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Assessment of a Posteriori Positivity-Preserving Limiters for a Modal Discontinuous Galerkin Method Applied to Multicomponent Flows

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Abstract

When computing high-order solutions of compressible multicomponent flows, numerical instabilities at shock waves and material interfaces must be mitigated without compromising accuracy. A widely adopted strategy is the use of *a posteriori* limiters to enforce physical admissibility at the cell level. In this work, we compare three different approaches: (i) pointwise correction of the solution followed by projection onto the polynomial space, a pragmatic strategy widely used in industrial practice; (ii) a variant of the Zhang and Shu moment limiter, based on the internal energy; and (iii) a more aggressive strategy based on local order reduction to recover positivity. To control spurious oscillations, an artificial-viscosity shock-capturing technique is employed, and pressure is computed via an L_2 projection of the ideal-gas equation of state. The three limiting strategies are assessed in several test cases, including Riemann problems with separated components and a multicomponent implosion. The results show that the moment limiter provides the best compromise between efficiency, robustness, and accuracy for multicomponent flows in the context of modal discontinuous Galerkin methods based on hierarchical and orthonormal bases, whereas the pointwise correction fails in the two-dimensional simulations, and the local order-reduction strategy is overly dissipative.

Keywords: discontinuous Galerkin; compressible multicomponent flows; diffuse interface model; stabilization techniques



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1. Introduction

The robust and accurate numerical solution of multicomponent flows is fundamental for simulating a wide range of industrial and scientific problems, for example, reacting flows [1] and high-speed mixing [2]. High-order methods, such as discontinuous Galerkin (dG) methods, are an attractive choice for resolving multicomponent flows due to their superior accuracy and geometric flexibility [3–5]. However, these methods may become unstable in the presence of discontinuities within cells, a problem that is accentuated when considering contact discontinuities between fluids with different thermodynamic properties in diffuse-interface models, where extremely steep gradients may arise. When addressing the solution of the multicomponent Euler model (MCE) using fully conservative methods, further complications may appear at material interfaces due to the well-known pressure oscillations caused by the inconsistent computation of mixture properties [6–8].

Therefore, one must consider adding stabilization techniques to the high-order framework. For the remainder of this paper, we will cast them into two categories: (i) *a priori* stabilization, where it is performed before advancing the solution in time, and (ii) *a posteriori* techniques, which will correct the newly obtained solution if it is considered “unsatisfactory”. For both philosophies, some kind of criterion is needed to mark a cell as “troubled” in order to take further action on it. These criteria are often referred to as detectors, and can also be classified into two broad categories: Numerical Admissibility Detectors (NADs), and Physical Admissibility Detectors (PADs). The physical admissibility is based for example on direct positivity constraints (e.g., the pressure and density must be positive) [9], or entropy conditions [10,11]; among the NADs, we highlight the Persson–Peraire shock sensor [12], the total variation sensors [13–15], and the data-driven sensors [16,17].

Among the *a priori* stabilization techniques, artificial viscosity stands as a common choice for high-order dG methods, either based on residual metrics [12], or on the solution itself [18]. In this work, we adopt the latter, which is slightly more complex but allows finer tuning of the viscosity and targets different phenomena, as demonstrated in [19]. To eliminate the pressure oscillations at material interfaces, the standard approaches are to use non-conservative methods [6–8], for example substituting the energy conservation equation by a pressure evolution equation [1]. Besides, these methods are not exactly conservative in a fully discrete sense, and according to Hou and Le Floch [20], this may lead to incorrect representation of shock waves. Recent advancements in the literature suggest that the pressure equilibrium can also be recovered with properly designed fluxes [21–23], while preserving conservation, but their high-order extension is not straightforward. Instead, we choose to apply a mitigation strategy which will reduce the magnitude of these instabilities, while not affecting possible entropy stability properties that can be imposed on the scheme. This can be achieved by “aligning” the pressure with the same functional space as the conservative set, a strategy that has been explored in the context of Finite Volume (FV) methods in [24] and its application to dG methods in [19]. This has been demonstrated to reduce the pressure oscillations and mitigate violations of the pressure equilibrium on constant pressure-velocity flows. Finally, we address the positivity violations from a third perspective, using *a posteriori* limiters. The comparison of the stabilization mechanisms at this last stage is the focus of this paper. This assignment of different numerical challenges to different strategies is depicted in Figure 1. We remark that the *a priori* and *a posteriori* techniques operate at different stages of the algorithm. The artificial-viscosity mechanism is activated through the NAD before advancing each Runge–Kutta stage, whereas the PAD is evaluated on the newly computed solution and immediately triggers the positivity-preserving limiter only if the updated state is found to be physically inadmissible. In this form, the *a posteriori* technique is used as a fallback in terms of physical admissibility; and the *a priori* stabilization is aimed at avoiding excessive triggering of these limiters, which may be computationally expensive and much more dissipative. Previous comparative studies on *a posteriori* dG stabilization have mainly focused on oscillation-control strategies, including TVB/WENO-type limiting [25,26] and *a posteriori* approaches based on the MOOD paradigm and subcell re-computation [14,27]. Instead, we compare three *a posteriori* strategies to directly enforce physical admissibility. First, we consider a simple pointwise clipping procedure (CL), in which non-admissible thermodynamic quantities are corrected locally at the quadrature points and then projected back onto the polynomial space. This approach is pragmatic and inexpensive to implement, and is representative of stabilization strategies often adopted in industrial solvers, although it does not preserve conservation exactly. Second, we consider a moment limiter (ML) inspired by the positivity-preserving framework of Zhang and Shu [9,28], here formulated in terms of the internal energy following [29]. We note that alternatives exist, such as using a simplified and approximate

correction coefficient arising from the condition on pressure [30]. This limiter rescales the high-order modes toward the cell average, preserving conservation while enforcing positivity of the density, partial densities, and internal energy. Third, we investigate a local order-reduction strategy (LO), similar in spirit to the robust fallback procedures used in MOOD-type approaches [13,15]. In this case, whenever the high-order polynomial violates the admissibility constraints, the local approximation degree is reduced until an admissible state is recovered. These three strategies represent different compromises between simplicity, conservation, robustness, and numerical dissipation, and therefore provide a useful basis for assessing the role of *a posteriori* stabilization in high-order multicomponent flow simulations.

The remainder of this paper is organized as follows. Section 2 introduces the multicomponent Euler model and the thermodynamic closure used throughout the work. Section 3 presents the numerical framework, including the modal dG discretization, the pressure projection, the artificial-viscosity shock-capturing approach, and the time-integration strategy. Section 4 describes the *a posteriori* positivity-preserving strategies considered in this study. Finally, Section 5 assesses the performance of the different limiters in one- and two-dimensional test cases, and the main conclusions are summarized in Section 6.

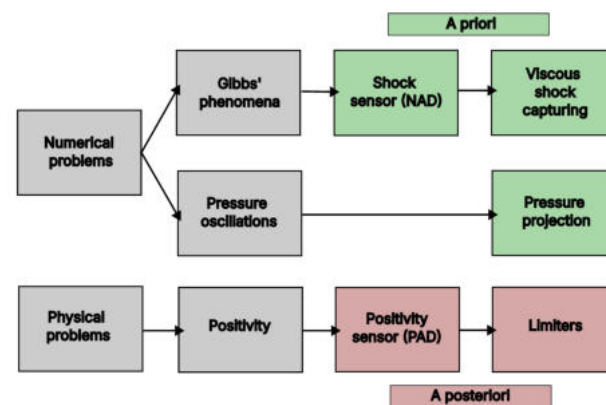


Figure 1. Schematic overview of the stabilization strategy adopted in this work. Numerical difficulties are split into Gibbs-type oscillations and pressure oscillations, which are addressed through *a priori* mechanisms: shock sensing with viscous shock capturing, and pressure projection, respectively. Positivity violations are treated separately through a *a posteriori* limiters activated by Physical Admissibility Detectors (PADs).

2. Governing Equations

The MCE model for an arbitrary number of components N in D spatial dimensions can be written in conservative form as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}_i(\mathbf{q})}{\partial x_i} = 0, \quad (1)$$

for $i = 1, \dots, D$, where repeated indices imply summation. The conservative set of variables $\mathbf{q} \in \mathbb{R}^M$ is defined as

$$\mathbf{q} = [\rho_1, \dots, \rho_N, \rho \mathbf{u}, \rho e_t]^\top, \quad (2)$$

where $\{\rho_k\}_{k=1}^N$ denotes the partial densities of the N fluids composing the mixture, $\mathbf{u} \in \mathbb{R}^D$ is the velocity vector in row form, and $M = N + D + 1$. The mixture density ρ and the total specific energy e_t are defined as

$$\rho = \sum_{k=1}^N \rho_k, \quad \rho e_t = \rho e + \frac{1}{2} \rho \|\mathbf{u}\|^2, \quad (3)$$

where e is the mixture specific internal energy and $\|\cdot\|$ denotes the Euclidean norm. The physical flux $\mathbf{F} \in \mathbb{R}^{M \times D}$ is given by

$$\mathbf{F} = \left[\rho_1 \mathbf{u}^\top, \dots, \rho_N \mathbf{u}^\top, \rho(\mathbf{u} \otimes \mathbf{u}) + p \mathbf{I}^D, (\rho e_t + p) \mathbf{u}^\top \right]^\top, \quad (4)$$

where p is the pressure, $\mathbf{I}^D \in \mathbb{R}^{D \times D}$ is the identity tensor, and $\otimes$ denotes the outer product. The model assumes mechanical and thermal equilibrium among the components. Supplying the ideal-gas equation of state (EOS) to both components, the mixture internal energy is obtained from

$$\rho e = \sum_{k=1}^N \rho_k e_k = T \sum_{k=1}^N \rho_k c_{v,k}, \quad (5)$$

where e_k is the specific internal energy of component k , the mixture temperature is T , and $c_{v,k}$ is the constant-volume specific heat of component k . The pressure is obtained from the ideal-gas equation of state (EOS) as

$$p = \sum_{k=1}^N (\gamma_k - 1) \rho_k e_k = \sum_{k=1}^N \rho_k r_k T, \quad (6)$$

where γ_k and r_k are the heat-capacity ratio and the specific gas constant of component k , respectively. The thermodynamic coefficients of the mixture depend on the partial densities as

$$\rho c_v = \sum_{k=1}^N \rho_k c_{v,k}, \quad \rho c_p = \sum_{k=1}^N \rho_k c_{p,k}, \quad \gamma = \frac{c_p}{c_v}, \quad (7)$$

where $c_{p,k}$ is the constant-pressure specific heat of component k . With the mixture properties, we can obtain its speed of sound as

$$c = \sqrt{\gamma(\gamma - 1)e}. \quad (8)$$

The adopted closure is valid as long as the molecular interaction effects are negligible; therefore, no very high-pressure gases or liquids can be considered [31]. The preferred alternatives in the literature to address such cases are the Stiffened-Gas EOS [32–34], its extension as Noble-Abel Stiffened-Gas EOS [15,35,36] and the Van der Waals-type EOS [37,38]. Finally, we remark that, in the results section, the composition of the mixture will often be expressed in terms of the mass fractions $\{Y_k = \rho_k / \rho\}_{k=1}^N$. Together with the mixture density ρ and the conservative momentum and energy variables, this representation fully defines the thermodynamic state of the fluid.

