



UNIVERSITÀ POLITECNICA DELLE MARCHE
Repository ISTITUZIONALE

A structurally damped σ -evolution equation with nonlinear memory

This is a pre print version of the following article:

Original

A structurally damped σ -evolution equation with nonlinear memory / D'Abbicco, Marcello; Girardi, Giovanni. - In: MATHEMATICAL METHODS IN THE APPLIED SCIENCES. - ISSN 0170-4214. - 47:13(2024), pp. 10872-10890. [10.1002/mma.6633]

Availability:

This version is available at: 11566/314981 since: 2024-09-28T16:33:13Z

Publisher:

Published

DOI:10.1002/mma.6633

Terms of use:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. The use of copyrighted works requires the consent of the rights' holder (author or publisher). Works made available under a Creative Commons license or a Publisher's custom-made license can be used according to the terms and conditions contained therein. See editor's website for further information and terms and conditions.

This item was downloaded from IRIS Università Politecnica delle Marche (<https://iris.univpm.it>). When citing, please refer to the published version.

(Article begins on next page)

A STRUCTURALLY DAMPED σ -EVOLUTION EQUATION WITH NONLINEAR MEMORY

MARCELLO D'ABBICCO AND GIOVANNI GIRARDI

ABSTRACT. In this paper we investigate the global (in time) existence of small data solutions to the Cauchy problem for the following structurally damped σ -evolution model with nonlinear memory term

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = \int_0^t (t-\tau)^{-\gamma} |u_t(\tau, \cdot)|^p d\tau,$$

with $\sigma > 0$. In particular, for $\gamma \in ((n-\sigma)/n, 1)$ we find the sharp critical exponent, under the assumption of small data in L^1 . Dropping the L^1 smallness assumption of initial data, we show how the critical exponent is consequently modified for the problem. In particular, we obtain a new interplay between the fractional order of integration $1-\gamma$ in the nonlinear memory term, and the assumption that initial data are small in L^m , for some $m > 1$.

1. INTRODUCTION

We consider the nonlinear Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = F(t, u_t), & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = 0, \quad u_t(0, x) = u_1(x), \end{cases} \quad (1)$$

where $\sigma > 0$, μ is a positive constant and the right-hand side is a memory term defined as

$$F(t, u_t) = \int_0^t (t-\tau)^{-\gamma} |u_t(\tau, \cdot)|^p d\tau, \quad (2)$$

for some $\gamma \in (0, 1)$ and $p > 1$. More in general, we may consider a memory term defined as

$$F(t, u_t) = \int_0^t (t-\tau)^{-\gamma} g(u_t(\tau, \cdot)) d\tau,$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function satisfying

$$g(0) = 0, \quad |g(u) - g(v)| \lesssim |u - v|(|u|^{p-1} + |v|^{p-1}), \quad \text{for some } p > 1. \quad (3)$$

The linear part of the equation in (1), i.e.

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = 0 \quad (4)$$

is a special case of a more general class of σ -evolution equations with structural damping

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad \sigma, \delta > 0. \quad (5)$$

For any $\delta > 0$ the equation in (5) is a dissipative σ -evolution equation; in particular, its energy

$$E(t) = \frac{1}{2} \|u_t(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|(-\Delta)^{\frac{\sigma}{2}} u(t, \cdot)\|_{L^2}^2,$$

is non-increasing, due to

$$E'(t) = -\mu \|(-\Delta)^{\frac{\delta}{2}} u_t(t, \cdot)\|_{L^2}^2.$$

Date: 31 January, 2020.

1991 Mathematics Subject Classification. Primary 35L15, 35L71; Secondary 35B33, 35A01.

Key words and phrases. semilinear evolution equations, structural damping, critical exponent, global in time existence, test function method.

