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Research paper

Modeling plant water deficit by a non-local root water uptake term in the unsaturated flow equation

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ABSTRACT

In this paper we present a novel way to mathematically frame the concept of *ecological memory* of plant water stress in the context of root water uptake in unsaturated flow equations. Inspired by recent eco-hydrological papers, we model the water absorption by roots with a non-local sink term, accounting also for a memory effect. In order to model such a memory effect, an integral equation is defined; the main purpose of this work is to provide sufficient conditions on the functions at play for ensuring existence and uniqueness of its solution. Finally, tailored numerical methods are implemented, and numerical simulations are also provided.

1. Introduction

Richards' equation represents the most commonly used model for soil moisture dynamics into the unsaturated zone, approximated as a porous medium. In particular, the soil water content changes across the unsaturated zone because of inflows of water via infiltration from the surface due, e.g. to rainfall and irrigation, and outflows caused by root water uptake (RWU). Such phenomena can be described requiring suitable boundary and initial conditions in Richards' equation and therein incorporating an appropriate sink term to describe the root water uptake dynamics; here, by RWU we mean the volume of water absorbed from a given soil volume over a given time. Several sink terms were investigated in the last decades to describe the root water uptake process (see [1–5]). Many authors proposed physically based empirical models, which are space and time dependent and are determined by plant characteristics, soil water and atmospheric conditions; in particular, they restricted their attention to the influence of concurrent soil water content in the root water uptake process.

Nevertheless, it is well known that explicitly considering the effects of antecedent conditions allows to get more accurate information about any ecological processes. In [6] the authors focused their attention on the notion of *ecological memory*, which is defined as "the capacity of past states or experiences to influence present or future responses of the community" [7], and as "the degree to which an ecological process is shaped by its past modifications of a landscape" [8]. Still in [6], three components are considered: (1) the *length* of the memory, "which quantifies the time period(s) over which antecedent conditions or states affect current processes or states", (2) the *temporal pattern* of the memory, "which is characterized by variation in the relative importance of conditions occurring at different times into the past, including potentially important time lags" and (3) the *strength* of the memory, "which describes the degree to which antecedent conditions affect the process of interest". Then, they consider four different case studies (stomatal conductance, soil respiration, ecosystem annual net primary production, and annual tree growth) to illustrate

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how the memory of antecedent conditions influences each ecological process in different ways, what features underlie memory and which mechanisms govern it. In order to answer to these questions the authors apply a stochastic approach: in each test case, an observed value $Y(t)$ measured at time t is defined as a probability distribution applied to a mean $\mu(t)$ that incorporates antecedent conditions X_{ant} and current conditions; in particular, X_{ant} is given by the sum of past conditions weighted by their relative importance, described by mean of a suitable weight function ω. The choice of an appropriate weight function ω is one of the most important challenges to obtain an accurate description of the process and it represents the research question of this paper. Indeed, applying their stochastic model to the four data sets, authors in [6] show that the length, temporal patterns, and strength of the memory can vary greatly among processes; for instance, in the study of annual net primary productivity of shortgrass steppe in Colorado, they show that the precipitation occurred 1–2 years before was significantly more important than that occurred during the year of production or 4 years before.

As in [9] we can summarize: "In some systems, the soil acts as a *capacitor*, and water from one year is transferred to the next one. In other systems, features of plant population dynamics (for example, initiation of buds in the previous growing season, a seed bank, establishment of a cohort of perennial plants) (see [10]), or plant structural factors related to changes in biomass, storage organs, or cover (for example, see [11]) may explain the memory of the system".

Some other authors in previous literature underlined the fact that memory might play an important role, for instance, in the functioning of semiarid rangelands by buffering fluctuations in phytomass production if wet, more productive years, alternate with dry, less productive years and by amplifying fluctuations if wet or dry sequences of several years take place [10]. Such carryover effects have been demonstrated for a few perennial grasslands and shrublands (for example, see [12]). Authors in [9] developed several regression models to predict long-term monthly phytomass production: in such models they proposed indices that summarize the effects of monthly rainfall, mean monthly soil temperature, and memory of past rainfall events, weighted with an exponentially decay factor.

Including memory improves the regression models; in particular, the results were consistent with many studies in arid and semiarid ecosystems: "in areas where there are no prolonged periods of dry and wet years, a dry year after a wet year will yield more production than expected by precipitation alone, and a wet year after a dry year will yield less production than expected by precipitation alone". Somehow, this pattern buffers the variability in precipitation.

An analogous issue was treated in [13,14], in which a model of rhizosphere dynamics is also proposed: several experiments show that during drying periods, the rhizosphere (i.e., the soil near the roots) is wetter than the bulk soil (see Figure 2 in [13]). Immediately after rewetting, the rhizosphere remains markedly dry. Afterward, the rhizosphere slowly rewetts and become wetter than the bulk soil. Such phenomena are justified by the presence of mucilage, a polimeric material in the rhizosphere: indeed, mucilage is capable of holding a large amount of water, but it contains also lipids that makes it hydrophobic when it dries.

The need to include memory terms in evolution models in order to make that more true to the reality is increasingly faced by using fractional calculus, for instance in areas like rheology, biology, engineering, mathematical physics, etc. (see for instance [15–20] and the reference given therein).

In [21–23] the authors proposed a physic-based generalized Richards' equation in which the integer order derivative (in time) is replaced by a fractional in time derivative allowing to consider a non-Brownian motion of water in an horizontal column of unsaturated soil, that can justify the non-Boltzmann scaling in the evolution of an horizontal wetting front, which was observed in several experiments. The use of nonlocal derivatives makes even more difficult the study of solutions to the fractional differential equations; thus, several authors investigates new numerical methods in order to approximate the solution of such models (see, for instance, [24–26]). On the other hand, some information about the existence and the long time behavior of the solution to fractional evolution models or classical PDEs with memory type nonlinearity can be investigated by using tools of Fourier and harmonic analysis (see, for instance, [27–31]).

In this work, nonlocal models are coupled with the standard tool to deal with variably saturated flow in porous media, the Richardson-Richards equation, from now on: Richards' equation. It is a non-linear degenerate parabolic partial differential equation, based on Darcy's law combined with mass conservation: the nonlinearity arises from the nature of hydraulic functions, whose form will be recalled in the following sections. For the sake of simplicity the soil is assumed to be isotropic, even though recent papers deal with the occurrence of discontinuities in soil profile (e.g., [32–35]). Even in presence of isotropic soils, several difficulties arise when attempting to numerically solve Richards' equation: it is worth recall classical papers on mixed finite elements or finite volumes with error estimates, as [36–39], discontinuous Galerkin [40] and the novel virtual element methods [41]. Another trend of numerical issues is related to appropriate numerical methods for dealing with nonlinearities, as the L-scheme [42], or Newton and its variants (e.g. [43–45]).

