SIGN-CHANGING SOLUTIONS FOR A FRACTIONAL KIRCHHOFF EQUATION

TERESA ISERNIA

Abstract. Using a minimization argument and a quantitative deformation lemma, we establish the existence of least energy sign-changing solutions for the following nonlinear Kirchhoff problem

$$(a+b[u]^2)(-\Delta)^s u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^3,$$

where $a,b>0$ are constants, $s\in(0,1)$, $(-\Delta)^s$ is the fractional Laplacian, V,K are continuous, positive functions, allowed for vanishing behavior at infinity, and f is a continuous function satisfying suitable growth assumptions. Moreover, when the nonlinearity f is odd, we obtain the existence of infinitely many nontrivial weak solutions not necessarily nodals.

1. Introduction

In this paper we are interested in the existence of least energy sign-changing (or nodal) solutions for the following nonlinear Kirchhoff problem

$$\begin{cases} (a+b[u]^2)(-\Delta)^s u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^3, \\ u\in \mathcal{D}^{s,2}(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a,b>0$ are constants, $s\in(0,1)$, and $(-\Delta)^s$ is the so-called fractional Laplacian which, up to a normalizing factor, may be defined for every $u\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ as

$$(-\Delta)^s u(x) = 2\lim_{r\to 0}\int_{\mathbb{R}^3\setminus\mathcal{B}_r(x)} \frac{u(x)-u(y)}{|x-y|^{3+2s}}\,dy \quad (x\in\mathbb{R}^3).$$

Here $\mathcal{D}^{s,2}(\mathbb{R}^3)$ is defined as the completion of $u\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ with respect to the Gagliardo semi-norm

$$[u]^2 := \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy.$$

Throughout the paper we will assume that the functions $V,K:\mathbb{R}^3\to\mathbb{R}$ are continuous, and we say that $(V,K)\in\mathcal{K}$ if the following conditions hold true (see [3]):

(VK_0) $V(x),K(x)>0$ for all $x\in\mathbb{R}^3$ and $K\in L^\infty(\mathbb{R}^3)$;

(VK_1) If $\{A_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure $m(A_n)$ is less than or equal to R, for all $n\in\mathbb{N}$ and some $R>0$, then

$$\lim_{r\to+\infty}\int_{A_n\cap\mathcal{B}_r^c(0)} K(x)\,dx = 0,$$

uniformly in $n\in\mathbb{N}$, where $\mathcal{B}_r^c(0) := \mathbb{R}^3\setminus\mathcal{B}_r(0)$.

Furthermore, one of the following conditions occurs

(VK_2) $K/V\in L^\infty(\mathbb{R}^3)$

or

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(VK_3) there exists $\nu \in (2, 2_s^*)$ such that
$$\frac{K(x)}{V(x)^{\frac{2_s^*-\nu}{2_s^*-2}}} \to 0 \text{ as } |x| \to +\infty,$$
where $2_s^* := \frac{6}{3-2s}$ is the fractional critical Sobolev exponent.

For what concerns the nonlinearity f, we assume that $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and fulfills the following conditions:

(f_1) $\lim_{|t|\to 0} \frac{f(t)}{|t|^3} = 0$ if (VK_2) holds, or

$(\tilde{f}_1)$ $\lim_{|t|\to 0} \frac{f(t)}{|t|^{\nu-1}} < +\infty$ if (VK_3) holds;

(f_2) f has a quasicritical growth at infinity, namely
$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{2_s^*-1}} = 0;$$

(f_3) F has a superquadratic growth at infinity, that is
$$\lim_{|t|\to+\infty} \frac{F(t)}{|t|^4} = +\infty, \qquad \text{where } F(t) := \int_0^t f(\tau) d\tau;$$

(f_4) the map $t \mapsto \frac{f(t)}{|t|^3}$ is strictly increasing for every $t \in \mathbb{R} \setminus \{0\}$.

When $a = 1$ and $b = 0$, and $\mathbb{R}^3$ is replaced by the more general space $\mathbb{R}^N$, problem (1.1) turns into the following fractional Schrödinger equation
$$(-\Delta)^s u + V(x)u = f(x, u) \quad \text{ in } \mathbb{R}^N, \tag{1.2}$$
which has been proposed by Laskin [30, 31] in fractional quantum mechanics as a result of extending the Feynman integrals from the Brownian like to the Lévy like quantum mechanical paths. We recall that in these years nonlinear problems involving nonlocal operators have received the attention of many mathematicians due to their intriguing structure and in view of several applications, therefore many papers appeared in the literature studying existence and multiplicity results of positive solutions for (1.2); see for instance [4, 5, 20, 22, 28] and the references therein.

On the other hand, only few results have been established for sign-changing solutions to (1.2); see [7, 9, 32, 42]. We point out that one of the main difficulty when we deal with sign-changing solutions to (1.2) is the nonlocal character of the fractional Laplacian. More precisely, in the fractional framework, the Gagliardo seminorm decomposes as follows
$$[u]^2 = [u^+]^2 + [u^-]^2 - 2 \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{3+2s}} \, dxdy,$$
where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$, which is in contrast with the local case, for which it holds the following decomposition
$$\|\nabla u\|_2^2 = \|\nabla u^+\|_2^2 + \|\nabla u^-\|_2^2.$$
Indeed, this decomposition is very useful when we deal with classical nonlinear Schrödinger equations and Dirichlet boundary value problems because permits to apply some variational and topological methods to obtain the existence of sign-changing solutions; see [12, 13, 14, 16, 44].

On the other hand, when $s = 1$, problem (1.1) reduces to the following Kirchhoff equation
$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + V(x)u = f(x, u) \quad \text{ in } \mathbb{R}^3, \tag{1.3}$$

related to the stationary analogue of the Kirchhoff equation [29]

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

for all $x \in (0, L)$ and $t \geq 0$. This equation is an extension of the classical D'Alembert wave equation taking into account the changes in the length of the strings produced by transverse vibrations. In (1.3), $u(x,t)$ is the lateral displacement of the vibrating string at the coordinate x and the time t, L is the length of the string, h is the cross-section area, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial axial tension.

The earliest studies dedicated to (1.3) can be found in [15, 37]. Anyway, only after the pioneering work by Lions [33], in which the author introduced a functional analysis approach to study a general Kirchhoff equation in arbitrary dimension with external force term, problem (1.3) began to attract the attention of many mathematicians; see for instance [1, 2, 23, 27] for positive solutions and [19, 25, 24, 36] for sign-changing solutions.

Recently, Fiscella and Valdinoci [26] proposed the following stationary Kirchhoff model driven by the fractional Laplacian

$$\begin{cases} -M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right) (-\Delta)^s u = \lambda\, f(x,u) + |u|^{2^*_s-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $2^*_s = \frac{2N}{N-2s}$, $N > 2s$, $s \in (0,1)$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function which behaves like $M(t) = a + b\,t$, with $b \geq 0$, and f is a continuous function. Based on a truncation argument and the mountain pass theorem, the authors established the existence of a non-negative solution to (1.4) for any $\lambda > \lambda^* > 0$, where λ^* is an appropriate threshold. Equation (1.4) models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string; see [26] for more physical background involving this subject. After their work, and due to the increasing interest toward fractional problems, many authors dealt with existence and multiplicity of solutions for (1.4); see [6, 8, 10, 11, 21, 34, 35, 38, 39].

On the other hand, only few results concerning the existence and multiplicity of sign-changing solutions for fractional Kirchhoff problems appear in the literature; see [17, 18]. In this case, the methods used to look for sign-changing solutions for (1.3) does not work due to the presence of two nonlocal terms, the fractional Laplacian $(-\Delta)^s$ and the fractional Kirchhoff term $[u]^2(-\Delta)^s u$, therefore a more accurate investigation is needed in this framework.

Motivated by the above results, in this paper we study the existence and multiplicity of sign-changing solutions for (1.1).

Now, we state the main result of this paper.

Theorem 1.1. *Suppose that $(V, K) \in \mathcal{K}$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ verifies either (f_1) or $(\tilde{f}_1)$ and $(f_2)-(f_4)$. Then, problem (1.1) possesses a least energy sign-changing weak solution. In addition, if the nonlinear term f is odd, then problem (1.1) has infinitely many nontrivial weak solutions not necessarily nodals.*

The proof of Theorem 1.1 is obtained by applying suitable variational techniques inspired by [7, 25]. In order to study (1.1), we consider the following functional $\mathcal{E} : \mathbb{X} \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}[u]^4 - \int_{\mathbb{R}^3} K(x) F(u)\, dx,$$

where

$$\mathbb{X} := \left\{ u \in \mathcal{D}^{s,2}(\mathbb{R}^3) \,:\, \int_{\mathbb{R}^3} V(x)|u|^2\, dx < +\infty \right\}$$

endowed with the norm

$$\|u\|^2 := a[u]^2 + \int_{\mathbb{R}^3} V(x)|u|^2 dx.$$

Clearly, $\mathcal{E} \in \mathcal{C}^1(\mathbb{X}, \mathbb{R})$, and its differential is given by

$$\langle \mathcal{E}'(u), \varphi \rangle = (a + b[u]^2) \iint_{\mathbb{R}^6} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{3+2s}} \, dxdy + \int_{\mathbb{R}^3} V(x) u \varphi \, dx - \int_{\mathbb{R}^3} K(x) f(u) \varphi \, dx,$$

for any $u, \varphi \in \mathbb{X}$.

