tau/\gamma^2} \Bigg[e^{-2v(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/(\gamma^2 s_{q,v,b})} e^{-M_{q,v}(\tilde{v}_q+v')}$$
$$M_{q,v} \left(\frac{v'}{\tilde{v}_q}\right)^{\nu_v/2} I_{\nu_v}\left(2M_{q,v}(\tilde{v}_q\, v')^{1/2}\right)\Bigg] \frac{1}{2\sqrt{\pi Q_2(\tau)}}\, e^{Q_0(\tau,k,r) - \frac{1}{2}\left(\frac{r' - \tilde{Q}_1(\tau,k,r)}{\sqrt{2Q_2(\tau)}}\right)^2}$$
$$(x, v, r),\, (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R},\, t, t' \geq 0,\, t' - t > 0,\, q \in \mathbb{R}. \qquad [7.120]$$

Finally, substituting Eq.(7.120) into Eq.(7.1), we obtain the representation formula of the probability density function $p_f(x, v, r, t, x', v', r', t')$ as follows.

$$p_f(x, v, r, t, x', v', r', t')$$
$$= \frac{e^{q(x-x')}}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v) L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r),$$
$$(x, v, r),\, (x', v', r') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\, t, t' \geq 0,\; q \in \mathbb{R},\, t' - t > 0. \qquad [7.121]$$

Motivated by the fact that the derivative pricing often requires the integration of the transition probability density function with respect to the future variance, we deduce the analytical expression of the marginal probability density function, $D_v(x, v, r, t, x', r', t')$, of the future variables (x', r') with respect $v_{t'}$, $t' > 0$, given that $x_t = x$, $v_t = v$, $r_t = r$, $t > 0$, $t < t'$. To derive a simple expression for D_v, we use the following results (see Erdely et al. Vol I, 1954, p. 197 formula (18)):

$$P_v(\tau, v, k) = \int_0^{+\infty} dv' (v')^{\nu_v/2} I_{\nu_v}(2\tilde{M}_{q,v}(\tilde{v}_q v')^{1/2}) e^{-M_{q,v}\, v'} = \frac{(\tilde{v}_q)^{\nu_v/2}}{M_{q,v}} e^{M_{q,v}\tilde{v}_q},$$
$$v > 0,\, k \in \mathbb{R}, \qquad [7.122]$$

Using (7.122), we integrate the joint probability density function over the future variance v' to find the marginal probability density function of x', that is:

$$D_{v,q}(x, v, r, t, x', r', t') = \int_0^{+\infty} dv'\, p_f(x, v, r, t, x', v', r', t')$$
$$= e^{q(x-x')} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{\imath k(x'-x)}\, W_{v,q}^0(t'-t, v, k; \underline{\Theta}_v)\, L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r), \; [7.123]$$

More specifically,

$$D_v(x,v,r,t,x',r',t') = \int_0^{+\infty} dv'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} e^{-2\chi v^* \ln(s_{q,v,b}/2\zeta_{q,v})/\gamma^2} e^{-2\chi v^*(\zeta_{q,v}+\mu_{q,v})(t'-t)/\gamma^2} \cdot$$
$$e^{-2v(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/(\gamma^2 s_{q,v,b})} \cdot \frac{1}{2\sqrt{\pi Q_2(\tau)}} e^{Q_0(\tau,k,r)-\frac{1}{2}\left(\frac{r'-\tilde{Q}_1(\tau,k,r)}{\sqrt{2Q_2(\tau)}}\right)^2}, \quad [7.124]$$
$$(x,v,r) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ (x',r') \in \mathbb{R}\times\mathbb{R}^+,\ t,t' \geq 0,\ t'-t>0,\ q\in\mathbb{R}.$$

Integrating the transition probability density function with respect to the future variables v', r', we obtain the marginal density of $x_{t'} = x'$, $t' > 0$ with respect to $v_{t'}$, $r_{t'}$ given that $x_t = x$, $v_t = v$, $r_t = r$, $0 < t < t'$:

$$D_{v,r,q}(x,v,r,t,x',t') = \int_0^{+\infty} dr' \int_0^{+\infty} dv'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)} W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v) W^0_{r,q}(t'-t,r,k;\underline{\Theta}_r)\,, \quad [7.125]$$

or more specifically

$$D_{v,r,q}(x,v,r,t,x',t') = \int_0^{+\infty} dv' \int_0^{+\infty} dr'\, p_f(x,v,r,t,x',v',r',t') =$$
$$e^{q(x-x')} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)}\, e^{-2\chi v^* \ln(s_{q,v,b}/2\zeta_{q,v})/\gamma^2} e^{-2\chi v^*(\zeta_{q,v}+\mu_{q,v})(t'-t)/\gamma^2}$$
$$e^{-2v(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/(\gamma^2 s_{q,v,b})} \cdot e^{Q_0(t'-t,k,r)}$$
$$(x,v,r) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ x' \in \mathbb{R},\ t,t' \geq 0,\ t'-t>0,\ q\in\mathbb{R}. \quad [7.126]$$

In fact, the marginal probability density function, $D_{v,q}$, can be used to price European call and put options with payoff functions independent of the variance process in the framework of stochastic interest rates. Thus, we use $D_{v,q}$ to deduce formulas for European call and put vanilla options. As stressed in Christoffersen et al. 2009, a multi-factor model is more flexible for conditional kurtosis and skewness. Thanks to formula (7.129), we can easily compute these indicators. Finally, a useful byproduct of these formulas are the moments and the mixed moments associated with the Heston model.