In this paper, we start considering a possible memory term introduced in [46], and we formalize in a mathematical framework their approach. We consider a non-local memory term in root water uptake described by an integral equation, which is coupled to Richards' equation in order to model plant water deficit. Sufficient conditions ensuring existence and uniqueness for such integral equation are provided in a wider class of weight functions. In particular, the solution to the integral equation can be obtained by applying an iterative method, which provides also an error estimate of this procedure. Furthermore, we apply a tailored numerical method for dealing with fractional memory terms in this problem, showing the effectiveness of the results, and comparing it with other more common weight functions simulations.

2. Encompassing plant water deficit with Richards equation

In the present paper we propose a Richards' equation with a root water uptake function which includes a memory term $\delta(t)$; namely, inspired by [46] we consider the model

$$C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\Big(K(h)\Big(\frac{\partial h}{\partial z} - 1\Big)\Big) - S(h,z,t), \quad (z,t) \in [0,L] \times [0,T] \tag{1}$$

where $C(h)$ is the soil water capacity (cm^{-1}), $K(h)$ is the soil hydraulic conductivity ($\mathrm{cm\ d}^{-1}$), h is the soil matric potential (cm), L is depth (cm), T is the period of observation of the process, and $S(h,z,t)$ is the root water uptake function (RWU) defined as

$$S(h,z,t) = \delta(t) S_c(h,z,t). \tag{2}$$

For our main result, we have not any hypothesis on C and K: in the classical literature on Richards' equation, it is enough to require them to be Lipschitz-continuous and monotonically increasing (see for instance [36,47]); $S_c(h,z,t)$ is a *classical* root water uptake function, depending only on the depth z and on the current time, for instance the Feddes RWU (see [1]) or the Jarvis RWU (see [3]); on the other hand, the function $\delta(t)$ satisfies an integral equation in the form

$$\delta(t) = \varphi\Big(\int_0^t \omega(t-\tau)\mathfrak{D}(\tau)d\tau\Big).$$

Originally, the function $\mathfrak{D}$ was defined as the ratio of water deficit to water demand (see for instance [48]). The plant water deficit index gives an estimation of water available for uptake, and then it provides an indications about the need to irrigate.

Plant water stress can be also investigated paying attention to other physiological indicators, like the leaf stomatal conductance [49], or the leaf water potential [50]; however, the application of such parameters is usually difficult due to their high variability during plant physiological processes; moreover, their measurements are often made unreliable by meteorological conditions.

In literature, [51], the plant water status is measured or calculated considering only the effects of soil without paying any attention to other characteristics of the plant; in particular, an arithmetic average of soil water content at various depths is used.

Nevertheless, recent studies highlighted that this approach has success only if water and roots have a uniform distribution in soil. In general, it seems more appropriate to apply a root-weighted approach (RWA), that is, a weighted average of soil water content over the root zone, evaluated by using a normalized root length density distribution (see, for instance, [51]); in particular, for all $t \geq 0$

$$\mathfrak{D}(t) = 1 - \gamma\big(\mathfrak{h}(t)\big), \tag{3}$$

where the function $\gamma(h)$ is a dimensionless soil water stress reduction function to quantify the effect of current soil water condition on RWU, and

$$\mathfrak{h}(t) := \int_0^1 h(t,z_n) L_{nrd}(z_n)\, dz_n;$$

here, $z_n := z/L$ is the normalized root depth (where L is the maximum rooting depth), L_{nrd} denotes the normalized root length density function; it is generally a function decreasing with soil depth, regardless of irrigation method and quantity, and in our numerical simulations it will be approximated (for instance, on the basis of [46]) by a polynomial function. Indeed, field experiments with various soil water and roots distribution conditions, showed that $\mathfrak{D}$ estimated with RWA improves agreement with theoretical values compared with the traditional one which considers only the arithmetic average of root-zone soil water content (see [48]).

In the present work we include in the plant water deficit index a memory term δ which keeps track of drought events occurred in the past; we define

$$\mathfrak{D}(t) := 1 - \delta(t)\sigma(t)$$

where $\delta : [0,1] \to [0,1]$ solves the following integral equation

$$\delta(t) = \varphi\Big(\int_0^t \omega(t-\tau)\mathfrak{D}(\tau)d\tau\Big), \tag{4}$$

and

$$\sigma(\tau) = \int_0^1 \gamma(h(z,\tau)) L_{nrd}(z_n)\, dz_n; \tag{5}$$

in particular, without loss of generality, in Section 4 we will consider $\gamma(h)$ equal to the very popular Feddes stress function; for the theoretical results in Section 3, we will only need that σ takes values between 0 and 1.

Summarizing, we are interested in describing the dynamics of soil water content θ in presence of plants roots absorption, by studying the following system, obtained coupling the Richards' equation (1) and the integral equation (4),

$$\begin{cases} C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\Big(K(h)\Big(\frac{\partial h}{\partial z} - 1\Big)\Big) - S(h,z,t), \quad (z,t) \in [0,L] \times [0,T], \\ \delta(t) = \varphi\Big(\int_0^t \omega(t-\tau)(1-\delta(\tau)\sigma(\tau))d\tau\Big), \end{cases} \tag{6}$$

where $S(h,z,t)$ is defined by (2); in particular, $S_c(h,z,t)$ is a *standard* root water uptake function, depending only on the depth z and on the current time, as for the Feddes and Jarvis models.

Additionally, we assume the initial condition

$$h(z,0) = \tilde{h}^0(z), \qquad 0 \le z \le L \tag{7}$$

together with a Neumann boundary condition at the top of the root zone

$$\frac{\partial}{\partial z}\Big[-K(h)\Big(\frac{\partial h}{\partial z} - 1\Big)\Big]_{z=0} = -E(t), \qquad t > 0, \tag{8}$$

and a Dirichlet boundary condition

$$h(L,t) = \tilde{h}_L(t), \qquad t > 0; \tag{9}$$

here, $\tilde{h}^0(z)$ is the initial distribution of soil matric potential (cm), $\tilde{h}_L(t)$ is the soil matric potential at the lower boundary and $E(t)$ is the soil surface evaporation rate; for simplicity, we suppose that surface evaporation is negligible, that is true for instance if the soil surface is covered [52]. We will give an explicit value to $\tilde{h}^0$ and $\tilde{h}_L$ in the numerical simulations in Section 4.