We recall that u is a sign-changing solution of (1.1) if $u \in \mathbb{X}$ is a weak solution to (1.1) and $u^{\pm} \neq 0$. Therefore, we define the nodal set

$$\mathcal{M} := \{w \in \mathcal{N} \,:\, w^{\pm} \neq 0, \, \langle \mathcal{E}'(w), w^+ \rangle = \langle \mathcal{E}'(w), w^- \rangle = 0\}.$$

where

$$\mathcal{N} := \{u \in \mathbb{X} \setminus \{0\} : \langle \mathcal{E}'(u), u \rangle = 0\}.$$

Then, we try to get sign-changing solutions for (1.1) by seeking minimizers of the functional $\mathcal{E}$ over the constraint $\mathcal{M}$. Since $(-\Delta)^s$ is nonlocal, we need some technical analysis to prove that $\mathcal{M} \neq \emptyset$ and the minimizer is indeed a sign-changing solution to (1.1). Anyway, several difficulties arise in the study of our problem. Indeed, as explained above, we have to take care of the presence of nonlocal terms, so some fine estimates will be done. Furthermore, the nonlinearity f is only continuous, so we cannot apply $\mathcal{C}^1$-Nehari manifold method but we borrow some ideas developed in [41]. Finally, to produce nodal solutions, instead of using the Miranda Theorem to get critical points of $h^u(\xi, \lambda) = \mathcal{E}(\xi u^+ + \lambda u^-)$, we use an iterative process to build a sequence which converges to a critical point of $h^u(\xi, \lambda)$.

The paper is organized as follows. In Section 2 we prove some useful results which allow us to overcome the lack of differentiability of the Nehari manifold in which we look for weak solutions to problem (1.1). In Section 3 we obtain some technical lemmas regarding the existence of a least energy nodal solution. In Section 4 we get the existence of sign-changing weak solutions by using minimization arguments and a variant of the Deformation Lemma.

Notations: We denote by $\mathcal{B}_R(x)$ the ball of radius R with center x and by $\mathcal{B}_R^c(x) = \mathbb{R}^3 \setminus \mathcal{B}_R(x)$. In the case $x = 0$, we set $\mathcal{B}_R = \mathcal{B}_R(0)$ and $\mathcal{B}_R^c = \mathbb{R}^3 \setminus \mathcal{B}_R(0)$. Let $1 \leq r \leq \infty$ and $A \subset \mathbb{R}^3$. We denote by $|u|_{L^r(A)}$ the $L^r(A)$-norm of $u : \mathbb{R}^3 \to \mathbb{R}$ and by $|u|_r$ its $L^r(\mathbb{R}^3)$-norm.

2. Preliminary results

We begin this section by proving the following results that allow us to overcome the non-differentiability of $\mathcal{N}$. Below, we denote by $\mathbb{S}$ the unit sphere on $\mathbb{X}$.

Lemma 2.1. *Suppose that $(V, K) \in \mathcal{K}$ and f verifies conditions $(f_1) - (f_4)$. Then, the following facts hold true:*

- (a) *For each $u \in \mathbb{X} \setminus \{0\}$, let $h_u : \mathbb{R}_+ \to \mathbb{R}$ be defined by $h_u(t) := \mathcal{E}(tu)$. Then, there is a unique $t_u > 0$ such that $h'_u(t) > 0$ in $(0, t_u)$ and $h'_u(t) < 0$ in $(t_u, +\infty)$;*
- (b) *There is $\tau > 0$, independent of u, such that $t_u \geq \tau$ for every $u \in \mathbb{S}$. Moreover, for each compact set $\mathcal{W} \subset \mathbb{S}$, there is $C_{\mathcal{W}} > 0$ such that $t_u \leq C_{\mathcal{W}}$ for every $u \in \mathcal{W}$;*
- (c) *The map $\hat{m} : \mathbb{X} \setminus \{0\} \to \mathcal{N}$ given by $\hat{m}(u) := t_u u$ is continuous and $m := \hat{m}|_{\mathbb{S}}$ is a homeomorphism between $\mathbb{S}$ and $\mathcal{N}$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|}$.*

Proof. (a) Firstly, assume that (VK_2) is in force. Then, from (f_1), (f_2), and the Sobolev embedding we get

$$\begin{aligned}\mathcal{E}(tu) &= a\frac{t^2}{2}[u]^2 + b\frac{t^4}{4}[u]^4 + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx - \int_{\mathbb{R}^3} K(x)F(tu)\,dx \\ &\geq \frac{t^2}{2}\|u\|^2 - \varepsilon\int_{\mathbb{R}^3} K(x)t^2u^2\,dx - C_\varepsilon\int_{\mathbb{R}^3} K(x)t^{2_s^*}|u|^{2_s^*}\,dx \\ &\geq \left(\frac{1}{2} - \varepsilon\,|K/V|_\infty\right)t^2\|u\|^2 - C_\varepsilon\, C|K|_\infty t^{2_\alpha^*}\|u\|^{2_\alpha^*}.\end{aligned} \tag{2.1}$$

Taking $\varepsilon \in (0, \frac{1}{2|K/V|_\infty})$, we find $t_0 > 0$ sufficiently small such that

$$0 < h_u(t) = \mathcal{E}(tu) \quad \text{for all } t < t_0. \tag{2.2}$$

Now, assume that (VK_3) is true. Then, there is a positive constant C_ν such that, for each $\varepsilon \in (0, C_\nu)$, we obtain $R > 0$ such that for every $u \in \mathbb{X}$

$$\int_{\mathcal{B}_R^c} K(x)|u|^\nu\,dx \leq \varepsilon\int_{\mathcal{B}_R^c}(V(x)|u|^2 + |u|^{2_s^*})\,dx. \tag{2.3}$$

Now, combining $(\tilde{f_1})$ with (f_2), the Sobolev embedding, (2.3) and the Hölder inequality, we can infer that

$$\begin{aligned}\mathcal{E}(tu) &\geq \frac{t^2}{2}\|u\|^2 - Ct^\nu\int_{\mathbb{R}^3} K(x)|u|^\nu\,dx - \tilde{C}t^{2_s^*}\int_{\mathbb{R}^3} K(x)|u|^{2_s^*}\,dx \\ &\geq \frac{t^2}{2}\|u\|^2 - Ct^\nu\varepsilon\int_{\mathcal{B}_R^c}(V(x)|u|^2 + |u|^{2_s^*})\,dx - Ct^\nu\int_{\mathcal{B}_R} K(x)|u|^\nu\,dx - \tilde{C}t^{2_s^*}|K|_\infty\int_{\mathbb{R}^3}|u|^{2_s^*}\,dx \\ &\geq \frac{t^2}{2}\|u\|^2 - Ct^\nu\varepsilon\int_{\mathcal{B}_R^c}(V(x)|u|^2 + |u|^{2_s^*})\,dx - Ct^\nu|K|_{L^{\frac{2_s^*}{2_s^*-\nu}}(\mathcal{B}_R)}\left(\int_{\mathcal{B}_R}|u|^\nu\,dx\right)^{\frac{\nu}{2_s^*}} \\ &\quad - \tilde{C}t^{2_s^*}|K|_\infty\int_{\mathbb{R}^3}|u|^{2_s^*}\,dx \\ &\geq \frac{t^2}{2}\|u\|^2 - C_1t^\nu\left(\varepsilon\|u\|^2 + \varepsilon C\|u\|^{2_s^*} + C|K|_{L^{\frac{2_s^*}{2_s^*-\nu}}(\mathcal{B}_R)}\|u\|^\nu\right) - Ct^{2_s^*}|K|_\infty\|u\|^{2_s^*},\end{aligned} \tag{2.4}$$

which implies that (2.2) is verified.

On the other hand, since $F(t) \geq 0$ for every $t \in \mathbb{R}$, we have

$$\mathcal{E}(tu) \leq a\frac{t^2}{2}[u]^2 + b\frac{t^4}{4}[u]^4 + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx - \int_A K(x)F(tu)\,dx,$$

where $A \subset \operatorname{supp} u$ is a measurable set with finite and positive measure. Hence,

$$\limsup_{t\to+\infty}\frac{\mathcal{E}(tu)}{\|tu\|^4} \leq \frac{b}{4} - \liminf_{t\to\infty}\left\{\int_A K(x)\left[\frac{F(tu)}{(tu)^4}\right]\left(\frac{u}{\|u\|}\right)^4\,dx\right\}.$$

In the light of (f_3) and using Fatou's lemma we can infer that

$$\limsup_{t\to+\infty}\frac{\mathcal{E}(tu)}{\|tu\|^4} \leq -\infty. \tag{2.5}$$

Therefore, there exists $R > 0$ sufficiently large such that $h_u(R) = \mathcal{E}(Ru) < 0$.