In the case $2\delta \in (0, \sigma)$ (effective damping, according to the definition in [6]) the solution to a Cauchy problem associated to (5) can be written as the sum of two terms, each one asymptotically behaving as the solution to a different diffusion problem (see [7] for more details). Instead, in the case $2\delta > \sigma$ in the asymptotic profile of the solution the wave structure appears and oscillations come into play (the case $\delta = \sigma = 1$ has been studied in details by R. Ikehata [15]). In [7] the authors study equation (5) in the case $0 < 2\delta < \sigma$; at first they obtain estimates for the solution to the linear Cauchy problem associated to (5) and then they apply those to study the critical exponent for the two corresponding nonlinear Cauchy problems

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = 0, & u_t(0, x) = u_1(x), \end{cases} \quad (6)$$

and

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u_t|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = 0, & u_t(0, x) = u_1(x). \end{cases} \quad (7)$$

The critical exponents are

$$p_0(n, \sigma, \delta) := 1 + \frac{2\sigma}{(n - 2\delta)_+}, \quad (8)$$

for Cauchy problem (6) and, respectively,

$$p_1(n, \sigma, \delta) := 1 + \frac{2\delta}{n}, \quad (9)$$

for Cauchy problem (7) (the critical exponents for (6) in the case $\sigma/2 < \delta \leq \sigma$ have been recently found in [8]). In both cases some restriction on the dimension appears: in fact, it is required $n < n_0 = n_0(\sigma, \delta)$ for (6), and, respectively, $n < n_1 = n_1(\sigma, \delta)$ for (7). Here, for $j = 0, 1$ the integer n_j is proportional to $1/(\sigma - 2\delta)$. In particular, in the limit case $2\delta = \sigma$ it holds $n_0 = n_1 = \infty$, i.e. (8) and (9) are critical exponents for any dimension $n \geq 1$ (see also [19]).

If the L^1 smallness assumption on the initial datum is replaced by L^m smallness assumption, for some $m \in (1, 2]$, then the two critical exponents in (8) and (9) become $p_0(n/m, \sigma, \delta)$ and $p_1(n/m, \sigma, \delta)$ (see [7, Section 2.5]).

In general, by critical exponent \bar{p} , in this paper we mean that

- if $p > \bar{p}$, then there exist global (in time) small data Sobolev or energy solutions;
- if $1 < p < \bar{p}$, there exist arbitrarily small initial data, such that there exists no global (in time) weak solution.

The difficulty in treating the higher space dimensions $n > n_j$ is related to the loss of regularity which appears when one deals with $L^q - L^q$ estimates, with $q \in (1, 2)$. Indeed, these estimates come into play when one considers power nonlinearities $|u|^p$ or $|u_t|^p$, when $p \in (1, 2)$, and the critical exponent eventually becomes smaller than 2 in high space dimension n (for instance, Fujita exponent $1 + 2/n$ is smaller than 2 in space dimension $n \geq 3$).

The loss of regularity for $L^q - L^q$ estimates, with $q \in (1, 2)$, is related to the wave structure of the equation at high frequencies (this is studied in detail in [17] where the model (5) is studied in the case $\sigma = 1$ and $\delta = 0$). However, the presence of the structural damping in (5), when $\delta > 0$, generates a smoothing effect on the solution, which does not appear for the classical damping u_t (see later Proposition 2.1). This smoothing effect allows us to recover the additional regularity by using estimates which are singular at $t = 0$. The singularity order is proportional to $n(\sigma - 2\delta)/2\delta$, and it vanishes at $\sigma = 2\delta$. This effect explains, roughly speaking, the possibility to employ these estimates in higher space dimensions when $\sigma/(2\delta)$ tends to 1. This motivates our choice to fix $\delta = \sigma/2$ to investigate in a better way the influence of the nonlinear memory term on the equation.

Recently, many authors investigated fractional PDEs from different points of view, since they are particularly interesting for the real world applications and the description of memory and hereditary process. In particular, it is of interest to understand how to treat nonlinear evolution problems in which the nonlinearity is represented by some memory term; for instance, one could consider a nonlinearity like $F(t, u_t)$

defined in (2), or even

$$G(t, u) = \int_0^t (t - \tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau.$$

We remark that

$$F(t, u_t) = \Gamma(1 - \gamma) J_{0|t}^{1-\gamma} |u_t(t, \cdot)|^p, \quad G(t, u) = \Gamma(1 - \gamma) J_{0|t}^{1-\gamma} |u(t, \cdot)|^p,$$

where Γ is the Euler function and $J_{0|t}^{1-\gamma}(v)$ denotes the fractional Riemann-Liouville integral of a function v . Hence, we have

$$\lim_{\gamma \rightarrow 1} \Gamma(1 - \gamma) F(t, u_t) = |u_t(t, \cdot)|^p$$

and, respectively

$$\lim_{\gamma \rightarrow 1} \Gamma(1 - \gamma) G(t, u) = |u(t, \cdot)|^p;$$

thus, one can expect some relations with the case of a power nonlinearity $|u_t|^p$ and, respectively $|u|^p$, as $\gamma \rightarrow 1$.