3. Existence and uniqueness of solutions to the integral equation

In this section we assume that φ and ω in (4) satisfy the following assumptions:

(H_1) $\varphi : [0,1] \to [0,1]$ is a non-increasing continuous positive function, which is Lipschitz in the interval $[0, 1-\mu]$, for some $\mu \in [0,1)$, with Lipschitz constant M_φ; additionally, it satisfies $\varphi(0) = 1$ and $\varphi(1) = 0$;

(H_2) $\omega : [0,T] \to \mathbb{R}_+$ is positive, Lebesgue integrable in $[0,T]$ with $\|\omega\|_{L^1([0,T])} < 1-\mu$; moreover, for any $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that for all $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $t_2 - t_1 \le \delta_0$ the following inequality is satisfied:

$$\int_0^{t_1} |\omega(t_1-\tau) - \omega(t_2-\tau)| d\tau \le \varepsilon_0; \tag{10}$$

furthermore, one of the following conditions holds:

(i) $\omega \in L^1$, and $\|\omega\|_{L^1([0,T])} < 1/M_\varphi$;
(ii) $\omega \in L^\infty([0,T])$.

Remark 1. We note that if $\omega \in C([0,T])$, then condition (10) follows immediately; indeed, by Heine–Cantor theorem we know that for any $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that for all $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $t_2 - t_1 \le \delta_0$, it holds

$$|\omega(t_1-\tau) - \omega(t_2-\tau)| \le \frac{\varepsilon_0}{T}, \qquad \text{for any } \tau \in [0,T];$$

thus,

$$\int_0^{t_1} |\omega(t_1-\tau) - \omega(t_2-\tau)|\, d\tau \le \varepsilon_0.$$

Remark 2. Assumption (10) is satisfied by

$$\omega(\tau) = \begin{cases} \tau^{-\alpha} & \text{for } \tau \in (0,T], \\ 0 & \text{for } \tau = 0, \end{cases}$$

for all $\alpha \in (0,1)$; indeed, let us fix $\varepsilon_0 > 0$ and $t_1 < t_2$; then, we have

$$\begin{aligned} \int_0^{t_1} |\omega(t_1-\tau) - \omega(t_2-\tau)|\, d\tau &= \int_0^{t_1} (t_1-\tau)^{-\alpha} - (t_2-\tau)^{-\alpha}\, d\tau \\ &= \frac{t_1^{1-\alpha}}{1-\alpha} - \frac{t_2^{1-\alpha}}{1-\alpha} + \frac{(t_2-t_1)^{1-\alpha}}{1-\alpha} \\ &\le \frac{(t_2-t_1)^{1-\alpha}}{1-\alpha}. \end{aligned}$$

The above estimate implies that we can take $\delta_0 = ((1-\alpha)\varepsilon_0)^{\frac{1}{1-\alpha}}$ in order to ensure that condition (10) is satisfied.

Theorem 1. *Let $\sigma : [0,T] \to [0,1]$; under the assumptions (H_1) on φ and (H_2) on ω there exists a unique solution δ to the integral equation*

$$\delta(t) = \varphi\Big(\int_0^t \omega(t-\tau)(1-\delta(\tau)\sigma(\tau))d\tau\Big), \quad t \in [0,T]. \tag{11}$$

In order to prove our main result we will rely on the following Banach fixed point theorem, given in [53].

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space, let $F : X \to X$ a function and F^n its nth iterate defined by*

$$F^n = \overbrace{F \circ F \circ \ldots \circ F}^{n \text{ times}};$$

suppose that there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers such that the series $\sum_{n\geq 1}\mu_n$ is convergent, and for all $x, y \in X$ and $n \in \mathbb{N}$ it holds $\|F^n x - F^n y\| \leq \mu_n \|x - y\|$. Then, F has a unique fixed point $u \in X$. Moreover, if $\bar{x}$ is an element in X, then for all $n \geq 1$ we have

$$\|F^n \bar{x} - u\| \leq \Big(\sum_{k=n}^{\infty} \mu_k\Big)\|F\bar{x} - \bar{x}\|.$$

In particular, $u = \lim_{n\to\infty} F^n \bar{x}$.

Proof of Theorem 1. We define the operator

$$N : B_1 \to B_1, \quad B_1 := \{u \in C([0,T], \mathbb{R}_+) : \|u\|_{L^\infty([0,T])} \leq 1\},$$

such that

$$Nu(t) = \varphi\Big(\int_0^t \omega(t-\tau)(1 - u(\tau)\sigma(\tau))d\tau\Big). \tag{12}$$

We first note that $(B_1, \|\cdot\|_{L^\infty([0,T])})$ is a Banach space, since B_1 is a closed subset of $C([0,T], \mathbb{R}_+)$ and $[0,T]$ is a compact interval. Moreover, under assumption (H2) on the weight ω the operator N is well defined; indeed, let $u \in B_1$; then, for any $t \in [0,T]$ we may estimate

$$0 < \int_0^t \omega(t-\tau)(1 - u(\tau)\sigma(\tau))\, d\tau \leq \|\omega\|_{L^1([0,T])} < 1 - \mu, \tag{13}$$

being $0 \leq \sigma(\tau) \leq 1$ for all $\tau \in [0,T]$. Thus, the function Nu is well defined in $[0,T]$ and it holds $\|Nu\|_{L^\infty} \leq 1$. In particular, the argument of φ in (12) varies in the interval $[0, 1-\mu]$, where φ is Lipschitz.

Let us prove that $Nu \in C([0,T], \mathbb{R}_+)$: let $\varepsilon > 0$; due to assumption (H_2) there exists $\delta_0 > 0$ such that, for all $0 \leq t_1 \leq t_2 \leq T$ with $t_2 - t_1 \leq \delta_0$, it holds

$$\int_0^{t_1} |\omega(t_1 - \tau) - \omega(t_2 - \tau)| d\tau \leq \frac{\varepsilon}{4M_\varphi};$$

moreover, since $\omega \in L^1([0,T])$, there exists $\delta_1 > 0$ such that

$$\int_{t_1}^{t_2} \omega(t_2 - \tau)\, d\tau \leq \frac{\varepsilon}{4M_\varphi},$$

for any $0 \leq t_1 \leq t_2 \leq T$ such that $t_2 - t_1 \leq \delta_1$.