3. Numerical Framework

The model is spatially discretized using a modal discontinuous Galerkin method with an orthonormal and hierarchical basis [3]. To control spurious oscillations arising at shock waves and material interfaces, an artificial-viscosity shock-capturing term is employed, and the pressure is computed via an L_2 projection of the ideal-gas equation of state into the dG space. In addition, positivity-preserving limiters are applied to ensure physical admissibility of the solution, as detailed in Section 4. The resulting semi-discrete system is advanced in time using a Strong Stability Preserving (SSP) explicit Runge–Kutta (RK) scheme [39].

3.1. Spatial Discretization

The spatial discretization is based on a modal dG method. We begin by deriving the weak formulation of the hyperbolic system (1). This is achieved by multiplying the

equation by a smooth test function $\Phi \in \mathbb{R}^M$, integrating over the computational domain $\Omega \in \mathbb{R}^D$, and applying integration by parts, which yields

$$\int_{\Omega} \Phi^{\top} \frac{\partial \mathbf{q}}{\partial t} d\Omega - \int_{\Omega} \left(\frac{\partial \Phi}{\partial x_i} \right)^{\top} \mathbf{F}_i(\mathbf{q}) d\Omega + \int_{\partial\Omega} \Phi^{\top} \mathbf{F}_i(\mathbf{q}) n_i d\sigma = 0, \quad (9)$$

where n_i represents the components of the outward-pointing unit normal to the domain boundary $\partial\Omega$. The domain Ω is then partitioned into a computational mesh $\mathcal{K}_h$ defined as a set of non-overlapping, arbitrarily shaped elements K , such that

$$\overline{\Omega}_h = \bigcup_{K \in \mathcal{K}_h} \overline{K}. \quad (10)$$

The numerical solution $\mathbf{q}_h$ and the test functions Φ_h are approximated with a finite-dimensional polynomial space, defined as

$$\mathbb{P}^{\kappa} = \{v_h \in L_2(\Omega) \mid v_h|_K \in \mathbb{P}^{\kappa}(K), \forall K \in \mathcal{K}_h\}, \quad (11)$$

where $\mathbb{P}^{\kappa}$ denotes the restriction to K of the polynomial functions of total degree at most $\kappa \in \mathbb{Z}_{\geq 0}$. For the basis of this space, we employ the set of orthonormal and hierarchical modal functions defined in the physical space $\{\phi_j\}_{j=1}^{N_{dof}} \forall K \in \mathcal{K}_h$, as proposed by Bassi et al. [3]. Here, $N_{dof} = \prod_{i=1}^D (\kappa + i) / i$ is the number of degrees of freedom per element and equation. Replacing the continuous variables with their discrete approximations within the vector space $\mathcal{V}_h = [\mathbb{P}^{\kappa}(\mathcal{K}_h)]^M$, each component $\{q_{h,s}\}_{s=1}^M$ of the solution $\mathbf{q}_h$ is expressed as a linear combination of these basis functions

$$q_{h,s} = \phi_j Q_{s,j} \quad \text{with} \quad j = 1, \dots, N_{dof}, \quad s = 1, \dots, M. \quad (12)$$

where $Q_{s,j}$ are the unknown degrees of freedom (DOFs). To complete the dG discretization, we define the set of all mesh faces $\mathcal{F}_h = \mathcal{F}_h^0 \cup \mathcal{F}_h^b$, which is partitioned into internal faces $\mathcal{F}_h^0$ and boundary faces $\mathcal{F}_h^b$. We also denote $\mathcal{F}_K = \{\mathcal{F}\}$ as the set of faces belonging to a single element $K \in \mathcal{K}_h$. By exploiting the orthonormality of the test and basis functions in physical space, the elemental mass matrices reduce to the identity. As a result, the semi-discrete system can be written directly in terms of the degrees of freedom. The semi-discrete dG formulation of the system (1) then consists of finding the solution $\mathbf{Q}$ such that

$$\frac{d\mathbf{Q}}{dt} - \sum_{K \in \mathcal{K}_h} \int_K \left(\frac{\partial \Phi_h}{\partial x_i} \right)^{\top} \mathbf{F}_i(\mathbf{q}_h) d\Omega + \sum_{F \in \mathcal{F}_h} \int_F \llbracket \Phi_h \rrbracket^{\top} \hat{\mathbf{F}}(\mathbf{q}_h^{\pm}, \mathbf{n}_F) d\sigma = 0, \quad (13)$$

where $\mathbf{n}_F = \{n_{F,i}\}_{i=1}^D$ is the uniquely defined unit vector normal at any point of a possibly curved face F . The notations $\circ^-$ and $\circ^+$ are the interior and exterior traces of $\circ$, where the normal $\mathbf{n}_F$ is defined to point from the interior (−) to the exterior (+) side of F . The jump operator $\llbracket \circ \rrbracket = \circ^+ - \circ^-$ accounts for the discontinuous nature of the polynomial space at each mesh face $F \in \partial K^+ \cup \partial K^-$; and $\hat{\mathbf{F}}$ is a suitable normal numerical flux function, chosen to be the HLLC flux in the remainder of the paper [33].

3.2. Pressure Projection

As in [19], to mitigate spurious oscillations, we “align” the pressure with the same functional space as the conservative variables. To do so, we compute it from the L_2 projection of the equation of state constitutive law

$$\int_K \Phi_h p_h d\Omega = \int_K \Phi_h p(\mathbf{q}_h) d\Omega. \quad (14)$$

This task is accomplished efficiently thanks to the orthogonality of the basis, so the DOFs related to the pressure P_j can be directly computed as

$$P_j = \int_K \phi_j p(\mathbf{q}_h) d\Omega. \quad (15)$$

3.3. Shock-Capturing Approach

To further control the spurious oscillations that may occur at shocks due to the high-order approximation, we employ the strategy proposed by Bassi et al. [18,40] and adapted to stabilize material interfaces between ideal gases by Franchina et al. [19]. The approach consists of adding an artificial diffusion term to the semi-discrete form of the model

$$\sum_{K \in \mathcal{K}_h} \int_K \varepsilon_S \left(\frac{\partial \Phi_h}{\partial x_i} \right)^\top \left(\frac{\partial \mathbf{q}_h}{\partial x_i} \right) d\Omega, \quad (16)$$

where the shock-capturing sensor ε_S is calculated using

$$\varepsilon_S = C_{SC} \left(h_K^{SC} \right)^2 \sum_{n=1}^{M-D} \frac{|c_{S,n}| + |d_{S,n}|}{\tilde{q}_n^*} f_S, \quad (17)$$

being C_{SC} a user-defined constant. When there are discontinuities in the flow problem, this constant is set to 1.0 in the present work. To tune the viscosity, which is sensitive to $\tilde{\mathbf{q}} \in \mathbb{R}^{M-D}$, we choose $\tilde{\mathbf{q}} = [\rho, T, Y_1, \dots, Y_{N-1}]^\top$. This choice is motivated by the fact that the spatial gradient of Y_k is highly sensitive to material interfaces, while shock waves will be detected mainly by gradients of density and temperature. As in ref. [19], we set $\tilde{\mathbf{q}}^* = [\rho, T, 1, \dots, 1]^\top$ in the denominator to avoid interrupting the computations for very small values of Y_k . The term c_S accounts for the difference between the numerical and the inner convective flux

$$c_{S,n} = \sum_{k=1}^M \frac{\partial \tilde{q}_n(\mathbf{q}_h)}{\partial q_{h,k}} \mathbf{c}, \quad (18)$$

where the term $\mathbf{c} \in [\mathcal{V}_h]^M$ comes from the solution of the auxiliary problem

$$\int_K \Phi_h^\top \mathbf{c} d\Omega = \sum_{F \in \mathcal{F}_h} \int_F \Phi_h^\top [\hat{\mathbf{F}}_i(\mathbf{q}_h^\pm, \mathbf{n}_F) - \mathbf{F}_i(\mathbf{q}_h) n_{F,i}] d\sigma. \quad (19)$$

Please note that thanks to the use of an orthonormal basis, we avoid the need to solve a linear system to obtain $\mathbf{c}$ [3]. The term $d_{S,n}$ accounts for the divergence of the inner physical flux

$$d_{S,n} = \frac{\partial \tilde{q}_n(\mathbf{q}_h)}{\partial \mathbf{q}_h} \frac{\partial \mathbf{F}_i(\mathbf{q}_h)}{\partial x_i}, \quad (20)$$

and in the implementation the divergence of $\mathbf{F}_i$ is computed using the chain rule

$$\frac{\partial \mathbf{F}_i(\mathbf{q}_h)}{\partial x_i} = \frac{\partial \mathbf{F}_i(\mathbf{q}_h)}{\partial \mathbf{q}_h} \frac{\partial \mathbf{q}_h}{\partial x_i}. \quad (21)$$

The f_S term in the RHS of Equation (17) rescales the rest of the shock-capturing sensor with the maximum normalized gradient of $\tilde{\mathbf{q}}_h$, reading

$$f_S = \max_k \left(\frac{1}{\tilde{q}_k(\mathbf{q}_h)} \sqrt{\frac{\partial \tilde{q}_k(\mathbf{q}_h)}{\partial x_i} \frac{\partial \tilde{q}_k(\mathbf{q}_h)}{\partial x_i}} \right) \left(\frac{h_K^{SC}}{\max(1, \kappa)} \right)^\kappa \quad \text{for } k = 1, \dots, M-D, \quad (22)$$

where the right-most term rescales the maximum normalized gradient with the polynomial degree of the approximation and the characteristic element size h_K^{SC} , which is also used in Equation (17). This is defined by

$$h_K^{SC} = \frac{1}{\sqrt{\left(\frac{1}{\Delta x_i}\right)\left(\frac{1}{\Delta x_i}\right)}}, \quad (23)$$

where for $D = 2$, $\{\Delta x_i\}_{i=1}^D$ represent the dimensions of the quadrilateral enclosing K , scaled such that their product equals the surface area of the cell. For $D = 1$, h_K^{SC} is set to the length of the segment. We refer the reader to [4] for the closed form for the divergence of $\mathbf{F}_i$, while the gradient of $\tilde{\mathbf{q}}$ with respect to the conservative variables is presented in Appendix A.