It is worth to highlight that, in formulas (5.28) and (7.126), x, v, r are, respectively, the initial value of the (log-)price, the initial value of the stochastic variance and the initial value of the stochastic interest rate. These two last parameters are not observable in the financial market and should be estimated by using an appropriate calibration procedure. The estimation of the initial stochastic volatility is a common practice as suggested by Bühler, 2002.

We conclude Appendix B by deriving an explicit expression of the moments $\mathcal{M}_m(S, v, r, t, t') = E(S_{t'}^m \,|S_t = S,\ v_t = v,\ r_t = r)$. Thank to the "trick" used in the derivation of the formula for the transition probability density function, this is done in a very simple way. In fact, using (7.126) with $q = m$ and $S' = Se^{x'}$ (i.e. S is the price observed at time t, $t < t'$), we have:

$$\mathcal{M}_m = E(S_{t'}^m) = S_0^m \int_{-\infty}^{+\infty} dx' e^{mx'} D_{v,r,q}(0, v_0, r_0, 0, x', t') =$$
$$S_0^m \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \left(\int_{-\infty}^{+\infty} dx' e^{mx'} e^{-qx'} e^{\imath k\, x'} \right) W_{v,q}^0(t', v_0, k; \underline{\Theta}_v)\, W_{r,q}^0(t', r_0, k; \underline{\Theta}_r)$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ q \in \mathbb{R},\ t' > 0. \qquad [7.127]$$

Choosing $m = q$, the integral in the bracket gives us a delta Dirac's function of the conjugate variable k which allows us to have the following explicit formula for the moments. By virtue of the Dirac delta function properties, we have:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx'\, e^{mx'} e^{m(0-x')} e^{\imath k(x'-0)} = \delta(k), \qquad (7.128)$$

and

$$\mathcal{M}_m = E(S_{t'}^m) = S_0^m W_{v,m}^0(t', v_0, 0; \underline{\Theta}_v)\, W_{r,m}^0(t', r_0, 0; \underline{\Theta}_r),$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0. \qquad [7.129]$$

Similarly, using the explicit formulas for the integrals (3.26) and (3.27), we obtain the following explicit formulas for the mixed moments:

$$E(S_{t'}^{m_1} r_{t'}^{m_2}) = S_0^{m_1} W_{v,m_1}^0(t', v_0, 0; \underline{\Theta}_v)\, W_{r,m_1}^{m_2}(t', r_0, 0; \underline{\Theta}_r),$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0, \qquad [7.130]$$

and

$$E(S_{t'}^{m_1} v_{t'}^{m_2}) = S_0^{m_1} W_{v,m_1}^{m_2}(t', v_0, 0; \underline{\Theta}_v)\, W_{r,m_1}^0(t', r_0, 0; \underline{\Theta}_r),$$
$$(S_0, v_0, r_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ t' > 0, \qquad [7.131]$$

where the functions $W_{v,q}^0$, $W_{r,q}^0$, $W_{v,q}^m$, $W_{r,q}^m$, $m = 1, 2, \ldots$ are elementary functions given by (7.116), (7.111), (7.117) and (7.115) respectively.

7.3 Appendix C: The Analytical Treatment Of Mutiscale Hybrid Heston Model Under Risk Neutral Measure

In Appendix C, let us derive an integral representation formula under risk neutral measure for the transition probability density function of the process described by Eqs.(6.118)-(6.120) and initial conditions (6.129) where we identify the initial condition of random variables $\tilde{x}_0^{i,j}$, $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$ with the points where they are concentrated. We assume $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$, χ_n, λ_m, γ_k, η_m, v_n^*, θ_m to be positive constant. Moreover, we assume $\frac{2\chi_n v_n^*}{\gamma_n^2} > 1$ and $\frac{2\lambda_m \theta_m}{\eta_m^2} > 1$. In addition, we deduce the moments of the volatility variables and the mixed moments. For simplicity, here we just delete the index Q from every parameter and Brownian motion, i.e. $W_{n,t}^{p,v} = W_{n,t}^{p,v(Q)}$, $W_{m,t}^{p,r} = W_{m,t}^{p,r(Q)}$, $\chi_n = \chi_n^Q$, $v_n^* = v_n^{*(Q)}$ and so on so forth.

