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Dr. Irina Belousova
Dottorato di Ricerca in Economia Politica
(XXIX ciclo)

Supervisor: Prof. Antonio Palestrini
PhD program coordinator: Prof. Riccardo Lucchetti

# THE ROLE OF ENDOGENOUS CAPITAL DEPRECIATION RATE IN DYNAMIC STOCHASTIC GENERAL EQUILIBRIUM MODELS

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Author's institutional e-mail: i.belousova@univpm.it

Author's institutional address:

Department of Economics and Social Sciences
Università Politecnica delle Marche
Piazzale Martelli, 8
60121 Ancona – Italia
tel. +39 071 220 7108
fax. +39 071 220 7102
web: http://www.dises.univpm.it

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**Abstract**

The main objective of the thesis is to investigate the optimal convergence dynamics of the main real economic variables in a Dynamic Stochastic General Equilibrium framework with variable depreciation rate of physical capital and maintenance and repair goods and services as a control variable of the agents. We define an explicit depreciation rate function which is positively related with capital utilization rate and negatively related with maintenance to capital ratio. Along the balanced growth path depreciation rate exhibits a growth trend given by the steady state value of the investment-specific technology progress. We include three types of technological progresses: the labor augmenting technology progress, the investment-specific technology progress (IST) and the marginal efficiency of investment technology progress (MEI). We compare the model with endogenous depreciation and maintenance sector to our baseline DSGE model, which is built following Justiniano et al. (2011). The estimation exercises of our maintenance model, performed on the Canadian economy, confirm the results of Justiniano et al. (2011) according to which the main driver of the business cycle fluctuations is the shock to the marginal efficiency of investment whereas the role of the IST shock is negligible. In response to the MEI shock our model is able to generate co-movement in all the considered real endogenous variables including consumption which in Justiniano et al. (2011) behaves countercyclically. The optimal paths result to be amplified and convergence is delayed as a consequence of increased depreciation due to obsolescence, which destroys part of installed capital. We as well include in our model a shock which affects the transformation process of the maintenance goods, named the maintenance-specific technology progress. In long run this shock is found to have no effect in the variations of the main real variables except of a low effect on the maintenance growth. On the contrary, it becomes the key-driver of maintenance growth in short run.

## Introduction

This thesis investigates about the possible role that an endogenous depreciation rate of capital could play in the propagation mechanism of the shocks to the economic business cycles. For this purpose, we build and estimate a Dynamic Stochastic General Equilibrium model (DSGE) and explicitly define an endogenous capital depreciation rate function. The production of maintenance goods is as well included in our model structure.

In general, the issue about a variable depreciation rate does not find much room in the endogenous growth models. In fact, it is largely used to be assumed that the capital stock depreciates at a constant rate over time. In support to this hypothesis the economic literature is extensive[1]. On the contrary, and specifically in the microeconomic studies[2], evidences arise in favor of a variable capital depreciation rate especially when the period of intensive technological progress is considered. It is found that over the past two decades the depreciation rate of capital goods has accelerated, with a persistent growth in the high-tech sectors. This happens because, as explained by Keynes (1936), when a new investment good becomes more efficient more capital with higher qualitative characteristics is produced, this induces a more intensive use of the already installed capital and, hence, an acceleration in its depreciation rate. Therefore, in the era of accelerated technological improvements the aspect of capital obsolescence is crucial. Two important issues for our research purposes stem from the Keynes' explanation. The first one is the evidence about the volatility of capital depreciation. The second one deals with the type of technology shock that contributes to amplify this volatility, that is, not all the shocks are responsible for the capital depreciation acceleration. In our DSGE model we assume three types of technology shocks already known in the related literature: the neutral labor-augmenting technological shock, the investment-specific technological shock (IST) and the shock to the marginal efficiency of investment (MEI). In line with Justiniano et al. (2011), we assume that the former one affects directly the production of intermediate goods and impacts on the marginal productivity of the production factors. The IST progress hits the production technology of new investment goods. Finally, the MEI shock impacts on the production of new capital goods. The model structure is decentralized, following Justiniano et al. (2011), and each sector deals with the proper optimization problem. Households acquire new capital from the new capital goods producers in order to transform it into installed capital and, subsequently, rent it to intermediate goods producers by choosing an optimal utilization rate. The new capital goods are produced using the investment goods, which, in turn, are obtained from the final output by the new investment goods producers. Both the markets are perfectly competitive. In addition to the model built in Justiniano et al. (2011), we assume that households can optimally choose the level of expenditures on maintenance and repair activity of installed capital. The maintenance goods producers operate in perfect competition and transform a fraction of final good into maintenance goods, subject to specific costs of adjustment. The maintenance goods are purchased by households given the depreciation path of capital stock. We assume the existence in this transformation process of a shock which we call the maintenance-specific technology shock (MST). According to our

[1] See, for example, Smets and Wouters (2007), Justiniano and Primiceri (2008), Justiniano et al. (2010, 2011) for DSGE models with constant depreciation rates. And Epstein and Denny (1980), Hulten and Wykoff (981a,b), Nadiri and Prucha (1996), Jorgenson (1996), Oliner (1996), Huang and Diewert (2011) for the estimations of the depreciation rates.

[2] See, for example, Tevlin and Whelan (2003), Doms et al. (2004), Geske et al. (2007) and Angelopoulou and Kalyvitis (2012).

estimation results, the MST progress plays a cardinal role in explaining the variations in maintenance solely, however only in the short run. Following Albonico et al. (2014), Boucekkine and Ruiz-Tamarit (2003), and Licandro and Puch (2000) we assume that maintenance to capital ratio and the utilization rate of capital both shape explicitly the function of the depreciation rate. Specifically, it is positively related with the rate of capital utilization, as has been shown in Greenwood et al. (1988), and negatively related with the maintenance to capital ratio, according to McGrattan and Schmitz (1999).

Given our model settings and assumptions it emerges that capital depreciation rate follows a trend which depends along the balanced growth path on the steady state level of the investment-specific technology progress. This equilibrium driven result is in line with the hypothesis dominating the issues about measurement of aggregated capital stock, according to which the rate of depreciation is inversely related with the relative price of new investment.[3] We perform Bayesian estimation exercises on both the baseline model and the maintenance model using data from the Canadian economy and compute the impulse response functions in order to compare the performances of the two models. Our baseline model delivers qualitative optimal behavior of the real endogenous variables similar to the model of Justiniano et al. (2011). From the comparison of the posterior estimates for the second moments and correlations of the real variables it emerges that the maintenance model mimics the respective values calculated on the actual data better than the baseline model. Overall, the maintenance model seems to perform fairly well as it is shown in the identification analysis on the structural parameters and in the multivariate convergence analysis.

Both the baseline model and the maintenance model confirm the results obtained in Justiniano et al. (2011) according to which the main driver of the economic business cycle is the shock to the marginal efficiency of investment. In fact, it explains more than 21% of variability in output in the baseline model and more than 61% in the maintenance model. The effects of the investment-specific technology shock are negligible, being responsible for 2.36% and 0.30% of output growth in the baseline and maintenance models, respectively. These are even lower than the figures estimated for the labor augmenting technology progress which explains 14.34% and 3.49% of output growth, respectively. Differently from the results obtained both in Justiniano et al. (2011) and in our baseline model, it emerges that the maintenance model is able to generate co-movement in response to the MEI shock for all the considered real variables including consumption, which in the other two models is countercyclical. In fact, the assumption about variable depreciation rate and the presence of nominal frictions imply, among others, higher variations in the mark-ups of the maintenance model which, in turn, allow consumption and labor to co-move. Both maintenance and capital depreciation rate are as well pro-cyclical. Finally, our estimation analysis confirms the literature findings according to which capital depreciation is more volatile than output.

The rest of the thesis is structured as follows. In chapter one we present the literature review which is relevant for understanding our research motivations and the basis of our assumptions about the model setting. In chapter two we describe analytically the structure of the maintenance model, compute the optimality conditions, log-linearize the model around the steady state and expose the linear rational expectations model which is used in our estimation analysis. The posterior estimation results are shown in chapter three, where we as well explain our estimation methodology and the construction of the dataset. Finally, we present a comparison between the performances of the maintenance and the baseline models resorting to the estimated impulse response functions.

[3] See, among others, Diewert and Schreyer (2006).

# Chapter 1

## 1 Literature Review

In this chapter the main research that have signed the route for the construction of endogenous capital depreciation rate and for the maintenance importance in the economic activity is presented. We start by exposing mainly the improvements in the modeling of the capital depreciation rate function. Two works are of a great interest in this literature, the first one is Greenwood et al. (1988), which stresses the importance of the factor hoarding assumption in reproducing the economic business cycles. The second one, McGrattan and Schmitz (1999), highlights the possible role of maintenance activity, which additionally may impact on the rate of capital depreciation, and can be listed as a decision policy of the economic agents. Then, the attention is delivered to the interpretation of the depreciation rate within the setting of vintage capital models. Insights on the economic behaviors of depreciation and maintenance in response to neutral and investment-specific technological progresses are presented. Finally, the recent findings with regard to the DSGE models with and without endogenously determined capital depreciation rate are highlighted.

The chapter is structured as follows. In section 1.1 we expose the main issues about the capital depreciation rate, with a particular attention to its analytical definition and methodologies of estimation. In section 1.2 are described the main models that gave importance to the utilization rate of capital in determining the depreciation rate. In section 1.3 we sketch the first steps in the Real Business Cycle modeling with depreciation depending both on utilization and maintenance expenditures. In section 1.4 several Vintage Capital models are presented which have analyzed the economic behavior of capital depreciation and maintenance expenditures in response to the technological progresses. In section 1.5 we describe two Dynamic Stochastic General Equilibrium models. The first one includes endogenous depreciation rate function. The second one is used as the baseline model for our research purposes.

### 1.1 Some facts on the depreciation rate of capital

Knowing how to better evaluate the depreciation rate of capital in order to obtain consistent measurements of it is important for the policy makers at least for two reasons. First of all, depending on the way the capital is evaluated, and hence, on how depreciation is computed, different taxations of capital may arise. Tax policy, in fact, is strictly related to the concepts of depreciation and revaluation of capital[4]. Second, depreciation is accounted for in the measurements of the total factor productivity growth, which is largely employed in the evaluation of the economic performances. Tevlin and Whelan (2003), in fact, in explaining the U.S. investment boom over the 1990s, conclude that "...an increasing

[4] Doms et al. (2004), for example, provide a detailed analytical and empirical evidence of the tight linkage between tax code and depreciation.

depreciation rate was of first-order importance in the extraordinary behavior of equipment investment (of the U.S.) in the 1990s...". The aggregate capital of an economy is used to be represented in the National Accounts Systems as a fixed reproducible tangible wealth, which is associated with the private and government equipment and structures and the durable goods owned by consumers. The measurements of fixed reproducible tangible wealth depend on the obtained a priori measurements of the capital depreciation rates. In general, as far as capital is a durable good, over time, the value of using it differs from that of owning it. Therefore, first of all, a distinction must be made between the capital stock and the flow of capital services. The latter ones are usually defined as those capital goods that are purchased or rented by the firms in order to constitute the actual input in the production process. Specifically, capital services are the flows of productive services accruing from the cumulative past investments. The prices of these capital goods are referred to as user costs or rental prices of capital.[5] The stock of capital, on the other hand, is defined as the discounted stream of future rental payments for capital services that it is expected to yield. The price of an $n$ years old asset, $P_t^n$, purchased at the beginning of period $t$, can be defined, using the terminology of Diewert and Schreyer (2006), as follows

$$P_t^n = f_t^n + \left[\frac{1+i_t}{1+r_t}\right] f_t^{n+1} + \left[\frac{1+i_t}{1+r_t}\right]^2 f_t^{n+2} + \left[\frac{1+i_t}{1+r_t}\right]^3 f_t^{n+3} + \cdots \tag{1.1}$$

with $n = 0, 1, 2...$, $f_t^n$ denoting the rental price (or user cost) of an $n$ years old asset at the beginning of period $t$, $i_t$ the expected rates of change of rental prices observed at the beginning of period $t$, and $1 + r_t$ the discount factor. Thus, $r_t^n$ defines the sequence of one-period nominal interest rate. The above equation can be re-expressed as

$$P_t^n = f_t^n + \left[\frac{1+i_t}{1+r_t}\right] P_t^{n+1}$$

or, else

$$\begin{aligned} f_t^n &= (1+r_t)^{-1}\left[P_t^n(1+r_t) - P_t^{n+1}(1+i_t)\right] = \\ &= \left[P_t^n - P_t^{n+1}\frac{1+i_t}{1+r_t}\right] \end{aligned} \tag{1.2}$$

The latter expression says that the user cost,$f_t^n$, is given by the difference between the purchase price of the $n$ years old asset at the beginning of the period $t$, $P_t^n$, and the value of its depreciation at the end of period $t$, i.e. $(1+i_t)\, P_t^{n+1} = P_{t+1}^{n+1}$, which must be divided by the discount rate, $(1 + r_t)$, as far as the loss in value due to depreciation may be accounted for only at the end of each period.

The depreciation rate, $\delta_t^n$, of an $n$ years old asset at the beginning of period $t$ is, therefore, defined as

$$\delta_t^n = 1 - \frac{P_t^{n+1}}{P_t^n}$$

Hence, given the sequence of $\{\delta_t^n\}$, and the price of a new asset in period $t$, one can determine the sequence of stock prices, $\{P_t^n\}$, as follows

$$P_t^n = \left(1-\delta_t^0\right)\left(1-\delta_t^1\right) ... \left(1-\delta_t^{n-1}\right) P_t^0 \tag{1.3}$$

[5] A thin distinction among the two concepts, however, exists. The rental price of capital, in fact, is the actual market price attributed to the use of the capital. The user cost, instead, is the cost that the capital-owner implicitly supports when using the capital.

with $n = 0, 1, 2....$ Therefore, the sequence of user costs $\{f_t^n\}$ can be expressed in terms of the new asset price at the beginning of period $t$, $P_t^0$, and the sequence of depreciation rates $\{\delta_t^n\}$, that is

$$\begin{aligned} f_t^n &= (1+r_t)^{-1}\left(1-\delta_t^0\right)...\left(1-\delta_t^{n-1}\right)\left[(1+r_t)-(1+i_t)\left(1-\delta_t^n\right)\right]P_t^0 \\ &= (1+r_t)^{-1}\left[r_t+\delta_t^n\left(1+i_t\right)-i_t\right]P_t^n \end{aligned} \tag{1.4}$$

with $n = 0, 1, 2....$

Given the above relationships, it is evident that the sequence of depreciation rates, the patterns of the user costs (or the rental prices) and the patterns of the asset prices are strictly correlated among them. Therefore, the more efficient the calculation of these values, the better the valuation of capital stocks and flows, which, in turn, are necessary for the calculations, as stated above, of the gross domestic and national products or of the total factor productivity or, else, for the implementation of capital taxation codes.

Diewert (1996) reports a detailed literature review about the role and the interpretation of the capital depreciation rate in the economy over the last decades. According to him, the first traces witnessing the need to account for losses in the value of capital stocks can be dated back to yearly 1880s, when the U.S. railroads had begun widely to spread out. Different approaches in defining and evaluation of capital depreciation rates have been followed, starting with simple appraisals in the value changes of the capital and ending with statistical estimations of the physical deteriorations and of the losses in productive lives of capital. Over the yearlies 1990s the problem of evaluating the depreciation rates has shifted towards that of "funding the future replacement of durable input" (See, for example, Taylor (1923) and Canning (1929)). This approach, named the sinking fund approach, sets the question about the retirement date of the capital asset and its replacement costs. However, it had been promptly discredited over the 1930s for several reasons (See Daniels (1933) and Gilman (1939)). Thereafter, the methodology advanced by Böhm-Bawerk (1891) has been deepened, according to which capital assets depreciate simply following a straight line path. Specifically, calling $P^0$ the initial cost of an asset, this must be allocated over the estimated $n$ periods of the asset service life.[6] The single period depreciation rate is, thus, calculated as a constant ratio $\delta = 1/n$ and is comprised between zero and unity, with the respective historical cost given by the sequence of $n$ allocations $(1/n)\,P^0$,..., $(1/n)\,P_0$, which, in current values, is equivalent to $(1/n)\,P_1$,..., $(1/n)\,P_n$. Akin to the straight line approach, the declining balance method assumes that the depreciation rate follows a constant geometric rate path, $\delta$, comprised between zero and unity, with the corresponding sequence of the periodic current cost given by $\delta P_1$, $\delta\,(1-\delta)\,P_2$, $\delta\,(1-\delta)^2\,P_3$, $\delta\,(1-\delta)^3\,P_4$, ..., $\delta\,(1-\delta)^n\,P_{n+1}$. Both the methods, however, suffer from the criticism about the *a priori* assumption regarding the asset service lifetime, which, indeed, is unobservable (see, for example, Preinreich (1938), Lutz and Lutz (1951)).

Over the second half of the 1990s calculations of capital depreciation based on the market value estimations, rather than on the historical cost estimations, have been highlighted (see Hicks (1939), Parker (1975), Griliches (1963), Edwards and Bell (1965)). According to this methodology, the decline in value of a durable asset can be computed by comparing the market prices of used assets with the original prices of the assets and to account for the differences among the two values, given that a second hand market for durable assets exists. Specifically, knowing the purchase price of a new durable asset, $P_0$, which is to be used for one period, and knowing the average market price in the next period of the same asset, $P_1$, the one period depreciation rate, $\delta_0$, is defined as

[6] Note that, this approach implies knowing a priori the total assets useful lifespan.

$P^1/P^0 = (1 - \delta^0)$

Generalizing for an $n$ years old asset it delivers the expression (1.3).

If the depreciation rate is assumed to be constant over time, i.e. $\delta^n = \delta^0$ for $n = 1, 2, ..., N - 1$, then equation (1.3) delivers the declining balance (or geometric) depreciation pattern. Empirical studies based on equation (1.3) have been carried out by Beidleman (1973, 1976), Hall (1971), Hulten and Wykoff (981a,b,c, 1996), Oliner (1996), and Jorgenson (1996). Of course, this approach is not free of criticisms (see Lachmann (1941), Baxter and Carrier (1971), and the same Hulten and Wykoff (1996))[7]. The most important one concerns the drawback of this methodology in accounting for the variations in the intensity of use of the durable assets. With regard to this, Edwards and Bell (1965) have suggested to construct sets of assets similar from the points of view of the intensity of use and maintenance policies and to estimate the set-related depreciation rates. This approach has been named by Jorgenson (1996) the "analysis of variance approach". Alternatively, to override this problem, Beidleman (1973, 1976) has used utilization and maintenance as explanatory variables in order to estimate equation (1.3). This latter approach Jorgenson (1996) has named the "hedonic approach".

Other estimation methodologies of the capital depreciation rate rely on some explicit economic assumptions. One of these approaches, the production model or factor demand model, is based on the estimation of the firms production function. So, for example, it is used to assume that a firm, over the period $t$, produces $y_t$ units of output using a vector $x_t$ of nondurable inputs, $I_t$ units of durable inputs purchased at the beginning of period $t$, and the remaining services of past durable inputs. A durable input lasts $N$ periods and every period is adjusted for the physical loss in efficiency. Moreover, it is usually assumed that unretired durable inputs are perfect substitutes in the production function $F$. Regardless the technological progress, the production function may be expressed as follows

$$y_t = F\left[x_t, I_t + (1 - \delta_0) I_{t-1} + (1 - \delta_0)(1 - \delta_1) I_{t-2} + \cdots + (1 - \delta_0) \cdots (1 - \delta_{N-1}) I_{t-N}\right] \tag{1.5}$$

Once the functional form of $F$ is defined, the stochastic processes are added, and provided that the data on output, and nondurable and durable inputs are available, the production function (1.5) is used to simultaneously estimate all the unknown parameters of the statistical functional form together with all the depreciation rates considered in the function. Estimations can also be carried out using the dual cost or profit function, or adding restrictions on the depreciation parameters, or, else, using assumptions on short run profit maximizing or cost minimizing behaviors which incorporate additional estimating equations with prices. These kinds of estimation methods have been implemented by Epstein and Denny (1980), Pakes and Griliches (1984), Prucha and Nadiri (1991), Nadiri and Prucha (1996) and Doms (1996). In general, this approach manifests some drawbacks as far as aggregating a wide range of outputs, and nondurable and durable inputs may result to be inefficient and cumbersome. Therefore, the resulting estimated values of the depreciation rates, in general, may differ considerably from the ones obtained using, for example, the used asset approach. However, different estimated values may be obtained also when the same methodology is implemented. In fact, the average economic depreciation rate in the U.S. total manufacturing sector over the period 1947-1971 is about 0.125 in the studies carried out by Epstein and Denny (1980) and Kollintzas and Choi (1985), which is slightly higher than the one obtained by Bischoff and Kokkelenberg (1987) over 1947-1978, which amounts to

[7] Wykoff (2003), for example, with regard to the used-asset prices based estimations carried out by Hulten and Wykoff (981a,b) has argued that the related estimated depreciation rates tend to be biased upward, thus the resulted capital stocks are biased downward.

0.106. The estimation obtained by Nadiri and Prucha (1996) is slightly higher than the one estimated by BEA over the period 1960-1988, which are 0.059 and 0.034, respectively. Prucha and Nadiri (1991) estimate the capital depreciation rate for the U.S. market assuming endogenous capital utilization in a dynamic factor demand model. According to their estimation results, capital depreciation rate ranges between 0.028 and 0.043 over the period 1959-1980 with a pick around 1973. The resulted average depreciation rate over the same period is about 3.8%. The depreciation rates estimated by Hulten and Wykoff (981c)[8] using the used market prices approach averaged, per year, 0.133 for equipment and 0.037 for structures, whereas those of BEA were, on average, 0.141 and 0.060, respectively. Considering specific sectors, the depreciation rate for autos estimated by Hulten and Wykoff (981c) was about 33%, that of BEA 13%, and those of Wykoff (1970), Cagan (1971) and Ackerman (1973) were between 17%-27%, 21%-30%, and 28%-34%, respectively. As to the machine tools the estimated depreciation rates were 12%, 13%, and in the range of 4%-21%, respectively, for Hulten and Wykoff (981c), BEA, and Beidleman (1976).

Another estimation approach is based on the observed rental prices of assets, assuming that a rental market for durables exists (see, for example, Jorgenson (1996)). In general, the functional form for the rental price, $f_t^n$, of an $n$ years old asset observed in period t is given by equation (1.4). This kind of expression can be used to compute the period related depreciation rates $\delta_t^n$, by assuming that the rental prices of durable assets are equivalent to the opportunity costs for the utilization of those assets over the same periods. However, as Diewert (1996) points out, the estimations of the capital depreciation rates generated by this approach may be inaccurate, especially when prices are highly volatile over the accounting periods. In fact, the rental prices of the assets are usually determined at the beginning of each period, whereas the prices of the used assets at the end of each period. For this reason an anticipated inflation rate for period $t$, $i_t$, is incorporated in the expression (1.4). So, computations of the depreciation rates using this approach require *a priori* assumptions about the anticipated rate of inflation, which, once again, generates uncertainty over the estimation procedure.

Finally, making use of equations (1.1)-(1.3) the one hoss shay depreciation estimation approach can be carried out according to which an asset yields constant levels of services over its useful life of $N$ years, i.e. $f_t^n/f_t^0 = 1$ for $n = 0, 1, 2, ..., N-1$ and zero thereafter. In contrast, the linear efficiency decline approach assumes that the sequence of relative efficiency declines with the asset's age, that is $f_t^n/f_t^0 = [N-n]/N$ for $n = 0, 1, 2, ..., N-1$, and $f_t^n/f_t^0 = 0$ for $n = N, N+1, N+2, ....$

Other methods for calculation of capital depreciation rates have been used in the literature. Beyond those listed above, the retirement approach focuses on the estimation of assets retirement, the investment approach is based on investment models, and the polynomial benchmark approach uses the perpetual inventory method. The most exhaustive studies about the capital depreciation rate based on used-asset prices, i.e. those of Hulten and Wykoff (981a,b), Koumanakos and Hwang (1988), and Coen (1975) conclude that the best fit in the manufacturing industries is provided by the geometric pattern [9]. BEA, in particular, makes use of the estimates carried out in Hulten and Wykoff (981a,b,c) for computation of depreciation rates in the manufacturing sector. With regard to the category of computers, the estimations made by Oliner (1992, 1993) are used. For automobiles the broad availability

[8] The authors have found that the declining balance rates for equipment and structures were, respectively, 1.65 and 0.91.

[9] Note that, under the geometric pattern assumption, the functions of efficiency and age-prices are the same, therefore the value of capital replacement equals capital depreciation.

of actual observations is used. Finally for nuclear fuel and missiles it uses the straight-line depreciation pattern in order to account for replacement and retirement aspects of these categories of assets.

Given the importance of a correctly estimated economic depreciation rate in the national accounts, over the 80s and 90s the attention of the U.S. statistical bureaus have been shifted towards the collection of as much as possible evidence about the service lives, rates of decay and economic depreciation of the capital assets. Basing on the gathered information, both the Bureau of Labor Statistics (BLS) and the Bureau of Economic Accounts (BEA) have revised their methodological measures of the capital depreciation rates and the estimations of their service lives. Specifically, BEA has started to assume a geometric depreciation path, rather than the straight-line one implemented till then. On the contrary, BLS introduced a hyperbolic age-efficiency pattern in order to produce consistent estimations of the U.S. productivity measures. The new improved methodologies used for the estimation of the economic depreciation rates exploit the observability of prices of used assets in the resale markets. Such resale prices have witnessed in favor of a geometric depreciation path, according to which capital assets are subject to a faster depreciation in the earlier years of their economic lives than in the later ones. According to the Systems of National Accounts depreciation rate of capital, which is usually labeled "consumption of fixed capital", reflects the cost deriving form the use of capital, therefore, it is crucial for the calculation of the net values of domestic and national products. Both the estimations of the capital net stock and of depreciation are based on the perpetual inventory method, which assumes that the net stock is given by the cumulative values of gross past investments minus the cumulative values of past depreciations. Moreover, since 1996 BEA has started to calculate all the real expenditure aggregates entering the National Income and Product Accounts (NIPA) using the Fisher chain aggregate, that is

$$g_t = \sqrt{\frac{\sum_{i=1}^{n} p_t^i q_t^i}{\sum_{i=1}^{n} p_t^i q_{t-1}^i} \frac{\sum_{i=1}^{n} p_{t-1}^i q_t^i}{\sum_{i=1}^{n} p_{t-1}^i q_{t-1}^i}}$$

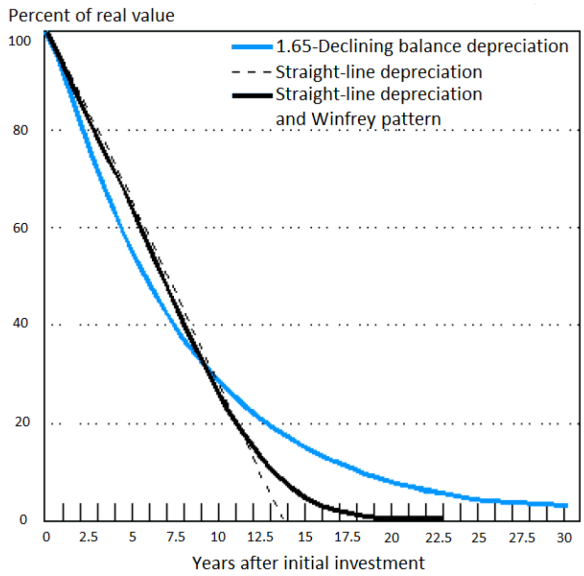
where $p_t^i$ and $q_t^i$ are, respectively, prices and quantities of the assets and $g_t$ determines the growth rate of the Fisher chain aggregate, which captures thus a mix of old and new quantities and prices of the assets.

As stated above, before the early 1990s BEA used to calculate the capital depreciation rates on the basis of the straight-line profile and of an explicitly assumed retirement pattern of the assets. Specifically, all the depreciation rates were assumed to decline to zero at constant rates each period. The service lives of the same asset types were assumed to be distributed about a mean service life of an asset according to one of the Winfrey's retirement patterns. The new improved methodology implemented thereafter, instead, estimates the depreciation rates assuming a geometric profile. Thus, given the properly calculated declining-balance rates[10], these are divided by the value of assumed assets service lives, in order to obtain the revised according to BEA values of depreciations. Moreover, as far as informations about changes in the assets lifetimes are often unavailable, these are used to be kept constant over time and periodically adjusted. Figure 1.1, taken from the BEA's official website, illustrates the different depreciation paths generated by a geometric estimation methodology, a straight-

[10] In general, the values used for the declining-balance rates were those calculated by Hulten and Wykoff (981a,b,c). The assumed declining-balance rate for equipment, for example, is 1.65, which according to BEA is common to many types of assets, whereas that for structures is 0.91.

line methodology, and a straight-line with Winfrey retirement patterns methodology for a typical type of equipment with an average service life of 15 years.

Fig. 1.1: Typical depreciation profiles for equipment with 15-year service life




U.S. Department of Commerce, Bureau of Economic Analysis

As it can be observed, over the first 10 years the geometric profile exhibits the most accelerated path, which becomes smoother in the latest years of the asset lifetime. The two types of the straight-line profiles, instead, are almost similar with exception of the latest periods, when the Winfrey retirement assumptions make the asset to survive longer.

BEA defines depreciation as "the decline in value (of an asset) due to wear and tear, obsolescence, accidental damage, and aging". This definition includes also the retirements and discards of the assets, which imply a withdrawal of the asset from the service. Therefore, depreciation is treated as a cost in the formation of gross domestic product or in the calculation of business income. Moreover, this definition is in line with the System of National Accounts, which defines depreciation as "the decline, during the course of the accounting period, in the current value of the stock of fixed assets owned and used by a producer as a result of physical deterioration, normal obsolescence, or normal accidental damage". BEA accounts for the obsolescence effect by incorporating the service lives of the assets in the calculation of the geometric depreciation rate, i.e. an asset which is still productive may be withdrawn from production due to the availability of new more productive assets. It as well uses hedonic or other quality-adjusted price indexes which are able to capture the effects of technological improvements. Of course, employing used-asset prices may generate distorted measures of depreciation either because of changes in economic conditions (for example changes in tax laws or interest rates) that affect prices or because the asset sample does not properly represent the whole asset population (especially when the asset retirement is to be accounted for).

The new measurements of depreciation for the U.S. National Income and Product Accounts, which are published by BEA, have been constructed by Fraumeni (1997). As stated above, these measurements make use of the revealed prices of used assets in the resale markets, which confirm geometric path as the best proxy. However, for some assets such as computers, autos, missiles and nuclear fuel empirical studies or technological progress have shown that a nongeometric depreciation pattern better suits the data.

Fernald (2014), in order to build up a new quarterly growth accounting database for the U.S. business sector, have aggregated the depreciation rates obtained in Fraumeni (1997) on the basis of geometric pattern. The aggregation technique follows the perpetual inventory method, according to which the stock of $j$-type capital in period $t$ is given by $K_{j,t} = (1-\delta_j)\,K_{j,t-1} + I_{j,t-1}$. The firm's cost-minimization problem delivers the following first order condition for the user cost of the $j$-type of capital $R_{j,t} = \left(i_t + \delta_j - \pi^e_{j,t+1}\right) P^I_{j,t}$, where $i_t$ is the nominal interest rate, $\pi^e_{j,t+1}$ is the expected rate of price appreciation, and $P^I_{j,t}$ is the purchase price of investment. According to this aggregation procedure the annual depreciation rate for equipment averages 13% per year . Among this, the average depreciation rate of computers and peripheral equipment is 31.5%, that of other information processing equipment is 13.4%, and of transportation equipment is 12.8%. The average annual depreciation rate of structures is 2.6%, among which that of manufacturing is 3.1%. As to the intellectual property capital it results to be about 15.8%. Specifically, the depreciation of softwares is 46% per year, and that of research and development is about 15%.

With regard to the Bureau of Labor Statistics (BLS) and its commitment in developing series of productivity for the U.S. industrial and manufacturing sectors, it is worth to mention the paper of Dean and Harper (2001). Spurred by the developments in the related literature, BLS have implemented several projects to improve its methodology of productivity measurements. One of these regards the aggregation concepts of tangible capital, which were basically distinguished between the aggregation based on relative efficiency of capital and the one based on the capital rental prices. The latter one, sustained by Hall (1968), explicitly accounts for the capital depreciation rates, as follows

$$f_t = p_t r_t + p_{t+1}\delta_t - \Delta p_{t+1} \tag{1.6}$$

where $f_t$ denotes the capital rental price in period $t$, $p_t$ is the price of new capital purchased in period $t$, $r_t$ the nominal discount rate, $p_{t+1}$ the price of new capital purchased in period $t+1$ and physically equivalent to the asset purchased in period $t$, $\delta_t$ the economic depreciation rate, and $\Delta p_{t+1}$ is the rate at which prices appreciate.

Similarly, Jorgenson and Griliches (1967) used the chained Tornqvist index to aggregate different types of assets, $k_{it}$, basing on the assumption that the discount rate is equivalent to the rate of return on capital, and which, in turn, implies that the implicit rents of capital account for the "property income", $\Phi_t$, as follows

$$\Phi_t = \sum_{i=1}^{n} K_{it} c_{it} = \sum_{i=1}^{n} k_{it}\left[p_{it} r_t + p_{it}\delta_i - (p_{it} - p_{it-1})\right]$$

According to the latter expression, assets that depreciate faster exhibit a higher weight, $c_{it}$, with respect to the long-lived ones. This suggests that an investor should apply higher rents on the short-lived assets in order to compensate their higher depreciation rate. With regard to the assets efficiency pattern, and differently from BEA, BLS has assumed a hyperbolic functional form, and calibrated it such that the asset's efficiency would decline more slowly over the first periods of its service life, and more rapidly over the later periods. The estimates for the assets service lives and for the discard patterns, however, were taken from the BEA datasets. So, contrary to BEA, BLS assumes a hyperbolic age-efficiency pattern because as ultimate goal it seeks to calculate the productivity measures of the U.S. economy, which require information about the age-related deterioration rather than the economic one.

The paper of Oliner and Sichel (2003) stresses the importance of the depreciation rates in the income shares of the high-tech assets. They build a neoclassical growth-accounting model, in order to measure the growth contribution of information technology capital to the output growth in the nonfarm business sector. In particular, they assume four types of capital stocks: computer hardware, software, communication equipment, and all the other tangible assets. They assume that all the markets are perfectly competitive and characterized by constant returns to scale. Labor and capital are perfectly mobile, and there are no adjustment costs for the integration of new inputs into production process. Given these assumptions, the income shares are equivalent to the respective output elasticities and all sum to unity. Thus, they obtain the following optimal capital income share, $j$,

$$\alpha_j = \frac{(R + \delta_j - \Pi_j)\, T_j p_j K_j}{pY}$$

where $R$ is the nominal net rate of return on capital, $\delta_j$ is the specific capital depreciation rate, $\Pi_j$ is the expected change in the capital value, $T_j$ is the taxation, $p_j K_j$ is the current-dollar capital stock, and $pY$ is the total nonfarm business income. According to the above expression capital depreciation rates positively contribute in the determination of capital income shares. Therefore, the authors perform estimation exercises relying mostly on the data available in BLS and BEA with regard to capital depreciation. For components of computers and peripheral equipment, however, they rely on Whelan (2000) methodology and estimate the depreciation rates of these assets following the geometric decay pattern finding that the personal computers depreciate at 30% per year. Given their estimation results the authors notice that, when considering high-tech assets such as computers, softwares and communication equipments, the estimated paths of the respective income shares are dominated by the "rapid trend rate of depreciation", which prevails even upon the considerable movements in the nominal rate of return on capital.

Thus, technological progress may have important implications not only in the growth of total factor productivity or price indexes, but also in the used asset prices, and depreciation and replacement of capital. In fact, when newly produced assets embody quality improvements, the shadow price of the existing assets may exhibit a decline. This decline can be attributed, among others, to a rise in the economic depreciation, which accounts not only for the mere wear and tear of the asset but also for its obsolescence effect. Recalling the expression for the user cost of capital (1.6) and assuming that the nominal interest rate, $r_t$, is given, the changes in the asset price are attributable to the economic depreciation rate $\delta_t$ and the price revaluation $\Delta p_{t+1}$. Both of them dealt with the vintage effect of capital. Depreciation, in fact, can be further more decomposed in obsolescence, which is the effect on the price changing of existing assets due to the availability of new vintages with higher qualitative characteristics, and into deterioration, which represents all the other effects of aging on price changes. Revaluation, on the other hand traces the passage of time, which simply makes the new vintages available. Following Wykoff (2003), the total change in the price of an asset can be expressed as follows

$$\begin{aligned} \Delta p(s,t) &= p(v,s,t) - p(v,s+1,t+1) \\ &= [p(v,s,t) - p(v-1,s+1,t)] - [p(v,s+1,t+1) - p(v-1,s+1,t)] \end{aligned} \tag{1.7}$$

where $s$ represents the age of the asset, $t$ is the time index, and $v = t - s$ is the vintage index. The first term in the brackets on the right hand side of the latter expression denotes depreciation, which is decomposed into the deterioration effect, $p(v, s, t)$, i.e. the price changes due to aging, and the obsolescence effect, $p(v - 1, s + 1, t)$, i.e. the prices changes due to the availability of quality improved new vintages. The second term in the brackets on the right hand side of the equation represents revaluation, which is similarly decomposed into the vintage effect, $p(v - 1, s + 1, t)$, and the passage of time effect (or "capital gain and loss", following the terminology of Wykoff (2003)), $p(v, s + 1, t + 1)$, which is assumed to reflect variations in the economic conditions such as declines in output, which cause prices of the input assets to change. A functional form for the price changes as expressed in (1.7) can be estimated making use of the regression techniques on the hedonic prices, which are designated to account for the quality improvements in the vintages characterized by frequent technological changes[11]. According to Wykoff (2003) estimation exercises msy be carried out by changing the functional form of the estimated equations, the number of observations, and the assets characteristics,in order to obtain different hedonic price regressions, or sequences of period by period prices that differ only in the vintage characteristics. Subsequently, patterns of obsolescence can be obtained net of the wear and tear and of the timing effects.

Finally, it is worth to note that, as far as quality changes (or obsolescence) tend to be positive over time, the prices of new assets are supposed to rise, whereas those of the old assets are supposed to decline.

## 1.2 The pioneer growth models with endogenous depreciation rate

The seminal steps in the real business cycle literature towards an endogenous modeling of the capital depreciation rate can be dated back to the 80s and 90s, when it was largely started to be recognized that the Solow-residual interpreted as the measure of the total factor productivity is no more satisfactory in order to explain the observed macroeconomic fluctuations of the economic business cycles. Therefore, new propagation mechanisms of the technology shocks into the real economy are developed exploiting the assumptions about the factor hoarding both in the labor and capital sectors, see, among others, Greenwood et al. (1988) and Burnside and Eichenbaum (1996). According to these assumptions, and in particular with regard to the capital sector, the agents are allowed to accumulate (hoard) capital in order to use it more intensively when the returns on capital are higher. So, agents can optimally decide about the level of capital utilization rate, which determines the effective level of capital services used in the economic production process. Therefore, the effective level of capital, assumed to be given by the product between the utilization rate and the level of capital stock, is the amount of capital which enters the production function of the final (or, in the case of DSGE models, intermediate) goods sector. In addition to the capital accumulation process, the two papers cited above make use of the *Depreciation-in-use* assumption, according to which capital deterioration depends on the intensity of use of capital. Specifically, the more intensively the capital is used the faster it depreciates. This implies that depreciation rate is an increasing and convex function of the utilization rate. Such a model setting creates a new channel for the technology shock propagation, which is, indeed, the rate of

[11] The first contributions in the economic literature on hedonic prices are to be ascribed to Griliches (1957, 1961, 1988, 1990).

capital utilization. Therefore, it has been found that the utilization rate is a good candidate through which a technology shock magnifies and propagates its effects into the real economy.

A brief exposition of the two papers cited above will help to fix the points.

**The model of J. Greenwood, Z. Hercowitz, G.W. Huffman (1988)**

Greenwood et al. (1988) investigate the transmission mechanism over the main aggregate economic variables of a shock to the marginal efficiency of investments. It is worth to say that, one important concern in the real business cycle literature is that the estimation of such kind of models does not properly mimic the observed economic behavior. The shortcoming stems in the lack of co-movement between the main aggregated variables, in particular between output, consumption and labor productivity. On the contrary, the model built in Greenwood et al. (1988) is able to generate pro-cyclical optimal responses of such variables. This result is achieved by exploiting the capital hoarding and the depreciation-in-use assumptions. Mainly, the authors hypothesize that the depreciation rate of capital is endogenously determined by the utilization rate of capital. The latter one is treated as a control variable. Then, when the shock is assumed to affect the future effective capital but not the already existing capital, i.e. they do not consider shocks that affect the production function directly, their model is able to generate co-movements between output and the other macroeconomic variables, in particular consumption, labor effort and labor productivity. This occurs because the presence of the utilization rate generates in the consumption optimal response path an additional term, which tends to shift the inter-temporal substitution effect from leisure towards consumption, and thus creates pro-cyclical movements in consumption and labor productivity. On the contrary, when the shock to the marginal efficiency of investment is assumed to alter the already installed capital, then, according to their model, all the propagation effects vanish.

Therefore, capital utilization rate plays a crucial role in their model. Additionally, it influences the path of the depreciation rate function. According to the depreciation in use assumption, a more intensive use of capital accelerates the rate at which it depreciates. Therefore, the depreciation rate is assumed to be strictly increasing in the use of capital, that is

$$\delta_u > 0,\ \delta_{uu} > 0 \text{ and } 0 \leq \delta \leq 1$$

In order to solve the dynamic programming problem of their model the authors assume the following simple functional form for the depreciation rate of capital

$$\delta(u) = \frac{1}{\omega} u^{\omega} \tag{1.8}$$

where $\omega > 1$ represents the elasticity of the depreciation rate with respect to capital utilization rate. They argue that, when a positive shock to the marginal efficiency of investment hits the economy, the cost of capital utilization declines, while its rate of utilization rises, which, in turn, implies higher levels in the marginal productivity of labor and employment due to the complementarity assumption between capital and labor in the production function of their model. Therefore, there exists a positive link between capital utilization and labor productivity. The main endogenous variables of the model, such as output, consumption, investment, employment and labor productivity, capital stock and the utilization rate all co-move in response to the shock on marginal efficiency of investment.

**The model of C. Burnside and M. Eichenbaum (1996)**

Similarly, Burnside and Eichenbaum (1996) have argued that a model incorporated with the capital utilization rate is able to better catch the characteristics of the post-war US aggregate time series with respect to the standard RBC model, explaining more than 50% of the output volatility, while the remaining is, as usual, directly attributed to the exogenous technology shock. An important result of their estimation exercise shows that capital utilization rate is more volatile with respect to capital stock. In addition, they have shown that the optimal response of the depreciation rate of capital is pro-cyclical together with the other endogenous variables. So, similarly to Greenwood et al. (1988) and following the depreciation-in-use hypothesis they assume the following functional form for the capital depreciation rate

$$\delta_t = \delta U_t^{\phi} \tag{1.9}$$

where $0 < \delta < 1$, $\phi > 1$, and $U_t$ is the capital utilization rate.

Both the papers stress the fact that, the rate of capital depreciation is not constant over time but, on the contrary, it is affected by the agents' decision policies with regard to capital utilization rate.

## 1.3 Endogenous depreciation rate, capital utilization rate and maintenance costs

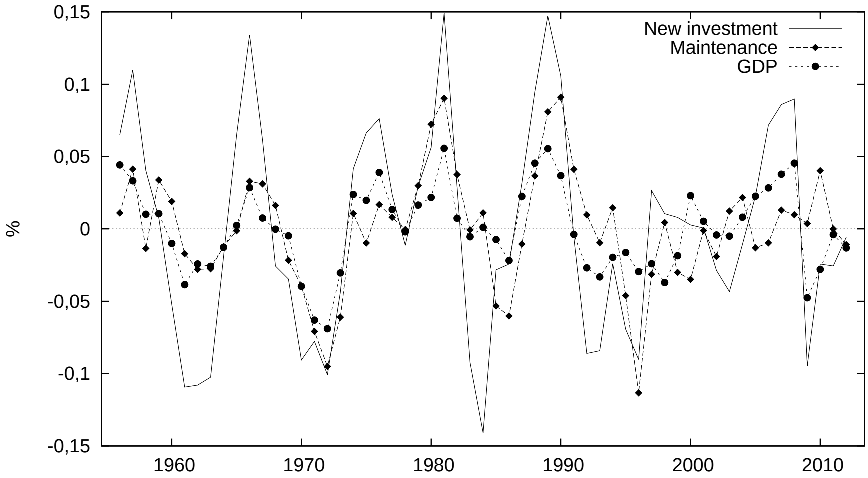
The following important step within the endogenous depreciation rate literature deals with the possible role of the maintenance and repair activity of physical capital. The paper of McGrattan and Schmitz (1999) entitled "Maintenance and Repair: Too big to ignore" is considered the milestone in this field[12]. In fact, the authors use the series from "Capital and Repair Expenditure" survey of Canada in order to highlight the importance of the maintenance and repair expenditures in the Canadian economy. This survey is held by the Canadian government statistic agency and queries the private and public sectors about the expenditures made in both the old and new capital. The old capital is defined in the survey as "non-capital maintenance and repair expenditures". It incorporates the spendings on the gross maintenance and repair of machinery and equipment, the non residential buildings and other structures, as well as reparation work in general. The new investment is named "capital expenditures" and it includes the major modifications and renovations, spendings on machinery and equipment, new buildings, engineering that are included in fixed asset accounts in addition to work in progress and other expenditures.

**The model of E. McGrattan and J. Schmitz (1999)**

As McGrattan and Schmitz (1999) have calculated, on average, expenditure on maintenance and repair in the whole economy was about 5.7% of Canadian GDP over the period 1981-1993, while that on education and research and development was 6.8% and 1.4%, respectively. Moreover, it accounted for 30% of spending in new physical capital, over the same period. According to Albonico et al. (2014), if considered only the manufacturing sector of Canada, the maintenance average share in new investment and in output amounted to 36.1% and 6% over the period 1956-2005, respectively. In general, we have calculated that, over the period 1956-2012 , the average share in GDP of the expenditures on

[12] Earlier literature, however, exists, which tries to outline a relationship between maintenance and depreciation, mainly concentrated in the fields of vintage capital models.

Fig. 1.2: Volatility: New Investment, Maintenance, GDP (averages, 1956-2012)




Source: "Annual Capital and Repair Expenditures Survey", Statistics Canada. Capital and repair expenditures on construction and machinery and equipment. Author's elaboration.

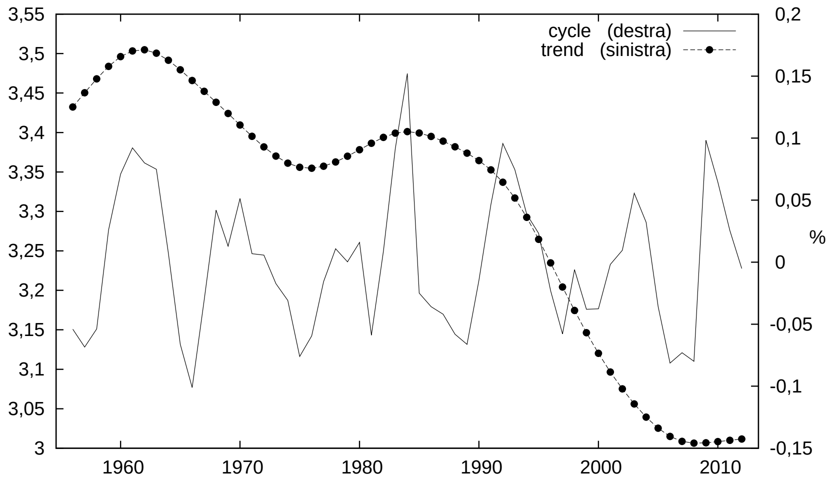
maintenance and repair of capital is almost the same as that found in McGrattan and Schmitz (1999). It is roughly the half (3.2%) if construction sector is excluded, i.e. only spendings in physical capital repairs are accounted for. Similarly, spendings on maintenance and repair is on average 27.7% of spendings on new physical capital, and the latter one is about 20% of GDP, over the period 1956-2012.

McGrattan and Schmitz (1999) have noted that new investment and maintenance are both pro-cyclical and more volatile than output. We can observe the same evidence in Fig.1.2 which shows the cyclical movements calculated using the Hodrick-Prescott filter [13] on the logs of the GDP at market prices, the expenditures in maintenance and the new physical capital, over the period 1956-2012. All the series exhibit strong positive correlations, which are 0.65% and 0.83% for that of GDP with maintenance and new investment, respectively, and 0.61% for the one of maintenance with new investment. Spending on new investment appears to be more volatile than spending on maintenance and both are more volatile than output. Moreover, it can be observed that maintenance to new investment ratio seems to exhibit no particular trends over the same period (see Fig.1.3).

Basing on the evidence from the Canadian survey about the behavior of the main macroeconomic aggregates, McGrattan and Schmitz (1999) estimate a simple neoclassical growth model enriched with the variable maintenance and repair services. They conjecture that this variable behaves as a substitute with respect to the new investment and analyze the economy's responses to an interest rate shock, namely an increase in the capital income tax rate, which impacts on the firms' capital intensity of use. Given their conjecture, they find that such a shock induces a smaller decrease in capital use than it would be in the absence of the maintenance variable. Therefore, according to their findings, maintenance tends to absorb the impact of a shock in the real economy and to lower the rate at which physical capital decreases . Excluding maintenance and repair from the economic analysis would lead,

[13] The coefficient parameter is set to 100.

Fig. 1.3: Maintenance to new investment ratio, HP detrended (1956-2012)




Source: "Annual Capital and Repair Expenditures Survey", Statistics Canada. Capital and repair expenditures on construction and machinery and equipment. Author's elaboration.

instead, to an underestimation of the capital stock and, consequently, to an overestimation of the total factor productivity. However, it is worth to note that, the quality of these results depends crucially on the share value of the maintenance expenditures with respect to the investment ones and on the degree of substitutability between the two control variables.

According to the McGrattan and Schmitz (1999) model setting, in fact, firms can choose the amount of maintenance and repair expenditures. This decision impacts on the depreciation rate of the capital stock. In fact, they assume that the depreciation rate is a decreasing function of maintenance per unit of capital stock, $\delta_t = \delta\left(M_t/K_t\right)$, which, therefore, may vary over time. It enters the law of motion of capital in the following way

$$K_{t+1} = \left[1 - \delta\left(\frac{M_t}{K_t}\right)\right] K_t + I_t$$

This specification leads, at optimum, to a positive relationship between the marginal benefit to capital stock deriving from maintenance activity and the relative price of maintenance with respect to investment. Specifically, if a firm increases its maintenance and repair activity, then the next period capital stock will increase as well, however at a lower extent than with respect to a rise in investment.

Accounting for maintenance and repair activity in the economic process may help, therefore, to better mimic the economic business cycles. This intuition by McGrattan and Schmitz (1999) has been, in particular, undertaken by the branch of the economic research that supports the idea according to which depreciation rate of capital can vary over time, which is definitely in contrast with the constant capital depreciation assumption dominating the analysis in the real business cycles.

**The model of O. Licandro and L.A. Puch (2000)**

Thus, Licandro and Puch (2000) combine the depreciation-in-use assumption and the McGrattan and Schmitz

(1999) results, by assuming a depreciation rate function depending both on the maintenance and utilization rates, $\delta_t = \delta(m_t, u_t)$. Specifically, the depreciation rate function is established to be decreasing in maintenance, increasing in utilization, and convex. Even if they do not provide an explicit specification for this functional form, they are able to carry out some quantitative and qualitative results of their real business cycle model extended with maintenance costs variable. Recall that, when the aggregate technology depends on the level of the effectively used capital and the depreciation rate is increasing in capital utilization, then it is shown that the utilization rate is pro-cyclical, see Greenwood et al. (1988). Moreover, it contributes to magnify and propagate the impact effects of a technology shock on output and other endogenous variables in the real economy. A direct consequence of there model setting is that the depreciation rate is pro-cyclical as well, which is in line with Burnside and Eichenbaum (1996). In addition, when the depreciation rate is negatively related with maintenance costs, then the behavior of the latter one depends crucially on the sign of the cross derivative of the depreciation rate function with respect to maintenance and utilization. Licandro and Puch (2000) assume the sign of the cross derivative to be positive, which, in this case, implies that maintenance is counter-cyclical, arguing that the cost to renounce to profits is relatively lower in recession times, and therefore, more resources can be allocated to maintenance and repair activity than to investment. This hypothesis is partly complementary to the McGrattan and Schmitz (1999) intuition according to which during the reparation times (but not necessarily the maintenance ones as well) the machines are not in process because, for example, broken.

As to the model equations, Licandro and Puch (2000) define the new aggregate technology as a function of the total effective level of capital, $U_tK_t$, total effective hours of work, $N_tl_t$, neutral technological shock, $X_t$, and maintenance costs, $M_t$, as follows

$$Y_t = (U_tK_t)^{1-\alpha} (N_tl_tX_t)^{\alpha} - m_tK_t$$

where $m_t$ is the maintenance cost ratio given by the fraction between the aggregate maintenance costs and capital stock. The law of motion of capital depends, among others, on the endogenous depreciation rate

$$K_{t+1} = [1 - \delta(m_t, u_t)] K_t + I_t$$

They carry out simple estimation exercises of their model and show that, just like the utilization rate, maintenance ratio contributes to propagate and magnify the effects of an exogenous technology shock in the real economy. So, according to their model analysis and in line with McGrattan and Schmitz (1999) the capital maintenance activity is, indeed, important for the study of the economic business cycles.

**The model of O. Licandro, L.A. Puch, J.R. Ruiz-Tamarit (2001)**

A more detailed analytical analysis of the equilibrium dynamics of growth models with endogenous depreciation, capital utilization and maintenance is offered in Licandro et al. (2001). The authors analytically establish a set of sufficient conditions for the existence and uniqueness of a local steady state equilibrium of their model. Under such conditions a unique interior "delta golden rule" exists, which they define as the value of capital stock consistent with the maximization of the steady state consumption with respect to capital, depreciation and utilization. The main assumptions with respect to the maintenance and investment activities are listed below:

i. maintenance activity is subject to maintenance cost technology, which is defined in terms of total depreciation and effective capital, decreasing in the first term and increasing in the second one; average maintenance cost, $m(\delta_t, u_t)$, is positive, quadratic, convex and linearly homogeneous on $\delta_t > 0$ and $u_t \in ]0, 1[$; moreover, $m_\delta(\delta_t, u_t) < 0$, $m_u(\delta_t, u_t) > 0$, $m_{\delta\delta}(\delta_t, u_t) > 0$, $m_{uu}(\delta_t, u_t) > 0$, $m_{\delta u}(\delta_t, u_t) < 0$; the explicit function is $m(\delta_t, u_t) = du_t^2/\delta_t$.

ii. maintenance activity is a substitute for investment, following McGrattan and Schmitz (1999).

iii. adjustment costs of investment is a linearly homogeneous function increasing in gross investment and decreasing in capital stock, $\Phi(I_t, K_t) = \phi(i_t)K_t$, where $\phi(i_t)$ exhibits the same properties of $m(\delta_t, u_t)$, and is represented bu the following explicit function $\phi(i_t) = (b/2)i_t^2$.

Hence, they assume a negative cross derivative of the maintenance cost function with respect to depreciation and utilization rates, which gives rise to pro-cyclical movements of maintenance, in contrast to Licandro and Puch (2000).

According to their model, the steady state equilibrium allows for an optimal underutilization of capital and maintenance. Next, the authors modify the delta golden rule such that the determination of the optimal capital stock requires the simultaneous determination of all the control variables, including maintenance. This way they find out that, a higher productivity of capital stock induces higher levels of investment, depreciation and utilization rate with respect to the original delta golden rule, in the long run. Moreover, the analysis of the dynamic properties of the convergence path confirms that the values of the parameters of the maintenance cost function and of the investment adjustment technology affect the steady state values of the main variables, however they do not impact on the speed of convergence of the model. In general, it results to be lower than the convergence rate obtained in the neoclassical growth models with investment adjustment costs only. As the authors argue, this is due to the presence of the variable utilization rate, which, together with the endogenous depreciation rate, generates an inter-temporal substitution effect between maintenance and investment and, thus, contributes to shape the saddle path convergence trajectories.

## 1.4 Endogenous capital depreciation rate within the vintage capital setting

Alternatively, a way to capture the endogenous aspect of capital depreciation rate makes use of the vintage capital growth theory. This models aim to capture the embodied aspect of the technological progress and are characterized by the hypothesis according to which the marginal contribution to output of new capital must be higher than the marginal contribution to output of the old one, see Solow (1960). This phenomenon is captured by a continuum-time representation of investment and produces an obsolescence effect on capital, as well as an optimal finite lifetime of capital stock, which defines the scrapping rate of capital. Note that, technological progress is said to be embodied when it contributes to increase productivity through a quality improvement of new investments. Otherwise, it is disembodied when productivity rises due to the marginal cost reduction of new investments. The necessity to capture in the economic growth models the embodied aspect of technological progress derives from the observed evidence about the acceleration in the technology growth rate, especially over the most recent decades.