Let us consider $\bar{\delta} := \min\{\delta_0, \delta_1\}$, and $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| < \bar{\delta}$. Without loss of generality, we suppose $t_1 \leq t_2$. Since φ is a Lipschitz function in $[0, 1-\mu]$ with Lipschitz constant $M_\varphi > 0$, and inequality (13) holds for all $t \in [0,T]$, we can estimate

$$\begin{aligned}
&|Nu(t_1) - Nu(t_2)| \\
&\quad = \Big|\varphi\Big(\int_0^{t_1} \omega(t_1 - \tau)(1 - u(\tau)\sigma(\tau))d\tau\Big) - \varphi\Big(\int_0^{t_2} \omega(t_2 - \tau)(1 - u(\tau)\sigma(\tau))d\tau\Big)\Big| \\
&\quad \leq M_\varphi \Big|\int_0^{t_1} (\omega(t_1 - \tau) - \omega(t_2 - \tau))(1 - u(\tau)\sigma(\tau))d\tau\Big| \\
&\quad + M_\varphi \Big|\int_{t_1}^{t_2} \omega(t_2 - \tau)(1 - u(\tau)\sigma(\tau))d\tau\Big| \\
&\quad \leq 2M_\varphi \int_0^{t_1} |\omega(t_1 - \tau) - \omega(t_2 - \tau)| d\tau + 2M_\varphi \int_{t_1}^{t_2} \omega(t_2 - \tau)\, d\tau \leq \varepsilon,
\end{aligned}$$

by using the definition of $\bar{\delta}$. We conclude that the operator N is well defined on $C([0,T], \mathbb{R}_+)$ and takes values in $C([0,T], \mathbb{R}_+)$.

Let us prove that the operator N satisfies the assumptions of Theorem 2. We first suppose that ω satisfies (i) of hypothesis (H_2). Since φ is a Lipschitz function in $[0, 1-\mu]$ and inequality (13) holds for all $u \in B_1$ and $t \in [0,T]$, we may estimate

$$\begin{aligned}
&|Nu(t) - Nv(t)| \\
&\quad = \Big|\varphi\Big(\int_0^{t} \omega(t - \tau)(1 - u(\tau)\sigma(\tau))d\tau\Big) - \varphi\Big(\int_0^{t} \omega(t - \tau)(1 - v(\tau)\sigma(\tau))d\tau\Big)\Big| \\
&\quad \leq M_\varphi \Big|\int_0^{t} \omega(t - \tau)(u(\tau) - v(\tau))\sigma(\tau)d\tau\Big| \qquad (14)\\
&\quad \leq M_\varphi \|u - v\|_{L^\infty([0,T])}\|\omega\|_{L^1([0,T])};
\end{aligned}$$

as a consequence, we get

$$\|Nu - Nv\|_{L^\infty} \leq (M_\varphi \|\omega\|_{L^1([0,T])})\|u - v\|_{L^\infty([0,T])};$$

applying an induction argument we obtain that for all $k \in \mathbb{N}$ the k−th iterate N^k of N satisfies

$$\|N^k u - N^k v\|_{L^\infty} \leq (M_\varphi \|\omega\|_{L^1([0,T])})^k \|u - v\|_{L^\infty([0,T])}.$$

Being $M_\varphi \|\omega\|_{L^1([0,T])} < 1$, by Theorem 2 the latter inequality allows to conclude that there exists a unique solution $\delta(t)$ to (11).

On the other hand, if ω satisfies (ii) of assumption (H_2), we may estimate

$$\begin{aligned} &|Nu(t) - Nv(t)| \\ &\quad = \left|\varphi\Big(\int_0^t \omega(t-\tau)(1-u(\tau)\sigma(\tau))d\tau\Big) - \varphi\Big(\int_0^t \omega(t-\tau)(1-v(\tau)\sigma(\tau))d\tau\Big)\right| \\ &\quad \leq M_\varphi \left|\int_0^t \omega(t-\tau)(u(\tau)-v(\tau))\sigma(\tau)d\tau\right| \\ &\quad \leq M_\varphi \|u-v\|_{L^\infty([0,T])}\|\omega\|_{L^\infty([0,T])}\|\sigma\|_{L^\infty([0,T])} t. \end{aligned}$$

Similarly, we get

$$\begin{aligned} &|N^2u(t) - N^2v(t)| \\ &\quad \leq M_\varphi \left|\int_0^t \omega(t-\tau)(Nu(\tau)-Nv(\tau))\sigma(\tau)d\tau\right| \\ &\quad \leq \|u-v\|_{L^\infty([0,T])}\ M_\varphi^2 \|\omega\|^2_{L^\infty([0,T])}\|\sigma\|^2_{L^\infty([0,T])} \int_0^t \tau\, d\tau \\ &\quad = \|u-v\|_{L^\infty([0,T])}\ M_\varphi^2 \|\omega\|^2_{L^\infty([0,T])}\|\sigma\|^2_{L^\infty([0,T])} \frac{t^2}{2}. \end{aligned}$$

By induction, for any $k \in \mathbb{N}$ we get

$$\|N^k u - N^k v\|_{L^\infty} \leq \frac{(M_\varphi \|\omega\|_{L^\infty([0,T])}\|\sigma\|_{L^\infty([0,T])}T)^k}{k!}\|u - v\|_{L^\infty([0,T])}. \tag{15}$$

Since the series

$$\sum_{k\geq 1} \frac{(M_\varphi \|\omega\|_{L^\infty([0,T])}\|\sigma\|_{L^\infty([0,T])}T)^k}{k!}$$

is convergent, with sum equal to $\exp(M_\varphi \|\omega\|_{L^\infty([0,T])}\|\sigma\|_{L^\infty([0,T])}T) - 1$, by Theorem 2 we conclude that there exists a unique fixed point $\delta \in B_1$ of the operator N, defined in (12); such fixed point coincide with the unique solution of Eq. (11). This conclude the proof of our result. □

Remark 3. Let N be defined as in (12). Thus, if ω satisfies (i) of assumption (H_2), then by (14) we may also estimate

$$\|N^k u - N^k v\|_{L^\infty} \leq M_\varphi^k \left(\sup_{t\in[0,T]} \int_0^t \omega(t-\tau)\sigma(\tau)\, d\tau\right)^k \|u - v\|_{L^\infty([0,T])}.$$

Therefore, thanks to Theorem 2, if $\delta = \delta(t)$ is the solution of (4) and $\bar{u} \in B_1$, for all $k \geq 1$ we may estimate

$$\|N^k \bar{u} - \delta\|_{L^\infty([0,T])} \leq \frac{(M_\varphi \kappa_1)^k}{(1 - M_\varphi \kappa_1)}\|N\bar{u} - \bar{u}\|_{L^\infty([0,T])},$$

where

$$\kappa_1 := \sup_{t\in[0,T]} \int_0^t \omega(t-\tau)\sigma(\tau)\, d\tau.$$

On the other hand, if ω satisfies condition (ii) of assumption (H_2), then estimate (15) holds; thus, for any $\bar{u} \in B_1$ and $k \in \mathbb{N}$ we may estimate

$$\|N^k \bar{u} - \delta\|_{L^\infty} \leq \|N\bar{u} - \bar{u}\|_{L^\infty([0,T])} \frac{1}{(k-1)!} \int_0^{\kappa_2} (\kappa_2 - \theta)^{k-1} e^\theta\, d\theta,$$

where

$$\kappa_2 := M_\varphi \|\omega\|_{L^\infty([0,T])}\|\sigma\|_{L^\infty([0,T])}T.$$

4. Numerical integration and simulations

In the following we solve numerically system (6). Regardless of the choice for the sink term S in (2), in order to discretize the Richards' equation (1), we make use of a standard method of lines; furthermore, the integral equation (4) will be tailor-made solved according to the sink term S in (2) with the corresponding choice of the weight function ω involved in it.