Let $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$, $(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2$, t, $t' \geq 0$, $t' - t > 0$, be the transition probability density function associated with the stochastic differential system (6.118),(6.119),(6.120), and (6.31), that is, the probability density function of having $x' = x_{t'}^{i,j}$, $\underline{v}' = (v_1', \ldots, v_d')^{\mathrm{T}}$, $\underline{r}' = (r_i', r_j')^{\mathrm{T}}$ given that $x = x_t^{i,j}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, when $t' - t > 0$. In analogy with Lipton (2001), this transition probability density function $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$ as a function of the "past" variables $(x, \underline{v}, \underline{r}, t)$ satisfies the following backward Kolmogorov equation:

$$
\begin{aligned}
-\frac{\partial p_f}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_f}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_f}{\partial r_i^2} \\
&+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_f}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_f}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_i^{2\alpha}} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_j} \\
&+\sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_f}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_f}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_f}{\partial r_j} \\
&+\left((r_i - r_j) - \frac{1}{2}\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha})\right)\frac{\partial p_f}{\partial x} \qquad [7.132] \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \leq t < t',
\end{aligned}
$$

with final condition:

$$\begin{aligned}
p_f(t,x,\underline{v},r,t'=t,x',\underline{v}',r') &= \delta(x'-x)\prod_{m=i}^{j}\delta(r'_m-r_m)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n) \\
&= \delta(x'-x)\cdot\delta(r'_i-r_i)\delta(r'_j-r_j)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n)\,, \qquad [7.133] \\
& \quad (x,v_n,r_m),(x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t\geq 0,
\end{aligned}$$

and the appropriate boundary conditions. Letting $\tau = t'-t$, we can introduce the function p_b defined as follows:

$$\begin{aligned}
&p_b(\tau,x,\underline{v},r,x',\underline{v}',r') = p_f(t,x,\underline{v},r,t',x',\underline{v}',r'), \\
&\qquad (x,\underline{v},\underline{r}),(x',\underline{v}',\underline{r}')\in\mathbb{R}\times(\mathbb{R}^+)^d\times(\mathbb{R}^+)^2,\ t'=t+\tau,\ \tau>0\,.
\end{aligned} \qquad (7.134)$$

The representation (7.134) holds since the coefficients of the Kolmogorov backward equation and condition (7.133) are invariant by time translation. Substituting the change of the time variable $\tau = t-t'$ into Eq.(7.132), it is worth nothing that p_b is the solution of the following problem:

$$\begin{aligned}
\frac{\partial p_b}{\partial\tau} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i-a_n^j)^2v_n+b_i^2r_i^{2\alpha}+b_j^2r_j^{2\alpha}\right]\frac{\partial^2p_b}{\partial x^2}+\frac{1}{2}\sum_{n=1}^{d}\gamma_n^2v_n\frac{\partial^2p_b}{\partial v_n^2}+\frac{1}{2}\eta_i^2r_i^{2\alpha}\frac{\partial^2p_b}{\partial r_i^2} \\
&+\frac{1}{2}\eta_j^2r_j^{2\alpha}\frac{\partial^2p_b}{\partial r_j^2}+\sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i-a_n^j)v_n\frac{\partial^2p_b}{\partial x\partial v_n}+\rho_{i,r}\eta_ib_ir_i^{2\alpha}\frac{\partial^2p_b}{\partial x\partial r_i^{2\alpha}}-\rho_{j,r}\eta_jb_jr_j^{2\alpha}\frac{\partial^2p_b}{\partial x\partial r_j} \\
&+\sum_{n=1}^{d}\chi_n(\overline{v}_n-v_n)\frac{\partial p_b}{\partial v_n}+\lambda_i(\theta_i-r_i)\frac{\partial p_b}{\partial r_i}+\lambda_j(\theta_j-r_j)\frac{\partial p_b}{\partial r_j} \\
&+\left((r_i-r_j)-\frac{1}{2}\sum_{n=1}^{d}(a_n^i-a_n^j)^2v_n-\frac{1}{2}(b_i^2r_i^{2\alpha}-b_j^2r_j^{2\alpha})\right)\frac{\partial p_b}{\partial x} \qquad [7.135] \\
&\qquad (x,v_n,r_i,r_j)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0\leq t<t',
\end{aligned}$$

with the initial condition:

$$\begin{aligned}
p_b(0,x,\underline{v},r,x',\underline{v}',r') &= \delta(x'-x)\prod_{m=i}^{j}\delta(r'_m-r_m)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n) \\
&= \delta(x'-x)\cdot\delta(r'_i-r_i)\delta(r'_j-r_j)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n)\,, \qquad [7.136] \\
& \quad (x,v_n,r_m),(x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t\geq 0,
\end{aligned}$$

and with the appropriate boundary conditions. For later convenience, let us consider the following change of dependent variable:

$$p_b(\tau, x, \underline{v}, r, x', \underline{v}', r') = e^{q(x-x')} p_q(\tau, x, \underline{v}, r, x', \underline{v}', r') \qquad [7.137]$$
$$(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2,\ t' = t + \tau,\ \tau > 0\,.$$