Yet, it is largely recognized that acceleration in the technological progress observed since the 1970's, and especially the progress in information technologies, attributes to the depreciation rate of capital not only a time-variable characteristic, but also an economic interpretation besides the mere physical one[14]. That is, the rate of depreciation is influenced both by physical deterioration of capital, such as wear and tear, and by obsolescence, which reflects embodied technological progress. At first, there have been the vintage capital models which tried to capture the obsolescence effect of capital, moreover, over the last decade, the inclusion of maintenance activity in these models has been deepened[15].

**The model of R. Boucekkine and J.R. Ruiz-Tamarit (2003)**

Boucekkine and Ruiz-Tamarit (2003) argue that expenditures in maintenance of capital should be central to the firms decision problem, as far as it impacts on the capital depreciation rate. They investigate the conditions under which a positive demand for maintenance activity exists, and the complementarity between the latter one and investments in a neoclassical growth model with vintage capital. They show that, the optimal investment decision is influenced by the amount of maintenance services undertaken by the firms. The existence of this relationship allows them to carry out comparative static analysis in the stationary equilibrium of the model. In line with Licandro et al. (2001), they enhance their model with quadratic investment adjustment technology and maintenance costs. They allow the firms to optimally choose, among others, the utilization and depreciation rates, and the level of maintenance and repair expenditures by maximizing the following discounted stream of profits

$$\pi_t = p_t z_t F(K_{t-1}, L_t) - w_t L_t - q_t M_t - p_t^k I_t - p_t \Phi(I_t) \tag{1.10}$$

subject to the following law of motion of capital

$$K_t = I_t + (1 - \delta(m_t, u_t)) K_{t-1} \tag{1.11}$$

where $z_t$ is the neutral technological progress, and $\delta(m_t, u_t)$ is the depreciation function depending on the maintenance ratio, $m_t = M_t/K_{t-1}$, $m_t \geq 0$, and on the utilization rate, $0 \leq u_t \leq 1$, and satisfying the following hypothesis:

- $\delta_t \geq \bar{\delta}$, $\forall(u_t, m_t)$; $\delta(\cdot, 0) = \bar{\delta}$ and $lim_{m_t \to \infty}\delta(m_t, u_t) = \bar{\delta}$
- $\delta_1(m_t, u_t) < 0$ and $\delta_{11}(m_t, u_t) > 0$, $\forall(m_t, u_t) \neq 0$
- $\delta_2(m_t, u_t) > 0$ and $\delta_{22}(m_t, u_t) > 0$, $\forall(m_t, u_t) \neq 0$

The first hypothesis sets a lower bound condition on the value of the depreciation rate, named the "natural rate" of depreciation, $\bar{\delta}$. This means that, given the level of maintenance expenditures, even when capital is not used it still keeps on to depreciate at a constant rate due to aging. Similarly, there exists a threshold above which maintenance improvements no more impact on the quality of capital, and, therefore, on its depreciation rate. The second hypothesis states that the depreciation rate function is decreasing and convex in maintenance costs, according to McGrattan and Schmitz (1999), and Licandro et al. (2001). The latter hypothesis recalls the depreciation in use assumption, which

[14] Regarding the obsolescence aspect of capital depreciation see, for example, Greenwood et al. (1997), Tevlin and Whelan (2003), Boucekkine and De La Croix (2003), Doms et al. (2004), and Geske et al. (2007).

[15] The key variable of these models, however, keeps to be the (finite) lifetime of capital, which is crucial for the determination of all the other endogenous variables of the model.

states that the depreciation rate function is increasing and convex with respect to capital utilization rate.

The optimality conditions of the firm's maximization problem with respect to maintenance and utilization are, respectively:

$$q_t = -\delta_1(m_t, u_t)\mu_t, \quad (1.12)$$
$$z_t F_1(u_t K_{t-1}, L_t) = \delta_2(m_t, u_t)\mu_t \quad (1.13)$$

where $\mu_t$ is the Lagrangian multiplier. The first condition states that, at the optimum, the cost of one additional unit of maintenance services equals the marginal benefit deriving from the reduced capital depreciation evaluated at the shadow price of capital. The second one equates the return from the marginal increment of the utilization rate to the marginal cost, induced by the higher depreciation, evaluated at the shadow price of capital. The authors show analytically that, when the cross derivative of the depreciation function with respect to utilization rate and maintenance costs is strictly positive, then, given the above hypothesis on the depreciation function, at optimum, the two variables move in opposite directions ($\frac{du_t}{dm_t} < 0$). This result is economically counter intuitive, since, as the authors argue, if a firm opts to worsen depreciation by the means of one of the control instruments (either increasing the utilization rate, or decreasing maintenance expenditures), then it will optimally choose the remaining control variable in such a way as to decrease the depreciation rate (that is, increasing maintenance activities or decreasing capital utilization, respectively). This intuition allows them to impose a negativity condition on the sign of the depreciation function cross derivative, as follows:

$$\delta_{12}(m_t, u_t) \leq 0, \forall (m_t, u_t) \quad (1.14)$$

which implies that, the marginal contribution of maintenance to the stock of capital by the means of lower depreciation rate is higher when the utilization rate, and hence depreciation, are high. At this point the authors explicitly define the functional form of the rate of depreciation of capital, which satisfies all the above cited conditions:

$$\delta(m_t, u_t) = a u_t^{1+\epsilon}(1 + m_t)^{-\eta} + \bar{\delta} \quad (1.15)$$

where the coefficients $\epsilon > 0$ and $\eta > 0$ measure the sensitivity of depreciation to changes in capital utilization and maintenance, respectively. Making use of the maximization conditions (1.12) and (1.13), the utilization rate is expressed as follows

$$\begin{aligned} u_t &= -\frac{1+\epsilon}{\eta}\frac{\delta_1(m_t, u_t)}{\delta_2(m_t, u_t)}(1 + m_t) \\ &= \frac{1+\epsilon}{\eta}\frac{q_t}{q^0(w_t, z_t)}(1 + m_t) \end{aligned} \quad (1.16)$$

where $q^0(w_t, z_t) = z_t F_1(u_t K_{t-1}, L_t)$. This expression implies that, at optimum, the utilization rate of capital depends on the sensitivity parameters of the depreciation rate function, and on the amount of maintenance costs net of the gains deriving from the marginal increase in capital utilization. Such a specification satisfies the desired property of $\frac{du_t}{dm_t} \geq 0$, that is, a marginal change in maintenance activity must produce positive marginal variations in the utilization rate of capital. Then, using the steady state relations, they define the expression for the long run maintenance ratio, as follows

$$\tau(m) = \frac{1+\epsilon}{\eta} \quad (1.17)$$

with

$$\tau(m) = \frac{1}{a\eta}\left[\frac{r+\bar{\delta}}{(qv^0(w,z))^{1+\epsilon}}(1+m)^{\eta-\epsilon-1} + a\left(1+\eta\frac{m}{1+m}\right)\right]$$

and

$$v^0(w,z) = \frac{1+\epsilon}{\eta q^0(w,z)}$$

and distinguish between the following three cases. First, when $\eta = \epsilon$ then no solution exists for the long run equilibrium of maintenance services. Second, when the depreciation function is more sensitive to changes in maintenance than to those in the capital utilization, then the existence and uniqueness of the stationary solution is ensured by the strict inequality $\tau(0) < \frac{1+\epsilon}{\eta}$ when $1+\epsilon \leq \eta$, and by the additional condition $\tau(0) < \frac{1+\epsilon}{\eta} < \frac{1+\eta}{\eta}$, when $\epsilon < \eta < 1+\epsilon$. In such a case, when the unit cost of maintenance,$q$, increases the equilibrium level of maintenance rises because of the shift to the right of the long run maintenance ratio function. But this implies that maintenance is positively related to its unit cost, which is economically counterintuitive, as far as it does not satisfy the Law of Demand in the maintenance sector. Finally, when $\eta < \epsilon$ (i.e. depreciation rate function is relatively more sensible to the utilization rate), then a unique long run stationary equilibrium solution exists if and only if $\frac{1+\eta}{\eta} < \frac{1+\epsilon}{\eta} < \tau(0)$. The authors show that, in this case, the long run maintenance ratio $m(q,w,r,z)$ depends negatively on the maintenance costs and on labor, and positively on the interest rate and on technological progress.

Recall that, when maintenance ratio is calibrated in order to produce a stronger effect, with respect to utilization, on the capital stock accumulation process, and hence on output, throughout the depreciation rate function, then the model goes against the fundamental economic theory. The authors, therefore, discard this solution and, on the contrary, enhance the other one explaining that, in equilibrium, the rate of utilization and maintenance ratio must move in the same direction, which implies $\frac{du_t}{dm_t} \geq 0$. Therefore, the optimal combination of the two control variables is to increase (or to decrease, in which case the effects on capital stock are opposite) both of them. This behavior generates one direct positive effect on the production function deriving from a higher capital utilization, and one indirect positive effect acting through the capital accumulation law of motion.

In general, the authors show that, both when $\eta < \epsilon$ and $\epsilon < \eta$, in the long run equilibrium capital utilization rate behaves always like maintenance ratio. The long run depreciation rate, however, behaves similarly in response to changes in technological progress and maintenance costs only. It moves in the opposite direction with respect to maintenance when the real interest rate varies. Moreover, when $\eta < \epsilon$, they find that, utilization, depreciation and maintenance all move pro-cyclically when a positive technology shock occurs, and counter-cyclically in the case of $\epsilon < \eta$. Furthermore, the long-run equilibrium of gross investment depends on the behavior of the shadow price of capital stock, but it is independent of the position of $\eta$ with respect to $\epsilon$. Therefore, the responses of gross investment to changes in the level of technology or maintenance costs are qualitatively the same as those of the two control variables and of the depreciation rate when $\eta < \epsilon$. The behavior of capital stock, however, depends crucially on the equilibrium position of the maintenance ratio. In fact, in order to obtain a positive optimal response of capital stock to variations in the cost of maintenance services, the maintenance ratio must satisfy an upper bound condition imposed by the positivity assumption of investments. When this condition is satisfied and $\eta < \epsilon$, then an increase in maintenance costs decreases the level of gross investments which, in turn, reduces the capital stock. At the same time, capital stock rises because of its lower rate of depreciation. The indirect positive effect deriving from a lower depreciation rate more than compensates the negative one induced by a drop in investments. This means that, when maintenance cost changes the maintenance ratio and capital stock move in opposite directions. Therefore, the optimal behavior of maintenance services, given by the product $M = mK$, is ambiguous as well. Nevertheless, they show that, thanks to the imposed positivity condition on the long-run investment, the behavior of maintenance ratio prevails on that of capital stock for small enough levels of the equilibrium maintenance ratio. So, when $\eta < \epsilon$, a rise in maintenance cost leads a representative firm to optimally decrease the demands both for new capital goods (gross investment) and for maintenance services, which suggests that maintenance and investment are complements and both pro-cyclical.

In conclusion, given the comparative statics analysis of their economic growth model implemented with an explicit function for capital depreciation rate and maintenance services, Boucekkine and Ruiz-Tamarit (2003) have found that , when the elasticity of the depreciation rate function is higher with respect to utilization rate than with respect to maintenance ratio, then the demands for maintenance services and investment co-move, which implies that they are complements, in response to changes in labor and maintenance costs, investment rate, and technological progress. On the contrary, they move in the opposite directions, and hence behave as substitutes, when the depreciation rate function is more elastic with respect to maintenance of capital then to its utilization rate. However, as noted above, this parametrization does not satisfy one fundamental economic law, that is, the negative relationship assumption between the demand for maintenance services and its relative price.

**The model of R. Boucekkine, F. del Rio, B. Martinez (2009)**

Contrary to them, Boucekkine et al. (2009) consider a two-sector vintage capital model, in which maintenance is no more a control variable and is explained through a positive relationship with the utilization rate of capital and the age of capital. They assume endogenous capital lifetime, which depends on the maintenance costs of vintages. Both the neutral and the investment-specific technological progresses are included in the model. The aim of their paper is to analyze the relationship standing between the deprecation rate of capital, and in particular its components, and the two types of technological shocks[16]. For this purpose they assume that the total rate of capital depreciation is given by three components. These are found implicitly from the equilibrium relationships, as far as they do not postulate any *ad hoc* functional form for depreciation. The first component represents the exogenous and constant physical depreciation rate ($e^{\omega}$). The second one is the age-related depreciation rate ($\delta_t$), which, as they show, depends on the obsolescence rate of capital determined by the investment-specific technological progress. Finally, the scrapping rate ($\xi_t$) represents the fraction of unprofitable capital that is discarded. Specifically, the authors assume that capital becomes unprofitable when it no longer compensates its maintenance costs.

The vintage characteristic of the model enters the capital production sector. This is a perfectly competitive market, which produces capital services using new capital goods according to the following production technology

$$S_{z,t} = I_z e^{\lambda z} e^{-\omega(t-z)} U_{z,t} \tag{1.18}$$

[16] Other studies about the investment and depreciation reactions to the different technology shocks are found also in Tevlin and Whelan (2003) and Saglam and Veliov (2008).

where $I_z$ represents the quality unadjusted investment at time $z$, $U_{z,t}$ is the utilization rate of vintage $z$ at time $t$, and $e^{\lambda z}$ is the embodied rate of investment-specific technological progress (i.e. a progress that increases the marginal productivity of new capital goods, as in contrast to the disembodied investment-specific technological progress which reduces the marginal costs of new investments). Hence, parameter $\lambda$ measures the quality improvements of capital goods.

The function for the unit maintenance cost of a vintage $z$ at time $t$ is explicitly given by

$$M_{z,t} = \left[\beta e^{\chi(t-z)} U_{z,t}^{\mu} + \eta\right] e^{-\phi z} \tag{1.19}$$

where $\beta, \eta > 0$ and $\mu > 1$. The term in the brackets represents the total maintenance cost $M_{z,t}(U_{z,t})$, satisfying $M'(U) > 0$, $M''(U) > 0$ and $M(0) > 0$, that is, maintenance is strictly increasing in the utilization rate and strictly positive. The first term inside the brackets represents maintenance variable costs, given by an indicator of the vintage utilization rate. The maintenance variable costs allow to distinguish between the use-related depreciation rate, depending on variations in capital utilization, and the scrapping rate, which instead depends on the capital lifetime. The second term is the fixed and strictly positive maintenance cost, $\eta$, which at optimum ensures a finite lifetime of capital, given the assumptions on the parameters $\chi \geq 0$, $\lambda \geq 0$ and $\phi \geq 0$. The term $e^{-\phi z}$ represents the disembodied investment-specific technological progress which, in perfect competition, is equivalent to the cost of producing a new unit of capital in terms of consumption. It is referred to as the quality-unadjusted relative price of investment. Then, the quality-adjusted relative price of investment is given by $e^{-\varepsilon t}$, with $\varepsilon = \lambda + \phi$ being the total investment-specific technological progress rate.

A representative firm maximizes the following profit function of vintage $z$ in time $t$, subject to the respective utilization rate in time $t$, $U_{z,t}$, new capital in time $z$, $I_z$, and the scrapping time $T_t$

$$\pi_{z,t} = R_t S_{z,t} - M_{z,t} e^{-\omega(t-z)} \tag{1.20}$$

where $R_t$ is the rental price of capital services at time $t$. Re-expressing the profit function in terms of the optimal utilization rate and imposing zero return condition on capital gives the optimal scrapping condition of vintage $z$ in time $z + J_z$, with $J_z = t - z$

$$\frac{\mu - 1}{\mu} e^{-(\delta+\varepsilon)J_z} U_{z+J_z, z+J_z} p_{z+J_z} = \eta \tag{1.21}$$

which states that, the lifetime of vintage $z$, $J_z$, must equal its scrapping time at time $t$, $T_t$. The term, $p_t = R_t e^{\varepsilon t}$ represents the quality-adjusted rental price of capital services, while the expression for the initial utilization rate $U_{t,t} = \left(\frac{1}{\mu\beta} p_t\right)^{\frac{1}{\mu-1}}$ is derived from the optimality condition for capital utilization which is given by

$$U_{z,t} = U_{t,t} e^{-\delta(t-z)} \tag{1.22}$$

The latter expression says that, because of aging, capital stock is used to a lesser extent, and this decline in the utilization rate of capital is measured by $\delta = \frac{\chi+\varepsilon}{\mu-1}$.

Given the above expressions, the stock of quality-adjusted capital services becomes

$$S_t = \int_{t-T_t}^{t} U_{t,t} I_z e^{\lambda z} e^{-(\omega+\delta)(t-z)} dz \tag{1.23}$$

where $I_z e^{\lambda z}$ is the quality-adjusted investment at time $z$. The market value of the capital stock, instead, is given by $K_t = S_t e^{-\varepsilon t}$, that is

$$K_t = \int_{t-T_t}^{t} U_{t,t} X_z e^{-(\varepsilon+\omega+\delta)(t-z)} dz \tag{1.24}$$

where $X_z = e^{-\phi z} I_z$ represents the expenditure in new capital at time $z$. It clearly emerges from expression (1.24) that, because of the particular setting of the model, capital at market value depreciates at a rate given by a linear combination of the total investment-specific technological progress ($\varepsilon$), the constant and exogenous physical capital depreciation ($\omega$) and the capital utilization rate ($\delta$).

The law of motion of the stock of quality-adjusted capital services is, thus

$$\dot{S}_t = e^{\lambda t} U_{t,t} I_t - (\omega + \delta_t + \xi_t) S_t \tag{1.25}$$

where $\delta_t$ represents the use-related depreciation rate depending on the deflated rental price of capital

$$\delta_t = \delta + \frac{1}{\mu - 1} \frac{dp_t}{dt} \frac{1}{p_t} \tag{1.26}$$

and $\xi_t$ is the scrapping rate, which is related to the decline rate of capital utilization due to aging

$$\xi_t = \left(1 - \frac{dT_t}{dt}\right) \frac{e^{\lambda(t-T_t)} U_{t-T_t,t} I_{t-T_t} e^{-\omega T_t}}{S_t} \tag{1.27}$$

Note that, $e^{\lambda t} U_{t,t}$ measures the productivity of the new capital goods.

The stock of quality-adjusted capital services is rented to the perfectly competitive final goods producing sector, according to the following Cobb-Douglas production function

$$Y_t = A e^{\gamma t} L_t^{1-\alpha} S_t^{\alpha} \tag{1.28}$$

with $0 < \alpha < 1$, and $A > 0$ being the neutral technology shock, which is assumed to grow at a positive constant rate $\gamma \geq 0$. The production technology in terms of labor and capital stock at market value becomes

$$Y_t = A e^{(1-\alpha) g t} L_t^{1-\alpha} K_t^{\alpha} \tag{1.29}$$

where $g = \frac{\gamma + \alpha\varepsilon}{1-\alpha}$ is the constant per capita growth rate along the balanced growth path of consumption, output, capital stock, investment expenditure and maintenance costs, given by a linear combination of investment-specific technological progress rate ($\varepsilon$) and neutral-technological progress rate ($\gamma$).

Finally, the representative household maximizes the following CRRA utility function with respect to consumption and labor

$$U_t = \int_0^{\infty} L_t e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt \tag{1.30}$$

where the discount parameter ($\rho$) and the intertemporal elasticity of substitution ($\sigma$) are both strictly positive. The Euler equation for per capita consumption is of the following form

$$\frac{dc_t}{dt} = \frac{1}{\sigma}(r_t - \rho) c_t \tag{1.31}$$

Aggregate output is distributed between consumption, investment expenditures and maintenance services, that is

$$Y_t = C_t + X_t + M_t \tag{1.32}$$

Finally, given the optimal path for capital utilization rate (1.22), the expression for the market value capital services (1.24), and the optimal condition for the demand of capital services, $\alpha Y_t / K_t = p_t$, the aggregate maintenance cost is re-expressed as follows

$$M_t = \frac{\alpha}{\mu} Y_t + \eta \int_{t-T_t}^{t} X_z e^{-\omega(t-z)} dz \tag{1.33}$$

according to which, in equilibrium, maintenance services are given by a fraction of final aggregate output and all past expenditures in new units of capital weighted by the rate of physical capital depreciation. How much of final output will be deserved to maintenance depends (positively) on the capital share in the production function and (negatively) on the sensibility of maintenance costs with respect to capital utilization rate.

As stated above, the main variables of the model all grow at the same constant rate $g = \frac{\gamma+\alpha\varepsilon}{1-\alpha}$, along the BGP. Additionally, the following conditions hold along the BGP: (i) the lifetime of vintages is constant, $J = T$; (ii) the initial utilization rate is constant ($U_{t,t} = U_0$) and equal to $\left(\frac{\alpha}{\beta\mu}\frac{y}{k}\right)^{1/(\mu-1)}$; (iii) $(1-\sigma)g < \rho$, in order to guarantee the boundness of the utility function. The authors show that, under these restrictions, a unique optimal solution for the lifetime of capital exists, which is necessary to calculate the stationary levels of the model variables, including that of capital depreciation rate. Therefore, the comparative statics analysis performed by the authors shows that the following relationships stand between the optimal lifetime of capital, the technological progresses, and the different components of the depreciation rate:

(i) optimal capital lifetime is an increasing function of neutral technological progress, because its positive response behavior serves to compensate the loss in profitability due to a higher interest rate;

(ii) optimal capital lifetime is a decreasing function of investment-specific technological progress, since, in this case, the interest rate effect is dominated by the acceleration in profitability decline, which, finally, shortens the lifetime of capital;

(iii) optimal lifetime of capital is increasing in physical depreciation rate ($\omega$), inter-temporal elasticity of substitution ($\sigma$), discount rate ($\rho$), and maintenance cost elasticity with respect to utilization ($\mu$), while it is decreasing in the rate of maintenance associated with aging of the vintage ($\chi$) and the maintenance fixed costs ($\eta$), finally, it is not affected by the level of neutral technology ($A$), the population growth rate ($n$), and the parameter of the maintenance function ($\beta$);

(iv) both the decline rate of capital utilization, that is the use-related depreciation, and the scrapping rate increase when the investment-specific technological progress accelerates;

(v) an acceleration in the neural technological progress increases the scrapping rate, while it does not impact on the use-related depreciation rate.

Recalling proposition (iv), the investment-specific technological progress affects the total depreciation rate throughout two components, i.e. the scrapping rate and the use-related depreciation rate. As the authors point out, this assertion justifies the opinion according to which economic depreciation rate of capital tends to be underestimated by the Systems of National Accounts, which would lead to an overestimation of the capital stock growth rate, and finally to underestimation of total factor productivity[17].

[17] See, for example, Mukoyama (2008) who analytically explains how, under an endogenous capital depreciation rate, an acceleration in the investment-specific technological progress tends to give distorted measures of capital stock.

In order to support the implications of their model, the authors construct analytical expressions for the economic and the age-related depreciation rates, in line with the definitions given by the US Bureau of Economic Analysis (BEA). They conclude that, both the rates increase in response to an acceleration in investment-specific technological progress, because of the decline in both the optimal lifetime of capital and its utilization rate. Similar behavior transpires from the actual BEA data on the depreciation rates of equipment and software and the corresponding decline rates of the relative prices. Note that, according to BEA, the economic depreciation rate is estimated using used-asset relative prices unadjusted for quality. Under perfect competition conditions, the decline rate of the quality-unadjusted relative price of investment corresponds to the disembodied investment-specific technological progress which, thus, affects the economic depreciation rate. Furthermore, the latter one is influenced by obsolescence as well, which is given by both the rate of embodied investment-specific technological progress and the age of capital. On the contrary, the construction of the age-related depreciation rate relies on the decline rate of quality-adjusted relative price of investment. Therefore, the age-related depreciation is affected by the total investment- specific technological progress, as well as by age, however, in contrast to the economic depreciation rate, it does not depend on the rate of obsolescence.

In conclusion, Boucekkine et al. (2009) tackle the issue about the way the depreciation rate is used to be estimated. As stated above, BEA estimates the depreciation rate of capital through the decline rate in the relative used-asset value and defines it as a combination of physical depreciation (or age-related depreciation), obsolescence, wear and tear, and accidental damage. Until the mid '90s the estimation methodology had implemented the straight-line depreciation schedule, which implies that the year-per-year depreciation rate is constant over the whole service life of the asset. Thereafter, BEA has switched to a new regime based on the geometric depreciation rate almost for all of the asset categories[18], which exhibits an accelerated pattern, as far as the depreciation rate is higher in the early years of the asset's service life than in the later years. The geometric depreciation rate is determined by the ratio between the Declining-Balance Rate of an asset and the corresponding assumed service life. In most cases it is calculated using the estimations carried out in Hulten and Wykoff (981b).

Boucekkine et al. (2009) find out important properties with regard to the geometric depreciation rate schedule rising up from their AK-vintage model. Specifically, they show that, when the lifetime of the asset is large enough, than the geometric schedule is a good proxy for the corresponding depreciation rate pattern. On the contrary, the true depreciation rate diverges from the geometric one as the asset lifetime gets shorter. This implies that, according to statements (i) and (iv), the neutral technological progress is more likely to generate a geometric depreciation path with respect to the investment-specific technological progress. Therefore, the depreciation rate path depends crucially on the assumption according to which the lifetime of capital stock is finite. In fact, when the investment-specific technological progress accelerates the capital lifetime decreases. But, as far as the lifetime of capital enters the expression for the quality-unadjusted relative price of capital, according to the following expression

$$\frac{\tilde{P}_{z,t}}{\tilde{P}_{t,t}} = \eta e^{-(\omega+\delta+\lambda)a}\left\{\frac{1}{\eta} - [e^{(\delta+\varepsilon)a} - 1]\int_0^{T-a} e^{-(r+\omega)s}ds - \int_{T-a}^{T} e^{-(r+\omega)s}[e^{-(\delta+\varepsilon)(s-T)} - 1]ds\right\}$$

then, a shortening in the asset lifetime rises the probability to get a distorted measure of the

[18] An exception represents the computer equipment and autos, and missiles and nuclear fuel rods for which a non geometric depreciation schedule is used.

depreciation rate estimated with the geometric approach. In fact, when the lifetime of capital is finite then its scrapping rate, which influences the depreciation rate, depends not only on the changes in the capital unit prices, but also on the amount of capital invested over the past periods. On the contrary, as it can be observed from the above expression, when capital lifetime tends to infinity the geometric approach turns out to be the best approximation, which implies that scrapping equals zero and the economic depreciation rate equals the decline rate of the quality-unadjusted relative price of used-assets. Moreover, note that, since the depreciation patterns for computer equipment and autos do not follow the geometric schedule of depreciation according to BEA's statistics, the observed acceleration in the depreciation rate over the last decades[19] may be attributed to a composition effect, given the rising weight of computers in the capital stock.

Finally, Boucekkine et al. (2009) perform some empirical exercises using the actual series from BEA on the US depreciation rate of non-residential private fixed equipment and software and on the corresponding decline rate of relative prices (NIPA), over the period 1029-2003. They highlight the evidence according to which, there has been an increase in the rate of depreciation of capital accompanied by a rise in the decline rate of its relative prices, since the 1960s. This relationship is further confirmed by a positive correlation between the two variables. Moreover, the decline rate of the relative prices appears to be negatively correlated with the lifetime of non-residential private equipment and software.

**The model of R. Boucekkine, Fabbri, Gozzi (2010)**

The last work that worths to be mentioned in this section returns to the complementarity and substitutability issue between investment and maintenance. This topic still remains open to controversial interpretation about the relationship among the two factors. The paper in question is of Boucekkine et al. (2010). They build a simplified vintage AK version of Boucekkine et al. (2009), considering only one sector with a neutral technological shock, and including replacement investment. The functional form of the depreciation rate is not given, but, instead, determined by endogenous scrapping decisions. This is in contrast with Boucekkine and Ruiz-Tamarit (2003), who have addressed the same issue, but with a depreciation rate function explicitly formulated. Moreover, and more importantly, in addition to previous literature Boucekkine et al. (2010) go beyond the mere steady state equilibrium analysis and carry out optimal short run dynamics of their vintage capital model with endogenous capital depreciation rate and, as stated above, without any *a priori* postulated analytical expression for it. The authors are able to find a closed-form solution of their model making use of the recent developments achieved in the optimal control problems with infinite dimensional systems, that characterizes this class of models, as far as they imply the resolution of a differential-difference state equation.[20] The main purpose of their paper is to investigate whether maintenance and investment co-move in response to a neutral technological shock in short-run, given their model assumptions.

For further reading about the model development and analysis we deliver the reader to the specific paper, and expose here its main findings. As it is usual in the vintage AK theory, detrended investment converges to its new steady state value by oscillations. Persistence is induced by the constant optimal lifetime of capital. The behavior of maintenance costs reflects the convergence dynamics of investment,

[19] See, for example, Tevlin and Whelan (2003) for an exhaustive explanation of this point.
[20] See, for further studies, Boucekkine et al. (2005), and Fabbri and Gozzi (008a,b).

as far as its current optimal path is given by a weighted integral of investment with time range endogenously depending on obsolescence. So, according to their model settings, when a positive neutral technology shock hits the economy, both investment and maintenance increase, which implies that they are pro-cyclical. Specifically, there is a rise in maintenance following an increase in the optimal lifetime of capital due to a positive neutral technology shock. A longer lasting capital stock necessitates a higher amount of maintenance services. Therefore, in short-run, maintenance and investment co-move, i.e. they are complements. On the contrary, obsolescence and replacement investment respond negatively to the shock and both converge to the new balanced growth path by oscillations thereafter. Their downward jump is, again, attributed to the instantaneous adjustment of the optimal lifetime of capital to its new higher value, while persistence derives from their lagged functional structure in line with the particular model setting. Therefore, contrary to new investment, replacement investment behaves as substitute for maintenance services.

## 1.5 Two dynamic stochastic general equilibrium models

In this section we explain two DSGE models which are of particular interest for our research purposes. The first one, Albonico et al. (2014), investigates the optimal dynamics of the main endogenous variables in a simplified DSGE model with a depreciation rate endogenously determined by capital utilization rate and maintenance. The second one, Justiniano et al. (2011), is used as a benchmark model in our analysis. It is a more exhaustive DSGE model and captures the most salient aspects of the recent literature in this class of models, with particular regard to the roles of the different technological progresses.

**The model of Albonico, Kalyvitis, Pappa (2014)**

Albonico et al. (2014) analyze a DSGE model specifying an explicit depreciation rate function in relation with capital utilization and maintenance. They estimate the model on the Canadian aggregate economy and on the US manufacturing sector over the period 1956-2005 and 1958-2009, respectively. For the former analysis, data on aggregate capital expenditures and maintenance from the Canadian "Capital and Repair Expenditures" survey are used. They carry out estimations of maintenance expenditures for both the economies[21] and use them to depict the correspondent depreciation rate paths. From their analysis it emerges that both the maintenance expenditures and maintenance to capital ratio are pro-cyclical, as suggested by McGrattan and Schmitz (1999), and the depreciation rate of capital is highly volatile and pro-cyclical both in the Canadian aggregate economy and in the US manufacturing sector, which is consistent with Burnside and Eichenbaum (1996). Moreover, according to their model estimations, the cross derivative of the depreciation rate function with respect to maintenance and utilization is negative, which confirms the analytical conjectures of Boucekkine and Ruiz-Tamarit (2003).

So, the model is composed of two sectors. A representative household owns the capital stock and rent the effective amount of capital stock ($U_t K_t$) to the final good producers at a rate $r_t$. A representative household maximizes the present value of the utility function with respect to consumption, hours

[21] In the case of Canada, they find that the estimated series obtained for the maintenance expenditures are consistent with the series collected from the empirical survey.

worked, capital utilization, investment and maintenance subject to the aggregate budget constraint and the capital stock accumulation law of motion, given the quadratic capital adjustment cost and the depreciation rate functions, as follows, respectively

$$\begin{aligned}
\max_{C_t,h_t,U_t,I_t,M_t} \quad & E\sum_{t=0}^{\infty}\beta^t\eta_t^u\left[\frac{C_t^{1-\sigma}}{\sigma}-\lambda_n\eta_t^h\frac{h_t^{1+\theta_n}}{1+\theta_n}\right] \\
s.t. \quad & C_t+I_t+M_t\leq w_th_t+r_tU_tK_t \\
& Z_tI_t=K_{t+1}-\left(1-\delta\left(U_t,\frac{M_t}{K_t}\right)\right)K_t+\nu\left(\frac{K_{t+1}}{K_t}\right)K_t \\
& \nu\left(\frac{K_{t+1}}{K_t}\right)=\frac{b}{2}\left(\frac{K_{t+1}}{K_t}-1\right)^2 \\
& \delta\left(U_t,\frac{M_t}{K_t}\right)=\xi\left[\psi U_t^{\phi}+(1-\psi)e^{-\gamma\frac{M_t}{K_T}}\right]^{\theta}
\end{aligned}$$

where $\sigma > 0$ is the risk aversion coefficient, $\theta_n > 0$ the inverse of the Frisch labor supply elasticity, $\lambda_n > 0$ the preference parameter, $0 < \beta < 1$ the discount factor, $b > 0$ the degree of capital adjustment cost, $\phi \geq 0$ the elasticity of the depreciation rate with respect to capital utilization, and $\gamma \geq 0$ the elasticity of the depreciation rate with respect to maintenance. The latter expression, which represents the depreciation rate of capital, nests the McGrattan and Schmitz (1999) specification when $\psi = 0$, i.e. capital utilization does not affect its depreciation rate, and the Burnside and Eichenbaum (1996) specification when $\psi = 1$. When, instead, maintenance is constant, the depreciation rate function becomes akin to those assumed in Greenwood et al. (1988) and Burnside and Eichenbaum (1996). Moreover, when capital stock is not utilized and maintenance expenditure is high, depreciation rate is assumed to approach zero, $\delta(0,\infty) = 0$. The assumption on the parameter $\gamma > 0$ implies that $\partial\delta/\partial M < 0$ and $\partial^2\delta/\partial M^2 > 0$, whereas when $\phi > 0$, then $\partial\delta/\partial U > 0$. The sign of the cross-derivative of the depreciation rate function, which determines the degree of substitutability between maintenance and investment, depends on the parameter $\theta$. In steady state its value is related to the deep parameters $\gamma$, $\phi$, $\alpha$, and it increases when the steady state real interest rate rises and/or the investment to capital ratio declines. Given the above postulated depreciation rate function, when $\theta > 1$, the sign of the depreciation function cross derivative is negative, which is the condition defended by Boucekkine and Ruiz-Tamarit (2003). On the contrary, when $\theta < 1$, then $\partial^2\delta/\partial U\partial M > 0$.

The following AR(1) processes with i.i.d. normal errors for the preference shock, $\eta_t^u$, labor supply shock, $\eta_t^h$, and investment specific technology shock, $Z_t$, are assumed, respectively

$$\begin{aligned}
\log(\eta_t^u/\eta^u) &= \rho_u\log(\eta_{t-1}^u/\eta^u)+\varepsilon_t^u \\
\log(\eta_t^h/\eta^h) &= \rho_h\log(\eta_{t-1}^h/\eta^h)+\varepsilon_t^h \\
\log(Z_t/Z) &= \rho_Z\log(Z_{t-1}/Z)+\varepsilon_t^Z
\end{aligned}$$

On the production side, firms minimize the total production costs choosing hours worked, $h_t$, and effective capital stock, $U_tK_t$, given the Cobb-Douglas production technology, as below

$$\begin{aligned}
\min_{h_t,U_tK_t} \quad & w_th_t+r_tU_tK_t \\
s.t. \quad & Y_t=(U_tK_t)^{1-\alpha}(X_th_t)^{\alpha}
\end{aligned}$$

where $X_t$ is the neutral labor-augmenting technology shock, which follows the AR(1) process with an i.i.d normal error, $\varepsilon_t^x$, according to $\log(X_t/X) = \rho_x\log(X_{t-1}/X) + \varepsilon_t^x$.

The model is closed by the following clearing condition in the goods market

$$Y_t = C_t + I_t + M_t + G_t$$

with $G_t$ being the AR(1) government spending shock, i.e. $\log(G_t/G) = \rho_g \log(G_{t-1}/G) + \varepsilon_t^g$.

The authors carry out the first order optimal conditions of their model, the steady state relations and the log-linearized optimality conditions around the steady state and estimate the detrended log-linearized model using Bayesian approach[22]. They estimate the values of the volatilities of the shocks, their autocorrelations, the labor share, $\alpha$, Frisch elasticity, $\theta_n$, relative risk aversion, $\sigma$, adjustment costs, $b$, and both the elasticities of the depreciation rate function, $\phi$, and $\gamma$. The values of the parameters $\theta$, $\xi$ and $\psi$ are calculated using the steady state relationships. While all the other structural parameters and the steady state ratios of investment and maintenance to capital, as well as government to output are calibrated by the authors. According to their assumptions, the steady state depreciation rate equals the steady state new investment to capital ratio, as far as capital adjustment costs equal zero.

According to the Albonico et al. (2014) estimation results, when there are no maintenance expenditures and capital is fully utilized, capital stock depreciation rate is about 19% per year for Canada and 16% for the US. The figures for the natural rates of depreciation (i.e. no maintenance expenditure and no capital utilization) are much lower, around 3.7% for Canada and 1.5% for US, whereas they amount to 4.4% and 3.6% per year, respectively, when capital stock is fully utilized and maintenance expenditures are high. The estimated value of the labor supply elasticity is higher in the US with respect to Canada, which implies that, due to complementarities in the production function, utilization increases relatively more. However, the relatively lower estimated value of $\phi$ in the US suggests that the utilization effect on depreciation is tenuous. So, given the optimal response of the US capital utilization rate, fewer movements of capital stock are required when a shock hits the economy and, therefore, the capital adjustment costs decline. On the other hand, the estimated $\gamma$ for the US is lower than the one for Canada. Hence, maintenance expenditures should increase relatively more in order to compensate the negative effects of utilization on depreciation in the US than in Canada.

From the comparison of their DSGE model with capital maintenance and the standard RBC model without capital maintenance the authors have found that the estimation results are similar for Canada, while significantly different for the US. The posterior estimated elasticities of the depreciation function with respect to utilization rate, $\phi$, are significantly different when comparing the two models in both the economies. Moreover, both tend to decline when maintenance expenditure is considered. In general, the impulse response functions are similar, with the exception that depreciation and utilization are relatively more volatile in the presence of maintenance expenditures.

The optimal responses to the total factor productivity shock show that all the endogenous variables of the model co-move a part those of the hours worked, this implies that maintenance expenditures and investment are complements. A positive investment specific technology shock busts investments and, hence, capital, hours worked and capital utilization, because of complementarities in production. Depreciation increases as well. Moreover, the relative price of maintenance grows up, which induces a fall in the maintenance to capital ratio and a further increase in the depreciation rate. Thus, in response to the investment-specific technology shock all the main endogenous variables are pro-cyclical

[22] For analytical developments and the characterization of the log-linearized model setting, please, refer to the relative paper.

with exception of maintenance to capital ratio, which is counter-cyclical. In this case maintenance expenditures and investment behave as substitutes. With regard to a negative labor supply shock and a positive preference shock maintenance, depreciation, capital utilization and investment are all pro-cyclical, suggesting that maintenance and investments are complements. Finally, the optimal paths in response to a positive government shock of maintenance, depreciation and capital utilization are pro-cyclical, while it is counter-cyclical for investments.

So, a great deal of this paper is that the authors, given the successful model estimation for the Canadian economy compared to the actual data available from the Canadian Survey on Capital and Repair Expenditures over the period 1956-2005, have been able to expand the same methodology approach to the US economy, where data on maintenance of capital are available only for the manufacturing sector since 2007. Thus, with regard to the US, the authors have spanned the estimation period over the 1958-2009. The estimation results show that maintenance to capital ratio and depreciation rate are pro-cyclical and highly volatile with respect to output. The same holds for maintenance to new investment ratio, however it exhibits a lower volatility with respect to output. Finally, they find that all the correlations relative to output trend are positive.

**The model of A. Justiniano, G.E. Primiceri, A. Tambalotti (2011)**

A different DSGE model setting is used in Justiniano et al. (2011). Here, the maintenance expenditures are not considered and the capital depreciation rate is assumed to be constant over time. However, it is a medium scale DSGE model augmented with the most common assumptions gained in the related literature. We choose this model for our research purposes mainly for two reasons. The first one, as stated just above, is that it is a well structured model, because accounts for the well established real and nominal rigidities assumptions like the Dixit-Stiglitz indexation, Calvo (1983) pricing, factor hoarding, and the labor supply heterogeneity. The second one is to be attributed to the nature of the shocks the authors consider. In fact, they distinguish between three types of technology shocks: the neutral labor-augmenting technology shock, the investment-specific technology shock (IST) and the shock to the marginal efficiency of investment (MEI). The first two are well known in the literature. Moreover, with regard to the second one, it is shown that, in a perfectly competitive equilibrium, the inverse of the investment-specific technology shock equals the relative price of investment goods. So, the IST shocks have been found to be good drivers of the business cycles fluctuations over the past decades.[23] On the contrary, some have argued that their role is negligible, instead.[24] Justiniano et al. (2011) argue that, measuring all the shocks to capital accumulation process by the relative price of investment can produce erroneous results. In fact, as their estimation results reflect, adding the relative price of investment among the observable variables produces almost no variations in the estimation outputs. This suggests that not all the investment shocks are to be simply attributed to the relative price of investment. In order to support these findings, they analytically develop an alternative model incorporating nominal rigidities both in the consumption and investment goods producing sectors. The presence of sticky prices and monopolistic competition in both the sectors draws a wedge between the inverse of investment-specific technology progress and the relative price of investment, given by the ratio between the consumption and investment real marginal costs. They show by contradiction that,

[23] See, for example, Greenwood et al. (1997), Fisher (2006), Smets and Wouters (2007).
[24] See Schmitt-Grohé and Uribe (2012) and Liu et al. (2009).

in such an economy, an equilibrium where the relative price of investment equals the inverse of the investment-specific technology shock is not feasible. The authors distinguish, therefore, between the shock that hits the transformation of consumption in investment goods (IST), which is assumed to reflect the relative price of investment, and the shock that influences the transformation of investment goods into physical capital (MEI). This distinction is achieved assuming a decentralized model setting. In fact, they suppose their economy consists of eight sectors. The agents of each sector aim to optimize their corresponding objective problems, as it is described below.

*Final good producing sector*

The final goods producing sector is inhabited by a continuum of perfectly competitive firms, which, combining a continuum of intermediate goods $Y_t(i)$, $i \in [0,1]$, produces a final good $Y_t$, according to the following technology

$$Y_t = \left[ \int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{p,t}}} \right]^{1+\lambda_{p,t}}$$

where $\lambda_{p,t}$ is the ARMA(1,1) price mark-up shock, i.e. the mark-up of price over intermediate firms marginal cost, as follows

$$\log\lambda_{p,t} = (1-\rho_p)\log\lambda_p + \rho_p \log\lambda_{p,t-1} + \varepsilon_{p,t} - \theta_p \varepsilon_{p,t-1}$$

with $\varepsilon_{p,t}$ being $i.i.d.N(0,\sigma_p^2)$. Final good $Y_t$ is then sold at a unit price $P_t$ both to households for consumption and to investment good producers. The zero profit condition and the profit maximization deliver the following aggregate price index, given by the CES aggregate of intermediate goods prices, $P_t(i)$, and the demand function for intermediate good, respectively

$$P_t = \left[ \int_0^1 P_t(i)^{-\frac{1}{\lambda_{p,t}}} \right]^{-\lambda_{p,t}}$$

$$Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} Y_t$$

*Intermediate good producing sector*

Firms operate in a monopolistic competition regime and produce the intermediate good $Y_t(i)$ using the effective amounts of capital and labor, $K_t(i)$ and $L_t(i)$, respectively. The optimization problem of the firm consists in maximizing profits with respect to effective capital and labor subject to its production function as follows

$$\max_{K_t(i),L_t(i)} \quad P_t(i)Y_t(i) - W_t L_t(i) - R_t^k K_t(i)$$
$$s.t. \quad Y_t(i) = A_t^{1-\alpha} K_t(i)^{\alpha} L_t(i)^{1-\alpha} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F$$

where $W_t$ and $R_t^k$ are nominal wage and rental rate of capital, respectively and $A_t$ is the non stationary exogenous labor-augmenting technology shock, with a stationary AR(1) growth rate, $z_t = \Delta\log A_t$, that is

$$z_t = (1-\rho_z)\gamma_z + \rho_z z_{t-1} + \varepsilon_{z,t}$$

with $\varepsilon_{z,t} i.i.d.N(0,\sigma_z^2)$, and $\Upsilon_t$ representing the investment-specific technology progress. Intermediate goods producing firms must support fixed costs of production, $F$. When these costs exceed the amount of output a firm is able to produce, given the combination of labor $L_t(i)$, effective capital, $K_t(i)$ and neutral technology, $A_t$, the total firm's production is turned to zero, such that $Y_t(i) = 0$.

The optimal conditions of the problem solution depict the capital to labor ratio and the marginal costs, at optimum.[25] Both are common across firms due to homogeneity in the factor markets. Moreover, given the homogeneity in the production function, marginal cost corresponds to the average variable cost. This condition is exploited in the price optimization problem. In fact, firms in this market sector are subject to the Calvo (1983) pricing system, according to which, every period a fraction of firms, $\xi_p$, resets its prices according to the indexation rule

$$P_t(i) = P_{t-1}(i)\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}$$

where $\pi_t = \frac{P_t}{P_{t-1}}$ and $\pi$ are the gross inflation and its steady state value, respectively. In equilibrium $\xi_p$ measures the degree of price stickiness. The remaining fraction of firms, $(1-\xi_p)$, is able to optimize the present discounted value of future profits with respect to their price, $\tilde{P}_t(i)$, subject to the optimal demand of intermediate goods, as follows

$$\max_{\tilde{P}_t(i)} \quad E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Lambda_{t+s}}{\Lambda_t} \left\{ \left[ \tilde{P}_t(i) \left( \prod_{j=0}^{s} \pi_{t-1+j}^{\iota_p} \pi^{1-\iota_p} \right) \right] Y_{t+s}(i) - \left[ W_t L_t(i) + R_t^k K_t(i) \right] \right\}$$
$$s.t. \quad \tilde{Y}_{t+s}(i) = \left( \frac{\tilde{P}_t(i)}{P_{t+s}} \pi_{t,t+s} \right)^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}} Y_{t+s}$$

where $\Lambda_{t+s}$ is the marginal utility of the nominal income of a representative household that owns a firm, and $\xi_p$ represents the index of price stickiness. The solution of the optimization problem returns the optimal price setting condition of the intermediate goods producers and the aggregate price index expressed in terms of a weighted aggregation of the optimally chosen price and the price set according to the indexation rule. Both are then used in the construction of the Phillips curve.

*Investment good producing sector*

Given the aggregate production technology, perfectly competitive firms produce $I_t$ efficiency units of investment goods using $Y_t^I$ units of final good, and sell them to the capital producing firms at a unit price $P_{It}$. Their profit maximization problem is, therefore

$$\max_{Y_t^I} \quad P_{It} I_t - P_t Y_t^I$$
$$s.t. \quad I_t = \Upsilon_t Y_t^I$$

which, at optimum, equals the relative price of investment with respect to consumption to the inverse of investment-specific technology progress. The term $\Upsilon_t$ is the non-stationary IST shock with the growth rate, $\upsilon_t = \Delta \log \Upsilon_t$ evolving according to the following stationary AR(1) process

$$\upsilon_t = (1-\rho_\upsilon)\gamma_\upsilon + \rho_\upsilon \upsilon_{t-1} + \varepsilon_{\upsilon,t}$$

where $\varepsilon_{\upsilon,t}$ is $i.i.d.N(0,\sigma_\upsilon^2)$.

[25] For the non stochastic steady state relationships and the log-linearized around the steady state value model with the detrended variables see Appendix A.

*Capital good producing sector*

Firms operate in perfect competition and produce installed capital, $i_t$, using investment goods $I_t$, purchased at the investment good producers. Installed capital is then sold to households at a unit price $P_{kt}$ in order to be transformed into effective capital. A representative firm maximizes the expected discounted stream of profits with respect to investment good $I_t$ subject to the production technology of installed capital as follows

$$\begin{aligned} \max_{I_t} \quad & E_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t+s} \left[ P_{k,t+s} i_{t+s} - P_{I,t+s} I_{t+s} \right] \\ s.t. \quad & i_{t+s} = \mu_{t+s} \left[ 1 - S\left( \frac{I_{t+s}}{I_{t+s-1}} \right) \right] I_{t+s} \end{aligned}$$

where $S(\cdot)$ is the adjustment cost of investments satisfying in steady state $S = S' = 0$ and $S'' > 0$. The shock to the marginal efficiency of investment, $\mu_t$, follows the following AR(1) process with an $i.i.d.N(0, \sigma_\mu^2)$ error term

$$\log \mu_t = \rho_\mu \log \mu_{t-1} + \varepsilon_{\mu,t}$$

The first order condition delivers the optimal capital good producing condition.[26]

*Households*

A continuum of households is supposed to be endowed with specialized labor, $L_t(j)$, $j \in [0, 1]$, and capital. They purchase consumption goods, $C_t$, from the final good producers and installed capital, $i_t$, from capital good producers. The stock of installed capital of period $t-1$, $\bar{K}_t$, is then transformed into effective capital in period $t$, $K_t$, through the rate of utilization, $u_t$, chosen by the households, that is $K_t = u_t \bar{K}_{t-1}$. Finally, the effective capital is rent to intermediate good producing firms at the rate $R_t^k$. Hence, a representative household maximizes a separable logarithmic utility function with respect to consumption, $C_t$, labor, $L(j)_t$, government bonds $B_t$, installed physical capital, $i_t$, physical capital stock, $\bar{K}_t$, and utilization rate subject to its budget constraint and physical capital law of motion, as follows, respectively

$$\begin{aligned} \max_{C_t, L_t, B_t, i_t, \bar{K}_t, u_t} \quad & E_t \sum_{s=0}^{\infty} \beta^s b_{t+s} \left[ \log(C_{t+s} - hC_{t+s-1}) - \varphi \frac{L_{t+s}(j)^{1+\upsilon}}{1+\upsilon} \right] \\ s.t. \quad & P_t C_t + P_{kt} i_t + T_t + B_t \leq R_{t-1} B_{t-1} Q_t(j) + \Pi_t + \\ & + W_t(j) L_t(j) + R_t^k u_t \bar{K}_{t-1} - P_t \frac{a(u_t)}{\Upsilon_t} \bar{K}_{t-1} \\ & \bar{K}_t = (1-\delta) \bar{K}_{t-1} + i_t \end{aligned}$$

where $\beta$ is the discount factor, $h$ the degree of habit formation, $\upsilon$ the Frisch elasticity, $T_t$ lump-sum taxes, $B_t$ government bonds, $R_t$ gross nominal interest rate, $Q_t(j)$ net cash flow from the state contingent security of the $j^{th}$ household, $\Pi_t$ the per-capita profit from owning a firm, and $P_t \frac{a(u_t)}{\Upsilon_t}$ the dollar cost of capital utilization per unit of physical capital, which is scaled by the IST shock in order to

[26] Note that, in a one-sector representation of the model, the zero profit condition implies that $P_{k,t} i_t = P_t \tilde{I}_t$, where $\tilde{I}_t$ represents real investment expressed in consumption units, $\tilde{I}_t = \frac{P_{I,t}}{P_t} I_t$, and is treated as observable in the model estimation. The law of motion of capital, in this case, is $\bar{K}_t = (1-\delta)\bar{K}_{t-1} + \mu_t \Upsilon_t (1 - S_t) \tilde{I}_t$, which clearly distinguishes between the two types of shocks that affect the accumulation process of capital.

ensure the existence of a balanced growth path. Moreover, it is assumed that, in steady state, $u_t = 1$, $a(1) = 0$ and $\chi = \frac{a''(1)}{a'(1)}$. The intertemporal preference shock, $b_t$, follows the following AR(1) process

$$\log b_t = \rho_b \log b_{t-1} + \varepsilon_{b,t}$$

where $\varepsilon_{b,t}$ is $i.i.d.N(0, \sigma_b^2)$. Moreover, a nominal rigidity is added to the labor supply sector, that is, every period a fraction $\xi_w$ of households sets its wages according to the indexation rule

$$W_t(j) = W_{t-1}(j) \left(\pi_{t-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha}\upsilon_t}\right)^{\iota_w} \left(\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_\upsilon}\right)$$

The remaining fraction $(1 - \xi_w)$ of households choses the level of optimal wage by maximizing the utility function subject to the optimal labor demand condition of employment agencies, as follows

$$\begin{aligned}
\max_{W_t(j)} \quad & E_t \sum_{s=0}^{\infty} \beta^s \xi_w^s \Bigg[\Lambda_{t+s} W_t(j) L_{t+s}(j) \left(\prod_{k=0}^{\infty} \left(\pi_{t+k-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha}\upsilon_t}\right)^{\iota_w} \left(\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_\upsilon}\right)^{1-\iota_w}\right) + \\
& - b_{t+s}\varphi \frac{L_{t+s}(j)^{1+\upsilon}}{1+\upsilon}\Bigg] \\
s.t. \quad & L_{t+s}(j) = \left[\frac{W_t(j)}{W_{t+s}} \pi_{t,t+s}^w\right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s}
\end{aligned}$$

where $L_t$ represents the aggregate labor demand. The solution of the latter maximization problem by the households designates the optimal wage setting condition and the aggregate wage index.

*Employment agencies*

In this sector perfectly competitive employment agencies purchase specialized labor $L_t(j)$ from monopolistic households at the 'price' $W_t(j)$ and transform it into homogeneous labor, $L_t$, which is supplied by the intermediate good firms in their production process at 'price' $W_t$. The agencies optimize their profits with respect to $L_t(j)$ given the aggregate index of labor, as follows

$$\begin{aligned}
\max_{L_t(j)} \quad & W_t L_t - \int_0^1 W_t(j) L_t(j) di \\
s.t. \quad & L_t = \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_{w,t}}}\right]^{1+\lambda_{w,t}}
\end{aligned}$$

where $\lambda_{w,t}$ is the mark-up of wage over the household's marginal rate of substitution, corresponding to the labor supply shock, and following the ARMA(1,1) process with $\varepsilon_{w,t}$ ~$i.i.d.N(0, \sigma_w^2)$

$$\log\lambda_{w,t} = (1 - \rho_w)\log\lambda_w + \rho_w \log\lambda_{w,t-1} + \varepsilon_{w,t} - \theta_w \varepsilon_{w,t-1}$$

Hence, the optimal labor demand function and the aggregate wage index are, respectively

$$L_t(j) = \left(\frac{W_t(j)}{W_t}\right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t$$

$$W_t = \left[\int_0^1 W_t(j)^{-\frac{1}{\lambda_{w,t}}}\right]^{-\lambda_{w,t}}$$

*Government*

The authors assume a Ricardian fiscal policy system, that is, government releases bonds in order to finance its budget deficit. The government budget constraint is, therefore

$$G_t = T_t + B_t - R_{t-1}B_{t-1} - Q_t(j)$$

where $G_t$ denotes government spendings depending on the exogenous parameter $g_t$ according to the following expression

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t$$

So, $g_t$ is the government spending shock following the AR(1) stochastic process

$$\log g_t = (1 - \rho_g) \log g_{t-1} + \varepsilon_{g,t}$$

where $\varepsilon_{g,t}$ is $i.i.d.N(0, \sigma_g^2)$. The market clearing condition

*Monetary policy authority*

Monetary authority chooses the level of nominal interest rate according to a policy rule, which relates the deviations of the latter one from its steady state value, $R_t/R$, with the deviations of inflation, the level of GDP gap and its growth rate from their corresponding steady states, $\frac{\pi_t}{\pi}$, $\frac{X_t}{X_t^\star}$ and $\frac{X_t/X_{t-1}}{X_t^\star/X_{t-1}^\star}$, respectively

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi}^{\phi_\pi}\right)\left(\frac{X_t}{X_t^\star}\right)^{\phi_X}\right]^{1-\rho_R} \left(\frac{X_t/X_{t-1}}{X_t^\star/X_{t-1}^\star}\right)^{\phi_{dX}} \varepsilon_{mp,t}$$

where $\varepsilon_{mp,t}$ is an $i.i.dN(0, \sigma_m p^2)$ monetary policy shock.