In [1] the authors consider the Cauchy problem for the heat equation

$$\begin{cases} u_t - \Delta u = G(t, u), & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = u_0(x), \end{cases} \quad (10)$$

and they prove that the critical exponent for (10) is given by

$$\max \left\{ \gamma^{-1}, 1 + \frac{2(2 - \gamma)}{n - 2(1 - \gamma)} \right\}. \quad (11)$$

Other diffusive models with nonlinear memory are treated in [13] and [20]; in particular, in this latter paper also fractional derivatives in time are considered in the linear part of the equation.

In [4] the author studies the nonlinear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \mu u_t = G(t, u), & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = 0, \quad u_t(0, x) = u_1(x), \end{cases} \quad (12)$$

and he proves that the critical exponent is again (11) (the nonexistence result was proved in [12], and in [11] for a damping with variable coefficients), as for the Cauchy problem (10), for any $n \leq 5$; this is reasonable since the solution to the linear Cauchy problem associated to (12), behaves asymptotically like the solution the linear Cauchy problem associated to (10) with a suitable initial datum v_0 (see, for instance [16, 17, 18]).

Furthermore, in [3] the first author considers the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}} u_t = G(t, u), & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = 0, \quad u_t(0, x) = u_1(x), \end{cases} \quad (13)$$

and he proves that global (in time) small data energy solutions exist for

$$p > \max \left\{ \gamma^{-1}, 1 + \frac{3 - \gamma}{n + \gamma - 2} \right\}, \quad (14)$$

for any $n \geq 2$; moreover, this exponent is optimal.

We remark that the threshold $(n - 2)/n$ which denotes the transition to the critical exponent γ^{-1} is the same for both the Cauchy problems (12) and (13). This suggests that if the nonlinear memory term is sufficiently strong in space dimension $n \geq 3$, then the influence from classical and structural damping is the same. Otherwise, the critical exponent is larger in the structural damping case, as it happens for the power nonlinearity $|u|^p$.

Finally, we recall that the beam equation has been investigated in the case of a nonlinear memory term in [9].

In this paper we want to investigate the critical exponent for the Cauchy problem (1). As discussed before, the special structure of equation (4) gives some benefit in the estimates for the solution to the

linear problem; in particular, it is possible to get non-singular estimates also for u_t which can be easily applied to study the nonlinear Cauchy problem. We will prove that, under L^1 smallness assumption for the initial datum u_1 , global (in time) energy solutions to (1) exist for any $p > \bar{p}_{\gamma,1}(\sigma, n)$, where

$$p > \bar{p}_{\gamma,1}(\sigma, n) := \begin{cases} p_\gamma(\sigma, n) & \text{if } \gamma \in [(n - \sigma)/n, 1) \\ \gamma^{-1} & \text{if } \gamma \in (0, (n - \sigma)/n) \end{cases}, \quad p_\gamma(\sigma, n) := 1 + \frac{(2 - \gamma)\sigma}{n - \sigma(1 - \gamma)}. \quad (15)$$

We are ready to state our first result.

Theorem 1.1. *Let $\sigma > 0$ and $\gamma \in (0, 1)$; let us assume $p > \bar{p}_{\gamma,1}(\sigma, n)$.*

Then, there exists $\epsilon > 0$, sufficiently small, such that for any

$$u_1 \in L^1 \cap L^\infty \quad \text{with} \quad \|u_1\|_{L^1 \cap L^\infty} \leq \epsilon, \quad (16)$$

there is a uniquely determined energy solution

$$u \in C^1([0, \infty), L^1 \cap L^\infty) \cap C([0, \infty), H^\sigma) \quad (17)$$

to (1). Furthermore, for $j = 0, 1$, the solution satisfies the following estimates:

$$\forall q \in [1, \infty] : \quad \|\partial_t^j u(t, \cdot)\|_{L^q} \lesssim \begin{cases} (1+t)^{2-j-\gamma-\frac{n}{\sigma}(1-\frac{1}{q})} \|u_1\|_{L^1 \cap L^\infty} & \text{if } \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) < 2 - j, \\ (1+t)^{-\gamma} \log(2+t) \|u_1\|_{L^1 \cap L^\infty} & \text{if } \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) = 2 - j, \\ (1+t)^{-\gamma} \|u_1\|_{L^1 \cap L^\infty} & \text{if } \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) > 2 - j. \end{cases} \quad (18)$$