Substituting Eq.(7.137) into (7.135)– (7.136), we obtain that p_q is the solution of the following problem:

$$\begin{aligned}
\frac{\partial p_q}{\partial \tau} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_q}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_q}{\partial v_n^2} \\
&+ \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_q}{\partial r_i^2} + \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_q}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_q}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_q}{\partial x \partial r_i} \\
&- \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_q}{\partial x \partial r_j} + \sum_{n=1}^{d}\left[\chi_n(\overline{v}_n - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial p_q}{\partial v_n} \\
&+ \left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial p_q}{\partial r_i} + \left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial p_q}{\partial r_j} \\
&+ \left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\,(2q-1) + \left(r_i + \frac{r_i^{2\alpha}}{2}b_i^2(2q-1)\right) + \left(-r_j + \frac{r_j^{2\alpha}}{2}b_j^2(2q-1)\right)\right]\frac{\partial p_q}{\partial x} \\
&+ \left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left(q^2 - q\right) + \left(qr_i + \frac{r_i^{2\alpha}}{2}b_i^2(q^2-q)\right) + \left(q(-r_j) + \frac{r_j^{2\alpha}}{2}b_j^2(q^2-q)\right)\right]p_q \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t', \qquad [7.138]
\end{aligned}$$

with the initial condition:

$$\begin{aligned}
p_q(0, x, \underline{v}, r, x', \underline{v}', r') &= e^{q(x-x')}\delta(x'-x)\prod_{m=i}^{j}\delta(r'_m - r_m)\cdot\prod_{n=1}^{d}\delta(v'_n - v_n) \\
&= e^{q(x-x')}\delta(x'-x)\cdot\delta(r'_i - r_i)\delta(r'_j - r_j)\cdot\prod_{n=1}^{d}\delta(v'_n - v_n)\,, \qquad [7.139] \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ \tau = 0,
\end{aligned}$$

Now we consider the following representation formula for p_q with a Fourier transform:

$$p_q(\tau, x, \underline{v}, r, x', \underline{v}', r') = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\, f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k),$$
$$(\underline{v}, \underline{r}) \in (\mathbb{R}^+)^d, (\underline{v}', \underline{r}') \times (\mathbb{R}^+)^2,\ k \in \mathbb{R},\ \tau > 0. \qquad [7.140]$$

This is possible since the coefficients (7.135) and the initial condition (7.136) are independent of translation in the log-price variable. Substituting Eq.(7.140) into (7.135), we obtain that the function f is the solution of the following problem:

$$\frac{\partial f}{\partial \tau} = \frac{-k^2}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right] f + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 f}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 f}{\partial r_i^2}$$
$$+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 f}{\partial r_j^2} + \sum_{n=1}^{d}(-\imath k)\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial f}{\partial v_n} + (-\imath k)\rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial f}{\partial r_i}$$
$$-(-\imath k)\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial f}{\partial r_j} + \sum_{n=1}^{d}\left[\chi_n(v_n^* - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial f}{\partial v_n}$$
$$+\left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial f}{\partial r_i} + \left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial f}{\partial r_j}$$
$$+\Bigg\{\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left[(q^2 - q) - \imath k\,(2q-1)\right] + \Bigg(r_i(q - \imath k) + \frac{r_i^{2\alpha}}{2}b_i^2\big[(q^2 - q)$$
$$-\imath k(2q-1)\big]\Bigg) + \left(r_j(-q + \imath k) + \frac{r_j^{2\alpha}}{2}b_j^2\left[(q^2 - q) - \imath k(2q-1)\right]\right)\Bigg\}f \qquad [7.141]$$
$$(x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t',$$

with the initial condition:

$$f(0, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \delta(r_i' - r_i)\delta(r_j' - r_j)\prod_{n=1}^{k}\delta(v_i' - v_i)\,, \qquad [7.142]$$
$$(v_i, r_i, r_j), (v_i', r_i', r_j') \in \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ k \in \mathbb{R}.$$

Now let us represent f as the inverse Fourier transform of the future variables $(\underline{v}', \underline{r}')$ whose conjugate variables are denoted by $(\underline{l}, \underline{\xi})$, that is:

$$f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \left(\frac{1}{2\pi}\right)^{d+2}\prod_{n=1}^{d}\int_{\mathbb{R}} dl_n\, e^{\imath l_n v_n'}\cdot\prod_{m=i}^{j}\int_{\mathbb{R}} d\xi_m\, e^{\imath \xi_m r_m'}\, g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}),$$
$$(v_n, r_m) \in \mathbb{R}^+\times\mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau > 0. \qquad [7.143]$$

It is easy to see that the function g satisfies Eq.(7.141) with the following initial condition:

$$g(0, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) = e^{-\imath \xi_i r_i} e^{-\imath \xi_j r_j} \prod_{n=1}^{k} e^{-\imath l_n v_n}, \quad [7.144]$$

$$= \prod_{m=i}^{j} e^{-\imath \xi_m r_m} \prod_{n=1}^{k} e^{-\imath l_n v_n}, \quad [7.145]$$

$$(v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0.$$

Please note that g is the Fourier transform with respect to the future variables $(\underline{v}', \underline{r}')$ of the function obtained by extending f, as a function of the variables $(\underline{v}, \underline{r})$, with zero when $v_n \notin \mathbb{R}^+$ and/or $r_m \notin \mathbb{R}^+$. The coefficients of the partial differential operator appearing on the right hand side of (7.135) are first degree polynomials in $\underline{v}$ and $\underline{r}$ so that we seek a solution of problem (7.141), (7.142) in the form (see Lipton, 2001):