Given the model settings and the assumptions made, the steady state growth rate of the composite technological progress, $A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}$, is a linear combination of the steady states of neutral technology and investment-specific technology progresses, that is

$$\gamma_\star = \gamma_z + \frac{\alpha}{1-\alpha}\gamma_\upsilon$$

The authors estimate their model for the US economy over the period 1954:III-2005:III, using the Bayesian approach and the following variables as observables

$$\left[\Delta \log X_t, \Delta \log C_t, \Delta \log \tilde{I}_t, \Delta \log L_t, \Delta \log \tfrac{W_t}{P_t}, \pi_t, R_t, \Delta \log \tfrac{P_{It}}{P_t}\right]$$

So, there are eight observables and eight exogenous shocks.[27] In order to investigate the role of the investment-specific technology shock, the authors use the relative price of investment as observable, which is defined as the ratio of the weighted chain deflators for consumption and investment. For investment price series, $P_{It}$, they rely on the actual data for deflators of durable consumption and

[27] For definitions of actual counterparts of the observable variables and the set of priors of the estimated parameters and standard deviations, please relate to the specific paper.

private investment provided by the National Income and Product Accounts (NIPA).[28] The obtained model estimations for correlations with respect to output and standard deviations relative to output of all the observables, as well as almost all the other second moments capture fairly well their actual counterpart values. However, the contemporaneous correlation between the growth rates of consumption and investment is significantly different. Moreover, the first order estimated autocorrelations of the growth rates of output, consumption and investment, as well as their corresponding volatilities are overpredicted. Nevertheless, the model fits fairly well the real economy. The impulse response functions of output, investment, hours, real wages, inflation and nominal interest rate to a shock in marginal efficiency of investment all exhibit an increasing hump-shaped pattern. Consumption, instead, starts to rise after a few periods. Moreover, the MEI shock accounts for 60, 85, 68 percent in volatility of output, investment and hours, respectively, while only for 8% in that of consumption. So, these shocks are found to be the "key drivers" of the business cycles fluctuations. As to IST shocks, the authors find that, contrary to Fisher (2006), their explanatory power in the business cycles is trivial, whereas, in line with Schmitt-Grohé and Uribe (2012), they are crucial in the long run growth.

Finally, the authors argue that, the MEI shock of the type they do consider in their model can also reflect disturbances that affect financial markets. In fact, they assume it hits the transformation process of investment goods into installed capital, reflecting, for example, the adjustment costs of the investment rate. Similarly, in an agency cost model, financial frictions influence the ability of the capital good producers to borrow from financial intermediaries in order to purchase investment goods for the production of installed capital. In both the cases installed capital declines in response to a negative shock affecting the capital good sector. The affinity between the two interpretations is furthermore enhanced by the observed strong negative correlation between the estimated mode of the MEI shock and the external finance premium, measured as the spread between returns on high-yield bonds and AAA corporate bonds.[29] To corroborate further their assertion, the authors show that the MEI shocks contribution to the 2008-2009 recession is considerable.

## 1.6 Concluding remarks

There exist many various insertions in the business cycles economic literature, which have progressively signed the evolving of these types of models both in their analytical structure as well as in the estimation methodology. In line with our research objective, we have delineated the literature evolution of modeling the real business cycle models, up to the DSGE models, with endogenously determined depreciation rate of capital. This topic has not collected great success in the related literature up today. In fact, the most notable DSGE models are used to be constructed with a capital depreciation rate constant over time, which is consistent with estimation results of several papers, such as, for example,

[28] The authors, moreover, check for robustness of their model estimation to a differently constructed price index, which combines NIPA deflators for consumption durables, and private investment in residential and non-residential structures, with the Gordon (1990) and Cummins and Violante (2002) price index series for private equipment and software. The only significant difference between the two appears in a faster decline rate of the latter series with respect to NIPA's ones. Both the relative prices present a trend break around 1982:I.

[29] The choice of high-yield bonds for the spread measure is justified by the fact that they are shown to be highly predictive, especially, for investment and employment. The same two variables are estimated by the authors to be mostly affected by the MEI shock.

Hulten and Wykoff (981a,b), Jorgenson (1996), Oliner (1996) and Huang and Diewert (2011). On the contrary, thanks to the Solow residual failure in explaining the real business cycles, a growing attention has been focused on the possible role of endogenous depreciation rate in the macroeconomic fluctuations. Specifically, the sprint has been given by Greenwood et al. (1988) and Burnside and Eichenbaum (1996), who have highlighted important transmission mechanisms of propagation of technology shocks throughout the capital utilization rate, which was supposed to endogenously model the function of capital depreciation rate. Models that assume a constant depreciation rate, but include the capital hoarding assumption, are only able to account for the direct effects of the utilization rate on the production function, thus, neglecting the indirect effects generated throughout the capital accumulation process. Later on, McGrattan and Schmitz (1999), basing on evidences from the Canadian "Capital and repair expenditure" survey, have detected maintenance and repair activity of capital stock as an additional important channel for the technology shocks propagation and amplification. They have calculated that, the maintenance and repair expenditures in the whole industry averaged 5.7 percent of GDP over the period 1961-1993, while excluding the construction sector (i.e. considering only structures and equipment) maintenance and repair spendings were about 28 percent of spending in new investment, over the same period. These figures suggest that ignoring activity of maintenance and repair of capital in the modeling macroeconomic models could give deceptive representation of the economy. Basing on this evidence and encouraged by the contrasted results carried out in Licandro and Puch (2000) and Licandro et al. (2001), who have found that in response to a positive TFP shock maintenance activity is countercyclical and pro-cyclical, respectively, a stream of research has addressed the topic of the qualitative behavior of maintenance. According to Boucekkine and Ruiz-Tamarit (2003) maintenance is pro-cyclical in response to a TFP shock only under certain parameter restrictions in the depreciation rate function and anyway this outcome is rejected because of the violation of the law of demand for maintenance. However, Boucekkine et al. (2010) show that maintenance may behave pro-cyclically in short-run. Conversely, Albonico et al. (2014) argue that the behavior of maintenance expenditures depends on the type of technological shock under consideration. According to them it is pro-cyclical in response to a TFP shock and countercyclical in response to the investment-specific technological progress. On the types of technological shocks depends also the behavior of the rate of depreciation according to Boucekkine et al. (2009), who claim that a positive investment-specific technological progress boosts both the age-related and use-related depreciations implying an acceleration in the economic depreciation rate. On the contrary, a positive TFP shock does not impact on the use-related depreciation, however the age-related depreciation decreases inducing the economic depreciation to decline. This is in contrast with Albonico et al. (2014) who have found that the depreciation rate is always pro-cyclical.

# Chapter 2

## 2 Model description

### Introduction

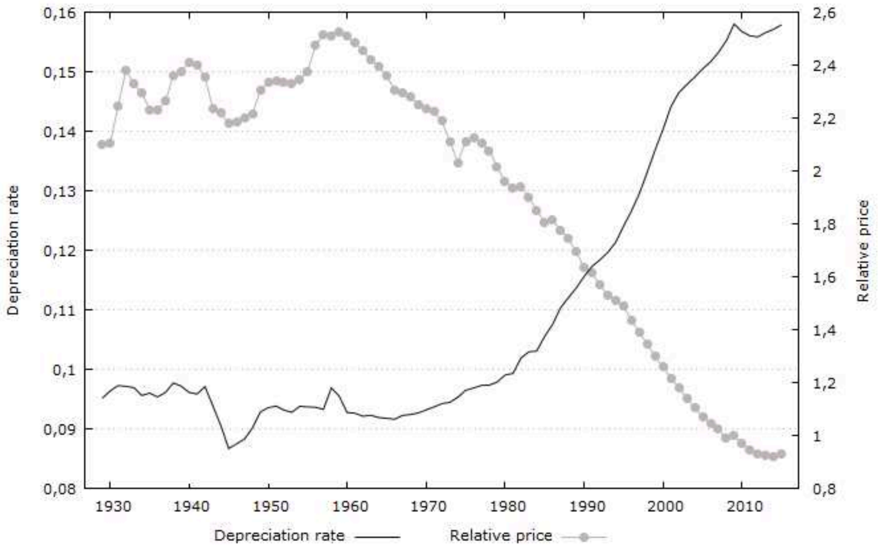
In the present chapter is described the structure of our model. The goal of our research is to determine whether, in a Dynamic Stochastic General Equilibrium framework, endogenously determined depreciation rate of capital may impact on the equilibrium levels and convergence dynamics of the main macroeconomic aggregates. On the basis of several empirical studies, we argue that capital depreciation rate has a growth trend. This behavior may be noticed in Fig.2.1. The ri-elaborated series are taken from the Bureau of Economic Analysis (BEA), which publishes unadjusted for quality prices including thus the effects of obsolescence of capital stock in the estimated series for prices. The figure shows that in the U.S., especially after the second half of the 1960s, capital depreciation rate has grown consistently. This surge is accompanied by a fall in the NIPA's relative price of investment[30]. In fact, the National Account Systems use to define economic depreciation rate as a decline in the asset value and estimate it according to the methodology of used-asset prices. This means that the lower the price of a used asset the higher the related depreciation.

Literature evidences using different estimation approaches come from Geske et al. (2007) who have computed estimations for the depreciation rate of computers using the used asset price approach over the period 1990-2001. According to them, on average, the value of used computers has declined by about 45% per year, which implies an age-related depreciation of about 3% per year and depreciation due to obsolescence of 8% on average per year. Moreover, their estimations show that, while the age-related depreciation is almost constant over the considered period, the rate of obsolescence ranges between 3% in 1990 and 1992 and 15% in 1999. Similarly, Tevlin and Whelan (2003), implementing the perpetual inventory approach on the U.S. actual data for chain-weighted investment and capital over the period 1965-1997 show that the aggregate economic depreciation rate exhibits a persistent increasing trend, rising from 8% in 1965 to 16% in 1997, with a more substantial growth over the latter ten years. An increase, even though less pronounced, is as well observed over the period 1965-1988 when computing-equipment is excluded or when fixed-weighted data are used, with a persistent increasing trend after the 1988. Given the evidences carried out by the above cited authors and basing on what emerges from the National Accounts estimates we set our model such that, in equilibrium, the path of the capital depreciation rate depends on the decline rate of the relative price of investment.

Our model as well includes maintenance activity following the claim of McGrattan and Schmitz (1999) "too big to ignore". It affects, together with capital utilization, the depreciation rate path and is treated as a control variable by the households.

[30] It is shown in Boucekkine et al. (2009) that the correlation between NIPA's relative price of investment and the depreciation rate is positive. Similar paths and conclusions can be made for the Canadian economy as well.

Fig. 2.1: Depreciation rate and relative price of private nonresidential equipment and software (1929-2015)




Source: BEA estimations. Author's elaboration.

The chapter is structured as follows. In section 2.1 we describe the setting of our model and the optimization problems of the agents. In section 2.2 we show the non-stochastic steady states and define the stationary equilibrium equations with the detrended endogenous variables. In section 2.3 the system of linear rational expectation equations is outlined which is obtained log-linearizing around the steady state our detrended optimal equations. For the analytical computations of all the equations we remainder to the appendixes.

## 2.1 Model structure

We implement a modified version of the Dynamic Stochastic General Equilibrium model presented in Justiniano et al. (2011), in which the most common assumptions with respect to consumption habits, technology and market structure are taken into account. It is a decentralized model with sector-specific optimization problem settings. Such a decentralization scheme allows the authors to clearly distinguish between the following three technological progresses: the neutral labor augmenting technological process, the investment specific technological process (IST), and the marginal efficiency of investment process (MEI). The first one affects the production of intermediate goods and enters the Cobb-Douglass production technology function. The IST shock affects the transformation process of final goods into new investment, while the MEI shock influences the transformation of new investment into new capital goods. An important result of their model is about the role of the IST progress in driving the macroeconomic business cycles. Specifically, when interpreted as the inverse of the relative price of investment with respect to consumption[31], the propagation effects of these shocks into the real economy appear to be negligible[32]. Their estimation analysis, in fact, displays that the contribution to the economic growth of the IST shock amounts to about 40% in contrast to the 60% ascribable to the marginal efficiency of investment technological progress.

In addition to Justiniano et al. (2011) model setting, we include the sector of capital maintenance activity. Moreover, we departure from the constant depreciation rate assumption, and allow for its endogenous determination. For this purposes we define the rate of depreciation as a function of capital utilization and capital maintenance, positively related with the first one and negatively with the latter one[33]. When capital is used in the production process more intensively depreciation accelerates, this is known as the depreciation-in-use assumption. However, the rate of depreciation is as well influenced by the technological progresses.

Several important attempts have been made in the related literature to explain the behaviors of the utilization and depreciation rates and the maintenance costs in response to technological progresses. Boucekkine and Ruiz-Tamarit (2003) argue that, when the sensitivity of the depreciation rate function is higher for capital utilization than for maintenance, then depreciation, maintenance and utilization are all procyclical in response to a neutral productivity shock. On the contrary, when depreciation function is more sensitive with respect to maintenance costs, then all of them are countercyclical. Boucekkine et al. (2010), assuming endogenous scrapping time in a vintage AK model, show that the scrapping of capital is positively related to neutral technological progresses, this induces maintenance costs to be procyclical in short run. The use-dependent depreciation and obsolescence, however, are both countercyclical. Boucekkine et al. (2009) incorporate the investment specific technological progress together with the neutral technology process in a two-sector vintage capital model. According to them, both the use-related and age-related depreciation rates are procyclical in response to the investment specific technological process, which implies that the economic depreciation rate co-moves with output. The effect of the neutral technological progress is null on the use-related depreciation and negative on the age-related one, which entails, in this case, a countercyclical response of the economic depreciation rate. Similarly, Albonico et al. (2014) highlight the procyclical behavior both of the maintenance costs and of the capital depreciation rate in response to the total factor productivity progress. However, maintenance is countercyclical when the investment-specific technology shock hits the economy.

In our model, similarly to Justiniano et al. (2011), there are three possible driving forces of the economic fluctuations, that are the marginal efficiency of investment technology progress (MEI), the investment-specific technology progress (IST) and the labor-augmenting technology progress. The distinction between the IST and MEI shocks for our research purposes is crucial. In fact, following the literature and the interpretation of Justiniano et al. (2011), in equilibrium IST is given by the inverse of the relative price of investment. As far as prices are unadjusted for quality, we interpret

[31] It is shown that, in a perfectly competitive environment, the investment-specific technological progress can be proxied by the inverse of the relative price of new investment.

[32] Other several papers have come out with the same arguments, attributing the driving forces of the macroeconomic fluctuations to other types of technological progresses, such as, the shock to the depreciation rate of capital or the shock to the quality of capital. For details see Schmitt-Grohé and Uribe (2012), Liu et al. (2009), Furlanetto and Seneca (2014).

[33] This assumption is in line with the related literature. See, for example, Greenwood et al. (1988) and McGrattan and Schmitz (1999).

this shock as the disembodied investment-specific technological progress, which reduces the marginal cost of production of one extra unit of investment. As to the MEI shock, it affects the transformation process of investment goods into installed capital. An additional interpretation given to this shock by Justiniano et al. (2011) is that it can represent some kind of financial disturbance which affects, for example, the transformation of households savings into new capital. However, we abstract from the financial sector in our model and rather consider the MEI shock as an embodied investment-specific technological progress, which increases the quality of one extra unit produced investment, and thus interpret it as obsolescence. Our consideration is supported by the fact that, when reducing the model to one sector representation, both the shocks enter linearly into the law of motion of capital stock, suggesting that they are elements of one composite total investment shock. The decentralization of the model allows to split the composite investment shock in two components, indeed, the IST and MEI progresses, or the disembodied and embodied specific-investment progresses, respectively. Other innovative element of our model is that our model assumptions entail that the economic depreciation rate exhibits a trend which is related to the disembodied IST progress and,therefore, to the relative price of investment.

So, given our model assumptions, our aim is to analyze the equilibrium dynamics of the endogenous depreciation rate of capital and of the maintenance expenditures, together with the other main macroeconomic variables in response to the shocks considered in the model. Following Justiniano et al. (2011), we assume that households purchase new capital from the perfectly competitive capital-good producers, and transform it into installed physical capital, which is used in the next period production process. The utilization rate of capital is a variable of choice of households. We assume that, they may furthermore decide to restore the depreciated capital demanding maintenance services to the perfectly competitive maintenance-services producers. The maintenance-services firms acquire units of final goods and, given positive adjustment costs, transform them into efficiency units of maintenance ready to "replace" the depreciated capital. Final good is produced in a perfectly competitive environment through a combination of intermediate goods and sold to households for consumption, to maintenance-services producers and to perfectly competitive investment-goods producers, who produce the new investment. New investment is then purchased by the perfectly competitive capital-goods producers, who transform it into new capital. New capital is purchased by households who transform it into installed capital and then rent to the intermediate-goods producers as effective capital. The latter ones, operating in a monopolistic competition, sell the intermediate goods to the final goods producers. Households, among others, choose the amounts of government bonds holdings. Each household is a monopolistic supplier of specialized labor, that is aggregated by employment agencies into homogeneous labor, which is then sold to intermediate good producers for their production process. Government implements Ricardian fiscal policy and the monetary authority sets the nominal interest rate according to the Taylor interest rate rule.

*The final good sector*

Perfectly competitive firms combine a continuum of intermediate goods $\{Y_t(i)\}_i$, $i \in [0,1]$, in order to produce the final good $Y_t$, given the Dixit and Stiglitz (1977) CES aggregate technology. Their profit maximization problem is, therefore

$$\begin{aligned}\max_{Y_t, Y_t(i)} \quad & P_t Y_t - \int_0^1 P_t(i) Y_t(i) di \\ s.t. \quad & Y_t = \left[ \int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{p,t}}} di \right]^{1+\lambda_{p,t}}\end{aligned} \tag{P1}$$

where $\lambda_{p,t}$ is the price mark-up shock following an exogenous stochastic ARMA(1,1) process, which, as stated in Justiniano et al. (2011), helps to capture the highly volatile inflation

$$\log \lambda_{p,t} = (1-\rho_p) \log \lambda_p + \rho_p \log \lambda_{p,t-1} + \varepsilon_{p,t} - \theta_p \varepsilon_{p,t-1} \tag{2.1.1}$$

with $\varepsilon_{p,t} \sim i.i.d.N(0, \sigma_p^2)$.
Combining the expression resulting from the profit maximization together with the zero profit condition it is obtained the optimal demand function for the intermediate good $i$, decreasing in its relative price, as well as the analytical expression for the final good price, which is represented by a CES aggregate of the intermediate goods prices $\{P_t(i)\}_i$, as follows, respectively[34]

$$Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} Y_t \tag{2.1.2}$$

$$P_t = \left[ \int_0^1 P_t(i)^{-\frac{1}{\lambda_{p,t}}} di \right]^{-\lambda_{p,t}} \tag{2.1.3}$$

Final goods are purchased by the households for consumption purposes, by the investment-goods producers, who transform them into efficiency units of new investment, and by the maintenance-services producers, who transform them into efficiency units of capital maintenance.

*The intermediate good sector*

In this sector firms are assumed to operate in a monopolistic regime as far as each one produces a diversified intermediate good by combining the amounts of effective capital $K_t(i)$ and effective labor $L_t(i)$, according to a Cobb-Douglas technology. The profit maximization problem is as follows

$$\begin{aligned}\max_{L_t(i), K_t(i)} \quad & P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t(i) \\ s.t. \quad & Y_t(i) = A_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F\end{aligned} \tag{P2}$$

where $W_t$ is the aggregate level of nominal wages and $R_t^k$ is the nominal return on capital. $A_t$ is a non-stationary process representing labor-augmenting technology shock. Its growth rate, $\Delta \log A_t = z_t$, follows the following stationary AR(1) process

$$z_t = (1-\rho_z)\gamma_z + \rho_z z_{t-1} + \varepsilon_{z,t} \tag{2.1.4}$$

with $\varepsilon_{z,t} \sim i.i.d.N(0, \sigma_z^2)$. Similarly, $\Upsilon_t$ represents a non-stationary process for the investment-specific technology progress, which can be considered either as quality or quantity improvement. Its growth rate, $\Delta \log \Upsilon_t = v_t$, follows the following stationary AR(1) process

$$v_t = (1-\rho_v)\gamma_v + \rho_v v_{t-1} + \varepsilon_{v,t} \tag{2.1.5}$$

with $\varepsilon_{v,t} \sim i.i.d.N(0, \sigma_v^2)$. Finally, $F$ represents the fixed costs. Its value is chosen such that in steady state profits are zero and it is multiplied by the composite technology factor $A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}$ in order

[34] For the analytical derivations of all the model equations please refer to Appendixes B, C, D, E.

to guarantee the existence of a balanced growth path. Note that, when fixed costs are high relative to the production capacity, which is given by the combination of capital and labor factors and neutral technology, the intermediate goods producer $i$ is constrained to exit the market so that its production output, $Y_t(i)$, is null.

The first order conditions with respect to capital and labor, combined together, give the optimal capital to labor ratio and the optimal nominal marginal cost, as follows, respectively

$$\frac{K_t}{L_t} = \frac{W_t}{R_t^k}\frac{\alpha}{1-\alpha} \tag{2.1.6}$$

$$MC_t = \left(\frac{W_t}{A_t}\right)^{1-\alpha}\left(R_t^k\right)^{\alpha}\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \tag{2.1.7}$$

The production technology of the intermediate goods producers satisfies the desirable property of duality. This property ensures that the capital to labor ratio and the marginal cost function are both common across all the monopolistic firms. Moreover, given the optimal relative demand for capital, obtained from the optimization problem, the average variable cost equals the nominal marginal cost.

Next, it is assumed a price optimization setting *a la* Calvo (1983). Namely, every period a fraction $\xi_p$ of intermediate firms resets its prices according to the following indexation rule

$$P_t(i) = P_{t-1}(i)\pi_{t-1}^{\iota_p}\pi^{1-\iota_p} \tag{2.1.8}$$

where $\pi_t$ represents the gross inflation and $\pi$ its steady state level. The fraction $\xi_p$, thus, represents the natural level of price stickiness. The remaining fraction of firms, $1-\xi_p$, is able to optimize for the price level $\tilde{P}_t(i)$ the present discounted value of future profits subject to the optimal intermediate goods demand function (2.1.2), that is

$$\begin{aligned} &\max_{\tilde{P}_t(i)} \quad E_t\sum_{t=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\left[\tilde{P}_t(i)\pi_{t,t+s} - MC_{t+s}\right]Y_{t+s}(i) \\ &s.t. \quad Y_{t+s}(i) = \left[\frac{\tilde{P}_t(i)}{P_{t+s}}\pi_{t,t+s}\right]^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}}Y_{t+s} \end{aligned} \tag{P3}$$

where $\pi_{t,t+s} = \prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}$.where $MC_t$ represents the nominal marginal cost and is substituted for the average variable cost due to their equivalence as stated above, while $\Lambda_t$ is the marginal utility of nominal income of the representative household who owns the firm. Therefore, the aggregate price index is given by the following expression

$$P_t = \left[\xi_p\left(P_{t-1}\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\right)^{-\frac{1}{\lambda_{p,t}}} + (1-\xi_p)\tilde{P}_t^{-\frac{1}{\lambda_{p,t}}}\right]^{-\lambda_{p,t}} \tag{2.1.9}$$

The optimal price setting condition relates the optimally decided price adjusted for inflation to the nominal marginal cost adjusted for the desired price mark-up, $\lambda_{p,t}$, as follows

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\Lambda_{t+s}\left[\tilde{P}_t(i)\pi_{t,t+s} - (1+\lambda_{p,t+s})MC_{t+s}\right]\tilde{Y}_{t+s}(i) = 0 \tag{2.1.10}$$

where $\tilde{P}_t$ is the level of the optimally chosen price common across firms and $\tilde{Y}_t$ is the demand these firms are facing.

*The maintenance services sector*

We assume that, perfectly competitive firms purchase units of final good, $Y_t^m$, in order to transform them into maintenance goods or services, $M_t$, which are then sold to households at the unit price of maintenance $P_t^m$. The transformation process from $Y_t^m$ to $M_t$ incurs some adjustment costs given by $f(Y_t^m/Y_{t-1}^m)$, which, in steady state, satisfy the following assumptions $f = f' = 0$ and $f'' > 0$[35]. An increase in the amount of final good designated for maintenance today, lowers the adjustment maintenance costs in the next period. Firms maximize the expected discounted value of future profits with respect to $Y_t^m$ and $M_t$ subject to the technology that transforms efficiency units of final goods into efficiency units of maintenance goods, as follows

$$\begin{aligned} \max_{M_t, Y_t^m} \quad & E_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t+s} \left[ P_{t+s}^m M_{t+s} - P_{t+s} Y_{t+s}^m \right] \\ s.t. \quad & M_{t+s} = d_{t+s} \left[ 1 - f\left( \frac{Y_{t+s}^m}{Y_{t+s-1}^m} \right) \right] Y_{t+s}^m \end{aligned} \tag{P4}$$

where we call $d_t$ the maintenance specific technology shock and assume it follows an AR(1) exogenous stochastic process

$$\log d_t = \rho_d \log d_{t-1} + \varepsilon_{d,t} \tag{2.1.11}$$

with $\varepsilon_{d,t} \sim i.i.d.N(0, \sigma_d^2)$. According to the technology for the production of efficiency units of maintenance an increase in the amount of the current final good used for maintenance, $Y_t^m$, decreases the expected level of the adjustment costs of maintenance, $f_{t+1}$.

Following the procedure implemented for investment in Justiniano et al. (2011), we call $\tilde{M}_t = (P_t^m/P_t)M_t$ the real maintenance in consumption units. From the zero profit condition of the firms we get $Y_t^m = (P_t^m/P_t)M_t$. Therefore, according to our model settings, the fraction of final good that is used as input in the production of maintenance goods equates real maintenance in consumption units, i.e. $Y_t^m = \tilde{M}_t$.

Optimization with respect to the efficiency units of maintenance, $M_t$, defines the equilibrium level of maintenance price as the shadow value of maintenance goods relative to the shadow value of consumption, as follows

$$P_t^m = \frac{\Gamma_t}{\Lambda_t} \tag{2.1.12}$$

The first order condition of the optimization problem with respect to $Y_t^m$ establishes the optimal supply of maintenance services, which depends on the maintenance adjustment technology and on the maintenance specific technology shock, as follows

$$\Lambda_t P_t = \Lambda_t P_t^m d_t \left[ 1 - f\left( \frac{Y_t^m}{Y_{t-1}^m} \right) - \frac{Y_t^m}{Y_{t-1}^m} f'\left( \frac{Y_t^m}{Y_{t-1}^m} \right) \right] + \beta E_t \Big\{ \Lambda_{t+1} P_{t+1}^m d_{t+1} \left( \frac{Y_{t+1}^m}{Y_t^m} \right)^2 f'\left( \frac{Y_{t+1}^m}{Y_t^m} \right) \Big\} \tag{2.1.13}$$

When the maintenance adjustment costs are zero, i.e. $f = f' = 0$, the relative price of maintenance services with respect to consumption equals the inverse of the maintenance specific technology shock, that is $P_t^m/P_t = d_t^{-1}$. Thus, a positive maintenance specific technology shock decreases the current

[35] The functional form for the maintenance adjustment costs and the underlying assumptions have been set following Christiano et al. (2005) and Justiniano et al. (2011), who resort to its flexible characteristics that we have mentioned in the main text.

relative price of maintenance and rises the current amount of efficiency units of maintenance. On the contrary, as it will be shown below, the real maintenance in consumption units declines in response to the same shock because of the inverse relationship among them.

*The investment good sector*

A fraction of final good $Y_t^I$ is purchased by perfectly competitive investment-goods producers in order to transform it into investment goods $I_t$ expressed in efficiency units, which are further sold to the capital-goods producers at the unit price $P_t^I$. These firms maximize their profit function subject to a production technology which accounts for the investment-specific technology progress, $\Upsilon_t$, as follows

$$\begin{aligned} &\max_{I_t, Y_t^I} \quad P_t^I I_t - P_t Y_t^I \\ &s.t. \quad I_t = \Upsilon_t Y_t^I \end{aligned} \tag{P4}$$

The optimization analysis in this sector draws out the common result according to which the relative price of investment equates the inverse of the investment-specific technology progress, that is

$$P_t^I / P_t = \Upsilon_t^{-1} \tag{2.1.14}$$

*The capital good sector*

Investment goods $I_t$ are purchased by the perfectly competitive capital goods producers, which transform them into installed capital $i_t^k$, that is further sold to households at the unit price $P_t^k$. Hence, firms maximize the expected discounted value of future profits subject to the technology for producing new capital

$$\begin{aligned} &\max_{I_t, i_t^k} \quad E_t \sum_{t=0}^{\infty} \beta^s \Lambda_{t+s} \left[ P_{t+s}^k i_{t+s}^k - P_{t+s}^I I_{t+s} \right] \\ &s.t. \quad i_{t+s}^k = \mu_{t+s} \left[ 1 - S\left( \frac{I_{t+s}}{I_{t+s-1}} \right) \right] I_{t+s} \end{aligned} \tag{P5}$$

where $\mu_t$ represents the marginal efficiency of investment shock and it follows an AR(1) exogenous stochastic process

$$\log \mu_t = \rho_\mu \log \mu_{t-1} + \varepsilon_{\mu,t} \tag{2.1.15}$$

with $\varepsilon_{\mu,t} \sim i.i.d.N(0, \sigma_\mu^2)$.

The transformation process of the investment goods into installed capital undergoes the adjustment costs $S(\cdot)$. The expected value of the investment adjustment costs decreases when the current amount of new investment increases. In steady state the following conditions are assumed to hold: $S = S' = 0$ and $S'' > 0$.

The optimality condition for the demand of new investment goods is derived as follows

$$P_t^I = P_t^k \mu_t \left[ 1 - S\left( \frac{I_t}{I_{t-1}} \right) - \frac{I_t}{I_{t-1}} S'\left( \frac{I_t}{I_{t-1}} \right) \right] + E_t \left\{ \frac{\beta \Lambda_{t+1}}{\Lambda_t} P_{t+1}^k \mu_{t+1} \left( \frac{I_{t+1}}{I_t} \right)^2 S'\left( \frac{I_{t+1}}{I_t} \right) \right\} \tag{2.1.16}$$

When there are no adjustment costs of investment, i.e. $S = S' = 0$, equation (2.1.16) reduces to $q_t = \mu_t^{-1}$, that is, the relative price of capital with respect to investment, which defines the Tobin's $q$, is equal to the inverse of the shock to marginal efficiency of investment, $\mu_t$.

The first order condition with respect to new capital, $i_t^k$, delivers an expression for the price of new capital as a ratio between the shadow value of installed new capital, $\Phi_t$, and the households Lagrangian multiplier

$$P_t^k = \frac{\Phi_t}{\Lambda_t}$$

substituting the latter expression back into the equation (2.1.16) gives the following optimal demand for new investment

$$\Lambda_t P_t^I = \Phi_t \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right)\right] + \beta E_t \left\{\Phi_{t+1}\mu_{t+1}\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\right\} \tag{2.1.17}$$

Recalling that the Tobin's $q$ is defined as the relative marginal value of installed capital with respect to investment, that is

$$q_t = \frac{\Phi_t}{P_t^I \Lambda_t}$$

where the denominator represents the replacement investment cost, the price for new capital becomes

$$P_t^k = q_t P_t \Upsilon_t^{-1}$$

According to the latter expression, the relative price of new capital is given by the Tobin's $q$ times the inverse of investment specific technology progress. In the absence of investment adjustment costs, instead, the relative price of new capital with respect to consumption can be re-expressed as

$$\frac{P_t^k}{P_t} = \Upsilon_t^{-1}\mu_t^{-1}$$

Hence, a positive shock to new investment makes the relative price of new capital to decline. The same happens with respect to the marginal efficiency of investment shock, $\mu$, when investment adjustment costs are excluded.

*The employment agencies sector*

Perfectly competitive employment agencies purchase specialized labor $L_t(j)$ from households at the specific wage level $W_t(j)$, and transform it into homogeneous labor, which is then sold to the intermediate-goods producers at the aggregate wage level, $W_t$. The employment agencies maximize their profits subject to the production function of homogeneous labor, that is

$$\begin{aligned} &\max_{L_t(j)} \quad W_t L_t - \int_0^1 W_t(j) L_t(j) dj \\ &s.t. \quad L_t = \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_{w,t}}} dj\right]^{1+\lambda_{w,t}} \end{aligned} \tag{P6}$$

where $\lambda_{w,t}$ represents the mark-up of the wage over the marginal rate of substitution of households, and it follows an AR(1) exogenous stochastic process

$$\log \lambda_{w,t} = (1-\rho_w)\log\lambda_w + \rho_w \log\lambda_{w,t-1} + \varepsilon_{w,t} - \theta_w \varepsilon_{w,t-1} \tag{2.1.18}$$

with $\varepsilon_{w,t} \sim i.i.d.N(0, \sigma_w^2)$.

The optimal demand for specialized labor and the aggregate wage level for homogeneous labor derived from the optimization problem are, respectively

$$L_t(j) = \left[\frac{W_t(j)}{W_t}\right]^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t \tag{2.1.19}$$

$$W_t = \left[\int_0^1 W_t(j)^{\frac{1}{\lambda_{w,t}}} dj\right]^{\lambda_{w,t}} \tag{2.1.20}$$

where $L_t$ represents the aggregate level of homogeneous labor.

*Households*

The economy is inhabited by a continuum of infinitely living households. A representative household maximizes the present value of the expected stream of logarithmic utility function with respect to current consumption, $C_t$, holdings of government bonds, $B_t$, capital utilization rate, $u_t$, physical capital stock, $\bar{K}_t$, and the efficiency units of maintenance, $M_t$, subject to the aggregate budget constraint, the law of motion of capital stock, the gross capital depreciation rate function and to the function describing the expenses in the efficiency maintenance units as follows, respectively

$$\begin{aligned}
\max_{C_t,B_t,\bar{K}_t,u_t,M_t} \quad & E_t \sum_{t=0}^{\infty} \beta^s b_{t+s} \left[\log\left(C_{t+s} - hC_{t+s-1}\right) - \varphi \frac{L_{t+s}(j)^{1+\nu}}{1+\nu}\right] \\
s.t. \quad & P_t C_t + P_t^k i_t^k + P_t^m M_t + T_t + B_t = \\
& = R_{t-1}B_{t-1} + Q_t(j) + \Pi_t + W_t(j)L_t(j) + R_t^k u_t \bar{K}_{t-1} - \frac{P_t}{\Upsilon_t} a(u_t)\bar{K}_{t-1} \\
& \bar{K}_t = (1 - \Upsilon_{t-1}^{-\sigma} D_t)\bar{K}_{t-1} + i_t^k \\
& D_t = \zeta u_t^{\eta} \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} + \Upsilon_t^{\sigma}\bar{\delta} \\
& M_t = \tau \Upsilon_t^{-1} u_t \bar{K}_{t-1} + A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M}
\end{aligned} \tag{P7}$$

where $h$ is the degree of habit formation, $\varphi$ is the share parameter of labor in the utility function, $\nu$ is the iverse Frisch elasticity, $T_t$ are lump-sum taxes, $R_t$ is the gross nominal interest rate, $Q_t(j)$ is the net cash flow of state contingent securities, which ensures that in equilibrium consumption and the asset holdings are the same across the households, and $\Pi_t$ is the per-capita profit accruing from the household's ownership of a firm. Moreover, $b_t$ represents the intertemporal preference shock which follows an AR(1) exogenous stochastic process according to

$$\log b_t = \rho_b \log b_{t-1} + \varepsilon_{b,t} \tag{2.1.21}$$

with $\varepsilon_{b,t} \sim i.i.d.N(0, \sigma_b^2)$.

The rate at which capital is utilized determines the amount of effective capital which is rented by households at rate $R_t^k$ to the firms of the intermediate-good producing sector, i.e. $K_t = u_t \bar{K}_{t-1}$. The adjustment costs of capital utilization, $a(u_t)$, following Justiniano et al. (2011), are evaluated at the dollar cost per unit of physical capital and scaled by the investment-specific technology progress in order to ensure the existence of the balanced growth path, that is $P_t a(u_t)/\Upsilon_t$. It is assumed that, in steady state, $u = 1$, $a(1) = 0$, and $\frac{a''(1)}{a'(1)} = \chi$.

We assume that the gross rate of capital depreciation, $D_t$, is endogenously determined by capital utilization rate, $u_t$, and by maintenance to capital ratio. Following the relevant literature, we assume that a more intensive use of capital leads to a faster depreciation of capital, on the contrary, a higher amount of maintenance activity reduces it. Parameters $\eta$ and $\sigma$ represent the sensitivity of the depreciation rate with respect to the utilization rate and maintenance to capital ratio, respectively. We assume as well the presence of a fixed cost of depreciation, $\bar{\delta}$, i.e. the natural rate of depreciation, which is multiplied by the investment specific technology progress, $\Upsilon_t^{\sigma}$, in oder to ensure the existence of a balanced growth path. The value of $\bar{\delta}$ is constant over time, however, its overall impact on the total gross depreciation rate is stronger when the IST progress occurs. Therefore, the first term on the right-hand side of the gross capital depreciation function captures the effects of the use-related depreciation rate, whereas the second one can be thought of as the obsolescence effect. That is, when new investment goods are available on the market more new capital goods are potentially produced, which makes the existent capital stock to depreciate on impact. Given our functional form of the capital depreciation rate, and the assumptions made about its relationship with capital utilization and maintenance, the following conditions must be satisfied: $\delta_u > 0$, $\delta_{uu} > 0$, $\delta_m < 0$, and $\delta_{mm} > 0$, which are in line with the assumptions grounded in the related literature[36]. Accordingly, the parameters of the depreciation rate function satisfy the following assumptions: $\eta > 1$, $\sigma > 0$, $\bar{\delta} > 0$ and $\zeta > 0$. We moreover assume that the depreciation rate is more sensitive to changes in maintenance expenses than to changes in the capital utilization rate, i.e. $\sigma > \eta$. This assumption is driven out from the optimality conditions of our model and is supported by the estimation results in Albonico et al. (2014). Finally, it has been shown that, the cross-derivative of the capital depreciation function with respect to maintenance and utilization must be negative, i.e. $\delta_{um} < 0$.[37] Given that we treat both the utilization rate and maintenance as control variables in our model, we adopt the following economic rational in order to defend the latter assumption: when a representative household decides to increase the rate of capital utilization, the rate of depreciation, given that $\delta_u > 0$, will also increase, hence, at optimum, a representative household will decide to increase the amount of maintenance too in order to reduce the depreciation rate, as far as $\delta_m < 0$, and vice versa; therefore, at optimum maintenance and utilization must move in the same direction but, since the two have an opposite effect on the depreciation rate, this implies that it must hold $\delta_{um} < 0$. Finally, note that, our model reduces to the baseline model of Justiniano et al. (2011) when both the sensibilities of the depreciation rate function with respect to maintenance and to utilization rate, $\sigma$ and $\eta$, respectively, tend to zero.

The law of motion of capital, contrary to Justiniano et al. (2011), includes the time dependent gross depreciation rate of capital, which is adjusted by the investment specific technology progress, $\Upsilon_t$. For the estimations of the capital depreciation rates it is commonly used to assume, in the National Account Systems, that the depreciation rate of capital is inversely related to the resale prices of used assets[38]. When the resale price tends to zero it means that the asset has been almost fully depreciated, vice versa when the resale price is relatively high the depreciation rate tends to zero, meaning that the respected asset is relatively 'new'. According to our model settings, the gross capital depreciation grows at the rate $\sigma\Upsilon_t$, which equates the inverse of the relative price of investment with respect to consumption weighted by the sensibility of depreciation with respect to maintenance. Thus, a positive IST shock decreses on impact the price of new investment and increases the current rate of depreciation,

[36] See, among others, Licandro et al. (2001), and Boucekkine et al. (2009).

[37] For the analytical derivation of the sign of the depreciation rate cross-derivative see Boucekkine and Ruiz-Tamarit (2003), and for estimation highlights consult Albonico et al. (2014).

[38] See, for example, the definition given for the rate of capital depreciation by the Bureau of Economic Analysis, www.bea.gov.

which, in turn, reduces the current stock of capital. Over the next period, however, due to $1/\Upsilon_{t-1}^{\sigma}$, the depreciation rate declines. This triggers off an adjustment mechanism for the path of the capital depreciation rate, which preserves from exploding values of the depreciation rate. Finally, the higher the sensibility of depreciation to maintenance expenses, the higher the volatility of the depreciation growth rate.

The latter function in our households optimization problem defines the expenses in maintenance. We assume there exists a positive relationship between efficient units of maintenance, $M_t$, and effective capital, $K_t = u_t\bar{K}_{t-1}$, imposing the positivity restriction on the marginal propensity to maintain, $\tau$, i.e. $\tau \geq 0$. That is, when capital is used more intensively in the production process a higher amount of maintenance is required. At the same time, more maintenance is needed when the amount of old capital is high. For what concerns the capital utilization rate, our assumption is in line with some macroeconomic studies, such as, for example, Licandro et al. (2001), Boucekkine et al. (2009) and Boucekkine et al. (2010). With regard to the stock of old installed capital, $\bar{K}_{t-1}$, instead, our assumption is enforced, among others, by the microeconomic evidence brought out by Bitros and Flytzanis (2004) and by Bitros (2016). The two works, in fact, show analytically and empirically, respectively, that maintenance expenses depend positively on the amount of scrapped capital. Effective capital is multiplied by the IST progress in order to ensure the existence of a balanced growth path, the same is done for the fixed costs of maintenance, $\bar{M}$, which are multiplied by $A_t\Upsilon_t^{\alpha/(1-\alpha)}$. We assume that there exists a strictly positive fixed cost of maintenance, $\bar{M} > 0$, in order to guarantee existance and uniqueness of the solution to our model. Such costs may capture, for example, those intrinsic maintenance activities accomplished by the households necessary for the physical capital assets to be usable. So, our maintenance expenditure function satisfies the following assumptions: $M_u > 0$, $M_{\bar{K}} > 0$, and $M(u \to 0, \bar{K} \to 0) \cong \bar{M}$. When capital utilization rate tends to zero, maintenance costs will approach their minimum level, $\bar{M}$, and the gross depreciation rate, $D_t$, will tend to the level of natural depreciation rate, $\Upsilon_t^{\sigma}\bar{\delta}$. When the stock of capital is fully utilized, i.e. $u_t$ tends to unity, then, for $\tau > 0$, maintenance expenses will tend to their maximum level, i.e. $M_t = \tau\Upsilon_t^{-1}\bar{K}_{t-1} + A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}$, and depreciation will be given by $D_t = \zeta\left(\tau\Upsilon_t^{-1} + A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}/\bar{K}_{t-1}\right)^{-\sigma} + \Upsilon_t^{\sigma}\bar{\delta}$. When, instead, capital is fully utilized and households decide to maintain the minimum required, i.e. $\tau = 0$ and $M_t = A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}$, the depreciation rate of capital will be given by $D_t = \zeta\left(A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}/\bar{K}_{t-1}\right)^{-\sigma} + \Upsilon_t^{\sigma}\bar{\delta}$. It is obvious that $\left(A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}/\bar{K}_{t-1}\right)^{-\sigma} > \left(\tau\Upsilon_t^{-1} + A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}/\bar{K}_{t-1}\right)^{-\sigma}$, so that in the latter case a representative household will tackle the highest level of depreciation with respect to the previous ones.

Note that, when $\eta \to 0$, $\sigma \to 0$ and $\tau \to 0$ our model reduces to the original model of Justiniano et al. (2011).

Recalling our definition of real maintenance and real investment in terms of consumption as $\tilde{M}_t = (P_t^m/P_t)M_t$ and $\tilde{I}_t = (P_t^I/P_t)I_t$, respectively, we can reduce our decentralized model to a one sector model, as follows

$$P_tC_t + P_t\tilde{I}_t + P_t\tilde{M}_t + T_t + B_t = R_{t-1}B_{t-1} + Q_t(j) + \Pi_t + W_t(j)L_t(j) + R_t^k u_t\bar{K}_{t-1} - \frac{P_t}{\Upsilon_t}a(u_t)\bar{K}_{t-1}$$

$$\bar{K}_t = (1 - \Upsilon_{t-1}^{-\sigma}D_t)\bar{K}_{t-1} + \mu_t\Upsilon_t(1 - S_t)\tilde{I}_t$$

$$D_t = \zeta u_t^\eta \left(\frac{d_t\tilde{M}_t\left[1 - f\left(\tilde{M}_t/\tilde{M}_{t-1}\right)\right]}{K_{t-1}}\right)^{-\sigma} + \Upsilon_t^\sigma\bar{\delta}$$

$$d_t\tilde{M}_t\left[1 - f\left(\tilde{M}_t/\tilde{M}_{t-1}\right)\right] = \tau\Upsilon_t^{-1}u_t\bar{K}_{t-1} + A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\bar{M}$$

In this case the law of motion of capital is subject to the composite shock $\mu_t\Upsilon_t$, which is used to be considered in the related literature as the investment-specific technology shock[39]. In our model, therefore, according to Justiniano et al. (2011), it is decomposed in two effects: the first one affects the efficiency of new investment ($\mu$), and the latter one affects the production of new investment ($\Upsilon$). Given the roles these shocks play in our decentralized economy, we interpret the IST shock, $\Upsilon$, as the disembodied investment-specific technological progress and identify it with the inverse of the relative price of investment. For the estimation purposes the relative price of investment is treated as observable which, thus, pins down the evolution of IST. The MEI shock, $\mu$, is interpreted as the embodied investment-specific technological progress, which explains the quality improvement of investment and therefore determines the rate of obsolescence of capital stock. Both of these concepts are strictly related with the economic depreciation rate, which accelerates when higher quality capital is available on the market (obsolescence). On the contrary, in response to a positive IST shock depreciation will decrease as a consequence of other stronger direct effects.

Turning back to the decentralized representation of the model, a representative household optimizes the discounted stream of expected utility with respect to current consumption, which gives the expression for the marginal utility of nominal income, as follows

$$\Lambda_tP_t = \frac{b_t}{C_t - hC_{t-1}} - \beta hE_t\left\{\frac{b_{t+1}}{C_{t+1} - hC_t}\right\} \tag{2.1.22}$$

The first order condition with respect to the holdings of government bonds, $B_t$, defines the consumption Euler equation, that is

$$\Lambda_t = \beta R_tE_t\{\Lambda_{t+1}\} \tag{2.1.23}$$

or, similarly

$$1 = E_t\left[\beta\frac{\Lambda_{t+1}P_{t+1}}{\Lambda_tP_t}R_t\frac{P_t}{P_{t+1}}\right]$$

where, the expression $\beta\frac{\Lambda_{t+1}P_{t+1}}{\Lambda_tP_t}$ defines the real stochastic discount factor.

The first order condition with respect to installed physical capital, $\bar{K}_t$, determines the optimal demand for physical capital stock, given by

[39] See, for example, Smets and Wouters (2007)

$$\Phi_t = \beta E_t \Big\{ \Lambda_{t+1} \left[ R^k_{t+1} u_{t+1} - \Upsilon^{-1}_{t+1} P_{t+1} a(u_{t+1}) \right] + (1 - \Upsilon_t^{-\sigma} D_{t+1}) \Phi_{t+1} + \\ - \sigma \left( D_{t+1} - \Upsilon^{-\sigma}_{t+1} \bar{\delta} \right) \Upsilon_t^{-\sigma} \Phi_{t+1} + \tau u_{t+1} \Upsilon^{-1}_{t+1} \Gamma_{t+1} \Big\} \tag{2.1.24}$$

where $\Phi_t$ is the shadow value of the new installed capital. The last two terms on the right-hand side of the above equation capture the effects of the variable depreciation rate and of the maintenance activity, respectively. The higher the expected maintenance the lower the expected depreciation rate and the higher the current shadow price of capital. On the contrary, the higher the expected depreciation rate the lower the current shadow price. The effect of the expected capital utilization rate, instead, is ambiguous. In fact, it will tend to increase the current value of the capital shadow price through higher maintenance activities but, at the same time, it will decrease the shadow price because of a higher expected depreciation.

The first order condition with respect to the rate of capital utilization, $u_t$, determines its optimal equilibrium demand, as follows

$$\Lambda_t R^k_t = \Lambda_t P_t \Upsilon_t^{-1} a'(u_t) + \eta \Phi_t \Upsilon^{-\sigma}_{t-1} u_t^{-1} \left( D_t - \Upsilon_t^{\sigma} \bar{\delta} \right) - \tau \Gamma_t \Upsilon_t^{-1} \tag{2.1.25}$$

According to the above expression, the rental price of capital, at optimum, must equate the marginal value of capital utilization costs plus the cost deriving from a higher capital depreciation rate due to a marginal increase in the utilization rate evaluated at the relative shadow value of new capital net of the marginal benefits accruing from a higher maintenance activity evaluated at the relative shadow value of maintenance.

The first order condition with respect to maintenance, $M_t$, is

$$\Lambda_t P^m_t = \sigma \Phi_t \Upsilon^{-\sigma}_{t-1} \left( \frac{M_t}{\bar{K}_{t-1}} \right)^{-1} \left( D_t - \Upsilon_t^{\sigma} \bar{\delta} \right) - \Gamma_t \tag{2.1.26}$$

according to which the optimal demand for maintenance to capital ratio is inversely related to the price of maintenance and positively related to the gross capital depreciation and to the relative shadow value of new capital. Note that, the demand for maintenance will be null when gross capital depreciation equals the natural rate of depreciation times the IST shock, which is the minimum level at which the depreciation rate may occur. So, at optimum, the price for maintenance must be equal to the marginal benefit accruing from the capital depreciation rate given an increase of one unit of maintenance evaluated at the relative shadow value of new capital minus the relative shadow value of maintenance.

Following Justiniano et al. (2011) we assume that, each household is a monopolistic supplier of a specialized labor, $L_t(j)$. Similarly to the price decision setting in the intermediate-goods sector, every period, a fraction $\xi_w$ of households sets their wage level according to the following indexation rule

$$W_t(j) = W_{t-1}(j) \left( \pi_{t-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha} v_t} \right)^{\iota_w} \left( \pi e^{\gamma_z + \frac{\alpha}{1-\alpha} \gamma_v} \right)^{1-\iota_w} \tag{2.1.27}$$

The remaining fraction of households, $1-\xi_w$, optimally chooses the wage level, $\tilde{W}_t(j)$, by maximizing the following present discounted value of future earnings subject to the optimal labor demand (2.1.19)

$$\max_{\tilde{W}_t(j)} \quad E_t \sum_{t=0}^{\infty} \beta^s \xi_w^s \left[ \Lambda_{t+s} \tilde{W}_t(j) L_{t+s}(j) \pi_{t,t+s}^w - b_{t+s} \varphi \frac{L_{t+s}(j)^{1+\nu}}{1+\nu} \right]$$
$$s.t. \quad L_{t+s}(j) = \left[ \frac{\tilde{W}_t(j)}{W_{t+s}} \pi_{t,t+s}^w \right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s} \tag{P8}$$

where $\pi_{t,t+s}^w = \prod_{k=0}^{s} \left( \pi_{t+k-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha} v_{t-1}} \right)^{\iota_w} \left( \pi e^{\gamma_z + \frac{\alpha}{1-\alpha} \gamma_v} \right)^{1-\iota_w}$.

The aggregate wage index is given by the following expression

$$W_t = \left( \xi_w \left[ W_{t-1}(j) \left( \pi_{t-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha} v_{t-1}} \right)^{\iota_w} \left( \pi e^{\gamma_z + \frac{\alpha}{1-\alpha} \gamma_v} \right)^{1-\iota_w} \right]^{-\frac{1}{\lambda_{w,t}}} + (1-\xi_w) \tilde{W}_t(j)^{-\frac{1}{\lambda_{w,t}}} \right)^{-\lambda_{w,t}} \tag{2.1.28}$$

The optimal wage setting condition is given by

$$E_t \sum_{s=0}^{\infty} \xi_w^s \beta^s \Lambda_{t+s} \tilde{L}_{t+s}(j) \left[ \pi_{t,t+s}^w \tilde{W}_t(j) - b_{t+s} \varphi \left(1 + \lambda_{w,t+s}\right) \frac{\tilde{L}_{t+s}(j)^{\nu}}{\Lambda_{t+s}} \right] = 0 \tag{2.1.29}$$

where $\tilde{L}_t(j)$ is the level of optimally supplied specialized labor.

*Public Sector*

According to the Ricardian fiscal policy, the public sector finances its budget deficit through short-term bonds releases. Government expenditures are assumed to be a fraction of GDP and are given exogenously by

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t \tag{2.1.30}$$

where $g_t$ is an exogenous stochastic process for government spendings

$$\log g_t = (1 - \rho_g) \log g + \rho_g \log g_{t-1} + \varepsilon_{g,t} \tag{2.1.31}$$

with $\varepsilon_{g,t} \sim i.i.d.N(0, \sigma_g^2)$.

*Monetary policy authority*

The monetary authority choses the level of the nominal interest rate according to the following interest rate rule

$$\frac{R_t}{R} = \left( \frac{R_{t-1}}{R} \right)^{\rho_R} \left[ \left( \frac{\pi_t}{\pi} \right)^{\phi_\pi} \left( \frac{X_t}{X_t^*} \right)^{\phi_X} \right]^{1-\rho_R} \left[ \frac{X_t / X_{t-1}}{X_t^* / X_{t-1}^*} \right]^{\phi_{dX}} \varepsilon_{mp,t} \tag{2.1.32}$$

where $R$ is the steady state value of the nominal interest rate, $X_t/X_t^*$ is the level of the GDP gap and $(X_t/X_{t-1})(X_t^*/X_{t-1}^*)$ is its growth rate, and $\varepsilon_{mp,t} \sim i.i.d.N(0, \sigma_{mp}^2)$ represents a monetary policy shock. Hence, according to this rule, the nominal interest rate responds to the deviations of inflation from its steady state level, to the level of the GDP gap and to its growth rate.

The model is closed by the expressions for the aggregate resource constraint and for the actual GDP, $X_t$, which are given by, respectively

$$C_t + \Upsilon_t^{-1} I_t + \Upsilon_t^{-1} a(u_t) \bar{K}_{t-1} + \tilde{M}_t = \frac{Y_t}{g_t} \tag{2.1.33}$$

$$X_t = \left(1 - \frac{1}{g_t}\right) Y_t + C_t + \Upsilon_t^{-1} I_t + \tilde{M}_t \tag{2.1.34}$$

Following Justiniano et al. (2011), the market clearing conditions are obtained from the aggregation of the households' and Government budget constraints combined with the zero profit conditions in the production sectors of final good, capital-good, investment-good, maintenance services and in the employment agencies sector. The functional form of the actual GDP, instead, is obtained by substitution of the definition for public spendings into the aggregate resource constraint, given that the actual GDP is defined as $X_t = G_t + C_t + \Upsilon_t^{-1} I_t + \tilde{M}_t$.

## 2.2 The stationary equilibrium and the non-stochastic steady state

As far as the levels of labor-augmenting technology and the investment-specific technology progresses have a unit root, the main macroeconomic variables of the model, that are output, consumption, investment, maintenance, capital and real wages, all fluctuate around a stochastic balanced growth path. The steady state growth rate is a linear combination of the composite technological progress $A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}$, that is

$$\gamma^* = \gamma_z + \frac{\alpha}{1-\alpha} \gamma_v$$

The stationary equilibrium model is achieved de-trending the model's variables by the composite growth trend. The non-stochastic steady state is then computed and the linear system of rational expectations equations is solved through a log-linear approximation of the model around the non-stochastic steady state. The lower case variables represent the normalized stationary variables and those with no timing subsciption are the respective steady state values. The main model variables have been de-trended as follows

- $y_t = \frac{Y_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}$
- $x_t = \frac{X_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}$
- $c_t = \frac{C_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}$
- $\bar{k}_t = \frac{\bar{K}_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}$
- $k_t = \frac{K_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}$
- $i_t = \frac{I_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}$
- $m_t = \frac{M_t}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}$
- $w_t = \frac{W_t}{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}$
- $\rho_t = \frac{R_t^k}{P_t}$
- $s_t = \frac{MC_t}{P_t}$
- $\tilde{p}_t = \frac{\tilde{P}_t}{P_t}$
- $\pi_t = \frac{P_t}{P_{t-1}}$
- $\lambda_t = \Lambda_t P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}$
- $\phi_t = \Phi_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}$

where $s_t$ is the real marginal cost and $\rho_t$ is the real return on capital. Moreover, the rate of capital depreciation is de-trended by the investment specific technological shock, that is $\delta_t = D_t/\Upsilon_t^\sigma$.

The stationary equilibrium model is described by the equations below.

