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ORIGINAL ARTICLE

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Dynare replication of "A Model of Secular Stagnation: Theory and Quantitative Evaluation" by Eggertsson et al. (2019)

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Abstract

This paper replicates the study "A Model of Secular Stagnation: Theory and Quantitative Evaluation" by Eggertsson et al. using the Dynare toolkit. Replication is important as it confirms the results of the original article, provides a user-friendly version using Dynare, and shows how to deal with large-scale models with occasionally binding constraints. The results show that the original Matlab code was fully replicated, but minor discrepancies were found between the paper's equations and the code. The two models produce similar dynamics but with small differences, particularly at the beginning of the simulation.

KEYWORDS

Dynare, occasionally binding constraints, OLG, secular stagnation

JEL CLASSIFICATION

E0, E43, C62

1 | INTRODUCTION

This paper replicates the large-scale overlapping generation model (OLG) of "A Model of Secular Stagnation: Theory and Quantitative Evaluation" by Eggertsson et al. (2019) using the latest version (5.3) of Dynare (Adjemian et al., 2022), a standard software to simulate and estimate dynamic general equilibrium models widely used in academies, central banks, and other institutions. The original model builds a large-scale OLG model with several occasionally binding constraints (OBCs) in order to capture the main features of the Secular Stagnation Hypothesis (Summers, 2015), with a focus on the long-run decline of interest rates. The authors show the quantitative importance of key drivers such as population aging and slowing productivity growth over the interest rate for the United States, computing transitional dynamics from 1970 to 2015.

Our replication exercise is intended to provide a valuable contribution along three dimensions. First, it confirms the results of a significant contribution well-known in macroeconomics. Indeed, the Secular Stagnation Hypothesis

Abbreviations: GFC, great financial crisis; OBCs, occasionally binding constraints; OLG, overlapping generation models; ZLB, zero lower bound.
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(Summers, 2015) and its quantitative relevance (Eggertsson et al., 2019), has been one of the most credited developments in macroeconomic thought since the Great Financial Crisis (GFC), especially for gaining insights on the causes of the GFC and on the role of monetary policy stuck at the zero lower bound. Second, the replication makes use of Dynare, which represents the state-of-the-art software for the simulation and estimation of dynamic general equilibrium models. Dynare provides a user-friendly platform that is easier to use with respect to the complex original Matlab code of Eggertsson et al. (2019), therefore reducing the entry barriers for those interested in large-scale OLG modeling and being helpful for the entire community of users. Third, the original model includes several OBCs that are particularly challenging to compute but extremely important from a policy standpoint. Modelers have traditionally dealt with the presence of OBCs by using toolkits such as Guerrieri and Iacoviello (2015) and Holden (2016). In this case, the presence of an OBC for each working-age generation makes them not feasible. By rewriting the constraints as in Swarbrick (2021), we show how to handle and treat several OBCs with a large-scale model and with the standard algorithms included in Dynare in an easier and faster way relative to the original Matlab code.[1]

The results from the replication substantially confirm the original outcome of Eggertsson et al. (2019). On one hand, following the original Matlab code available on the *American Economic Journal: Macroeconomics* repository, we are able to fully replicate the original results with the exception of fig. 8 of the original paper. However, we found the equations of the original Matlab code to be slightly different from the ones reported in the text. Once we rewrite the model as in the paper, small differences between the transition dynamics of the two models emerge, especially in the first 30 periods of the simulation. All in all, the replication exercise confirms the original results of the paper.

This replication paper is organized as follows: In Section 2, we compare the results from the original Matlab code with our Dynare implementation, whereas in Section 3 we show the same Dynare replication but making use of the equations delivered in the original paper. Section 4 concludes.

2 | REPLICATION OF THE ORIGINAL MATLAB CODE

In this section, we focus on the replication of the original Matlab code. A salient feature of the original model is the presence of several occasionally binding collateral constraints, one for every 40 working-generations (j) contained in the model. The collateral constraint is as follows:

$$a^j \geq D^j \cdot w \cdot hc^j \tag{1}$$

OccBin (Guerrieri & Iacoviello, 2015) and DynareOBC (Holden, 2016) toolkits, typically used in Dynare to deal with OBCs cannot deal with a large numbers of constraints. The simplest way to workaround the problem is therefore using a brute-force approach with the min/max function in a perfect foresight environment along the lines proposed by Swarbrick (2021). Therefore, Equation (1) can be transformed as follows:

$$\min\left(\lambda^j, a^j - D^j \cdot w \cdot hc^j\right) = 0 \tag{2}$$

where λ^j represents the lagrange multiplier, a^j the asset of each generation, $D^j \leq 0$ the individual debt limit of each generation, and hc^j the human capital profile which shapes the wage w profile among generations. However, in our Dynare code as well as in the Matlab implementation by Eggertsson et al. (2019), the debt limit is written with positive values, $D^j \geq 0$, therefore each constraint will be:

$$\min\left(\lambda^j, a^j + D^j \cdot w \cdot hc^j\right) = 0 \tag{3}$$

2.1 | The Matlab equations

The results of this section are given by the direct translation from the original Matlab code provided by the authors into Dynare notation. The full set of the equation used in this section can be found in Appendix A.[2]

2.2 | Results

We start our replication exercise by computing the 1970 steady-state values for the baseline calibration.[3] Results are reported in Table 1 where the last four rows should be compared with the ones reported in tab. 5 on page 39 of the original paper. The other variables are not presented in the original paper but can be easily retrieved from the Matlab code.

Figure 1 instead, reports the comparison of the transitional dynamics of several endogenous variables of our Dynare code with the original Matlab code, again for the main calibration made by the authors. There are basically no discrepancies between the two simulations, where many differences arrive to a magnitude of 10^{-13}. The dynamics of these aggregate variables are not reported in the original paper but can be found on the Matlab code.[4]

Figure 2 is the replication of fig. 8 on page 42 of the original paper. The simulations obtained from the Dynare code and the Matlab code are filtered using the two-side Hodrick-Prescott (HP) filter and they are identical to each other. However, the two simulations differ from fig. 8 in the original paper (the dotted line in fig. 8, Model national rate). The original paper does not report any details on the filtering methodology or if the data are filtered at all. Our intuition is that, since the Fed funds rate is plotted using the HP trend, the simulated series are also filtered in order to maintain comparability. However, also by doing this, we find some discrepancies in fig. 8, it seems to us that the dynamics reported are shifted down by a constant, relative to what is written in the text and what comes from the Matlab code of the authors. In particular, the description of fig. 8 on page 41 basically does not correspond with fig. 8 itself. The steady-state at 1970 which is around 2% in fig. 8, is lower than what is reported in the text and what is coming from the Matlab code of the authors, 2.55%. Moreover, data about the figure and the plot of the figure itself are not present in any part of their replication material. However, despite these small inconsistencies, the overall storytelling of the paper remains unaltered.

3 | PAPER REPLICATION

In this section, we implement the Dynare code using the equations as written in the original paper, both for the steady state and the transition dynamics. We highlight the differences in red. Appendix B presents the full set of equations and C provides the full derivation of the model.

TABLE 1 1970 steady state for the baseline calibration.

Variables	Steady state 1970
Capital	48.69
Labor	36.58
Population	34.03
Income	52.72
Consumption	31.49
Aggregate profit	6.65
Investment	10.03
Rental rate	0.195
Wage tax	0.301
Bequest	2.504
Population growth rate	0.014
Debt	0.458
Public expenditure	11.22
Investments to output ratio	**19.0%**
Interest rate	**2.55%**
Labor share	**72.4%**
Consumer-debt-to-output-ratio	**4.20%**

Note: Bold variables are presented in tab. 5 on page 39 of the original paper.