Let us first consider a spatial mesh constituted by $N+1$ nodes equally spaced $z_0, \ldots, z_N$; thus, we are looking for the vector function $h : [0,T] \to \mathbb{R}^{N-1}$ such that

$$h(t) = [h_1(t), \ldots, h_{N-1}(t)]^\top,$$

where for any $i \in \{0, \ldots, N\}$ we are denoting $h_i(t) := h(z_i, t)$. Following [54] we apply the method of lines to Eq. (1) and we obtain that its solution $h = h(z,t)$ can be approximated by the solution to the ODEs system:

$$C(h)\frac{dh}{dt} = A(h)h + b(h), \tag{16}$$

where the matrices $C(h)$, $A(h)$ are defined as in [54,55] and the vector $b(h)$ accounts also for the root water uptake; in particular, it reads as

$$b(h) := \frac{1}{2\Delta z}\begin{pmatrix} (K_0 + K_1)h_0/\Delta z + (K_2 - K_0) - 2\Delta z S_1 \\ (K_3 - K_1) - 2\Delta z S_2 \\ (K_4 - K_2) - 2\Delta z S_3 \\ \vdots \\ (K_N + K_{N-1})h_N/\Delta z + (K_N - K_{N-2}) - 2\Delta z S_{N-1} \end{pmatrix};$$

here, we are denoting $K_i := K(h_i(t))$, $S_i = S(h_i(t), z_i, t)$, and Δz the discretization step-size on space.

Thus, if we consider a uniform time discretization $t_j = j\Delta t$ for any $j \in \{0, \ldots, J\}$, with $J > 0$ a natural number and $\Delta t := T/J$, then (16) can be discretized by a standard Picard iteration scheme applied to the implict Euler method as in [56]:

$$C(h^{j,m})\frac{h^{j+1,m+1} - h^j}{\Delta t} = A(h^{j+1,m})h^{j+1,m} + b(h^{j+1,m}), \quad \text{for any} \quad j \in \{0, \ldots, J-1\}, \tag{17}$$

where m represents the index for the Picard iteration, and

$$h^j := [h_1(t_j), \ldots, h_{N-1}(t_j)]^\top.$$

We remark that the vector h^0 is assigned, due to our initial condition (7), and it can be written as

$$h^0 = [\tilde{h}^0(z_1), \ldots, \tilde{h}^0(z_{N-1})]^\top;$$

moreover, the boundary values $h_0(t_j)$ and $h_N(t_j)$ are known for any $j \in \{0, \ldots, J\}$ as a consequence of the Neumann condition (8) and the Dirichlet condition (9); indeed, replacing Eq. (8), with $E(t) \equiv 0$, in Eq. (1) we get

$$C(h(z_0,t))\left.\frac{\partial h}{\partial t}\right|_{z=z_0} = -S(h_0(t), z_0, t), \quad t \in [0,T];$$

then, applying the time discretization, the following identity holds

$$h_0(t_{j+1}) = h_0(t_j) - \frac{\Delta t S(h_0(t_j), z_0, t_j)}{C(h_0(t_j))} \tag{18}$$

for any $j \in \{0, \ldots, J-1\}$; finally,

$$h_N(t_j) = \tilde{h}_L(t_j), \quad \text{for any } j \in \{0, \ldots, J-1\}. \tag{19}$$

Thus, for each $i \in \{0, \ldots, N-1\}$ and $j \in \{1, \ldots, J\}$ we obtain the approximated solution $h(z_i, t_j)$ to the Richards equation in (1) simply by solving the system (17).

In particular, from [46], in (2) we consider the Feddes RWU as root water uptake function S_c, i.e.

$$S_c(h,z,t) = \gamma(h(z,t))\frac{T_p}{L}L_{nrd}(z/L),$$

where T_p denotes the potential transpiration rate (assumed equal to 1.38 cmd^{-1} as in [57]) and γ is defined as

$$\gamma(h) = \begin{cases} 0 & \text{if } h_H < h \le 0 \\ 1 & \text{if } h_L < h \le h_H \\ 1 - \left(\frac{h - h_L}{h_W - h_L}\right)^\rho & \text{if } h_W < h \le h_L \\ 0 & \text{if } h \le h_W; \end{cases} \tag{20}$$

here, $\rho \in \mathbb{R}$ and h_H, h_L, $h_W \in \mathbb{R}_-$ are, respectively, the higher and lower thresholds of optimal soil water condition and the wilting point (cm), assumed, respectively, equal to -50 cm, -400 cm and -15000 cm, and $\rho = 1.2$.

Inspired by [46], L_{nrd} is approximated as

$$L_{nrd}(z_n) = (1 - z_n)^p, \tag{21}$$

with $p = 2.85$, and function $\varphi : [0,1] \to [0,1]$ in (4) is defined as

$$\varphi(x) = (1-x)^\mu, \qquad \mu \ge 0,$$

which satisfies assumption (H_1); indeed, it is Lipschitz in any interval $[0, K]$ for any $K < 1$, with Lipschitz constant $M = \mu$; therefore, we propose two possible weight functions ω satisfying assumption (H_2); in both cases we compare the solution to (6) with the solution to the Richards equation with classical root water uptake function ($\delta \equiv 1$), and we represent the plant water deficit index function $\mathfrak{D}$.