$$g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) = e^{A(\tau,k,\underline{l},\underline{\xi})} \prod_{m=i}^{j} e^{-r_m B_{r_m}(\tau,k,\xi_m)} \prod_{n=1}^{d} e^{-v_n B_{v_n}(\tau,k,l_n)},$$

$$= e^{A(\tau,k,\underline{l},\underline{\xi})} e^{-r_i B_{r_i}(\tau,k,\xi_i)} e^{-r_j B_{r_j}(\tau,k,\xi_j)} e^{-\sum_{n=1}^{d} v_n B_{v_n}(\tau,k,l_n)}, \quad [7.146]$$

$$(v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, \underline{l}, \underline{\xi}) \in \mathbb{R} \times (\mathbb{R})^d \times (\mathbb{R})^2,\ \tau > 0.$$

Substituting Eq.(7.146) into Eq.(7.141) and setting $\alpha = \frac{1}{2}$, we obtain that the functions $A(\tau, k, \underline{l}, \underline{\xi})$, $B_{v_n}(\tau, k, l_n)$, and $B_{r_i}(\tau, k, \xi_i)$, $B_{r_j}(\tau, k, \xi_j)$ must satisfy the following ordinary differential equations:

$$\frac{dA}{d\tau}(\tau, k, \underline{l}, \underline{\xi}) = -\lambda_i \theta_i B_{r_i}(\tau, k, \xi_i) - \lambda_j \theta_j B_{r_j}(\tau, k, \xi_j) - \sum_{n=1}^{d} \chi_n v_n^* B_{v_n}(\tau, k, l_n)$$

$$(k, l_n, \xi_i, \xi_j) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0, \quad [7.147]$$

$$\begin{aligned} &\frac{dB_{v_n}}{d\tau}(\tau, k, l_n) \\ &= \frac{k^2}{2}(a_n^i - a_n^j)^2 - \frac{(a_n^i - a_n^j)^2}{2}\left[\left(q^2 - q\right) - \imath k\,(2q-1)\right] \\ &\quad - \left[\chi_n + (\imath k - q)\gamma_n \rho_{n,v}(a_n^i - a_n^j)\right] B_{v_n} - \frac{\gamma_n^2}{2} B_{v_n}^2 \\ &= \varphi_q^{v_n}(k)(a_n^{i,j})^2 - (\chi_n + (\imath k - q)\gamma_n \tilde{\rho}_{n,v}) B_{v_n}(\tau, k, l_n) - \frac{\gamma_n^2}{2} B_{v_n}^2(\tau, k, l_n), \quad [7.148] \end{aligned}$$

$$(k, l_n) \in \mathbb{R} \times \mathbb{R}, \tau > 0$$

where $\varphi_q^{v_n}(k) = \dfrac{k^2}{2} - \dfrac{1}{2}\left[(q^2-q) - \imath k\,(2q-1)\right]$, $\tilde{\rho}_{n,v} = \rho_{n,v}(a_n^i - a_n^j)$.

$$\begin{aligned}\frac{dB_{r_i}}{d\tau}(\tau,k,\xi_i) &= \frac{k^2}{2}b_i^2 + (\imath k - q) - \frac{b_i^2}{2}\left[(q^2-q) - \imath k(2q-1)\right] \\ &\quad - \left[\lambda_i + (\imath k - q)\eta_i\rho_{i,r}b_i\right]B_{r_i}(\tau,k,\xi_i) - \frac{\eta_i^2}{2}B_{r_i}^2(\tau,k,\xi_i) \\ &= \varphi_q^{r_i}(k)b_i^2 + (\imath k - q) - \left[\lambda_i + (\imath k - q)\eta_i\rho_{i,r}b_i\right]B_{r_i}(\tau,k,\xi_i) - \frac{\eta_i^2}{2}B_{r_i}^2(\tau,k,\xi_i) \\ &\qquad (k,\xi_i) \in \mathbb{R}\times\mathbb{R}, \tau > 0, \end{aligned} \qquad [7.149]$$

where $\varphi_q^{r_i}(k) = \dfrac{k^2}{2} - \dfrac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right]$.

$$\begin{aligned}\frac{dB_{r_j}}{d\tau}(\tau,k,\xi_i) &= \frac{k^2}{2}b_j^2 + (q - \imath k) - \frac{b_j^2}{2}\left[(q^2-q) - \imath k(2q-1)\right] \\ &\quad - \left[\lambda_j + (\imath k - q)\eta_j\rho_{j,r}(-b_j)\right]B_{r_j} - \frac{\eta_j^2}{2}B_{r_j}^2 \\ &= \varphi_q^{r_j}(k)b_j^2 + (q - \imath k) - \left[\lambda_j + (q - \imath k)\eta_j\rho_{i,r}b_j\right]B_{r_j}(\tau,k,\xi_j) - \frac{\eta_j^2}{2}B_{r_j}^2(\tau,k,\xi_j) \\ &\qquad (k,\xi_j) \in \mathbb{R}\times\mathbb{R}, \tau > 0, \end{aligned} \qquad [7.150]$$

where $\varphi_q^{r_j}(k) = \dfrac{k^2}{2} - \dfrac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right]$. It is worth to highlight that, in comparison with the analytical treatment under physical measure in Section 5.2, the following equality holds. Indeed, this equality simplifies our computation of the FX option pricing formulas as well as correlating SDEs.