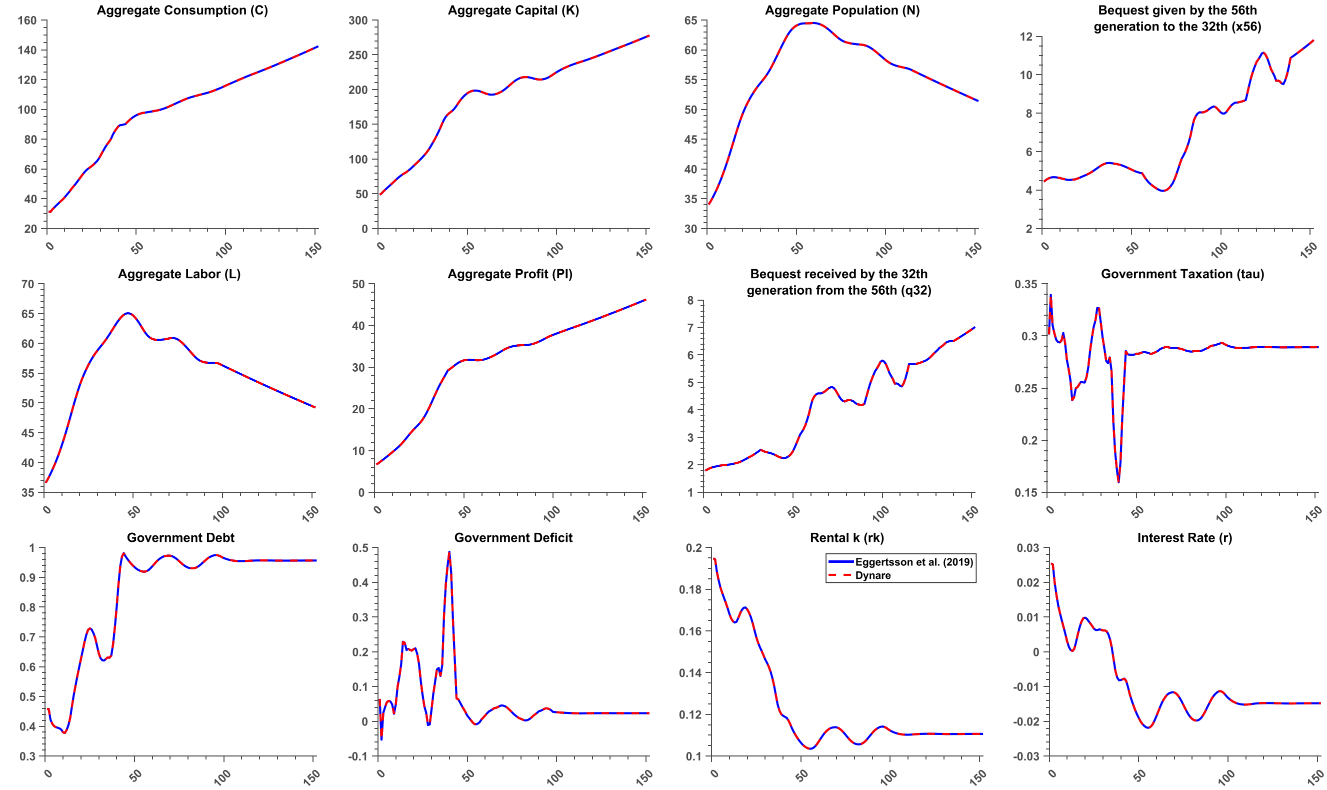

FIGURE 1 Aggregate variables dynamics from Eggertsson et al. (2019) Matlab code (blue line) and the Dynare implementation (dashed red line).

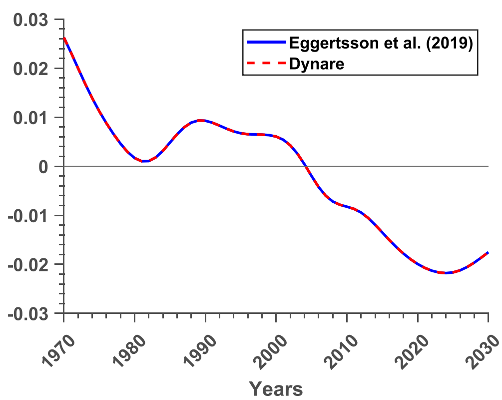

FIGURE 2 Natural interest rate (HP trend). HP, Hodrick-Prescott.

Assuming as a benchmark their Matlab code, we find differences about some points:

1. The Equations (B3–B12) and (B36–B41) for the various budget constraints written in the original paper differ from the ones written in the function "opt_lb_alt.m" of the Matlab code with respect to (a) the relative price of capital goods (e) which is not presented at all in the Matlab function and cannot be simplified with the no-arbitrage condition, at least when considering the transitional dynamics for what is written in the paper; (b) the bequest received (q^j) differently from what is written in the paper, is left out of the multiplication with $(rk+\epsilon(1-\delta))$ in Matlab;
2. The Equations (B13) and (B42) for the borrowing constraints used in the functions "opt_lb_alt.m" and "create_profile.m" of the Matlab code, is different from what is written in the paper. Instead of using $a^j \geq \frac{D^j}{1+r}$, they used $a^j \geq D^j \cdot w \cdot hc^j$;

3. The Equations (B30) and (B58) for the asset market clearing conditions are different from what is written in “repeatfunc.m” in the Matlab code.

3.1 | Results

The results obtained from the paper equations are reported in Figures 3 and 4. The results are essentially the same compared to the ones obtained with the original Matlab code. However, there are a few discrepancies at the beginning of the sample, especially for the bequest variables (q32 and x56), but at the end of the day the endogenous variables' dynamics are fully captured by our Dynare implementation.

Figure 4 compares the dynamics of the interest rates filtered using the two-side HP filter obtained from the original Matlab implementation and our Dynare code written with the paper equations. The replication is not as perfect as in Figure 2 but the dynamic is basically the same. The problems emerged in the previous Section remain.