Since h_u is a continuous function and exploiting (f_4), there is $t_u > 0$ which is a global maximum of h_u with $t_uu \in \mathcal{N}$. Next we prove that t_u is the unique critical point of h_u. Assume by contradiction that there are $t_1 > t_2 > 0$ critical points of h_u. Then we have

$$\frac{\|u\|^2}{t_1^2} + b[u]^4 - \int_{\mathbb{R}^3} K(x)\frac{f(t_1u)}{(t_1u)^3}u^4\,dx = 0,$$
$$\frac{\|u\|^2}{t_2^2} + b[u]^4 - \int_{\mathbb{R}^3} K(x)\frac{f(t_2u)}{(t_2u)^3}u^4\,dx = 0.$$

From which, taking into account (f_4), we deduce

$$0 > \left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right)\|u\|^2 - \int_{\mathbb{R}^3} K(x)\left[\frac{f(t_1u)}{(t_1u)^3} - \frac{f(t_2u)}{(t_2u)^3}\right]u^4\,dx > 0,$$

which leads a contradiction.

(b) By (a) there exists $t_u > 0$ such that $\langle \mathcal{E}'(t_uu), t_uu\rangle = 0$. Arguing as in (2.1) and (2.4), we obtain that there exists $\tau > 0$, independent of u, such that $t_u \geq \tau$.

On the other hand, let $\mathcal{W} \subset \mathbb{S}$ be a compact set. Assume by contradiction that there exists $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{W}$ such that $t_n := t_{u_n} \to \infty$. Therefore, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in $\mathbb{X}$. From (2.5), we have

$$\mathcal{E}(t_nu_n) \to -\infty \text{ in } \mathbb{R}. \tag{2.6}$$

We notice that by (f_4) it follows that the real function

$$t \mapsto \frac{1}{4}f(t)t - F(t)$$

is strictly increasing for every $t > 0$ and strictly decreasing for every $t < 0$. Hence, we have that for any $v \in \mathcal{N}$

$$\mathcal{E}(v) = \mathcal{E}(v) - \frac{1}{4}\langle \mathcal{E}'(v), v\rangle = \frac{1}{4}\|v\|^2 + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{4}f(v)v - F(v)\right]dx \geq 0, \tag{2.7}$$

Since $\{t_{u_n}u_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$, we conclude from (2.6) that (2.7) is not true, which is a contradiction.

(c) Let us note that $\hat{m}$, m and m^{-1} are well defined. Actually, for each $u \in \mathbb{X}\setminus\{0\}$, by (a) there is a unique $m(u) \in \mathcal{N}$. On the other hand, if $u \in \mathcal{N}$ then $u \neq 0$, so $m^{-1}(u) = \frac{u}{\|u\|} \in \mathbb{S}$ and m^{-1} is well defined. Moreover, since

$$m^{-1}(m(u)) = m^{-1}(t_uu) = \frac{t_uu}{\|t_uu\|} = u \quad \text{for all } u \in \mathbb{S}$$

and

$$m(m^{-1}(u)) = m\left(\frac{u}{\|u\|}\right) = t_{\left(\frac{u}{\|u\|}\right)}\left(\frac{u}{\|u\|}\right) = u \quad \text{for all } u \in \mathcal{N},$$

we can deduce that m is bijective and m^{-1} is continuous. Next we verify that $\hat{m}$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ and let $u \in \mathbb{X}\setminus\{0\}$ be such that $u_n \to u$ in $\mathbb{X}$. From (b) there exists $t_0 > 0$ such that $t_{u_n}\|u_n\| = t_{(u_n/\|u_n\|)} \to t_0$. Hence, $t_{u_n} \to \frac{t_0}{\|u\|}$. From $t_{u_n}u_n \in \mathcal{N}$ we deduce that

$$(a + bt_{u_n}^2[u_n]^2)t_{u_n}^2[u_n]^2 + t_{u_n}^2\int_{\mathbb{R}^3} V(x)|u_n|^2\,dx = \int_{\mathbb{R}^3} K(x)f(t_{u_n}u_n)t_{u_n}u_n\,dx.$$

Letting $n \to \infty$ we have

$$a\frac{t_0^2}{\|u\|^2}[u]^2 + b\frac{t_0^4}{\|u\|^4}[u]^4 + \frac{t_0^2}{\|u\|^2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx = \int_{\mathbb{R}^3} K(x)f\left(\frac{t_0}{\|u\|}u\right)\frac{t_0}{\|u\|}u\,dx,$$

that is $\frac{t_0}{\|u\|}u \in \mathcal{N}$ and $t_u = \frac{t_0}{\|u\|}$, that shows $\hat{m}(u_n) \to \hat{m}(u)$. Hence, $\hat{m}$ and m are continuous functions. ∎

Let us define the maps

$$\hat{\psi} : \mathbb{X} \to \mathbb{R} \quad \text{and} \quad \psi : \mathbb{S} \to \mathbb{R},$$

by $\hat{\psi}(u) := \mathcal{E}(\hat{m}(u))$ and $\psi := \hat{\psi}|_{\mathbb{S}}$.

The next result is a consequence of Lemma 2.1 (see [41]).

Proposition 2.1. *Suppose that $(V, K) \in \mathcal{K}$ and f fulfills $(f_1) - (f_4)$. Then, one has the following assertions:*

(a) $\hat{\psi} \in \mathcal{C}^1(\mathbb{X} \setminus \{0\}, \mathbb{R})$ *and*

$$\langle \hat{\psi}'(u), v\rangle = \frac{\|\hat{m}(u)\|}{\|u\|}\langle \mathcal{E}'(\hat{m}(u)), v\rangle \text{ for all } u \in \mathbb{X} \setminus \{0\} \text{ and } v \in \mathbb{X},$$

(b) $\psi \in \mathcal{C}^1(\mathbb{S}, \mathbb{R})$ *and* $\langle \psi'(u), v\rangle = \|m(u)\|\langle \mathcal{E}'(m(u)), v\rangle$, *for every* $v \in T_u\mathbb{S}$, *where*

$$T_u\mathbb{S} := \left\{ v \in \mathbb{X} : \langle v, u\rangle = a \iint_{\mathbb{R}^6} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{3+2s}}\, dxdy + \int_{\mathbb{R}^3} V(x)uv\, dx = 0 \right\},$$

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(\mathrm{PS})_d$ *sequence for* ψ, *then* $\{m(u_n)\}_{n\in\mathbb{N}}$ *is a* $(\mathrm{PS})_d$ *sequence for* $\mathcal{E}$. *Moreover, if* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}$ *is a bounded* $(\mathrm{PS})_d$ *sequence for* $\mathcal{E}$, *then* $\{m^{-1}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(\mathrm{PS})_d$ *sequence for the functional* ψ,

(d) u *is a critical point of* ψ *if and only if* $m(u)$ *is a nontrivial critical point for* $\mathcal{E}$. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}} \psi(u) = \inf_{u\in\mathcal{N}} \mathcal{E}(u).$$

Remark 2.1. We notice that the following equalities hold true:

$$d_\infty := \inf_{u\in\mathcal{N}} \mathcal{E}(u) = \inf_{u\in\mathbb{X}\setminus\{0\}} \max_{t>0} \mathcal{E}(tu) = \inf_{u\in\mathbb{S}} \max_{t>0} \mathcal{E}(tu). \tag{2.8}$$

In particular, relations (2.1), (2.5) and (2.8) imply that

$$d_\infty > 0. \tag{2.9}$$

3. Technical lemmas

The aim of this section is to prove some technical lemmas related to the existence of a least energy nodal solution.

For each $u \in \mathbb{X}$ with $u^\pm \not\equiv 0$, we introduce the function $h^u : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ defined as

$$h^u(\xi, \lambda) := \mathcal{E}(\xi u^+ + \lambda\, u^-).$$

Let us observe that its gradient $\Phi^u : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ is defined by

$$\begin{aligned} \Phi^u(\xi, \lambda) &:= (\Phi^u_1(\xi, \lambda), \Phi^u_2(\xi, \lambda)) \\ &= \left(\frac{\partial h^u}{\partial \xi}(\xi, \lambda), \frac{\partial h^u}{\partial\, \lambda}(\xi, \lambda)\right) \\ &= \left(\langle \mathcal{E}'(\xi u^+ + \lambda\, u^-), u^+\rangle, \langle \mathcal{E}'(\xi u^+ + \lambda\, u^-), u^-\rangle\right). \end{aligned} \tag{3.1}$$

Lemma 3.1. *Suppose that $(V, K) \in \mathcal{K}$ and f fulfills $(f_1) - (f_4)$. Then, it follows that*

(i) *The pair* (ξ, λ) *is a critical point of* h^u *with* $\xi, \lambda > 0$ *if, and only if,* $\xi u^+ + \lambda\, u^- \in \mathcal{M}$;

(ii) *The map* h^u *has a unique critical point* (ξ_+, λ_-), *with* $\xi_+ = \xi_+(u) > 0$ *and* $\lambda_- = \lambda_-(u) > 0$, *which is the unique global maximum point of* h^u;

(iii) *The maps* $a_+(r) := \Phi^u_1(r, \lambda_-)r$ *and* $a_-(r) := \Phi^u_2(\xi_+, r)r$ *are such that*

$$a_+(r) > 0 \text{ if } r \in (0, \xi_+) \quad \text{and} \quad a_+(r) < 0 \text{ if } r \in (\xi_+, +\infty)$$
$$a_-(r) > 0 \text{ if } r \in (0, \lambda_-) \quad \text{and} \quad a_-(r) < 0 \text{ if } r \in (\lambda_-, +\infty).$$

Proof. (i) For all $\xi, \lambda > 0$, from (3.1) it follows that

$$\Phi^u(\xi,\lambda) = \left(\frac{1}{\xi}\langle \mathcal{E}'(\xi u^+ + \lambda\, u^-), \xi u^+\rangle, \frac{1}{\lambda}\langle \mathcal{E}'(\xi u^+ + \lambda\, u^-), \lambda\, u^+\rangle\right).$$

Then, $\Phi^u(\xi,\lambda) = 0$ if, and only if,

$$\langle \mathcal{E}'(\xi u^+ + \lambda\, u^-), \xi u^+\rangle = 0 \quad \text{and} \quad \langle \mathcal{E}'(\xi u^+ + \lambda\, u^-), \lambda\, u^-\rangle = 0,$$

and this implies that $\xi u^+ + \lambda\, u^- \in \mathcal{M}$.