Moreover, $\|(-\Delta)^{\frac{\sigma}{2}} u(t, \cdot)\|_{L^2}$ verifies the same estimates verified by $\|u_t(t, \cdot)\|_{L^2}$ in (18). Consequently,

$$E(t) \lesssim \begin{cases} (1+t)^{2(1-\gamma)-\frac{n}{\sigma}} \|u_1\|_{L^1 \cap L^\infty}^2 & \text{if } n < 2\sigma, \\ (1+t)^{-2\gamma} (\log(2+t))^2 \|u_1\|_{L^1 \cap L^\infty}^2 & \text{if } n = 2\sigma, \\ (1+t)^{-2\gamma} \|u_1\|_{L^1 \cap L^\infty}^2 & \text{if } n > 2\sigma. \end{cases}$$

Remark 1.1. In estimates (18) it does not appear a decay rate better than $(1+t)^{-\gamma}$. This restriction comes from the influence of the nonlinear memory term (see Lemma 4.2). In the best case scenario, a loss of decay $(1+t)^{1-\gamma}$ appears with respect to the estimates for the Sobolev solutions to the corresponding linear Cauchy problem with $F \equiv 0$ (see Proposition 2.1). This phenomenon has been described in [4].

If we drop the L^1 smallness assumption for the initial datum u_1 , replacing it by L^m smallness, for some $m > 1$, we expect that this loss of information on the datum does not influence the critical exponent for (1), for ‘‘sufficiently small’’ γ , with respect to m . This expectation is motivated by the fact that the presence of the nonlinear memory term induces a loss of decay rate which amounts to $(1+t)^{1-\gamma}$, or even larger, with respect to the estimates for the Sobolev solutions to the corresponding linear Cauchy problem with $F \equiv 0$, with L^1 data. On the other hand, assuming initial data in L^m , with $m > 1$, leads to a loss of decay rate which amounts to $(1+t)^{\frac{n}{\sigma}(1-\frac{1}{m})}$, with respect to assume initial data in L^1 . Consequently, the critical exponent is modified only when this latter loss is greater than the loss related to the nonlinear memory term. This phenomenon has been recently investigated in [5] for models (10) and (12).

In particular, if initial data are assumed to be small in L^m , for some

$$1 < m < \frac{n^2}{(n^2 - \sigma^2)_+},$$

then we show that global (in time) energy or Sobolev solutions exist for any $p > \bar{p}_{\gamma,m}(\sigma, n)$, where

$$\bar{p}_{\gamma,m}(\sigma, n) := \max\{\gamma^{-1}, p_\gamma(\sigma, n), \tilde{p}_m(\sigma, n)\}, \quad \tilde{p}_m(\sigma, n) := 1 + \frac{(2 - \gamma)\sigma m}{n}.$$

Explicitly, we may compute

$$\bar{p}_{\gamma,m}(\sigma, n) = \begin{cases} \tilde{p}_m(\sigma, n) & \text{if } \gamma \in [\bar{\gamma}, 1), \\ p_\gamma(\sigma, n) & \text{if } \gamma \in [(n - \sigma)/n, \bar{\gamma}), \\ \gamma^{-1} & \text{if } \gamma \in (0, (n - \sigma)/n), \end{cases}$$

where

$$\bar{\gamma} = \bar{\gamma}(m) = 1 - \frac{n}{\sigma} \left(1 - \frac{1}{m}\right).$$

We notice that $\bar{\gamma} > (n - \sigma)/n$ due to the assumption $m < n^2/(n^2 - \sigma^2)$.