$$\varphi_q(k) := \varphi_q^{v_n}(k) = \varphi_q^{r_i}(k) = \varphi_q^{r_j}(k) = \frac{k^2}{2} - \frac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right] \qquad [7.151]$$

with initial condition:

$$A(0,k,\underline{l},\underline{\xi}) = 0, \quad B_{v_n}(0,k,l_n) = \imath\, l_n, \quad B_{r_i}(0,k,\xi_i) = \imath\,\xi_i, \quad B_{r_j}(0,k,\xi_j) = \imath\,\xi_j,$$
$$\text{with } (k,\underline{l},\underline{\xi}) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^2.$$

Eqs.(7.148), (7.149) and (7.150) are Riccati equdations that can be solved elementarily by substituting their solutions into Eq.(7.147) and integrating with respect to τ to obtain $A(\tau,k,\underline{l},\underline{\xi})$. The following steps are exactly the same with Section 6.2.1. We could get the joint probability density function (pdf) in Eq.(7.165) and the pdf over the future variance $\underline{v}'$ to find the marginal density for $(x',\underline{r}')$ in Eq.(7.166).

Let us derive the joint transition probability density function p_f in the case $\alpha = 1/2$, that is when the CIR interest rate model is considered. Considering integration on Eq.(7.147) for $A(\tau, k, \underline{l}, \underline{\xi})$, we obtain:

$$\begin{aligned}
&A(\tau, k, \underline{l}, \underline{\xi}) \\
&= -\sum_{n=1}^{d} \frac{2\chi_n v_n^*}{\gamma_n^2} \ln C_{v_n}(\tau, k, l_n) \; - \; \sum_{m=i}^{j} \frac{2\lambda_m \theta_m}{\eta_m^2} \ln C_{r_m}(\tau, k, \xi_m), \\
&= -\sum_{n=1}^{d} \frac{2\chi_n v_n^*}{\gamma_n^2} \ln C_{v_n}(\tau, k, l_n) \; - \; \frac{2\lambda_i \theta_i}{\eta_i^2} \ln C_{r_i}(\tau, k, \xi_i) - \; \frac{2\lambda_j \theta_j}{\eta_j^2} \ln C_{r_j}(\tau, k, \xi_j) \qquad [7.152]
\end{aligned}$$

Hence, the function $g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi})$ in Eq.(7.144) is given by:

$$\begin{aligned}
g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) = \prod_{n=1}^{d} &\left(e^{-\frac{2\chi_n v_n^*}{\gamma_n^2} \ln C_{v_n}(\tau,k,l_n)} \, e^{-\frac{2v_n}{\gamma_n^2} \frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \right) \prod_{m\in\{i,j\}} \Bigg(\\
& e^{-\frac{2\lambda_m \theta_m}{\eta_m^2} \ln C_{r_m}(\tau,k,\xi_m)} e^{-\frac{2r_m}{\eta_m^2} \frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \Bigg), \qquad [7.153] \\
& (v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+, \; (k, l_n, \xi_m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \; \tau > 0.
\end{aligned}$$

In order to obtain an explicit expression for $f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k)$ in Eq.(7.143), that is the inverse Fourier transform of $g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi})$ with respect to the variable $\underline{v}'$ and $\underline{r}'$, we have to compute the following integrals:

$$\begin{aligned}
&L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dl_n e^{\imath l_n v_n'} \, e^{-\frac{2\chi_n \bar{v}_n}{\gamma_n^2} \ln C_{v_n}(\tau,k,l_n)} \, e^{-\frac{2v_n}{\gamma_n^2} \frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \qquad [7.154] \\
&L_{r_m}(\tau, r_m, r_m', k \mid \underline{\Theta}_{r_m}) \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi_m \, e^{\imath \xi_m r_m'} e^{-\frac{2\lambda_m \theta_m}{\eta_m^2} \ln C_{r_m}(\tau,k,\xi_m)} e^{-\frac{2r_m}{\eta_m^2} \frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \qquad [7.155]
\end{aligned}$$

Let us show how to compute the integral appearing in (7.154) and (7.155) analogously by using Eq.(6.59) and (6.63) with the change of variable $l_n' = -l_n \frac{(\gamma_n)^2}{2} \frac{s_{q,v_n,g}}{s_{q,v_n,b}}$ and the following equality:

$$s_{q,v_n,d} s_{q,v_n,b} = \left(\zeta_{q,v_n}^2 - \mu_{v_n}^2\right) s_{q,v_n,g}^2 - \imath \, l_n' \left(\frac{8\zeta_{v_n}^2 e^{-2\zeta_{v_n}\tau}}{1 - 2\imath \, l_n'} \right), \qquad [7.156]$$