(ii) Firstly we verify the existence of a critical point for h^u. For each $u \in \mathbb{X}$ with $u^\pm \neq 0$ and λ_0 fixed, we define the function $h_1 : [0,+\infty) \to [0,+\infty)$ by

$$h_1(\xi) := h^u(\xi, \lambda_0).$$

Arguing as in the proof of Lemma 2.1-(a), we can infer that there exists a unique $\xi_0 = \xi_0(u, \lambda_0) > 0$ such that

$$\begin{aligned} &h_1'(\xi) > 0 \ \text{ if } \xi \in (0,\xi_0)\\ &h_1'(\xi_0) = 0\\ &h_1'(\xi) < 0 \ \text{ if } \xi \in (\xi_0,+\infty). \end{aligned}$$

Thus, the map $\phi_1 : [0,+\infty) \to [0,+\infty)$ defined by $\phi_1(\lambda) := \xi(u,\lambda)$, where $\xi(u,\lambda)$ satisfies the properties just mentioned with λ in place of λ_0, is well defined.

By the definition of h_1 we have

$$h_1'(\phi_1(\lambda)) = \Phi_1^u(\phi_1(\lambda),\lambda) = 0 \quad \forall\, \lambda \geq 0, \tag{3.2}$$

that is

$$\begin{aligned} &(a + b[\phi_1(\lambda)u^+ + \lambda\, u^-]^2) \iint_{\mathbb{R}^6} \frac{((\phi_1(\lambda)u^+ + \lambda\, u^-)(x) - (\phi_1(\lambda)u^+ + \lambda\, u^-)(y))\phi_1(\lambda)(u^+(x) - u^+(y))}{|x-y|^{3+2s}}\, dxdy\\ &\quad + \int_{\mathbb{R}^3} V(x)(\phi_1(\lambda)u^+)^2\, dx = \int_{\mathbb{R}^3} K(x) f(\phi_1(\lambda)u^+)\phi_1(\lambda)u^+\, dx. \end{aligned} \tag{3.3}$$

Observing that

$$[\phi_1(\lambda)u^+ + \lambda\, u^-]^2 = \phi_1^2(\lambda)[u^+]^2 + \lambda^2[u^-]^2 - 2\,\lambda\,\phi_1(\lambda) \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{3+2s}}\, dxdy$$

and

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{((\phi_1(\lambda)u^+ + \lambda\, u^-)(x) - (\phi_1(\lambda)u^+ + \lambda\, u^-)(y))\phi_1(\lambda)(u^+(x) - u^+(y))}{|x-y|^{3+2s}}\, dxdy =\\ &\quad = \phi_1^2(\lambda)[u^+]^2 - 2\,\lambda\,\phi_1(\lambda) \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{3+2s}}\, dxdy, \end{aligned}$$

equation (3.3) can be rewritten as

$$\begin{aligned} &a\left(\phi_1^2(\lambda)[u^+]^2 - 2\,\lambda\,\phi_1(\lambda) \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{3+2s}}\, dxdy\right)\\ &\quad + b\left(\phi_1^2(\lambda)[u^+]^2 + \lambda^2[u^-]^2 - 2\,\lambda\,\phi_1(\lambda) \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{3+2s}}\, dxdy\right) \times\\ &\quad\quad \times \left(\phi_1^2(\lambda)[u^+]^2 - 2\,\lambda\,\phi_1(\lambda) \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{3+2s}}\, dxdy\right)\\ &\quad + \int_{\mathbb{R}^3} V(x)(\phi_1(\lambda)u^+)^2\, dx = \int_{\mathbb{R}^3} K(x) f(\phi_1(\lambda)u^+)\phi_1(\lambda)u^+\, dx. \end{aligned} \tag{3.4}$$

Now, we prove some properties of ϕ_1.

$a)$ The map ϕ_1 is continuous.

Let $\lambda_n \to \lambda_0$ as $n \to \infty$ in $\mathbb{R}$. We want to prove that $\{\phi_1(\lambda_n)\}_{n\in\mathbb{N}}$ is bounded. Assume by contradiction that there is a subsequence, again denoted by $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $\phi_1(\lambda_n) \to +\infty$ as $n \to \infty$. So, for n large enough, $\phi_1(\lambda_n) \geq \lambda_n$. Let us point out that, for $\phi_1(\lambda_n) \geq \lambda_n$ we find

$$\begin{aligned}(a+b[\phi_1(\lambda_n)u]^2)[\phi_1(\lambda_n)u]^2 \\ \geq (a+b[\phi_1(\lambda_n)u^+ + \lambda_n u^-]^2)\times \\ \times \iint_{\mathbb{R}^6} \frac{((\phi_1(\lambda_n)u^+ + \lambda_n u^-)(x) - (\phi_1(\lambda_n)u^+ + \lambda_n u^-)(y))\phi_1(\lambda_n)(u^+(x)-u^+(y))}{|x-y|^{3+2s}}\,dxdy.\end{aligned} \tag{3.5}$$

Therefore, combining (3.3) with (3.5) we obtain

$$\begin{aligned}&(a+b[\phi_1(\lambda_n)u]^2)[\phi_1(\lambda_n)u]^2 + \int_{\mathbb{R}^3} V(x)|\phi_1(\lambda_n)u^+|^2\,dx \geq \\ &\geq (a+b[\phi_1(\lambda_n)u^+ + \lambda_n u^-]^2)\times \\ &\quad \times \iint_{\mathbb{R}^6} \frac{((\phi_1(\lambda_n)u^+ + \lambda_n u^-)(x) - (\phi_1(\lambda_n)u^+ + \lambda_n u^-)(y))\phi_1(\lambda_n)(u^+(x)-u^+(y))}{|x-y|^{3+2s}}\,dxdy \\ &\quad + \int_{\mathbb{R}^3} V(x)(\phi_1(\lambda_n)u^+)^2\,dx \\ &= \int_{\mathbb{R}^3} K(x)f(\phi_1(\lambda_n)u^+)\phi_1(\lambda_n)u^+\,dx,\end{aligned}$$

and in particular,

$$\frac{a[u]^2}{\phi_1^2(\lambda_n)} + b[u]^4 + \frac{1}{\phi_1^2(\lambda_n)}\int_{\mathbb{R}^3} V(x)(u^+)^2\,dx \geq \int_{\mathbb{R}^3} K(x)\frac{f(\phi_1(\lambda_n)u^+)}{(\phi_1(\lambda_n)u^+)^3}(u^+)^4\,dx.$$

Taking into account that $\lambda_n \to \lambda_0$, $\phi_1(\lambda_n) \to +\infty$ as $n \to \infty$, assumptions (f_3) and (f_4), and Fatou's lemma, we get a contradiction. Hence, $\{\phi_1(\lambda_n)\}_{n\in\mathbb{N}}$ is a bounded sequence.

Therefore there exists $\xi_0 \geq 0$ such that $\phi_1(\lambda_n) \to \xi_0$. Now, consider (3.4) with $\lambda = \lambda_n$, and letting $n \to \infty$ we have

$$\begin{aligned}&a\left(\xi_0^2[u^+]^2 - 2\,\lambda_0\,\xi_0 \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right) \\ &\quad + b\left(\xi_0^2[u^+]^2 + \lambda_0^2[u^-]^2 - 2\,\lambda_0\,\xi_0 \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right)\times \\ &\quad \times \left(\xi_0^2[u^+]^2 - 2\,\lambda_0\,\xi_0 \iint_{\mathbb{R}^6} \frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right) \\ &\quad + \int_{\mathbb{R}^3} V(x)(\xi_0 u^+)^2\,dx = \int_{\mathbb{R}^3} K(x)f(\xi_0 u^+)\xi_0 u^+\,dx\end{aligned}$$

that is $h_1'(\xi_0) = \Phi_1^u(\xi_0, \lambda_0) = 0$. Consequently, $\xi_0 = \phi_1(\lambda_0)$, i.e. ϕ_1 is continuous.

$b)$ There holds $\phi_1(0) > 0$.