3.4. Time Integration

After numerically evaluating the integrals in Equation (13) and assembling the elemental contributions, the formulation becomes a system of Ordinary Differential Equations (ODEs) governing the evolution in time of the discrete solution

$$\frac{d\mathbf{Q}}{dt} = \mathbf{R}(\mathbf{Q}), \quad (24)$$

We integrate Equation (24) using the third-order, five-stage Strong-Stability-Preserving Runge–Kutta (SSPRK) scheme proposed by Spiteri and Ruuth [39]. Its formulation can be written in compact form as

$$\left. \begin{aligned} \mathbf{Y}_0 &= \mathbf{Q}^n, \\ \mathbf{Y}_i &= \mathbf{Y}_0 + \Delta t \sum_{j=1}^{i-1} a_{ij} \mathbf{R}(\tilde{\mathbf{Y}}_j), \quad i = 1, \dots, s, \\ \mathbf{Q}^{n+1} &= \mathbf{Q}^n + \Delta t \sum_{j=1}^s b_j \mathbf{R}(\tilde{\mathbf{Y}}_j). \end{aligned} \right\} \quad (25)$$

Here, s is the number of stages, and a_{ij} and b_j are the scheme's coefficients from the Butcher tableau. We remark that at the end of each stage we may apply one of the limiting strategies explained in Section 4, yielding the limited state $\tilde{\mathbf{Y}}_i$. This strategy will also be applied to $\mathbf{Q}^{n+1}$. The time step size is determined in two different ways depending on the case. It can be either imposed as a fixed value, based on physical arguments and the stability constraints, or can be computed adaptively based on the Courant–Friedrich–Lewy (CFL) condition $\Delta t = \min_{K \in \mathcal{K}_h} \{\text{CFL } h_K / (\bar{c}_K + \|\bar{\mathbf{u}}_K\|)\}$ for a fixed CFL value. Here, the elemental convective eigenvalue in the denominator is computed from the cell K mean values $\bar{c}_K$ and $\bar{\mathbf{u}}_K$, and h_K corresponds to the cell size.

4. Positivity-Preserving Strategies

At each Runge–Kutta stage, the dG solution can be passed element-wise through a limiting step to ensure physical admissibility. For a dG method, it is very challenging to ensure this condition at the whole element. Therefore, we enforce it at the volume quadrature points (used for volume integrals) and the surface quadrature points used for integrals at the faces. We denote the solution at a generic quadrature point in this set as $\mathbf{q}_h^g$ where $g = 1, \dots, N_G$ and N_G is the total number of quadrature points. The limiters ensure that a new, limited solution at these points $\tilde{\mathbf{q}}_h^g$ remains within the set of admissible states $\mathcal{G}$. We investigated several strategies to enforce this constraint: a simple clipping of variables; a variation of the linear scaling limiter presented in [9]; and an order-reducing strategy.

4.1. Clipping the Values to Admissible Quantities (CL)

A commonly adopted pragmatic strategy is to enforce physical admissibility by directly clipping selected variables to prescribed bounds. While straightforward to implement, this approach does not preserve conservation nor the high-order consistency of the dG approximation. Furthermore, it is restricted to the quadrature points used for volume integration. To limit a scalar quantity ϕ within the range $(\epsilon_{min}, \epsilon_{max})$, we compute its modified value $\tilde{\phi}$ as

$$\tilde{\phi}(\phi, \epsilon_s, \epsilon_{min}, \epsilon_{max}) = \begin{cases} \epsilon_{min} + C & \text{if } \phi < \epsilon_{min} \\ \phi + C & \text{if } \epsilon_{min} \leq \phi \leq \epsilon_{max} \\ \epsilon_{max} + C & \text{if } \epsilon_{max} < \phi \end{cases} \quad (26)$$

where the smooth correction term is defined as

$$C = \epsilon_s \exp\left(-\frac{|\phi - \epsilon_{min}|}{2\epsilon_s}\right) - \epsilon_s \exp\left(-\frac{|\phi - \epsilon_{max}|}{2\epsilon_s}\right). \quad (27)$$

This function is conceptually described in Figure 2. The parameters $\epsilon_{min} = \phi_{min} + 10^{-10}$ and $\epsilon_{max} = \phi_{max} - 10^{-10}$ are chosen slightly within the admissible bounds (ϕ_{min}, ϕ_{max}) . The parameter ϵ_s controls the smoothness of the transition and is set to 10^{-9} in the remainder of this paper. For quantities that are only lower-bounded, this function is used in the limit of arbitrarily large ϵ_{max} .

The clipping is applied pointwise at each volume quadrature point to the following thermodynamic quantities ρ , ρe , and $\{\rho_k\}_{k=1}^N$ in this order. The first two are constrained to be greater than ϵ_{min} , while the partial densities satisfy $\tilde{\rho}_k \in (\epsilon_{min}, \tilde{\rho}(\mathbf{q}_h^s))$. This procedure yields an admissible state at each volume quadrature point. The modified values are then projected back onto the polynomial basis to obtain the updated degrees of freedom $\tilde{\mathbf{Q}}$.

In the present framework, this procedure does not ensure positivity at the face quadrature points. We discuss how to avoid this failure mode in Section 5.3, after it is shown for the two-dimensional results.

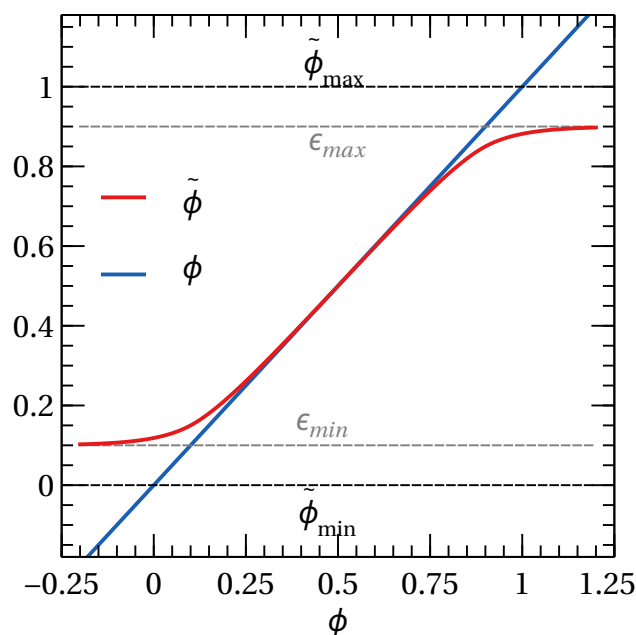


Figure 2. Sketch of $\tilde{\phi}$ as a function of ϕ using the smoothed ramp. We used very large parameters for illustrative purposes.

4.2. Moment Limiter Based on Internal Energy (ML)

Based on the work of Zhang and Shu [9], we apply a positivity-preserving strategy based on rescaling the high-order solution towards the cell-average solution $\bar{\mathbf{q}}_h$ in each element, which, following the proof in [9] and given that the SSPRK advancement is computed as a convex combination of the stages, is physically admissible, i.e., $\bar{\mathbf{q}}_h \in \mathcal{G}$ under the CFL condition in Section 3.4. From this point on, we drop the subscript K , with the understanding that the limiters presented are strictly element-wise. The procedure is applied sequentially to the mixture density, the partial densities, and the internal energy to ensure that they are larger than a small value ϵ , which may be problem-dependent and is taken as $\epsilon = 10^{-12}$ for the remainder of this work. We consider the version of this limiter based on the internal energy since it is more efficient than the original [9] and is almost equivalent when considering the ideal-gas equation of state. The algorithm consists of

1. finding the minimal mixture density among the quadrature points

$$\rho|_{\min} = \min_{g=1,\dots,N_G} \left(\rho(\mathbf{q}_h^g) \right), \quad (28)$$

and if $\rho|_{\min} > \epsilon$, set $\theta^{(1)} = 1$. Otherwise, the correction factor is computed as

$$\theta^{(1)} = \frac{\rho(\bar{\mathbf{q}}_h) - \epsilon}{\rho(\bar{\mathbf{q}}_h) - \rho|_{\min}}. \quad (29)$$

Thanks to the orthonormality of the basis, the classical formulation of this moment limiter

$$\tilde{q}_{h,k} = \theta^{(1)} q_{h,k} + (1 - \theta^{(1)}) \bar{q}_{h,k} \quad \text{for } k = 1, 2 \quad (30)$$

is exactly equivalent to rescaling the modal coefficients $Q_{k,j}$ to obtain corrected coefficients $\tilde{Q}_{k,j}$:

$$\tilde{Q}_{k,j}^{(1)} = \theta^{(1)} Q_{k,j} \quad \text{for } k = 1, 2 \quad \text{and } j = 2, \dots, N_{\text{dof}}, \quad (31)$$

so that the average of the cell is preserved $\tilde{Q}_{k,1} = Q_{k,1}$, thus guaranteeing conservation. After applying Equation (31), the modal coefficients $\mathbf{Q} = \tilde{\mathbf{Q}}$ are updated, and the solution is re-evaluated at the quadrature points.

2. For each component k , finding the minimum partial density

$$q_{h,k}|_{\min} = \min_{g=1,\dots,N_G} \left(q_{h,k}^g \right). \quad (32)$$

If $q_{h,k}|_{\min} > \epsilon$, set $\theta^{(2,k)} = 1$. Otherwise, the correction for this component is computed as

$$\theta^{(2,k)} = \frac{\bar{q}_{h,k} - \epsilon}{\bar{q}_{h,k} - q_{h,k}|_{\min}}, \quad (33)$$

and applied directly to the modal coefficients

$$\tilde{Q}_{k,j}^{(2,k)} = \theta^{(2,k)} Q_{k,j} \quad \text{for } j = 2, \dots, N_{\text{dof}}. \quad (34)$$

With the corrected solution obtained, the modal coefficients $\mathbf{Q} = \tilde{\mathbf{Q}}$ are updated, and the solution at the points is re-evaluated. The procedure is repeated until all partial densities have been checked. We remark that this stage of the procedure further modifies the value of the mixture density in a pointwise sense, but conservation is ensured thanks to the preservation of the average within the cell.