The stationary counterparts of the production function in (P2), of the capital to labor ratio (2.1.6) and of the nominal marginal cost (2.1.7) of a representative intermediate good producer $i$ are, respectively

$$y_t(i) = k_t(i)^\alpha L_t(i)^{1-\alpha} - F \tag{2.2.1}$$

$$\frac{k_t}{L_t} = \frac{w_t}{\rho_t}\frac{\alpha}{1-\alpha} \tag{2.2.2}$$

$$s_t = w_t^{1-\alpha}\rho_t^\alpha \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \tag{2.2.3}$$

where equation (2.2.3) describes the de-trended real marginal cost. The stationary equations for the intermediate firms that are able to optimize for the price level are given by

$$\tilde{y}_{t+s} = \left(\tilde{p}_t\tilde{\pi}_{t,t+s}\right)^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}}\tilde{y}_{t+s} \tag{2.2.4}$$

$$\tilde{\pi}_{t,t+s} = \prod_{j=0}^{s}\left(\frac{\pi_{t+j-1}}{\pi}\right)^{\iota_p}\left(\frac{\pi_{t+j}}{\pi}\right)^{-1} \tag{2.2.5}$$

$$1 = \left\{\xi_p\left[\left(\frac{\pi_{t-1}}{\pi}\right)^{\iota_p}\left(\frac{\pi_t}{\pi}\right)^{-1}\right]^{-\frac{1}{\lambda_{p,t}}} + (1-\xi_p)\,\tilde{p}_t^{-\frac{1}{\lambda_{p,t}}}\right\}^{-\lambda_{p,t}} \tag{2.2.6}$$

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\lambda_{t+s}\left[\tilde{p}_t(i)\tilde{\pi}_{t,t+s} - (1+\lambda_{p,t+s})\,s_{t+s}\right]\tilde{y}_{t+s}(i) = 0 \tag{2.2.7}$$

Equation (2.2.4) is the stationary optimal demand for the intermediate good $i$. Equation (2.2.5) is the index for inflation from time $t$ to time $t+s$. Equation (2.2.6) is the stationary aggregate price index and equation (2.2.7) is the stationary optimal price setting condition, corresponding to equations (2.1.9) and (2.1.10), respectively.

In order to obtain the detrended optimal condition for maintenance goods, substitute the optimal demand for maintenance activity (2.1.26) from the households optimization problem (P7) into the expression for the optimal supply of maintenance (2.1.13) from the maintenance services optimization problem (P4) so that

$$\begin{aligned}\Lambda_t P_t = &\left[\Phi_t\Upsilon_{t-1}^{-\sigma}\sigma\left(D_t - \Upsilon_t^\sigma\bar{\delta}\right)\left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-1} - \Gamma_t\right]d_t\left[1 - f\left(\frac{Y_t^m}{Y_{t-1}^m}\right) - \frac{Y_t^m}{Y_{t-1}^m}f'\left(\frac{Y_t^m}{Y_{t-1}^m}\right)\right] + \\ &+\beta E_t\left\{\left[\Phi_{t+1}\Upsilon_t^{-\sigma}\sigma\left(D_{t+1} - \Upsilon_{t+1}^\sigma\bar{\delta}\right)\left(\frac{M_{t+1}}{\bar{K}_t}\right)^{-1} - \Gamma_{t+1}\right]d_{t+1}\left(\frac{Y_{t+1}^m}{Y_t^m}\right)^2 f'\left(\frac{Y_{t+1}^m}{Y_t^m}\right)\right\}\end{aligned}$$

De-trending the latter expression and substituting $y_t^m$ with $m_t$ we get the optimal demand for maintenance services in terms of stationary variables as follows

$$\lambda_t = \left[\sigma\phi_t\left(\frac{m_t}{\bar{k}_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)^{-1}\left(\delta_t-\bar{\delta}\right)e^{\sigma v_t}-\varsigma_t\right]\times$$
$$\times d_t\left[1-f\left(\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}\right)-\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}f'\left(\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}\right)\right]+ \tag{2.2.8}$$
$$+\beta E_t\left\{\left[\sigma\phi_{t+1}\left(\frac{m_{t+1}}{\bar{k}_t}e^{z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}}\right)^{-1}\left(\delta_{t+1}-\bar{\delta}\right)e^{\sigma v_{t+1}}-\varsigma_{t+1}\right]d_{t+1}e^{-\left(z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}\right)}\times\right.$$
$$\left.\times\left(\frac{\tilde{m}_{t+1}}{\tilde{m}_t}e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\right)^2 f'\left(\frac{\tilde{m}_{t+1}}{\tilde{m}_t}e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\right)\right\}$$

The marginal utility of nominal income (2.1.22) and the consumption Euler equation (2.1.23) in terms of de-trended variables are, respectively

$$\lambda_t = \frac{e^{z_t+\frac{\alpha}{1-\alpha}v_t}b_t}{e^{z_t+\frac{\alpha}{1-\alpha}v_t}c_t-hc_{t-1}}-\beta hE_t\left\{\frac{b_{t+1}}{e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}c_{t+1}-hc_t}\right\} \tag{2.2.9}$$

$$\lambda_t = \beta R_t E_t\left\{\frac{\lambda_{t+1}}{e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\pi_{t+1}}\right\} \tag{2.2.10}$$

De-trending the optimal demand for capital utilization rate given in (2.1.25) we obtain the expression for the stationary optimal capital utilization rate as follows

$$\lambda_t\rho_t = \lambda_t a'\left(u_t\right)+\eta\phi_t\left(\delta_t-\bar{\delta}\right)e^{\sigma v_t}u_t^{-1}-\tau\varsigma_t \tag{2.2.11}$$

The stationary expression for the optimal demand for new investment (2.1.17) becomes

$$\lambda_t = \phi_t\mu_t\left[1-S\left(\frac{i_t}{i_{t-1}}e^{z_t+(\frac{\alpha}{1-\alpha}+1)v_t}\right)+\right.$$
$$\left.-\frac{i_t}{i_{t-1}}e^{z_t+(\frac{\alpha}{1-\alpha}+1)v_t}S'\left(\frac{i_t}{i_{t-1}}e^{z_t+(\frac{\alpha}{1-\alpha}+1)v_t}\right)\right]+$$
$$+\beta E_t\left\{\phi_{t+1}\mu_{t+1}e^{(-z_{t+1}-(\frac{\alpha}{1-\alpha}+1)v_{t+1})}\left(\frac{i_{t+1}}{i_t}e^{z_{t+1}+(\frac{\alpha}{1-\alpha}+1)v_{t+1}}\right)^2\times\right.$$
$$\left.\times S'\left(\frac{i_{t+1}}{i_t}e^{z_{t+1}+(\frac{\alpha}{1-\alpha}+1)v_{t+1}}\right)\right\} \tag{2.2.12}$$

And the stationary optimal demand for installed physical capital (2.1.24) is

$$\phi_t = \beta E_t\left\{\lambda_{t+1}\left[\rho_{t+1}u_{t+1}-a\left(u_{t+1}\right)\right]e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}+\right.$$
$$+\tau\varsigma_{t+1}u_{t+1}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}+\phi_{t+1}\left(1-\delta_{t+1}e^{\sigma v_{t+1}}\right)e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}+$$
$$\left.-\sigma\phi_{t+1}\left(\delta_{t+1}-\bar{\delta}\right)e^{\sigma v_{t+1}}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}\right\} \tag{2.2.13}$$

The stationary expressions for the definition of effective capital, for the law of motion of physical capital, for the definition of the capital depreciation rate and for the definition of maintenance costs are given by, respectively

$$k_t = u_t \bar{k}_{t-1} e^{-z_t - (\frac{\alpha}{1-\alpha}+1)v_t} \tag{2.2.14}$$

$$\bar{k}_t = (1 - e^{\sigma\gamma_v}\delta_t) e^{-z_t-(\frac{\alpha}{1-\alpha}+1)v_t}\bar{k}_{t-1} + \mu_t \left[1 - S\left(\frac{i_t}{i_{t-1}} e^{z_t+(\frac{\alpha}{1-\alpha}+1)v_t}\right)\right] i_t \tag{2.2.15}$$

$$\delta_t = \zeta u_t^{\eta} \left[\frac{m_t}{\bar{k}_{t-1}} e^{z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right]^{-\sigma} + \bar{\delta} \tag{2.2.16}$$

$$m_t = \tau u_t \bar{k}_{t-1} e^{-z_t - \left(\frac{\alpha}{1-\alpha}+1\right)v_t} + \bar{M} \tag{2.2.17}$$

With regard to households that are able to renegotiate their wages, the stationary expressions of the optimal wage setting equations are as follows

$$\tilde{L}_{t+s}(j) = \left[\tilde{w}_t \tilde{\pi}^w_{t,t+s}\right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s} \tag{2.2.18}$$

$$\tilde{\pi}^w_{t,t+s} = \prod_{k=0}^{s} \left(\frac{\pi_{t+k-1} e^{z_{t+k-1} + \frac{\alpha}{1-\alpha} v_{t+k-1}}}{\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}}\right)^{\iota_w} \left(\frac{\pi_{t+k} e^{z_{t+k} + \frac{\alpha}{1-\alpha} v_{t+k}}}{\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}}\right)^{-1} \tag{2.2.19}$$

$$E_t \sum_{s=0}^{\infty} \xi_w^s \beta^s \lambda_{t+s} \tilde{L}_{t+s} \left[\tilde{\pi}^w_{t,t+s} \tilde{w}_t - b_{t+s}\varphi \left(1 + \lambda_{w,t+s}\right) \frac{\tilde{L}^{\nu}_{t+s}}{\lambda_{t+s}}\right] = 0 \tag{2.2.20}$$

$$w_t = \left\{\xi_w \left[w_{t-1} \left(\frac{\pi_{t-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha} v_{t-1}}}{\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}}\right)^{\iota_w} \left(\frac{\pi_t e^{z_t + \frac{\alpha}{1-\alpha} v_t}}{\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}}\right)^{-1}\right]^{-\frac{1}{\lambda_{w,t}}} + (1-\xi_w)\, \tilde{w}_t^{-\frac{1}{\lambda_{w,t}}}\right\}^{-\lambda_{w,t}} \tag{2.2.21}$$

Equation (2.2.18) drafts the de-trended optimal demand for specialized labor in (2.1.19). Equation (2.2.19) describes the wage inflation that occurs from time $t$ to time $t+s$. Equation (2.2.20) is the stationary optimal wage setting condition (2.1.29) and equation (2.2.21) is the de-trended expression for the aggregate wage index (2.1.28).

The stationary expression for the monetary policy rule (2.1.32) is

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi} \left(\frac{x_t}{x_t^\star}\right)^{\phi_X}\right]^{1-\rho_R} \left[\frac{x_t/x_{t-1}}{x_t^\star/x_{t-1}^\star}\right]^{\phi_{dX}} \varepsilon_{mp,t} \tag{2.2.22}$$

Finally, the de-trended counterparts of the aggregate resource constraint in (2.1.33) and of the definition of GDP in (2.1.34) are, respectively

$$c_t + i_t + \tilde{m}_t + a(u_t)\bar{k}_{t-1} e^{-z_t - (\frac{\alpha}{1-\alpha}+1)v_t} = \frac{y_t}{g_t} \tag{2.2.23}$$

$$x_t = \left(1 - \frac{1}{g_t}\right) y_t + c_t + i_t + \tilde{m}_t \tag{2.2.24}$$

Next, we describe the steady state relationships that characterize our model.

Using the expression (2.2.1) and the zero profit condition of the intermediate goods producers we get

$$\frac{y}{L} = \left(\frac{k}{L}\right)^{\alpha} - \frac{F}{L} \tag{2.2.25}$$

The capital to labor ration (2.2.2) and the real marginal cost (2.2.3) in steady state are, respectively

$$\frac{k}{L} = \frac{w}{\rho}\frac{\alpha}{1-\alpha} \tag{2.2.26}$$

$$s = w^{1-\alpha}\rho^{\alpha}\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \tag{2.2.27}$$

Given the expression for the de-trended optimal price setting condition (2.2.7) and exploiting the fact that in steady state $\tilde{p} = 1$ and $\pi = 1$, we get a steady state relationship between the real marginal cost and the price mark-up as follows

$$s = \frac{1}{1+\lambda_p} \tag{2.2.28}$$

As far as, in steady state, $S = S' = 0$, from the expression (2.2.12) we get that $\lambda = \phi$. It follows that, the steady state expression of the de-trended optimal demand for maintenance (2.2.8) is given by

$$\lambda = \sigma\phi e^{\sigma\gamma_v}\left(\delta - \bar{\delta}\right)\left[\frac{\tilde{m}}{\bar{k}}e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right]^{-1} - \varsigma \tag{2.2.29}$$

The optimal marginal utility of nominal income (2.2.9) in steady state becomes

$$\lambda c = \frac{e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v} - \beta h}{e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v} - h} \tag{2.2.30}$$

While the steady state quarterly real interest rate is achieved from the de-trended Euler equation (2.2.10) as follows

$$R = \frac{e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}}{\beta} - 1 \tag{2.2.31}$$

Assuming that, in steady state, the capital is fully utilized, i.e. $u = 1$, and that $a(1) = 0$, the stationary optimal demand for capital utilization (2.2.11) and the stationary optimal demand for physical capital (2.2.13), in steady state, become, respectively

$$\rho = a'(1) + \eta\frac{\phi}{\lambda}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v} - \tau\frac{\varsigma}{\lambda} \tag{2.2.32}$$

$$\phi = \frac{\lambda\beta(\rho - \tau)}{e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v} - \beta\left(1 - \delta e^{\sigma\gamma_v}\right) + \beta\sigma\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v}\left[1 - \tau\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\right]} \tag{2.2.33}$$

The steady state relations of the definition of capital input, (2.2.14), and of the law of motion of capital, (2.2.15) are, respectively

$$k = \bar{k} e^{-\gamma_z - (\frac{\alpha}{1-\alpha}+1)\gamma_v} \tag{2.2.34}$$

$$\bar{k} = \frac{i}{1 - (1 - \delta e^{\sigma\gamma_v})\, e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}} \tag{2.2.35}$$

Whereas, in steady state, the definitions of the capital depreciation rate (2.2.16) and of the maintenance costs (2.2.17) are, respectively

$$\delta = \zeta \left( \frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v} \right)^{-\sigma} + \bar{\delta} \tag{2.2.36}$$

$$\tilde{m} = \tau \bar{k} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} + \bar{M} \tag{2.2.37}$$

Given the de-trended optimal wage setting (2.2.20) and the steady state equalities $\tilde{\pi}^w = 1$ and $\tilde{w} = 1$, the following expression for the steady state labor arises

$$L = \left( \frac{\lambda}{\varphi} \frac{w}{1+\lambda_w} \right)^{\frac{1}{\nu}} \tag{2.2.38}$$

The aggregate resource constraint (2.2.23) and the definition of GDP (2.2.24) in steady state become, respectively

$$c + i + \tilde{m} = (1/g)\, y \tag{2.2.39}$$

$$x = (1 - 1/g)\, y + c + i + \tilde{m} \tag{2.2.40}$$

Combining the definition for the gross depreciation rate of capital with the definition for the real maintenance costs in terms of consumption, both given in the maximization problem (P7), we get that, along the balanced growth path, the following relationship holds

$$M_t = \frac{\tau\sigma}{\eta} \Upsilon_t^{-1} u_t \bar{K}_{t-1}$$

which, after being de-trended and using equation (2.2.34), in steady state becomes

$$\frac{\tilde{m}}{k} = \frac{\tau\sigma}{\eta} \tag{2.2.41}$$

Combining equation (2.2.41) with equation (2.2.37) we obtain the following two equalities

$$\frac{\bar{M}}{k} = \left( \frac{\sigma}{\eta} - 1 \right) \tau \tag{2.2.42}$$

$$\frac{\bar{M}}{\tilde{m}} = 1 - \frac{\eta}{\sigma} \tag{2.2.43}$$

which define, along the balanced growth path, the ratio of fixed maintenance costs over effective capital and real maintenance, respectively.

Combining equation (2.2.41) with equations (2.2.29) and (2.2.34), and recalling that in steady state $\lambda = \phi = \varsigma$, we get

$$\delta = 2e^{-\sigma\gamma_v} \frac{\tau}{\eta} + \bar{\delta} \tag{2.2.44}$$

Combining the latter expression with the steady state depreciation rate givein in (2.2.36), we define the following expression

$$\zeta = 2e^{-\sigma\gamma_v}\left(\frac{\tau}{\eta}\right)^{1+\sigma}\sigma^{\sigma} \tag{2.2.45}$$

Finally, we can combine the equations (2.2.35), (2.2.36) and (2.2.34) in order to obtain a steady state relationship between the real maintenance to capital ratio and the investment to capital ratio as follows

$$\left(\frac{\tilde{m}}{k}\right)^{\sigma} = \frac{\zeta e^{\sigma\gamma_v}}{1+\frac{i}{k}-e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}-\bar{\delta}e^{\sigma\gamma_v}}$$

According to the latter expression the higher the steady state level of investment the lower the steady state amount of maintenance undertaken by the households in order to restore the capital stock.

## 2.3 Linear rational expectations model

We compute the log-linear approximation of our model around the non-stochastic steady state. The variables denoted by a hat represent the log-linear deviations from their respective steady states, that is, given a generic variable $H$, $\hat{H}_t = \log H_t - \log H$. An exception of this rule is applied to the IST and the MEI shocks as well as to the price and wage mark-up shocks, for which the log-linear deviations from the steady state are defined as follows

$$\hat{\lambda}_{p,t+s} = \log(1+\lambda_{p,t+s}) - \log(1+\lambda_p)$$
$$\hat{\lambda}_{w,t+s} = \log(1+\lambda_{w,t+s}) - \log(1+\lambda_w)$$
$$\hat{z}_{t+s} = z_{t+s} - \gamma_z$$
$$\hat{v}_{t+s} = v_{t+s} - \gamma_v$$

Our model is composed of 20 endogenous variables in the sticky price-wage economy and of 19 endogenous variables, which are denoted by a 'star', in the flexible price-wage economy with null mark-up shocks.

$$\begin{bmatrix} \hat{y}_t & \hat{\bar{k}}_t & \hat{k}_t & \hat{c}_t & \hat{i}_t & \hat{\tilde{m}}_t & \hat{L}_t & \hat{\delta}_t & \hat{\rho}_t & \hat{w}_t & \hat{\tilde{w}}_t & \hat{s}_t & \hat{R}_t & \hat{\lambda}_t & \hat{\varsigma}_t & \hat{u}_t & \hat{\phi}_t & \hat{x}_t & \hat{g}_{w,t} & \hat{\pi}_t \\ \hat{y}^\star_t & \hat{\bar{k}}^\star_t & \hat{k}^\star_t & \hat{c}^\star_t & \hat{i}^\star_t & \hat{\tilde{m}}^\star_t & \hat{L}^\star_t & \hat{\delta}^\star_t & \hat{\rho}^\star_t & \hat{w}^\star_t & \hat{\tilde{w}}^\star_t & \hat{s}^\star_t & \hat{R}^\star_t & \hat{\lambda}^\star_t & \hat{\varsigma}^\star_t & \hat{u}^\star_t & \hat{\phi}^\star_t & \hat{x}^\star_t & \hat{g}^\star_{w,t} & \end{bmatrix}$$

The following 20 equations describe the linear system of rational expectations equations of the sticky price-wage economy.

$$\hat{y}_t = \frac{y+F}{y}\alpha\hat{k}_t + \frac{y+F}{y}(1-\alpha)\hat{L}_t \tag{2.3.1}$$

$$\hat{\rho}_t = \hat{w}_t - \hat{k}_t + \hat{L}_t \tag{2.3.2}$$

$$\hat{s}_t = (1-\alpha)\hat{w}_t + \alpha\hat{\rho}_t \tag{2.3.3}$$

$$\hat{\pi}_t = \frac{\iota_p}{1+\beta\iota_p}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta\iota_p}E_t\{\hat{\pi}_{t+1}\} + \frac{(1-\xi_p\beta)(1-\xi_p)}{\xi_p(1+\beta\iota_p)}(\hat{s}_t+\lambda_{p,t}) \tag{2.3.4}$$

$$\begin{aligned}\hat{\lambda}_t = & \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)}\hat{c}_{t-1} - \frac{e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}+\beta h^2}{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)}\hat{c}_t + \\ & + \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\beta h}{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)}E_t\{\hat{c}_{t+1}\} + \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\beta h\rho_z - e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}h}{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)}\hat{z}_t + \\ & + \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\beta h\rho_v - e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}h}{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)}\frac{\alpha}{1-\alpha}\hat{v}_t + \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\rho_b}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h}\hat{b}_t\end{aligned} \tag{2.3.5}$$

$$\hat{\lambda}_t = \hat{R}_t - \rho_z\hat{z}_t - \rho_v\frac{\alpha}{1-\alpha}\hat{v}_t + E_t\{\hat{\lambda}_{t+1}-\hat{\pi}_{t+1}\} \tag{2.3.6}$$

$$\hat{\rho}_t = \left\{\frac{\chi}{\rho}\left[\rho-\bar{B}+\tau\left(\bar{A}-1\right)\right]-\frac{\bar{B}}{\rho}\right\}\hat{u}_t + \frac{\sigma}{\rho}\bar{B}\hat{v}_t + \frac{\bar{B}}{\rho}\left(\hat{\phi}_t-\hat{\lambda}_t\right) - \frac{\tau}{\rho}\left(\bar{A}-1\right)\left(\hat{\varsigma}_t-\hat{\lambda}_t\right) + \frac{\delta}{\rho\left(\delta-\bar{\delta}\right)}\bar{B}\hat{\delta}_t \tag{2.3.7}$$

$$\begin{aligned}\hat{\phi}_t = & \hat{\lambda}_{t+1} - \rho_z\hat{z}_t - \left[\frac{\alpha}{1-\alpha}+1+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\sigma\bar{C}\right]\rho_v\hat{v}_t + \beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\bar{B}\hat{u}_{t+1} + \frac{\rho}{\rho-\tau}\bar{D}\hat{\rho}_{t+1} \\ & - \beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\bar{C}+e^{\sigma\gamma_v}\sigma\bar{\delta}\right)\hat{\delta}_{t+1} + \left[1-\bar{D}-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A}\right]\left(\hat{\phi}_{t+1}-\hat{\lambda}_{t+1}\right) + \\ & + \left[\frac{\tau}{\rho-\tau}\bar{D}-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A}\right]\left(\hat{\varsigma}_{t+1}-\hat{\lambda}_{t+1}\right)\end{aligned} \tag{2.3.8}$$

$$\begin{aligned}\hat{\lambda}_t = & \hat{\phi}_t + \hat{\mu}_t + S''e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{i}_{t-1} - (1+\beta)S''e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{i}_t + \\ & + \beta S''e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}E_t\{\hat{i}_{t+1}\} + (\beta\rho_z-1)S''e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{z}_t + \\ & + (\beta\rho_v-1)S''e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t\end{aligned} \tag{2.3.9}$$

$$\hat{k}_t = \hat{u}_t + \hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t \tag{2.3.10}$$

$$\begin{aligned}\hat{\bar{k}}_t = & (1-\delta e^{\sigma\gamma_v})e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{\bar{k}}_{t-1} - \delta e^{\sigma\gamma_v}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{\delta}_t + \\ & + \left[1-(1-\delta e^{\sigma\gamma_v})e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\right]\left(\hat{\mu}_t+\hat{i}_t\right) - (1-\delta e^{\sigma\gamma_v})e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{z}_t + \\ & - \left[(1-\delta e^{\sigma\gamma_v})\left(\frac{\alpha}{1-\alpha}+1\right)+\sigma\delta e^{\sigma\gamma_v}\right]e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{v}_t\end{aligned} \tag{2.3.11}$$

$$\hat{\delta}_t = \frac{\zeta}{\zeta + \bar{\delta}\left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right)^{\sigma}} \left[\sigma \hat{\bar{k}}_{t-1} - \sigma \hat{\tilde{m}}_t - \sigma \hat{d}_t - \sigma \hat{z}_t - \sigma \left(\frac{\alpha}{1-\alpha}+1\right) \hat{\upsilon}_t + \eta \hat{u}_t\right] \tag{2.3.12}$$

$$\hat{\tilde{m}}_t = \tau \left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1} \left[\hat{u}_t + \hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{\upsilon}_t\right] - \hat{d}_t \tag{2.3.13}$$

$$\begin{aligned}
&\left[(1+\beta)\, e^{2\left(\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v\right)} f'' + \bar{A}\right] \hat{\tilde{m}}_t = e^{2\left(\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v\right)} f'' \hat{\tilde{m}}_{t-1} + \beta e^{2\left(\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v\right)} f'' \hat{\tilde{m}}_{t+1} + \bar{A}\hat{\bar{k}}_{t-1} + \bar{A}\frac{\delta}{\delta - \bar{\delta}}\hat{\delta}_t + \\
&+ \left(1-\bar{A}\right)\hat{d}_t + \left(1-\bar{A}\right)\left(\hat{\varsigma}_t - \hat{\lambda}_t\right) + \bar{A}\left(\hat{\phi}_t - \hat{\lambda}_t\right) - \left[\left(1-\beta\rho_z\right) e^{2\left(\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v\right)} f'' + \bar{A}\right]\hat{z}_t + \\
&- \left[\frac{\alpha}{1-\alpha}\left(1-\beta\rho_v\right) e^{2\left(\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v\right)} f'' + \left(\frac{\alpha}{1-\alpha} + 1 - \sigma\right)\bar{A}\right]\hat{\upsilon}_t
\end{aligned} \tag{2.3.14}$$

$$\frac{1}{g}\hat{y}_t - \frac{1}{g}\hat{g}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{\tilde{m}}{y}\hat{\tilde{m}}_t + \left\{\rho - \bar{B} + \tau\left[\bar{A} - 1\right]\right\}\frac{\bar{k}}{y} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t \tag{2.3.15}$$

$$\hat{x}_t = \hat{y} - \left\{\rho - \bar{B} + \tau\left[\bar{A} - 1\right]\right\}\frac{\bar{k}}{y} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t \tag{2.3.16}$$

$$\hat{g}_{w,t} = \nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_t + \nu\hat{L}_t + \hat{b}_t - \hat{\lambda}_t + \hat{\lambda}_{w,t} \tag{2.3.17}$$

$$\hat{w}_t = \left(1-\xi_w\right)\hat{\tilde{w}}_t + \xi_w\left(\hat{w}_{t-1} - \hat{\pi}_t - \hat{z}_t - \frac{\alpha}{1-\alpha}\hat{\upsilon}_t + \iota_w\hat{\pi}_{t-1} + \iota_w\hat{z}_{t-1} + \iota_w\frac{\alpha}{1-\alpha}\hat{\upsilon}_{t-1}\right) \tag{2.3.18}$$

$$\hat{\tilde{w}}_t = \xi_w\beta\hat{\tilde{w}}_{t+1} - \xi_w\beta\hat{\pi}_t + \xi_w\beta\hat{\pi}_{t+1} + \xi_w\beta\left(\rho_z - \iota_w\right)\hat{z}_t + \xi_w\beta\left(\rho_v - \iota_w\right)\frac{\alpha}{1-\alpha}\hat{\upsilon}_t + \frac{1-\xi_w\beta}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\hat{g}_{w,t} \tag{2.3.19}$$

$$\hat{R}_t = \rho_R\hat{R}_{t-1} + \left(1-\rho_r\right)\left[\phi_\pi\hat{\pi}_t + \phi_X\left(\hat{x}_t - \hat{x}_t^*\right)\right] + \phi_{dX}\left[\left(\hat{x}_t - \hat{x}_{t-1}\right) - \left(\hat{x}_t^* - \hat{x}_{t-1}^*\right)\right] + \hat{\eta}_{mp,t} \tag{2.3.20}$$

where the constants have been defined as follows

$$\bar{A} = \sigma\left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta - \bar{\delta}\right) e^{\sigma\gamma_v} \geq 0$$

$$\bar{B} = \eta e^{\sigma\gamma_v}\left(\delta - \bar{\delta}\right) \geq 0$$

$$\bar{C} = e^{\sigma\gamma_v}\left[\delta + \sigma\left(\delta - \bar{\delta}\right)\right] > 0$$

$$\bar{D} = 1 - \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(1 - e^{\sigma\gamma_v}\delta\right) + \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\frac{\sigma}{\eta}\bar{B} - \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A} > 0$$

$\bar{A}$ equals the steady state expression of the depreciation rate first order condition with respect to the maintenance to effective capital ratio, that is $-\delta_{m/k}e^{\sigma\gamma_v}$, while $\bar{B}$ is the steady state first order condition of capital depreciation with respect to utilization rate, that is $\delta_u e^{\sigma\gamma_v}$. The constant $\bar{D}$ represents the steady state marginal product of capital net of the marginal propensity to maintain, i.e. $\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}(\rho-\tau)$. The term $\hat{q}_t = \hat{\phi}_t - \hat{\lambda}_t$ is the Tobin's q. Equation (2.3.1) describes the log-linearized production function for intermediate good producers. The capital to labor ratio given in equation (2.3.2) sets that the marginal returns on effective capital and labor must equal, respectively, the return on capital and the real wage. According to equation (2.3.3) the real marginal cost is given by the sum of the return on capital and real wage weighted by the share of capital and labor in the production function, respectively. Equation (2.3.4) is the New Keynesian Phillips curve for prices, which depends on past and expected inflation as well as on real marginal costs and the price mark-up shock. Equations (2.3.5) and (2.3.6) are the log-linearized households marginal utility from income and the Euler equation for consumption, respectively. Equation (2.3.7) defines the log-linearized optimal demand for capital utilization rate, which determines the convergence path for the rental price of capital. Note that, differently from Justiniano et al. (2011) where the convergence path of $\hat{\rho}_t$ is described by the capital utilization rate only, in our model capital rental price is positively related, among others, to the capital depreciation rate, $\hat{\delta}_t$. This is in line with Jorgenson and Griliches (1967) who suggest that assets that exhibit a faster depreciation should be rented at higher prices in order to recover the costs of depreciation. The current shadow value of maintenance, $\varsigma_t$, impacts negatively on $\hat{\rho}_t$ as far as the first term in squared brackets is greater than one, as it emerges from the steady state relations. Therefore, an increase in $\hat{\varsigma}_t$ tears down the level of maintenance rendering capital stock less attractive and thus less worthy. The effects of the shadow value of consumption, $\hat{\lambda}_t$ and of the capital utilization rate are ambiguous and depend crucially on the parameters for the elasticity of capital depreciation rate with respect to maintenance, $\sigma$, and utilization, $\eta$, and on the marginal propensity to maintain, $\tau$. Equation (2.3.8) describes the optimal convergence path of the demand for new physical capital. Accordingly, when the depreciation rate is expected to rise, the current shadow price of installed capital, $\hat{\phi}_t$, decreases. The same occurs when expected relative shadow value of maintenance, $\left(\hat{\varsigma}_{t+1} - \hat{\lambda}_{t+1}\right)$, increases. That is, agents anticipate the expected rise in the relative cost of maintenance by increasing maintenance expenditures today which, in turn, lowers the current shadow value of capital. The expected rate of utilization, $\hat{u}_{t+1}$, impacts positively on the current shadow value of capital as far as more capital stock is expected to be used in the production process. As to the expected shadow value of consumption, $\hat{\lambda}_{t+1}$, and to the expected rental price of capital, $\hat{\rho}_{t+1}$, both impact positively on $\hat{\phi}_t$, whereas the direct effects of the neutral technology and IST shocks are negative. A rise in Tobin's Q, $\left(\hat{\phi}_{t+1} - \hat{\lambda}_{t+1}\right)$, is expected to increase capital current shadow price however, the positive relationship is not so obvious. In fact, it depends on the deep parameters of the depreciation rate and of the maintenance expenditures functions, as well as on the steady states of the level of depreciation and its growth rate. It may occur, for example, that when the elasticity of the capital depreciation rate with respect to maintenance is very high (or $\sigma \to \infty$), than a small change in the amount of maintenance expenditures produces large shifts in the rate of depreciation. In such an environment the price of the maintenance goods and services will tend to be law, which implies relatively high price of investments and therefore discourages investment in new capital goods thus, finally, lowering the current value of capital. Equations (2.3.9) and (2.3.10) are log-linearizations of the optimal demand for new investment and of the definition of effective capital, respectively. Both remain unchanged with respect to the baseline model. Equation (2.3.11) is the log-linearized law of motion of capital. Differently from Justiniano et al. (2011), we observe here that the current depreciation rate of capital enters directly the equation for the convergence path of capital stock and impacts negatively on it. Equation (2.3.12) describes the optimal convergence path of the capital depreciation rate, which increases in the current utilization rate and past physical capital, while decreases in current maintenance and in the maintenance-specific technological progress. Note that, a positive MST shock increases the amount of maintenance expressed in efficiency units, whereas decreases the amount of real maintenance in consumption units. In fact, the latter one is defined as the product between efficiency units of maintenance and the relative price of maintenance in terms of consumption, which, in turn, is negatively related with the maintenance-specific technology progress. Thus, a positive MST shock decreases the relative price of maintenance, that consequently makes real maintenance to decline. Equation (2.3.13) is the log-linearized around the steady state expression for the definition of maintenance expenses. Accordingly, a higher intensity of utilization and a higher amount of 'old' capital both tend to increase real maintenance, on the contrary, the MST shock decreases it. In fact, as stated above, by construction this shock impacts positively on maintenance expressed in efficiency units, i.e. $\hat{m}_t$, so, given our definition for the maintenance technology production, it holds that, $\hat{m}_t = \hat{\tilde{m}}_t + \hat{d}_t$. Equation (2.3.14) defines the log-linearized optimal demand for maintenance goods, which is increasing both in the past and expected maintenance. MST shock and the shadow value of maintenance affect it negatively, whereas an increase in depreciation requires a higher maintenance demand. Equations (2.3.15) and (2.3.16) are the log-linearizations of the aggregate resource constraint and of the actual GDP, respectively. Differently form the baseline model, both depend, among others, on the optimal path of real maintenance and the impact of the optimal utilization rate of capital depends, in addition, on the sensibility parameters of the depreciation rate function, the marginal propensity to maintain, and the steady states of the level of depreciation and its growth rate. Equations (2.3.17)-(2.3.19), if conveniently combined, deliver an expression for the wage Phillips curve. The term $\hat{g}_{w,t}$ defines the marginal utility of labor, while $\hat{\tilde{w}}_t$ is an auxiliary variable used to define the behavior of real wages. Finally, equation (2.3.20) describes the log-linearized optimal monetary policy rule. The latter four expressions do not present changes with respect to the baseline model.

## 2.4 Concluding remarks

We have built a Dynamic Stochastic General Equilibrium model following Justiniano et al. (2011). We take advantage from the decentralized scheme of their model in order to introduce a new sector which produces maintenance goods and services employing a fraction of final goods. Maintenance goods are purchased by households, who detain capital stock, and use them for the reparations of damaged or obsolete capital due to wear and tear (usage), aging and quality innovations. The level of maintenance expenditures is controlled by households. Capital stock is rented to the firms producing intermediate goods. Differently from Justiniano et al. (2011), our model is characterized by a time varying depreciation rate, which is influenced by the rate of capital utilization and by maintenance expenditures. The optimization routine rules out a growth trend for depreciation which depends on the investment-specific technological progress weighted by the sensibility of depreciation to changes in maintenance. The existence in our model of a growing path of depreciation is supported by the U.S. and Canadian actual data, especially when a period over the latest 50 years is considered, and by empirical research carried out, for example, by Tevlin and Whelan (2003). Nonetheless, our work is the first one being characterized by a growing trend in the depreciation rate. The only estimation analysis in a DSGE framework with endogenous depreciation is made in Albonico et al. (2014) however they abstract from many salient features of the DSGE models and do not incorporate any growth trend in their model. According to our settings, instead, the main macroeconomic variables grow following a composite growth trend given by a linear combination of the labor-augmenting progress and of the investment-specific technical process, whereas depreciation is driven by the latter one only. As a consequence of our model assumptions, several optimal paths of the respective endogenous variables are clearly directly linked to the model's new variables and parameters characterizing the functions of the depreciation rate and of the maintenance expenditures. It emerges that, the optimal rental price of capital, for example, is positively influenced by depreciation, which suggests that when depreciation accelerates the respective capital asset is more likely to be discarded sooner. The agents, therefore, are prompted to increase the rental price of capital in order to recover the costs of increasing depreciation. Conversely, the shadow value of maintenance influences negatively capital rental price as far as, when the former increases maintenance decreases, which implies that less capital is repaired and that it worths less. The optimal shadow value of capital, as it should be obvious, depends negatively on the expected depreciation rate and positively on the shadow value of maintenance. The optimal demand for maintenance is increasing in the old capital stock and current depreciation, that is as well intuitive. Depreciation rate accelerates when capital is used more intensively and declines when more maintenance is implemented. As a consequence of our model settings some steady state key relationships differ from the relative ones in the baseline model as well. We furthermore have assumed the presence of one more shock in our model with respect to the baseline model which impacts on the production technology of new maintenance units in the maintenance production sector and which we have labeled the maintenance-specific technological progress.

# Chapter 3

## 3 Estimations and results

### Introduction

In this chapter we expose our estimation analysis and results based on the Bayesian approach. For these purposes we use the database of the Canadian economy (CANSIM) which has available series for maintenance expenditures in the "Capital and repair expenditures" survey. The data used in the estimations are expressed in quarters and cover the period over 1981Q2-2015Q1. Performing the Random Walk Metropolis Algorithm we carry out estimations for the parameters values both of our model and of the baseline model. In order to analyze the differences between the two models and to verify the contribution of the novelties introduced in our model we compute the impulse response functions of the main macroeconomic variables.

The chapter is structured as follows. In section 3.1 we describe the construction of the dataset used for the estimation exercises and the distribution forms of the priors of parameters. In section 3.2 we briefly describe the posterior results of the maintenance and the baseline models focusing on the estimated modes of the respective parameters. In section 3.3 convergence diagnostics analysis of the model and identification analysis of the parameters for the maintenance model are presented in order to detect the goodness of the estimation exercises. In section 3.4 we compare the impulse response functions of the maintenance model relative to the baseline model and analyze the optimal convergence dynamics. In section 3.5 we present the variance decomposition analysis in order to investigate the roles of exogenous shocks in explaining the real business cycle fluctuations. In section 3.6 we describe the behavior of the two models by comparing some estimated moments with the statistics measured in the actual data.

### 3.1 Data and Priors

*Data*

The estimation exercise of our research consists in the Bayesian inference of our model and of the baseline model following the main approach adopted in Justiniano et al. (2011), in order to be able to make comparisons of the posteriors results of the two models. We consider eight observable variables[40]

$$\left[\Delta \log X_t \quad \Delta \log C_t \quad \Delta \log \tilde{I}_t \quad \log L_t \quad \Delta \log \frac{W_t}{P_t} \quad \pi_t \quad R_t \quad \Delta \log \frac{P_t^I}{P_t}\right]$$

[40] For a detailed description of the construction of our dataset please refer to Appendix F.

where $\Delta \log X_t$, $\Delta \log C_t$ and $\Delta \log \tilde{I}_t$ are differences in logarithms of real GDP, real consumption and real investment, respectively. The latter two are defined as ratios between the corresponding nominal series and the series of the implicit price indexes for consumption of non durables, semi-durables and services and are all expressed in per-capita terms. As usual, nominal consumption defines the expenditures on non durables, semi-durables and services, while nominal investment defines the expenditures on durables and gross private domestic investment. Per-capita real GDP is expressed in chained 2007 dollars. The observable $\log L_t$ is the logarithm of per-capita worked hours, while $\Delta \log \frac{W_t}{P_t}$ is the difference in logarithms of per-capita real wages, both in the non-farm business sector. Inflation, $\pi_t$, is the quarterly difference in logarithms of consumption deflator and the nominal interest rate, $R_t$, is the three-months treasury bill rate. Finally, the difference in logarithms of the relative price of investment, $\Delta \log \frac{P_t^I}{P_t}$, is given by the ratio of the deflators for investment and consumption. The series for the price for investment is the average of the implicit price index on durables and gross private domestic investment. All the series are taken from the CANSIM Statistics of Canada. The dataset covers a period going from 1981Q2 to 2015QI and is expressed on quarterly basis. Note that, differently from Justiniano et al. (2011) we do not consider any structural break in our model for two reasons. First of all, our dataset for the observables is starting in 1981 in order to have coherent data for all the observable series, as far as several methodological changes have occurred in the CANSIM database both with regard to computational approaches and to data gathering. Justiniano et al. (2011), instead, cover a period over 1954QIII-2009QI and the structural break for the U.S. is set in 1982, which is the year when the path of the relative price of investment changes its slope. Therefore, given the affinity of the two economies and as it emerges by observing the actual data, it is plausible to suppose that our dataset starts after the relevant changes in the path of the relative prices of investment have been occurred.

*Priors*

We set all the priors of the parameters common for the two models following Justiniano et al. (2011), with exception of the steady state of hours worked. All the figures are summarized in Table 3.1. All the parameters characterizing the persistences of the shock processes, including the parameters of the moving averages in the ARMA processes, are described by a Beta distribution. The standard errors of the innovations are described by an Inverse-gamma distribution. Prior mean of the capital share is set to 0.30 that is a value broadly used in this literature. We set the prior means of the steady state composite growth rate, $\gamma^\star$, and the steady state IST growth rate, $\gamma^v$, respectively to 0.30 and 0.60, which correspond to the respective values after the structural break period in Justiniano et al. (2011). Both the priors follow the Normal distribution. The prior means of the price and wage stickiness, $\xi_p$ and $\xi_w$, are both set such that only one third of the intermediate firms and of households can set their optimal price and optimal wage, respectively. We set the prior mean of the steady state of hours worked, $\log L^{ss}$, to 0.30 contrary to 0.00 of Justiniano et al. (2011), while keep unchanged the value of the prior standard deviation and the prior distribution form. Our choice is driven by the fact that the value for the hours worked in the actual data of our dataset is on average positive. On the contrary, In Justiniano et al. (2011) the respective average value is negative over the first sub-sample, i.e. before the structural break, and positive thereafter. The prior mean of 2.00 for the inverse Frisch elasticity is relatively high however broadly into the ranges found in the literature. We have five more parameters estimated in the maintenance model for which we assume very dispersive priors. The sensitivity parameters of the depreciation rate function, $\eta$ and $\sigma$, and the elasticity of the maintenance adjustment costs, $f''$, are all assumed to follow a Gamma distribution. The prior means of the sensitivity with respect to utilization, $\eta$, and with respect to maintenance, $\sigma$, are respectively 9.00 and 10.00 and the respective prior standard deviations are set to 7.00 and 10.00. The prior support of the elasticity of maintenance adjustment costs, $f''$, is slightly broader with respect to the one of the investment adjustment costs, $S''$, with prior mean equal to 3.00 and prior standard deviation equal to 2.00, against 4.00 and 1.00, respectively. In fact, given our prior mean assumptions on $f''$ and $S''$, we believe that a marginal change in the respective relative inputs induces more variation in the maintenance adjustment cost than in investment adjustment cost. The smooth parameter of the maintenance-specific technological progress, $\rho_d$, follows a Beta distribution with prior mean 0.6 and prior standard deviation 0.2. The standard deviation of the innovation of maintenance-specific technological progress is described by an Inverse-gamma distribution with prior mean and prior standard deviation of 0.1 and 1, respectively. The priors set for the remaining parameters are in general in line with the related literature. The parameters of the marginal propensity to maintain, $\tau$, and of the depreciation rate function, $\zeta$, are internally determined by the steady state relations. Moreover, as far as we are interested in the comparison of the baseline model with our model which includes endogenous depreciation rate and maintenance activity, we set a restriction condition on the natural rate of depreciation, that is $\bar{\delta} = 0.025 - \zeta$. Thus, when all the new parameters of the maintenance model tend to zero $\zeta$ tends to zero, too, and the depreciation rate becomes constant and equal to 0.025 as it is in the baseline model. Anyway, given the flexible structure of our model, such a restriction condition can be relaxed and both the values of the steady state depreciation rate and of the natural rate of depreciation can be determined entirely by the steady state driven relations and by the estimated parameters. Otherwise, the arbitrary value of 0.025 may be set following, for example, the planned obsolescence literature. For example, one could be interested in to assume that the capital stock at most approaches 30 years of useful life, which would imply an average quarterly natural rate of depreciation, $\bar{\delta}$, of about 0.0083. In our case, since we have assumed that $\bar{\delta}$ at most equals 0.025 implies that the average useful life of capital is around 10 years when it depreciates according to the natural rate of depreciation. Finally, we calibrate the value of the steady state government spending to GDP ratio, i.e. $(1 - 1/g)$, in order to match the average value of $G_t/Y_t$ in the actual data and set it to 0.25. As to the calibration of the ratio of fixed maintenance costs to installed capital we assume an arbitrary value of 0.01. This means that households spend in fixed maintenance on average 1% of the value of current capital stock.

In Justiniano et al. (2011), the intertemporal preference shock and the price and wage mark-up shocks are normalized so that they enter the equations of marginal utility of nominal income and the price and wage Phillips curves, respectively, with a unit coefficient. These normalizations, as the authors explain, are convenient for the definition of the priors for the standard deviations of the shocks and for the estimation purposes when the Metropolis-Hasting algorithm is implemented. Normalization requires the definition of new exogenous variables, which are denoted by a 'star', for the three shocks as follows, respectively

$$\hat{b}_t^\star = \frac{(1-\rho_b)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v} - h\beta\rho_b\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v} - h\right)}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}h + e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)} + \beta h^2}\hat{b}_t$$

$$\hat{\lambda}_{p,t}^\star = \frac{(1-\xi_p\beta)(1-\xi_p)}{\xi_p(1+\beta\iota_p)}\hat{\lambda}_{p,t}$$

$$\hat{\lambda}_{w,t}^\star = \frac{(1-\beta\xi_w)(1-\xi_w)}{\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)(1+\beta)\xi_w}\hat{\lambda}_{w,t}$$

As far as marginal utility of nominal income, and the price and wage Phillips curves remain unchanged with respect to the baseline model, we implement the normalization procedure of the respective shocks following Justiniano et al. (2011).

## 3.2 Posterior estimation results

We perform estimation exercises both for the baseline model and for the maintenance model using the free software DYNARE. The Random Walk Metropolis-Hastings algorithm within the Bayesian estimation framework is used. The posterior estimates are obtained from 2 chains of 80,000 draws and following Justiniano et al. (2011) we discard the initial 50,000 draws and keep one every five subsequent draws imposing the scale for the jumping distribution to 0.31 for both the models. The estimated modes, respective estimated standard deviations and five degree acceptance ranges of the structural parameters and standard deviations of the innovations are listed in Table 3.2. As we can observe, overall posterior estimates confirm our model assumptions. In particular, we have assumed that the elasticity of the depreciation rate function with respect to utilization, $\eta$, is greater than unity. In fact, the estimated mode of $\eta$ is 1.327. Additionally we assumed that this elasticity must be lower than the elasticity of depreciation with respect to maintenance, $\sigma$, which is as well satisfied, being the estimated mode of $\sigma$ equal to 5.116. All the other estimates are, in general, within the ranges of those found in the related literature for Canada. The estimated habit degree in consumption is 0.851 and 0.860 in the maintenance and baseline models, respectively, which is lower than 0.94 estimated by, for example, Dorich et al. (2013)[41]. The estimated intertemporal elasticity of substitution, $\beta$, is 0.99 in both the models whereas it is 0.88 in Dorich et al. (2013). This suggests that, with respect to the model of Dorich et al. (2013), in our economy consumption is more sensitive and thus responds faster to movements in the interest rate. With respect to the baseline model, the composite steady state growth trend, $\gamma^\star$, in the maintenance model is slightly higher, 0.318 against 0.310. Inversely, in the maintenance model the steady state growth rate of IST progress, $\gamma^v$, is slightly lower, being 0.573 against the 0.578 of the baseline model. Moreover, the estimated posterior mode of capital share in the intermediate goods production function is higher in the maintenance model than in the baseline model, being 0.183 and 0.141, respectively. Recall that the composite growth trend in steady state is given by $\gamma^\star = \gamma^z + \frac{\alpha}{1-\alpha}\gamma^v$, hence, given the posterior mode for $\alpha$, the contribution to the composite growth rate of the IST growth is almost double in the maintenance model with respect to the baseline

[41] Dorich et al. (2013) estimate a Terms-of-Trade Economic Model for Canada with the full information estimation technique. The model is implemented by the Bank of Canada for its quarterly projection forecasts.

Tab. 3.1: Prior distributions of structural parameters and standard deviations of the shocks

| Parameter | Description | Prior shape | Prior mean | Prior Std |
|---|---|---|---|---|
| $\alpha$ | Capital share | N | 0.30 | 0.05 |
| $\iota_p$ | Price indexation | B | 0.50 | 0.15 |
| $\iota_w$ | Wage indexation | B | 0.50 | 0.15 |
| $\gamma^\star$ | SS composite technology | N | 0.30 | 0.025 |
| $\gamma^v$ | SS IST growth rate | N | 0.60 | 0.025 |
| $h$ | Consumption habit | B | 0.50 | 0.10 |
| $\lambda_p$ | SS prices mark-up | N | 0.15 | 0.05 |
| $\lambda_w$ | SS wages mark-up | N | 0.15 | 0.05 |
| $\log L^{ss}$ | SS hours | N | 0.30 | 0.50 |
| $100(\pi-1)$ | SS quarterly inflation | N | 0.50 | 0.10 |
| $100(\beta^{-1}-1)$ | Discount factor | G | 0.25 | 0.10 |
| $\nu$ | Inverse Frisch elasticity | G | 2.00 | 0.75 |
| $\xi_p$ | Price stickiness | B | 0.66 | 0.10 |
| $\xi_w$ | Wage stickiness | B | 0.66 | 0.10 |
| $\chi$ | Utilization cost elasticity | G | 5.00 | 1.00 |
| $S''$ | Investment adjustment costs | G | 4.00 | 1.00 |
| $\phi_p$ | Inflation Taylor rule | N | 1.70 | 0.30 |
| $\phi_y$ | Output Taylor rule | N | 0.13 | 0.05 |
| $\phi_{dy}$ | Output growth Taylor rule | N | 0.125 | 0.05 |
| $\eta$ | Elasticity of depreciation wrt utilization | G | 9.00 | 7.00 |
| $\sigma$ | Elasticity of depreciation wrt maintenance | G | 10.00 | 10.00 |
| $f''$ | Maintenance adjustment costs | G | 3.00 | 2.00 |
| $\rho_d$ | Persistence maintenance cost | B | 0.60 | 0.20 |
| $\rho_R$ | Persistence Taylor rule | B | 0.60 | 0.20 |
| $\rho_z$ | Persistence Neutral technology | B | 0.40 | 0.20 |
| $\rho_g$ | Persistence government spending | B | 0.60 | 0.20 |
| $\rho_v$ | Persistence IST | B | 0.20 | 0.10 |
| $\rho_p$ | Persistence price mark-up | B | 0.60 | 0.20 |
| $\rho_w$ | Persistence wage mark-up | B | 0.60 | 0.20 |
| $\rho_b$ | Persistence intertemporal preference | B | 0.60 | 0.20 |
| $\rho_\mu$ | Persistence MEI | B | 0.60 | 0.20 |
| $\theta_p$ | MA price mark-up | B | 0.50 | 0.20 |
| $\theta_w$ | MA wage mark-up | B | 0.50 | 0.20 |
| $\sigma_{mp}$ | Monetary policy Std | I | 0.10 | 1.00 |
| $\sigma_z$ | Neutral technology growth Std | I | 0.50 | 1.00 |
| $\sigma_g$ | Government spending Std | I | 0.50 | 1.00 |
| $\sigma_v$ | IST growth Std | I | 0.50 | 1.00 |
| $\sigma_\mu$ | MEI growth Std | I | 0.50 | 1.00 |
| $\sigma_p$ | Price mark-up Std | I | 0.10 | 1.00 |
| $\sigma_w$ | Wage mark-up Std | I | 0.10 | 1.00 |
| $\sigma_b$ | Intertemporal preference Std | I | 0.10 | 1.00 |
| $\sigma_d$ | Maintenance specific growth Std | I | 0.10 | 1.00 |

Tab. 3.2: Posterior estimates of the baseline and maintenance model

| | Baseline model | | | Maintenance model | | |
|---|---|---|---|---|---|---|
| Parameter | Mode | s.d. | [5, 95] | Mode | s.d. | [5, 95] |
| $\alpha$ | 0.141 | 0.018 | [0.11, 0.17] | 0.183 | 0.020 | [0.14, 0.23] |
| $\iota_p$ | 0.267 | 0.078 | [0.10, 0.40] | 0.280 | 0.096 | [0.12, 0.47] |
| $\iota_w$ | 0.087 | 0.035 | [0.04, 0.17] | 0.100 | 0.041 | [0.04, 0.19] |
| $\gamma^\star$ | 0.310 | 0.024 | [0.26, 0.35] | 0.318 | 0.024 | [0.27, 0.35] |
| $\gamma^v$ | 0.578 | 0.025 | [0.53, 0.63] | 0.573 | 0.025 | [0.53, 0.62] |
| $h$ | 0.860 | 0.024 | [0.80, 0.90] | 0.851 | 0.025 | [0.78, 0.90] |
| $\lambda_p$ | 0.224 | 0.043 | [0.13, 0.30] | 0.227 | 0.043 | [0.13, 0.31] |
| $\lambda_w$ | 0.147 | 0.049 | [0.05, 0.23] | 0.154 | 0.045 | [0.06, 0.24] |
| $\log L^{ss}$ | 0.163 | 0.429 | [-0.63, 0.99] | 0.415 | 0.471 | [-0.51, 1.22] |
| $100(\pi - 1)$ | 0.544 | 0.086 | [0.39, 0.71] | 0.529 | 0.085 | [0.36, 0.69] |
| $100(\beta^{-1} - 1)$ | 0.413 | 0.096 | [0.24, 0.62] | 0.454 | 0.111 | [0.26, 0.69] |
| $\nu$ | 3.884 | 0.843 | [2.50, 5.79] | 3.144 | 0.922 | [1.85, 5.16] |
| $\xi_p$ | 0.910 | 0.021 | [0.84, 0.94] | 0.823 | 0.058 | [0.74, 0.90] |
| $\xi_w$ | 0.693 | 0.073 | [0.55, 0.84] | 0.652 | 0.073 | [0.52, 0.84] |
| $\chi$ | 4.819 | 0.976 | [3.20, 7.07] | 5.007 | 0.988 | [3.32, 7.46] |
| $S''$ | 4.756 | 0.902 | [3.40, 7.15] | 4.136 | 0.799 | [3.03, 6.30] |
| $\phi_p$ | 1.689 | 0.239 | [1.28, 2.16] | 2.026 | 0.201 | [1.55, 2.42] |
| $\phi_y$ | 0.176 | 0.042 | [0.09, 0.26] | 0.112 | 0.061 | [0.04, 0.24] |
| $\phi_{dy}$ | 0.134 | 0.038 | [0.06, 0.20] | 0.097 | 0.040 | [0.03, 0.19] |
| $\eta$ | | | | 1.327 | 1.534 | [0.01, 7.69] |
| $\sigma$ | | | | 5.116 | 1.985 | [3.19, 13.44] |
| $f''$ | | | | 1.592 | 1.426 | [0.12, 5.61] |
| $\rho_d$ | | | | 0.668 | 0.272 | [0.24, 0.97] |
| $\rho_R$ | 0.869 | 0.018 | [0.83, 0.90] | 0.853 | 0.023 | [0.82, 0.89] |
| $\rho_z$ | 0.061 | 0.049 | [0.005, 0.17] | 0.056 | 0.045 | [0.01, 0.15] |
| $\rho_g$ | 0.982 | 0.005 | [0.97, 0.99] | 0.986 | 0.005 | [0.98, 0.99] |
| $\rho_v$ | 0.413 | 0.070 | [0.28, 0.54] | 0.435 | 0.071 | [0.28, 0.55] |
| $\rho_p$ | 0.950 | 0.040 | [0.65, 0.99] | 0.959 | 0.030 | [0.86, 0.99] |
| $\rho_w$ | 0.915 | 0.053 | [0.70, 0.97] | 0.837 | 0.064 | [0.59, 0.92] |
| $\rho_b$ | 0.183 | 0.087 | [0.05, 0.39] | 0.151 | 0.079 | [0.04, 0.37] |
| $\rho_\mu$ | 0.699 | 0.086 | [0.54, 0.85] | 0.979 | 0.013 | [0.93, 0.99] |
| $\theta_p$ | 0.930 | 0.051 | [0.46, 0.97] | 0.839 | 0.100 | [0.62, 0.94] |
| $\theta_w$ | 0.864 | 0.079 | [0.55, 0.95] | 0.733 | 0.105 | [0.32, 0.86] |
| $\sigma_{mp}$ | 0.250 | 0.017 | [0.22, 0.29] | 0.242 | 0.016 | [0.22, 0.28] |
| $\sigma_z$ | 0.860 | 0.058 | [0.75, 0.99] | 0.905 | 0.065 | [0.79, 1.04] |
| $\sigma_g$ | 0.661 | 0.041 | [0.60, 0.76] | 0.627 | 0.039 | [0.56, 0.72] |
| $\sigma_v$ | 0.727 | 0.044 | [0.65, 0.84] | 0.723 | 0.044 | [0.65, 0.82] |
| $\sigma_\mu$ | 4.762 | 1.058 | [3.28, 7.79] | 7.873 | 1.144 | [6.22, 11.28] |
| $\sigma_p$ | 0.346 | 0.028 | [0.27, 0.39] | 0.319 | 0.031 | [0.25, 0.38] |
| $\sigma_w$ | 0.265 | 0.028 | [0.21, 0.31] | 0.275 | 0.031 | [0.21, 0.33] |
| $\sigma_b$ | 0.139 | 0.024 | [0.09, 0.18] | 0.142 | 0.022 | [0.09, 0.18] |
| $\sigma_d$ | | | | 0.247 | 0.098 | [0.13, 0.79] |
| *log Posterior* | -1141.65 | | | -1139.74 | | |

model. Hours worked in steady state, $\log L^{ss}$, are 0.415 in the maintenance model and 0.163 in the baseline model. Thus, when accounting for the maintenance activity and endogenous depreciation rate higher amounts of factor inputs are, in general, used in the production of intermediate-goods. The labor supply elasticities, $1/\nu$, in the maintenance and baseline models are, respectively, 0.318 and 0.257. The estimations of Albonico et al. (2014), instead, are found to be higher amounting to 0.49 and 0.47, respectively, in the models with and without maintenance activity. The estimated elasticity of the utilization cost, $\chi$, in the baseline model is smaller, being 4.819 against the 5.007 in the maintenance model. Therefore, with respect to the baseline one in the maintenance model adjustment mechanism to shocks at margin has more room of play in the sense of utilization costs and labor supply. However, given the estimated value for the elasticity of the depreciation rate function with respect to utilization, $\eta$, that is 1.327 (1.08 in Albonico et al. (2014)), this implies that recovery through adjustments in utilization is counterproductive in terms of higher depreciation. On the contrary, an increase in the utilization rate of capital raises maintenance expenditures which, in turn, downturns depreciation rate. The estimated mode for the elasticity of depreciation rate with respect to maintenance, $\sigma$, is relatively high amounting to 5.116 (whereas Albonico et al. (2014) have obtained an estimate of 19.19) suggesting that a rise in maintenance expenditures of one unit induces a marginal decrease in depreciation of about 5%. When maintenance sector is included in the model, estimated investment adjustment costs decline.[42] In fact, the estimated parameter for the adjustment cost of investment, $S''$, is higher in the baseline model amounting to 4.756 than the 4.136 in the maintenance model. This suggests that, in the maintenance model, the direct impact of the MEI shock in the investment Euler equation is stronger than in the baseline one. In the maintenance model the estimated adjustment cost of maintenance, $f''$, is relatively lower (1.592) than the adjustment cost of investment, therefore it results to be more costly to adjust maintenance to exogenous shocks than investment (that is $1/f''$ with respect to $1/S''$). Similarly to the model of Dorich et al. (2013) our Phillips curve for consumption prices is more forward looking than backward looking with the respective coefficients for $\hat{\pi}_{t-1}$ and $\hat{\pi}_{t+1}$ of 0.22 and 0.78 in the maintenance model, given the estimated values of $\iota_p$ and $\beta$. The contributions to the price Phillips curve of real marginal cost and price mark-up disturbance, given high estimated value of price stickiness, $\xi_p$, are very low (0.03), which is as well in line with Dorich et al. (2013). The estimated value for the wage stickiness, $\xi_w$, is slightly lower in the maintenance model with respect to baseline model being, respectively, 0.652 and 0.693 (0.59 in Dorich et al. (2013)). The difference is more pronounced for the parameter of price stickiness, $\xi_p$, being respectively 0.823 and 0.910 (0.75 in Dorich et al. (2013)). This suggests that, when depreciation is endogenous and the amount of maintenance can be optimally chosen a higher fraction of intermediate-goods producing firms and of households is able to optimally set the price level and wage level, respectively. On the contrary, both the estimates of the price and wage indexations, $\iota_p$ and $\iota_w$, respectively, are higher in the maintenance model with respect to the baseline ones, which are, respectively, 0.280 against 0.267 for $\iota_p$, and 0.100 against 0.087 for $\iota_w$. Lower value for $\iota_p$ has been estimated in Dorich et al. (2013) which amounts to 0.06, while $\iota_w$ is 0.11. Therefore, in particular with regard to the maintenance model, firms that do not optimize for price and follow, instead, a general indexation rule are prompted to pay more attention to the dynamics of past price inflation relatively to the model of Dorich et al. (2013). In fact, as far as depreciation rate is time-varying, nominal interest rate, and hence inflation, will be influenced by its path impacting therefore on the dynamics of the prices. The estimated smoothing parameter of the interest rate rule

[42] Note that, a similar result has been achieved in Angelopoulou and Kalyvitis (2012).

is slightly lower in the maintenance model than in the baseline one, 0.853 and 0.869, respectively, and both are lower than the 0.9 estimated by Dorich et al. (2013). In the maintenance model the estimated mode of the Taylor rule response to deviations of inflation, $\phi_p$, is definitely higher, 2.026 against the 1.689 of the baseline model, and is broadly in line with the estimated 2.14 of Dorich et al. (2013). Responses to deviations of output gap level, $\phi_y$, and its growth rate, $\phi_{dy}$ are relatively lower, 0.112 and 0.097 with respect to 0.176 and 0.134 in the baseline model. Maintenance model estimates for Taylor rule response to output gap level is as well broadly in line with the estimated value of 0.076 in Dorich et al. (2013). The estimated smoothing parameters of the labor augmenting technological process, $\rho_z$, of the wage mark-up shock, $\rho_w$, of the intertemporal preference shock, $\rho_b$, and of both the moving average parameters, $\theta_p$ and $\theta_w$, are lower in the maintenance model with respect to the baseline one. The estimated standard deviation of the marginal efficiency of investment technological progress, $\sigma_\mu$, is much larger in the maintenance model than in the baseline model, 7.873 and 4.762, respectively. The estimated standard deviation of the maintenance specific technological progress, $\sigma_d$, is 0.247.