Assume that there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\phi_1(\lambda_n) \to 0^+$ and $\lambda_n \to 0$ as $n \to \infty$. Using the fact that

$$\begin{aligned}
&(a + b[\phi_1(\lambda_n)u^+ + \lambda_n u^-]^2)\times \\
&\quad \times \iint_{\mathbb{R}^6} \frac{((\phi_1(\lambda_n)u^+ + \lambda_n u^-)(x) - (\phi_1(\lambda_n)u^+ + \lambda_n u^-)(y))\phi_1(\lambda_n)(u^+(x) - u^+(y))}{|x-y|^{3+2s}}\,dxdy \\
&\quad + \int_{\mathbb{R}^3} V(x)(\phi_1(\lambda_n)u^+)^2\,dx \\
&\geq a\phi_1^2(\lambda_n)[u^+]^2 + b\phi_1^4(\lambda_n)[u^+]^4 + \int_{\mathbb{R}^3} V(x)(\phi_1(\lambda_n)u^+)^2\,dx \\
&= \phi_1^2(\lambda_n)\|u^+\|^2 + b\phi_1^4(\lambda_n)[u^+]^4,
\end{aligned}$$

equation (3.3) and assumption (f_1), we can see that

$$b[u^+]^4 \leq \frac{\|u^+\|^2}{\phi_1^2(\lambda_n)} + b[u^+]^4 \leq \int_{\mathbb{R}^3} K(x)\frac{f(\phi_1(\lambda_n)u^+)}{(\phi_1(\lambda_n)u^+)^3}(u^+)^4\,dx \to 0, \qquad \text{as } n \to \infty$$

and this fact gives a contradiction. So we deduce that $\phi_1(0) > 0$.

c) Now we show that $\phi_1(\lambda) \leq \lambda$ for λ large.

Arguing as in the proof of a), we can see that it is not possible to find any sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \to +\infty$ and $\phi_1(\lambda_n) \geq \lambda_n$ for all $n \in \mathbb{N}$. This implies that $\phi_1(\lambda) \leq \lambda$ for λ large.

Similarly, fixed ξ_0, we introduce the map $h_2 : [0,\infty) \to [0,\infty)$ defined as $h_2(\lambda) := h^u(\xi_0, \lambda)$ and, as a consequence, we can find a map ϕ_2 such that for any $\xi \geq 0$

$$h_2'(\phi_2(\xi)) = \Phi_2^u(\xi, \phi_2(\xi)). \tag{3.6}$$

The maps ϕ_2 fulfills the properties a), b) and c).

By c) we can find a positive constant C_1 such that for every $\xi, \lambda \geq C_1$

$$\phi_1(\lambda) \leq \lambda \quad \text{ and } \quad \phi_2(\xi) \leq \xi.$$

Let

$$C_2 := \max\left\{\max_{\lambda\in[0,C_1]} \phi_1(\lambda), \max_{\xi\in[0,C_1]} \phi_2(\xi)\right\}$$

and set $C := \max\{C_1, C_2\}$.

We define $\mathcal{T} : [0,C]\times[0,C] \to \mathbb{R}^2$ by

$$\mathcal{T}(\xi,\lambda) := (\phi_1(\lambda), \phi_2(\xi)).$$

Let us note that, since ϕ_1 and ϕ_2 are continuous functions, we deduce that $\mathcal{T}$ is a continuous map. Moreover,

$$\mathcal{T}([0,C]\times[0,C]) \subset [0,C]\times[0,C].$$

Indeed, for every $\xi \in [0,C]$, we have that

$$\begin{cases} \phi_2(\xi) \leq \xi \leq C & \text{if } \xi \geq C_1 \\ \phi_2(\xi) \leq \max\limits_{\xi\in[0,C_1]} \phi_2(\xi) \leq C_2 & \text{if } \xi \leq C_1 \end{cases}.$$

Similarly, we can see that $\phi_1(\lambda) \leq C$ for all $\lambda \in [0,C]$.

Then, by the Brouwer fixed point theorem, there exists $(\xi_+, \lambda_-) \in [0,C]\times[0,C]$ such that

$$\mathcal{T}(\xi_+, \lambda_-) = (\phi_1(\lambda_-), \phi_2(\xi_+)) = (\xi_+, \lambda_-).$$

Owing to this fact and recalling that $\phi_i > 0$ for $i = 1,2$, we have $\xi_+ > 0$ and $\lambda_- > 0$. By (3.2) and (3.6) we have

$$\Phi_1^u(\xi_+, \lambda_-) = \Phi_2^u(\xi_+, \lambda_-) = 0,$$

that is (ξ_+, λ_-) is a critical point of h^u.

Next we prove the uniqueness of (ξ_+, λ_-). Let $w \in \mathcal{M}$. Then we find

$$\Phi^w(1,1) = \big(\langle \mathcal{E}'(w^+ + w^-), w^+\rangle, \langle \mathcal{E}'(w^+ + w^-), w^-\rangle\big) = (0,0),$$

that is, $(1,1)$ is a critical point of h^w.

Now, assume that (ξ_0, λ_0) is a critical point of h^w, with $0 < \xi_0 \le \lambda_0$. Then from (3.1) we get

$$\langle \mathcal{E}'(\xi_0 w^+ + \lambda_0\, w^-), \xi_0 w^+\rangle = 0 \;\text{ and }\; \langle \mathcal{E}'(\xi_0 w^+ + \lambda_0\, w^-), \lambda_0\, w^-\rangle = 0.$$

From $\langle \mathcal{E}'(\xi_0 w^+ + \lambda_0\, w^-), \lambda_0\, w^-\rangle = 0$, we deduce that

$$\begin{aligned}
&a\,\lambda_0^2[w^-]^2 + b\,\lambda_0^4[w^-]^4 + \lambda_0^2\int_{\mathbb{R}^3} V(x)(w^-)^2\,dx - 2\,a\,\xi_0\,\lambda_0 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&- 2\,b\,\xi_0\,\lambda_0^3\,[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b\,\xi_0^2\,\lambda_0^2[w^+]^2[w^-]^2\\
&- 2\,b\,\xi_0^3\;\lambda_0[w^+]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&- 2\,b\,\xi_0\;\lambda_0^3[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&+ 4\,b\,\xi_0^2\,\lambda_0^2\left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2 = \int_{\mathbb{R}^3} K(x) f(\lambda_0\, w^-)\,\lambda_0\, w^-\,dx
\end{aligned}$$

and dividing by $\lambda_0^4 > 0$ we obtain

$$\begin{aligned}
&\frac{a}{\lambda_0^2}[w^-]^2 + b[w^-]^4 + \frac{1}{\lambda_0^2}\int_{\mathbb{R}^3} V(x)(w^-)^2\,dx - 2\,a\frac{\xi_0}{\lambda_0^3} \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&- 2\,b\frac{\xi_0}{\lambda_0}\,[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b\frac{\xi_0^2}{\lambda_0^2}[w^+]^2[w^-]^2\\
&- 2\,b\frac{\xi_0^3}{\lambda_0^3}[w^+]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&- 2\,b\frac{\xi_0}{\lambda_0}[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&+ 4\,b\frac{\xi_0^2}{\lambda_0^2}\left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2 = \int_{\mathbb{R}^3} K(x)\frac{f(\lambda_0\, w^-)}{(\lambda_0\, w^-)^3}(w^-)^4\,dx.
\end{aligned}$$

Using the fact that $0 < \xi_0 \le \lambda_0$ we can see that

$$\begin{aligned}
&\frac{a}{\lambda_0^2}[w^-]^2 + b[w^-]^4 + \frac{1}{\lambda_0^2}\int_{\mathbb{R}^3} V(x)(w^-)^2\,dx - 2\frac{a}{\lambda_0^2} \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&\quad- 2\,b[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b[w^+]^2[w^-]^2\\
&\quad- 2\,b[w^+]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy - 2\,b[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\\
&\quad+ 4\,b\left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2\\
&\ge \int_{\mathbb{R}^3} K(x)\frac{f(\lambda_0\, w^-)}{(\lambda_0\, w^-)^3}(w^-)^4\,dx.
\end{aligned}$$

Since $w \in \mathcal{M}$, we also have

$$\begin{aligned}
&a[w^-]^2 + b[w^-]^4 + \int_{\mathbb{R}^3} V(x)(w^-)^2\,dx - 2a \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b[w^+]^2[w^-]^2 \\
&\quad - 2b[w^+]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy - 2b[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad + 4b \left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \right)^2 \\
&= \int_{\mathbb{R}^3} K(x) \frac{f(w^-)}{(w^-)^3} (w^-)^4\,dx.
\end{aligned}$$

Subtracting we have

$$\begin{aligned}
&\left(\frac{1}{\lambda_0^2} - 1\right) \|w^-\|^2 - 2a \left(\frac{1}{\lambda_0^2} - 1\right) \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad \geq \int_{\mathbb{R}^3} K(x) \left(\frac{f(\lambda_0\, w^-)}{(\lambda_0\, w^-)^3} - \frac{f(w^-)}{(w^-)^3} \right) (w^-)^4\,dx,
\end{aligned}$$

which together with (f_4) yields $0 < \xi_0 \leq \lambda_0 \leq 1$.