3. The last correction is applied based on the positivity of the internal energy. Let $\rho e|_{min} = \min_{g=1, \dots, N_G} \rho e(\mathbf{q}_h^g)$, and if $(\rho e)|_{min} > \epsilon$ set $\theta^{(3)} = 1$. Otherwise, the correction factor of this stage is computed as

$$\theta^{(3)} = \frac{\rho e(\bar{\mathbf{q}}_h) - \epsilon}{\rho e(\bar{\mathbf{q}}_h) - \rho e|_{min}}, \quad (35)$$

and is applied to the modal coefficients as

$$\tilde{Q}_{k,j}^{(3)} = \theta^{(3)} Q_{k,j} \quad \text{for } k = 1, \dots, M \quad \text{and } j = 2, \dots, N_{dof}. \quad (36)$$

With these new coefficients, update the solution one last time $\mathbf{Q} = \tilde{\mathbf{Q}}$ and terminate the limiting procedure.

We remark that the pressure DOFs do not need to be corrected by the limiter, since they will be recomputed in the pressure projection of the following stage of the RK scheme.

4.3. Polynomial Order Limiter (LO)

A common and simple limiter for high-order methods is to perform an element-wise reduction of the polynomial degree of the solution to ensure positivity when needed. In this work, the following physical admissibility criteria

$$\rho > \epsilon; \quad \rho_k > \epsilon \quad \forall k = 1, \dots, N; \quad p > \epsilon \quad (37)$$

are checked at all quadrature points $g = 1, \dots, N_G$, and ϵ is taken as 10^{-12} for the remainder of this paper. If any of the admissibility criteria is violated within an element, the local polynomial degree is reduced by one, and the solution is re-evaluated at the quadrature points. This procedure is applied recursively until an admissible solution is obtained, i.e., $\tilde{\mathbf{q}}_h^g \in \mathcal{G}$ for all $g = 1, \dots, N_G$. This process is performed efficiently thanks to the hierarchical structure of the basis, which allows obtaining the degrees of freedom of a lower-order solution by simply zeroing the higher ones. Thanks to the preservation of the first-order mode, the strategy does not compromise conservation, but the local order of accuracy is reduced in troubled regions.

5. Numerical Results

In this section, we assess the accuracy, robustness, and computational efficiency of the different positivity-preserving strategies presented. Unless otherwise stated, all simulations employ the pressure projection and the artificial-viscosity shock-capturing approach described in Section 3. The time step is computed from a convective CFL condition using different values for the restriction, which are specified for each case. The numerical assessment is organized as follows. First, an advection problem with a smooth exact solution is considered to verify the accuracy of the discretization and evaluate the effect of the pressure projection. Next, a set of one-dimensional multicomponent Riemann problems is employed to investigate the behavior of the limiters in the presence of shocks, rarefactions, and material interfaces. Finally, a two-dimensional multicomponent implosion problem is used to assess the robustness of the stabilization framework and its capability to resolve complex flow structures on unstructured meshes.

5.1. Advection of Density Waves

We assess the spatial accuracy of the solver by simulating the advection of density waves in uniform velocity and pressure fields (ADW) [41]. We consider a unit-length 1D domain with periodic boundary conditions. The initial condition is

$$\rho_1(x) = \left[\frac{\sin(2\pi x)}{2} + 1 \right] \left[\frac{\sin(4\pi x)}{4} + \frac{1}{2} \right], \quad (38)$$

$$\rho_2(x) = - \left[\frac{\sin(2\pi x)}{2} + 1 \right] \left[\frac{\sin(4\pi x)}{4} - \frac{1}{2} \right], \quad (39)$$

$$u(x) = 1, \quad (40)$$

$$p(x) = 1, \quad (41)$$

so the pressure and velocity of the exact solution remain constant in time. The thermodynamic coefficients of the EOS are set to $\gamma_1 = 1.6$, $\gamma_2 = 1.4$, $c_{v,1} = 5/3$ and $c_{v,2} = 5/6$. The simulation is advanced in time using a fixed time step size of $\Delta t = 5 \times 10^{-5}$ up to the end time $t = 1$, where the time step is small enough to ensure that the time-integration error remains below the spatial discretization error. To study the latter, we consider the following metric

$$\eta(\circ, \bullet_{ref}) = |\Omega_h|^{1/2} \|\circ - \bullet_{ref}\|_{L_2}, \quad (42)$$

which compares the discrete solution to the exact solution projected to the dG space, in order to remove the contribution of the projection error. In Figure 3a, the distribution of the error in non-dimensional pressure at the end time is shown, in order to assess the violation of the pressure equilibrium preservation. The corresponding errors associated with the experiments in this figure are $\eta(p, p_{ref}) = 1.035 \times 10^{-5}$ when using pressure projection, and $\eta(p, p_{ref}) = 1.591 \times 10^{-5}$ when not using it. This pressure equilibrium property has not been considered in our numerical analysis, but it becomes relevant for the stability of long-time simulations and is an active research topic [1,22,42–46]. The results suggest that the framework using the L_2 projection of the EOS mitigates the pressure equilibrium preservation errors with respect to using the exact EOS, as also observed in [24] in the context of FV schemes. We regard this as encouraging evidence of another advantage of using this framework, which features a very computationally efficient pressure projection, and we aim to study it in a future publication. The results of the spatial refinement test are shown in Figure 3b. For the coarser meshes, some of the observed rates remain below the design order because the solutions have not yet reached the asymptotic convergence regime. Upon further mesh refinement, the expected order $\kappa + 1$ is recovered for $\kappa = 0, \dots, 4$. The error reaches a plateau around 10^{-9} , as also remarked by [4], and we conjecture that it is due to the pressure equilibrium violations [47].

5.2. Riemann Problems

In this section, we are interested in the performance of the limiters in the presence of flow discontinuities, particularly material interfaces between gases. We study the Riemann problems (RPs) with the initial condition

$$\mathbf{w}(x) = \begin{cases} \mathbf{w}_L & \text{if } x < x_0 \\ \mathbf{w}_R & \text{if } x \geq x_0 \end{cases}, \quad (43)$$

in the domain $\Omega \in [0, 1]$, where x_0 is the initial position of the discontinuity. We consider separated gases for each problem, so we impose the left state to be filled with the first component ($Y_{L,1} = 1 - \delta$), and the right state to be filled with the second component ($Y_{R,1} = \delta$)

in all the RPs. As commonly done in diffuse interface models, we add a contamination factor to each pure phase $\delta = 1 \times 10^{-5}$ at each state. Table 1 indicates the initial conditions of the problems, the coordinate x_0 , the end time t_{end} , and the thermodynamic coefficients of the fluids. We use 200 elements and a fourth-order approximation for all the numerical solutions presented in this section, and provide the results as non-dimensional quantities. Problem RP0, adapted from [41], results in a subsonic shock-contact-rarefaction pattern and is computed using $CFL = 5 \times 10^{-2}$. The RP1 develops a similar structure, but the flow is supersonic; it is adapted from [48,49] and is simulated using $CFL = 1 \times 10^{-2}$. We also investigate a variant of the receding flow problem in RP2 ($CFL = 2 \times 10^{-2}$), and we add the solution obtained using 2000 elements and a fourth-order approximation to assess the convergence of the solver to the exact solution. Finally, we present a shock-contact-shock case featuring hypersonic flow in RP3, adapted from [50] and simulated here using $CFL = 5 \times 10^{-2}$. All the solutions presented in Figures 4b, 5 and 6 use the pressure projection and the viscous shock-capturing strategy with $C_{SC} = 1$ for *a priori* stabilization. This combination has been shown to be a successful approach by Franchina et al. [19]. We remark that the *a priori* stabilization is fundamental to reduce the oscillations at discontinuities and obtain numerically admissible solutions. The studied *a posteriori* limiters are not primarily designed to control spurious oscillations of the solution, since they are based exclusively on physical admissibility criteria. The results shown in Figure 4a for RP1, where we test the solver with and without pressure projection (PP) and shock capturing (SC), support these considerations. In this figure, it can be observed that when only *a posteriori* stabilization is used, the solution becomes more oscillatory. In particular, stronger oscillations are observed in the vicinity of the contact discontinuity.

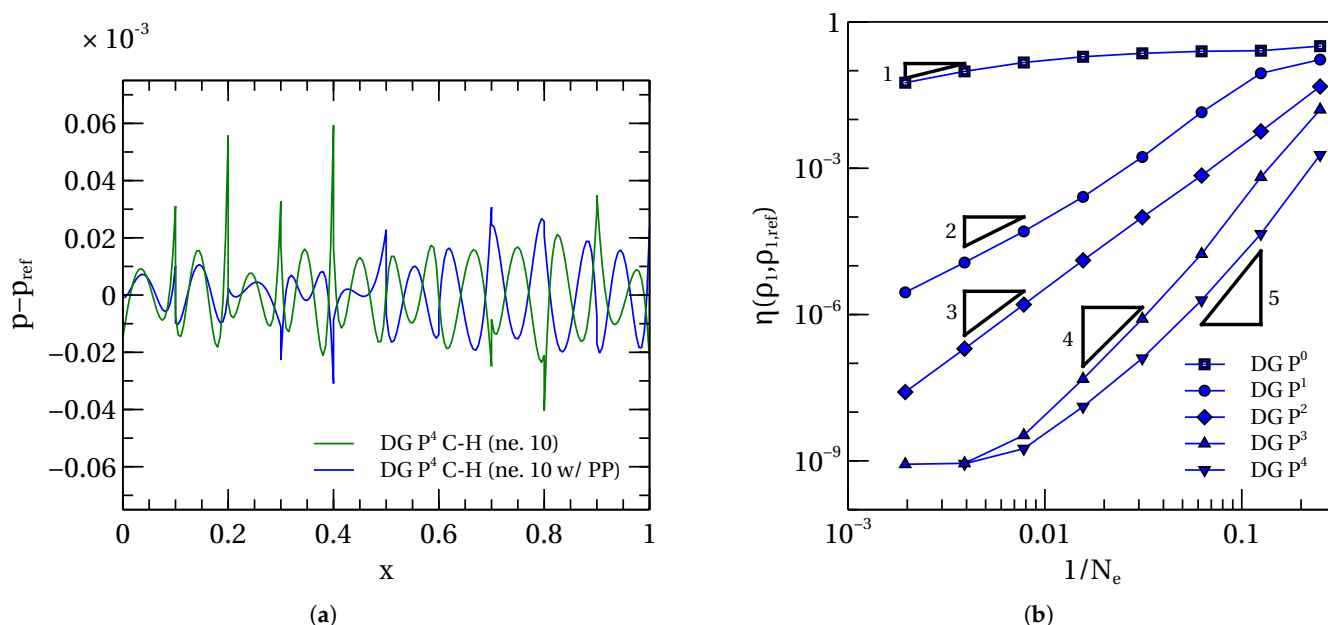


Figure 3. ADW—(a) Error in pressure at $t = 1$, where “w/o PP” indicates that pressure has been computed directly from the EOS. (b) Results of the spatial refinement test using pressure projection.