## 3.3 Maintenance model behavior

In order to evaluate the behavior of the maintenance model we have performed the multivariate convergence diagnostics of the Monte Carlo Markov Chains shown in Fig. 3.1. When more Metropolis chains are run in parallel, the univariate diagnostics analysis performed by DYNARE is based on Brooks and Gelman (1998). In general, they introduce an alternative graphical approach for detecting the existence of convergence both in the univariate and multivariate environments. Assuming there are simulated in parallel $m \geq 1$ chains, each one of length $2n$, of which the first $n$ are discarded, vector $(\phi_{j1}, \phi_{j2}, ..., \phi_{jn})$, with $j = 1, ..., m$, represents the observation outputs of the simulation for a given parameter. Each simulation starts at a different point, which is found using a simple mode-finding algorithm, i.e. high density regions are detected first from which are then picked up the initial values using a mixture of $t$-distributions located at the found modes. Their originality consists in dividing the $m$ chains into $k$ sets of length $b$ where the value of the latter one must be equaled to $n/20$, as the authors suggest. So, denoting in a multivariate setting any $i$th simulated scalar summary $\phi^i_{jt}$ of the parameter vector $\phi$ as the $t$th of the $n$ iterations in chain $j$, and assuming $\bar{\phi}_{..}$ equals some unbiased moment estimator, then the between sequence variance $B/n$ and the within-sequence variance $W$ are calculated as follows

$$B/n = \frac{1}{m-1} \sum_{j=1}^{m} \left(\bar{\phi}_{j.} - \bar{\phi}_{..}\right) \left(\bar{\phi}_{j.} - \bar{\phi}_{..}\right)'$$

$$W = \frac{1}{m(n-1)} \sum_{j=1}^{m} \sum_{t=1}^{n} \left(\phi_{jt} - \bar{\phi}_{j.}\right) \left(\phi_{jt} - \bar{\phi}_{j.}\right)'$$

The estimated posterior variance-covariance matrix is given by

$$\hat{V} = \frac{n-1}{n} W + \left(1 + \frac{1}{m}\right) B/n$$

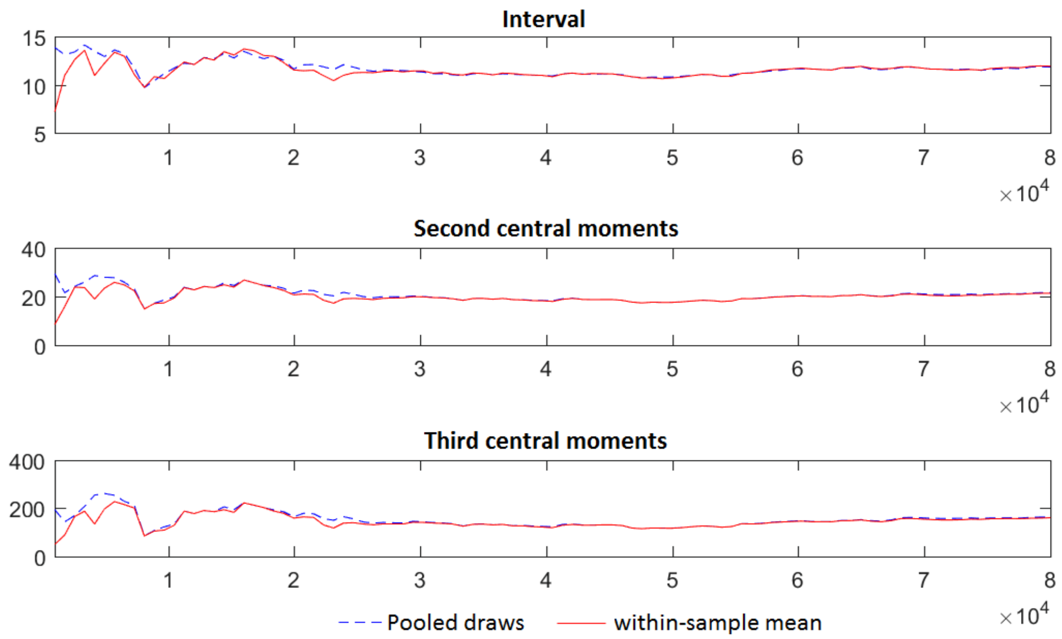
The multivariate potential scale reduction factor (MPSRF), which measures the convergence diagnostic, is then calculated as follows

$$\hat{R} = \max_a \frac{a'\hat{V}a}{a'Wa} = \frac{n-1}{n} + \left(\frac{m+1}{m}\right)\lambda$$

where $\lambda$ represents the largest eigenvalue of the symmetric and positive definite matrix $W^{-1}B/n$. In general, according to Brooks and Gelman (1998), convergence occurs when $\hat{R}$ approaches 1, which implies that $\hat{V}$ and $W$ joint each other. When $\hat{R} > 1$ convergence may be improved by, for example, increasing the number of simulations in order to either reduce $\hat{V}$ or increase $W$. Additionally, convergence is accomplished when both $\hat{V}$ and $W$ stabilize as the number of iterations $n$ increases, i.e. $\hat{V}$ decreases with $n$ and $W$ becomes, in general, less than $\hat{V}$.

In Fig. 3.1 we display the multivariate convergence diagnostics. In performing this analysis DYNARE slightly departures from the Brooks and Gelman (1998) approach in that it is computed using a range of log-posterior functions rather than individual parameters, which are then aggregated

Fig. 3.1: Multivariate convergence diagnostics: maintenance model




using posterior kernel into a scalar statistic. Hence, the dashed lines represent the pooled draws from all the Monte Carlo Chains, in our case two sequences. The continuous lines represent the mean interval range of the posterior likelihood function from the draws of individual chains. The first panel shows the convergence diagnostic for the length interval with the highest probability density over 80% of the quantile range of the posterior distribution. The second panel displays the convergence diagnostic for the squared deviations in absolute values from the respective means (pooled and within-sample), while the latter panel depicts convergence diagnostic for the respective cubed absolute deviations. In general, we assert that the estimation performance of our model is good as far as the multivariate convergence criteria are satisfied[43]. In fact, both the pooled and within-sample mean interval ranges stabilize over the window interval and get close to each other after about 30,000 iterations.

For further analysis we have plotted the distributions of the parameters priors and posteriors of the maintenance model. Fig. 3.2 displays these distributions, where the dotted lines represent the prior densities, the continuous lines represent the posterior density distributions and the vertical dashed lines are the estimated posterior modes. As it can be observed posterior densities fit the estimated modes for all the estimated parameters. The priors we have set are almost all informative, however, for some parameters such as the steady state composite growth rate, the maintenance-specific technology process and its persistence, the steady state mark-up for wages and the steady state of hours, the steady state quarterly inflation, output gap and output Taylor rules, the wage stickiness parameter as well as utilization cost elasticity, and finally the maintenance adjustment cost the priors and posteriors are completely or nearly the same. The overlapping of the two density distributions is used to be attributed to either a very accurate prior definition such that it precisely reflects the information provided by the actual data or, conversely, to a weak identification of these parameters of which priors can not be updated using the data information. In general, the Dynamic Stochastic General Equilibrium models suffer from a weak identification problem of some parameters. In order to deeper investigate the roles of our priors we have computed the identification and sensitivity analysis for the estimated parameters which are shown in Fig. 3.3. For this purposes DYNARE implements the procedure highlighted in Ratto and Iskrev (2011) and the related literature. Specifically, the strength of identification, $s_i$ for any parameter $\theta_i$ is carried out using the Fischer information matrix, $\mathcal{I}_T(\boldsymbol{\theta})$, computed on the vector of deep parameters, that is

$$s_i = \sqrt{\theta_i^2 / \left(\mathcal{I}_T(\boldsymbol{\theta})^{-1}\right)_{(i,i)}}$$

where the sensitivity components is given by

$$\Delta_i = \sqrt{\theta_i^2 \times \mathcal{I}_T(\boldsymbol{\theta})_{(i,i)}}$$

Parameters $\theta_i$ are evaluated at the prior mean, as shown above, and at the prior standard deviation, $\sigma(\theta_i)$, as follows, respectively

$$s_i = \sigma\left(\theta_i\right) / \sqrt{\left(\mathcal{I}_T(\boldsymbol{\theta})^{-1}\right)_{(i,i)}}$$

and

$$\Delta_i = \sigma\left(\theta_i\right) \times \sqrt{\mathcal{I}_T(\boldsymbol{\theta})_{(i,i)}}$$

In general, it is said that any given parameter will be locally identifiable if the Jacobian matrix $J(q) = \partial \boldsymbol{m}_q / \partial \boldsymbol{\theta}'$, which maps from the vector of the deep parameters, $\boldsymbol{\theta}$ to the vector of, typically, the first two moments of the data, $\boldsymbol{m}_T$ has a full column rank. For the calculation of the Jacobian matrix DYNARE proceeds through two steps, which are summarized by the following expression

$$J(q) = \frac{\partial \boldsymbol{m}_q}{\partial \boldsymbol{\tau}'} \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\theta}'}$$

[43] The univariate convergence diagnostics computed on individual parameters, displayed in Appendix G, additionally enhances our assertion showing that convergence has been successful for almost all the estimated parameters.

Fig. 3.2: Priors and posteriors: maintenance model

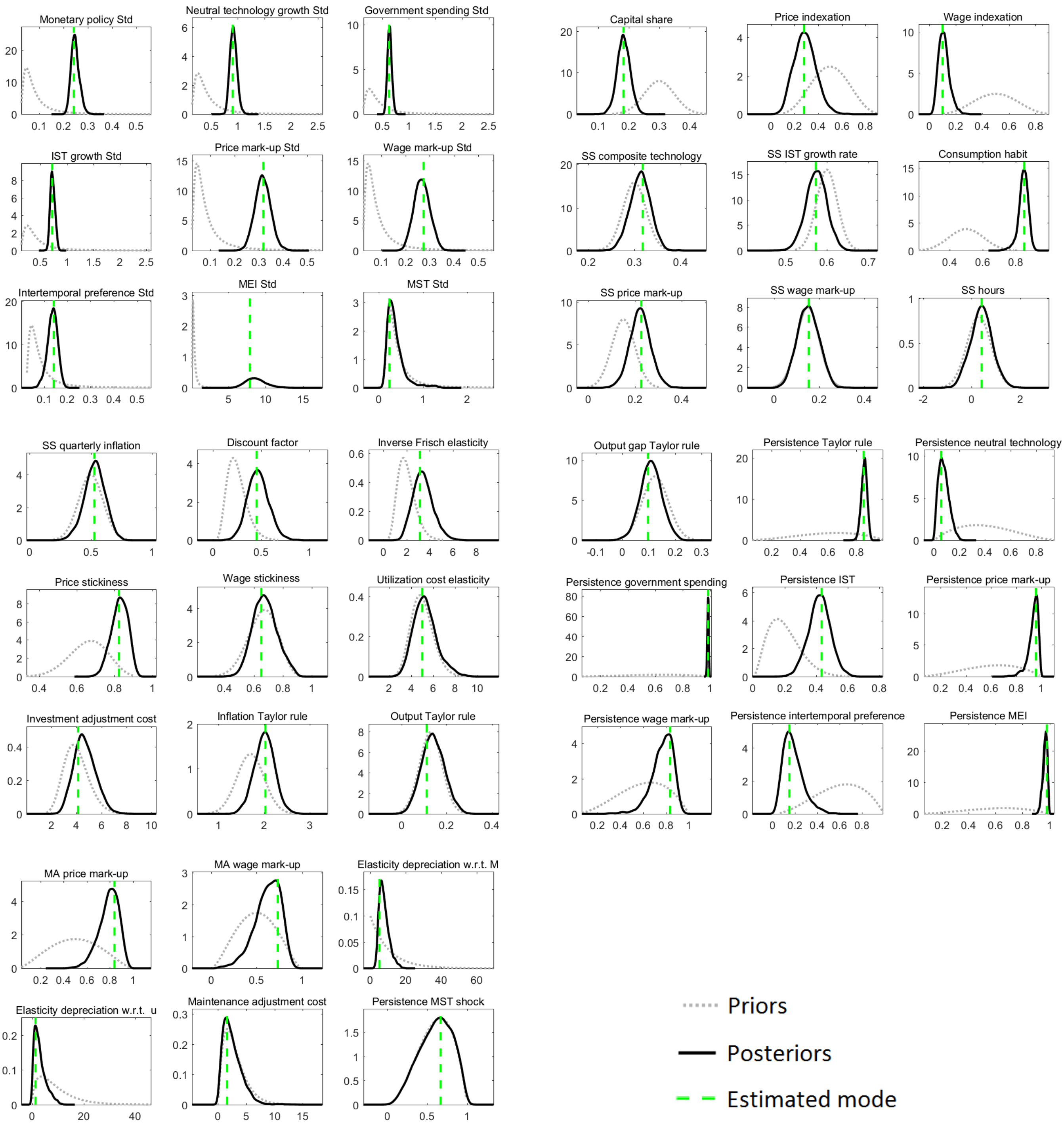
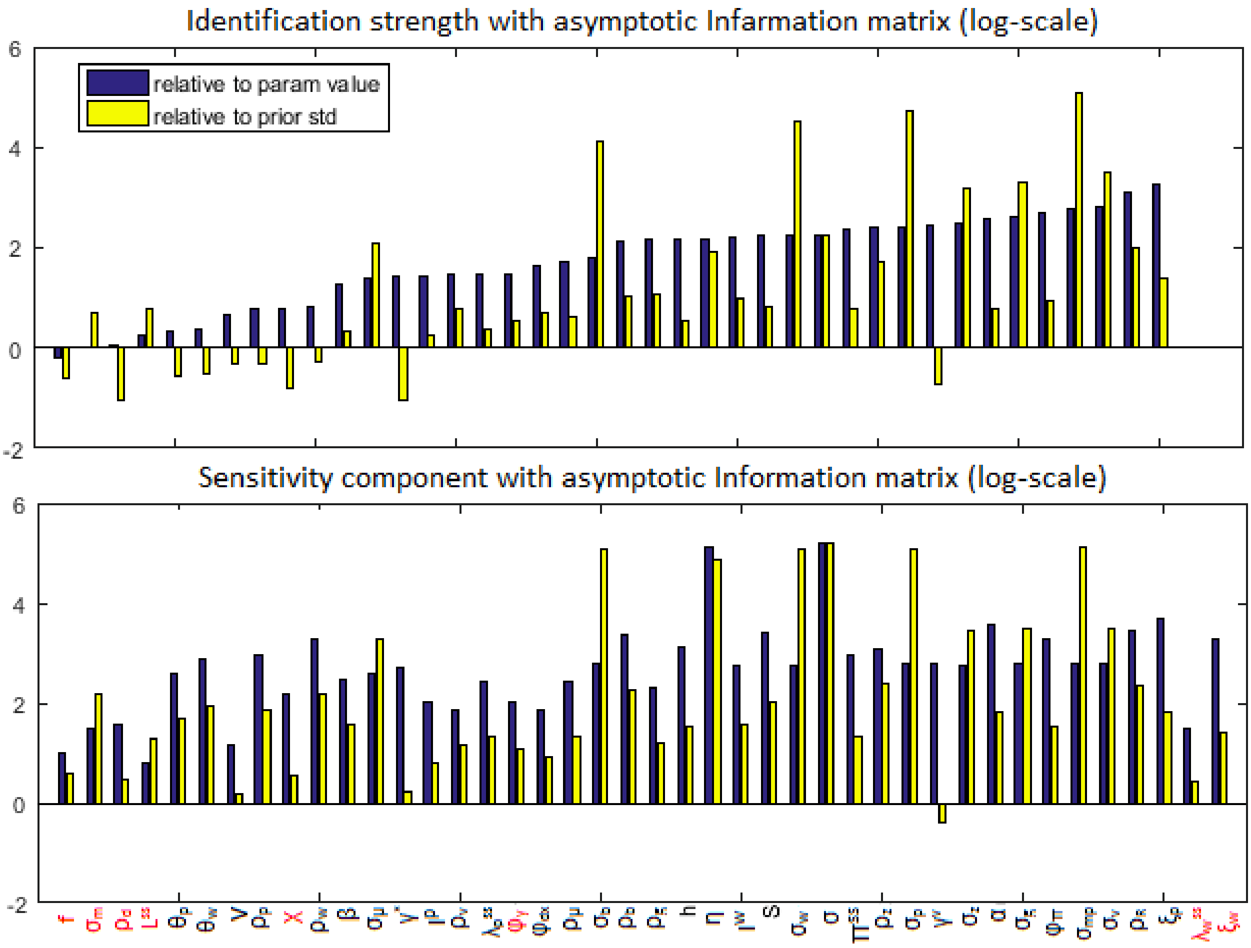


Fig. 3.3: Identification analysis of the parameters: maintenance model




where $\boldsymbol{\tau}$ represents the vector of all the elements, which appear in the transition equation of the state space form representation, depending on $\boldsymbol{\theta}$. Therefore, an additional necessary condition arises for the local identifiability which requires $J_2 = \partial\boldsymbol{\tau}/\partial\boldsymbol{\theta}'$ to be of a full rank[44]. When $J_2$ has a deficient rank at $\boldsymbol{\theta}_i$ it means that the specific parameter is not identified in that particular point of the parameter space. When the rank of $J_2$ is full whereas it is not that of $J(q)$ then $\boldsymbol{\theta}_i$ is unidentifiable given the observation variables and the total number of observations.

So, in the upper and lower panels of Fig. 3.3 the bars represent the strength of identification and the sensitivity components of the maintenance model parameters, respectively, computed on the basis of the Fisher information matrix normalized by the prior standard deviations and the prior means. The upper panel displays the identification strength of the estimated parameters. The higher the bars the stronger the identification. So, we can observe that the parameters of the elasticity of depreciation with

[44] In the DYNARE output $J(q)$ is labeled $J$ while $J_2$ is labeled $H$.

respect to maintenance and utilization, $\sigma$ and $\eta$, respectively, are strongly identified. The standard deviation of the maintenance-specific technology process, $\sigma_d$, and its persistence parameter, $\rho_d$, the steady state of hours, $L^{ss}$, the utilization cost elasticity, $\chi$, and the maintenance adjustment cost, $f$, (on the left-hand side of the upper panel) are, effectively, very weakly identified. As to the wage stickiness, $\xi_w$, and the steady state mark-up for wages, $\lambda_w^{ss}$, (on the right-hand side of the upper panel) the likelihood function in the direction of this parameters is completely flat, i.e. the identification strength is zero. However, in the bottom panel it can be observed that all the parameters are identified at least at the local mean as far as for all of them the sensitivity component is high. This means that all of the parameters contribute effectively to shape the posterior log-likelihood function. Therefore, since the identification strength for $\xi_w$ and $\lambda_w^{ss}$ is zero but their respective sensitivity components are not, these two parameters are pairwise collinear and thus their effects on the posterior log-likelihood compensate each other.

## 3.4 Maintenance model convergence dynamics

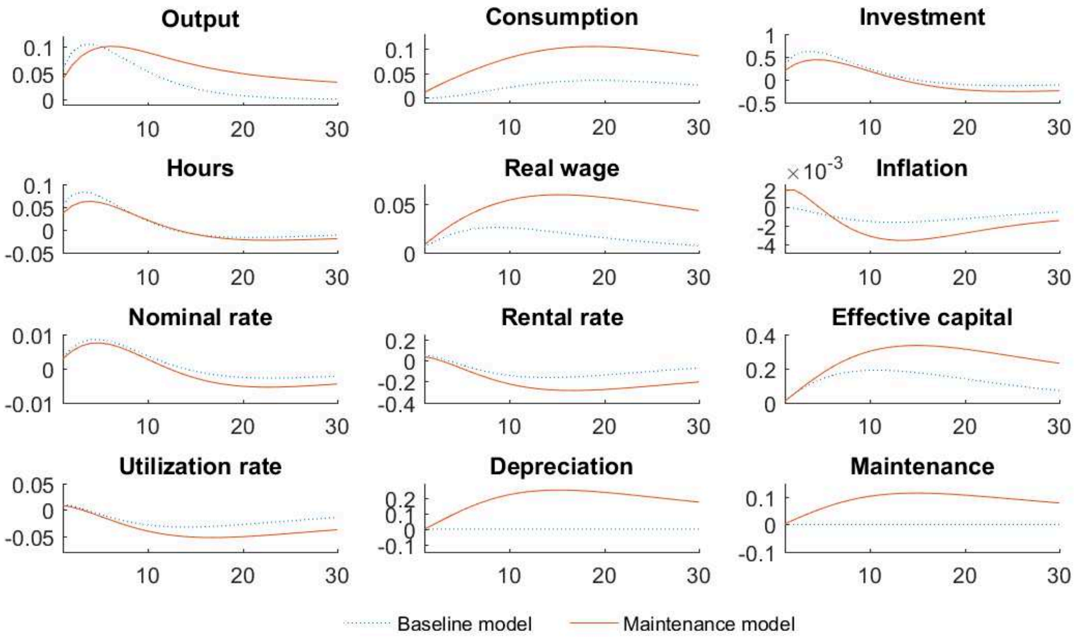
In order to investigate the dynamics of our model we have computed the impulse response functions of both the baseline and maintenance models to all the shocks included in the two models. We have calibrated the parameters using the estimated median values of the respective models and normalized to one the standard deviations of the shocks. Over all the related figures the dotted lines describe the optimal paths of the baseline model and the continuous lines those of the maintenance model.

Fig. 3.4 displays the impulse response functions of the main real variables to a positive one standard deviation shock on the marginal efficiency of investment technology process. Both the models deliver well hump-shaped curves. A positive shock to the MEI process increases on impact the amount of new capital above its steady state level. As a consequence, in equilibrium the cost of new capital declines inducing an impact reduction in the Tobin's q, $\hat{q}_t = \hat{\phi}_t - \hat{\lambda}_t$. Given the following equilibrium condition for the real interest rate

$$\begin{aligned}\hat{R}_t - \hat{\pi}_{t+1} =& -\hat{q}_t + \left[1 - \bar{D} - \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A}\right]\hat{q}_{t+1} + \frac{\rho}{\rho-\tau}\bar{D}\hat{\rho}_{t+1} + \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\bar{B}\hat{u}_{t+1} + \\ & - \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\bar{C} + e^{\sigma\gamma_v}\sigma\bar{\delta}\right)\hat{\delta}_{t+1} - \left[1 + \beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\sigma\bar{C}\right]\rho_v\hat{v}_t + \\ & + \left[\beta e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A} - \frac{\tau}{\rho-\tau}\bar{D}\right]\left(\hat{\varsigma}_{t+1} - \hat{\lambda}_{t+1}\right)\end{aligned} \tag{3.4.1}$$

the inverse relation with the Tobin's q induces real interest rate to increase on impact, which in turn attracts new investment. In the baseline model the interest rate effect is strong enough to generate a substitution effect away from consumption. Hence, agents find it optimal to postpone consumption of final goods and to increase investment. Consequently, consumption decreases on impact of the shock delivering an increasing path and overshoots above the steady state level after a few periods, following the recovery of the real interest rate. The optimal level of labor increases on impact of the shock given a higher marginal utility of labor on the supply side, and a lower price mark-up of the intermediate goods producers, because of increased real marginal costs, on the demand side. On the contrary, in the maintenance model current consumption increases on impact. This is explained by a relatively stronger wealth effect combined with a weaker interest rate effect, given higher posterior estimates both of the inverse of investment adjustment costs and of the steady state composite growth

Fig. 3.4: Impulse response functions to one standard deviation shock to MEI

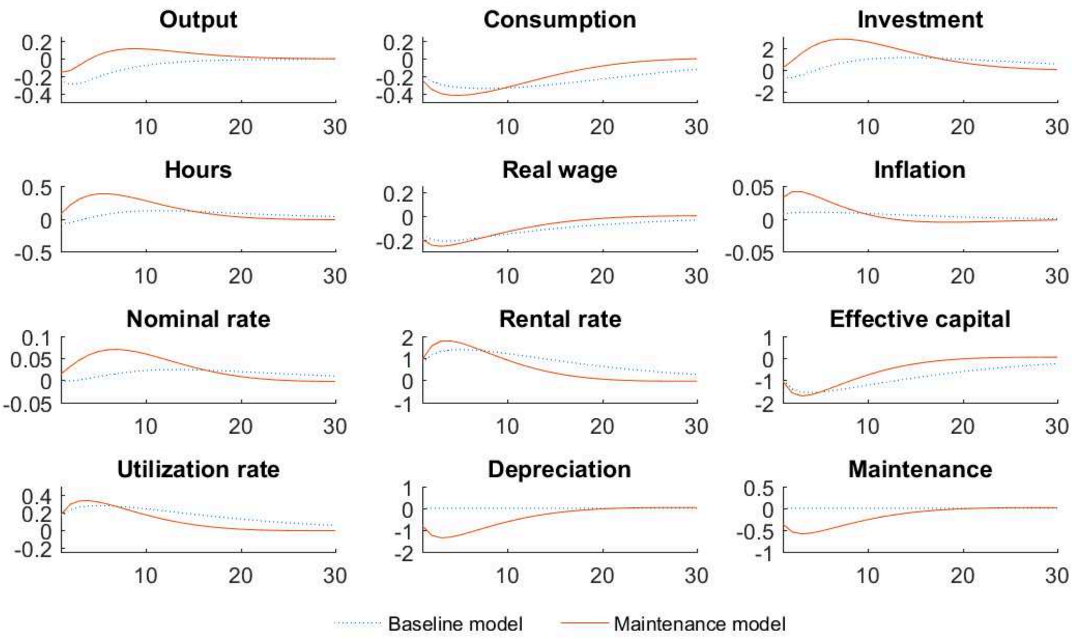



trend. Therefore the optimal level of investment increases on impact of the shock though less than in the baseline model. The increased economy's productive capacity delivers a slightly positive impact response of the capital utilization rate which, in turn, rises on impact, although by a very small amount, both effective capital and its marginal productivity. Therefore, given that real marginal costs of the intermediate goods producers immediately rise, and that the presence of nominal rigidities do not allow the firms to suddenly adjust for the prices, the price mark-up declines. Hence, the co-movement of consumption and hours in the maintenance model is ensured through the labor market equilibrium condition given the countercyclical behavior of the mark-ups. The increase in the level of new capital rises on impact the stock of installed capital which over the next periods affects effective capital. Thus, after a weak positive response on impact of the shock due to the impact increase in the utilization rate, the optimal effective capital keeps on growing and reverts back to its steady state level after several periods. Consequently, the marginal product of capital, after the immediate slightly positive response, declines below its steady state level in order to invert its route as the effective capital recovers slowly to equilibrium. Capital utilization rate depicts a hump-shaped path similar to the rental rate of capital. Hence, the positive wealth effect and the relatively weaker substitution effect in the maintenance model allows for co-movements in consumption, output, investment and hours. On the contrary, in the baseline model the co-movement in investment, hours and output comes out at the cost of countercyclical consumption. Moreover, the persistent increase in the capital stock creates in the maintenance model a slightly positive impact response in maintenance and raises the next period capital depreciation rate which further pushes up the demand for maintenance.

Overall, endogenous depreciation and the presence of maintenance sector amplify the convergence dynamics of the equilibrium paths in response to a positive MEI shock with respect to the baseline model. Convergence for almost all the considered real variables is delayed in part because of a relatively higher estimated persistence in the shock with respect to the baseline model. The optimal response paths of effective capital, consumption, real wages and output are significantly amplified although the estimated nominal and real frictions in the maintenance model are relatively lower. The lower estimated level of the investment adjustment cost in the maintenance model smooths the optimal path of investment. Similarly, a higher estimated steady state elasticity of the utilization costs delivers a stronger response path for the utilization rate. On the contrary, given the relatively higher estimates of the baseline model nominal rigidities, the adjustment of wages and prices towards their desired levels is slower. So, agents necessitate to increase relatively more the hours of work with respect to the maintenance model in order to recover to equilibrium. As to the speed of convergence, in the baseline model output reaches its equilibrium level after something more than 20 periods. In the maintenance model convergence is not completed before than 30 periods. Optimal investment in the maintenance model, after a positive response on impact of about 0.20%, falls below its steady state level after 14 periods, thereafter recovery to equilibrium is very slow. In the baseline model overshooting occurs after 15 periods and recovery is slightly faster. Delayed convergence occurs in the maintenance model also with regard to the utilization rate and to the capital rental rate. After a positive almost of the same magnitude response on impact of the shock, overshooting for the nominal interest rate occurs over the periods 13 and 14 in the maintenance and baseline models, respectively, with a slower convergence in the former one. The impulse response function of inflation in the maintenance model is mildly cyclical as a consequence of the accelerated depreciation rate and a relatively more volatile price mark-up. All the main real variables are pro-cyclical in both the models, with exception of consumption which is countercyclical in the baseline model. Hence, in response to a positive shock to the marginal efficiency of investment, maintenance and investment are complements.

Fig. 3.5 displays the impulse response functions to a positive one standard deviation shock to investment-specific technology process. The IST shock (or disembodied investment specific technology shock) equates, in equilibrium, the inverse of the relative price of investment which thus decreases. Following the decline in the relative price of investment the Tobin's q rises on impact of the shock which, combined with the direct effect of the IST shock, generates a decline in the real interest despite a rise in the expected marginal product of capital. In the baseline model the strong interest rate effect generates an impact decline in the real investment, which depicts an increasing path and overshoots above its steady state value after several periods. The impact decline in real investment induces a fall in new capital and, hence, the level of capital stock falls below its steady state level whereas the cost of installed capital increases. Moreover, the positive investment-specific technology shock generates an immediate decline in effective capital, which keeps on decreasing over the next several periods despite an impact positive response of the utilization rate as far as part of installed capital is destroyed. Given the equilibrium path of the effective capital the marginal product of capital rises on impact to the shock as well as over the following periods and reverts its path downward as effective capital starts to recover towards its steady state level. Similarly, capital utilization rate rises above its steady state level on impact of the shock. Give the reduction in the economy's productive capacity output declines on impact. The monetary authority, in order to balance the deviations both of output from its steady state

Fig. 3.5: Impulse response functions to one standard deviation shock to IST




level and of inflation from its target level, reduces the nominal interest rate although by a very small amount. Given the sharp decline in output, the negative wealth effect dominates over the interest rate effect so that agents find it optimal to reduce current consumption on impact of the shock. The price of consumption immediately grows up which in the presence of nominal rigidities pushes downward real wages and increases inflation. As a consequence of the wage and price stickiness both the mark-ups increase and the intermediate goods producers find it optimal to reduce the demand for labor. Hence, in the baseline model output, consumption, investment, hours, real wages, nominal interest rate and effective capital all co-move in response to a positive investment-specific technology progress.

With regard to the maintenance model the interest rate effect is dominated by the substitution effect away from consumption and maintenance. The agents, hence, find it optimal to increase investment on impact of the shock in order to rebuild the destroyed stock of capital. However, the low magnitude in the impact response of real investment and the decline in capital depreciation rate are not sufficient to overhang the negative impact of the IST shock and to push up immediately the amount of capital stock. Moreover, the higher is the elasticity of capital depreciation rate with respect to maintenance the stronger the negative impact of the IST shock on the stock of capital. Therefore, similarly to the baseline model, effective capital declines both on impact of the shock and over the subsequent periods whereas its rental price increases following a hump-shaped path. The impact increase of the capital rental price is additionally boosted by the impact rise in the Tobin's q. As a consequence of a higher marginal product of capital the utilization rate rises on impact following a hump-shaped path. Depreciation rate declines on impact of the IST shock as far as its direct effect dominates the positive effect induced by the increase in utilization. Furthermore, the impact decline in capital stock and its equilibrium dynamics generate a hump-shaped convergence path for the rate of depreciation. Maintenance declines on impact below its steady state level, whereas its relative price increases, and reverts its path over the subsequent periods as the amount of existing capital stock rises and the rate of depreciation accelerates. Differently from the baseline model, the impact response of the nominal interest rate is positive driven by the strong response of the monetary authority to higher variations in inflation. Output decreases on impact as in the baseline model and overshoots its steady state equilibrium following the sharp peak of investment. Differently from the baseline model, consumption and hours move in the opposite directions. Indeed their optimal paths are mirrored through the optimal labor market condition up to the countercyclical effects of the nominal rigidities. So, driven by the negative wealth effect at optimum agents decide to recover the reduction in capital by postponing consumption and reducing leisure and thus increasing the hours of work, as well as investment. In the maintenance model, hence, the optimal convergence paths in response to a positive IST shock deliver countercyclical behavior for investment and hours while consumption, depreciation, effective capital and maintenance all co-move with output.

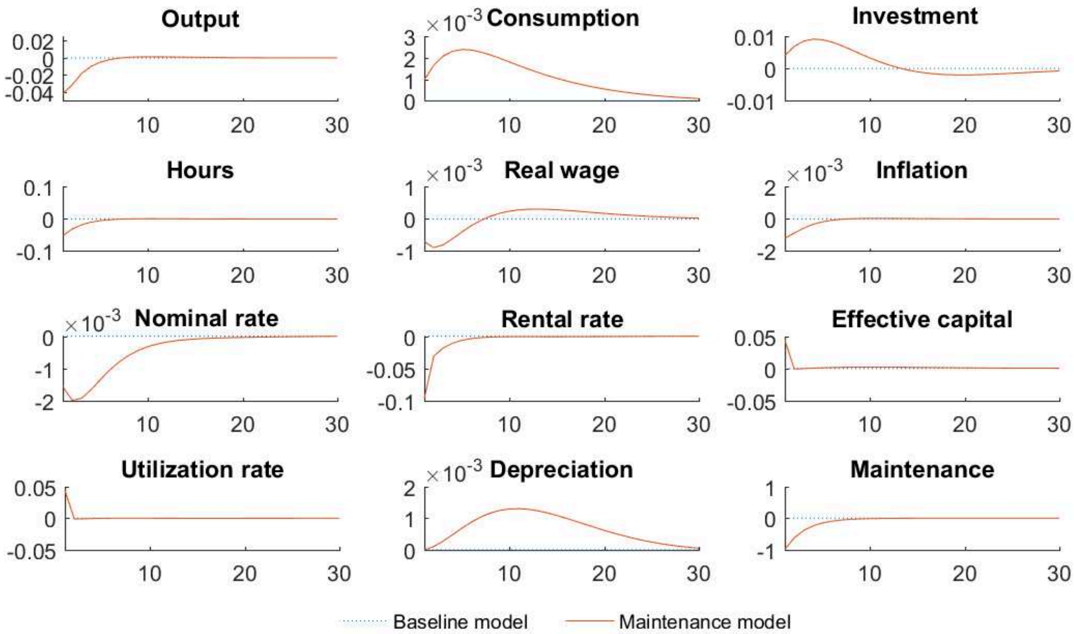
As Fig. 3.5 shows, the impulse response functions of all the considered real variables in the maintenance model depict more hump-shaped paths than those in the baseline model. Moreover, convergence dynamics are slightly faster despite a relatively higher estimated persistency in the maintenance model IST shock. This qualitative behavior could be explained by the presence of the maintenance sector which, together with endogenous depreciation rate, serves as an additional transition mechanism for the propagation of the shocks. Moreover, differently from the convergence dynamics depicted in the responses to MEI shock, in this case we can assert that maintenance and investment behave as substitutes one to each other. Our claim supports the findings of Albonico et al. (2014) in which the response functions of the main endogenous variables to a positive IST shock are all pro-cyclical (and positive) with exception of maintenance to capital ratio suggesting thus that maintenance and investment are substitutes.

In general, when the IST shock is equated to the inverse of the relative price of investment it is interpreted as the disembodied investment-specific technology progress[45]. This shock aims to reduce the unit cost of production of new investment. In the maintenance model, a positive impact of this shock creates new investment and delivers an increasing optimal path for the capital stock which, given the nature of the shock, is qualitatively comparable to the already existing one. Hence, in the long-run aggregate economy there will be available on the market more capital stock which is qualitatively equivalent and thus, in aggregate, rises the optimal lifetime of capital[46]. As a consequence of the positive disembodied IST shock, the capital depreciation rate decelerates. Differently from the disembodied investment-specific technology progress and according to the literature interpretation the embodied progress increases the marginal productivity of new investment. Hence, we claim that the marginal efficiency of investment technology progress which affects in our model the transformation

[45] See, for example, Greenwood et al. (1997) and Boucekkine et al. (2009).

[46] Boucekkine et al. (2009) have shown that the optimal capital lifetime is a decreasing function of the total investment-specific technology progress, both embodied and disembodied, under certain restrictions on the steady state relations of their model parameters. This, in turn, delivers a positive relation between the total IST shock and the capital depreciation rate. In our model, instead, we distinguish between the two types of the IST progresses which allows us to assert that there exists a positive relation between the (disembodied) IST progress and the capital optimal lifetime and a negative relation between the latter one and the (embodied) MEI progress.

Fig. 3.6: Impulse response functions to one standard deviation shock to MST




process of new investment into new capital can reflect the embodied investment shocks. In this case, a positive MEI shock aims to improve the quality of the new capital available on the market. The existing capital becomes more obsolete which thus accelerates the depreciation path.

The optimal convergence paths to a positive one standard deviation shock to the maintenance-specific technology progress are exhibited in Fig. 3.6. This shock affects the transformation process of the units of final good, used as input by the maintenance goods producers, into efficiency units of maintenance and is inversely related to the relative price of maintenance. Thus, a positive MST shock reduces the cost of producing one unit of maintenance good on the impact of the shock. Nevertheless, real maintenance declines on impact depicting an increasing path. This occurs, on the one hand, because of a direct negative effect of the MST shock on current real maintenance. On the other hand, because of a sharp decline both in the price of consumption and in the price of new investment, the strong substitution effect shifts expenditures away from maintenance. Therefore, despite a negative wealth effect generated by an impact decline in real output households find it optimal to increase consumption and leisure. The rise in Tobin's q makes the real interest rate to fall below its steady state level. Nonetheless, real investment rises on impact as far as the weak interest rate effect is dominated by the substitution effect. Thus, both real consumption and real investment increase on impact and follow a hump-shaped path. Increasing consumption induces households to increase leisure thus hours worked decline on impact. Marginal utility of labor declines which, in turn, drops real wages below their equilibrium level. The marginal product of capital declines on the impact of the shock due to a strong negative effect of the maintenance relative shadow value enhanced by an increase in the effective capital. As a consequence, real marginal costs and hence inflation decrease on impact. Given the deviation of inflation from its steady state level and the higher output gap, the monetary authority reduces nominal interest rate. Moreover, the decline in real maintenance accelerates capital depreciation rate which, following a hump-shaped path, destroys current and part of the future capital stock. Moreover, there occurs a one-off event on the capital utilization rate which increases on the impact of the shock due to a temporary positive gap between the economy's aggregate supply and demand. Consequently, effective capital rises above its steady state level. However, it returns to its equilibrium level over the next period as far as the weak effect of increased real investment is contrasted by the increasing path in the depreciation rate and is thus not able to boost up enough the amount of capital stock.

In general, as it will be detailed in the next section, the role of the MST shock in explaining the long run real business cycle fluctuations is completely irrelevant. In fact, in the maintenance model the variability of all the considered real variables is mostly governed by the shock to the marginal efficiency of investment and the role played by the IST technology progress results to be negligible.

The optimal convergence dynamics in response to the remaining shocks included in the models are displayed in Appendix H. All the real variables perform the same qualitative behavior in response to the labor augmenting technology progress in both the models. Therefore, all the variables, including depreciation and real maintenance in the maintenance model, co-move with real output. In general, the speed of convergence is slightly faster in the maintenance model with respect to the baseline model whereas the difference becomes significant for real investment, capital rental price and utilization rate.

Similarly, all the real variables including depreciation and maintenance and with exception of the nominal interest rate co-move with real output in response to a positive one standard deviation shock to the monetary policy. Convergence dynamics are significantly different in the optimal responses of investment, wages, effective capital, the marginal product of capital and the utilization rate. On the contrary the magnitude of the impact responses are almost the same.

With regard to the positive government spending shock consumption and investment move countercyclically with respect to real output. The behavior of this variables is explained by the assumption of a fully Recardian government policy, according to which government finances its spendings issuing short term bonds. The capital depreciation rate decreases on impact as well, although by a very small amount, and exhibits a well hump-shaped path due to a relatively stronger effect of a decrease in real maintenance with respect to an increase in capital utilization. The convergence dynamics of effective capital, real investment and real wages are significantly faster in the maintenance model as the capital depreciation rate decelerates.

The impulse response functions to a positive one standard deviation intertemporal preference shock are very similar between the two models. All the real variables are pro-cyclical with exception of investment which decreases as a result of a strong substitution effect towards consumption as well as towards maintenance in the maintenance model. In the latter one depreciation moves countercyclically governed by the optimal equilibrium dynamics of real maintenance.

In response to a positive price mark-up shock all the real variables are pro-cyclical, including depreciation and maintenance, with exception of inflation and nominal interest rate. An increase in the price mark-up, in fact, rises inflation on impact which, in turn, requires a strong positive interest rate response from the monetary authority. A significant difference in the optimal dynamics between the two models occurs for real consumption. As far as both investment and maintenance decline in the maintenance model, agents are willing to renounce to consumption today more than in the baseline model. In general, convergence is slightly faster in the baseline model despite a relatively higher estimated price friction and almost the same estimated shock persistence.

Finally, with respect to the baseline model, the convergence dynamics in response to a positive wage mark-up shock result to be more hump-shaped in the maintenance model. Convergence is significantly faster for almost all the real variables. In general, a relatively lower estimated persistence parameter and the presence of an additional adjustment mechanism accelerates the convergence dynamics in the maintenance model. Depreciation and maintenance move in opposite directions, the former one pro-cyclically and the latter one countercyclically. Moreover, differently from the baseline model, real investment in the maintenance model is pro-cyclical. In fact, in the baseline model the slight increase in real investment is explained by the substitution effect which moves away from consumption. In the maintenance model, instead, the increased demand for maintenance followed by a rise in the stock of capital tears both real consumption and investment down below their steady state levels.

## 3.5 Roles of the shocks in the real business cycles

In Tables 3.3 and 3.4 we display the percentage contributions of the shocks to the variations of the main endogenous variables in the baseline and maintenance models, respectively. With regard to the baseline model the marginal efficiency of investment technology progress is the one that mostly explains the variations in output and investment, 21.66% and 50.44%, respectively. Moreover, it explains 17.32% of variability in hours worked, although the government spending is the most important in this case accounting for 25.32% of hours growth. The MEI shock appears negligible in the fluctuations of consumption, which are mainly attributable to the government spending shock (55.14%), in those of real wages and inflation which are instead driven by the shock to the mark-up of prices (47.48% and 64.84%, respectively), as well as in the fluctuations of the nominal interest rate which are mostly explained by the monetary policy shock amounting to 39.18%. The contribution of the investment-specific technology progress is only relevant in explaining the dynamics of the relative price of investment accounting for 100% of its variation whereas its impact on the remaining main variables is even lower than the one of the labor augmenting technology progress. These findings are in line with Justiniano et al. (2011) who have found that the MEI shock is responsible for 60%, 68%, and 85% of respectively output, hours and investment growths in the US economy and, similarly, explains very little of variability in consumption. Moreover, our findings as well confirm their estimation results about the negligible role of the IST shock in explaining the business cycle fluctuations. Similarly, when maintenance and endogenous depreciation rate are considered variations in output and investment are mostly due to the MEI shock with its contribution of 61.24% and 68.60%, respectively. Variabilities in consumption and hours worked are as well mainly to be attributed in the maintenance model to the MEI progress, which explains 66.82% and 27.68% of them, respectively. Furthermore, the MEI shock is responsible for 93.04% and 94.63% of variability in maintenance growth and depreciation, respectively. As in the baseline model, fluctuations in real wages and inflation are mainly due to the price mark-up shock (55.86% and 64.25%, respectively) however, the impact of the MEI shock is higher amounting

Tab. 3.3: Variance decomposition in the baseline model (percent)

| | TFP | IST | MEI | Monetary policy | Government | Price mark-up | Wage mark-up | Intertemporal preference |
|---|---|---|---|---|---|---|---|---|
| Output | 14.34 | 2.36 | 21.66 | 11.07 | 17.86 | 12.46 | 14.30 | 5.95 |
| Consumption | 14.82 | 8.19 | 4.26 | 2.06 | 55.14 | 1.36 | 7.24 | 6.93 |
| Investment | 10.09 | 9.46 | 50.44 | 9.41 | 2.08 | 13.75 | 3.57 | 1.20 |
| Hours | 11.10 | 2.32 | 17.32 | 11.09 | 25.32 | 9.22 | 16.19 | 7.45 |
| Wages | 17.54 | 1.56 | 1.63 | 1.56 | 0.14 | 47.48 | 30.00 | 0.080 |
| Inflation | 2.44 | 0.18 | 0.23 | 0.79 | 0.73 | 64.84 | 30.71 | 0.07 |
| Interest rate | 4.89 | 1.72 | 4.89 | 39.18 | 5.31 | 13.97 | 26.82 | 3.22 |
| Relative price | 0 | 100 | 0 | 0 | 0 | 0 | 0 | 0 |

*Note*: The figures represent percentage contributions of the shocks to the endogenous variables computed at the posterior mean from the unconditional variance decomposition at infinity horizon and sum up to 100 across the columns.

Tab. 3.4: Variance decomposition in the maintenance model (percent)

| | TFP | IST | MEI | Monetary policy | Government | Price mark-up | Wage mark-up | Intertemporal preference | MST |
|---|---|---|---|---|---|---|---|---|---|
| Output | 3.49 | 0.30 | 61.24 | 1.63 | 9.01 | 14.97 | 7.46 | 1.90 | 0 |
| Consumption | 3.25 | 1.87 | 66.82 | 0.41 | 19.72 | 3.28 | 3.16 | 1.48 | 0 |
| Investment | 6.88 | 11.49 | 68.60 | 1.27 | 0.04 | 9.34 | 2.28 | 0.11 | 0 |
| Hours | 4.62 | 6.11 | 27.68 | 2.91 | 20.64 | 19.31 | 14.41 | 4.32 | 0 |
| Wages | 4.52 | 0.72 | 34.10 | 0.15 | 0.06 | 55.86 | 4.57 | 0.02 | 0 |
| Inflation | 4.43 | 1.32 | 3.71 | 1.65 | 0.89 | 64.25 | 23.38 | 0.37 | 0 |
| Interest rate | 3.53 | 5.11 | 16.80 | 23.45 | 3.29 | 26.50 | 19.46 | 1.87 | 0 |
| Relative price | 0 | 100 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Maintenance | 1.17 | 2.83 | 93.04 | 0.09 | 0.01 | 2.14 | 0.23 | 0.01 | 0.49 |
| Depreciation | 0.97 | 3.14 | 94.63 | 0.06 | 0 | 1.00 | 0.19 | 0.01 | 0 |

*Note*: The figures represent percentage contributions of the shocks to the endogenous variables computed at the posterior mean from the unconditional variance decomposition at infinity horizon and sum up to 100 across the columns.

to 34.10% and 3.71%, respectively, against 1.63% and 0.23% in the baseline model. The price mark-up shock becomes the major driver of the fluctuations in the nominal interest rate as well, accounting for 26.50% of them, while the impact of the MEI shock increases to 16.80% from 4.89% in the baseline model. Variations in the relative price of investment in the maintenance model continue to be caused entirely by the IST shock. The estimation results of the maintenance model as well confirm that the IST progress plays an irrelevant role in driving the business cycle fluctuations, while the contribution of the labor augmenting technology progress has diminished with respect to the baseline model. According to our estimation results, the new shock we consider in the maintenance model, i.e. the maintenance-specific technology progress, in long run has a null effect with respect to all the considered variables except of a very low impact of 0.49% on the variability of maintenance. Anyway, it is the fifth shock in order of importance in explaining the long run variability in maintenance among the all considered shocks. On the contrary, in short run the MST shock is the key-driver of the variability of real maintenance, explaining more than 40% of its growth in the first period after the shock. Its effect vanishes after more than ten periods.

The above exposed results are broadly in line with the mainstream DSGE literature which considers the investment shocks, and in general shocks to the capital accumulation process, to be the key drivers of the business cycle fluctuations. In particular, Fisher (2006), for example, states that the IST shock is responsible for most of variations in output and hours. However, as highlighted in Justiniano et al. (2011), this result is generated by the fact that he excludes the price of durable consumption from the measure of investment deflator. Therefore, by considering only the price of equipment which is always countercyclical the Fisher (2006) model attributes more importance to the IST progress in explaining real business cycles. In Justiniano et al. (2011) the shock which explains the most of variability in output, investment and hours is the marginal efficiency of investment technology progress. However, their model fails to generate co-movement in consumption and, similarly, this occurs in our baseline model. Gertler and Karadi (2011) highlight the importance of the shock to the quality of capital in contributing to the business cycle fluctuations. Moreover, their model is able to generate co-movements of output, consumption, investment and hours. Nevertheless, the qualitative behavior of the real variables in their model depends crucially on the calibration of the nominal rigidities and of the autoregressive coefficient in the shock process to the quality of capital. Furlanetto and Seneca (2014) compare the roles of a shock to the quality of capital, an investment-specific shock and a shock to the capital depreciation rate in a DSGE model setting. They find that, the shocks to capital depreciation are the most important drivers in the macroeconomic fluctuations. However, co-movements in consumption, output, investment and hours are achieved only when the capital depreciation shock is highly persistent. Therefore, in order to validate the shock high persistence, they assert that the most plausible economic interpretation for these shocks would be provided by the disturbances that affect the ability of the financial intermediaries to finance the investment projects. A similar interpretation Justiniano et al. (2011) have suggested for the shock to the marginal efficiency of investment. Therefore, the Furlanetto and Seneca (2014) shock to the depreciation rate of capital could be related to our MEI shock, as the one hitting the transformation process of the investment goods into new capital goods. Our model, instead, as it has been highlighted in the previous section, generates co-movements in response to a positive MEI shock between all the real variables and especially in consumption though the estimated autocorrelation of the MEI process is relatively higher with respect to the baseline model.

Notice that, adopting a standard neoclassical model setting enriched with a variable capital utilization rate and the depreciation-in-use hypothesis, Greenwood et al. (1988) show through a quantitative analysis that in response to a shock affecting the marginal efficiency of the newly produced capital goods only, consumption is pro-cyclical. On the contrary, as they argue, when the same type of shock hits the standard neoclassical model framework, in which capital utilization is fixed, consumption fails to co-move due to an intertemporal substitution effect away from leisure and consumption. However, the subsequent generation of models up to the DSGE ones, in which the hypothesis about variable capacity utilization has become standard, often tackle the problem of countercyclical consumption anyway. Therefore, we assert that the modifications in the transmission mechanism necessary to generate an optimal behavior in consumption in line with the observed evidence must be ascribed to the endogenous treatment of the capital depreciation rate.

Tab. 3.5: Standard deviations of actual and estimated series

| | Standard deviation | | | Standard deviation relative to output | | |
|---|---|---|---|---|---|---|
| | Data | Baseline model | Maintenance model | Data | Baseline model | Maintenance model |
| | | Median | Median | | Median | Median |
| Output growth | 0.73 | 0.99 | 0.73 | | | |
| Consumption growth | 0.56 | 0.70 | 0.50 | 0.77 | 0.71 | 0.68 |
| Investment growth | 2.27 | 2.33 | 2.50 | 3.10 | 2.35 | 3.44 |
| Hours | 1.98 | 2.43 | 2.24 | 2.70 | 2.45 | 3.08 |
| Wage growth | 0.78 | 0.82 | 0.87 | 1.07 | 0.83 | 1.19 |
| Inflation | 0.65 | 0.63 | 0.60 | 0.89 | 0.64 | 0.82 |
| Interest rate | 1.09 | 0.54 | 0.58 | 1.49 | 0.54 | 0.78 |
| Relative price | 0.72 | 0.85 | 0.66 | 0.99 | 0.86 | 0.90 |
| Maintenance growth | | | 0.80 | | | 1.10 |
| Depreciation growth | | | 3.75 | | | 5.16 |

*Note*: Empirical moments are computed using Canadian data over 1981Q2-2015Q1. The moments of the models are computed at the median of 80,000 draws from estimated posterior distribution.