Hereinafter, we consider the well-known van Genuchten–Mualem type hydraulic functions:

$$\text{a)}\quad K(h) = \frac{(1-(\alpha h)^{n-1}(1+(\alpha h)^n)^{-m})^2}{(1+(\alpha h)^n)^{\frac{m}{2}}}, \qquad m = 1 - \frac{1}{n},$$

with $n = 1.629$ and $\alpha = 9 \cdot 10^{-3}$ as in [46];

$$\text{b)}\quad \theta(h) = \frac{\alpha(\theta_s - \theta_r)}{(1+|\alpha h|^n)^m} + \theta_r,$$

which is equivalent to

$$h(\theta) = -\frac{1}{\alpha}\left(\left(\frac{\theta_s - \theta_r}{\theta - \theta_r}\right)^{\frac{1}{m}} - 1\right)^{\frac{1}{n}},$$

with the following choice of saturated and residual water content: $\theta_s = 0.46$; $\theta_r = 0.084$.

Moreover, in our numerical simulations we set the maximum root depth $L = 50$ cm and a spatial mesh $\Delta z = 2.5$ cm, i.e. $N = 20$; we study the numerical solution to (1) with $\mathtt{t} \in [0, \mathtt{T}]$ where $\mathtt{T} = 6$ days, taking a time mesh $\Delta\mathtt{t} = 0.05$ d, i.e. $J = 121$. We also assume initial conditions

$$\mathtt{h}^0 = [\mathtt{h}(\mathtt{theta0}(2)), \dots, \mathtt{h}(\mathtt{theta0}(\mathtt{N}-1))]^{\top},$$

where, for any $i = 0, \dots, N$, the variable `theta0(i)` denotes the i-th component of the vector `theta0` containing the water content at the initial time $\mathtt{t} = 0$ in each point z_i of the mesh; here we have obtained this vector as an interpolation from the values reported in [46].

Moreover, we assume to know the values assumed by the hydraulic potential $h_N(t)$ at the bottom of the root zone at each time $t = 0$, $t = 1$, $t = 2$, $t = 4$ and $t = 6$ days, given by `hbottom = h(thetabottom)`, where

$$\mathtt{thetabottom} = [0.192, 0.182, 0.1645, 0.128, 0.116];$$

then, we obtain the Dirichlet condition (19) by using a linear interpolation of data in [46].

Finally, since the Neumann condition (8) with $E(t) \equiv 0$ is assumed at the top of the root zone, we can derive the value $\mathtt{h}_0(\mathtt{t}_j)$ for any $j \in \{1, \dots, J\}$ by using (18), starting from $\mathtt{h}_0(\mathtt{t}_0) = \mathtt{h}(\mathtt{theta0}(1))$.

4.1. A memory term with exponential weight

For any $\lambda \in (0, 1)$ and $c > 0$ the weight

$$\omega(\tau) = -\log(\lambda)\lambda^{\tau}, \tag{22}$$

satisfies assumption (H_2), since it belongs to $C([0, T])$, (see Remark 1).

In order to evaluate the numerical solution $\delta_{\exp} = \delta_{\exp}(t)$ to the integral equation (11) we firstly apply the composite trapezoidal rule in order to approximate the integral term: for any $j = 0, 1, 2, \dots, J$, we define $t_j = j\Delta t$; thus, on the one hand, being $\delta_{\exp}(0) = 1$ we may approximate

$$\int_0^{t_1} \omega(t_1 - \tau)(1 - \delta_{\exp}(\tau)\sigma(\tau))d\tau \sim \frac{\Delta t}{2}(\omega(0)(1 - \delta_{\exp}(t_1)\sigma(t_1)) + \omega(t_1)(1 - \sigma(0)));$$

moreover, for any $j = 2, \dots J$ we can approximate

$$\int_0^{t_j} \omega(t_j - \tau)(1 - \delta_{\exp}(\tau)\sigma(\tau))d\tau \sim \Delta t \sum_{i=1}^{j-1} \omega(t_j - t_i)(1 - \delta_{\exp}(t_i)\sigma(t_i)) + \frac{\Delta t}{2}(\omega(0)(1 - \delta_{\exp}(t_j)\sigma(t_j)) + \omega(t_j)(1 - \sigma(0)));$$

after defining

$$d_1 := \frac{\Delta t}{2}\omega(0)\sigma(t_j),$$

$$d_2 := 1 - \Delta t \sum_{i=1}^{k-1} \omega(t_j - t_i)(1 - \delta_{\exp}(t_i)\sigma(t_i)) - \frac{\Delta t}{2}\omega(0) - \frac{\Delta t}{2}\omega(t_j)(1 - \sigma(0)),$$

we apply an iteration procedure in order to obtain $\delta_{\exp}(t_j)$ for any $j \in \{2, \dots, J\}$:

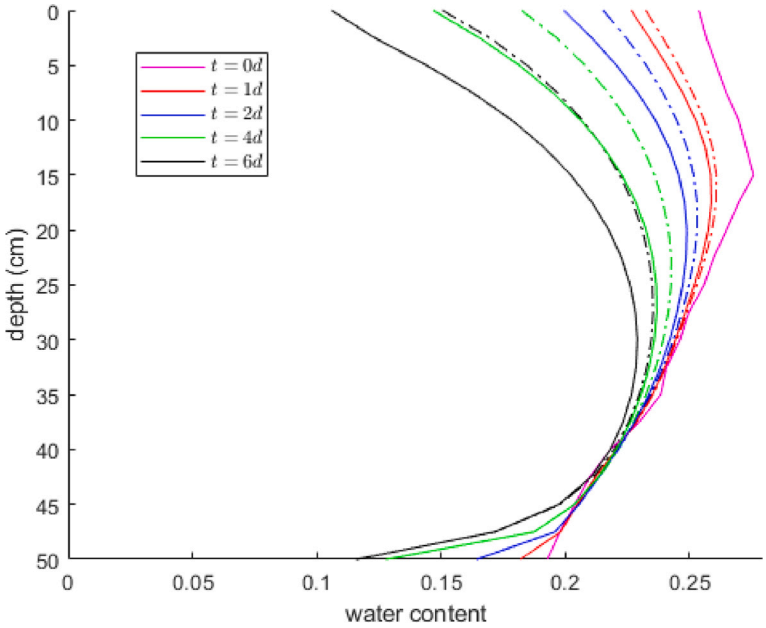

Fig. 1. Water content profiles considering Feddes root water uptake with exponential weighted memory (dashed line) and without memory (continuous line): parameters are $\lambda = 0.01$, $\mu = 1.1$, and $\mathtt{tol} = 0.05$.

```
for j=2:J
        deltaiter1=deltaexp(t_{j-1});
        deltaiter2=0;
        while abs(deltaiter1-deltaiter2)>tol
                deltaiter2=(d_2+d_1*deltaiter1)^mu;
        end
        deltaexp(t_j)=deltaiter2;
end
```

here, by `deltaexp(t_0)` we denote the initial condition $\delta_{\exp}(0)$ which is equal to 1 independently by the values assumed by ω and φ, being $\varphi(0) = 1$; moreover, the parameter `tol` denotes the error tolerance.