Next we prove that $\xi_0 \geq 1$. From $\langle \mathcal{E}'(\xi_0 w^+ + \lambda_0\, w^-), \xi_0 w^+ \rangle = 0$ we deduce that

$$\begin{aligned}
&a\xi_0^2[w^+]^2 + b\xi_0^4[w^+]^4 + \xi_0^2 \int_{\mathbb{R}^3} V(x)(w^+)^2\,dx - 2a\xi_0\,\lambda_0 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b\xi_0^3\,\lambda_0[w^+]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b\xi_0^2\,\lambda_0^2[w^-]^2[w^+]^2 \\
&\quad - 2b\xi_0\,\lambda_0^3[w^-]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b\xi_0^3\,\lambda_0[w^+]^2 \iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad + 4b\xi_0^2\,\lambda_0^2 \left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \right)^2 \\
&= \int_{\mathbb{R}^3} K(x) f(\xi_0 w^+) \xi_0 w^+\,dx
\end{aligned}$$

and dividing by $\xi_0^4 > 0$ we obtain

$$\begin{aligned}
&\frac{a}{\xi_0^2}[w^+]^2 + b[w^+]^4 + \frac{1}{\xi_0^2}\int_{\mathbb{R}^3} V(x)(w^+)^2\,dx - 2a\frac{\lambda_0}{\xi_0^3}\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b\frac{\lambda_0}{\xi_0}[w^+]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b\frac{\lambda_0^2}{\xi_0^2}[w^-]^2[w^+]^2 \\
&\quad - 2b\frac{\lambda_0^3}{\xi_0^3}[w^-]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b\frac{\lambda_0}{\xi_0}[w^+]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad + 4b\frac{\lambda_0^2}{\xi_0^2}\left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2 \\
&= \int_{\mathbb{R}^3} K(x)\frac{f(\xi_0 w^+)}{(\xi_0 w^+)^3}(w^+)^4\,dx.
\end{aligned}$$

Exploiting the fact that $0 < \xi_0 \leq \lambda_0$ we find

$$\begin{aligned}
&\frac{a}{\xi_0^2}[w^+]^2 + b[w^+]^4 + \frac{1}{\xi_0^2}\int_{\mathbb{R}^3} V(x)(w^+)^2\,dx - 2\frac{a}{\xi_0^2}\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b[w^+]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b[w^-]^2[w^+]^2 \\
&\quad - 2b[w^-]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b[w^+]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad + 4b\left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2 \\
&\leq \int_{\mathbb{R}^3} K(x)\frac{f(\xi_0 w^+)}{(\xi_0 w^+)^3}(w^+)^4\,dx.
\end{aligned}$$

Now, using $w \in \mathcal{M}$, we also get

$$\begin{aligned}
&a[w^+]^2 + b[w^+]^4 + \int_{\mathbb{R}^3} V(x)(w^+)^2\,dx - 2a\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b[w^+]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy + b[w^+]^2[w^-]^2 \\
&\quad - 2b[w^-]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad - 2b[w^+]^2\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy \\
&\quad + 4b\left(\iint_{\mathbb{R}^6} \frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2 \\
&= \int_{\mathbb{R}^3} K(x)\frac{f(w^+)}{(w^+)^3}(w^+)^4\,dx.
\end{aligned}$$

Subtracting we obtain

$$\left(1-\frac{1}{\xi_0^2}\right)\|w^+\|^2-2a\left(1-\frac{1}{\xi_0^2}\right)\iint_{\mathbb{R}^6}\frac{w^+(x)w^-(y)+w^-(x)w^+(y)}{|x-y|^{3+2s}}\,dxdy$$
$$\geq\int_{\mathbb{R}^3}K(x)\left(\frac{f(w^+)}{(w^+)^3}-\frac{f(\xi_0 w^+)}{(\xi_0 w^+)^3}\right)(w^+)^4\,dx,$$

and from (f_4) we find $\xi_0\geq 1$. Therefore, $\xi_0=\lambda_0=1$, and this proves that $(1,1)$ is the unique critical point of h^w with positive coordinates.

Now, let $u\in\mathbb{X}$ be such that $u^\pm\neq 0$, and let $(\xi_1,\lambda_1),(\xi_2,\lambda_2)$ be critical points of h^u with positive coordinates. From (i) we have

$$\xi_1 u^+ +\lambda_1\, u^-\in\mathcal{M}\quad\text{ and }\quad \xi_2 u^+ +\lambda_2\, u^-\in\mathcal{M}.$$

Set $w_1:=\xi_1 u^+ +\lambda_1\, u^-$ and $w_2:=\xi_2 u^+ +\lambda_2\, u^-$. Then, $w_1\in\mathbb{X}$ is such that $w_1^\pm\neq 0$, and

$$w_2=\left(\frac{\xi_2}{\xi_1}\right)\xi_1 u^+ +\left(\frac{\lambda_2}{\lambda_1}\right)\lambda_1\, u^- =\frac{\xi_2}{\xi_1}w_1^+ +\frac{\lambda_2}{\lambda_1}w_1^-\in\mathcal{M}.$$

Again from (i) we can infer that $\left(\frac{\xi_2}{\xi_1},\frac{\lambda_2}{\lambda_1}\right)$ is a critical point for h^{w_1} with positive coordinates.

Taking into account that $w_1=w_1^+ + w_1^-\in\mathcal{M}$, we deduce that $\frac{\xi_2}{\xi_1}=\frac{\lambda_2}{\lambda_1}=1$. Hence $\xi_1=\xi_2$ and $\lambda_1=\lambda_2$.

Finally we prove that h^u has a maximum global point $(\bar\xi,\bar\lambda)\in(0,+\infty)\times(0,+\infty)$. Let $A^+\subset\operatorname{supp}u^+$ and $A^-\subset\operatorname{supp}u^-$ positive with finite measure. Using (f_3) and $F(t)\geq 0$ for every $t\in\mathbb{R}$ we can see that

$$\begin{aligned}h^u(\xi,\lambda)&\leq\frac{1}{2}\|\xi u^+ +\lambda\, u^-\|^2+\frac{b}{4}[\xi u^+ +\lambda\, u^-]^4-\int_{A^+}K(x)F(\xi u^+)\,dx-\int_{A^-}K(x)F(\lambda\, u^-)\,dx\\ &=\frac{\xi^2}{2}\|u^+\|^2+\frac{\lambda^2}{2}\|u^-\|^2-a\,\xi\,\lambda\iint_{\mathbb{R}^6}\frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\\ &\quad+\frac{b}{4}\left(\xi^2[u^+]^2+\lambda^2[u^-]^2-2\,\xi\,\lambda\iint_{\mathbb{R}^6}\frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2\\ &\quad-\int_{A^+}K(x)F(\xi u^+)\,dx-\int_{A^-}K(x)F(\lambda\, u^-)\,dx.\end{aligned}$$

Assume that $|\xi|\geq|\,\lambda\,|>0$. Then, recalling that $F(t)\geq 0$ for every $t\in\mathbb{R}$, we get

$$\begin{aligned}h^u(\xi,\lambda)&\leq(\xi^2+\lambda^2)\left(\frac{1}{2}\|u^+\|^2+\frac{1}{2}\|u^-\|^2-\frac{1}{2}\iint_{\mathbb{R}^6}\frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right)\\ &\quad+\frac{b}{4}(\xi^2+\lambda^2)^2\left([u^+]^2+[u^-]^2-\iint_{\mathbb{R}^6}\frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2\\ &\quad-\int_{A^+}K(x)F(\xi u^+)\,dx-\int_{A^-}K(x)F(\lambda\, u^-)\,dx\\ &\leq(\xi^2+\lambda^2)\left(\frac{1}{2}\|u^+\|^2+\frac{1}{2}\|u^-\|^2-\frac{1}{2}\iint_{\mathbb{R}^6}\frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right)\\ &\quad+\frac{b}{4}(\xi^2+\lambda^2)^2\left([u^+]^2+[u^-]^2-\iint_{\mathbb{R}^6}\frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{3+2s}}\,dxdy\right)^2\\ &\quad-\int_{A^+}K(x)F(\xi u^+)\,dx.\end{aligned}$$

Hence, assumption (f_3), Fatou's lemma and the fact that $0<\xi^2+\lambda^2\leq 2\xi^2$, allow us to say that

$$\limsup_{|(\xi,\lambda)|\to\infty}\frac{h^u(\xi,\lambda)}{(\xi^2+\lambda^2)^2}\leq \bar{C}-\frac{1}{4}\liminf_{|\xi|\to\infty}\int_{A^+}K(x)\frac{F(\xi u^+)}{(\xi u^+)^4}(u^+)^4\,dx=-\infty,$$

where $\bar{C}$ is a positive constant which depends on u^+ and u^-, from which, in particular, we deduce that $h^u(\xi,\lambda)\to-\infty$ as $|(\xi,\lambda)|\to\infty$. Since h^u is a continuous function, we deduce that h^u has a maximum global point $(\bar{\xi},\bar{\lambda})\in(0,+\infty)\times(0,+\infty)$.

Using the linearity of F and the positivity of K we find

$$\int_{\mathbb{R}^3}K(x)(F(\xi u^+)+F(\lambda u^-))\,dx=\int_{\mathbb{R}^3}K(x)F(\xi u^++\lambda u^-)\,dx,$$

which combined with $[\xi u^++\lambda u^-]^2\geq \xi^2[u^+]^2+\lambda^2[u^-]^2$ yields

$$h^u(\xi,0)+h^u(0,\lambda)=\mathcal{E}(\xi u^+)+\mathcal{E}(\lambda u^-)\leq\mathcal{E}(\xi u^++\lambda u^-)=h^u(\xi,\lambda)$$

for all $u\in\mathbb{X}$ such that $u^\pm\neq 0$ and for every $\xi,\lambda\geq 0$. Then,

$$\max_{\xi\geq 0}h^u(\xi,0)<\max_{\xi,\lambda>0}h^u(\xi,\lambda)\ \text{ and }\ \max_{\lambda\geq 0}h^u(0,\lambda)<\max_{\xi,\lambda>0}h^u(\xi,\lambda),$$

and this proves that $(\bar{\xi},\bar{\lambda})\in(0,+\infty)\times(0,+\infty)$.