Tab. 3.6: Correlations of actual and estimated series

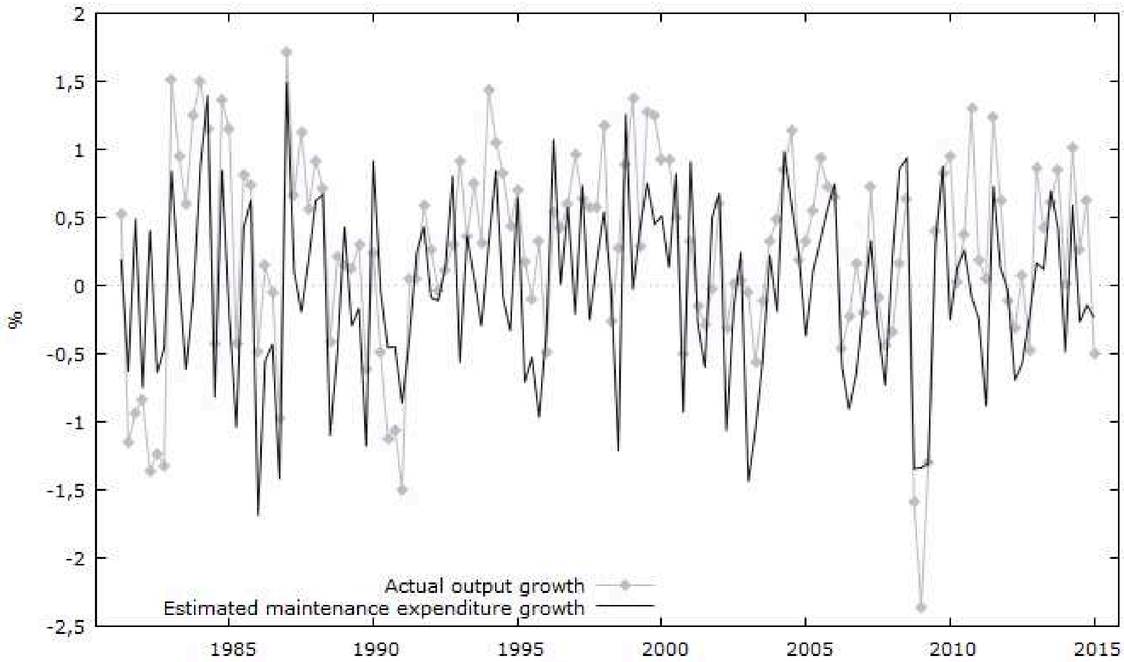
| | Autocorrelation | | | Correlation with output growth | | |
|---|---|---|---|---|---|---|
| | Data | Baseline model | Maintenance model | Data | Baseline model | Maintenance model |
| | | Median | Median | | Median | Median |
| Output growth | 0.43 | 0.23 | 0.58 | | | |
| Consumption growth | 0.002 | 0.26 | 0.73 | 0.45 | 0.56 | 0.75 |
| Investment growth | 0.52 | 0.65 | 0.77 | 0.61 | 0.42 | 0.74 |
| Hours | 0.95 | 0.92 | 0.95 | 0.04 | 0.11 | 0.08 |
| Wage growth | 0.24 | 0.33 | 0.35 | 0.21 | 0.24 | 0.36 |
| Inflation | 0.58 | 0.58 | 0.70 | -0.20 | -0.19 | -0.41 |
| Interest rate | 0.98 | 0.88 | 0.92 | -0.25 | -0.23 | -0.25 |
| Relative price | 0.30 | 0.41 | 0.42 | 0.21 | -0.06 | 0.06 |
| Maintenance growth | | | 0.16 | | | 0.50 |
| Depreciation growth | | | 0.42 | | | 0.03 |

*Note*: Empirical moments are computed using Canadian data over 1981Q2-2015Q1. The moments of the models are computed at the median of 80,000 draws from estimated posterior distribution.

## 3.6 Models fit

In the present section we compare some estimated statistics both for the maintenance and the baseline models with those generated by the actual data for the variables treated as observables and expose also the estimates for maintenance growth and the depreciation rate. In Table 3.5 are presented the posterior medians of the standard deviations in absolute values as well as those relative to output growth. In Table 3.6 we refer to the autocorrelation values and to the cross-correlations with output growth. In general, it can be observed that the maintenance model mimics the real Canadian economy much better with respect to the baseline model. The estimated volatility of output growth perfectly matches the respective empirical moment in the maintenance model (0.73), which instead is overpredicted by the baseline model (0.99). Consumption growth volatility is as well better fitted by the maintenance

Fig. 3.7: Actual real output and estimated real maintenance expenditures growth rates for Canada, (1981:II-2015:I)




Source: Real output growth rate from Statistics of Canada (CANSIM). Growth rate of real maintenance expenditures from estimation results.

model though underpredicted (0.50) while again overpredicted by the baseline model (0.70). The latter one performs slightly better with respect to volatilities of real investment (2.33), wage growth (0.82), and inflation (0.63). On the contrary, the empirical volatilities of hours worked, nominal interest rate and the relative investment price are better fitted by the estimates in the maintenance model which are, respectively, 2.24, 0.58, and 0.66. The estimated relative standard deviations in the maintenance model better fit the respective empirical ones in the cases of real investment growth (3.44), real wage growth (1.19), inflation (0.82), nominal interest rate (0.78), and the relative price of investment (0.90). All of the estimated relative volatilities are underpredicted by the baseline model. The maintenance model, on the contrary, overpredicts the empirical relative volatilities of real investment, hours worked, and wage growth. In the maintenance model the estimated autocorrelations, which are displayed in 3.6, resemble better those of the actual data for output growth, hours worked and the nominal interest rate. In the case of investment relative price the estimated autocorrelation is fitted slightly better by the baseline model, 0.41, being 0.42 in the maintenance model and 0.30 in the actual data. Similarly, the baseline model matches slightly better the persistence of real wage growth, which is 0.33 against 0.35 in the maintenance model and 0.24 in the data. The empirical autocorrelation of consumption growth, 0.002, is highly overpredicted by both the models with a much higher estimated persistence in the maintenance model (0.73). The autocorrelation of hours worked is perfectly matched by the maintenance model, wheres that of inflation is perfectly matched by the baseline model. The overall estimated contemporaneous cross-correlations with output growth are performed better in the maintenance model as well. In particular, a better fit occurs in the case of investment growth which is estimated to amount to 0.74, while it is 0.61 in the data. The baseline model, on the contrary, underpredicts the cross-correlation for investment which is around 0.42. The empirical cross-correlation of hours worked (0.04) is as well better suited by the maintenance model (0.08), whereas it is highly overpredicted by the baseline model (0.11). The latter one fits much better with respect to the maintenance model the cross-correlations of real wage growth (0.24) and inflation (−0.19). On the contrary, the maintenance model perfectly matches the empirical cross-correlation of the nominal interest rate (−0.25). Finally, the baseline model totally fails to capture the cross-correlation of the relative price of investment which is estimated to be negative, −0.06, although the respective one in the maintenance model, 0.06, highly underpredicts the empirical one, 0.21. Hence, overall, the estimated moments in the maintenance model fit the respective empirical ones much better in comparison to the baseline model. With regard to the estimates produced for the maintenance and the depreciation growth rates in the maintenance model no direct comparisons can be made with the respective empirical measures because of the lack of actual data. However, it could be inferred that the maintenance model preforms fairly well as far as it almost confirms the theoretical insights. The estimated volatility of maintenance growth rate, 0.80, is higher, in fact, with respect to both the estimated and empirical volatilities of real output growth, 0.73. Furthermore, as the relative literature suggests[47], real maintenance growth rate is less volatile than the growth rate of real investment (2.50). Our estimation results as well confirm the literature insights according to which depreciation growth rate is highly volatile, which is estimated to be around 3.75[48]. Moreover, the estimated persistency of capital depreciation resembles the one estimated for the investment relative price which, as a consequence of our model assumptions, determines the trend of the optimal depreciation path. Real maintenance growth rate is found to be the less persistent endogenous variable in our economy with an estimated autocorrelation of about 0.16. On the contrary, the contemporaneous cross-correlation of maintenance with output is estimated at 0.50. The contemporaneous cross-correlation between depreciation growth and output growth is almost null (0.03). This estimation result could be attributed to the assumption of factor hoarding in the production function. In fact, factor hoarding implies that the current capital stock part of which, according to our model settings, is destroyed by the current depreciation, is used in the production of the intermediate goods over the next period. This implies that the final output produced in period $t$ will be affected by the capital depreciation rate which occurs in period $t-1$. Even though, the estimated at the median autocorrelation function between capital depreciation growth rate and output growth rate, $(dy_t d\delta_{t-k})$, delivers relatively low figures, averaging around 0.08, 0.07, 0.05, and 0.03 for the first, second, third, and fourth order cross-correlations, respectively. On the contrary, our estimation results confirm the existence of a high positive correlation between capital depreciation rate and the relative price of investment which in the median is around 0.98[49]. Finally, given the posterior mean values of our estimated parameters, the average quarterly depreciation rate results to be 2.95 which implies an annual average depreciation of about 11.80%. We should notice that this value does not departure too much from the 10% assumed in Justiniano et al. (2011) which could be attributed to the restriction we have imposed in order to be able to switch from one model to the other. We have

[47] See, for example, the highlights in McGrattan and Schmitz (1999) or Fig. 1.2 in Chapter 1: Literature review.
[48] See, for example, Albonico et al. (2014).
[49] See, among others, Boucekkine et al. (2009).

furthermore computed an estimation exercise of our model estimating the parameter of the steady state depreciation rate. For this purpose we have assumed that the steady state depreciation rate follows a normal distribution with prior mean and prior standard deviation both of 0.05. In this case the estimation results do not change qualitatively though the annualized posterior mean of depreciation amounts to almost 14%.

In Fig. 3.7 we plot the estimated series for the maintenance expenditures growth rate and the actual growth rate of real GDP. As it can be observed, most of the peaks and troughs in real output variability are fairly well captured by the movements in real maintenance.

## 3.7 Concluding remarks

We have estimated a DSGE model with endogenous capital depreciation rate and maintenance sector and compared it to the baseline model for the Canadian economy. The Bayesian technique with Metropolis-Hasting algorithm has been used for the estimation purposes. The sampling dataset is in quarters, covers the period over 1981Q2-2015Q1 and includes as observables the growth rates of real per capita output, consumption, investment, real wage, and relative price of investment, as well as hours worked, inflation and nominal interest rate. We consider nine stochastic shocks in the maintenance model and eight in the baseline model, where the maintenance-specific technology process is excluded. The estimation results show that accounting for endogenous depreciation rate and maintenance activity generates significant differences in the optimal response paths to the exogenous shocks and in particular with regard to some of the key real endogenous variables. For what it concerns the shock to the marginal efficiency of investment technology process the convergence paths towards equilibrium are amplified for output, consumption, real wage and effective capital. Inflation exhibits a slightly cyclical path. As far as this shock affects the quality innovations of new investment and, hence, of capital, it accelerates the rate of depreciation through the obsolescence effect. As a result the demand for maintenance increases. All the considered real variables, including utilization, depreciation, consumption, investment and maintenance behave pro-cyclically. In response to a positive investment-specific technology process, on the contrary, both depreciation and maintenance decline whereas utilization rate increases. The behavior of depreciation is explained by the fact that this type of shock increases the average service life of existing capital as far as it affects the marginal cost of production of one extra unit of capital, which implies that more capital of the same quality is available. So, in the maintenance model depreciation, maintenance, effective capital and consumption are pro-cyclical whereas utilization, investment and hours worked are countercyclical. Moreover, the convergence of the optimal paths towards equilibrium is accelerated with respect to the baseline model. Similarly, a slightly faster convergence occurs in response to the labor augmenting technology process. In this case all the variables behave pro-cyclically and the optimal convergence paths of real investment, capital rental price, utilization and inflation diverge significantly from the baseline model. With regard to both the government spending shock and the intertemporal preference shock maintenance and utilization are pro-cyclical whereas depreciation is not as far as the positive effects of a higher utilization rate are dominated by the negative effects of an increase in maintenance. The wage mark-up shock generates countercyclical responses both for utilization and maintenance whereas depreciation co-moves with output. On the contrary, all of them behave pro-cyclically in response to a price mark-up shock and to the monetary policy shock. Finally, in the maintenance model the optimal responses to a positive maintenance-specific technology shock of real consumption, investment, effective capital, utilization and depreciation are all countercyclical due to a strong intertemporal substitution effect away from real maintenance. It is found that in long-run this shock explains almost nothing of the variations in the main macroeconomic real variables. However, it becomes the main driver of the real maintenance growth in short run. In fact, the estimation results show that most of the variations in the growth rates of our main real variables are explained by the marginal efficiency of investment technology progress. Specifically, the MEI shock explains more than 60% of variation in output, consumption and investment, and more than 90% of maintenance and depreciation. Variations in hours worked are also driven by this shock which accounts for about 28%. These results are in line with Justiniano et al. (2011) except for consumption growth which, in their model, is driven by the shock to intertemporal preference. Moreover, similarly to them, we have found that the role of investment-specific technology progress is negligible. As stated above, the maintenance-specific technology progress explains in long run only about 0.50% of variations in real maintenance and plays virtually no role in all the remaining variables. In short run, instead, it explains on average more than 25% per period of real maintenance variation over the subsequent five periods after the shock. Overall we have found that our model performs fairly well and, with respect to the baseline model, fits better the estimated theoretical moments of the smoothed endogenous variables in comparison to the respective empirical moments of the actual series.

## Conclusions

In this thesis we have shown that accounting for a time-varying capital depreciation rate and including expenditures for the maintenance of capital among the control variables a Dynamic Stochastic General Equilibrium model is able to generate co-movement of the main real endogenous variables, especially for what it concerns real consumption. The DSGE model in question, in fact, offers an additional transmission mechanism for the propagation of technology shocks throughout the variable depreciation rate which is endogenously determined by the capital utilization rate and the maintenance expenditures. Both the (positive) direct effect of the capital utilization rate operating on the aggregated production process by the means of the hoarding assumption and the (negative) indirect effect operating through the capital accumulation process are thus accounted for. The novelty characterizing our model setting, however, concerns the fact that capital depreciation rate exhibits an increasing trend path. The model's optimal conditions, indeed, rule out a growth rate for depreciation which depends on the investment-specific technology progress weighted by the sensibility of depreciation to changes in maintenance. Over the latest decades it has been observed a persistent decrease in the relative price of investment coupled with an increasing path of the capital depreciation rate. Thus, we capture in our model such a behavior of the depreciation rate basing on the assumption that in a perfectly competitive market the relative price of investment is inversely related to the investment-specific technology progress. The estimation exercises conducted on the Canadian economy show that the qualitative behaviors of capital depreciation rate, maintenance and investment differ in response to the types of the shocks that hit real economy. When considering the shock to marginal efficiency of investment, which is found to be the major driver of the real business cycles explaining more than sixty percent of the long-run output growth, real investment and maintenance behave as complements. Both of them, together with real consumption and the rate of depreciation, co-move with real output. On the contrary, in response to the investment-specific technology progress maintenance and investment act as substitutes. The rate of depreciation, real consumption and maintenance are all pro-cyclical in this case. The labor-augmenting technology shock produces co-movement in all the considered real variables, which thus suggests that maintenance and investment are complements. Our model considers as well a shock that hits the technology of production of maintenance goods, which generates a countercyclical behavior in real consumption, investment and the capital depreciation rate suggesting that, in this case, maintenance and investment act as substitutes. In general, the estimation exercises of our model confirm the hints advanced in the related literature with regard to the qualitative behavior of the capital depreciation rate. In particular, we have shown that there exists a positive correlation between the declining rate of the relative price of investment and the depreciation rate of capital. Moreover, the rate of depreciation is found to be more volatile than investment. However, our model fails to predict the strong pro-cyclicality between depreciation and output.

The model setting implemented in this work may represent a good starting point for the kind of research willing to capture variations in capital depreciation rate in the endogenous growth models. As far as more and more evidence arises about the increasing trend over time of the depreciation rate, our model structure provides useful instruments able to tackle the implied accumulation process of capital stock with the time-dependent depreciation. Moreover, following our model key assumptions, co-movement among the main macroeconomic real variables, such as output, labor productivity and consumption is accomplished. Such a model structure is particularly useful, for example, in the assessment of the economic performance as far as it is used to be captured by the economy's total factor productivity which, in turn, accounts for the capital depreciation rate. The availability of consistent estimates of depreciation are as well crucial for the capital taxation codes. In fact, as it is stressed in, for example, Doms et al. (2004) biased estimates of depreciation may result in misleading tax policy for capital income. Finally, the strategic investment decisions undertaken at the firm level are affected by the path of the depreciation rate especially during technological booms. Tevlin and Whelan, 2003, for example, assess that when an acceleration in the depreciation rate occurs firms need to invest more in order to preserve a given level of capital stock. Therefore, we argue that a variable depreciation rate must be taken in consideration by the model builders, as far as it is able to generate an additional adjustment mechanism in the optimal behavior of the endogenous variables and thus helps to better capture the salient features of the real economy.

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# A APPENDIX: The model of Justiniano et al. (2011)

We present here the steady state relationships and the log-linearized equilibrium expressions of the model Justiniano et al. (2011). Endogenous variables are detrended by the composite technological progress. Variables with no time subscription represent the steady state values, while upper-hat variables are the log deviations from their respective steady state values. Among others, $s_t$ is the detrended real marginal cost, $\rho_t$ is detrended return on capital.

*Steady state relationships*

$$\frac{y}{L} = \left(\frac{k}{L}\right)^{\alpha} - \frac{F}{L}$$

$$s = w^{1-\alpha}\rho^{\alpha}\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$$

$$\frac{k}{L} = \frac{w}{\rho}\frac{\alpha}{1-\alpha}$$

$$s = \frac{1}{1+\lambda_p}$$

$$\lambda = \frac{1}{c}\frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon} - \beta h}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon} - h}$$

$$\beta = \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon}}{R}$$

$$\rho = a'(1)$$

$$\rho = \frac{e^{\gamma_z+(\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon}}{\beta} - (1-\delta)$$

$$k = \bar{k}e^{-(\gamma_z+(\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}$$

$$\bar{k} = \frac{i}{1-e^{-(\gamma_z+(\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}(1-\delta)}$$

$$c + i = \frac{1}{g}y$$

$$x = (1-\frac{1}{g})y + c + i$$

*Log-linearized approximations*

$$\hat{y}_t = \frac{y+F}{y}\alpha\hat{k}_t + \frac{y+F}{y}(1-\alpha)\hat{L}_t$$

$$\hat{k}_t - \hat{L}_t = \hat{w}_t - \hat{\rho}_t$$

$$\hat{s}_t = (1-\alpha)\hat{w}_t + \alpha\hat{\rho}_t$$

$$\hat{\pi}_t = \frac{\iota_p}{1+\beta\iota_p}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta\iota_p}E_t\{\hat{\pi}_{t+1}\} + \frac{(1-\xi_p\beta)(1-\xi_p)}{\xi_p(1+\beta\iota_p)}(\hat{s}_t + \hat{\lambda}_{p,t})$$

$$\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon} - \beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon} - h\right)\hat{\lambda}_t =$$

$$= e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon}h\hat{c}_{t-1} - \left(e^{2(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon)} + \beta h^2\right)\hat{c}_t + e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon}\beta h E_t\{\hat{c}_{t+1}\}+$$

$$+ (\beta h\rho_z - h)\, e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon}\hat{z}_t + (\beta h\rho_\upsilon - h)\, e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon}\frac{\alpha}{1-\alpha}\hat{\upsilon}_t+$$

$$+ \left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon} - \beta h\rho_b\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_\upsilon} - h\right)\hat{b}_t$$

$$\hat{\lambda}_t = \hat{R}_t - \rho_z \hat{z}_t - \rho_\upsilon \frac{\alpha}{1-\alpha}\hat{\upsilon}_t + E_t\{\hat{\lambda}_{t+1} - \hat{\pi}_{t+1}\}$$

$$\hat{\rho}_t = \chi \hat{u}_t$$

$$\hat{\phi}_t = -\rho_z \hat{z}_t - \rho_\upsilon (\frac{\alpha}{1-\alpha}+1)\hat{\upsilon}_t + e^{-(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}\beta(1-\delta)E_t\{\hat{\phi}_{t+1}\}+$$
$$+\left[1 - e^{-(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}\beta(1-\delta)\right]E_t\{(\hat{\rho}_{t+1} + \hat{\lambda}_{t+1})\}$$

$$\hat{\lambda}_t = \hat{\phi}_t + \hat{\mu}_t + e^{2(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}S''(\beta\rho_z - 1)\hat{z}_t + e^{2(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}S''(\beta\rho_\upsilon - 1)\hat{\upsilon}_t +$$
$$-e^{2(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}S''(\beta+1)\hat{i}_t + e^{2(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}S''\hat{i}_{t-1}+$$
$$+e^{2(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}S''\beta E_t\{\hat{i}_{t+1}\}$$

$$\hat{\bar{k}}_t = \hat{u}_t + \hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{\upsilon}_t$$

$$\hat{\bar{k}}_t = e^{-(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}(1-\delta)\left[\hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{\upsilon}_t\right]+$$
$$+\left[1 - e^{-(\gamma_z + (\frac{\alpha}{1-\alpha}+1)\gamma_\upsilon)}(1-\delta)\right]\left(\hat{i}_t + \hat{\mu}_t\right)$$

$$\frac{1}{g}\hat{y}_t - \frac{1}{g}\hat{g}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{\rho k}{y}\hat{u}_t$$

$$\hat{x}_t = \hat{y}_t - \frac{\rho k}{y}\hat{u}_t$$

$$\hat{w}_t = -\frac{1}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\frac{(1-\beta\xi_w)(1-\xi_w)}{\xi_w(1+\beta)}\left[\hat{w}_t - \hat{\lambda}_{w,t} - \left(\nu\hat{L}_t + \hat{b}_t - \hat{\lambda}_t\right)\right]+$$
$$+\frac{1}{1+\beta}\hat{w}_{t-1} + \frac{\beta}{1+\beta}E_t\{\hat{w}_{t+1}\}+$$
$$+\frac{\iota_w}{1+\beta}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta}E_t\{\hat{\pi}_{t+1}\} - \frac{1+\beta\iota_w}{1+\beta}\hat{\pi}_t+$$
$$+\frac{\iota_w}{1+\beta}\hat{z}_{t-1} - \frac{1+\beta\iota_w - \beta\rho_z}{1+\beta}\hat{z}_t+$$
$$+\frac{\iota_w}{1+\beta}\frac{\alpha}{1-\alpha}\hat{\upsilon}_{t-1} - \frac{1+\beta\iota_w - \beta\rho_\upsilon}{1+\beta}\frac{\alpha}{1-\alpha}\hat{\upsilon}_t$$

$$\hat{r}_t = \rho_R \hat{r}_{t-1} + (1-\rho_R)\left[\phi_\pi \hat{\pi}_t + \phi_y\left(\hat{x}_t - \hat{x^\star}_t\right)\right]+$$
$$+\phi_{dy}\left[(x_t - x_{t-1}) - (x_t^\star - x_{t-1}^\star)\right] + mp_t$$

## B APPENDIX: Optimality conditions

In the present section of the appendix we analytically derive all the first order conditions of our model.

*Final good sector*
Recalling the problem setting (P1) we write down the Lagrangian in the following form

$$\mathcal{L} = P_t Y_t - \int_0^1 P_t(i) Y_t(i) di + \Lambda_t \left\{ \left[ \int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{p,t}}} di \right]^{1+\lambda_{p,t}} - Y_t \right\} \tag{B.1}$$

The first order condition (FOC) with respect to $Y_t$ is, therefore

$$\frac{\partial \mathcal{L}}{\partial Y_t} = P_t - \Lambda_t$$

$$P_t - \Lambda_t = 0$$

$$P_t = \Lambda_t \tag{B.2}$$

where the latter expression simply equates the consumption price to its shadow value.

The FOC with respect to $Y_t(i)$, using (B.2), is as follows

$$\frac{\partial \mathcal{L}}{\partial Y_t(i)} = -P_t(i) + \Lambda_t \left[ \int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{p,t}}} di \right]^{1+\lambda_{p,t}-1} Y_t(i)^{\frac{1}{1+\lambda_{p,t}}-1}$$

$$P_t(i) = P_t Y_t(i)^{-\frac{\lambda_{p,t}}{1+\lambda_{p,t}}} \left\{ \left[ \int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{p,t}}} di \right]^{1+\lambda_{p,t}} \right\}^{\frac{\lambda_{p,t}}{1+\lambda_{p,t}}}$$

$$Y_t(i) = \left[ \frac{P_t(i)}{P_t} \right]^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} Y_t \tag{B.3}$$

Expression (B.3) defines the optimal intermediate good demand function, which negatively relates it to the relative price of consumption.

Starting with the zero profit condition and substituting for equation (B.3), the following passages derive the optimal final good aggregate price, $P_t$,

$$P_t Y_t - \int_0^1 P_t(i) \left[ \frac{P_t(i)}{P_t} \right]^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} Y_t di = 0$$

$$P_t - P_t^{\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} \int_0^1 P_t(i) P_t(i)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} di$$

$$P_t = \left[ \int_0^1 P_t(i)^{-\frac{1}{\lambda_{p,t}}} di \right]^{-\lambda_{p,t}} \tag{B.4}$$

*Intermediate good sector*
Recalling problem setting P2 from the main text, the Lagrangian is written as follows

$$\mathcal{L} = P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t(i) + MC_t(i) \left[ A_t^{1-\alpha} K_t(i)^{\alpha} L_t(i)^{1-\alpha} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F - Y_t(i) \right] \tag{B.5}$$

The FOCs with respect to labor, $L_t(i)$, and effective capital $K_t(i)$, are, respectively

$$\frac{\partial \mathcal{L}}{\partial L_t(i)} = -W_t + MC_t(i)\,(1-\alpha)\,A_t^{1-\alpha} K_t(i)^{\alpha} L_t(i)^{-\alpha} = 0$$

$$MC_t(i) = \frac{W_t}{1-\alpha} A_t^{-(1-\alpha)} \left[\frac{K_t(i)}{L_t(i)}\right]^{-\alpha} \tag{B.6}$$

$$\frac{\partial \mathcal{L}}{\partial K_t(i)} = -R_t^k + MC_t(i)\alpha A_t^{1-\alpha} K_t(i)^{\alpha-1} L_t(i)^{1-\alpha} = 0$$

$$MC_t(i) = \frac{R_t^k}{\alpha} A_t^{-(1-\alpha)} \left[\frac{K_t(i)}{L_t(i)}\right]^{1-\alpha} \tag{B.7}$$

where the Lagrangian shadow value, $MC_t(i)$, represents the nominal marginal cost, which is common to all the intermediate sector firms, i.e. $MC_t(i) = MC_t$. This implies that capital to labor ratio is the same among all the intermediate good producers, and it is obtained by combining (B.7) with (B.6), so that

$$\frac{K_t}{L_t} = \frac{W_t}{R_t^k}\frac{\alpha}{1-\alpha} \tag{B.8}$$

Then, substituting back (B.8) into either of the two FOCs for capital and labor we obtain the following expression for nominal marginal cost, $MC_t$,

$$MC_t = \frac{W_t}{1-\alpha} A_t^{-(1-\alpha)} \left[\frac{W_t}{R_t^k}\frac{\alpha}{1-\alpha}\right]^{-\alpha} =$$

$$= \left(\frac{W_t}{A_t}\right)^{1-\alpha} \left(R_t^k\right)^{\alpha} \frac{1}{\alpha^{\alpha}\,(1-\alpha)^{1-\alpha}} \tag{B.9}$$

Next, we can show that, given the duality property of the production function, the nominal marginal cost, $MC_t$, equals the nominal average variable cost, $AVC_t$. Specifically, we substitute the relative demand for capital (B.8) into the aggregate production function and explicit all for labor $L_t(i)$, in order to obtain the following expression

$$L_t(i) = Y_t(i) A_t^{-(1-\alpha)} \left(\frac{W_t}{R_t^k}\frac{\alpha}{1-\alpha}\right)^{-\alpha} + A_t^{-(1-\alpha)} \left(\frac{W_t}{R_t^k}\frac{\alpha}{1-\alpha}\right)^{-\alpha} A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F$$

Then, using the above expression together with (B.8) inside the total cost function $TC_t = W_t L_t(i) + R_t^k K_t(i)$ we obtain

$$TC_t = \left(\frac{W_t}{A_t}\right)^{1-\alpha} \left(R_t^k\right)^{\alpha} \frac{1}{\alpha^{\alpha}\,(1-\alpha)^{1-\alpha}} Y_t(i) + \left(\frac{W_t}{A_t}\right)^{1-\alpha} \left(R_t^k\right)^{\alpha} \frac{1}{\alpha^{\alpha}\,(1-\alpha)^{1-\alpha}} A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F$$

where the first term on the right-hand side of the equation represents the total variable cost, which divided by $Y_t(i)$ gives the average variable cost clearly equal to the nominal marginal cost $MC_t$, given in equation (B.9).

Next, we are going to derive the optimality condition of the fraction $1-\xi_p$ of firms that are allowed to choose their optimal price level in a given period. Note first that the remaining fraction $\xi_p$ of firms that cannot optimally choose their price will reset it accordingly to the following indexation rule

$$P_t(i) = P_{t-1}(i)\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}$$

where $\pi$ is the level of inflation in steady state. Hence, inflation can be defined as follows

$$\pi_t = \frac{P_t(i)}{P_{t-1}(i)} = \pi_{t-1}^{\iota_p}\pi^{1-\iota_p}$$

Re-expressing the above equation for the infinite sum we obtain the total inflation generated from period $t$ to period $t+s$, $\pi_{t,t+s}$

$$\pi_{t,t+s} = \frac{P_{t+s}(i)}{P_t(i)} = \prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}$$

Hence, as far as the average variable cost equals the nominal marginal cost, we can express the present discounted value of future profits as follows

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\left[\tilde{P}_t(i)\left(\prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}\right)Y_{t+s}(i) - W_{t+s}L_{t+s}(i) - R_{t+s}^k K_{t+s}(i)\right]$$

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\left[\tilde{P}_t(i)\left(\prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}\right) - \frac{W_{t+s}L_{t+s}(i) - R_{t+s}^k K_{t+s}(i)}{Y_{t+s}(i)}\right]Y_{t+s}(i)$$

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\left[\tilde{P}_t(i)\left(\prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}\right) - AVC_{t+s}\right]Y_{t+s}(i)$$

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\left[\tilde{P}_t(i)\left(\prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}\right) - MC_{t+s}\right]Y_{t+s}(i)$$

The latter expression is, then, maximized with respect to the optimal price $\tilde{P}_t(i)$ subject to the optimal demand for intermediate good given in problem setting (P3). Namely, we derive the FOC of the following Lagrangian equation, as follows

$$\mathcal{L} = E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\left[\tilde{P}_t(i)\left(\prod_{j=0}^{s}\pi_{t+j-1}^{\iota_p}\pi^{1-\iota_p}\right) - MC_{t+s}\right]\left[\frac{\tilde{P}_t(i)}{P_{t+s}}\pi_{t,t+s}\right]^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}}Y_{t+s} \tag{B.10}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_t(i)} = E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\frac{\Lambda_{t+s}}{\Lambda_t}\Bigg[-\frac{1}{\lambda_{p,t+s}}\tilde{P}_t(i)^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}}\pi_{t,t+s}+$$

$$-MC_{t+s}\left(-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}\right)\tilde{P}_t(i)^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}-1}\Bigg]\left(\frac{\pi_{t,t+s}}{P_{t+s}}\right)^{-\frac{1+\lambda_{p,t+s}}{\lambda_{p,t+s}}}Y_{t+s}$$

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\Lambda_{t+s}\left[\tilde{P}_t(i)\pi_{t,t+s} - (1+\lambda_{p,t+s})\,MC_{t+s}\right]\tilde{Y}_{t+s}(i) = 0 \tag{B.11}$$

Equation (B.11) describes the optimal price setting condition of the firms allowed to choose their price level. Recalling the optimal price function for the intermediate good $i$ in (B.4) and the indexation rule common to the firms that are not optimizing for the price, the aggregate price index is defined as follows

$$P_t = \left[\xi_p \left(P_{t-1}\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\right)^{-\frac{1}{\lambda_{p,t}}} + (1-\xi_p)\,\tilde{P}_t^{-\frac{1}{\lambda_{p,t}}}\right]^{-\lambda_{p,t}} \tag{B.12}$$

*Maintenance good sector*

In order to calculate the optimality conditions in this sector we express the Lagrangian of the problem setting (P4), as follows

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t+s} \left\{ P_{t+s}^m d_{t+s} Y_{t+s}^m \left[1 - f\left(\frac{Y_{t+s}^m}{Y_{t+s-1}^m}\right)\right] - P_{t+s} Y_{t+s}^m \right\} \tag{B.13}$$

The FOC with respect to $Y_t^m$ is computed as follows

$$\frac{\partial \mathcal{L}}{\partial Y_t^m} = -\Lambda_t P_t + \Lambda_t P_t^m d_t \left[1 - f\left(\frac{Y_t^m}{Y_{t-1}^m}\right) - \frac{Y_t^m}{Y_{t-1}^m} f'\left(\frac{Y_t^m}{Y_{t-1}^m}\right)\right] + \beta E_t \left\{ \Lambda_{t+1} d_{t+1} P_{t+1}^m \left(\frac{Y_{t+1}^m}{Y_t^m}\right)^2 f'\left(\frac{Y_{t+1}^m}{Y_t^m}\right) \right\}$$

$$\Lambda_t P_t = \Lambda_t P_t^m d_t \left[1 - f\left(\frac{Y_t^m}{Y_{t-1}^m}\right) - \frac{Y_t^m}{Y_{t-1}^m} f'\left(\frac{Y_t^m}{Y_{t-1}^m}\right)\right] + \beta E_t \left\{ \Lambda_{t+1} d_{t+1} P_{t+1}^m \left(\frac{Y_{t+1}^m}{Y_t^m}\right)^2 f'\left(\frac{Y_{t+1}^m}{Y_t^m}\right) \right\} \tag{B.14}$$

Equation (B.14) defines the optimal condition for maintenance goods supply.

*Investment good sector*

Firms producing investment goods are assumed to maximize with respect to the fraction of final good, $Y_t^I$, which is used for the production of investment goods. Their profit function, $\Pi_t^I$, is given by

$$\Pi_t^I = P_t^I \Upsilon_t Y_t^I - P_t Y_t^I$$

which is obtained simply substituting the technology for production of investment goods into the revenue function in problem setting (P4). Thus, the FOC is derived as follows

$$\frac{\partial \Pi_t^I}{\partial Y_t^I} = P_t^I \Upsilon_t - P_t$$

$$\frac{P_t^I}{P_t} = \Upsilon_t^{-1}$$

where the latter expression defines the relative price of investment as the inverse of the IST progress, $\Upsilon_t$.

*Capital good sector*

Given the problem setting (P5) of the firms producing capital goods, we maximize the following expression of the discounted stream of profits with respect to $I_t$

$$\max_{I_t} \quad E_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t+s} \left[ P_{t+s}^k \mu_{t+s} \left[1 - S\left(\frac{I_{t+s}}{I_{t+s-1}}\right)\right] I_{t+s} - P_{t+s}^I I_{t+s} \right]$$

The following passages describe the analytical derivation of the FOC with respect to $I_t$

$$E_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t+s} \left[ P^k_{t+s}\mu_{t+s} - P^k_{t+s}\mu_{t+s} S\left(\frac{I_{t+s}}{I_{t+s-1}}\right) - P^k_{t+s}\mu_{t+s}\frac{I_{t+s}}{I_{t+s-1}} S'\left(\frac{I_{t+s}}{I_{t+s-1}}\right) - P^I_{t+s}\right] +$$
$$+ E_t \sum_{s=0}^{\infty} \beta^{s+1} \Lambda_{t+s+1} P^k_{t+s+1}\mu_{t+s+1}\left(\frac{I_{t+s+1}}{I_{t+s}}\right)^2 S'\left(\frac{I_{t+s+1}}{I_{t+s}}\right) = 0$$

$$\Lambda_t P^k_t \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right)\right] - \Lambda_t P^I_t + \beta E_t \Big\{\Lambda_{t+1} P^k_{t+1}\mu_{t+1}\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\Big\} = 0$$
$$P^I_t = P^k_t \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right)\right] + \beta E_t \Big\{\frac{\Lambda_{t+1}}{\Lambda_t} P^k_{t+1}\mu_{t+1}\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\Big\} \quad \text{(B.15)}$$

Equation (B.15) defines the optimal supply curve for capital goods.

*Employment agencies sector*

Recalling the problem setting (P6) the Lagrangian is written as follows

$$\mathcal{L} = W_t L_t - \int_0^1 W_t(j) L_t(j) dj + \Lambda_t \Big\{ \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_{w,t}}} dj\right]^{1+\lambda_{w,t}} - L_t \Big\} \quad \text{(B.16)}$$

The FOC with respect to $L_t$ gives $W_t = \Lambda_t$, while, the FOC with respect to $L_t(j)$ is calculated as follows

$$\frac{\partial \mathcal{L}}{\partial L_t(i)} = -W_t(j) + \Lambda_t \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_{w,t}}} dj\right]^{1+\lambda_{w,t}-1} L_t(j)^{\frac{1}{1+\lambda_{w,t}}-1}$$
$$W_t(j) = W_t L_t(j)^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}} \left\{ \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_{w,t}}} dj\right]^{1+\lambda_{w,t}} \right\}^{\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}$$
$$L_t(j) = \left[\frac{W_t(j)}{W_t}\right]^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t \quad \text{(B.17)}$$

Expression (B.17) defines the optimal demand function for heterogeneous labor, which is supplied by the households, and is transformed by the employment agencies into homogeneous labor purchased, thereafter, by the intermediate goods producers at the cost $W_t$.

Starting with the zero profit condition, which implies that $W_t(j)L_t(j) = W_t L_t$, and substituting for equation (B.17), the following passages derive the optimal aggregate wage function, $W_t$,

$$W_t L_t - \int_0^1 W_t(j) \left[\frac{W_t(j)}{W_t}\right]^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t dj = 0$$
$$W_t - W_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \int_0^1 W_t(j) W_t(j)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} dj = 0$$
$$W_t = \left[\int_0^1 W_t(j)^{-\frac{1}{\lambda_{w,t}}} dj\right]^{-\lambda_{w,t}} \quad \text{(B.18)}$$

*Households*

Given the problem setting (P7), we define the Lagrangian as follows

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \beta^s \Bigg\{ b_{t+s} \left[ \log\left(C_{t+s} - hC_{t+s-1}\right) - \varphi \frac{L_{t+s}(j)^{1+\nu}}{1+\nu} \right] +$$

$$+ \Lambda_{t+s} \Bigg[ R_{t+s-1} B_{t+s-1} + Q_{t+s}(j) + \Pi_{t+s} + W_{t+s}(j) L_{t+s}(j) + R^k_{t+s} u_{t+s} \bar{K}_{t+s-1} +$$

$$- \frac{P_{t+s}}{\Upsilon_{t+s}} a(u_{t+s}) \bar{K}_{t+s-1} - P_{t+s} C_{t+s} - P^k_{t+s} i^k_{t+s} - P^m_{t+s} M_{t+s} - T_{t+s} - B_{t+s} \Bigg] + \tag{B.19}$$

$$+ \Phi_{t+s} \left[ \left[ 1 - \Upsilon^{-\sigma}_{t+s-1} \left( \zeta u^{\eta}_{t+s} \left( \frac{M_{t+s}}{\bar{K}_{t+s-1}} \right)^{-\sigma} + \Upsilon^{\sigma}_{t+s} \bar{\delta} \right) \right] \bar{K}_{t+s-1} + i^k_{t+s} - \bar{K}_{t+s} \right] +$$

$$+ \Gamma_{t+s} \left[ \tau \Upsilon^{-1}_{t+s} u_{t+s} \bar{K}_{t+s-1} + A_{t+s} \Upsilon^{\frac{\alpha}{1-\alpha}}_{t+s} \bar{M} - M_{t+s} \right] \Bigg\}$$

Next, we calculate the FOC with respect to current consumption, $C_t$, as follows

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{b_t}{C_t - hC_{t-1}} - \Lambda_t P_t - \beta h E_t \Big\{ \frac{b_{t+1}}{C_{t+1} - hC_t} \Big\}$$

$$\Lambda_t P_t = \frac{b_t}{C_t - hC_{t-1}} - \beta h E_t \Big\{ \frac{b_{t+1}}{C_{t+1} - hC_t} \Big\} \tag{B.20}$$

The FOC with respect to current bonds holdings, $B_t$, is

$$\frac{\partial \mathcal{L}}{\partial B_t} = -\Lambda_t + \beta R_t E_t \Big\{ \Lambda_{t+1} \Big\}$$

$$\Lambda_t = \beta R_t E_t \Big\{ \Lambda_{t+1} \Big\} \tag{B.21}$$

The FOC with respect to current labor, $L_t(j)$, is

$$\frac{\partial \mathcal{L}}{\partial L_t} = -b_t \varphi L_t(j)^{\nu} + \Lambda_t W_t(j)$$

$$\Lambda_t W_t(j) = b_t \varphi L_t(j)^{\nu} \tag{B.22}$$

where the latter expression equates wages in terms of the shadow price for consumption to the marginal utility of labor.

The FOC with respect to new capital, $i^k_t$, is

$$\frac{\partial \mathcal{L}}{\partial i^k_t} = -\Lambda_t P^k_t + \Phi_t$$

$$P^k_t = \Phi_t / \Lambda_t \tag{B.23}$$

which says that, in equilibrium, the price of capital equals the relative shadow value of capital with respect to the shadow value of consumption. Substitute equation (B.23) into the optimal condition for capital supply (B.15) in order to obtain the following expression

$$\Lambda_t P_t^I = \Phi_t \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right)\right] + \beta E_t \Big\{ \Phi_{t+1}\mu_{t+1} \left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right) \Big\} \tag{B.24}$$

The FOC with respect to maintenance, $M_t$, is derived as follows

$$\frac{\partial \mathcal{L}}{\partial M_t} = -\Lambda_t P_t^m - \Phi_t \Upsilon_{t-1}^{-\sigma} \bar{K}_{t-1} \zeta u_t^{\eta} (-\sigma) \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma-1} \frac{1}{\bar{K}_{t-1}} - \Gamma_t$$

$$\Lambda_t P_t^m = \sigma \zeta \Phi_t \Upsilon_{t-1}^{-\sigma} u_t^{\eta} \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma-1} - \Gamma_t$$

which, in terms of gross depreciation rate, is equivalent to

$$\Lambda_t P_t^m = \sigma \Phi_t \Upsilon_{t-1}^{-\sigma} \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-1} \left(D_t - \Upsilon_t^{\sigma} \bar{\delta}\right) - \Gamma_t \tag{B.25}$$

So, in equilibrium, the demand for maintenance goods is negatively related to its price, $P_t^m$, and positively related to the gross depreciation rate of capital. Moreover, as far as we have imposed the conditions on the parameters $\sigma > 0$, $0 < \zeta < 1$, and $\eta > 1$, the former equation suggests that maintenance demand is also positively related to the capital utilization rate, $u_t$.

Substituting equation (B.25) into (B.14) we define the following optimal condition for maintenance goods

$$\Lambda_t P_t = \left[\sigma \Phi_t \Upsilon_{t-1}^{-\sigma} \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-1} \left(D_t - \Upsilon_t^{\sigma} \bar{\delta}\right) - \Gamma_t\right] d_t \left[1 - f\left(\frac{Y_t^m}{Y_{t-1}^m}\right) - \frac{Y_t^m}{Y_{t-1}^m} f'\left(\frac{Y_t^m}{Y_{t-1}^m}\right)\right] +$$

$$+ \beta E_t \Big\{ \left[\sigma \Phi_{t+1} \Upsilon_t^{-\sigma} \left(\frac{M_{t+1}}{\bar{K}_t}\right)^{-1} \left(D_{t+1} - \Upsilon_{t+1}^{\sigma} \bar{\delta}\right) - \Gamma_{t+1}\right] d_{t+1} \left(\frac{Y_{t+1}^m}{Y_t^m}\right)^2 f'\left(\frac{Y_{t+1}^m}{Y_t^m}\right) \Big\} \tag{B.26}$$

A representative household, further more, chooses the level of capital utilization rate, $u_t$. The optimal condition for utilization is derived as follows

$$\frac{\partial \mathcal{L}}{\partial u_t} = \Lambda_t R_t^k \bar{K}_{t-1} - \Lambda_t P_t \Upsilon_t^{-1} a'(u_t) \bar{K}_{t-1} - \zeta \eta \Phi_t \Upsilon_{t-1}^{-\sigma} \bar{K}_{t-1} \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} u_t^{\eta-1} + \tau \Gamma_t \Upsilon_t^{-1}$$

$$\Lambda_t R_t^k = \Lambda_t P_t \Upsilon_t^{-1} a'(u_t) + \zeta \eta \Phi_t \Upsilon_{t-1}^{-\sigma} \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} u_t^{\eta-1} + \tau \Gamma_t \Upsilon_t^{-1}$$

Substituting back our assumed depreciation rate function, the latter expression becomes

$$\Lambda_t R_t^k = \Lambda_t P_t \Upsilon_t^{-1} a'(u_t) + \eta \Phi_t \Upsilon_{t-1}^{-\sigma} u_t^{-1} \left(D_t - \Upsilon_t^{\sigma} \bar{\delta}\right) + \tau \Gamma_t \Upsilon_t^{-1} \tag{B.27}$$

Next, we derive the FOC with respect to capital stock at time $t$, $\bar{K}_t$, as follows

$$\frac{\partial \mathcal{L}}{\partial \bar{K}_t} = \beta \Lambda_{t+1} R_{t+1}^k u_{t+1} - \beta \Lambda_{t+1} P_{t+1} \Upsilon_{t+1}^{-1} a(u_{t+1}) - \Phi_t + \beta \Phi_{t+1} \left(1 - \Upsilon_t^{-\sigma} D_{t+1}\right) +$$

$$+\beta\tau\Gamma_{t+1}\Upsilon_{t+1}^{-1}u_{t+1}-\beta\Phi_{t+1}\Upsilon_t^{-\sigma}\bar{K}_t\zeta\left(-\sigma\right)u_{t+1}^{\eta}\left(\frac{M_{t+1}}{\bar{K}_t}\right)^{-\sigma-1}\left(-\frac{M_{t+1}}{\bar{K}_t^2}\right)$$

Rearranging and substituting for the equation of the depreciation rate in the latter term on the right-hand side of the above equation, we obtain the following optimal expression for the demand of capital stock

$$\Phi_t = \beta E_t\Bigg\{\Lambda_{t+1}\left[R_{t+1}^k u_{t+1}-P_{t+1}\Upsilon_{t+1}^{-1}a\left(u_{t+1}\right)\right]+\Phi_{t+1}\left(1-\Upsilon_t^{-\sigma}D_{t+1}\right)+$$
$$+\tau\Gamma_{t+1}\Upsilon_{t+1}^{-1}u_{t+1}-\sigma\Phi_{t+1}\Upsilon_t^{-\sigma}\left(D_{t+1}-\Upsilon_{t+1}^{\sigma}\bar{\delta}\right)\Bigg\} \tag{B.28}$$

Finally, the fraction $1-\xi_w$ of representative households optimally set the level of their wages by optimizing with respect to $\tilde{W}_t(j)$ the following expression of the discounted stream of future earnings, given in the problem setting (P8)

$$\max_{\tilde{W}_t(j)} \quad E_t\sum_{s=0}^{\infty}\beta^s\xi_w^s\Bigg\{\Lambda_{t+s}\tilde{W}_t(j)^{1-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}\left(\frac{\pi_{t,t+s}^w}{W_{t+s}}\right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}L_{t+s}\pi_{t,t+s}^w+$$
$$-b_{t+s}\frac{\varphi}{1+\nu}\tilde{W}_t(j)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}(1+\nu)}\left[\left(\frac{\pi_{t,t+s}^w}{W_{t+s}}\right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}L_{t+s}\right]^{1+\nu}\Bigg\}$$

The FOC is, therefore

$$E_t\sum_{s=0}^{\infty}\beta^s\xi_w^s\Bigg\{-\frac{1}{\lambda_{w,t}}\Lambda_{t+s}\tilde{W}_t(j)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}\left(\frac{\pi_{t,t+s}^w}{W_{t+s}}\right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}L_{t+s}\pi_{t,t+s}^w+$$
$$-b_{t+s}\varphi\left(-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}\right)\tilde{W}_t(j)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}(1+\nu)-1}\left[\left(\frac{\pi_{t,t+s}^w}{W_{t+s}}\right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}L_{t+s}\right]^{1+\nu}\Bigg\}=0$$

Which, rearranging, gives the following optimal wage setting condition

$$E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\Lambda_{t+s}\tilde{L}_{t+s}(j)\left[\pi_{t,t+s}^w\tilde{W}_t(j)-b_{t+s}\varphi\left(1+\lambda_{w,t+s}\right)\frac{\tilde{L}_{t+s}(j)^{\nu}}{\Lambda_{t+s}}\right]=0 \tag{B.29}$$

Given the indexation rule followed by the fraction $\xi_w$ of households, who does not optimize for wages, as

$$W_t(j)=W_{t-1}(j)\left(\pi_{t-1}e^{z_{t-1}+\frac{\alpha}{1-\alpha}v_{t-1}}\right)^{\iota_w}\left(\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)^{1-\iota_w}$$

The aggregate wage index is straightforward

$$W_t=\left(\xi_w\left[W_{t-1}\left(\pi_{t-1}e^{z_{t-1}+\frac{\alpha}{1-\alpha}v_{t-1}}\right)^{\iota_w}\left(\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)^{1-\iota_w}\right]^{-\frac{1}{\lambda_{w,t}}}+(1-\xi_w)\tilde{W}_t(j)^{-\frac{1}{\lambda_{w,t}}}\right)^{-\lambda_{w,t}} \tag{B.30}$$

## C APPENDIX: Trends

In this section of the appendix we will analytically derive the expressions of the trends of our main endogenous variables. We will denote by $g$ the growth rates of the respective enodgenous variables of our model.

*Intermediate good production function*

Recall the aggregate production technology of the intermediate good, that is

$$Y_t(i) = max\Big\{A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha} - A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}F;0\Big\}$$

Assume a representative firm decides not to produce then, the aggregate production technology becomes

$$A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha} - A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}F = 0$$

Differentiating the latter expression with respect to time and assuming that hours worked, $L_t$, exhibit no growth over time, we obtain

$$(1-\alpha)\,A_t^{1-\alpha-1}K_t^{\alpha}L_t^{1-\alpha}\frac{\partial A_t}{\partial t} + \alpha A_t^{1-\alpha}K_t^{\alpha-1}L_t^{1-\alpha}\frac{\partial K_t}{\partial t} - \Upsilon_t^{\frac{\alpha}{1-\alpha}}F\frac{\partial A_t}{\partial t} - \frac{\alpha}{1-\alpha}A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}-1}F\frac{\partial \Upsilon_t}{\partial t} = 0$$

$$(1-\alpha)\,A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha}g_A + \alpha A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha}g_K - A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}Fg_A - \frac{\alpha}{1-\alpha}A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}Fg_\Upsilon = 0$$

$$A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha}\left[(1-\alpha)\,g_A + \alpha g_K\right] - A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}F\left[g_A + \frac{\alpha}{1-\alpha}g_\Upsilon\right] = 0$$

$$(1-\alpha)\,g_A + \alpha g_K = g_A + \frac{\alpha}{1-\alpha}g_\Upsilon$$

$$g_K = g_A + \left(\frac{\alpha}{1-\alpha} + 1\right)g_\Upsilon \tag{C.1}$$

Thus, the growth rate of effective capital, $g_K$, is given by a linear combination of the growth rates of neutral technology progress, $g_A$, and the investment- specific technology progress, $g_\Upsilon$, which, in steady state, are given by $\gamma_z$ and $\gamma_v$, respectively.

On the contrary, when $Y_t(i) > 0$, differentiating with respect to time we obtain

$$\frac{\partial Y(i)_t}{\partial t} = (1-\alpha)\,A_t^{1-\alpha-1}K_t^{\alpha}L_t^{1-\alpha}\frac{\partial A_t}{\partial t} + \alpha A_t^{1-\alpha}K_t^{\alpha-1}L_t^{1-\alpha}\frac{\partial K_t}{\partial t} - \Upsilon_t^{\frac{\alpha}{1-\alpha}}F\frac{\partial A_t}{\partial t} - \frac{\alpha}{1-\alpha}A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}-1}F\frac{\partial \Upsilon_t}{\partial t}$$

$$Y_t(i)g_Y = A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha}\left[(1-\alpha)\,g_A + \alpha g_K\right] - A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}F\left[g_A + \frac{\alpha}{1-\alpha}g_\Upsilon\right]$$

Using (C.1) in the above expression, it follows that

$$Y_t(i)g_Y = A_t^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha}\left[g_A + \frac{\alpha}{1-\alpha}g_\Upsilon\right] - A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}F\left[g_A + \frac{\alpha}{1-\alpha}g_\Upsilon\right]$$

$$g_Y = g_A + \frac{\alpha}{1-\alpha}g_\Upsilon \tag{C.2}$$

The growth rate of investment in efficiency units, $I_t$, can be found by exploiting the respective production technology

$$I_t = \Upsilon_t Y_t^I$$

$$\frac{\partial I_t}{\partial t} = Y_t^I \frac{\partial \Upsilon_t}{\partial t} + \Upsilon_t \frac{\partial Y_t^I}{\partial t}$$

$$I_t g_I = \Upsilon_t Y_t^I g_\Upsilon + \Upsilon_t Y_t^I g_Y$$

$$g_I = g_\Upsilon + g_Y$$

$$g_I = g_K = g_A + \left(\frac{\alpha}{1-\alpha} + 1\right) g_\Upsilon \tag{C.3}$$

Given the definition of capital utilization rate, and assuming that the capital utilization rate, $u_t$, does not grow over time, it follows that

$$K_t = u_t \bar{K}_{t-1}$$

$$\frac{\partial K_t}{\partial t} = u_t \frac{\partial \bar{K}_{t-1}}{\partial t}$$

$$K_t g_K = u_t \bar{K}_{t-1} g_{\bar{K}}$$

$$g_K = g_{\bar{K}} = g_A + \left(\frac{\alpha}{1-\alpha} + 1\right) g_\Upsilon \tag{C.4}$$

Following our assumptions with respect to the maintenance cost function we find the growth rate of maintenance as follows

$$M_t = \tau \Upsilon_t^{-1} u_t \bar{K}_{t-1} + A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M}$$

$$\frac{\partial M_t}{\partial t} = -\tau \Upsilon_t^{-2} u_t \bar{K}_{t-1} \frac{\partial \Upsilon_t}{\partial t} + \tau \Upsilon_t^{-1} u_t \frac{\partial \bar{K}_{t-1}}{\partial t} + \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M} \frac{\partial A_t}{\partial t} + \frac{\alpha}{1-\alpha} A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}-1} \bar{M} \frac{\partial \Upsilon_t}{\partial t}$$

$$M_t g_M = -\tau \Upsilon_t^{-1} u_t \bar{K}_{t-1} g_\Upsilon + \tau \Upsilon_t^{-1} u_t \bar{K}_{t-1} g_{\bar{K}} + A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M} g_A + \frac{\alpha}{1-\alpha} A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M} g_\Upsilon$$

$$g_M = g_Y = g_A + \frac{\alpha}{1-\alpha} g_\Upsilon \tag{C.5}$$

Recall the following functional form for the gross rate of depreciation, $D_t$,

$$D_t = \zeta u_t^\eta \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} + \Upsilon_t^\sigma \bar{\delta}$$

By the total differentiation of the above expression we obtain

$$\frac{\partial D_t}{\partial t} = -\sigma \zeta u_t^\eta \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} \left(\frac{1}{M_t}\frac{\partial M_t}{\partial t} - \frac{1}{\bar{K}_{t-1}}\frac{\partial \bar{K}_{t-1}}{\partial t}\right) + \sigma \bar{\delta} \Upsilon_t^{\sigma-1} \frac{\partial \Upsilon_t}{\partial t}$$

$$D_t g_\delta = \left[\zeta u_t^\eta \left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} + \Upsilon_t^\sigma \bar{\delta}\right] \sigma g_\gamma$$

$$g_\delta = \sigma g_\Upsilon \tag{C.6}$$

Therefore, given our model assumptions, equation (C.6) shows that the gross capital depreciation rate exhibits a trend, which follows the IST shock, $\Upsilon_t$ and it moreover depends on the sensibility of the depreciation function with respect to maintenance to capital ratio, $\sigma$.