In Fig. 1 we represent the numerical solution to (6) where ω is an exponential weight defined by (22) with $\lambda = 0.01$; in particular, we compare it with the numerical solution to the classical Richards' equation (1) with Feddes root water uptake function.

4.2. A memory term with fractional weight

In this section we consider a singular weight; this choice allows to write the integral equation as a fractional ordinary differential equation. For any $c > \max\{1, \mu\}$ and any $\gamma \in (0, 1)$ the function

$$\omega(\tau) = \tau^{-\gamma} \frac{1-\gamma}{cT^{1-\gamma}}, \tag{23}$$

satisfies assumption (H_2). Thus, a unique solution to (11) exists. In this case, Eq. (11) reads as

$$\delta_{RL}(t) = \left(1 - C \int_0^t (t-\tau)^{-\gamma}(1 - \delta_{RL}(\tau)\sigma(\tau))d\tau\right)^{\mu},$$

where $C = C(T, \gamma) = (1-\gamma)/(cT^{1-\gamma})$ is constant with respect to t, that is

$$\delta_{RL}(t)^{\frac{1}{\mu}} = 1 - C\Gamma(1-\gamma)J_0^{1-\gamma}(1 - \delta_{RL}(\cdot)\sigma(\cdot))(t); \tag{24}$$

here, by $J_0^{1-\gamma}$ we denote the Riemann–Liouville integral operator (see, for instance [58]), defined by

$$J_0^{1-\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} f(\tau)\, d\tau; \tag{25}$$

in particular, δ_{RL} can be interpreted as the solution to the following nonlinear fractional equation,

$$^{C}D_0^{1-\gamma}(\delta_{RL}(\cdot)^{\frac{1}{\mu}})(t) = C\Gamma(1-\gamma)(\delta_{RL}(t)\sigma(t) - 1), \quad \delta_{RL}(0) = 1,$$

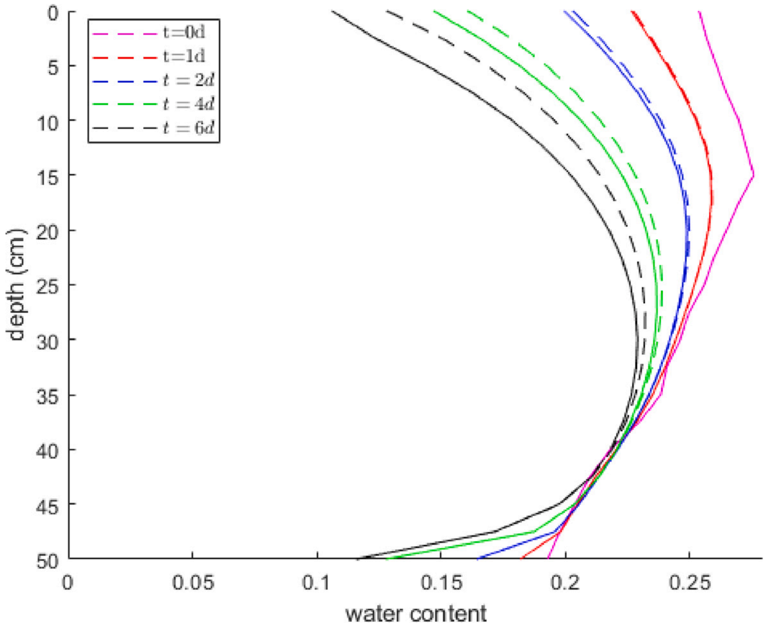

Fig. 2. Water content profiles considering Feddes root water uptake with singular weight (dashed line) and without memory (continuous line). The parameters are $\gamma = 0.6$, $\mu = 1.1$ and $\texttt{tol} = 0.05$.

where ${}^C D_0^{1-\gamma}$ is the Caputo fractional derivative of order $1-\gamma$, defined by

$$ {}^C D_0^{1-\gamma} f(t) = (J_0^{\gamma} f')(t). \tag{26}$$

The discretization applied in Section 4.1 is not suitable to obtain a numerical solution to Eq. (24), due to the singularity of the weight $\omega(\tau)$ in $\tau = 0$. Following [59] we obtain a generalized trapezoidal rule which can be used in order to obtain an approximate solution to (24); indeed, for a given grid $t_k := k\Delta t$ ($k = 0, 1, \dots, J$) of the interval $[0, T]$, with constant step-size $\Delta t = T/J$ with $J \in \mathbb{N}$ and for any continuous function $g(\tau, y)$ the integral

$$G(t_k) = \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k - \tau)^{\alpha-1} g(\tau, y(\tau)) d\tau$$

can be approximated for any $k = 1, \dots, J$ by

$$G(t_k) \sim (\Delta t)^{\alpha} \left(\tilde{a}_k^{(\alpha)} g_0 + \sum_{j=1}^{k} a_{k-j}^{(\alpha)} g_j \right) \tag{27}$$

where $g_j = g(t_j, y(t_j))$ and

$$\tilde{a}_i^{(\alpha)} := \frac{(i-1)^{\alpha+1} - i^{\alpha}(i - \alpha - 1)}{\Gamma(\alpha+2)},$$

$$a_i^{(\alpha)} := \begin{cases} \frac{1}{\Gamma(\alpha+2)} & \text{if } i = 0 \\ \frac{(i-1)^{\alpha+1} - 2i^{\alpha+1} + (i+1)^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } i = 1, 2, \dots \end{cases}$$

By applying formula (27) to the Riemann–Liouville integral in (24) we get for any $k = 1, 2, \dots,$

$$\delta_{RL}(t_k)^{\frac{1}{\mu}} = 1 - C\Gamma(1-\gamma)\Delta t^{1-\gamma} \left(\tilde{a}_k^{(1-\gamma)}(1 - \sigma(0)) + \sum_{j=1}^{k} a_{k-j}^{(1-\gamma)}(1 - \delta_{RL}(t_j)\sigma(t_j)) \right)$$

Thus, defining

$$c_1 := C\Gamma(1-\gamma) a_0^{(1-\gamma)} \delta(t_k) \sigma(t_k),$$

$$c_2 := 1 - C\Gamma(1-\gamma)\Delta t^{1-\gamma} \left(\tilde{a}_k^{(1-\gamma)}(1 - \sigma(0)) + \sum_{j=1}^{k-1} a_{k-j}^{(1-\gamma)}(1 - \delta(t_j)\sigma(t_j)) + a_0^{(1-\gamma)} \right),$$

similarly to the case with exponential memory, we obtain the numerical solution to (24) by means of an iterative procedure analogous to the one used in Section 4.1.