(iii) This is a direct consequence of Lemma 2.1-(a). □

Lemma 3.2. *If $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{M}$ and $u_n\rightharpoonup u$ in $\mathbb{X}$, then $u\in\mathbb{X}$ and $u^\pm\neq 0$.*

Proof. Let us observe that there exists $\beta>0$ such that

$$\beta\leq\|v^\pm\|\qquad\text{for all } v\in\mathcal{M}. \tag{3.7}$$

Indeed, if $v\in\mathcal{M}$, then

$$\|v^\pm\|^2+b[v^\pm]^4=\int_{\mathbb{R}^3}K(x)f(v^\pm)v^\pm\,dx. \tag{3.8}$$

If (VK_2) is true, then, combining (3.8) with (f_1), (f_2) and the Sobolev inequality, we get

$$\|v^\pm\|^2\leq\varepsilon\left|\frac{K}{V}\right|_\infty\|v^\pm\|^2+C_\varepsilon C_*|K|_\infty\|v^\pm\|^{2^*_s}.$$

Choosing $\varepsilon\in\left(0,\frac{1}{|K/V|_\infty}\right)$, we can find $\beta_1>0$ such that $\|v^\pm\|>\beta_1$.

Now, assume that (VK_3) holds. Then, using (3.8), $(\tilde{f}_1)$, (f_2), the Sobolev embedding and the Hölder inequality we obtain

$$\|v^\pm\|^2\leq C\varepsilon\|v^\pm\|^2+C\,C_*(\varepsilon+C|K|_\infty)\|v^\pm\|^{2^*_s}+|K|_{L^{\frac{2^*_\alpha}{2^*_s-\nu}}(\mathcal{B}_R)}C_*\|v^\pm\|^\nu.$$

Since $\nu\in(2,2^*_s)$, we can choose ε sufficiently small such that there exists $\beta_2>0$ such that $\|v^\pm\|>\beta_2$. Therefore, setting $\beta:=\min\{\beta_1,\beta_2\}$, we deduce that (3.7) holds true.

Hence, if $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{M}$, then

$$\beta^2\leq\int_{\mathbb{R}^3}K(x)f(u_n^\pm)u_n^\pm\,dx\qquad\text{for all } n\in\mathbb{N}.$$

Letting $n\to\infty$ in the above relation, and using [9, Lemma 2.2] we have

$$0<\beta^2\leq\int_{\mathbb{R}^3}K(x)f(u^\pm)u^\pm\,dx,$$

from which we deduce the assertion. □

Let us denote by c_∞ the number

$$c_\infty:=\inf_{u\in\mathcal{M}}\mathcal{E}(u).$$

Since $\mathcal{M}\subset\mathcal{N}$, we deduce that $c_\infty\geq d_\infty>0$.

4. Proof of Theorem 1.1

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ be such that

$$\mathcal{E}(u_n) \to c_\infty \quad \text{in } \mathbb{R}. \tag{4.1}$$

Let us point out that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$. Indeed, assume by contradiction that there is a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that $\|u_n\| \to +\infty$ as $n \to \infty$. Set $v_n := \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Then, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$, so there exists $v \in \mathbb{X}$ such that

$$v_n \rightharpoonup v \quad \text{in } \mathbb{X}, \tag{4.2}$$

and by [9, Lemma 2.1] we can infer that $v_n \to v$ a.e. in $\mathbb{R}^3$.

Using Lemma 3.1-(i) and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ we have that $\xi_+(v_n) = \lambda_-(v_n) = \|u_n\|$. By the definition of v_n and Lemma 3.1-(i) we also have that for any $\xi > 0$ and $n \in \mathbb{N}$

$$\mathcal{E}(u_n) = \mathcal{E}(\|u_n\|\, v_n) \geq \mathcal{E}(\xi\, v_n) = \frac{\xi^2}{2} + \frac{b\,\xi^4}{4}[v_n]^4 - \int_{\mathbb{R}^3} K(x)F(\xi v_n)\, dx. \tag{4.3}$$

Suppose that $v = 0$. Then, by (4.2) and [9, Lemma 2.2] we deduce that

$$\int_{\mathbb{R}^3} K(x)F(\xi v_n) \to 0 \quad \text{for all } \xi > 0. \tag{4.4}$$

Taking the limit as $n \to \infty$ in (4.3), and using (4.1) and (4.4) we get a contradiction. Hence, $v \neq 0$.

On the other hand, by the definition of $\mathcal{E}$ it follows that

$$\frac{\mathcal{E}(u_n)}{\|u_n\|^4} = \frac{1}{2\|u_n\|^2} + \frac{b}{4}\,\frac{[u_n]^4}{\|u_n\|^4} - \int_{\mathbb{R}^3} K(x)\frac{F(v_n\|u_n\|)}{(v_n\|u_n\|)^4}\,(v_n)^4\, dx. \tag{4.5}$$

Using the facts that $\|u_n\| \to \infty$, $v_n \to v$ a.e. in $\mathbb{R}^3$, assumption (f_3) and Fatou's lemma we obtain

$$\int_{\mathbb{R}^3} K(x)\frac{F(v_n\|u_n\|)}{(v_n\|u_n\|)^4}\,(v_n)^4\, dx \to \infty.$$

Hence, taking the limit as $n \to \infty$ in (4.5) we get a contradiction. Therefore $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$, so there exists $u \in \mathbb{X}$ such that $u_n \rightharpoonup u$ in $\mathbb{X}$. Applying Lemma 3.2 we can deduce that $u^\pm \neq 0$. By Lemma 3.1, we can find two positive constants $\xi_+, \lambda_- > 0$ such that $\xi_+ u^+ + \lambda_-\, u^- \in \mathcal{M}$. Next, we prove that $\xi_+, \lambda_- \in (0, 1]$. Combining $u_n \rightharpoonup u$ in $\mathbb{X}$ with [9, Lemma 2.2], we deduce the following relations of limit

$$\begin{gathered} \int_{\mathbb{R}^3} K(x)f(u_n^\pm)u_n^\pm\, dx \to \int_{\mathbb{R}^3} K(x)f(u^\pm)u^\pm\, dx \\ \text{and} \\ \int_{\mathbb{R}^3} K(x)F(u_n^\pm)\, dx \to \int_{\mathbb{R}^3} K(x)F(u^\pm)\, dx. \end{gathered} \tag{4.6}$$

From these relations, $u_n \rightharpoonup u$ in $\mathbb{X}$ and Fatou's lemma we find

$$\langle \mathcal{E}'(u), u^\pm \rangle \leq \liminf_{n\to\infty} \langle \mathcal{E}'(u_n), u_n^\pm \rangle = 0.$$

Without loss of generality, let us assume that $0 < \xi_+ < \lambda_-$. Arguing as in the proof of (ii)-Lemma 3.1, it is possible to show that $0 < \xi_+, \lambda_- \leq 1$.

Now, putting together the definition of c_∞, $0 < \xi_+, \lambda_- \le 1$, assumption (f_4), $u_n \rightharpoonup u$ in $\mathbb{X}$ and (4.6) we deduce that

$$
\begin{aligned}
c_\infty &\le \mathcal{E}(\xi_+ u^+ + \lambda_- u^-) = \mathcal{E}(\xi_+ u^+ + \lambda_- u^-) - \frac{1}{4}\langle \mathcal{E}'(\xi_+ u^+ + \lambda_- u^-), \xi_+ u^+ + \lambda_- u^-\rangle \\
&= \frac{1}{4}\|\xi_+ u^+ + \lambda_- u^-\|^2 + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(\xi_+ u^+ + \lambda_- u^-)(\xi_+ u^+ + \lambda_- u^-) - F(\xi_+ u^+ + \lambda_- u^-)\right) dx \\
&= \frac{1}{4}\|\xi_+ u^+ + \lambda_- u^-\|^2 + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(\xi_+ u^+)\xi_+ u^+ - F(\xi_+ u^+)\right) dx \\
&\quad + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(\lambda_- u^-)\,\lambda_- u^- - F(\lambda_- u^-)\right) dx \\
&\le \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(u^+)u^+ - F(u^+)\right) dx + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(u^-)u^- - F(u^-)\right) dx \\
&= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(u)u - F(u)\right) dx \\
&= \lim_{n\to\infty} \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{4} f(u_n)u_n - F(u_n)\right) dx \\
&= \lim_{n\to\infty} \mathcal{E}(u_n) - \frac{1}{4}\langle \mathcal{E}'(u_n), u_n\rangle = c_\infty,
\end{aligned}
$$

that is we have proved that $\mathcal{E}(\xi_+ u^+ + \lambda_- u^-) = c_\infty$ and that $\xi_+ = \lambda_- = 1$.

Next, our purpose is to show that the minimum point is a critical point of $\mathcal{E}$, that is $u = u^+ + u^-$ is a critical point of the functional $\mathcal{E}$. We argue by contradiction, and we suppose that $\mathcal{E}'(u) \neq 0$. By continuity, we can find $\delta, \mu > 0$ such that for all $v \in \mathbb{X}$ satisfying $\|v - u\| \le 3\delta$, it holds $\mu \le |\mathcal{E}'(v)|$.