We next find the growth rate of new investment goods, $i_t^k$, using the expression for capital accumulation

$$\bar{K}_t = \left(1 - \Upsilon_{t-1}^{-\sigma} D_t\right) \bar{K}_{t-1} + i_t^K$$

$$\frac{\partial \bar{K}_t}{\partial t} = \left(1 - \Upsilon_{t-1}^{-\sigma} D_t\right) \frac{\bar{K}_{t-1}}{\partial t} - \sigma \Upsilon_{t-1}^{-\sigma-1} D_t \bar{K}_{t-1} \frac{\partial \Upsilon_{t-1}}{\partial t} - \Upsilon_{t-1}^{-\sigma} \bar{K}_{t-1} \frac{\partial D_t}{\partial t} + \frac{\partial i_t^K}{\partial t}$$

$$\bar{K}_t g_{\bar{K}} = \left(1 - \Upsilon_{t-1}^{-\sigma} D_t\right) \bar{K}_{t-1} g_{\bar{K}} + \sigma \Upsilon_{t-1}^{-\sigma} D_t \bar{K}_{t-1} g_\Upsilon - \Upsilon_{t-1}^{-\sigma} D_t \bar{K}_{t-1} g_\delta + i_t^K g_{i^K}$$

$$\bar{K}_t g_{\bar{K}} = \left(1 - \Upsilon_{t-1}^{-\sigma} D_t\right) \bar{K}_{t-1} g_{\bar{K}} + i_t^K g_{i^K}$$

$$g_{i^K} = g_K = g_{\bar{K}} = g_I = g_A + \left(\frac{\alpha}{1-\alpha} + 1\right) g_\Upsilon \tag{C.7}$$

Recalling the aggregate resource constraint, we find the growth rate of consumption as follows

$$C_t + \Upsilon_t^{-1} I_t + \Upsilon_t^{-1} a(u_t) \bar{K}_{t-1} = \frac{Y_t}{g_t}$$

where $g_t$ is a stationary government spending shock.

$$\frac{\partial C_t}{\partial t} - \Upsilon_t^{-2} I_t \frac{\partial \Upsilon_t}{\partial t} + \Upsilon_t^{-1} \frac{\partial I_t}{\partial t} - \Upsilon_t^{-2} a(u_t) \bar{K}_{t-1} \frac{\partial \Upsilon_t}{\partial t} + \Upsilon_t^{-1} a(u_t) \frac{\partial \bar{K}_{t-1}}{\partial t} = \frac{1}{g_t} \frac{\partial Y_t}{\partial t}$$

$$C_t g_C - \Upsilon_t^{-1} I_t g_\Upsilon + \Upsilon_t^{-1} I_t g_I - \Upsilon_t^{-1} a(u_t) \bar{K}_{t-1} g_\Upsilon + \Upsilon_t^{-1} a(u_t) \bar{K}_{t-1} g_{\bar{K}} = \frac{1}{g_t} Y_t g_Y$$

$$C_t g_C - \Upsilon_t^{-1} I_t \left[g_\Upsilon - g_I\right] - \Upsilon_t^{-1} a(u_t) \bar{K}_{t-1} \left[g_\Upsilon - g_{\bar{K}}\right] = \frac{1}{g_t} Y_t g_Y$$

$$C_t g_C = \left[\frac{1}{g_t} Y_t - \Upsilon_t^{-1} I_t - \Upsilon_t^{-1} a(u_t) \bar{K}_{t-1}\right] \left(g_A + \frac{\alpha}{1-\alpha} g_\Upsilon\right)$$

$$g_C = g_M = g_Y = g_A + \frac{\alpha}{1-\alpha} g_\Upsilon \tag{C.8}$$

which implies that actual GDP, $X_t$, grows at the same rate

$$g_X = g_C = g_M = g_Y = g_A + \frac{\alpha}{1-\alpha} g_\Upsilon \tag{C.9}$$

## D APPENDIX: Stationary equilibria and steady states

Given the normalization conditions obtained in the previous section of the appendix, we derive here the model equilibrium conditions in terms of stationary variables.

*Intermediate good sector*
Recalling the production function of the intermediate goods producers

$$Y_t(i) = A_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F$$

we obtain its stationary expression as follows

$$Y_t(i)\frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} = A_t^{1-\alpha}\left[K_t(i)\frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}\right]^\alpha L_t(i)^{1-\alpha} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F$$

$$y_t(i) A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} = A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} k_t(i)^\alpha L_t(i)^{1-\alpha} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} F$$

$$y_t(i) = k_t(i)^\alpha L_t(i)^{1-\alpha} - F \tag{D.1}$$

The steady state relation of equation (D.1) is

$$y = k^\alpha L^{1-\alpha} - F$$

The zero profit condition implies

$$y - \rho k - wL = k^\alpha L^{1-\alpha} - F - \rho k - wL = 0$$

$$\left(\frac{k}{L}\right)^\alpha \frac{L}{L} - \frac{F}{L} - \rho\frac{k}{L} - w = 0$$

$$\left(\frac{k}{L}\right)^\alpha - \frac{F}{L} = \frac{y}{L} \tag{D.2}$$

The detrended optimal relative demand for capital given in (B.8) is obtained as follows

$$\frac{K_t}{L_t}\ \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}\ \frac{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}\ \frac{P_t \Upsilon_t^{-1}}{P_t \Upsilon_t^{-1}}$$

$$\frac{k_t}{L_t}\ A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1} = \frac{\alpha}{1-\alpha}\frac{w_t}{\rho_t}\ \frac{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_t \Upsilon_t^{-1}}$$

$$\frac{k_t}{L_t} = \frac{\alpha}{1-\alpha}\frac{w_t}{\rho_t} \tag{D.3}$$

The latter expression, in steady state, becomes

$$\frac{k}{L} = \frac{\alpha}{1-\alpha}\frac{w}{\rho} \tag{D.4}$$

Given the expression for the nominal marginal cost (B.9), call $s$ the stationary real marginal cost, which is found as follows

$$MC_t \frac{P_t}{P_t} = \left( \frac{W_t}{A_t} \frac{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \right)^{1-\alpha} \left( R_t^k \frac{P_t \Upsilon_t^{-1}}{P_t \Upsilon_t^{-1}} \right)^{\alpha} \frac{1}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}}$$

$$s_t P_t = \frac{1}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}} w_t^{1-\alpha} \rho_t^{\alpha} P_t^{1-\alpha} \Upsilon_t^{\alpha} P_t^{\alpha} \Upsilon_t^{-\alpha}$$

$$s_t = \frac{1}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}} w_t^{1-\alpha} \rho_t^{\alpha} \tag{D.5}$$

The steady state expression of the detrended real marginal cost is as follows

$$s = \frac{1}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}} w^{1-\alpha} \rho^{\alpha} \tag{D.6}$$

We derive as follows the stationary equilibrium of the optimal price setting condition given in (B.11)

$$E_t \sum_{s=0}^{\infty} \xi_p^s \beta^s \Lambda_{t+s} \frac{P_{t+s} A_{t+s} \Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}{P_{t+s} A_{t+s} \Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}} \left[ \tilde{P}_t(i) \frac{P_t(i)}{P_t(i)} \pi_{t,t+s} - (1+\lambda_{p,t+s}) MC_{t+s} \frac{P_{t+s}}{P_{t+s}} \right] \tilde{Y}_{t+s}(i) \frac{A_{t+s} \Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}{A_{t+s} \Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}} = 0$$

$$E_t \sum_{s=0}^{\infty} \xi_p^s \beta^s \lambda_{t+s} \left[ \tilde{p}_t(i) \pi_{t,t+s} \frac{P_t(i)}{P_{t+s}} - (1+\lambda_{p,t+s}) s_{t+s} \right] \tilde{y}_{t+s}(i) = 0$$

$$E_t \sum_{s=0}^{\infty} \xi_p^s \beta^s \lambda_{t+s} \left[ \tilde{p}_t(i) \tilde{\pi}_{t,t+s} - (1+\lambda_{p,t+s}) s_{t+s} \right] \tilde{y}_{t+s}(i) = 0 \tag{D.7}$$

where the stationary inflation, $\tilde{\pi}_{t,t+s}$, is derived as follows

$$\pi_{t,t+s} = \prod_{j=0}^{s} \pi_{t+j-1}^{\iota_p} \pi^{1-\iota} = \prod_{j=0}^{s} \left( \frac{\pi_{t+j-1}}{\pi} \right)^{\iota_p} \pi$$

so,

$$\tilde{\pi}_{t,t+s} = \prod_{j=0}^{s} \left( \frac{\pi_{t+j-1}}{\pi} \right)^{\iota_p} \pi \frac{P_t(i)}{P_{t+j}} = \prod_{j=0}^{s} \left( \frac{\pi_{t+j-1}}{\pi} \right)^{\iota_p} \left( \frac{\pi_{t+j}}{\pi} \right)^{-1}$$

Given that, in steady state $\tilde{p} = \tilde{P}/P = 1$, and, therefore, $\tilde{\pi}_{t,t+s} = 1$, expression (D.7), in steady state, becomes

$$\xi_p \beta \lambda \left[ \tilde{p} \tilde{\pi} - (1+\lambda_p) s \right] \tilde{y} = 0$$

$$1 - (1+\lambda_p) s = 0$$

$$s = \frac{1}{1+\lambda_p} \tag{D.8}$$

The aggregate price index given in (B.12), in terms of stationary variables, becomes

$$P_t = \left[ \xi_p \left( P_{t-1} \frac{P_t}{P_t} \pi_{t-1}^{\iota_p} \pi^{1-\iota_p} \right)^{-\frac{1}{\lambda_{p,t}}} + (1-\xi_p) \left( \tilde{P}_t \frac{P_t}{P_t} \right)^{-\frac{1}{\lambda_{p,t}}} \right]^{-\lambda_{p,t}}$$

$$P_t = \left[ P_t^{-\frac{1}{\lambda_{p,t}}} \xi_p \left( \frac{P_{t-1}}{P_t} \pi_{t-1}^{\iota_p} \pi^{1-\iota_p} \right)^{-\frac{1}{\lambda_{p,t}}} + P_t^{-\frac{1}{\lambda_{p,t}}} (1-\xi_p) \left( \frac{\tilde{P}_t}{P_t} \right)^{-\frac{1}{\lambda_{p,t}}} \right]^{-\lambda_{p,t}}$$

$$1 = \left\{ \xi_p \left[ \left( \frac{\pi_{t-1}}{\pi} \right)^{\iota_p} \left( \frac{\pi_t}{\pi} \right)^{-1} \right]^{-\frac{1}{\lambda_{p,t}}} + (1-\xi_p) \tilde{p}_t^{-\frac{1}{\lambda_{p,t}}} \right\}^{-\lambda_{p,t}} \tag{D.9}$$

which in steady state reduces to

$$1 = \left\{ \xi_p \left[ (1)^{\iota_p} (1)^{-1} \right]^{-\frac{1}{\lambda_{p,t}}} + (1-\xi_p) 1^{-\frac{1}{\lambda_{p,t}}} \right\}^{-\lambda_{p,t}}$$

$$1 = (\xi_p + 1 - \xi_p)^{-\lambda_{p,t}} = 1$$

*Households*

The marginal utility for consumption given in (B.20) is detrended as follows

$$\Lambda_t P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} = A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \frac{b_t}{C_t - hC_{t-1}} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \beta h E_t \left\{ \frac{b_{t+1}}{C_{t+1} - hC_t} \right\}$$

$$\lambda_t = A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \frac{b_t}{\frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} C_t - h \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}} C_{t-1}} - A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \beta h E_t \left\{ \frac{b_{t+1}}{\frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}} C_{t+1} - h \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} C_t} \right\}$$

$$\lambda_t = \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}} \frac{b_t}{\frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}} c_t - h c_{t-1}} - \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \beta h E_t \left\{ \frac{b_{t+1}}{\frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} c_{t+1} - h c_t} \right\}$$

Now, note that, the growth rates of the two non stationary shocks, i.e. the IST shock, $v_t$ and the labor-augmenting technology shock, $z_t$, can be expressed as follows, respectively

$$v_t = \Delta \log \Upsilon_t = \log \Upsilon_t - \log \Upsilon_{t-1} = \log \frac{\Upsilon_t}{\Upsilon_{t-1}}$$

$$z_t = \Delta \log A_t = \log A_t - \log A_{t-1} = \log \frac{A_t}{A_{t-1}}$$

which implies that

$$\frac{\Upsilon_t}{\Upsilon_{t-1}}^{\frac{\alpha}{1-\alpha}} = e^{\frac{\alpha}{1-\alpha} v_t}$$

$$\frac{A_t}{A_{t-1}} = e^{z_t}$$

Therefore, the detrended marginal utility of consumption becomes

$$\lambda_t = \frac{e^{z_t + \frac{\alpha}{1-\alpha} v_t} b_t}{e^{z_t + \frac{\alpha}{1-\alpha} v_t} c_t - h c_{t-1}} - \beta h E_t \left\{ \frac{b_{t+1}}{e^{z_{t+1} + \frac{\alpha}{1-\alpha} v_{t+1}} c_{t+1} - h c_t} \right\} \tag{D.10}$$

Next, note that the steady state expressions of the intertemporal preference shock, $b_t$, and of the growth rates of the IST shock, $v_t$, and the labor-augmenting technology shock, $z_t$, are, respectively

$$\log b_t = \rho_b \log b_{t-1} + \varepsilon_{b,t} \qquad \Rightarrow \qquad b = e^{\varepsilon_b} = e^0 = 1,$$
$$v_t = (1-\rho_v)\,\gamma_v + \rho_v v_{t-1} + \varepsilon_{v,t} \qquad \Rightarrow \qquad e^v = e^{\gamma_v},$$
$$z_t = (1-\rho_z)\,\gamma_z + \rho_z z_{t-1} + \varepsilon_{z,t} \qquad \Rightarrow \qquad e^z = e^{\gamma_z}$$

Therefore, equation (D.10) in steady state becomes

$$\lambda = \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc} - \beta h \frac{1}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc}$$
$$\lambda c = \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h} \tag{D.11}$$

The stationary Euler equation (B.21) is calculated as follows

$$\Lambda_t P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} = P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \beta R_t E_t \left\{ \frac{P_{t+1}A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{P_{t+1}A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}} \Lambda_{t+1} \right\}$$
$$\lambda_t = \beta R_t E_t \left\{ \lambda_{t+1} e^{-\left(z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}\right)} \pi_{t+1}^{-1} \right\} \tag{D.12}$$

which in steady state becomes

$$\beta = \frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{R} \tag{D.13}$$

The optimal capital supply condition (B.24), given that $P_t^I = P_t \Upsilon_t^{-1}$, is detrended as follows

$$\Lambda_t P_t \Upsilon_t^{-1} A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} = A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \Phi_t \mu_t \left[ 1 - S\left( \frac{I_t}{I_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} \right) + \right.$$
$$\left. - \frac{I_t}{I_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} S'\left( \frac{I_t}{I_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} \right) \right] +$$
$$+ A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \beta E_t \left\{ \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}} \Phi_{t+1}\mu_{t+1} \left( \frac{I_{t+1}}{I_t} \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \right)^2 \times \right.$$
$$\left. \times S'\left( \frac{I_{t+1}}{I_t} \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \right) \right\}$$
$$\lambda_t = \phi_t \mu_t \left[ 1 - S\left( \frac{i_t}{i_{t-1}} e^{z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t} \right) - \frac{i_t}{i_{t-1}} e^{z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t} S'\left( \frac{i_t}{i_{t-1}} e^{z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t} \right) \right] +$$
$$+ \beta E_t \left\{ \phi_{t+1}\mu_{t+1} e^{-\left(z_{t+1} + \left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)} \left( \frac{i_{t+1}}{i_t} e^{z_{t+1} + \left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}} \right)^2 S'\left( \frac{i_{t+1}}{i_t} e^{z_{t+1} + \left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}} \right) \right\} \tag{D.14}$$

Given that, in steady state $S = S' = 0$, $S'' > 0$, and that the MEI shock equals unity, i.e. $\mu = 1$, equation (D.14), in steady state, becomes

$$\lambda = \phi \left[ 1 - 0 - 0 \right] + \beta * 0$$

$$\lambda = \phi \tag{D.15}$$

Before detrending the optimal condition for maintenance goods, note that, the zero profit condition of the maintenance goods producing sector implies that

$$P_t^m M_t = P_t Y_t^m$$

$$Y_t^m = \frac{P_t^m}{P_t} M_t$$

where $Y_t^m = \tilde{M}_t$ represents the real maintenance in consumption units. Therefore, detrending the production technology for maintenance we obtain

$$M_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} = d_t Y_t^m \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \left[ 1 - f\left( \frac{Y_t^m}{Y_{t-1}^m} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \right) \right]$$

$$m_t = d_t \tilde{m}_t \left[ 1 - f\left( \frac{\tilde{m}_t}{\tilde{m}_{t-1}} e^{z_t + \frac{\alpha}{1-\alpha} v_t} \right) \right] \tag{D.16}$$

which, recalling that in steady state $f = f' = 0$ and $d = 1$, gives the following identity in steady state

$$m = \tilde{m}$$

Hence, we calculate the detrended counterpart of the maintenance optimal condition, (B.26), as follows

$$\Lambda_t P_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} = \left[ \sigma \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \Phi_t \Upsilon_{t-1}^{-\sigma} \left( \frac{M_t}{\bar{K}_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} \right)^{-1} \left( D_t \frac{\Upsilon_t^\sigma}{\Upsilon_t^\sigma} - \Upsilon_t^\sigma \bar{\delta} \right) - \Gamma_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \right] \times$$

$$\times\, d_t \left[ 1 - f\left( \frac{\tilde{M}_t}{\tilde{M}_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}} \right) - \frac{\tilde{M}_t}{\tilde{M}_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}} f' \left( \frac{\tilde{M}_t}{\tilde{M}_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}} \right) \right] +$$

$$+\, \beta E_t \Big\{ \Big[ \sigma \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}} \Phi_{t+1} \Upsilon_t^{-\sigma} \left( \frac{M_{t+1}}{\bar{K}_t} \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \right)^{-1} \left( \frac{\Upsilon_{t+1}^\sigma}{\Upsilon_{t+1}^\sigma} D_{t+1} - \Upsilon_{t+1}^\sigma \bar{\delta} \right) +$$

$$-\, \Gamma_{t+1} \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}} \Big] \times d_{t+1} \left( \frac{\tilde{M}_{t+1}}{\tilde{M}_t} \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \right)^2 f' \left( \frac{\tilde{M}_{t+1}}{\tilde{M}_t} \frac{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_{t+1} \Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} \right) \Big\}$$

which becomes

$$\lambda_t = \left[ \sigma \phi_t \left( \frac{m_t}{\bar{k}_{t-1}} e^{z_t + \left( \frac{\alpha}{1-\alpha} + 1 \right) v_t} \right)^{-1} \left( \delta_t - \bar{\delta} \right) e^{\sigma v_t} - \varsigma_t \right] \times$$

$$\times\, d_t\left[1 - f\left(\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}\right) - \frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}f'\left(\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}\right)\right]+ \tag{D.17}$$

$$+\,\beta E_t\Big\{\left[\sigma\phi_{t+1}\left(\frac{m_{t+1}}{\bar{k}_t}e^{z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}}\right)^{-1}\left(\delta_{t+1}-\bar{\delta}\right)e^{\sigma v_{t+1}} - \varsigma_{t+1}\right]d_{t+1}e^{-\left(z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}\right)}\times$$

$$\times\left(\frac{\tilde{m}_{t+1}}{\tilde{m}_t}e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\right)^2 f'\left(\frac{\tilde{m}_{t+1}}{\tilde{m}_t}e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\right)\Big\}$$

The above expression in steady state becomes

$$\lambda = \sigma\phi e^{\sigma\gamma_v}\left(\delta-\bar{\delta}\right)\left[\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right]^{-1} - \varsigma \tag{D.18}$$

Next, we detrend the optimal capital utilization rate condition, (B.27), that is

$$\Lambda_t R_t^k \frac{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_t A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}\frac{P_t\Upsilon_t^{-1}}{P_t\Upsilon_t^{-1}} = \Lambda_t P_t \frac{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}\Upsilon_t^{-1}a'\left(u_t\right)+$$

$$+\,\eta\Phi_t\frac{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}\Upsilon_{t-1}^{-\sigma}u_t^{-1}\left(D_t\frac{\Upsilon_t^\sigma}{\Upsilon_t^\sigma} - \Upsilon_t^\sigma\bar{\delta}\right) - \tau\Gamma_t\frac{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}\Upsilon_t^{-1}$$

$$\lambda_t\rho_t = \lambda_t a'\left(u_t\right) + \eta\phi_t\left(\delta_t-\bar{\delta}\right)e^{\sigma v_t}u_t^{-1} - \tau\varsigma_t \tag{D.19}$$

Given the steady state assumptions $u_t = u = 1$, $a(1) = 0$, and $a'(1) > 0$, the above expression becomes

$$\lambda\rho = \lambda a'\left(1\right) + \eta\phi\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v} - \tau\varsigma$$

$$\rho = a'\left(1\right) + \eta\frac{\phi}{\lambda}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v} - \tau\frac{\varsigma}{\lambda} \tag{D.20}$$

The detrended optimal demand for capital stock, (B.28), is derived as follows

$$\Phi_t\frac{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} = \beta\Lambda_{t+1}\frac{P_{t+1}A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{P_{t+1}A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}\left[R_{t+1}^k u_{t+1}\frac{P_{t+1}\Upsilon_{t+1}^{-1}}{P_{t+1}\Upsilon_{t+1}^{-1}} - P_{t+1}\Upsilon_{t+1}^{-1}a\left(u_{t+1}\right)\right]+$$

$$+\,\beta\Phi_{t+1}\frac{A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}\left(1-\Upsilon_t^{-\sigma}D_{t+1}\frac{\Upsilon_{t+1}^\sigma}{\Upsilon_{t+1}^\sigma}\right) + \beta\tau\Gamma_{t+1}\frac{A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}{A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}}}\Upsilon_{t+1}^{-1}u_{t+1}+$$

$$-\,\beta\sigma\Phi_{t+1}\frac{A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t+1}\Upsilon_{t+1}^{\frac{\alpha}{1-\alpha}+1}}\Upsilon_t^{-\sigma}\left(D_{t+1}\frac{\Upsilon_{t+1}^\sigma}{\Upsilon_{t+1}^\sigma} - \Upsilon_{t+1}^\sigma\bar{\delta}\right)$$

$$\phi_t = \beta E_t\Big\{\lambda_{t+1}\left[\rho_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right]e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)} + \tau\varsigma_{t+1}u_{t+1}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}+$$

$$+\phi_{t+1}\left(1-\delta_{t+1}e^{\sigma v_{t+1}}\right)e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}-\sigma\phi_{t+1}\left(\delta_{t+1}-\bar{\delta}\right)e^{\sigma v_{t+1}}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}\Bigg\} \tag{D.21}$$

which, using the expression (D.18), in steady state, becomes

$$\phi\left[e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}-\beta\left(1-\delta e^{\sigma\gamma_v}\right)+\beta\sigma e^{\sigma\gamma_v}\left(\delta-\bar{\delta}\right)\right]=\beta\lambda\rho+$$

$$+\beta\tau\left[\sigma\phi e^{\sigma\gamma_v}\left(\delta-\bar{\delta}\right)\left(\frac{\tilde{m}}{\tilde{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}-\lambda\right]$$

$$\phi=\frac{\lambda\beta\left(\rho-\tau\right)}{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}-\beta\left(1-\delta e^{\sigma\gamma_v}\right)+\beta\sigma\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\left[1-\tau\left(\frac{\tilde{m}}{\tilde{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\right]} \tag{D.22}$$

Recalling the optimal wage setting condition given in (B.29), we detrend it as follows

$$E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\Lambda_{t+s}\frac{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}\tilde{L}_{t+s}(j)\left[\pi_{t,t+s}^{w}\tilde{W}_t(j)\frac{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}+\right.$$

$$\left.-\,b_{t+s}\varphi\left(1+\lambda_{w,t+s}\right)\frac{\tilde{L}_{t+s}(j)^{\nu}}{\Lambda_{t+s}}\frac{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}\right]=0$$

$$E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\lambda_{t+s}\tilde{L}_{t+s}(j)\left[\tilde{w}_t(j)\pi_{t,t+s}^{w}\frac{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}-b_{t+s}\varphi\left(1+\lambda_{w,t+s}\right)\frac{\tilde{L}_{t+s}(j)^{\nu}}{\lambda_{t+s}}\right]=0$$

$$E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\lambda_{t+s}\tilde{L}_{t+s}(j)\left[\tilde{w}_t(j)\tilde{\pi}_{t,t+s}^{w}-b_{t+s}\varphi\left(1+\lambda_{w,t+s}\right)\frac{\tilde{L}_{t+s}(j)^{\nu}}{\lambda_{t+s}}\right]=0 \tag{D.23}$$

where $\tilde{\pi}_{t,t+s}^{w}$ is defined as follows

$$\tilde{\pi}_{t,t+s}^{w}=\prod_{k=0}^{s}\left[\left(\pi_{t+k-1}e^{z_{t+k-1}+\frac{\alpha}{1-\alpha}v_{t+k-1}}\right)^{\iota_w}\left(\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)^{1-\iota_w}\frac{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_{t+k}A_{t+k}\Upsilon_{t+k}^{\frac{\alpha}{1-\alpha}}}\right]$$

$$=\prod_{k=0}^{s}\left[\left(\frac{\pi_{t+k-1}e^{z_{t+k-1}+\frac{\alpha}{1-\alpha}v_{t+k-1}}}{\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}\right)^{\iota_w}\left(\frac{\pi_{t+k}e^{z_{t+k}+\frac{\alpha}{1-\alpha}v_{t+k}}}{\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}\right)^{-1}\right]$$

In steady state, expression (D.23) becomes

$$\lambda\tilde{L}(j)\left[\tilde{w}(j)-\varphi\left(1+\lambda_w\right)\frac{\tilde{L}(j)^{\nu}}{\lambda}\right]=0$$

$$\tilde{L}(j)^{\nu}=\frac{\lambda}{\varphi}\frac{\tilde{w}(j)}{1+\lambda_w} \tag{D.24}$$

Recall that the optimal demand for labor is given by

$$L_{t+s}(j) = \left[\frac{W_t(j)}{W_{t+s}}\pi^w_{t,t+s}\right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s}$$

Detrending it we obtain

$$L_{t+s}(j) = \left[\frac{W_t(j)}{W_{t+s}} \; \frac{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}}{P_{t+s}A_{t+s}\Upsilon_{t+s}^{\frac{\alpha}{1-\alpha}}} \; \frac{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}} \; \pi^w_{t,t+s}\right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s}$$

$$L_{t+s}(j) = \left[\frac{w_t(j)}{w_{t+s}}\tilde{\pi}^w_{t,t+s}\right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s} \tag{D.25}$$

The aggregate wage index (B.30) in the stationary equilibrium is derived as follows

$$\frac{W_t}{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}} = \left\{\xi_w\left[\frac{W_{t-1}}{P_{t-1}A_{t-1}\Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}}\frac{P_{t-1}A_{t-1}\Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}}}{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}\left(\pi_{t-1}e^{z_{t-1}+\frac{\alpha}{1-\alpha}v_{t-1}}\right)^{\iota_w}\left(\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)^{1-\iota_w}\right]^{-\frac{1}{\lambda_{w,t}}} + \right.$$

$$\left. + (1-\xi_w)\left[\frac{\tilde{W}_t(j)}{P_tA_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}\right]^{-\frac{1}{\lambda_{w,t}}}\right\}^{-\lambda_{w,t}}$$

$$w_t = \left\{\xi_w\left[w_{t-1}\left(\frac{\pi_{t-1}e^{z_{t-1}+\frac{\alpha}{1-\alpha}v_{t-1}}}{\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}\right)^{\iota_w}\left(\frac{\pi_te^{z_t+\frac{\alpha}{1-\alpha}v_t}}{\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}\right)^{-1}\right]^{-\frac{1}{\lambda_{w,t}}} + (1-\xi_w)\,\tilde{w}_t(j)^{-\frac{1}{\lambda_{w,t}}}\right\}^{-\lambda_{w,t}} \tag{D.26}$$

The definition of capital utilization, $K_t = u_t\bar{K}_{t-1}$, in stationary equilibrium is given by

$$K_t\frac{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} = u_t\frac{A_{t-1}\Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1}\Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}\bar{K}_{t-1}$$

$$k_t = u_t\bar{k}_{t-1}e^{-\left(z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)} \tag{D.27}$$

which in steady state becomes

$$k = \bar{k}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \tag{D.28}$$

The definition of the capital depreciation rate is given by

$$D_t = \zeta u_t^\eta\left(\frac{M_t}{\bar{K}_{t-1}}\right)^{-\sigma} + \Upsilon_t^\sigma\bar{\delta}$$

which in the stationary equilibrium model becomes

$$D_t\frac{\Upsilon_t^\sigma}{\Upsilon_t^\sigma} = \zeta u_t^\eta\left(\frac{M_t}{\bar{K}_{t-1}}\frac{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}}\frac{A_{t-1}\Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1}\Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}\right)^{-\sigma} + \Upsilon_t^\sigma\bar{\delta}$$

$$\delta_t = \zeta u_t^\eta\left(\frac{m_t}{\bar{k}_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)^{-\sigma} + \bar{\delta} \tag{D.29}$$

and in steady state, recalling that $m = \tilde{m}$, it is given by

$$\delta = \zeta \left( \frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v} \right)^{-\sigma} + \bar{\delta} \tag{D.30}$$

The definition of maintenance costs is given by

$$M_t = \tau \Upsilon_t^{-1} u_t \bar{K}_{t-1} + A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M}$$

In the stationary equilibrium it becomes

$$M_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} = \tau \Upsilon_t^{-1} u_t \bar{K}_{t-1} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} + A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} \bar{M}$$

$$m_t = \tau u_t \bar{k}_{t-1} e^{-\left(z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)} + \bar{M} \tag{D.31}$$

while in steady state is given by

$$\tilde{m} = \tau \bar{k} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} + \bar{M} \tag{D.32}$$

The law of motion of capital, substituting for the technology to produce new capital goods, $i_t^k = \mu_t \left[1 - S\left(I_t / I_{t-1}\right)\right] I_t$ is given by

$$\bar{K}_t = \left(1 - \Upsilon_{t-1}^{-\sigma} D_t\right) \bar{K}_{t-1} + \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t$$

which is detrended as follows

$$\bar{K}_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} = \left(1 - \Upsilon_{t-1}^{-\sigma} D_t \frac{\Upsilon_t^{\sigma}}{\Upsilon_t^{\sigma}}\right) \bar{K}_{t-1} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} +$$

$$+ \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}\right)\right] I_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}+1}}$$

$$\bar{k}_t = \left(1 - \delta_t e^{\sigma v_t}\right) \bar{k}_{t-1} e^{-\left(z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)} + \mu_t \left[1 - S\left(\frac{i_t}{i_{t-1}} e^{z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)\right] i_t \tag{D.33}$$

and in steady state it becomes

$$\bar{k} = \left(1 - \delta e^{\sigma \gamma_v}\right) \bar{k} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} + [1 - 0]\, i$$

$$\bar{k} = \frac{i}{1 - \left(1 - \delta e^{\sigma \gamma_v}\right) e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}} \tag{D.34}$$

The aggregate resource constraint is defined as follows

$$C_t + \Upsilon_t^{-1} I_t + a\left(u_t\right) \Upsilon_t^{-1} \bar{K}_{t-1} + \tilde{M}_t = \left(1/g_t\right) Y_t$$

where $\tilde{I}_t = \Upsilon_t^{-1} I_t = (P_t^I / P_t) I_t$, and $\tilde{M}_t = (P_t^m / P_t) M_t$ are the real investment and real maintenance in consumption units, respectively, and $\bar{K}_{t-1}$ is multiplied by $\Upsilon_t^{-1}$ in order to ensure the balanced growth path. Detrending the above expression we obtain

$$C_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} + \Upsilon_t^{-1} I_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} + a(u_t)\, \Upsilon_t^{-1} \bar{K}_{t-1} \frac{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}}{A_{t-1} \Upsilon_{t-1}^{\frac{\alpha}{1-\alpha}+1}} + \tilde{M}_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}} = (1/g_t)\, Y_t \frac{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}{A_t \Upsilon_t^{\frac{\alpha}{1-\alpha}}}$$

$$c_t + i_t + a(u_t)\, \bar{k}_{t-1} e^{-\left(z_t + \left(\frac{\alpha}{1-\alpha}+1\right) v_t\right)} + \tilde{m}_t = (1/g_t)\, y_t \qquad \text{(D.35)}$$

which gives the following expression in steady state

$$c + i + \tilde{m} = (1/g)\, y \qquad \text{(D.36)}$$

Finally, the actual GDP, $X_t$ is given by

$$X_t = (1 - 1/g_t)\, Y_t + C_t + \Upsilon_t^{-1} I_t + \tilde{M}_t$$

which detrended becomes

$$x_t = (1 - 1/g_t)\, y_t + c_t + i_t + \tilde{m}_t \qquad \text{(D.37)}$$

and in steady state it is

$$x = (1 - 1/g)\, y + c + i + \tilde{m} \qquad \text{(D.38)}$$

## E APPENDIX: Linear rational expectations model

In the present section we log linearize our equilibrium model equations around the steady state. The variables denoted by a hat represent the deviations form their respective steady state values. Moreover, following Justiniano et al. (2011), we apply the following rules of the deviations from steady states of the shocks

$$\hat{\lambda}_{p,t+s} = \ln(1+\lambda_{p,t+s}) - \ln(1+\lambda_p) \qquad \hat{\lambda}_{w,t+s} = \ln(1+\lambda_{w,t+s}) - \ln(1+\lambda_w)$$
$$\hat{d}_t = d_t - 1 \qquad \hat{g}_t = g_t - g$$
$$\hat{b}_t = b_t - 1 \qquad \hat{\mu}_t = \mu_t - 1$$
$$\hat{z}_t = z_t - \gamma_z \qquad \hat{v}_t = v_t - \gamma_v$$

Moreover, the expectations of the stationary shocks are given by

$$\ln E_t\{b_{t+1}\} = \rho_b \ln b_t \qquad \ln E_t\{d_{t+1}\} = \rho_d \ln d_t$$
$$E_t\{z_{t+1}\} = \rho_z z_t \qquad \ln E_t\{\mu_{t+1}\} = \rho_\mu \ln \mu_t$$
$$E_t\{v_{t+1}\} = \rho_v v_t \qquad \ln E_t\{g_{t+1}\} = \rho_g \ln g_t$$

Recall the detrended aggregate production function of the intermediate goods producers, (D.1), and its steady state expression, (D.2). We log-linearize it as follows

$$\ln[y_t(i)] = \ln\left[k_t(i)^\alpha L_t(i)^{1-\alpha} - F\right]$$
$$y + \hat{y}_t = y + \frac{1}{y}\alpha k^{\alpha-1} L^{1-\alpha}(k_t - k) + \frac{1}{y}(1-\alpha) k^\alpha L^{1-\alpha-1}(L_t - L)$$

$$\hat{y}_t = \frac{y+F}{y}\alpha \hat{k}_t + \frac{y+F}{y}(1-\alpha)\hat{L}_t \tag{E.1}$$

The optimal relative demand for capital given in (D.3), using its steady state expression, (D.4), is log-linearized as follows

$$\ln\left[\frac{k_t}{L_t}\right] = \ln\left[\frac{\alpha}{1-\alpha}\frac{w_t}{\rho_t}\right]$$
$$\ln k_t - \ln L_t = \ln w_t - \ln \rho_t + \ln\frac{\alpha}{1-\alpha}$$

$$\hat{k}_t - \hat{L}_t = \hat{w}_t - \hat{\rho}_t \tag{E.2}$$

Given the optimal detrended nominal marginal costs, (D.5), and the respective steady state relation, (D.6), the log-linearization is as follows

$$\ln\left[s_t\right] = \ln\left[\frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}} w_t^{1-\alpha}\rho_t^{\alpha}\right]$$

$$\ln s_t = \ln\frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}} + (1-\alpha)\ln w_t + \alpha\ln\rho_t$$

$$\hat{s}_t = (1-\alpha)\hat{w}_t + \alpha\hat{\rho}_t \tag{E.3}$$

The log-linearization of the detrended optimal price setting, given equations (D.7), and (D.8), is

$$\ln\left[E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\lambda_{t+s}\quad \tilde{p}_t(i)\tilde{\pi}_{t,t+s}\tilde{y}_{t+s}(i)\right] = \ln\left[E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\lambda_{t+s}\quad \left(1+\lambda_{p,t+s}\right)s_{t+s}\tilde{y}_{t+s}(i)\right]$$

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\left(\ln\tilde{p}_t + \ln\tilde{\pi}_{t,t+s}\right) = E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\left[\ln\left(1+\lambda_{p,t+s}\right) + \ln s_{t+s}\right]$$

$$\frac{1}{1-\xi_p\beta}\hat{\tilde{p}}_t + E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\hat{\tilde{\pi}}_{t,t+s} = E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\left(\hat{\lambda}_{p,t+s} + \hat{s}_{t+s}\right)$$

$$\frac{1}{1-\xi_p\beta}\hat{\tilde{p}}_t = E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\left(-\hat{\tilde{\pi}}_{t,t+s} + \hat{\lambda}_{p,t+s} + \hat{s}_{t+s}\right)$$

$$= -\hat{\tilde{\pi}}_{t,t} + \hat{\lambda}_{p,t} + \hat{s}_t + E_t\sum_{s=1}^{\infty}\xi_p^s\beta^s\left(-\hat{\tilde{\pi}}_{t,t+s} + \hat{\lambda}_{p,t+s} + \hat{s}_{t+s}\right)$$

$$= 0 + \hat{\lambda}_{p,t} + \hat{s}_t + \xi_p\beta E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\left(-\hat{\tilde{\pi}}_{t,t+s+1} + \hat{\lambda}_{p,t+s+1} + \hat{s}_{t+s+1}\right)$$

$$= \hat{\lambda}_{p,t} + \hat{s}_t - \xi_p\beta E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\hat{\tilde{\pi}}_{t,t+1} + \xi_p\beta E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\left(-\hat{\tilde{\pi}}_{t+1,t+s+1} + \hat{\lambda}_{p,t+s+1} + \hat{s}_{t+s+1}\right)$$

$$= \hat{\lambda}_{p,t} + \hat{s}_t - \frac{\xi_p\beta}{1-\xi_p\beta}\hat{\tilde{\pi}}_{t,t+1} + \frac{\xi_p\beta}{1-\xi_p\beta}\hat{\tilde{p}}_{t+1}$$

Therefore,

$$\frac{1}{1-\xi_p\beta}\hat{\tilde{p}}_t = \hat{\lambda}_{p,t} + \hat{s}_t + \frac{\xi_p\beta}{1-\xi_p\beta}\left(\hat{\tilde{p}}_{t+1} - \hat{\tilde{\pi}}_{t,t+1}\right) \tag{E.4}$$

The log-linear detrended aggregate price index, (D.9) is given by

$$1^{-\frac{1}{\lambda_{p,t}}} = \xi_p\left[\left(\frac{\pi_{t-1}}{\pi}\right)^{\iota_p}\left(\frac{\pi_t}{\pi}\right)^{-1}\right]^{-\frac{1}{\lambda_{p,t}}} + \left(1-\xi_p\right)\tilde{p}_t^{-\frac{1}{\lambda_{p,t}}}$$

$$\ln 1 = \ln\left\{\xi_p\left[\left(\frac{\pi_{t-1}}{\pi}\right)^{\iota_p}\left(\frac{\pi_t}{\pi}\right)^{-1}\right]^{-\frac{1}{\lambda_{p,t}}} + \left(1-\xi_p\right)\tilde{p}_t^{-\frac{1}{\lambda_{p,t}}}\right\}$$

$$0 = -\frac{1}{\lambda_{p,t}}\xi_p\left[\left(\frac{\pi}{\pi}\right)^{\iota_p}\left(\frac{\pi}{\pi}\right)^{-1}\right]^{-\frac{1}{\lambda_{p,t}}-1}\left(\iota_p\hat{\pi}_{t-1}-\hat{\pi}_t\right) - \frac{1}{\lambda_{p,t}}\left(1-\xi_p\right)\left(\tilde{p}_t - 1\right)$$

$$0 = \xi_p\left(\iota_p\hat{\pi}_{t-1}-\hat{\pi}_t\right) + \left(1-\xi_p\right)\hat{\tilde{p}}_t \qquad \text{(E.5)}$$

Combining equation (E.4) with (E.5), both in the current and forward one period forms, and using the log-linearized expression for the definition of inflation, i.e. $\hat{\tilde{\pi}}_{t,t+s} = \sum_{j=0}^{s}\left(\iota_t\hat{\pi}_{t+j-1}-\hat{\pi}_{t+j}\right)$, we obtain the new Phillips curve, as follows

$$\frac{1}{1-\xi_p\beta}\left[-\frac{\xi_p}{1-\xi_p}\left(\iota_t\hat{\pi}_{t-1}-\hat{\pi}_t\right)\right] = \hat{\lambda}_{p,t} + \hat{s}_t + \frac{\xi_p\beta}{1-\xi_p\beta}\left[-\frac{\xi_p}{1-\xi_p}\left(\iota_t\hat{\pi}_t-\hat{\pi}_{t+1}\right)\right] - \frac{\xi_p\beta}{1-\xi_p\beta}\left(\iota_t\hat{\pi}_t-\hat{\pi}_{t+1}\right)$$

$$\frac{\xi_p+\xi_p\beta\iota_p}{\left(1-\xi_p\beta\right)\left(1-\xi_p\right)}\hat{\pi}_t = \hat{\lambda}_{p,t} + \hat{s}_t + \frac{\xi_p\iota_p}{\left(1-\xi_p\beta\right)\left(1-\xi_p\right)}\hat{\pi}_{t-1} + \frac{\xi_p\beta}{\left(1-\xi_p\beta\right)\left(1-\xi_p\right)}\hat{\pi}_{t+1}$$

$$\hat{\pi}_t = \frac{\left(1-\xi_p\beta\right)\left(1-\xi_p\right)}{\xi_p\left(1+\beta\iota_p\right)}\left(\hat{\lambda}_{p,t}+\hat{s}_t\right) + \frac{\iota_p}{1+\beta\iota_p}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta\iota_p}\hat{\pi}_{t+1} \qquad \text{(E.6)}$$

Next, we log-linearize the detrended marginal utility of consumption given in (D.10), using its steady state expression (D.11), as follows

$$\ln\lambda_t = \ln\left[\frac{e^{z_t+\frac{\alpha}{1-\alpha}v_t}b_t}{e^{z_t+\frac{\alpha}{1-\alpha}v_t}c_t - hc_{t-1}} - \beta h E_t\left\{\frac{b_{t+1}}{e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}c_{t+1}-hc_t}\right\}\right]$$

$$\begin{aligned}
\hat{\lambda}_t = c\frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h}\Bigg\{&\left[ce^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\beta h\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\right]\left(z_{t+1}-\gamma_z\right)+\\
&+\left[e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-1} - ce^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\right]\left(z_t-\gamma_z\right)+\\
&+\frac{\alpha}{1-\alpha}\left[e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-1} - ce^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\right]\left(v_t-\gamma_v\right)+\\
&+\frac{\alpha}{1-\alpha}\left[ce^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\beta h\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\right]\left(v_{t+1}-\gamma_v\right)+\\
&+e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-1}\left(b_t-1\right) - \beta h\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-1}\left(b_{t+1}-1\right)+\\
&-e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\left(c_t-c\right) - \beta h^2\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\left(c_t-c\right)+\\
&+he^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\left(c_{t-1}-c\right) + \beta he^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}c-hc\right)^{-2}\left(c_{t+1}-c\right)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\hat{\lambda}_t = \frac{1}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h}\Bigg\{&\beta h\frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\hat{z}_{t+1} + \left[e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v} - \frac{e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\right]\hat{z}_t+\\
&+\frac{\alpha}{1-\alpha}\beta h\frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\hat{v}_{t+1} + \frac{\alpha}{1-\alpha}\left[e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v} - \frac{e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\right]\hat{v}_t+
\end{aligned}$$

$$+ e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\hat{b}_t - \beta h\hat{b}_{t+1} - \frac{e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\hat{c}_t - \beta h^2\frac{1}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\hat{c}_t +$$

$$+ h\frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\hat{c}_{t-1} + \beta h\frac{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}{e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h}\hat{c}_{t+1}\Bigg\}$$

ending up with

$$\hat{\lambda}_t = \frac{1}{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)}\Bigg\{\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-\beta h\rho_b\right)\left(e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}-h\right)\hat{b}_t +$$

$$- \left(e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}+\beta h^2\right)\hat{c}_t + he^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\hat{c}_{t-1} + \beta he^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\hat{c}_{t+1} + \tag{E.7}$$

$$+ \left(\beta h\rho_z e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v} - he^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)\hat{z}_t + \frac{\alpha}{1-\alpha}\left(\beta h\rho_v e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v} - he^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)\hat{v}_t\Bigg\}$$

Log-linearizing the detrended Euler equation (D.12) we obtain

$$\ln\lambda_t = \ln\left[\beta R_t E_t\left\{\lambda_{t+1}e^{-\left(z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}\right)}\pi_{t+1}^{-1}\right\}\right]$$

$$\ln\lambda_t = \ln\beta + \ln R_t + \ln\lambda_{t+1} - z_{t+1} - \frac{\alpha}{1-\alpha}v_{t+1} - \ln\pi_{t+1}$$

$$\hat{\lambda}_t = \hat{R}_t + \hat{\lambda}_{t+1} - \hat{z}_{t+1} - \frac{\alpha}{1-\alpha}\hat{v}_{t+1} - \hat{\pi}_{t+1}$$

$$\hat{\lambda}_t = \hat{R}_t - \rho_z\hat{z}_t - \frac{\alpha}{1-\alpha}\rho_v\hat{v}_t + E_t\left\{\hat{\lambda}_{t+1} - \hat{\pi}_{t+1}\right\} \tag{E.8}$$

The detrended optimal capital good supply condition, (D.14), making use of (D.15), is log-linearized as follows

$$\ln\lambda_t = \ln\Bigg\{\phi_t\mu_t\left[1 - S\left(\frac{i_t}{i_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right) - \frac{i_t}{i_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}S'\left(\frac{i_t}{i_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)\right] +$$

$$+ \beta E_t\left\{\phi_{t+1}\mu_{t+1}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}\left(\frac{i_{t+1}}{i_t}e^{z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}}\right)^2 S'\left(\frac{i_{t+1}}{i_t}e^{z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}}\right)\right\}\Bigg\}$$

$$\hat{\lambda}_t = \frac{1}{\phi}\Bigg\{(\phi_t - \phi) + \phi(\mu_t - 1) - \phi\frac{i}{i}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}S''\frac{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}}{i}(i_t - i) +$$

$$- \beta\phi e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\frac{i}{i}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^2 S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}i^{-1}(i_t - i) +$$

$$+ \phi\frac{i}{i}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}i^{-1}(i_{t-1} - i) +$$

$$+ \beta\phi e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\frac{i}{i}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^2 S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}i^{-1}(i_{t+1} - i) +$$

$$
\begin{aligned}
&-\phi\frac{i}{i}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\frac{i}{i}\left(z_t-\gamma_z\right)+\\
&+\beta\phi e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\left(\frac{i}{i}\right)^2 S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\frac{i}{i}\left(z_{t+1}-\gamma_z\right)+\\
&-\phi\frac{\alpha}{1-\alpha}\frac{i}{i}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\frac{i}{i}\left(v_t-\gamma_v\right)+\\
&+\frac{\alpha}{1-\alpha}\beta\phi e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\left(\frac{i}{i}\right)^2 S''e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\frac{i}{i}\left(v_{t+1}-\gamma_v\right)\Bigg\}
\end{aligned}
$$

which finally gives

$$
\begin{aligned}
\hat{\lambda}_t =&\hat{\phi}_t+\hat{\mu}_t-(1+\beta)\,e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}S''\hat{\imath}_t+e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}S''\hat{\imath}_{t-1}+\\
&+\beta e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}S''E_t\{\hat{\imath}_{t+1}\}-(1-\beta\rho_z)\,e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}S''\hat{z}_t+\\
&-\left(\frac{\alpha}{1-\alpha}+1\right)(1-\beta\rho_v)\,e^{2\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}S''\hat{v}_t
\end{aligned}
\tag{E.9}
$$

We next log-linearize the detrended optimality condition for maintenance supply given in (D.17)

$$
\begin{aligned}
\ln\lambda_t =& \ln\Bigg\{\left[\sigma\phi_t\left(\frac{m_t}{\bar{k}_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)^{-1}\left(\delta_t-\bar{\delta}\right)e^{\sigma v_t}-\varsigma_t\right]d_t\Bigg[1-f\left(\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}\right)+\\
&-\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}f'\left(\frac{\tilde{m}_t}{\tilde{m}_{t-1}}e^{z_t+\frac{\alpha}{1-\alpha}v_t}\right)\Bigg]+\\
&+\beta E_t\Bigg\{\left[\sigma\phi_{t+1}\left(\frac{m_{t+1}}{\bar{k}_t}e^{z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}}\right)^{-1}\left(\delta_{t+1}-\bar{\delta}\right)e^{\sigma v_{t+1}}-\varsigma_{t+1}\right]d_{t+1}e^{-\left(z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}\right)}\times\\
&\times\left(\frac{\tilde{m}_{t+1}}{\tilde{m}_t}e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\right)^2 f'\left(\frac{\tilde{m}_{t+1}}{\tilde{m}_t}e^{z_{t+1}+\frac{\alpha}{1-\alpha}v_{t+1}}\right)\Bigg\}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_t =& \frac{1}{\lambda}\Bigg\{\sigma\phi\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\Big[\hat{\phi}_t-\hat{m}_t+\hat{\bar{k}}_{t-1}-\hat{z}_t+\\
&-\left(\frac{\alpha}{1-\alpha}+1-\sigma\right)\hat{v}_t\Big]+\sigma\phi\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\delta e^{\sigma\gamma_v}\hat{\delta}_t-\varsigma\hat{\varsigma}_t+\\
&-\left[\sigma\phi\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\varsigma\right]\left(\frac{\tilde{m}}{\tilde{m}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}\right)\gamma_v}\right)^2 f''\Big[\hat{\tilde{m}}_t-\hat{\tilde{m}}_{t-1}+\hat{z}_t+\frac{\alpha}{1-\alpha}\hat{v}_t\Big]+\\
&+\beta\left[\sigma\phi\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\varsigma\right]e^{-\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}\times
\end{aligned}
$$

$$\times\left(\frac{\tilde{m}}{\tilde{m}}e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}\right)^3 f''\left[\hat{\tilde{m}}_{t+1}-\hat{\tilde{m}}_t+\hat{z}_{t+1}+\frac{\alpha}{1-\alpha}\hat{v}_{t+1}\right]+$$
$$+\left[\sigma\phi\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\varsigma\right]\hat{d}_t\Bigg\}$$

making use of the log-linearized expression of (D.16), $\hat{m}_t = \hat{d}_t + \hat{\tilde{m}}_t$, we obtain the following expression for optimal maintenance condition

$$\hat{\lambda}_t = e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''\hat{\tilde{m}}_{t-1}+\beta e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''\hat{\tilde{m}}_{t+1}+$$
$$-\left[\left(1+\beta\right)e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''+\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\right]\hat{\tilde{m}}_t+$$
$$+\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\left(\hat{\phi}_t+\hat{\bar{k}}_{t-1}\right)+\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\delta e^{\sigma\gamma_v}\hat{\delta}_t+$$
$$+\left[1-\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\right]\left(\hat{d}_t+\hat{\varsigma}_t\right)+$$
$$-\left[\left(1-\beta\rho_z\right)e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''+\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\right]\hat{z}_t+$$
$$-\left[\frac{\alpha}{1-\alpha}\left(1-\beta\rho_v\right)e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''+\left(\frac{\alpha}{1-\alpha}+1-\sigma\right)\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\right]\hat{v}_t$$

and rearranging the terms we end up with

$$\left[\left(1+\beta\right)e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''+\bar{A}\right]\hat{\tilde{m}}_t = e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''\hat{\tilde{m}}_{t-1}+\beta e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''\hat{\tilde{m}}_{t+1}+\bar{A}\hat{\bar{k}}_{t-1}+\bar{A}\frac{\delta}{\delta-\bar{\delta}}\hat{\delta}_t+$$
$$+\left(1-\bar{A}\right)\hat{d}_t+\left(1-\bar{A}\right)\left(\hat{\varsigma}_t-\hat{\lambda}_t\right)+\bar{A}\left(\hat{\phi}_t-\hat{\lambda}_t\right)-\left[\left(1-\beta\rho_z\right)e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''+\bar{A}\right]\hat{z}_t+ \quad \text{(E.10)}$$
$$-\left[\frac{\alpha}{1-\alpha}\left(1-\beta\rho_v\right)e^{2\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)}f''+\left(\frac{\alpha}{1-\alpha}+1-\sigma\right)\bar{A}\right]\hat{v}_t$$

where the constant $\bar{A}=\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}$ equals the steady state expression of the depreciation rate first order condition with respect to the maintenance to effective capital ratio, that is $-\delta_{m/k}e^{\sigma\gamma_v}$, and $\hat{q}_t=\hat{\phi}_t-\hat{\lambda}_t$ is the Tobin's q.

Next, it follows the log-linearization of the detrended optimal condition for the capital utilization rate, (D.19), making use of (D.20). Re-express, first, both the equations as follows, respectively, making use of (D.18) for the second one

$$a'\left(u_t\right)=\rho_t-\eta\frac{\phi_t}{\lambda_t}\left(\delta_t-\bar{\delta}\right)e^{\sigma v_t}u_t^{-1}+\tau\frac{\varsigma_t}{\lambda_t}$$
$$a'(1)=\rho-\eta\frac{\phi}{\lambda}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}+\tau\left[\sigma\frac{\phi}{\lambda}\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-1\right] \quad \text{(E.11)}$$

Call $\chi = a''(1)/a'(1)$ the value of the relative elasticity of the utilization costs of capital in steady state, and log-linearize (D.19) as follows

$$\ln\left[a'\left(u_t\right)\right] = \ln\left[\rho_t - \eta\frac{\phi_t}{\lambda_t}\left(\delta_t - \bar{\delta}\right)e^{\sigma v_t}u_t^{-1} + \tau\frac{\varsigma_t}{\lambda_t}\right]$$

$$\frac{a''(1)}{a'(1)}\hat{u}_t = \frac{1}{a'(u)}\left\{\rho\hat{\rho}_t - \eta\frac{\phi}{\lambda}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v}\left[\hat{\phi}_t - \hat{\lambda}_t - \hat{u}_t + \sigma\hat{v}_t\right] - \eta\frac{\phi}{\lambda}\delta e^{\sigma\gamma_v}\hat{\delta}_t + \tau\frac{\varsigma}{\lambda}\left(\hat{\varsigma}_t - \hat{\lambda}_t\right)\right\}$$

$$\left\{\chi\left\{\rho - \eta\frac{\phi}{\lambda}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v} + \tau\left[\sigma\frac{\phi}{\lambda}\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v} - 1\right]\right\} - \eta\frac{\phi}{\lambda}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v}\right\}\hat{u}_t =$$

$$= \rho\hat{\rho}_t - \eta\frac{\phi}{\lambda}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v}\left[\hat{\phi}_t - \hat{\lambda}_t + \sigma\hat{v}_t\right] - \eta\frac{\phi}{\lambda}\delta e^{\sigma\gamma_v}\hat{\delta}_t +$$

$$+ \tau\left[\sigma\frac{\phi}{\lambda}\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v} - 1\right]\left(\hat{\varsigma}_t - \hat{\lambda}_t\right)$$

Recalling that, in steady state, $\lambda = \phi$, the latter expression becomes

$$\rho_t = \left\{\frac{\chi}{\rho}\left[\rho - \bar{B} + \tau\left(\bar{A} - 1\right)\right] - \frac{\bar{B}}{\rho}\right\}\hat{u}_t + \frac{\sigma}{\rho}\bar{B}\hat{v}_t + \frac{\bar{B}}{\rho}\left(\hat{\phi}_t - \hat{\lambda}_t\right) - \frac{\tau}{\rho}\left(\bar{A} - 1\right)\left(\hat{\varsigma}_t - \hat{\lambda}_t\right) + \frac{\delta}{\rho\left(\delta - \bar{\delta}\right)}\bar{B}\hat{\delta}_t \tag{E.12}$$

where the constant $\bar{B} = \eta e^{\sigma\gamma_v}\left(\delta - \bar{\delta}\right)$ is the steady state first order condition of capital depreciation with respect to utilization rate, that is $\delta_u e^{\sigma\gamma_v}$.