In all the studied problems, the non-limited solution persistently exhibits non-physical values of Y_1 near the contact discontinuity. These values do not cause solver failure in the present setup, since they do not produce a negative pressure. Nevertheless, they correspond to erroneous solutions and may lead to failures when additional physics is considered, such as chemical reactions. The under-resolution of the weak contact wave in Figure 6 (left) is particularly remarkable. When the resolution is increased, the discon-

tinuity is better resolved, but the large undershoot persists. These results are consistent with those reported in [51], where this behavior is attributed to a spurious generation of thermodynamic entropy.

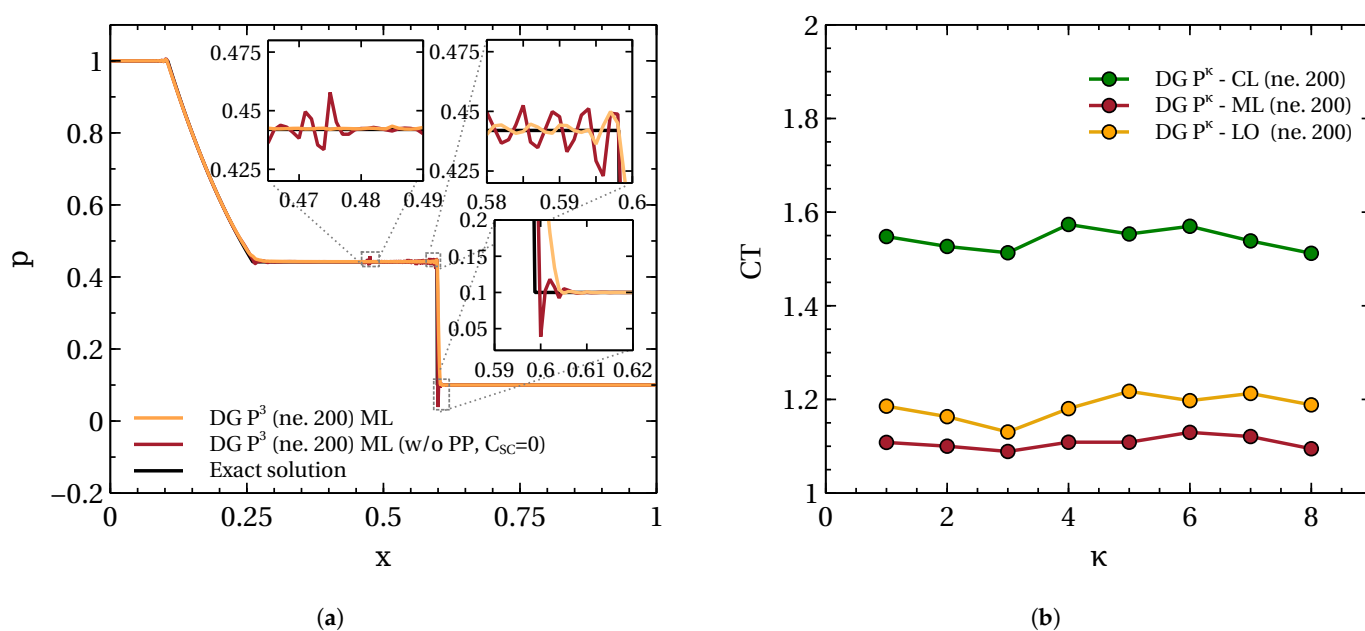


Figure 4. (a) RP1—Comparison of the solution with and without pressure projection “PP” and shock capturing “SC” for *a priori* stabilization. (b) RP0—Computational time made dimensionless with the equivalent non-limited simulation “CT” for different polynomial approximations. These results were obtained with the in-house parallel Fortran CPU code MIGALE on an Intel® Core™ i7-12700H processor.

The ML and CL limiters generally produce similar solutions, which in some cases coincide with the non-limited solution (*cf.* velocity in Figure 5 left and Figure 6 left). A noticeable difference appears in the mixture densities in Figure 5, where the CL slightly reduces the accuracy; we conjecture that this is caused by the loss of conservation. By contrast, the LO limiter reduces systematically the resolution in all the test cases and for all types of waves, particularly in the region upstream of the contact discontinuity in Figure 5 (right). It is also noticeable that the limited solutions in Figure 5 are slightly more oscillatory at the shock fronts. In RP1 and RP2, the tails of the rarefaction waves are much more affected by the limited resolution employed than the heads. In RP3, shown in Figure 6 (right), all solutions are dominated by oscillations in the region between the shock waves. Nevertheless, the capability of solving a hypersonic problem without failure at the contact discontinuity highlights the robustness of the stabilization employed.

To monitor the error in conservation of a quantity, we define the following scalar metric

$$\varepsilon_k(\mathbf{q}_h, t) = |\Omega_h|^{-1} \left(\int_{\Omega_h} q_{h,k}(t) d\Omega - \int_{\Omega_h} q_{h,k}^0 d\Omega + \int_0^t \int_{\partial\Omega_h} \hat{F}_k(\mathbf{q}_h^\pm(\tau), \mathbf{n}_F) d\sigma d\tau \right), \quad (44)$$

where $q_{h,k}^0$ is the projection of the initial condition of the k -th conservative variable onto the discrete space, and the spatial integrals are evaluated using the high-order quadrature rules employed by the dG scheme. To eliminate the temporal dependence, we define the following scalar quantity for each component k of the system of conservation laws

$$\mu_k = \sqrt{\frac{1}{t_{end}} \int_0^{t_{end}} \varepsilon_k^2(\mathbf{q}_h, \tau) d\tau}, \quad (45)$$

where the temporal integrals are evaluated using the same SSP Runge–Kutta method presented in Section 3.4. Table 2 shows, using μ_1 , that the error in the conservation of ρ_1 is at least eight orders of magnitude larger for the CL limiter, while the other limiters are shown to be conservative with an error on the order of machine precision. In Figure 4 (right), we present the results of a study of the computational-time scaling with respect to the polynomial degree of the approximation. The CL is about 50% more expensive than the non-limited solution, and our results suggest that approximately 60% of the CL cost is attributable to the evaluation of Equation (26). The ML is about 10% more efficient than the LO limiter, because it does not require iterative reductions of the polynomial degree. Instead, its main cost lies in finding the minimum over the quadrature points at each stage of the limiter. Therefore, we observe that both the ML and the LO limiters greatly benefit from the properties of the basis, which allow them to operate directly on the modal coefficients and thus avoid expensive polynomial projections. These trends are consistent across different RPs, as shown in Table 3, which also indicates that the less intrusive limiters are substantially more active than the more intrusive ones. It should nevertheless be emphasized that the relatively satisfactory behavior of the CL limiter in the present one-dimensional RPs does not extend to all test cases. In particular, the CL strategy fails in the more demanding two-dimensional implosion problem discussed in Section 5.3.

Table 1. Initial conditions of the Riemann problems, interface position x_0 , end time t_{end} and thermodynamic coefficients.

Case	ρ_L	u_L	p_L	ρ_R	u_R	p_R	x_0	t_{end}	γ_1	$c_{v,1}$	γ_2	$c_{v,2}$
RP0	1.0	0	1.0	0.125	0	0.1	0.5	0.2	1.5	2.0	1.3	2.0
RP1	2.0	0.75	1.0	1.5	0	2.0	0.5	0.2	1.5	2.0	1.3	2.0
RP2	0.45	0	1.0	1.15	0	1.0	0.5	1.0	1.4	2.5	1.6	2.0
RP3	5.9992	19.580	460.89	5.9924	−6.1963	46.095	0.4	0.035	1.4	2.5	1.6	2.0

Table 2. Time-integrated conservation metric for the first component μ_1 for the different strategies adopted to simulate the studied RPs.

Case	Limiter	Baseline	CL	ML	LO
	RP0	2.06×10^{-14}	7.00×10^{-5}	2.06×10^{-14}	2.02×10^{-14}
	RP1	1.07×10^{-14}	5.15×10^{-5}	1.10×10^{-14}	1.20×10^{-14}
	RP2	1.61×10^{-14}	1.36×10^{-6}	2.16×10^{-14}	1.84×10^{-14}
	RP3	8.94×10^{-12}	8.49×10^{-3}	1.08×10^{-11}	9.22×10^{-12}

Table 3. Number of activations of the limiter “NT” and computational time normalized with the equivalent non-limited solution “CT” for the studied RPs.

Case	CL		ML		LO	
	NT	CT	NT	CT	NT	CT
RP0	4679	1.51	1640	1.09	229	1.13
RP1	25,848	1.50	10,284	1.11	370	1.22
RP2	8781	1.48	2393	1.08	90	1.20
RP3	27×10^4	1.52	11×10^4	1.19	359	1.21