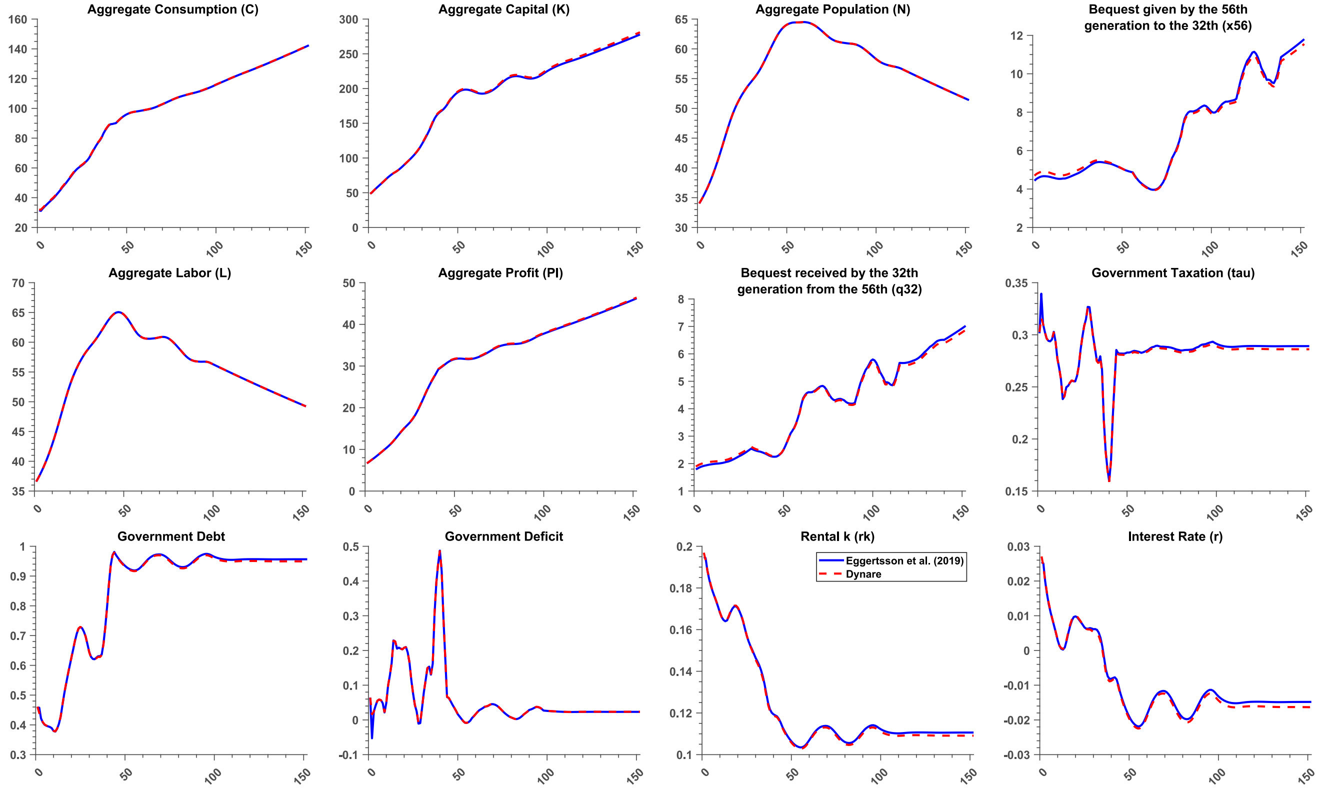

FIGURE 3 Aggregate variables dynamics from Eggertsson et al. (2019) Matlab code (blue line) and the paper Dynare implementation (dashed red line).

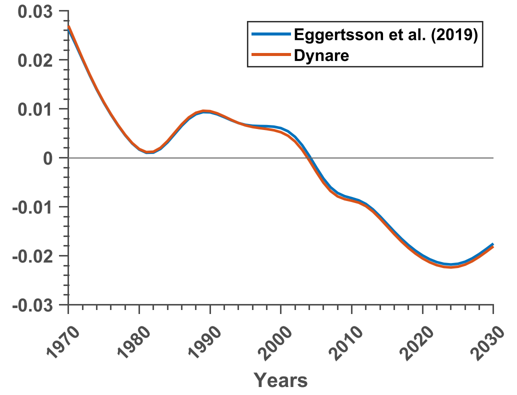

FIGURE 4 HP trend interest rate. HP, Hodrick-Prescott.

4 | CONCLUSION

In this article, we provide a successful replication of the paper "A Model of Secular Stagnation: Theory and Quantitative Evaluation" by Eggertsson et al. (2019) using the free software Dynare. The results are almost identical to the ones presented in the original paper. A few discrepancies emerge between the equations presented in the paper and the ones available in the original Matlab code. However, these small differences have negligible effects on the dynamics of the aggregate variable.

Our Dynare implementation has several strengths with respect to the original MatLab implementation. In particular, the easiness of our code can facilitate the works of other scholars in the field of OLG modeling tearing down the entry barriers usually very high, helping them in developing a complete quantitative research going from the calibration to the simulation procedures. Finally, our code shows how to deal with several OBCs in Dynare providing a fast, yet accurate, way to produce consistent results.

DATA AVAILABILITY STATEMENT

The replication code is openly available in openICPSR at https://doi.org/10.3886/E193461V2 (Giri, 2023).

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ENDNOTES

[1] Of course, many models cannot be handled by Dynare, and each one may require its own modeling techniques. However, when the constraints are well-behaved (Swarbrick, 2021), we show that Dynare is also able to solve non-linear large-scale models with multiple occasionally binding constraints (Eggertsson et al., 2019) in a very efficient way.

[2] The model's derivations are basically the same as detailed in Appendix C for the model written as in the paper, but with the discrepancies in the equations as outlined in Section 3.

[3] Inside the replication kit, the user will also find the code to replicate the calibration of the model using Dynare.

[4] In the folder "Alt_Calibrations" of our replication material, you also have access to the alternative calibrations made by Eggertsson et al. (2019) (look at the Online Appendix from page A.28) and the relative comparisons with our Dynare code. Again, the results are identical with many differences in the order of 10^{-13}.

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SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

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APPENDIX A: THE MATLAB EQUATIONS

A.1 | Steady state equations

$$j \in \{26, \ldots, J = 81\} \qquad n^{26} = 1 \tag{A1}$$

$$n^{j+1} = \frac{s^j \cdot n^j}{1+n} \tag{A2}$$

$$for \quad j \in \{26, J-1\} \qquad \frac{1}{\beta} = \left(\frac{c^{j+1}}{c^j}\right)^{-\frac{1}{\gamma}} \cdot (1+r) + \lambda^{j+1}\frac{(c^j)^{-\frac{1}{\gamma}}}{su^j\beta^j} \tag{A3}$$

$$for \quad j \in \{26, J-1\} \qquad x^J = \left(\frac{\Gamma}{\mu}\right)^{-\gamma} \cdot c^J \tag{A4}$$

$$for \quad j \in \{J\} \qquad a^j = 0 \tag{A5}$$

$$for \quad j \in \{26\} \qquad a^{j+1} = \frac{(1+r)\cdot a^j}{sv^j} + \left((1-\tau)\cdot w \cdot hc^j + \pi^j\right)\cdot\left(1+AL_{growth}\right)^j - c^j \tag{A6}$$

$$for \quad j \in \{26, \ldots, 56\} \qquad a^{j+1} = \frac{(1+r)\cdot a^j}{sv^j} + \left((1-\tau)\cdot w \cdot hc^j + \pi^j\right)\cdot\left(1+AL_{growth}\right)^j + q^{j+1}\cdot\left(1+AL_{growth}\right)^{j+1} - c^j \tag{A7}$$

$$for \quad j \in \{56\} \qquad q^j = \frac{x^J \cdot \Gamma \cdot n^J}{n^j} \tag{A8}$$

$$for \quad j \in \{57\} \qquad a^{j+1} = \frac{(1+r)\cdot a^j}{sv^j} + \left((1-\tau)\cdot w \cdot hc^j + \pi^j\right)\cdot\left(1+AL_{growth}\right)^j - c^j \tag{A9}$$

$$for \quad j \in \{57, \ldots, 65\} \qquad a^{j+1} = \frac{(1+r)\cdot a^j}{sv^j} - c^j \tag{A10}$$

$$for \quad j \in \{66, \ldots, 80\} \qquad c^j = \frac{(1+r)\cdot a^j}{sv^j} - \Gamma \cdot x^j \tag{A11}$$