Indeed, it is easy to show that for a fixed $m \in (1, \frac{n^2}{(n^2 - \sigma^2)_+})$, the statement of Theorem 1.1 remains valid if $\gamma \in (0, \bar{\gamma}]$ and $p > \tilde{p}_{\gamma,1}(\sigma, n)$, as in (15), replacing the initial datum assumption (16) by the weaker condition

$$u_1 \in L^m \cap L^\infty \quad \text{with} \quad \|u_1\|_{L^m \cap L^\infty} \leq \epsilon, \quad (19)$$

and the solution space (17) by

$$u \in \mathcal{C}^1([0, \infty), L^m \cap L^\infty) \cap C([0, \infty), H^\sigma) \quad (20)$$

if $m \in (1, 2]$, and by

$$u \in \mathcal{C}^1([0, \infty), L^m \cap L^\infty) \quad (21)$$

if $m > 2$. The proof also requires only minor modifications with respect to the proof of Theorem 1.1. In particular, we stress that $p > m$, due to

$$p > p_\gamma(\sigma, n) \geq \tilde{p}_m(\sigma, n) = 1 + \frac{(2 - \gamma)\sigma m}{n} \geq 1 + \frac{(2 - \bar{\gamma})\sigma m}{n} = m \left(1 + \frac{\sigma}{n}\right) > m,$$

which follows from the assumption $\gamma \leq \bar{\gamma}$.

Having this in mind, we may focus to the case $\gamma \in (\bar{\gamma}, 1)$, in which the critical exponent $\tilde{p}_m(\sigma, n)$ comes into play.

Theorem 1.2. *Let $\sigma > 0$, $m \in (1, n^2/(n^2 - \sigma^2)^+)$ and $\gamma \in (\bar{\gamma}, 1)$; let us assume $p \geq \tilde{p}_m(\sigma, n)$. Then, there exists $\epsilon > 0$, sufficiently small, such that for any initial data as in (19) there is a uniquely determined energy solution*

$$u \in \mathcal{C}^1([0, \infty), L^m \cap L^\infty) \cap C([0, \infty), H^\sigma)$$

to (1), if $m \leq 2$, or a uniquely determined Sobolev solution

$$u \in \mathcal{C}^1([0, \infty), L^m \cap L^\infty)$$

to (1), if $m > 2$. Furthermore, the solution satisfies the following estimates for $j = 0, 1$ and $q \in [m, \infty]$:

$$\|\partial_t^j u(t, \cdot)\|_{L^q} \lesssim \begin{cases} C(1+t)^{1-j-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{q})} \|u_1\|_{L^m \cap L^\infty} & \text{if } \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) < 2 - j, \\ C(1+t)^{-1+\frac{n}{\sigma}(1-\frac{1}{m})} \log(2+t) \|u_1\|_{L^m \cap L^\infty} & \text{if } \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) = 2 - j, \\ C(1+t)^{-1+\frac{n}{\sigma}(1-\frac{1}{m})} \|u_1\|_{L^m \cap L^\infty} & \text{if } \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) > 2 - j. \end{cases} \quad (22)$$

Moreover, $\|(-\Delta)^{\frac{\sigma}{2}} u(t, \cdot)\|_{L^2}$ verifies the same estimate verified by $\|u_t(t, \cdot)\|_{L^2}$ in (22), if $m \leq 2$. Namely,

$$E(t) \leq \begin{cases} (1+t)^{-\frac{n}{\sigma}(\frac{2}{m}-1)} \|u_1\|_{L^m \cap L^\infty}^2 & \text{if } n < 2\sigma, \\ (1+t)^{-2+\frac{2n}{\sigma}(1-\frac{1}{m})} (\log(2+t))^2 \|u_1\|_{L^m \cap L^\infty}^2 & \text{if } n = 2\sigma, \\ (1+t)^{-2+\frac{2n}{\sigma}(1-\frac{1}{m})} \|u_1\|_{L^m \cap L^\infty}^2 & \text{if } n > 2\sigma. \end{cases}$$

Remark 1.2. We remark that the assumption $m < n^2/(n^2 - \sigma^2)^+$ also implies that $m < n/(n - \sigma)_+$, so that $p > m$ as a consequence of

$$p \geq \tilde{p}_m(\sigma, n) = 1 + \frac{(2 - \gamma)\sigma m}{n} > 1 + \frac{\sigma m}{n} > m.$$

Indeed, the latter is trivial if $\sigma \geq n$, whereas it is equivalent to $m < n/(n - \sigma)$ otherwise.

Remark 1.3. The main difference in the decay rate profile in Theorem 1.2 with respect to Theorem 1.1 is that no loss of decay rate is caused by the presence of the nonlinear memory term in (22), when

$$\frac{n}{\sigma} \left(1 - \frac{1}{q}\right) < 2 - j,$$

with respect to the linear Cauchy problem (see Proposition 2.1).