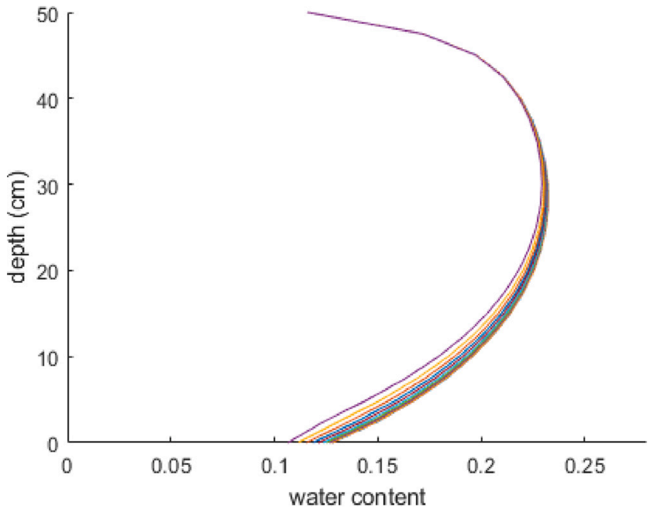

Fig. 3. Water content after six days obtained by using a singular weight $(t-s)^{-\gamma}$ with γ varying from 0.1 (on the far right-hand side) to 0.99 (on the far left-hand side).

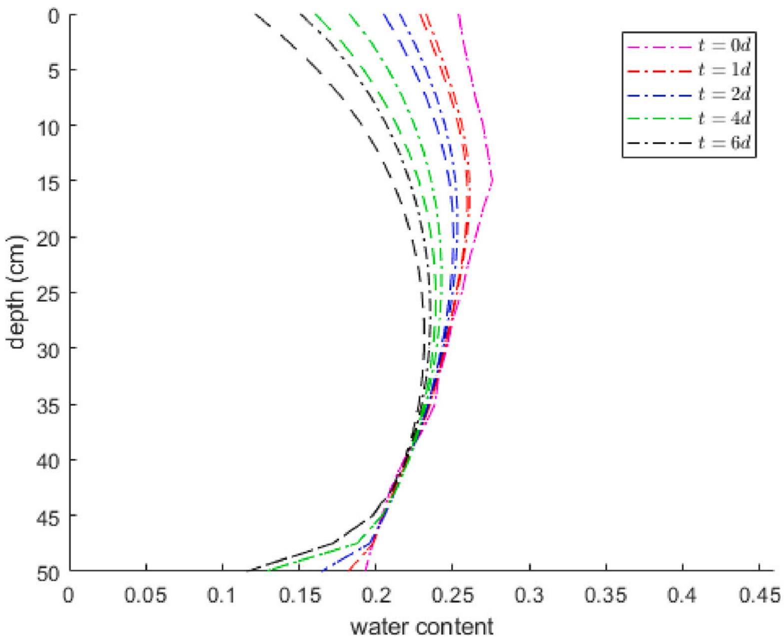

Fig. 4. Comparison between the water content at different time with Feddes root water uptake with exponential memory (dashed–dotted line) and with singular weight (dashed line), depending on the depth.

In Fig. 2 we represent the numerical solution to (6) where ω is a singular weight defined by (23) with $\gamma = 0.6$; in particular, we compare it with the numerical solution to the classical Richards' equation (1) with Feddes root water uptake function.

From Fig. 3 we can understand that the choice of a greater γ allows to give more importance to the effect of plant water stress.

Fig. 4 highlights the differences between the two different weights exemplified in this paper; in particular, we can notice that the fractional weight emphasizes the effect of plant water stress on the water content dynamics: this can also be deduced by observing different plant water deficit index in Fig. 5.

5. Conclusions and future work

In the present paper, inspired by [46], we propose a general non-local in time Richards' equation which aims to model the memory of plant water stress. Focusing on water absorption by roots, we provide a soil water content prediction more comprehensive than classical local models, holding more potential for accurately capturing observed dynamics.

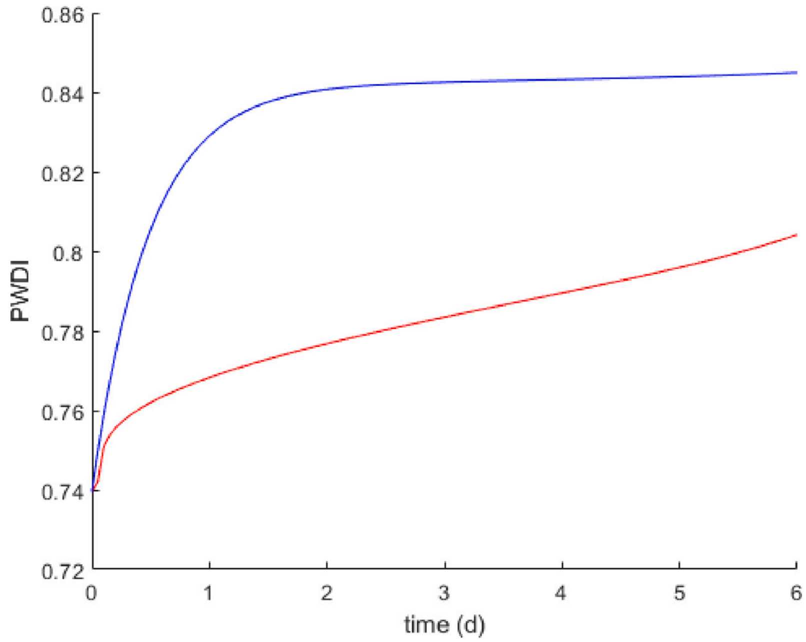

Fig. 5. Comparison between the plant water deficit index with exponential weight (blue line) and with singular weight (red line).

Here, the non-local term appears in the root water uptake function and takes into account the plant water stress caused by the alternation of dry and wet periods. Such model of plant water stress is also crucial for suitably optimizing irrigation in a water scarce environment, in the framework of control approaches (see for instance [60–62]). On the other hand, with different techniques (e.g. [63]), the spatial non-locality could be also addressed, modeled in this context in [64,65].

The proposed model paves the way to numerous future studies. Field experiments would allow to identify appropriate functions φ and ω according to site-specific field conditions, and suitably fitting the corresponding model parameters. A deeper numerical investigation could be carried out for coupling error estimates for the integral equation, provided in Remark 3 with Richards' equation solution ones.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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