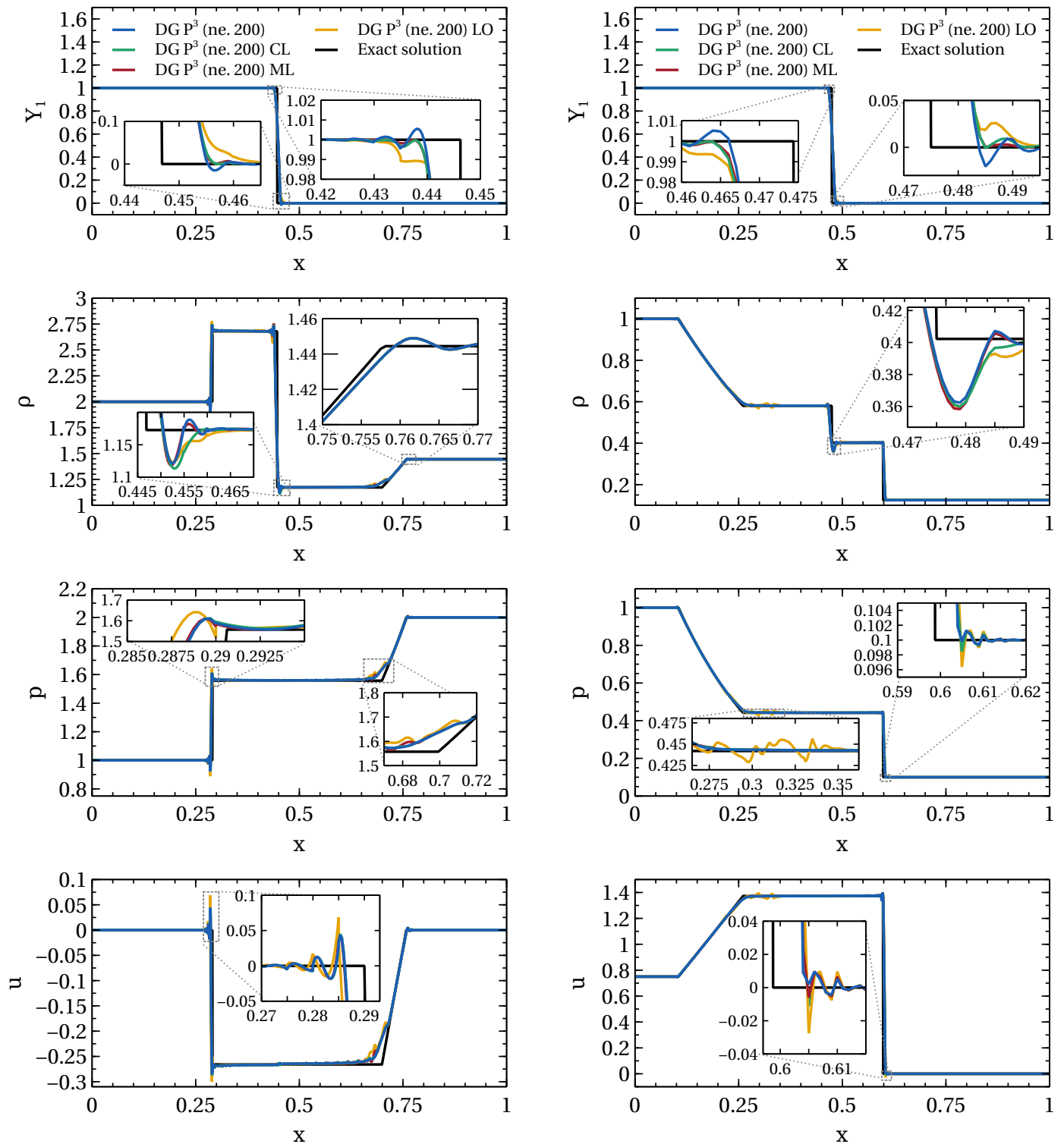


Figure 5. RP0 (left), RP1 (right)—DG P^3 solution on 200 cells.

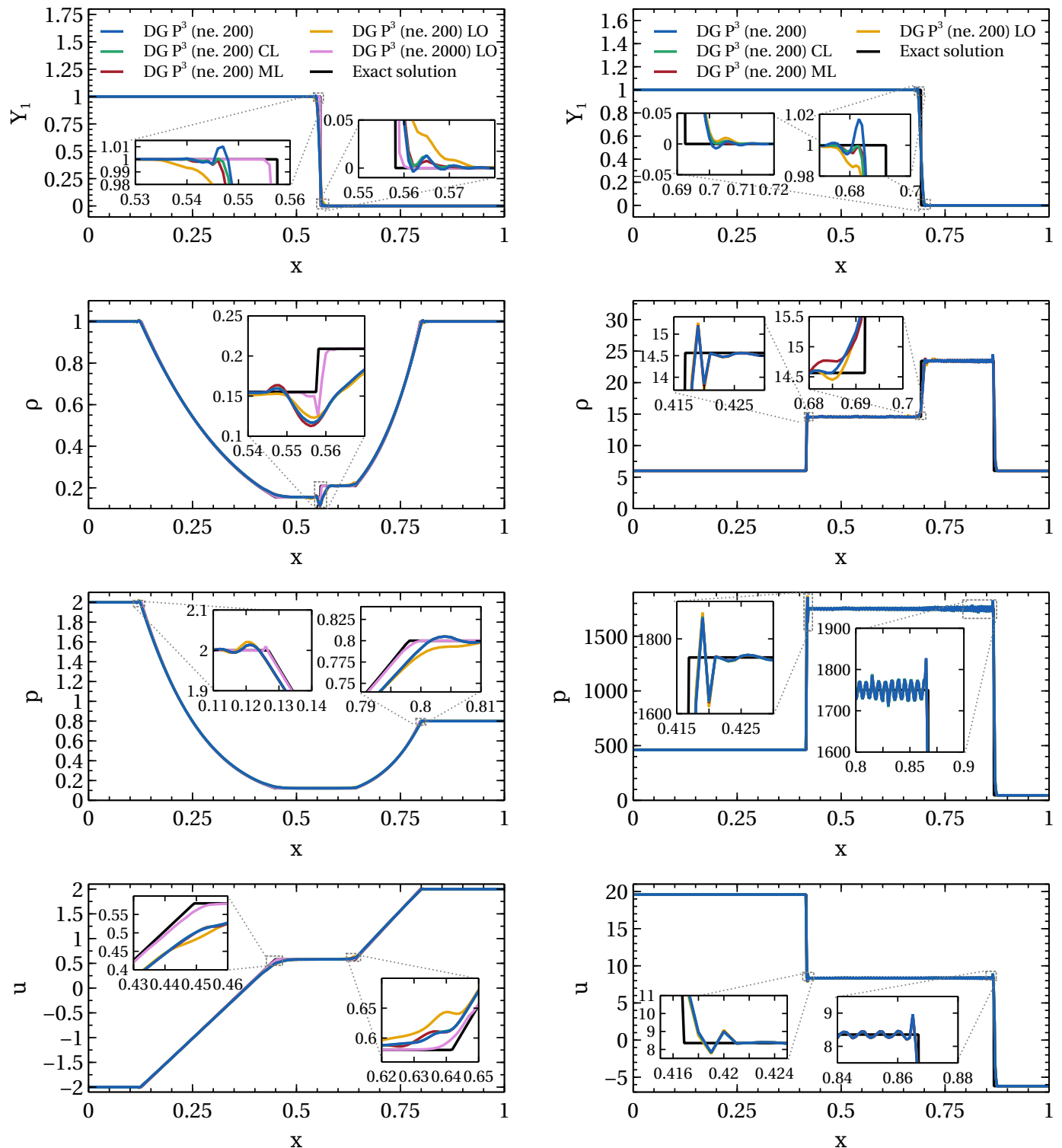


Figure 6. RP2 (left), RP3 (right)—DG P^3 solution on 200 cells. Additionally, we show a solution using P^3 and 2000 elements to assess the convergence to the exact solution of RP2.

5.3. Multicomponent Implosion

To assess the capability of the solver to capture discontinuities and small-scale flow structures as the polynomial degree is increased, we consider a multicomponent extension of the two-dimensional implosion problem proposed in [50,52] (MIM). The original test

case consists of a square domain containing a centrally located low-pressure and low-density region, rotated by an angle of $\pi/4$ with respect to the outer boundaries. The interaction between the resulting converging shock waves produces a complex flow field with strong discontinuities and fine-scale features, making the problem particularly suitable for evaluating high-order numerical methods. Exploiting the symmetry of the configuration, the computation is performed only on the upper-right quadrant of the domain, namely $\Omega \in [0, 0.3] \times [0, 0.3]$. To introduce multicomponent effects, we define an additional region in the upper-right part of the sub-domain to be occupied by a different gas, allowing shock-interface interactions to be investigated. Symmetry boundary conditions are imposed on all boundaries, and the initial condition is defined as

$$[Y_1, \rho, u_1, u_2, p] = \begin{cases} [\delta, 0.125, 0, 0, 0.1] & \text{if } 0 \leq x_1 + x_2 \leq 0.15 \\ [1 - \delta, 1, 0, 0, 1] & \text{if } 0.15 < x_1 \text{ and } 0.15 < x_2, \\ [\delta, 1, 0, 0, 1] & \text{otherwise} \end{cases} \quad (46)$$

with the components having the thermodynamic coefficients $\gamma_1 = 1.5$, $\gamma_2 = 1.3$, and $c_{v,1} = c_{v,2} = 2$. A contamination factor is set to $\delta = 1 \times 10^{-5}$. A sketch of the configuration is shown in Figure 7. The mesh employed for the κ -refinement study consists of 915 elements arranged as shown in Figure 8a, where triangular elements are introduced only in the vicinity of the diagonal initial discontinuity in order to represent it sharply. We also present the results obtained using an unstructured mesh composed of 506 elements and featuring randomly curved parabolic faces, shown in Figure 8b, to highlight the flexibility of the numerical method. We perform all the simulations in this section with a constant $CFL = 5 \times 10^{-2}$ in order to ease the comparison between methods, meshes and polynomial orders. We present the non-dimensional solution at different time levels obtained with the ML in Figure 9. At early stages, it forms a shock-contact-rarefaction pattern, and it can be observed how the rarefaction advances until impinging on the material interface at $t = 0.1$, at which point the limiter becomes active to prevent solver failure. At later times, the material interface becomes unstable due to the repeated impingement of waves on it. A similar evolution of the flow can be observed in Figure 10 on mesh (b), which is significantly coarser and therefore yields a more dissipative solution. This case validates the applicability of the solver on unstructured and curved meshes. Figure 11 shows the non-dimensional solution at $t = 2.5$ obtained with the ML for three increasing polynomial degrees on mesh (a). Since no surface tension or physical viscosity is present to stabilize the material interface, its geometry becomes increasingly complex as the polynomial degree κ is increased, due to the reduced numerical dissipation of the discretization. It is also remarkable that the instabilities developing in the low-density region (in the bottom-left of the domain) become progressively finer and more complex as the discretization is refined. The ejected vortical structures are deflected by the presence of the interface upstream, as also noticed by [4]. Confirming the evidence collected in the previous section, we observe in Figure 12 that the solutions obtained with the LO limiter are systematically more dissipative, with the discrepancy becoming more pronounced as the polynomial degree κ increases, particularly at the material interface, where the limiter is active. Finally, all simulations employing the CL limiter eventually failed. We attribute this behavior to the fact that the limited solution is reconstructed through a volumetric projection, while admissibility needs to be enforced at both volume and face quadrature points. As mentioned in Section 4.1, positivity violations occurring exclusively at face quadrature points may not be removed by the projection procedure, eventually leading to solver failure. We propose that, in the context of dG methods, this problem can be circumvented in two different ways: (i) employing a volume quadrature rule whose node set includes the face quadrature points, which in

general will require more quadrature points; (ii) over-constraining the projection to include the face quadrature points and solving it under some criterion (e.g., minimization of the least-squares error with non-linear constraints), which in general is expensive to solve.