$$for \quad j \in \{81\} \qquad \min\left(\lambda^j, a^j + \left(D^j \cdot w \cdot hc^j\right)\cdot\left(1+AL_{growth}\right)^j\right) = 0 \tag{A12}$$

$$\begin{aligned}&for \quad j \in \{26,\dots,65\}\\&\min\left(\lambda^{j},a^{j}\right)=0\end{aligned} \tag{A13}$$

$$\begin{aligned}&for \quad j \in \{66,\dots,81\}\\&\pi^{j}=\frac{hc^{j}\cdot\Pi}{L}\end{aligned} \tag{A14}$$

$$for \quad j \in \{26,\dots,65\}$$

$$\frac{p^{int}}{P}=\frac{\theta-1}{\theta} \tag{A15}$$

$$A_{adj}=\frac{p^{int}}{P}\cdot\left(\alpha\cdot(AK\cdot K)^{\frac{\sigma-1}{\sigma}}+(1-\alpha)\cdot(AL\cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\cdot(1-\alpha)\cdot AL^{\frac{\sigma-1}{\sigma}}\cdot L^{-\frac{1}{\sigma}} \tag{A16}$$

$$w=1 \tag{A17}$$

$$rk=\frac{\frac{p^{int}}{P}\cdot\left(\alpha\cdot(AK\cdot K)^{\frac{\sigma-1}{\sigma}}+(1-\alpha)\cdot(AL\cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\cdot\alpha\cdot AK^{\frac{\sigma-1}{\sigma}}\cdot K^{-\frac{1}{\sigma}}}{A_{adj}} \tag{A18}$$

$$Y=\frac{\left(\alpha\cdot(AK\cdot K)^{\frac{\sigma-1}{\sigma}}+(1-\alpha)\cdot(AL\cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{A_{adj}} \tag{A19}$$

$$r=\frac{rk+(1-\delta)\epsilon}{\epsilon}-1 \tag{A20}$$

$$\Pi=\frac{Y}{\theta} \tag{A21}$$

$$gov^{deficit}\cdot gov^{rev}=\left(\left(1+AL_{growth}\right)\cdot(1+n)-1\right)\cdot\left(gov^{debt}\cdot K\right) \tag{A22}$$

$$gov^{debt}=b\cdot\frac{Y}{K} \tag{A23}$$

$$gov^{rev}=\left(g\cdot Y+r\cdot gov^{debt}\cdot K\right) \tag{A24}$$

$$\tau=\frac{\left(gov^{rev}\cdot\left(1-gov^{deficit}\right)\right)}{w\cdot L} \tag{A25}$$

$$N=\sum_{j=26}^{J}n^{j} \tag{A26}$$

$$L=\sum_{j=26}^{J}n^{j}hc^{j} \tag{A27}$$

$$C=\sum_{j=26}^{J}n^{j}c^{j}\left(1+AL_{growth}\right)^{j} \tag{A28}$$

$$K=\frac{\left(\sum_{j=26}^{J}\frac{n^{j}a^{j}}{sv^{j}\cdot\left(1+AL_{growth}\right)^{j}}\right)}{\epsilon+gov^{debt}} \tag{A29}$$

A.1.1 | Transitional dynamics

$$j \in \{26,\ldots,J=81\}$$
$$n_t^{26} = \frac{n_{t-1}^{25}}{su_{t-1}^{25}} \cdot \Gamma_t \tag{A30}$$

$$n_t^{j+1} = s_{t-1}^j \cdot n_{t-1}^j \tag{A31}$$

$$for \quad j \in \{26, J-1\}$$
$$\frac{1}{\beta} = \left(\frac{c_{t+1}^{j+1}}{c_t^j}\right)^{-\frac{1}{\gamma}} \cdot (1+r_{t+1}) + \lambda_t^{j+1} \frac{\left(c_t^j\right)^{-\frac{1}{\gamma}}}{su_t^j \beta^j} \tag{A32}$$

$$for \quad j \in \{26, J-1\}$$
$$x_t^J = \left(\frac{\Gamma_{t-J+26}}{\mu}\right)^{-\gamma} \cdot c_t^J \tag{A33}$$

$$for \quad j \in \{J\}$$
$$a_t^j = 0 \tag{A34}$$

$$for \quad j \in \{26\}$$
$$a_{t+1}^{j+1} = \frac{(1+r_t) \cdot a_t^j}{sv_t^j} + (1-\tau_t) \cdot w_t \cdot hc^j + \pi_t^j - c_t^j \tag{A35}$$

$$for \quad j \in \{26,\ldots,56\}$$
$$a_{t+1}^{j+1} = \frac{(1+r_t) \cdot a_t^j}{sv_t^j} + (1-\tau_t) \cdot w_t \cdot hc^j + \pi_t^j + q_{t+1}^{j+1} - c_t^j \tag{A36}$$

$$for \quad j \in \{56\}$$
$$q^j = \frac{x_{t-1}^J \cdot \Gamma_{t-56} \cdot n_{t-1}^J}{n_t^j} \tag{A37}$$

$$for \quad j \in \{57\}$$
$$a_{t+1}^{j+1} = \frac{(1+r_t) \cdot a_t^j}{sv_t^j} + (1-\tau_t) \cdot w_t \cdot hc^j + \pi_t^j - c_t^j \tag{A38}$$

$$for \quad j \in \{57,\ldots,65\}$$
$$a_{t+1}^{j+1} = \frac{(1+r_t) \cdot a_t^j}{sv_t^j} - c_t^j \tag{A39}$$

$$for \quad j \in \{66,\ldots,80\}$$
$$c_t^j = \frac{(1+r_t) \cdot a_t^j}{sv_t^j} - \Gamma_{t-55} \cdot x_t^j \tag{A40}$$

$$for \quad j \in \{81\}$$
$$\min\left(\lambda_t^j, a_t^j + \left(D_{t+1}^j \cdot w_{t+1} \cdot hc^j\right)\right) = 0 \tag{A41}$$

$$\begin{gathered} for \quad j \in \{26, \ldots, 65\} \\ \min\left(\lambda_t^j, a_t^j\right) = 0 \end{gathered} \tag{A42}$$

$$\begin{gathered} for \quad j \in \{66, \ldots, 81\} \\ \pi_t^j = \frac{hc^j \cdot \Pi_t}{L_t} \end{gathered} \tag{A43}$$

$$for \quad j \in \{26, \ldots, 65\}$$

$$\frac{p_t^{int}}{P_t} = \frac{\theta - 1}{\theta} \tag{A44}$$

$$w_t = \frac{\frac{p_t^{int}}{P_t} \cdot \left(\alpha \cdot (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot (1-\alpha) \cdot AL_t^{\frac{\sigma-1}{\sigma}} \cdot L_t^{-\frac{1}{\sigma}}}{A_{adj}} \tag{A45}$$

$$rk_t = \frac{\frac{p_t^{int}}{P_t} \cdot \left(\alpha \cdot (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot \alpha \cdot AK_t^{\frac{\sigma-1}{\sigma}} \cdot K_t^{-\frac{1}{\sigma}}}{A_{adj}} \tag{A46}$$

$$Y_t = \frac{\left(\alpha \cdot (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{A_{adj}} \tag{A47}$$