Finally, we will employ the test function method to prove that no weak global (in time) solution exists if $1 < p < \max\{p_\gamma(\sigma, n), \tilde{p}_m(\sigma, n)\}$, under suitable sign assumption on initial data in L^m .

This allows us to conclude that $\bar{p}_{\gamma, m}(\sigma, n)$ is really the critical exponent for any $\gamma \in ((n - \sigma)/n, 1)$. It remains an open problem to prove a satisfying nonexistence result when γ belongs to the interval $(0, (n - \sigma)/n)$.

Theorem 1.3. *Let us assume $u_1 \in L^1$ is positive. Then, there exists no global (in time) weak solution to (1) in the following cases:*

- if $n \leq (1 - \gamma)\sigma$ for any $\gamma \in (0, 1)$ and $p > 1$;
- if $n > (1 - \gamma)\sigma$, for any $\gamma \in (0, 1)$ and $1 < p < p_\gamma(\sigma, n)$.

Moreover, if $\sigma/2 \in \mathbb{N}$, no global (in time) weak solution exists also in the case

- $n > (1 - \gamma)\sigma$, for any $\gamma \in ((n - \sigma)/n, 1)$ and $p = p_\gamma(\sigma, n)$.

Theorem 1.4. *Let us assume $u_1 \in L^m$ such that*

$$u_1(x) \geq c|x|^{-\frac{n}{m}}(\log(|x|))^{-1}, \quad \forall |x| \gg 1. \quad (23)$$

Then, there exists no global (in time) weak solution to (1) in the following cases:

- if $n \leq (1 - \gamma)\sigma$ for any $\gamma \in (0, 1)$ and $p > 1$;
- if $n > (1 - \gamma)\sigma$, for any $\gamma \in (0, 1)$ and $1 < p < \max\{p_\gamma(\sigma, n), \tilde{p}_m(\sigma, n)\}$.

In view of the previous result we can conclude that $\bar{p}_{\gamma, m}(\sigma, n)$ is the critical exponent for any $\gamma \in ((n - \sigma)/n, 1)$. In particular we remark that, as we expected, for $\gamma \rightarrow 1$ the critical exponent $p_\gamma(\sigma, n)$ tends to $p_1(\sigma, \sigma/2, n)$, as defined in (9), and the critical exponent $\tilde{p}_m(\sigma, n)$ tends to $p_1(\sigma, \sigma/2, n/m)$, as $\gamma \rightarrow 1$.

However, it is interesting to remark that the critical exponent for the L^m theory, when $m > 1$, is not obtained by dividing the space dimension n by m , as it happens for the structurally damped evolution equation with power nonlinearity $|u_t|^p$ in [7], but it is smaller than this latter, due to

$$\tilde{p}_m(\sigma, n) = 1 + \frac{(2 - \gamma)\sigma m}{n} < 1 + \frac{(2 - \gamma)\sigma m}{n - m\sigma(1 - \gamma)} = p_\gamma(\sigma, n/m).$$

This phenomenon is due to the interesting interplay between the loss of decay rate due to the nonlinear memory, and the loss of decay rate due to the different space assumption for the initial data. This interplay has a less negative influence on the decay rate profile, than one may expect.

2. PROOF OF THEOREM 1.1

In order to prove our existence results, we need to employ decay estimates for the linear Cauchy problem associated to (1), that is

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\sigma}{2}} u_t = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = 0, \quad u_t(0, x) = u_1(x). \end{cases} \quad (24)$$

In [7, 10, 19] the authors prove the following decay estimates for (24).

Proposition 2.1. Let $j = 0, 1$. Then, for any $1 \leq m \leq q \leq \infty$ the solution $u = u(t, x)$ to (24) satisfies the following estimates:

$$\|\partial_t^j u(t, \cdot)\|_{L^q} \lesssim t^{-\frac{n}{\sigma}(\frac{1}{m} - \frac{1}{q}) + 1 - j} \|u_1\|_{L^m}, \quad (25)$$

and, as a consequence,

$$\|\partial_t^j u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{n}{\sigma}(\frac{1}{m} - \frac{1}{q}) + 1 - j} \|u_1\|_{L^m \cap L^q}. \quad (26)$$

Moreover, we have

$$\|(-\Delta)^{\frac{\sigma}{2}} u(t, \cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{q})} \|u_1\|_{L^m \cap L^q}. \quad (27)$$

Remark 2.1. We remark that in estimate (26) the assumption of additional regularity L^q for the datum allows to avoid the singularity which appears in (25) when $m < q$.