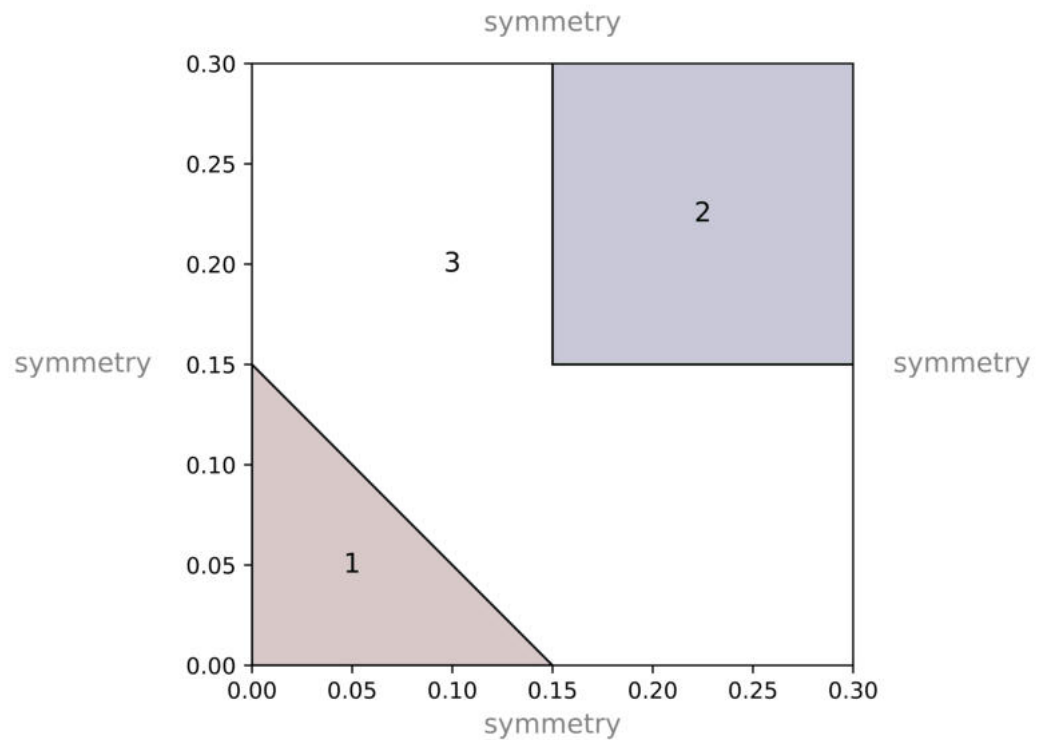


Figure 7. MIM—Initial condition of the multicomponent implosion problem. Region 1 corresponds to the low-density and low-pressure region, while a material interface separates regions 2 and 3.

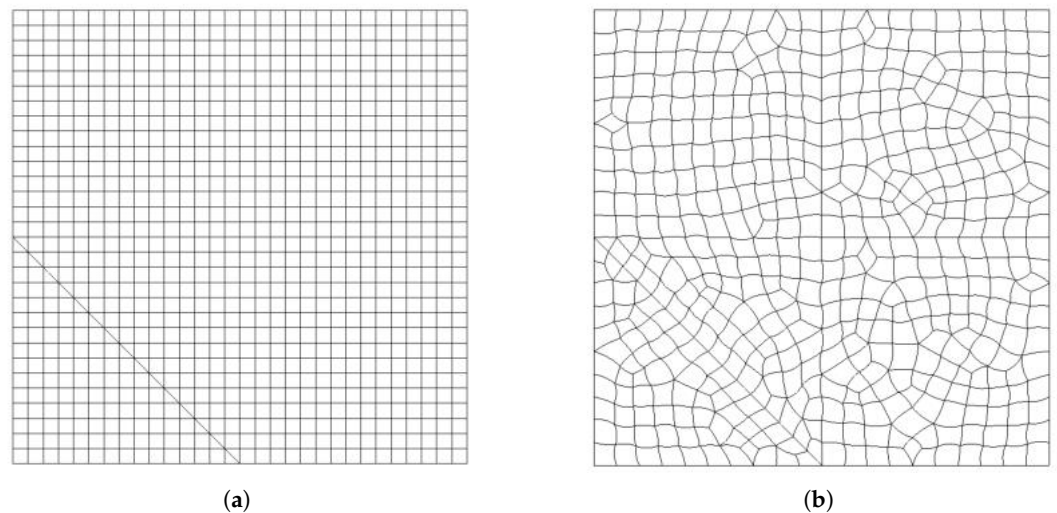


Figure 8. MIM—Computational mesh (a), employed for the κ -refinement study, composed of 915 elements. Triangular elements are used only in the vicinity of the diagonal initial discontinuity to represent it sharply. Computational mesh (b), with curved edges, is used to demonstrate the flexibility of the solver, and consists of 506 elements.

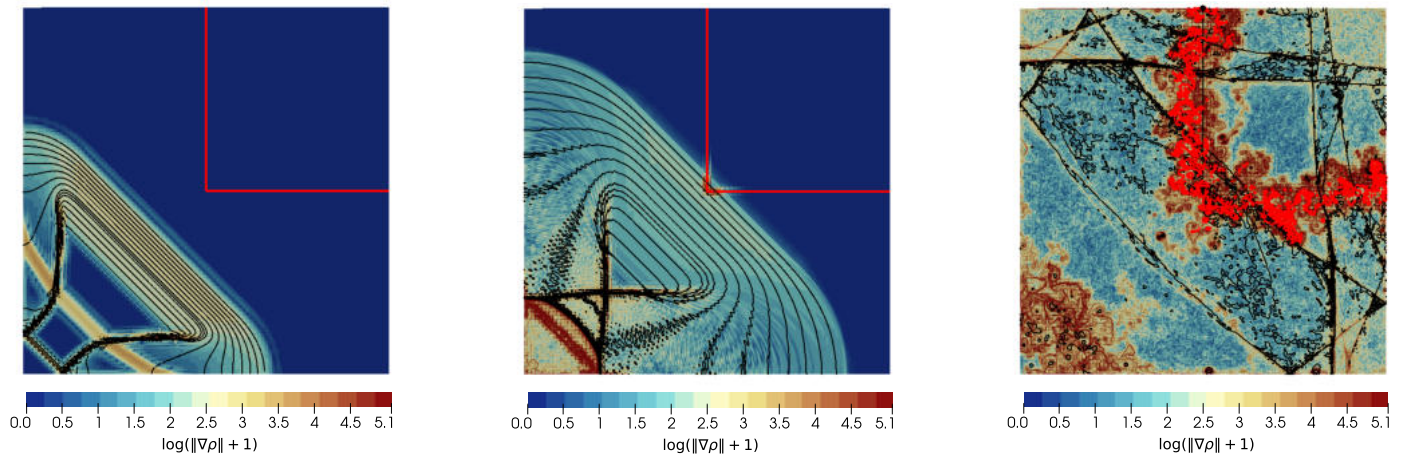


Figure 9. MIM—From left to right, solutions obtained at $t = 0.045, 0.1, 2.5$ with the ML limiter using a $\mathbb{P}^6$ approximation over mesh (a), which is composed of 915 elements. We display the pseudo-Schlieren field and superimpose 20 pressure iso-contours (in black), and the line $Y_1 = 0.5$ that marks the interface (in red).

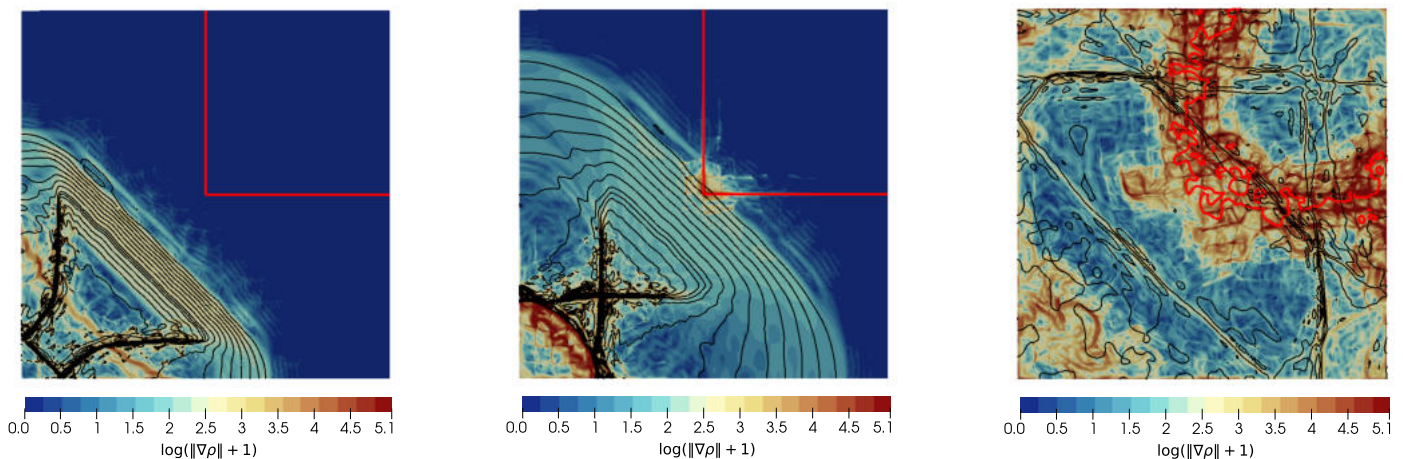


Figure 10. MIM—From left to right, solutions obtained at $t = 0.045, 0.1, 2.5$ with the ML limiter using a $\mathbb{P}^6$ approximation over mesh (b), which is composed of 506 elements. We display the pseudo-Schlieren field and superimpose 20 pressure iso-contours (in black), and the line $Y_1 = 0.5$ that marks the interface (in red).