The detrended optimal demand for capital stock in (D.21), using its steady state expression (D.22), is log-linearized below

$$\ln\phi_t = \ln\left\{\beta E_t\left\{\lambda_{t+1}\left[\rho_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right]e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)} +\right.\right.$$

$$+ \phi_{t+1}\left(1 - \delta_{t+1}e^{\sigma v_{t+1}}\right)e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)} + \tau\varsigma_{t+1}u_{t+1}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)} +$$

$$\left.\left. - \sigma\phi_{t+1}\left(\delta_{t+1} - \bar{\delta}\right)e^{\sigma v_{t+1}}e^{-\left(z_{t+1}+\left(\frac{\alpha}{1-\alpha}+1\right)v_{t+1}\right)}\right\}\right\}$$

$$\hat{\phi}_t = \frac{1}{\phi}\left\{\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left[\lambda\rho + \phi\left(1 - \delta e^{\sigma\gamma_v}\right) - \phi\sigma\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v} + \tau\varsigma\right]\left[-\hat{z}_{t+1} - \left(\frac{\alpha}{1-\alpha} + 1\right)\hat{v}_{t+1}\right] +\right.$$

$$+ \beta\lambda\rho e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left[\hat{\lambda}_{t+1} + \hat{\rho}_{t+1} + \hat{u}_{t+1}\right] - \beta\lambda a'(1)e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_{t+1} +$$

$$+ \beta\phi\left(1 - \delta e^{\sigma\gamma_v}\right)e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{\phi}_{t+1} - \beta\phi\delta e^{\sigma\gamma_v}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left[\hat{\delta}_{t+1} + \sigma\hat{v}_{t+1}\right] +$$

$$- \beta\sigma\phi\delta e^{\sigma\gamma_v}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{\delta}_{t+1} - \beta\sigma\phi\left(\delta - \bar{\delta}\right)e^{\sigma\gamma_v}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\hat{\phi}_{t+1} + \sigma\hat{v}_{t+1}\right) +$$

$$\left. + \beta\tau\varsigma e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left[\hat{\varsigma}_{t+1} + \hat{u}_{t+1}\right]\right\}$$

$$\hat{\phi}_t = -\hat{z}_{t+1} - \left[\left(\frac{\alpha}{1-\alpha}+1\right) + \beta\delta\sigma e^{\sigma\gamma_v} e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} + \beta\sigma^2\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\right]\hat{v}_{t+1}+$$

$$+\frac{\rho}{\rho-\tau}\Big\{1-\beta e^{-\gamma_z-\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\left(1-\delta e^{\sigma\gamma_v}\right)+$$

$$+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\sigma\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\left[1-\tau\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\right]\Big\}\left[\hat{\lambda}_{t+1}+\hat{\rho}_{t+1}\right]+$$

$$+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left[\left(1-\delta e^{\sigma\gamma_v}\right)-\sigma\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}\right]\hat{\phi}_{t+1}+$$

$$-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left[\delta e^{\sigma\gamma_v}+\sigma\delta e^{\sigma\gamma_v}\right]\hat{\delta}_{t+1}+$$

$$+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\left[\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\frac{\lambda}{\phi}\right]\hat{\varsigma}_{t+1}+$$

$$+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\Big\{\eta\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\tau\left[\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\frac{\lambda}{\phi}\right]\Big\}\hat{u}_{t+1}$$

$$+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\left[\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-\frac{\lambda}{\phi}\right]\hat{u}_{t+1}$$

Rearranging the latter expression we obtain the following log-linear optimal capital stock condition

$$\begin{aligned}
\hat{\phi}_t &= \hat{\phi}_{t+1}-\rho_z\hat{z}_t-\left[\frac{\alpha}{1-\alpha}+1+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\sigma\bar{C}\right]\rho_v\hat{v}_t+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\bar{B}\hat{u}_{t+1}+\frac{\rho}{\rho-\tau}\bar{D}\hat{\rho}_{t+1}\\
&-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(\bar{C}+e^{\sigma\gamma_v}\sigma\bar{\delta}\right)\hat{\delta}_{t+1}-\left[\bar{D}+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A}\right]\left(\hat{\phi}_{t+1}-\hat{\lambda}_{t+1}\right)+\\
&+\left[\frac{\tau}{\rho-\tau}\bar{D}-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A}\right]\left(\hat{\varsigma}_{t+1}-\hat{\lambda}_{t+1}\right)
\end{aligned} \tag{E.13}$$

where the constants have been defined as follows

$$\bar{C}=e^{\sigma\gamma_v}\left[\delta+\sigma\left(\delta-\bar{\delta}\right)\right]$$

$$\bar{D}=1-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\left(1-e^{\sigma\gamma_v}\delta\right)+\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\frac{\sigma}{\eta}\bar{B}-\beta e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\tau\bar{A}$$

The detrended optimal wage setting condition (D.23) is log-linearized as follows

$$\ln\left\{E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\lambda_{t+s}\tilde{L}_{t+s}(j)\tilde{w}_t(j)\tilde{\pi}^w_{t,t+s}\right\}=\ln\left\{E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\lambda_{t+s}\tilde{L}_{t+s}(j)b_{t+s}\varphi\left(1+\lambda_{w,t+s}\right)\frac{\tilde{L}_{t+s}(j)^{\nu}}{\lambda_{t+s}}\right\}$$

$$\ln\left\{E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\tilde{w}_t(j)\tilde{\pi}^w_{t,t+s}\right\}=\ln\left\{E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s b_{t+s}\varphi\left(1+\lambda_{w,t+s}\right)\frac{\tilde{L}_{t+s}(j)^{\nu}}{\lambda_{t+s}}\right\}$$

$$E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\left[\ln\tilde{w}_t(j)+\ln\tilde{\pi}^w_{t,t+s}\right]=E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\left[\ln b_{t+s}+\ln\varphi+\ln\left(1+\lambda_{w,t+s}\right)+\ln\tilde{L}_{t+s}(j)^{\nu}-\ln\lambda_{t+s}\right]$$

obtaining

$$\frac{1}{1-\xi_w\beta}\hat{\tilde{w}}_t(j) = E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\left[\hat{b}_{t+s}+\hat{\lambda}_{w,t+s}+\nu\hat{\tilde{L}}_{t+s}(j)-\hat{\lambda}_{t+s}-\hat{\tilde{\pi}}_{t,t+s}^w\right] \tag{E.14}$$

Next, we log-linearize as follows the definition of wage inflation

$$\ln\tilde{\pi}_{t,t+s}^w = \ln\left\{\prod_{k=0}^{s}\left[\left(\frac{\pi_{t+k-1}e^{z_{t+k-1}+\frac{\alpha}{1-\alpha}v_{t+k-1}}}{\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}\right)^{\iota_w}\left(\frac{\pi_{t+k}e^{z_{t+k}+\frac{\alpha}{1-\alpha}v_{t+k}}}{\pi e^{\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v}}\right)^{-1}\right]\right\}$$

$$\ln\tilde{\pi}_{t,t+s}^w = \sum_{k=0}^{s}\left\{\iota_w\ln\pi_{t+k-1}+\iota_w\left(z_{t+k-1}+\frac{\alpha}{1-\alpha}v_{t+k-1}\right)-\iota_w\ln\pi-\iota_w\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)+\right.$$

$$\left.-\left(z_{t+k}+\frac{\alpha}{1-\alpha}v_{t+k}\right)+\ln\pi+\left(\gamma_z+\frac{\alpha}{1-\alpha}\gamma_v\right)-\ln\pi_{t+k}\right\}$$

which gives

$$\hat{\tilde{\pi}}_{t,t+s}^w = \sum_{k=0}^{s}\left[\iota_w\left(\hat{\pi}_{t+k-1}+\hat{z}_{t+k-1}+\frac{\alpha}{1-\alpha}\hat{v}_{t+k-1}\right)-\left(\hat{\pi}_{t+k}+\hat{z}_{t+k}+\frac{\alpha}{1-\alpha}\hat{v}_{t+k}\right)\right] \tag{E.15}$$

The log-linearized detrended labor demand, (D.25), becomes

$$\ln L_{t+s}(j) = \ln\left\{\left[\frac{w_t(j)}{w_{t+s}}\tilde{\pi}_{t,t+s}^w\right]^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}}L_{t+s}\right\}$$

$$\ln L_{t+s}(j) = -\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}\ln\left[\frac{w_t(j)}{w_{t+s}}\tilde{\pi}_{t,t+s}^w\right]+\ln L_{t+s}$$

therefore,

$$\hat{L}_{t+s}(j) = -\frac{1+\lambda_w}{\lambda_w}\left[\hat{\tilde{w}}_t(j)-\hat{w}_{t+s}+\hat{\tilde{\pi}}_{t,t+s}^w\right]+\hat{L}_{t+s} \tag{E.16}$$

Now, combine equations (E.14) and (E.16) as follows

$$\frac{1}{1-\xi_w\beta}\hat{\tilde{w}}_t(j) = E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\left\{\hat{b}_{t+s}+\hat{\lambda}_{w,t+s}-\hat{\lambda}_{t+s}-\hat{\tilde{\pi}}_{t,t+s}^w+\nu\left[\hat{L}_{t+s}-\frac{1+\lambda_w}{\lambda_w}\left(\hat{\tilde{w}}_t(j)-\hat{w}_{t+s}+\hat{\tilde{\pi}}_{t,t+s}^w\right)\right]\right\}$$

$$\frac{1}{1-\xi_w\beta}\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\hat{\tilde{w}}_t(j) = E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s\left[\nu\hat{L}_{t+s}+\nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_{t+s}+\hat{b}_{t+s}+\hat{\lambda}_{w,t+s}-\hat{\lambda}_{t+s}+\right.$$

$$\left.-\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\hat{\tilde{\pi}}_{t,t+s}^w\right] =$$

$$= \xi_w^0\beta^0\left[\nu\hat{L}_t+\nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_t+\hat{b}_t+\hat{\lambda}_{w,t}-\hat{\lambda}_t-\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\hat{\tilde{\pi}}_{t,t}^w\right]+$$

$$+ E_t \sum_{s=1}^{\infty} \xi_w^s \beta^s \Big[ \nu \hat{L}_{t+s} + \nu \frac{1+\lambda_w}{\lambda_w} \hat{w}_{t+s} + \hat{b}_{t+s} + \hat{\lambda}_{w,t+s} - \hat{\lambda}_{t+s} - \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \hat{\tilde{\pi}}^w_{t,t+s} \Big] =$$

$$= \nu \hat{L}_t + \nu \frac{1+\lambda_w}{\lambda_w} \hat{w}_t + \hat{b}_t + \hat{\lambda}_{w,t} - \hat{\lambda}_t + \xi_w \beta E_t \sum_{s=0}^{\infty} \xi_w^s \beta^s \Big[ \nu \hat{L}_{t+s+1} + \nu \frac{1+\lambda_w}{\lambda_w} \hat{w}_{t+s+1} + \hat{b}_{t+s+1} + \hat{\lambda}_{w,t+s+1} +$$

$$- \hat{\lambda}_{t+s+1} - \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \hat{\tilde{\pi}}^w_{t+1,t+s+1} \Big] - \xi_w \beta E_t \sum_{s=0}^{\infty} \xi_w^s \beta^s \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \hat{\tilde{\pi}}^w_{t,t+1} =$$

$$= \nu \hat{L}_t + \nu \frac{1+\lambda_w}{\lambda_w} \hat{w}_t + \hat{b}_t + \hat{\lambda}_{w,t} - \hat{\lambda}_t + \frac{\xi_w \beta}{1-\xi_w \beta} \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \hat{\tilde{w}}_{t+1}(j) - \frac{\xi_w \beta}{1-\xi_w \beta} \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \hat{\tilde{\pi}}^w_{t,t+1}$$

which gives

$$\frac{1}{1-\xi_w \beta} \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \hat{\tilde{w}}_t(j) = \frac{\xi_w \beta}{1-\xi_w \beta} \left(1 + \nu \frac{1+\lambda_w}{\lambda_w}\right) \left(\hat{\tilde{w}}_{t+1}(j) - \hat{\tilde{\pi}}^w_{t,t+1}\right) +$$

$$+ \nu \hat{L}_t + \nu \frac{1+\lambda_w}{\lambda_w} \hat{w}_t + \hat{b}_t + \hat{\lambda}_{w,t} - \hat{\lambda}_t \tag{E.17}$$

The latter expression characterizes the optimal wage setting in terms of the optimal demand for labor.

The log-linearized aggregate wage index in (D.26) is given by

$$- \frac{1}{\lambda_w} \ln w_t = \ln \left\{ \xi_w \left[ w_{t-1} \left( \frac{\pi_{t-1} e^{z_{t-1} + \frac{\alpha}{1-\alpha} v_{t-1}}}{\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}} \right)^{\iota_w} \left( \frac{\pi_t e^{z_t + \frac{\alpha}{1-\alpha} v_t}}{\pi e^{\gamma_z + \frac{\alpha}{1-\alpha}\gamma_v}} \right)^{-1} \right]^{-\frac{1}{\lambda_{w,t}}} + (1-\xi_w)\, \tilde{w}_t(j)^{-\frac{1}{\lambda_{w,t}}} \right\}$$

$$- \frac{1}{\lambda_w} \hat{w}_t = \frac{1}{w^{-1/\lambda_w}} \Bigg\{ - \xi_w \frac{1}{\lambda_w} w^{-\frac{1}{\lambda_w}-1} \Big[ (w_{t-1} - w) + w \iota_w \frac{\pi_{t-1} - \pi}{\pi} - w \frac{\pi_t - \pi}{\pi} + w \iota_w (z_{t-1} - \gamma_z) +$$

$$- w (z_t - \gamma_z) + w \iota_w \frac{\alpha}{1-\alpha} (v_{t-1} - \gamma_v) - w \frac{\alpha}{1-\alpha} (v_t - \gamma_v) \Big] - \frac{1}{\lambda_w} (1-\xi_w)\, w^{-\frac{1}{\lambda_w}-1} \left[ \tilde{w}_t(j) - w \right] \Bigg\}$$

$$\hat{w}_t = (1-\xi_w)\, \hat{\tilde{w}}_t(j) + \xi_w \left[ \hat{w}_{t-1} - \hat{\pi}_t - \hat{z}_t - \frac{\alpha}{1-\alpha} \hat{v}_t + \iota_w \hat{\pi}_{t-1} + \iota_w \hat{z}_{t-1} + \iota_w \frac{\alpha}{1-\alpha} \hat{v}_{t-1} \right] \tag{E.18}$$

Combine the above expression with the log-linearized wage inflation given in (E.15) in order to obtain the following log-linearized aggregate wage index in terms of wage inflation

$$\hat{w}_t = (1-\xi_w)\, \hat{\tilde{w}}_t(j) + \xi_w \left[ \hat{w}_{t-1} + \hat{\tilde{\pi}}^w_{t-1,t} \right] \tag{E.19}$$

Next, combine equation (E.17) with equation (E.19) expressed both in current period and forward one period, as follows

$$\frac{1}{1-\xi_w\beta}\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\left[\frac{1}{1-\xi_w}\hat{w}_t-\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}-\frac{\xi_w}{1-\xi_w}\hat{\tilde{\pi}}^w_{t-1,t}\right]=$$
$$=\frac{\xi_w\beta}{1-\xi_w\beta}\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\left[\frac{1}{1-\xi_w}\hat{w}_{t+1}-\frac{\xi_w}{1-\xi_w}\hat{w}_t-\frac{\xi_w}{1-\xi_w}\hat{\tilde{\pi}}^w_{t,t+1}-\hat{\tilde{\pi}}^w_{t,t+1}\right]+$$
$$+\nu\hat{L}_t+\nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_t+\hat{b}_t+\hat{\lambda}_{w,t}-\hat{\lambda}_t$$

Collect the terms, and add $-\hat{w}_t$ on both the sides of the above equation

$$\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\left[\frac{1}{(1-\xi_w\beta)(1-\xi_w)}+\frac{\xi_w^2\beta}{(1-\xi_w\beta)(1-\xi_w)}\right]\hat{w}_t-\nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_t-\hat{w}_t=$$
$$=-\hat{w}_t+\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\frac{\xi_w}{(1-\xi_w\beta)(1-\xi_w)}\hat{w}_{t-1}+\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\frac{\xi_w\beta}{(1-\xi_w\beta)(1-\xi_w)}\hat{w}_{t+1}+$$
$$+\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\frac{\xi_w}{(1-\xi_w\beta)(1-\xi_w)}\hat{\tilde{\pi}}^w_{t-1,t}-\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\frac{\xi_w\beta}{1-\xi_w\beta}\left(\frac{\xi_w}{1-\xi_w}+1\right)\hat{\tilde{\pi}}^w_{t,t+1}+$$
$$+\left(\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t\right)+\hat{\lambda}_{w,t}$$

$$\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\frac{\xi_w(1+\beta)}{(1-\xi_w\beta)(1-\xi_w)}\hat{w}_t=-\left[\hat{w}_t-\left(\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t\right)\right]+\hat{\lambda}_{w,t}+$$
$$+\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\frac{\xi_w}{(1-\xi_w\beta)(1-\xi_w)}\left[\hat{w}_{t-1}+\beta\hat{w}_{t+1}+\hat{\tilde{\pi}}^w_{t-1,t}-\beta\hat{\tilde{\pi}}^w_{t,t+1}\right]$$

$$\hat{w}_t=\frac{1}{1+\beta}\left(\hat{w}_{t-1}+\hat{\tilde{\pi}}^w_{t-1,t}\right)+\frac{\beta}{1+\beta}\left(\hat{w}_{t+1}-\hat{\tilde{\pi}}^w_{t,t+1}\right)+$$
$$-\frac{1}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\frac{(1-\xi_w\beta)(1-\xi_w)}{\xi_w(1+\beta)}\left[\hat{w}_t-\left(\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t\right)-\hat{\lambda}_{w,t}\right]$$

Substitute in the above expression equation (E.15) for log-linearized wage inflation, so that

$$\hat{w}_t=\frac{1}{1+\beta}\hat{w}_{t-1}+\frac{\iota_w}{1+\beta}\left(\hat{\pi}_{t-1}+\hat{z}_{t-1}+\frac{\alpha}{1-\alpha}\hat{v}_{t-1}\right)-\frac{1}{1+\beta}\left(\hat{\pi}_t+\hat{z}_t+\frac{\alpha}{1-\alpha}\hat{v}_t\right)+$$
$$+\frac{\beta}{1+\beta}\hat{w}_{t+1}-\frac{\beta\iota_w}{1+\beta}\left(\hat{\pi}_t+\hat{z}_t+\frac{\alpha}{1-\alpha}\hat{v}_t\right)+\frac{\beta}{1+\beta}\left(\hat{\pi}_{t+1}+\hat{z}_{t+1}+\frac{\alpha}{1-\alpha}\hat{v}_{t+1}\right)+$$
$$-\frac{1}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\frac{(1-\xi_w\beta)(1-\xi_w)}{\xi_w(1+\beta)}\left[\hat{w}_t-\left(\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t\right)-\hat{\lambda}_{w,t}\right]$$

Rearranging the terms we obtain

$$\hat{w}_t=\frac{1}{1+\beta}\hat{w}_{t-1}+\frac{\beta}{1+\beta}\hat{w}_{t+1}-\frac{1}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\frac{(1-\xi_w\beta)(1-\xi_w)}{\xi_w(1+\beta)}\left[\hat{w}_t-\left(\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t\right)\right]+$$

$$+\frac{\iota_w}{1+\beta}\hat{\pi}_{t-1}+\frac{\beta}{1+\beta}\hat{\pi}_{t+1}-\frac{1+\beta\iota_w}{1+\beta}\hat{\pi}_t+\frac{\iota_w}{1+\beta}\hat{z}_{t-1}-\frac{1+\beta\iota_w-\beta\rho_z}{1+\beta}\hat{z}_t+ \tag{E.20}$$

$$+\frac{\iota_w}{1+\beta}\frac{\alpha}{1-\alpha}\hat{v}_{t-1}-\frac{1+\beta\iota_w-\beta\rho_v}{1+\beta}\frac{\alpha}{1-\alpha}\hat{v}_t+\frac{1}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\frac{(1-\xi_w\beta)(1-\xi_w)}{\xi_w(1+\beta)}\hat{\lambda}_{w,t}$$

The above expression represents the log-linearized wage Phillips curve, and the term $\hat{w}_t-\left(\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t\right)$ is the log-linearized marginal utility of labor.

We are know going to re-express the wage Phillips curve in terms of the optimally chosen wage, $\tilde{w}_t$, by the households. For this purpose, combine equations (E.17) and (E.15) and rearrange the terms

$$\frac{1}{1-\xi_w\beta}\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\hat{\tilde{w}}_t(j)=\nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_t+\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t+\hat{\lambda}_{w,t}+$$

$$+\frac{\xi_w\beta}{1-\xi_w\beta}\left(1+\nu\frac{1+\lambda_w}{\lambda_w}\right)\left(\hat{\tilde{w}}_{t+1}(j)-\iota_w\hat{\pi}_t-\iota_w\hat{z}_t-\iota_w\frac{\alpha}{1-\alpha}\hat{v}_t+\hat{\pi}_{t+1}+\hat{z}_{t+1}+\frac{\alpha}{1-\alpha}\hat{v}_{t+1}\right)$$

in order to obtain

$$\hat{\tilde{w}}_t(j)=\xi_w\beta\hat{\tilde{w}}_{t+1}(j)-\xi_w\beta\iota_w\hat{\pi}_t+\xi_w\beta\hat{\pi}_{t+1}+\xi_w\beta\left(\rho_z-\iota_w\right)\hat{z}_t+\xi_w\beta\left(\rho_v-\iota_w\right)\frac{\alpha}{1-\alpha}\hat{v}_t+$$

$$+\frac{1-\xi_w\beta}{1+\nu\frac{1+\lambda_w}{\lambda_w}}\left(\nu\frac{1+\lambda_w}{\lambda_w}\hat{w}_t+\nu\hat{L}_t+\hat{b}_t-\hat{\lambda}_t+\hat{\lambda}_{w,t}\right) \tag{E.21}$$

Given the definition of detrended capital utilization, (D.27), and its steady state (D.28), we log-linearize it as follows

$$\ln k_t=\ln\left[u_t\bar{k}_{t-1}e^{-\left(z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)}\right]$$

$$\hat{k}_t=\frac{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}}{\bar{k}}\Big[\frac{\bar{k}}{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}}\left(u_t-1\right)+\frac{1}{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}}\left(\bar{k}_{t-1}-\bar{k}\right)-\frac{\bar{k}}{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}}\left(z_t-\gamma_z\right)+$$

$$-\frac{\bar{k}}{e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}}\left(\frac{\alpha}{1-\alpha}+1\right)\left(v_t-\gamma_v\right)\Big]$$

which gives

$$\hat{k}_t=\hat{u}_t+\hat{\bar{k}}_{t-1}-\hat{z}_t-\left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t \tag{E.22}$$

Given equations (D.29) and (D.30), and the relations $\hat{m}_t=\hat{\tilde{m}}_t+\hat{d}_t$, and $m=\tilde{m}$, capital depreciation rate is log-linearized as follows

$$\ln\delta_t=\ln\left[\zeta u_t^{\eta}\left(\frac{m_t}{\bar{k}_{t-1}}e^{z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)^{-\sigma}+\bar{\delta}\right]$$

$$\hat{\delta}_t = \frac{\left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right)^{\sigma}}{\zeta + \bar{\delta}\left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right)^{\sigma}} \left\{ \zeta\eta \left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right)^{-\sigma} \hat{u}_t + \right.$$

$$\left. - \zeta\sigma \left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right)^{-\sigma-1} \left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right) \left[\hat{m}_t - \hat{\bar{k}}_{t-1} + \hat{z}_t + \left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t\right]\right.$$

which gives

$$\hat{\delta}_t = \frac{\zeta}{\zeta + \bar{\delta}\left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)}\right)^{\sigma}} \left[\sigma\hat{\bar{k}}_{t-1} - \sigma\hat{\tilde{m}}_t - \sigma\hat{d}_t - \sigma\hat{z}_t - \sigma\left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t + \eta\hat{u}_t\right]$$

The definition of maintenance costs given in (D.31), using (D.32), is log-linearized as follows

$$\ln m_t = \ln\left[\tau u_t \bar{k}_{t-1} e^{-\left(z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)} + \bar{M}\right]$$

$$\hat{m}_t = \frac{1}{\tilde{m}}\tau\bar{k}e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \left[\hat{u}_t + \hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t\right]$$

Rearranging and recalling that $\hat{m}_t = \hat{\tilde{m}}_t + \hat{d}_t$, we obtain

$$\hat{\tilde{m}}_t = -\hat{d}_t + \tau\left(\frac{\tilde{m}}{\bar{k}} e^{\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1} \left[\hat{u}_t + \hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t\right] \quad \text{(E.23)}$$

Next, we log-linearize the law of motion of capital, given (D.33) and (D.34), as follows

$$\ln \bar{k}_t = \ln\left\{ \left(1 - \delta_t e^{\sigma v_t}\right)\bar{k}_{t-1} e^{-\left(z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)} + \mu_t\left[1 - S\left(\frac{i_t}{i_{t-1}} e^{z_t + \left(\frac{\alpha}{1-\alpha}+1\right)v_t}\right)\right] i_t \right\}$$

$$\hat{\bar{k}}_t = \frac{1}{\bar{k}}\left\{ \left(1 - \delta e^{\sigma\gamma_v}\right)\bar{k}e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \left[\hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t\right] + \right.$$

$$\left. - \delta e^{\sigma\gamma_v}\bar{k}e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \left[\hat{\delta}_t + \sigma\hat{v}_t\right] + i\left(\hat{\mu}_t + \hat{i}_t\right) \right\}$$

$$\hat{\bar{k}}_t = \left(1 - \delta e^{\sigma\gamma_v}\right) e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \left[\hat{\bar{k}}_{t-1} - \hat{z}_t - \left(\frac{\alpha}{1-\alpha}+1\right)\hat{v}_t\right] - \delta e^{\sigma\gamma_v} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \left[\hat{\delta}_t + \sigma\hat{v}_t\right] +$$

$$+ \left[1 - \left(1 - \delta e^{\sigma\gamma_v}\right) e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\right] \left(\hat{\mu}_t + \hat{i}_t\right)$$

$$\hat{\bar{k}}_t = \left(1 - \delta e^{\sigma\gamma_v}\right) e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \hat{\bar{k}}_{t-1} - \delta e^{\sigma\gamma_v} e^{-\left(\gamma_z + \left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)} \hat{\delta}_t +$$

$$+\left[1-\left(1-\delta e^{\sigma\gamma_v}\right)e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\right]\left(\hat{\mu}_t+\hat{i}_t\right)-\left(1-\delta e^{\sigma\gamma_v}\right)e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{z}_t+ \tag{E.24}$$
$$-\left[\left(1-\delta e^{\sigma\gamma_v}\right)\left(\frac{\alpha}{1-\alpha}+1\right)+\sigma\delta e^{\sigma\gamma_v}\right]e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{v}_t$$

We log-linearize the aggregate resource constraint, given (D.35) and (D.36), as follows

$$\ln\left[c_t+i_t+a\left(u_t\right)\bar{k}_{t-1}e^{-\left(z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)}+\tilde{m}_t\right]=\ln\left[\left(1/g_t\right)y_t\right]$$
$$\hat{y}_t-\hat{g}_t=\frac{g}{y}\left[c\hat{c}_t+i\hat{i}_t+\tilde{m}\hat{\tilde{m}}_t+a'(1)\bar{k}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t\right]$$

Substituting the expression for the steady state of the optimal capital utilization rate given in (E.11), and rearranging, the latter expression becomes

$$\frac{1}{g}\hat{y}_t-\frac{1}{g}\hat{g}_t=\frac{c}{y}\hat{c}_t+\frac{i}{y}\hat{i}_t+\frac{\tilde{m}}{y}\hat{\tilde{m}}_t+$$
$$+\left\{\rho-\eta\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}+\tau\left[\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-1\right]\right\}\frac{\bar{k}}{y}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t \tag{E.25}$$

Before log-linearizing the definition of actual GDP, combine equations (D.35) and (D.37) in order to obtain the following expressions

$$x_t=\left(1-1/g_t\right)y_t+\frac{y_t}{g_t}-a\left(u_t\right)\bar{k}_{t-1}e^{-\left(z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)}$$
$$x_t=y_t-a\left(u_t\right)\bar{k}_{t-1}e^{-\left(z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)}$$

In steady state, the latter one becomes

$$x=y$$

which, recalling (E.11), will be used for the log-linearization computations as follows

$$\ln x_t=\ln\left[y_t-a\left(u_t\right)\bar{k}_{t-1}e^{-\left(z_t+\left(\frac{\alpha}{1-\alpha}+1\right)v_t\right)}\right]$$
$$\hat{x}_t=\frac{1}{y}\left[y\hat{y}-a'(1)\bar{k}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t\right]$$
$$\hat{x}_t=\hat{y}-a'(1)\frac{\bar{k}}{y}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t$$
$$\hat{x}_t=\hat{y}-\left\{\rho-\eta\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}+\tau\left[\sigma\left(\frac{\tilde{m}}{\bar{k}}e^{\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v}\right)^{-1}\left(\delta-\bar{\delta}\right)e^{\sigma\gamma_v}-1\right]\right\}\frac{\bar{k}}{y}e^{-\left(\gamma_z+\left(\frac{\alpha}{1-\alpha}+1\right)\gamma_v\right)}\hat{u}_t \tag{E.26}$$

Finally, denoting by $x^\star$ the stationary GDP gap, the nominal interest rate rule is log-linearized as follows

$$\ln\left(\frac{R_t}{R}\right)=\ln\left\{\left(\frac{R_{t-1}}{R}\right)^{\rho_R}\left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi}\left(\frac{x_t}{x_t^\star}\right)^{\phi_X}\right]^{1-\rho_R}\left(\frac{x_t/x_{t-1}}{x_t^\star/x_{t-1}^\star}\right)^{\phi_{dX}}\varepsilon_{mp,t}\right\}$$

$$\ln R_t - \ln R = \rho_R \left(\ln R_{t-1} - \ln R\right) + \left(1-\rho_R\right)\left[\phi_\pi \left(\ln \pi_t - \ln \pi\right) + \phi_X \left(\ln x_t - \ln x_t^\star\right)\right] +$$
$$+ \phi_{dX} \left(\ln x_t - \ln x_{t-1} - \ln x_t^\star + \ln x_{t-1}^\star\right) + \ln \varepsilon_{mp,t}$$
$$\hat{R}_t = \rho_R \hat{R}_{t-1} + \left(1-\rho_R\right)\left(\phi_\pi \hat{\pi}_t + \phi_X \hat{x}_t - \phi_X \hat{x}_t^\star\right) + \phi_{dX}\left(\hat{x}_t - \hat{x}_{t-1} - \hat{x}_t^\star + \hat{x}_{t-1}^\star\right) + \hat{\varepsilon}_{mp,t}$$

obtaining, finally

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + \left(1-\rho_R\right)\phi_\pi \hat{\pi}_t + \left[\left(1-\rho_R\right)\phi_X + \phi_{dX}\right]\hat{x}_t +$$
$$- \left[\left(1-\rho_R\right)\phi_X + \phi_{dX}\right]\hat{x}_t^\star - \phi_{dX}\hat{x}_{t-1} + \phi_{dX}\hat{x}_{t-1}^\star + \hat{\varepsilon}_{mp,t} \tag{E.27}$$

## F APPENDIX: Data construction

For the construction of our database we have used the series available on the CANSIM database of Canadian Statistics. For the estimation of our model we use eight observables, which are

$$\left[\Delta \log X_t \quad \Delta \log C_t \quad \Delta \log \tilde{I}_t \quad \log L_t \quad \Delta \log \frac{W_t}{P_t} \quad \pi_t \quad R_t \quad \Delta \log \frac{P_t^I}{P_t}\right]$$

The above variables are defined as follows

- Output: $\log X_t = \ln(GDP/POPindex) \times 100$
- Consumption: $\log C_t = \ln[(CONS/IMPC)/POPindex] \times 100$
- Investment: $\log \tilde{I}_t = \ln[(INV/IMPC)/POPindex] \times 100$
- Hours: $\log L_t = \ln[(H \times EMPindex/100)/POPindex] \times 100$
- Real wage: $\log W_t/P_t = \ln(COMPH/IMPC) \times 100$
- Inflation: $\pi_t = [\ln(IMPC/IMPC(-1))] \times 100$
- Nominal rate: $R_t = NOMR/4$
- Relative price of investment: $\log(P_t^I/P_t) = \ln(IMPI/IMPC) \times 100$

The terminology on the left-hand side of the above expressions is explained below.

$GDP$: Real Gross Domestic Product, billions of chained 2007 dollars, seasonally adjusted

$CONS$: Households final consumption expenditure in goods (non-durables and semi-durables) and services, billions of dollars at current prices, seasonally adjusted

$INV$: Households final consumption expenditure in durable goods plus business gross fixed capital formation, billions of dollars at current prices, seasonally adjusted

$IMPC$: GDP implicit price index (2007Q3=100), average of non-durables, semi-durables and services, seasonally adjusted

$IMPI$: GDP implicit price index (2007Q3=100), average of durables and business gross fixed capital formation, seasonally adjusted

$POP$: Labor force 15 years and over, number in thousands of civilian, non-institutionalized persons, seasonally adjusted

$POPindex$: $POP$(2007Q3)=1

$EMP$: Civilian employment 15 years and over, thousands, seasonally adjusted

$EMPindex$: 100×$EMP$(2007Q3)=1

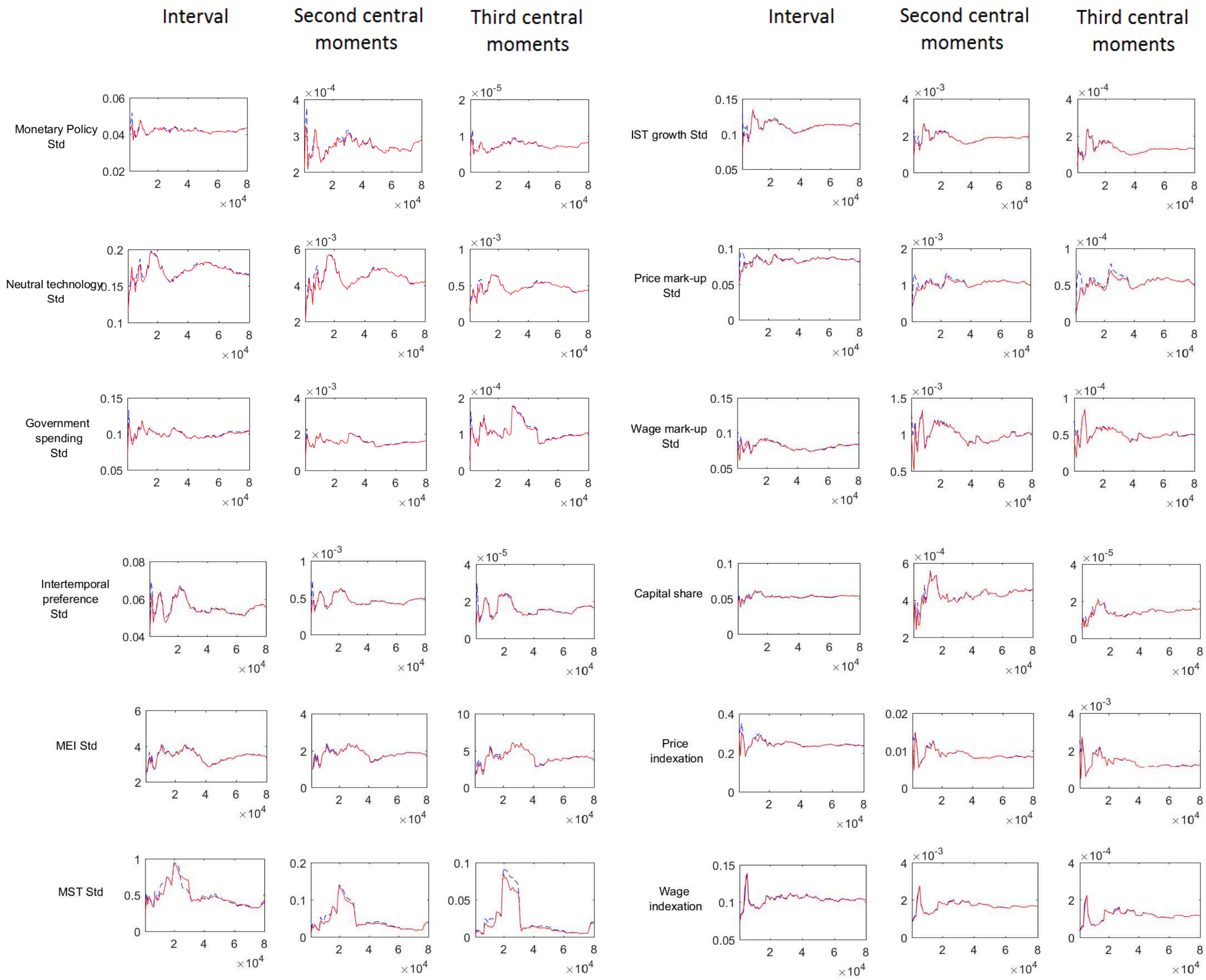
$H$: Average of hours worked (2007Q3=100), business sector, seasonally adjusted

$COMPH$: Total compensation per hour worked (2007Q3=100), business sector, seasonally adjusted

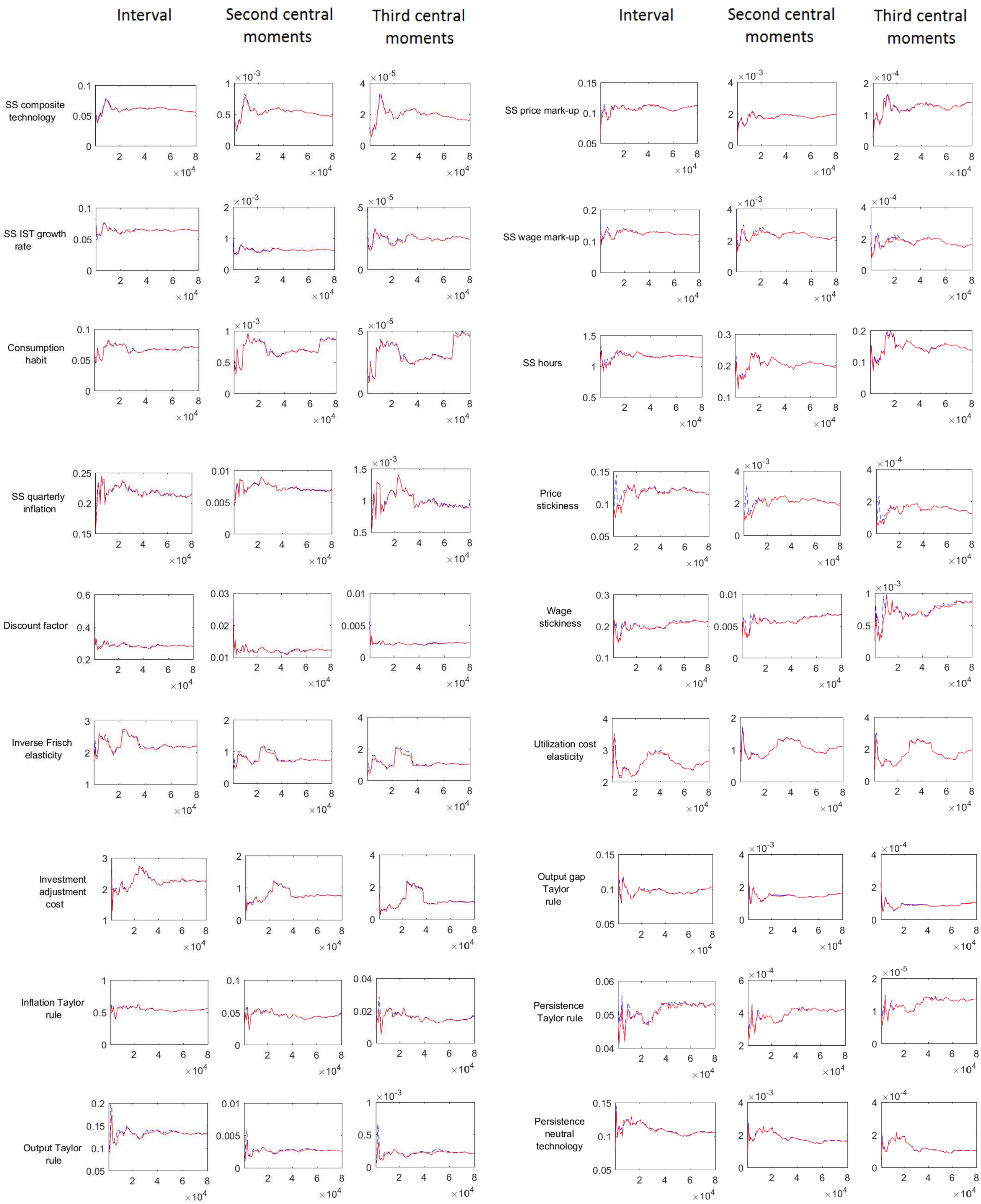
$NOMR$: 3-month treasury bills, percent of seven-day average
Source: Financial market statistics, Bank of Canada

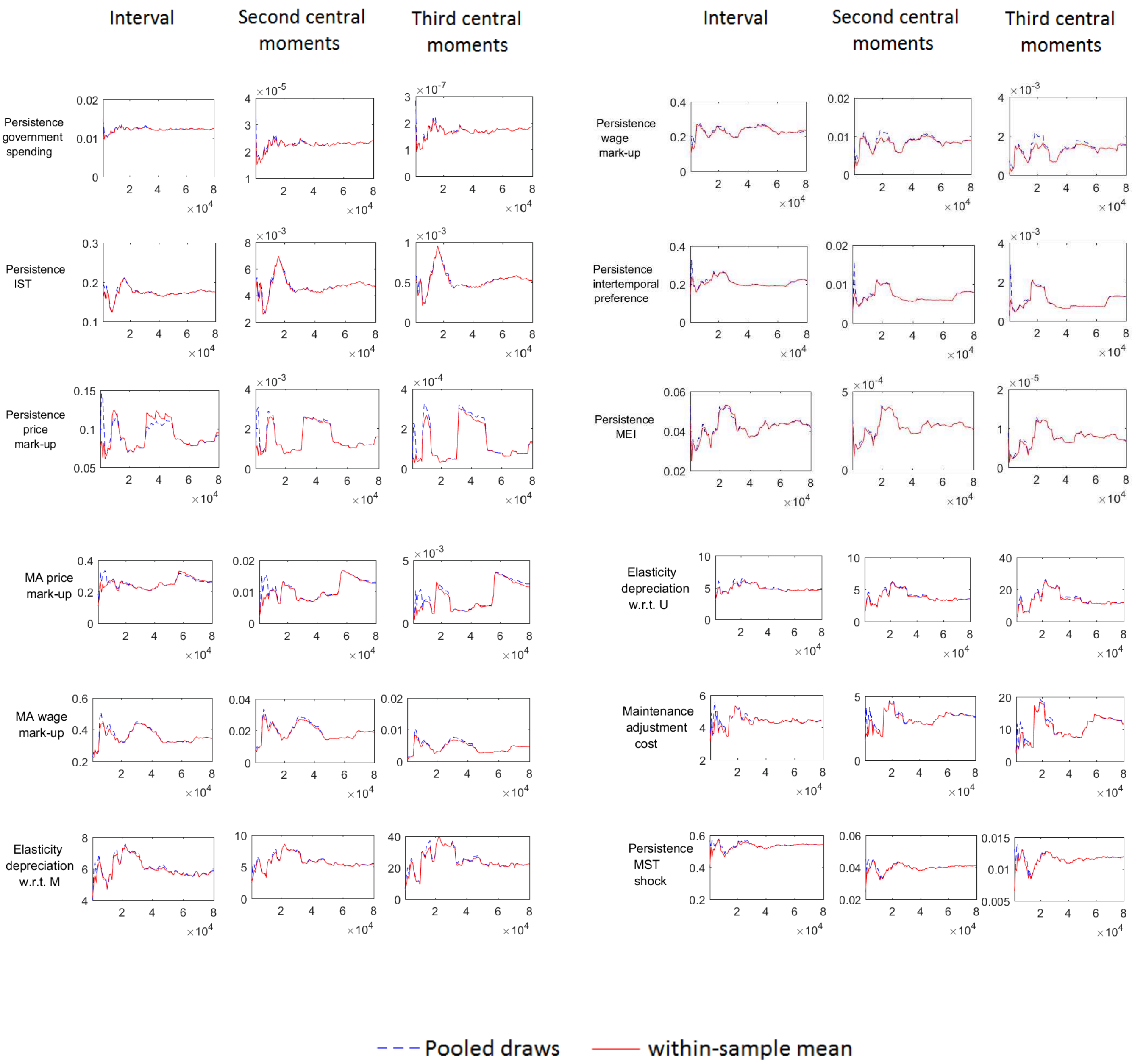
# G APPENDIX: Univariate convergence diagnostics

Fig. G.1: Univariate convergence diagnostics: maintenance model




...continue


Interval
Second central moments
Third central moments
Interval
Second central moments
Third central moments
SS composite technology
SS IST growth rate
Consumption habit
SS quarterly inflation
Discount factor
Inverse Frisch elasticity
Investment adjustment cost
Inflation Taylor rule
Output Taylor rule
SS price mark-up
SS wage mark-up
SS hours
Price stickiness
Wage stickiness
Utilization cost elasticity
Output gap Taylor rule
Persistence Taylor rule
Persistence neutral technology
$\times 10^4$


Interval
Second central moments
Third central moments
Interval
Second central moments
Third central moments
Persistence government spending
Persistence IST
Persistence price mark-up
Persistence wage mark-up
Persistence intertemporal preference
Persistence MEI
MA price mark-up
MA wage mark-up
Elasticity depreciation w.r.t. M
Elasticity depreciation w.r.t. U
Maintenance adjustment cost
Persistence MST shock
Pooled draws
within-sample mean

## H APPENDIX: Impulse response functions

Fig. H.1: Impulse response functions to one standard deviation shock to labor augmenting technology

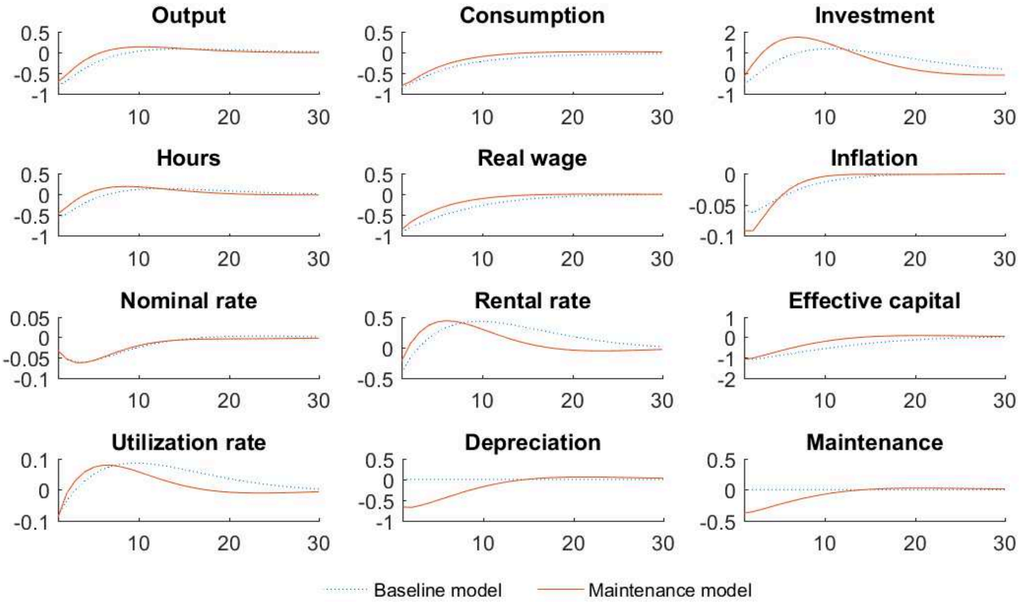



Fig. H.2: Impulse response functions to one standard deviation shock to monetary policy

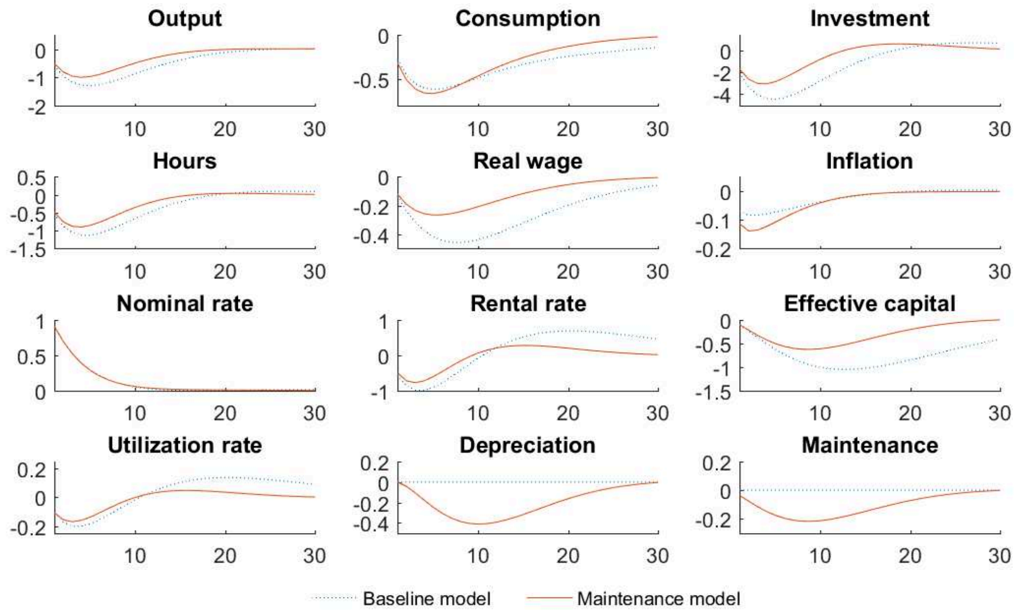


Fig. H.3: Impulse response functions to one standard deviation shock to government spending

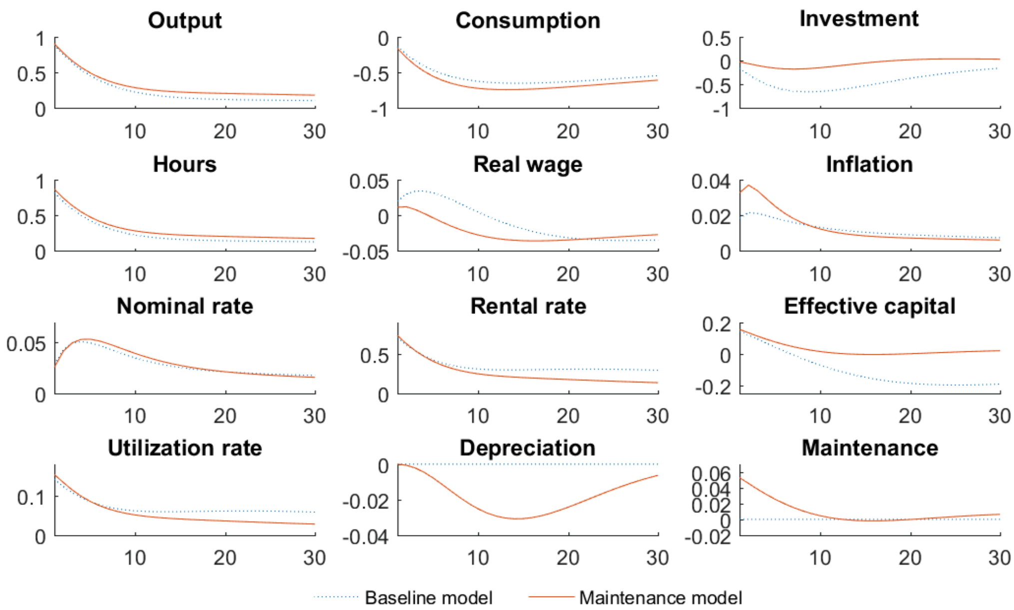


Fig. H.4: Impulse response functions to one standard deviation shock to intertemporal preferences

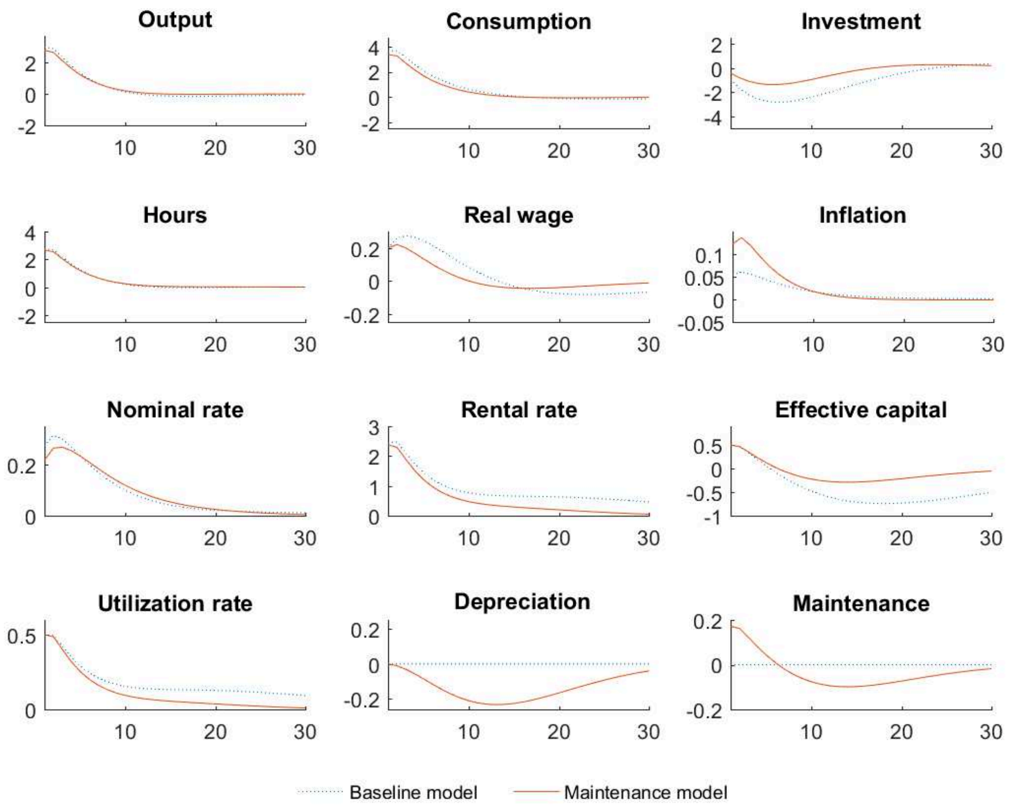


Fig. H.5: Impulse response functions to one standard deviation shock to price mark-up

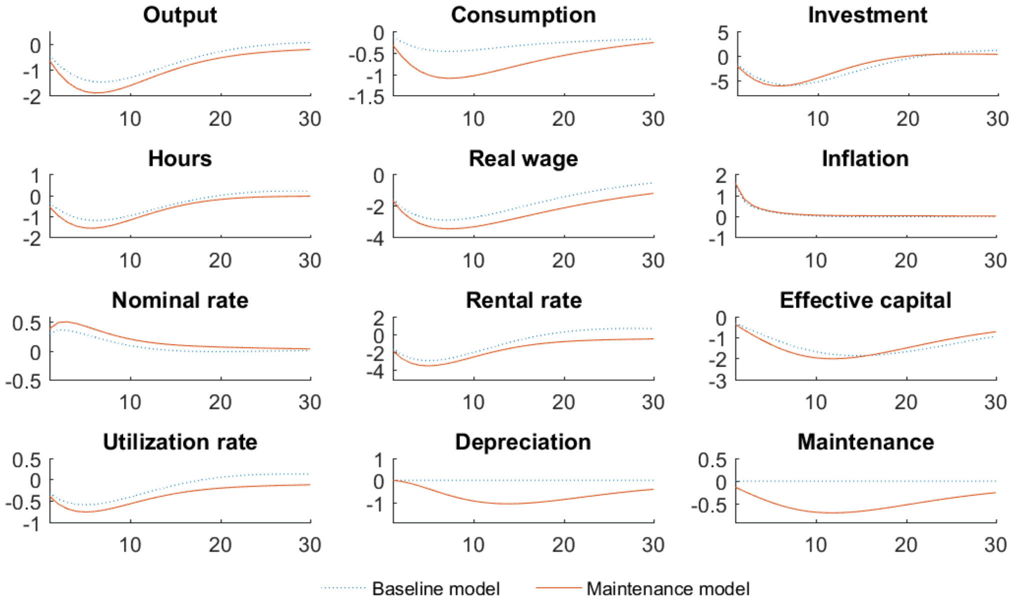



Fig. H.6: Impulse response functions to one standard deviation shock to wage mark-up