$$r_t = \frac{rk_t + (1-\delta)\epsilon_t}{\epsilon_{t-1}} - 1 \tag{A48}$$

$$\Pi_t = \frac{Y_t}{\theta} \tag{A49}$$

$$gov_t^{rev} = g_t \cdot Y_t + r_t \cdot gov_t^{debt} \cdot K_t \tag{A50}$$

$$gov_t^{debt} = \frac{\left(gov_{t-1}^{debt} K_{t-1} \cdot (1 + r_{t-1}) + g_{t-1} \cdot Y_{t-1} - gov_{t-1}^{rev} \cdot \left(1 - gov_{t-1}^{deficit}\right)\right)}{K_t} \tag{A51}$$

$$gov_t^{deficit} = \frac{\left(b_{t+1} \cdot Y_{t+1} - gov_t^{debt} \cdot K_t\right)}{gov_t^{rev}} \tag{A52}$$

$$\tau_t = \frac{gov_t^{rev} \cdot \left(1 - gov_t^{deficit}\right)}{w_t L_t} \tag{A53}$$

$$N_t = \sum_{j=26}^{J} n_t^j \tag{A54}$$

$$L_t = \sum_{j=26}^{J} n_t^j hc^j \tag{A55}$$

$$C_t = \sum_{j=26}^{J} n_t^j c_t^j \tag{A56}$$

$$K_t = \frac{\left(\sum_{j=26}^{J} \frac{n_t^j a_{t-1}^j}{sv_t^j}\right)}{\epsilon_{t-1} + gov_t^{debt}} \tag{A57}$$

APPENDIX B: THE PAPER EQUATIONS

B.1 | Steady state equations

$$\begin{gathered} j \in \{26, ..., J = 81\} \\ n^{26} = 1 \end{gathered} \tag{B1}$$

$$n^{j+1} = \frac{s^j \cdot n^j}{1+n} \tag{B2}$$

$$\begin{gathered} for \quad j \in \{26, J-1\} \\ \frac{1}{\beta} = \left(\frac{c^{j+1}}{c^j}\right)^{-\frac{1}{\gamma}} \cdot (1+r) + \lambda^{j+1} \cdot \frac{sv^{j+1}(c^j)^{\frac{1}{\gamma}}}{su^j \beta^j \epsilon} \end{gathered} \tag{B3}$$

$$\begin{gathered} for \quad j \in \{26, J-1\} \\ x_{t+80}^{81} = \left(\frac{\Gamma_{t-80}^{26}}{\mu}\right)^{-\gamma} c_{t+80}^{81} \end{gathered} \tag{B4}$$

$$x^J = \left(\frac{\Gamma}{\mu}\right)^{-\gamma} \cdot c^J \tag{B5}$$

$$\begin{gathered} for \quad j \in \{J\} \\ a^j = 0 \end{gathered} \tag{B6}$$

$$\begin{gathered} for \quad j \in \{26\} \\ \epsilon \cdot a^{j+1} = \frac{(rk + \epsilon(1-\delta)) \cdot a^j}{sv^j} + \left((1-\tau) \cdot w \cdot hc^j + \pi^j\right) \cdot \left(1 + AL_{growth}\right)^j - c^j \end{gathered} \tag{B7}$$

$$\begin{gathered} for \quad j \in \{26, ..., 56\} \\ \epsilon \cdot a^{j+1} = \frac{(rk + \epsilon(1-\delta)) \cdot \left(a^j + sv^j q^{j+1} \cdot \left(1 + AL_{growth}\right)^{j+1}\right)}{sv^j} + ... \\ ... + \left((1-\tau) \cdot w \cdot hc^j + \pi^j\right) \cdot \left(1 + AL_{growth}\right)^j - c^j \end{gathered} \tag{B8}$$

$$\begin{gathered} for \quad j \in \{56\} \\ q^j = \frac{x^J \cdot \Gamma \cdot n^J}{n^j} \end{gathered} \tag{B9}$$

$$\begin{gathered} for \quad j \in \{57\} \\ \epsilon \cdot a^{j+1} = \frac{(rk + \epsilon(1-\delta)) \cdot a^j}{sv^j} + \left((1-\tau) \cdot w \cdot hc^j + \pi^j\right) \cdot \left(1 + AL_{growth}\right)^j - c^j \end{gathered} \tag{B10}$$

$$\begin{gathered} for \quad j \in \{57,\ldots,65\} \\ \epsilon \cdot a^{j+1} = \frac{(rk + \epsilon(1-\delta)) \cdot a^j}{sv^j} - c^j \\ for \quad j \in \{66,\ldots,80\} \end{gathered} \tag{B11}$$

$$c^j = \frac{(rk + \epsilon(1-\delta)) \cdot a^j}{sv^j} - \Gamma \cdot x^j \tag{B12}$$

$$\begin{gathered} for \quad j \in \{81\} \\ \min\left(\lambda^j, a^j + \frac{D^j}{1+r} \cdot \left(1 + AL_{growth}\right)^j\right) = 0 \end{gathered} \tag{B13}$$

$$\begin{gathered} for \quad j \in \{26,\ldots,65\} \\ \min\left(\lambda^j, a^j\right) = 0 \end{gathered} \tag{B14}$$

$$\begin{gathered} for \quad j \in \{66,\ldots,81\} \\ \pi^j = \frac{hc^j \cdot \Pi}{L} \end{gathered} \tag{B15}$$

$$\begin{gathered} for \quad j \in \{26,\ldots,65\} \\ \frac{p^{int}}{P} = \frac{\theta-1}{\theta} \end{gathered} \tag{B16}$$

$$A_{adj} = \frac{p^{int}}{P} \cdot \left(\alpha \cdot (AK \cdot K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL \cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot (1-\alpha) \cdot AL^{\frac{\sigma-1}{\sigma}} \cdot L^{-\frac{1}{\sigma}} \tag{B17}$$

$$w = \frac{\frac{p^{int}}{P} \cdot \left(\alpha \cdot (AK \cdot K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL \cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot (1-\alpha) \cdot AL^{\frac{\sigma-1}{\sigma}} \cdot L^{-\frac{1}{\sigma}}}{A_{adj}} = 1 \tag{B18}$$

$$rk = \frac{\frac{p^{int}}{P} \cdot \left(\alpha \cdot (AK \cdot K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL \cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot \alpha \cdot AK^{\frac{\sigma-1}{\sigma}} \cdot K^{-\frac{1}{\sigma}}}{A_{adj}} \tag{B19}$$

$$Y = \frac{\left(\alpha \cdot (AK \cdot K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL \cdot L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{A_{adj}} \tag{B20}$$

$$r = \frac{rk + (1-\delta)\epsilon}{\epsilon} - 1 \tag{B21}$$

$$\Pi = \frac{Y}{\theta} \tag{B22}$$

$$b \cdot Y \cdot \left(\left(1 + AL_{growth}\right) \cdot (1+n) - 1\right) = g \cdot Y + (1+r) \cdot b \cdot Y - \tau \cdot w \cdot L \tag{B23}$$

$$gov^{rev} = (g \cdot Y + r \cdot b \cdot Y) \tag{B24}$$

$$gov^{deficit} = \frac{\left(\left(1 + AL_{growth}\right) \cdot (1+n) - 1\right) \cdot (b \cdot Y)}{gov^{rev}} \tag{B25}$$

$$gov^{debt} = b \cdot \frac{Y}{K} \tag{B26}$$

$$N = \sum_{j=26}^{J} n^j \tag{B27}$$

$$L = \sum_{j=26}^{J} n^j hc^j \tag{B28}$$

$$C = \sum_{j=26}^{J} \frac{n^j c^j}{\left(1 + AL_{growth}\right)^j} \tag{B29}$$

$$\epsilon \cdot K = \left(\sum_{j=26}^{J} \frac{\epsilon \cdot n^j a^j}{\left(1 + AL_{growth}\right)^j} \right) - b \cdot Y \tag{B30}$$