Thus, $L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n})$ can be written as follows:

$$\begin{aligned}
&L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) = \frac{1}{2\pi} M_{q,v_n} e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n}) + (\mu_{q,v_n} + \zeta_{q,v_n})\tau]} \; . \\
&e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2 - \mu_{q,v_n}^2) s_{q,v_n,g}/s_{q,v_n,b}} \int_{-\infty}^{+\infty} dl_n' \, e^{-(\imath \, l_n') M_{q,v_n}} e^{-(2\chi_n v_i^*/\gamma_n^2) \ln(1 - \imath \, l_n')} e^{\frac{(M_{q,v_n} \bar{v}_{q,n}) \imath \, l_n'}{1 - \imath \, l_n'}}, \qquad [7.157]
\end{aligned}$$

where

$$M_{q,v_n} = \frac{2}{\gamma_n^2}\frac{s_{q,v_n,b}}{s_{q,v_n,g}}, \quad \tilde{v}_{q,n} = \frac{4(\zeta_{q,v_n})^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s^2_{q,v_n,b}}\,,\; M_{q,v_n}\tilde{v}_{q,n} = \frac{8}{\gamma_n^2}\frac{\zeta^2_{q,v_n} v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,g}s_{q,v_n,b}}\,. \qquad [7.158]$$

Now using formula n.34 on p.156 in Oberhettinger 1973, we obtain:

$$\begin{aligned}&L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n})\\ &= e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n})+(\mu_{q,v_n}+\zeta_{q,v_n})\tau]}\, e^{-(2v_n/\gamma_n^2)(\zeta^2_{q,v_n}-\mu^2_{q,v_n})s_{q,v_n,g}/s_{q,v_n,b}} M_{q,v_n}\\ &\quad (M_{q,v_n}\tilde{v}_{q,n})^{-\nu_{q,v_n}/2}\,(M_{q,v_n}v_n')^{\nu_{q,v_n}/2}\, e^{-M_{q,v_n}\tilde{v}_{q,n}} e^{-M_{q,v_n}v_n'} I_{\nu_{q,v_n}}(2\tilde{M}_{q,v_n}(\tilde{v}_{q,n}v_n')^{1/2}),\\ &\qquad ,\text{with},\; v_n, v_n' > 0,\; k\in\mathbb{R}. \qquad [7.159]\end{aligned}$$

where $\nu_{q,v_n} = 2\chi_n v_n^*/\gamma_n^2 - 1$ and $I_{\nu_{q,v_n}}$ is the modified Bessel function of order ν_{q,v_n} (see, for example, Abramowitz and Stegun, 1970). Similarly, we obtain:

$$\begin{aligned}&L_{r_m}(\tau, r_m, r_m', k \mid \underline{\Theta}_{r_m})\\ &= e^{-(2\lambda_m\theta_m/\eta_m^2)[\ln(s_{q,r_m,b}/2\zeta_{q,r_m})+(\mu_{q,r_m}+\zeta_{q,r_m})\tau]}\, e^{-(2r_m/\eta_m^2)(\zeta^2_{q,r_m}-\mu^2_{q,r_m})s_{q,r_m,g}/s_{q,r_m,b}} M_{q,r_m}\\ &\quad (M_{q,r_m}\tilde{r}_{q,n})^{-\nu_{q,r_m}/2}\,(M_{q,r_m}r_m')^{\nu_{q,r_m}/2}\, e^{-M_{q,r_m}\tilde{r}} e^{-M_{q,r_m}r_m'} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,m}r_m')^{1/2}),\\ &\qquad \text{with},\; r_m, r_m' > 0,\; k\in\mathbb{R}, \qquad [7.160]\end{aligned}$$

where

$$M_{q,r_m} = \frac{2}{\eta_m^2}\frac{s_{q,r_m,b}}{s_{q,r_m,g}}, \quad \tilde{r}_{q,m} = \frac{4(\zeta_{q,r_m})^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s^2_{q,r_m,b}}\,, M_{q,r_m}\tilde{r}_{q,m} = \frac{8}{\eta_m^2}\frac{\zeta^2_{q,r_m} r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,g}s_{q,r_n,b}}\,,$$
$$\text{and } \nu_{q,r_m} = 2\lambda_m\theta_m/\eta_m^2 - 1\,, \quad m = i, j \qquad [7.161]$$

The following results are remarkable (see Abramowitz and Stegun 1970 pp. 375 and 486):

$$\begin{aligned}&P_{p,v_n}(\tau, v_n, k)\\ &= \int_0^{+\infty} dv_n'(v_n')^{\nu_{q,v_n}/2} I_{\nu_{q,v_n}}(2M_{q,v_n}(\tilde{v}_{q,v_n}v_n')^{1/2})e^{-M_{q,v_n}\,v_n'} = \frac{(\tilde{v}_{q,n})^{\nu_{q,v_n}/2}}{M_{q,v_n}} e^{M_{q,v_n}\tilde{v}_{q,n}},\\ &\qquad v_n > 0,\; k\in\mathbb{R}, \qquad [7.162]\end{aligned}$$

$$\begin{aligned}&P_{p,r_m}(\tau, r_m, k)\\ &= \int_0^{+\infty} dr_m'(r_m')^{\nu_{q,r_m}/2} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,r_m}r_m')^{1/2})e^{-M_{q,r_m}\,r_m'} = \frac{(\tilde{r}_{q,m})^{\nu_{q,r_m}/2}}{M_{q,r_m}} e^{M_{q,r_m}\tilde{r}_{q,m}}\,,\\ &\qquad r_m > 0,\; k\in\mathbb{R}, \qquad [7.163]\end{aligned}$$