Let us now introduce some notation for the proof of the global (in time) existence of small data solutions. Throughout this section, we denote by $K_1 = K_1(t, x)$ the fundamental solution to the linear Cauchy problem (1) with initial data $u_0 = 0$ and $u_1 = \delta_0$, where δ_0 is the Dirac distribution in $x = 0$ with respect to the spatial variables. As a consequence, we may represent the solution to the Cauchy problem (24) in the form

$$u^{\text{lin}}(t, x) = K_1(t, x) *_{(x)} u_1(x).$$

We may introduce the operator

$$P : u \in X(T) \rightarrow Pu(t, x) := u^{\text{lin}}(t, x) + Nu(t, x),$$

where $X(T)$ is an evolution space which we will define in a suitable way. Then, we define Nu as an integral operator having the following representation

$$Nu(t, x) \doteq \int_0^t K_1(t-s, x) *_{(x)} F(s, u_t(\cdot, x)) ds. \quad (28)$$

According to the Duhamel's principle, we can consider a global (in time) solution to (1) as a fixed point of the operator P . Hence, in order to get the global (in time) existence and uniqueness of the solution in $X(T)$, we need to prove the following two crucial estimates:

$$\|Pu\|_{X(T)} \lesssim \|u_1\|_{L^m \cap L^\infty} + \|u\|_{X(T)}^p, \quad (29)$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right), \quad (30)$$

uniformly with respect to T . Thus, we show the desired property for the operator P providing that $\|u_1\|_{L^m \cap L^\infty} = \epsilon$ is sufficiently small. As a consequence of the Banach fixed point theorem, the conditions (29) and (30) guarantee the existence of a uniquely determined solution u to (1), that is, $u = u^{\text{lin}} + Nu$. We gain the global (in time) existence of small data solutions. In the proof of our result, it will be useful to apply the following straightforward estimates (see, for instance, [2]).

Proposition 2.2. Let $\alpha \in \mathbb{R}$, $\beta > 1$ and $\gamma \in (0, 1)$. Then,

$$\int_0^t (1+t-s)^{-\alpha} \int_0^s (s-\tau)^{-\gamma} (1+\tau)^{-\beta} d\tau ds \lesssim \begin{cases} (1+t)^{-\gamma} & \text{if } \alpha > 1, \\ (1+t)^{-\gamma} \log(2+t) & \text{if } \alpha = 1, \\ (1+t)^{1-\alpha-\gamma} & \text{if } \alpha < 1. \end{cases}$$

Proposition 2.3. Let $\alpha \in \mathbb{R}$ and $\beta, \gamma \in (0, 1)$. Then,

$$\int_0^t (1+t-s)^{-\alpha} \int_0^s (s-\tau)^{-\gamma} (1+\tau)^{-\beta} d\tau ds \lesssim \begin{cases} (1+t)^{1-\beta-\gamma} & \text{if } \alpha > 1, \\ (1+t)^{1-\beta-\gamma} \log(2+t) & \text{if } \alpha = 1, \\ (1+t)^{2-\alpha-\beta-\gamma} & \text{if } \alpha < 1. \end{cases}$$

We are now ready to prove Theorem 1.1.

Proof. [Theorem 1.1] If $n > \sigma$, we define

$$q_1 = \frac{n}{n-\sigma},$$

that is, the solution to

$$\frac{n}{\sigma} \left(1 - \frac{1}{q} \right) = 1.$$

If $n > 2\sigma$, we also define

$$q_0 = \frac{n}{n - 2\sigma},$$

that is, the solution to

$$\frac{n}{\sigma} \left(1 - \frac{1}{q}\right) = 2.$$

Moreover, we put $q_1 = \infty$ if $n = \sigma$ and $q_0 = \infty$ if $n = 2\sigma$.