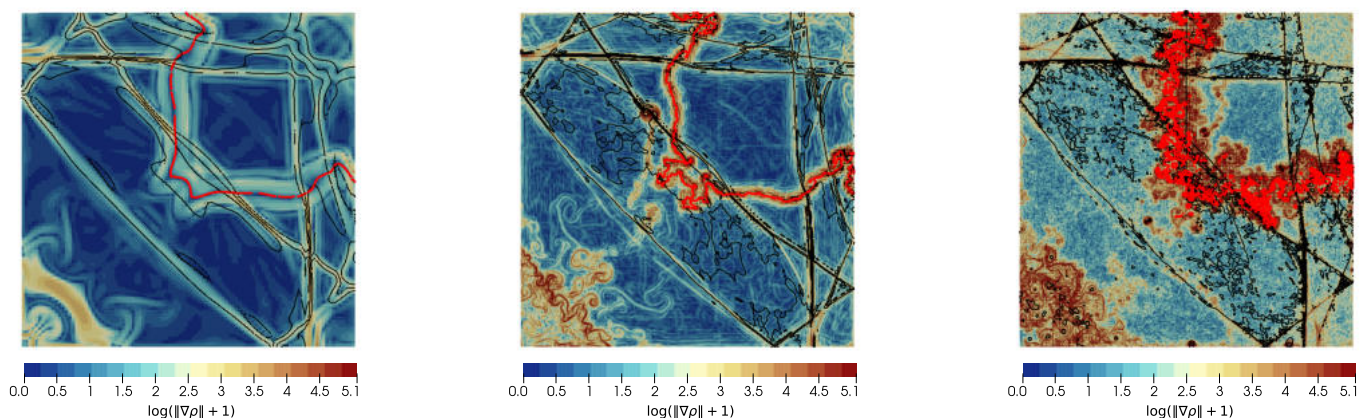


Figure 11. MIM—Results with ML using three different polynomial degrees $\mathbb{P}^k$ for $k = 1, 3, 6$ (from left to right) using mesh (a), which is composed of 915 elements. We display the pseudo-Schlieren field and superimpose 20 pressure iso-contours (in black), and the line $Y_1 = 0.5$ that marks the interface (in red).

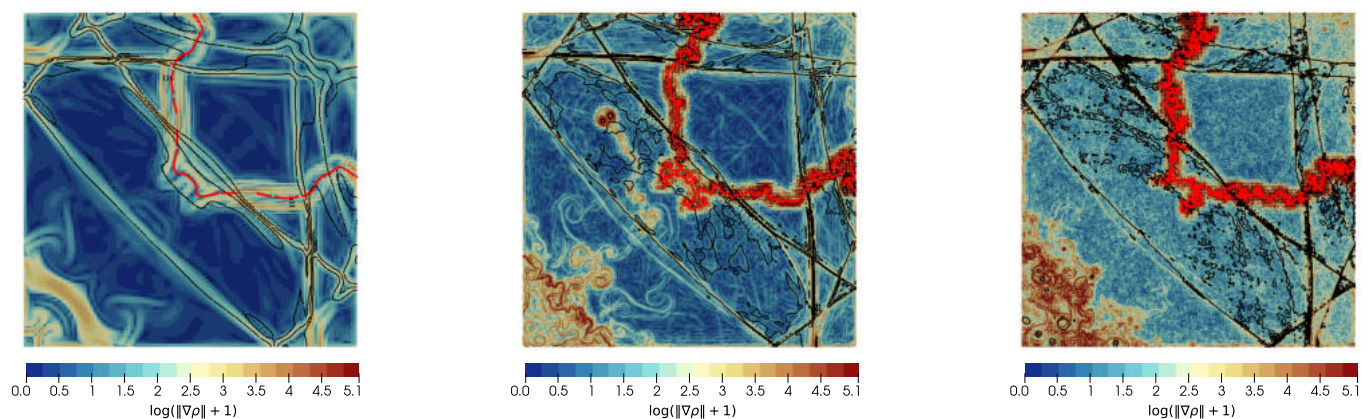


Figure 12. MIM—Results with LO using three different polynomial degrees $\mathbb{P}^k$ for $k = 1, 3, 6$ (from left to right). We display the pseudo-Schlieren field and superimpose 20 pressure iso-contours (in black) and the line $Y_1 = 0.5$ that marks the interface (in red).

6. Conclusions

The comparison between the *a posteriori* limiting techniques for ensuring physical admissibility in the solution shows that the moment limiting technique by Zhang and Shu [9] is the appropriate choice for the framework studied, applied to inviscid compressible flows featuring separated ideal gases. The qualitative conclusions of the present study are summarized in Table 4. The clipping technique has been shown to be ineffective for a multi-dimensional case in the present framework due to the projection of the limited solution being unconstrained at the face quadrature points. We suggested two alternatives to circumvent this problem. Moreover, the smoothing function applied is a remarkably expensive kernel that introduces around a 50% computational overhead for the one-dimensional problems. Iteratively removing higher-order modes until positivity is achieved has been proved to be an effective strategy, but has consistently been shown to be more dissipative and to require approximately 10% more computational time for one-dimensional problems. Therefore, the ML technique has been shown to be the most efficient and most accurate technique for the collection of benchmark problems studied. Although the simulation was completed, the non-limited solution systematically produced negative mass fractions around the material interfaces in the one-dimensional problems. In the present numerical method, the three studied limiters can be formulated for arbitrarily shaped elements. This capability is demonstrated on an unstructured mesh composed of triangular and quadrilateral elements with curved parabolic edges. The combination of viscous shock capturing and pressure projection for the *a priori* stabilization has been confirmed to be an effective strategy for mitigating unphysical oscillations in the high-order solution, arising from both the Gibbs phenomenon and equilibrium thermodynamics. Furthermore, it has not been shown to introduce excessive dissipation at discontinuities. The choice of a high-order numerical method has been shown to successfully resolve small-scale features of the flow, particularly instabilities appearing at material interfaces. In a possible three-dimensional extension, we would expect similar results for the presented stabilization mechanisms, except for a decrease in efficiency of the LO limiter due to the iterative procedure it features when more quadrature points are needed. Future work will focus on extending the present stabilization framework to multiphase flows governed by stiffened-gas equations of state. Particular attention will be devoted to investigating pressure equilibrium preservation and whether the separation between *a priori* and *a posteriori* stabilization mechanisms remains effective in the presence of stronger thermodynamic non-linearities and larger density ratios. In this context, we expect the conclusions of this manuscript to not be directly extensible to

more complex systems, particularly since the ML can induce oscillations when activated due to the non-linearity of the EOS. We will also consider the viscous extension of the model following [53,54], with emphasis on assessing if the presented stabilization mechanisms are too dissipative in the presence of the natural stabilization from the viscous terms.

Table 4. Comparison of the investigated *a posteriori* limiting strategies in the present framework based on the results of the present test suite. Comparative ratings are marked as favorable (+), neutral (○) and unfavorable (−), while ✓ and × indicate whether the method preserves conservation.

Criterion \ Limiter	CL	ML	LO
Ease of implementation	+	+	○
Robustness	−	○	+
Computational efficiency	−	+	○
Conservation	×	✓	✓
Accuracy	○	+	−
Main limitation	Failures at faces	Minimum search	Numerical dissipation
Recommended role	Not recommended	Preferred	Robust fallback

Author Contributions: Conceptualization, D.R.R.; methodology, A.C. (Alessandro Colombo), A.C. (Andrea Crivellini), F.M. and D.R.R.; software, A.C. (Alessandro Colombo), A.C. (Andrea Crivellini), F.M. and D.R.R.; validation, D.R.R. and F.M.; formal analysis, A.C. (Alessandro Colombo) and D.R.R.; investigation, D.R.R.; resources, A.C. (Alessandro Colombo) and A.C. (Andrea Crivellini); data curation, D.R.R. and F.M.; writing—original draft preparation, D.R.R.; writing—review and editing, D.R.R., A.C. (Andrea Crivellini) and A.C. (Alessandro Colombo); visualization, D.R.R., F.M. and A.C. (Alessandro Colombo); supervision, A.C. (Alessandro Colombo); project administration, A.C. (Alessandro Colombo). All authors have read and agreed to the published version of the manuscript.

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Abbreviations

The following abbreviations are used in this manuscript:

ADW	Advection of Density Waves
CFL	Courant–Friedrich–Lewy
CL	Clipping Limiter
CT	Normalized Computational Time
DOF	Degree of Freedom
EOS	Equation of State
FV	Finite Volume
RK	Runge–Kutta
RP	Riemann Problem
LO	Lower-Order Limiter
NAD	Numerical Admissibility Detector
NT	Number of Activations of the Limiter During the Simulation
MIM	Multicomponent Implosion
ML	Moment Limiter
MOOD	Multi-dimensional Optimal Order Detection

ODE	Ordinary Differential Equation
PAD	Physical Admissibility Detector
PP	Pressure Projection
SC	Shock Capturing
SSP	Strong Stability Preserving
TVB	Total Variation Bounded
WENO	Weighted Essentially Non-Oscillatory
dG	Discontinuous Galerkin

Appendix A. Jacobian of $\tilde{\mathbf{q}}$ with Respect to $\mathbf{q}$

In this appendix, we present the Jacobian of the change in variables required to compute the terms $c_{S,n}$ and $d_{S,n}$ in Section 3.3. We are interested in obtaining closed-form expressions for $\partial\tilde{\mathbf{q}}/\partial\mathbf{q}$, where

$$\tilde{\mathbf{q}} = [\rho, T, Y_1, \dots, Y_{N-1}]^\top \quad (\text{A1})$$

and $\mathbf{q}$ denotes the conservative set of variables. The derivative of the mass fraction of the component k with respect to its corresponding partial density ρ_k reads

$$\frac{\partial Y_k}{\partial \rho_k} = \frac{1}{\rho}(1 - Y_k). \quad (\text{A2})$$

When the derivative is with respect to the partial density of another component, the result is

$$\frac{\partial Y_k}{\partial \rho_l} = -\frac{Y_k}{\rho} \quad \text{for } k \neq l \text{ and } l = 1, \dots, N. \quad (\text{A3})$$

For the remaining conservative variables, the derivatives vanish:

$$\frac{\partial Y_k}{\partial u_i} = \frac{\partial Y_k}{\partial \rho e_t} = 0. \quad (\text{A4})$$

Regarding the gradient of the mixture density ρ , its derivative with respect to the partial densities is unity

$$\frac{\partial \rho}{\partial q_k} = 1, \quad (\text{A5})$$

while it is zero for the remaining conservative variables:

$$\frac{\partial \rho}{\partial u_i} = \frac{\partial \rho}{\partial \rho e_t} = 0. \quad (\text{A6})$$

The gradient of the temperature with respect to the partial densities reads

$$\frac{\partial T}{\partial \rho_k} = \frac{1}{\rho c_v} \left(\frac{1}{2} \|u\|^2 - c_{v,k} T \right), \quad (\text{A7})$$

while the derivatives with respect to the momentum components are

$$\frac{\partial T}{\partial \rho u_i} = -\frac{u_i}{\rho c_v}. \quad (\text{A8})$$

Finally, the derivative of the temperature with respect to the total energy is

$$\frac{\partial T}{\partial \rho e_t} = \frac{1}{\rho c_v}. \quad (\text{A9})$$

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