Set $D := [\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]$, $\mathbb{X}^\pm := \{u \in \mathbb{X} : u^\pm \neq 0\}$, and define $g : D \to \mathbb{X}^\pm$ as $g(\xi, \lambda) = \xi u^+ + \lambda u^-$. It follows from Lemma 3.1 that $\mathcal{E}(g(1,1)) = c_\infty$, and $\mathcal{E}(g(\xi,\lambda)) < c_\infty$ in $D \setminus \{(1,1)\}$. Define $\beta := \max_{(\xi,\lambda)\in\partial D} \mathcal{E}(g(\xi,\lambda))$, then $\beta < c_\infty$.

Applying [43, Theorem 2.3] with $\tilde{S} := \{v \in \mathbb{X} : \|v - u\| \le \delta\}$, $c := c_\infty$ and choosing $\varepsilon := \min\left\{\frac{c_\infty - \beta}{4}, \frac{\mu\delta}{8}\right\}$, there exists a deformation $\eta \in \mathcal{C}([0,1] \times \mathbb{X}, \mathbb{X})$ such that the following assertions hold true:

(a) $\eta(\xi, v) = v$ if $v \in \mathcal{E}^{-1}([c_\infty - 2\varepsilon, c_\infty + 2\varepsilon])$;
(b) $\mathcal{E}(\eta(1,v)) \le c_\infty - \varepsilon$ for each $v \in \mathbb{X}$ with $\|v - u\| \le \delta$ and $\mathcal{E}(v) \le c_\infty + \varepsilon$;
(c) $\mathcal{E}(\eta(1,v)) \le \mathcal{E}(v)$ for all $u \in \mathbb{X}$.

Using (b) and (c) we deduce that

$$
\max_{(\xi,\lambda)\in\partial D} \mathcal{E}(\eta(1, g(\xi,\lambda))) < c_\infty. \tag{4.7}
$$

To complete the proof it suffices to prove that

$$
\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset \tag{4.8}
$$

because the definition of c_∞ and (4.8) contradict (4.7)

Let us define

$$
\begin{aligned}
&h(\xi,\lambda) := \eta(1, g(\xi,\lambda)), \\
&\psi_0(\xi,\lambda) := \left(\langle \mathcal{E}'(g(\xi,1)), \xi u^+\rangle, \langle \mathcal{E}'(g(1,\lambda)), \lambda u^-\rangle\right) \\
&\psi_1(\xi,\lambda) := \left(\frac{1}{\xi}\langle \mathcal{E}'(h(\xi,1)), h^+(\xi,1)\rangle, \frac{1}{\lambda}\langle \mathcal{E}'(h(1,\lambda)), h^-(1,\lambda)\rangle\right).
\end{aligned}
$$

Using (iii)-Lemma 3.1 we deduce that the function $\gamma_+(\xi) = h^u(\xi,1) \in \mathcal{C}^1$ has a unique global maximum point $\xi = 1$ (let us observe that $\xi\gamma_+'(\xi) = \langle \mathcal{E}(g(\xi,1)), \xi u^+\rangle$).

By density, given $\varepsilon > 0$ small enough, there is $\gamma_{+,\varepsilon} \in \mathcal{C}^\infty([\frac{1}{2},\frac{3}{2}])$ such that $\|\gamma_+ - \gamma_{+,\varepsilon}\|_{\mathcal{C}^1([\frac{1}{2},\frac{3}{2}])} < \varepsilon$ with ξ_+ being the unique maximum global point of $\gamma_{+,\varepsilon}$ in $[\frac{1}{2},\frac{3}{2}]$. Therefore, $\|\gamma_+' - \gamma_{+,\varepsilon}'\|_{\mathcal{C}([\frac{1}{2},\frac{3}{2}])} < \varepsilon$, $\gamma_{+,\varepsilon}'(1) = 0$ and $\gamma_{+,\varepsilon}''(1) < 0$.

Similarly, setting $\gamma_-(\lambda) = h^u(1,\lambda)$, there exists $\gamma_{-,\varepsilon} \in \mathcal{C}^\infty([\frac{1}{2},\frac{3}{2}])$ such that $\|\gamma_-' - \gamma_{-,\varepsilon}'\|_{\mathcal{C}([\frac{1}{2},\frac{3}{2}])} < \varepsilon$, $\gamma_{+,\varepsilon}'(1) = 0$ and $\gamma_{+,\varepsilon}''(1) < 0$.

Let us define $\psi_\varepsilon \in \mathcal{C}^\infty(D)$ by $\psi_\varepsilon(\xi,\lambda) := (\xi\gamma_{+,\varepsilon}'(\xi), \lambda\,\gamma_{-,\varepsilon}'(\lambda))$. We note that $\|\psi_\varepsilon - \psi_0\|_{\mathcal{C}(D)} < \frac{3\sqrt{2}}{2}\varepsilon$, $(0,0) \notin \psi_\varepsilon(\partial D)$, and $(0,0)$ is a regular value of ψ_ε in D.

Since $(1,1)$ is the unique solution of $\psi_\varepsilon(\xi,\lambda) = (0,0)$ in D, by the definition of Brouwer's degree, we can infer that, for ε small enough, it holds

$$\deg(\psi_0, D, (0,0)) = \deg(\psi_\varepsilon, D, (0,0)) = \operatorname{sgn}\operatorname{Jac}(\psi_\varepsilon)(1,1), \tag{4.9}$$

where $\operatorname{Jac}(\psi_\varepsilon)$ is the Jacobian determinant of ψ_ε and sgn denotes the sign function. We note that

$$\operatorname{Jac}(\psi_\varepsilon)(1,1) = [\gamma_{+,\varepsilon}'(1) + \gamma_{+,\varepsilon}''(1)] \times [\gamma_{-,\varepsilon}'(1) + \gamma_{-,\varepsilon}''(1)] = \gamma_{+,\varepsilon}''(1) \times \gamma_{-,\varepsilon}''(1) > 0, \tag{4.10}$$

so combining (4.9) with (4.10) we find

$$\deg(\psi_0, D, (0,0)) = \operatorname{sgn}[\gamma_{+,\varepsilon}''(1) \times \gamma_{-,\varepsilon}''(1)] = 1.$$

By the definition of β we have that for any $(\xi,\lambda) \in \partial D$

$$\mathcal{E}(g(\xi,\lambda)) \leq \beta < \frac{\beta + c_\infty}{2} = c_\infty - 2\left(\frac{c_\infty - \beta}{4}\right) \leq c_\infty - 2\varepsilon.$$

This and (a) yields that $g = h$ on ∂D. Therefore, $\psi_1 = \psi_0$ on ∂D and consequently

$$\deg(\psi_1, D, (0,0)) = \deg(\psi_0, D, (0,0)) = 1,$$

which shows that $\psi_1(\xi,\lambda) = (0,0)$ for some $(\xi,\lambda) \in D$.

Now, in order to verify that (4.8) holds true, we prove that

$$\psi_1(1,1) = \big(\mathcal{E}'(h(\xi,1))h(1,1)^+, \mathcal{E}'(h(1,1))h(1,1)^-\big) = 0. \tag{4.11}$$

As a matter of fact, (4.11) and the fact that $(1,1) \in D$, yield $h(1,1) = \eta(1, g(1,1)) \in \mathcal{M}$.

We argue as follows. If the zero (ξ,λ) of ψ_1 obtained above is equal to $(1,1)$ there is nothing to do. Otherwise, we take $0 < \delta_1 < \min\{|\xi - 1|, |\lambda - 1|\}$ and consider

$$D_1 := \left[1 - \frac{\delta_1}{2}, 1 + \frac{\delta_1}{2}\right] \times \left[1 - \frac{\delta_1}{2}, 1 + \frac{\delta_1}{2}\right].$$

Then, $(\xi,\lambda) \in D \setminus D_1$. Hence, we can repeat for D_1 the same argument used for D, so that we can find a couple $(\xi_1,\lambda_1) \in D_1$ such that $\psi_1(\xi_1,\lambda_1) = 0$. If $(\xi_1,\lambda_1) = (1,1)$, there is nothing to prove. Otherwise, we can continue with this procedure and find in the n-th step that (4.11) holds, or produce a sequence $(\xi_n,\lambda_n) \in D_{n-1} \setminus D_n$ which converges to $(1,1)$ and such that

$$\psi_1(\xi_n,\lambda_n) = 0, \quad \text{for every } n \in \mathbb{N}. \tag{4.12}$$

Thus, taking the limit as $n \to \infty$ in (4.12) and using the continuity of ψ_1 we get (4.11). Therefore, $u := u^+ + u^-$ is a critical point of $\mathcal{E}$.

Finally, we consider the case when f is odd. Clearly, the functional ψ is even. In the light of (2.9) and $c_\infty \geq d_\infty > 0$ we can see that ψ is bounded from below in $\mathbb{S}$. Moreover, using [9, Proposition 2.1 and Lemma 2.2], we deduce that ψ satisfies the Palais-Smale condition on $\mathbb{S}$. Hence, applying Proposition 2.1 and [40], we conclude that $\mathcal{E}$ has infinitely many critical points.

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Teresa Isernia
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche, 12
60131 Ancona (Italy)
E-mail address: t.isernia@univpm.it