B.1.1 | Transitional dynamics

$$\begin{aligned} & j \in \{26, \ldots, J = 81\} \\ & n_t^{26} = \frac{n_{t-1}^{25}}{su_{t-1}^{25}} \cdot \Gamma_t \end{aligned} \tag{B31}$$

$$n_t^{j+1} = s_{t-1}^j \cdot n_{t-1}^j \tag{B32}$$

$$\begin{aligned} & for \quad j \in \{26, J-1\} \\ & \frac{1}{\beta} = \left(\frac{c_{t+1}^{j+1}}{c_t^j} \right)^{-\frac{1}{\gamma}} \cdot (1 + r_{t+1}) + \lambda_t^{j+1} \cdot \frac{sv_t^{j+1} \left(c_t^j\right)^{\frac{1}{\gamma}}}{su_t^j \beta^j \epsilon_t} \end{aligned} \tag{B33}$$

$$\begin{aligned} & for \quad j \in \{26, J-1\} \\ & x_t^J = \left(\frac{\Gamma_{t-J+26}}{\mu} \right)^{-\gamma} \cdot c_t^J \end{aligned} \tag{B34}$$

$$\begin{aligned} & for \quad j \in \{J\} \\ & a_t^j = 0 \end{aligned} \tag{B35}$$

$$\begin{aligned} & for \quad j \in \{26\} \\ & \epsilon_t \cdot a_{t+1}^{j+1} = \frac{(rk_t + \epsilon_t(1-\delta)) \cdot a_t^j}{sv_t^j} + (1-\tau_t) \cdot w_t \cdot hc^j + \pi_t^j - c_t^j \end{aligned} \tag{B36}$$

$$\begin{aligned} & for \quad j \in \{26, \ldots, 56\} \\ & \epsilon_t \cdot a_{t+1}^{j+1} = \frac{(rk_t + \epsilon_t(1-\delta)) \cdot \left(a^j + sv_t^j q_{t+1}^{j+1}\right)}{sv_t^j} + (1-\tau_t) \cdot w_t \cdot hc^j + \pi_t^j - c_t^j \end{aligned} \tag{B37}$$

$$\begin{aligned} & for \quad j \in \{56\} \\ & q^j = \frac{x_{t-1}^J \cdot \Gamma_{t-56} \cdot n_{t-1}^J}{n_t^j} \end{aligned} \tag{B38}$$

$$for \quad j \in \{57\}$$

$$\epsilon_t \cdot a_{t+1}^{j+1} = \frac{(rk_t + \epsilon_t(1-\delta)) \cdot a_t^j}{sv_t^j} + (1-\tau_t) \cdot w_t \cdot hc^j + \pi_t^j - c_t^j \tag{B39}$$

$$for \quad j \in \{57,\ldots,65\}$$

$$\epsilon_t \cdot a_{t+1}^{j+1} = \frac{(rk_t + \epsilon_t(1-\delta)) \cdot a_t^j}{sv_t^j} - c_t^j \tag{B40}$$

$$for \quad j \in \{66,\ldots,80\}$$

$$c_t^j = \frac{(rk_t + \epsilon_t(1-\delta)) \cdot a_t^j}{sv_t^j} - \Gamma_{t-55} \cdot x_t^j \tag{B41}$$

$$for \quad j \in \{81\}$$

$$\min\left(\lambda_t^j, a_t^j + \frac{D_t^j}{1+r_t}\right) = 0 \tag{B42}$$

$$for \quad j \in \{26,\ldots,65\}$$

$$\min\left(\lambda_t^j, a_t^j\right) = 0 \tag{B43}$$

$$for \quad j \in \{66,\ldots,81\}$$

$$\pi_t^j = \frac{hc^j \cdot \Pi_t}{L_t} \tag{B44}$$

$$for \quad j \in \{26,\ldots,65\}$$

$$\frac{p_t^{int}}{P_t} = \frac{\theta-1}{\theta} \tag{B45}$$

$$w_t = \frac{\frac{p_t^{int}}{P_t} \cdot \left(\alpha \cdot (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot (1-\alpha) \cdot AL_t^{\frac{\sigma-1}{\sigma}} \cdot L_t^{-\frac{1}{\sigma}}}{A_{adj}} \tag{B46}$$

$$rk_t = \frac{\frac{p_t^{int}}{P_t} \cdot \left(\alpha \cdot (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot \alpha \cdot AK_t^{\frac{\sigma-1}{\sigma}} \cdot K_t^{-\frac{1}{\sigma}}}{A_{adj}} \tag{B47}$$

$$Y_t = \frac{\left(\alpha \cdot (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \cdot (AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{A_{adj}} \tag{B48}$$

$$r_t = \frac{rk_t + (1-\delta)\epsilon_t}{\epsilon_{t-1}} - 1 \tag{B49}$$

$$\Pi_t = \frac{Y_t}{\theta} \tag{B50}$$

$$b_{t+1}\, Y_{t+1} = g_t\, Y_t + (1+r_t) \cdot b_t\, Y_t - \tau_t w_t L_t \tag{B51}$$

$$gov_t^{rev} = g_t\, Y_t + r_t b_t\, Y_t \tag{B52}$$

$$gov_t^{deficit} = \frac{b_{t+1}\,Y_{t+1} - b_t\,Y_t}{gov_t^{rev}} \tag{B53}$$

$$gov^{debt} = \frac{b_t\,Y_t}{K_t} \tag{B54}$$

$$N_t = \sum_{j=26}^{J} n_t^j \tag{B55}$$

$$L_t = \sum_{j=26}^{J} n_t^j hc^j \tag{B56}$$

$$C_t = \sum_{j=26}^{J} n_t^j c_t^j \tag{B57}$$

$$\epsilon_t \cdot K = \left(\sum_{j=26}^{J} \epsilon_t n_t^j a_{t-1}^j\right) - b_t\,Y_t \tag{B58}$$

APPENDIX C: FULL MODEL'S DERIVATIONS

We follow closely Eggertsson et al. (2019) and we report the main derivations of the model to help the reader. We invite interested readers to follow the original paper for a complete description of the model's equations.