Substituting Eqs.(7.159) and (7.160) into Eq.(7.142), we obtain:

$$f(\tau,\underline{v},\underline{r},\underline{v}',\underline{r}',k)=\prod_{n=2}^{d}L_{v_n}(\tau,v_n,v_n',k)\cdot L_{r_i}(\tau,r_i,r_i',k)L_{r_j}(\tau,r_j,r_j',k)$$
$$e^{-2\sum_{n=1}^{d}\chi_n v_n^*\ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\,e^{-2\sum_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})\tau/\gamma_n^2}\,e^{-2\sum_{m=i}^{j}\lambda_m\theta_m\ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}$$
$$\cdot e^{-2\sum_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})\tau/\eta_m^2}\cdot\Bigg\{e^{-2\sum_{n=1}^{d}v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\cdot$$
$$e^{-\sum_{n=1}^{d}M_{q,v_n}(\tilde{v}_{q,n}+v_n')}\prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v_n'}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2}\cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\,v_n')^{1/2}\right)\right]\Bigg\}$$
$$\cdot\Bigg\{e^{-2\sum_{m=i}^{j}r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\,e^{-\sum_{m=i}^{j}M_{q,r_m}(\tilde{r}_{q,m}+r_m')}\cdot$$
$$\prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r_m'}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\,r_m')^{1/2}\right)\right]\Bigg\},\qquad[7.164]$$
$$(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\,t,t'\geq 0,\,t'-t>0.$$

Substituting Eq.(7.164) into Eq.(7.140), we obtain the probability density function $p_f(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')$ as follows:

$$p_f(\underline{x},\underline{v},\underline{r},t,\underline{x}',\underline{v}',\underline{r}',t')=e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}}dk\,e^{\imath k(x'-x)}\cdot\Bigg\{$$
$$\cdot e^{-2\sum_{n=1}^{d}\chi_n v_n^*\ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\,e^{-2\sum_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2}$$
$$\cdot e^{-2\sum_{m=i}^{j}\lambda_m\theta_m\ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\,e^{-2\sum_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2}$$
$$\cdot\Bigg\{e^{-2\sum_{n=1}^{d}v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\,e^{-\sum_{n=1}^{d}M_{q,v_n}(\tilde{v}_{q,n}+v_n')}\cdot$$
$$\prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v_n'}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2}\cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\,v_n')^{1/2}\right)\right]\Bigg\}$$
$$\cdot\Bigg\{e^{-2\sum_{m=i}^{j}r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\,e^{-\sum_{m=i}^{j}M_{q,r_m}(\tilde{r}_{q,m}+r_m')}\cdot$$
$$\prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r_m'}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\,r_m')^{1/2}\right)\right]\Bigg\}\Bigg\},\qquad[7.165]$$
$$(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\,t,t'\geq 0,\,t'-t>0.$$

Thanks to Eq.(7.162), we integrate the joint probability density function in Eq.(7.165) over the future variance $\underline{v}'$ to find the marginal density for $(x', \underline{r}')$ as follows:

$$D_v(x, \underline{v}, \underline{r}, t, \underline{x}', \underline{r}', t') = \prod_{n=1}^{d} \left(\int_0^{+\infty} dv'_n \, p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t') \right)$$

$$= e^{q(x-x')} \frac{1}{2\pi} \int_{\mathbb{R}} dk \, e^{\imath k(x'-x)} \cdot \Bigg\{$$

$$\cdot e^{-2 \sum_{n=1}^{d} \chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2} e^{-2 \sum_{n=1}^{d} \chi_n v_n^* (\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2}$$

$$\cdot e^{-2 \sum_{m=i}^{j} \lambda_m \theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2} e^{-2 \sum_{m=i}^{j} \lambda_m \theta_m (\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2}$$

$$\cdot \left\{ e^{-2 \sum_{n=1}^{d} v_n (\zeta_{q,v_n}^2 - \mu_{q,v_n}^2) s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})} \right\}$$

$$\cdot \Bigg\{ e^{-2 \sum_{m=i}^{j} r_m (\zeta_{q,r_m}^2 - \mu_{q,r_m}^2) s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})} e^{-\sum_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m} + r'_m)} \cdot$$

$$\prod_{m=i}^{j} \left[M_{q,r_m} \left(\frac{r'_m}{\tilde{r}_{q,m}} \right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}} \left(2 M_{q,r_m} (\tilde{r}_{q,m} \, r'_m)^{1/2} \right) \right] \Bigg\} \Bigg\}, \qquad [7.166]$$

(x, v_n, r_m), $(x', v'_n, r'_m) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t, t' \geq 0$, $t' - t > 0$.

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