C.1 | Demographics

The population growth rate is determined by the total fertility rate of every household (Γ) and by the probability of dying before arriving at the maximum age $J = 81$ years, which is set stochastically. The probability of surviving between age j and $j + 1$ is given by s_j and it's called conditional, instead, the probability of arriving at age j is given by s^j and it's called unconditional probability. The total population alive at any given time, N_t, is the sum of the population of the individual ages, n_t^j. The population size of a given generation n_t^j is the population of the generation the previous year that has survived, except for the generation $j = 26$ years, which is the first generation in the model. That is given by the total population of their parents which entered the economic maturity at time $t - 25$, multiplied by the total fertility rate of their parent's generation at that time (Γ_{t-25}) and discounted for the unconditional probability of survival. In sum, the total population evolves in the model according to the law of motions and aggregates given below:

$$N_t = \sum_{j=26}^{J} n_t^j \tag{C1}$$

$$n_{t+1}^{j+1} = s_j n_t^j \quad \text{for } j \in \{26, J-1\} \tag{C2}$$

$$n_t^{26} = \frac{n_{t-25}^{26}\,\Gamma_{t-25}}{su^{26}} \tag{C3}$$

where:

$$\Gamma_{t-25} = (1 + n_{t-25})^{\frac{1}{25}}$$

Households do not receive wage income after retirement, set at age $j = 65$. Labor is supplied inelastically, but it depends on the individual age-specific exogenous labor productivity hc^j. Thus the total labor supply at a given time t is given by:

$$L_t = \sum_{j=26}^{J} n_t^j hc^j \tag{C4}$$

C.2 | Households problem

Each generation j of the population maximizes the following intertemporal utility function:

$$\max_{\left\{c_{t+j-1}^j, x_{t+j-1}^j\right\}} U_t = \frac{1}{\left(1-\frac{1}{\gamma}\right)} \left[\left(\sum_{j=26}^{J} su^j \beta^{j-1} u\left(c_{t+j-1}^j\right) \right) + su^J \beta^{J-1} \mu v\left(x_{t+J-1}^J\right) \right] \tag{C5}$$

subject to:

$$c_t^j + \epsilon_t a_{t+1}^{j+1} + \Gamma_{t-j+26}^{26} x_t^j = (1-\tau^w) w_t hc^j + \pi_t^j + \left(r_t^k + \epsilon_t (1-\delta)\right) \cdot \left(a_t^j + q_{t+1}^{j+1} + \frac{1-sv_j}{sv_j} a_t^j \right) \tag{C6}$$

$$a_t^j \geq \frac{D_t}{1+r_t} \tag{C7}$$

$$c_t^j \geq 0 \tag{C8}$$

$$a_t^{26} = 0 \tag{C9}$$

$$a_t^{J+1} = 0 \tag{C10}$$

$$q_t^j = \frac{n_{t-1}^J x_{t-1}^J \Gamma_{t-J+26}^{26}}{n_t^{57}} \tag{C11}$$

where:

$$su^j = \prod_{m=26}^{j-1} sv_m$$

$$D_t^j \leq 0 \quad \text{for } j \leq 65$$

$$D_t^j = 0,\ hc^j = 0,\ \pi_t^j = 0 \quad \text{for } j > 65$$

$$q_t^j = 0 \quad \text{for } j \neq 57$$

$$x_t^j = 0 \quad \text{for } j \neq 81$$

The utility and bequest are constant elasticity of substitution (CES) function:

$$u\left(c_{t+j-1}^j\right) = \left(c_{t+j-1}^j\right)^{\left(1-\frac{1}{\gamma}\right)}$$

$$v\left(x_{t+J-1}^J\right) = \left(x_{t+J-1}^J\right)^{\left(1-\frac{1}{\gamma}\right)}$$

The non-negativity constraint for consumption (C8) can be omitted. Substituting the consumption $c_{t+j-1}^{j,i}$ into the utility function (C5) using the equality constraint (C6), using the financial (occasionally binding) constraint (C7) and taking care of all the other conditions, we can form the lagrangian to be maximized as follows:

$$\max_{\left\{a_{t+j}^{j+1}, x_{t+j-1}^{j}, \lambda_{t+j-1}^{j}\right\}} \quad \mathcal{L}_t = \frac{1}{\left(1-\frac{1}{\gamma}\right)}\left\{\sum_{j=26}^{J}\left(\prod_{m=26}^{j-1} sv_m\right)\cdot\beta^{j-1}\cdot ...\right.$$

$$\left[-\epsilon_t a_{t+j}^{j+1} - \Gamma_{t-j+26}^{26} x_t^j + (1-\tau) w_t hc^j + \pi_{t+j-1}^j + ...\right.$$

$$\left.\left. ... + (rk_t + \epsilon_t(1-\delta))\cdot\left(a_t^j + q_{t+1}^{j+1} + \frac{1-sv_j}{sv_j}a_t^j\right)\right]^{1-\frac{1}{\gamma}}\right\} + ...$$

$$... + \frac{1}{\left(1-\frac{1}{\gamma}\right)}\left\{\left(\prod_{m=26}^{J-1} sv_m\right)\cdot\beta^{J-1}\mu\left[x_{t+J-1}^{J}\right]^{1-\frac{1}{\gamma}}\right\} + ...$$

$$... + \sum_{j=26}^{J}\lambda_{t+j-1}^{j}\left(a_{t+j-1}^{j} - \frac{D_t^j}{1+r_t}\right)$$

subject to:

$$a_t^{26} = 0$$

$$a_t^{J+1} = 0$$

$$q_t^j = \frac{n_{t-1}^J x_{t-1}^J \Gamma_{t-J+26}^{26}}{n_t^{57}}$$

where:

$$D_t^j \leq 0 \quad \text{for } j \leq 40$$

$$D_t^j = 0,\ hc^j = 0,\ \pi_t^j = 0 \quad \text{for } j > 40$$

$$q_t^j = 0 \quad \text{for } j \neq 57$$

$$x_t^j = 0 \quad \text{for } j \neq 81$$

Deriving with respect to a_{t+j}^{j+1}, x_{t+j-1}^{j} and considering the complementary slackness conditions, we get the first-order conditions (FOCs):

$$\bullet\frac{\partial\mathcal{L}}{\partial a_{t+j}^{j+1}} =$$

$$su^j\beta^{j-1}\left(c_{t+j-1}^{j}\right)^{-\frac{1}{\gamma}}\cdot -\epsilon_t + su^{j+1}\beta^j\left(c_{t+j}^{j+1}\right)^{-\frac{1}{\gamma}}\frac{(rk_{t+1}+\epsilon_{t+1}(1-\delta))}{sv_j} + \lambda_{t+j}^{j+1} = 0$$

$$for \quad j \in \{26,...,80\}$$

$$su^j\beta^{j-1}\left(c_{t+j-1}^{j}\right)^{-\frac{1}{\gamma}}\cdot 0 = 0$$

$$for \quad j \in \{81\}$$

$$\bullet\frac{\partial\mathcal{L}}{\partial x_{t+j-1}^{j}} =$$

$$su^j\beta^{j-1}\left(c_{t+j-1}^{j}\right)^{-\frac{1}{\gamma}}\cdot -\Gamma_{t-j+1}^{26} + su^j\beta^{j-1}\mu\left(x_{t+j-1}^{j}\right)^{-\frac{1}{\gamma}} = 0$$

$$for \quad j \in \{81\}$$

•*Slackness conditions*:

$$\lambda^{j}_{t+j-1}\left(a^{j}_{t+j-1}-\frac{D^{j}_{t}}{1+r_t}\right)=0$$
$$for \quad j\in\{26,\ldots,65\} \quad and, \quad a^{j}_{t+j-1}\geq 0$$
$$\lambda^{j}_{t+j-1}\left(a^{j}_{t+j-1}\right)=0$$
$$for \quad j\in\{66,\ldots,81\} \quad and, \quad a^{j}_{t+j-1}\geq 0$$

Rewriting using the no-arbitrage condition below (Equation C29) we get:

$$\bullet\frac{\partial\mathcal{L}}{\partial a^{j+1}_{t+j}}:$$

$$\frac{1}{\beta}=\left(\frac{c^{j+1}_{t+1}}{c^{j}_{t}}\right)^{-\frac{1}{\gamma}}\cdot(1+r_{t+1})+\lambda^{j+1}_{t}\cdot\frac{sv^{j+1}_{t}\left(c^{j}_{t}\right)^{\frac{1}{\gamma}}}{su^{j}_{t}\beta^{j}\epsilon_t} \tag{C12}$$
$$for \quad j\in\{26,\ldots,80\}$$

$$\bullet\frac{\partial\mathcal{L}}{\partial x^{j}_{t+j-1}}:$$

$$x^{81}_{t+80}=\left(\frac{\Gamma^{26}_{t-80}}{\mu}\right)^{-\gamma}c^{81}_{t+80} \tag{C13}$$
$$for \quad j\in\{81\}$$

$\bullet$*Slackness conditions*:

$$\lambda^{j}_{t+j-1}\left(a^{j}_{t+j-1}-\frac{D^{j}_{t}}{1+r_t}\right)=0 \tag{C14}$$

$$for \quad j\in\{26,\ldots,65\} \quad and, \quad a^{j}_{t+j-1}\geq 0$$
$$\lambda^{j}_{t+j-1}\left(a^{j}_{t+j-1}\right)=0 \tag{C15}$$

$$for \quad j\in\{66,\ldots,81\} \quad and, \quad a^{j}_{t+j-1}\geq 0$$

We follow Swarbrick (2021, p. 8) and we summarize conditions (C7), (C14) and, (C15) making use of the minimum function to handle the financial OBCs. The resulting two expressions are the following:

$$\min\left(\lambda^{j}_{t+j-1},a^{j}_{t+j-1}-\frac{D^{j}_{t}}{1+r_t}\right)=0 \tag{C16}$$

$$for \quad j\in\{26,\ldots,65\}$$
$$\min\left(\lambda^{j}_{t+j-1},\, a^{j}_{t+j-1}\right)=0 \tag{C17}$$

$$for \quad j\in\{66,\ldots,81\}$$

C.3 | Firms problem

C.3.1 | Final goods firms

The final goods firms choose real prices $\frac{p_t(i)}{P_t}$ to maximize real profits:

$$\max_{\left\{\frac{p_t(i)}{P_t}\right\}} \Pi_t = \frac{p_t(i)}{P_t} y_t^f(i) - \frac{p_t^{int}}{P_t} y_t^f(i)$$

subject to the following demand curve constraint:

$$y_t^f(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta_t}$$

where θ_t is a time-varying shock to the firm's market power. An increase in θ_t decreases a firm's market power and lowers equilibrium markups. Then, the lagrangian is given by:

$$\max_{\left\{\frac{p_t(i)}{P_t}\right\}} \mathcal{L}_t = \frac{p_t(i)}{P_t} Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta_t} - \frac{p_t^{int}}{P_t} Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta_t}$$

Deriving with respect to $\frac{p_t(i)}{P_t}$ we get the FOC:

$$\frac{p_t(i)}{P_t} = \frac{\theta_t}{\theta_t - 1} \frac{p_t^{int}}{P_t} \tag{C18}$$

The nominal price index implies the following expression for the price of intermediate goods:

$$P_t = \left(\int p_t(i)^{1-\theta_t di}\right)^{\frac{1}{1-\theta_t}}$$

Since the price of intermediate good is the same, all final goods firms make the same pricing decisions (no pricing frictions), and thus $p_t(i) = P_t$, yielding to:

$$\frac{p_t^{int}}{P_t} = \frac{\theta_t - 1}{\theta_t} \tag{C19}$$

Substituting, $\frac{p_t^{int}}{P_t}, \frac{p_t(i)}{P_t}, y_t^f(i)$ into Π_t we get the aggregate profit:

$$\Pi_t = \frac{Y_t}{\theta_t} \tag{C20}$$

Profits from monopolistically competitive firms are distributed according to wage income, $\pi_t^j = hc^j \frac{\Pi_t}{L_t}$. In equilibrium, the total distributed profit must equal total profits:

$$\Pi_t = \sum_{j=26}^{65} n_t^j \pi_t^j \tag{C21}$$

C.3.2 | Intermediate goods firms

This is a perfectly competitive market in which intermediate firms rent capital K_t from the capital market at rk_t, hire labor L_t from the labor market at w_t, and sell their production Y_t to the final firms at a real price $\frac{p_t^{int}}{P_t}$ taken as given. They maximize the following profit function:

$$\max_{\{K_t, L_t\}} \Pi_t^{int} = \frac{p_t^{int}}{P_t} Y_t - w_t L_t - rk_t K_t$$

subject to the production constraint, given by a CES production function:

$$Y_t = \left(\alpha (AK_t K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL_t L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

The lagrangian for the Intermediate Firms problem is:

$$\max_{\{K_t, L_t\}} \mathcal{L}_t = \frac{p_t^{int}}{P_t}\left(\alpha (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - w_t L_t - rk_t K_t$$

Deriving with respect to L_t, K_t we get the FOCs:

$$w_t = \frac{p_t^{\text{int}}}{P_t}(1-\alpha)(AL_t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{L_t}\right)^{\frac{1}{\sigma}} \tag{C22}$$

$$rk_t = \frac{p_t^{\text{int}}}{P_t}(\alpha)(AK_t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{K_t}\right)^{\frac{1}{\sigma}} \tag{C23}$$

$$Y_t = \left(\alpha (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{C24}$$

Taking w_t as a numeraire, we define $A_{adj} = w$ as a parameter at its steady-state value, and we divide w_t, rk_t, Y_t for A_{adj} to get:

$$A_{adj} = \frac{p_t^{\text{int}}}{P_t}(1-\alpha)(AL_t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{L_t}\right)^{\frac{1}{\sigma}} \tag{C25}$$

$$w_t = \frac{\frac{p_t^{\text{int}}}{P_t}(1-\alpha)(AL_t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{L_t}\right)^{\frac{1}{\sigma}}}{A_{adj}} \tag{C26}$$

$$rk_t = \frac{\frac{p_t^{\text{int}}}{P_t}(\alpha)(AK_t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{K_t}\right)^{\frac{1}{\sigma}}}{A_{adj}} \tag{C27}$$

$$Y_t = \frac{\left(\alpha (AK_t \cdot K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL_t \cdot L_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{A_{adj}} \tag{C28}$$

Eventually, we have the no-arbitrage condition which relates the risk-free real rate with the return on capital:

$$1 + r_t = \frac{rk_t + (1-\delta)\epsilon_t}{\epsilon_{t-1}} \tag{C29}$$

C.4 | Government

The government spends an exogenous G_t and may issue debt. The following equations describe the main government variables:

$$G_t = g \cdot Y_t \tag{C30}$$

$$T_t = \tau_t w_t L_t \tag{C31}$$

$$b_{t+1}\,Y_{t+1} = g_t\,Y_t + (1 + r_t) \cdot b_t\,Y_t - \tau_t w_t L_t \tag{C32}$$

$$gov_t^{rev} = g_t\,Y_t + r_t b_t\,Y_t \tag{C33}$$

$$gov_t^{deficit} = \frac{b_{t+1}\,Y_{t+1} - b_t\,Y_t}{gov_t^{rev}} \tag{C34}$$

$$gov^{debt} = \frac{b_t\,Y_t}{K_t} \tag{C35}$$

C.5 | Aggregates

Besides the other aggregates, such as (C1), (C4), (C20), and (C28), we have:

$$C_t = \sum_{26}^{J} n_t^j c_t^j \tag{C36}$$

$$\epsilon_t \cdot K_t = \left(\sum_{j=26}^{J} \epsilon_t n_t^j a_{t-1}^j \right) - b_t\,Y_t \tag{C37}$$