We now consider the solution space

$$X(T) = \mathcal{C}^1([0, T], L^1 \cap L^\infty) \cap \mathcal{C}([0, T], H^\sigma),$$

equipped with the norm

$$\|u\|_{X(T)} := \|u\|_{X_0(T)} + \sup_{s \in [0, T]} \left\{ (1+s)^{\gamma-1} (\|u_t(s, \cdot)\|_{L^1} + (1+s)^{\frac{n}{\sigma}} \|u_t(s, \cdot)\|_{L^\infty}) \right\}$$

if $n < \sigma$, or

$$\|u\|_{X(T)} := \|u\|_{X_0(T)} + \sup_{s \in [0, T]} \left\{ (1+s)^{\gamma-1} \|u_t(s, \cdot)\|_{L^1} + (1+s)^\gamma \ell(s)^{-1} \|u_t(s, \cdot)\|_{L^\infty} \right\}$$

if $n = \sigma$, or

$$\|u\|_{X(T)} := \|u\|_{X_0(T)} + \sup_{s \in [0, T]} \left\{ (1+s)^{\gamma-1} \|u_t(s, \cdot)\|_{L^1} + (1+s)^\gamma \ell(s)^{-1} \|u_t(s, \cdot)\|_{L^{q_1}} + \|u_t(s, \cdot)\|_{L^\infty} \right\}$$

if $n > \sigma$, where $\ell(t) := \log(2+t)$ and $X_0(T)$ denotes the evolution space $C([0, T], L^1 \cap L^\infty \cap H^\sigma)$ equipped with the norm

$$\|u\|_{X_0(T)} := \sup_{s \in [0, T]} \left\{ (1+s)^{\gamma-2} (\|u(s, \cdot)\|_{L^1} + (1+s)^{\frac{n}{\sigma}} \|u(s, \cdot)\|_{L^\infty} + (1+s)^{\frac{n}{2\sigma}+1} \|(-\Delta)^{\frac{\sigma}{2}} u(s, \cdot)\|_{L^2}) \right\},$$

if $n < 2\sigma$, or

$$\|u\|_{X_0(T)} := \sup_{s \in [0, T]} \left\{ (1+s)^{\gamma-2} \|u(s, \cdot)\|_{L^1} + (1+s)^\gamma \ell(s)^{-1} (\|u(s, \cdot)\|_{L^\infty} + \|(-\Delta)^{\frac{\sigma}{2}} u(s, \cdot)\|_{L^2}) \right\},$$

if $n = 2\sigma$, or

$$\begin{aligned} \|u\|_{X_0(T)} := & \sup_{s \in [0, T]} \left\{ (1+s)^{\gamma-2} \|u(s, \cdot)\|_{L^1} + (1+s)^\gamma \ell(s)^{-1} \|u(s, \cdot)\|_{L^{q_0}} \right. \\ & \left. + (1+s)^\gamma (\|u(s, \cdot)\|_{L^\infty} + \|(-\Delta)^{\frac{\sigma}{2}} u(s, \cdot)\|_{L^2}) \right\}, \end{aligned}$$

if $n > 2\sigma$.

Applying Proposition 2.1, it follows immediately

$$\|u^{\text{lin}}\|_{X(T)} \lesssim \|u_1\|_{L^1 \cap L^\infty}, \quad (31)$$

and so we conclude $u^{\text{lin}} \in X(T)$.

In the remaining part of the proof we will estimate Nu in the $X(T)$ norm. To do this, we first remark that for any $t \in [0, T]$ it is possible to estimate by interpolation

$$\|u_t(t, \cdot)\|_{L^q} \lesssim (1+t)^{\left(1 - \frac{n}{\sigma} \left(1 - \frac{1}{q}\right)\right)^+ - \gamma} \ell(t) \|u\|_{X(T)}, \quad (32)$$

for any $q \in [1, \infty]$. In particular, the decay becomes $(1+t)^{-\gamma}$ for any $q > q_1$, when $n > \sigma$.

By using the derived $(L^1 \cap L^q) - L^q$ estimates stated in (26), applying the Minkowski's integral inequality, we get

$$\|\partial_t Nu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1+t-s)^{-\frac{n}{\sigma} \left(1 - \frac{1}{q}\right)} \int_0^s (s-\tau)^{-\gamma} \| |u_t(\tau, \cdot)|^p \|_{L^1 \cap L^q} d\tau ds,$$

for $q = 1, \infty$, and for $q = q_1$ if $n > \sigma$. By (32) we know that

$$\| |u_t(\tau, \cdot)|^p \|_{L^1} = \|u_t(\tau, \cdot)\|_{L^p}^p \lesssim (1+\tau)^{\left(p - \frac{n}{\sigma} (p-1)\right)^+ - p\gamma} \ell(\tau)^p \|u\|_{X(T)}^p; \quad (33)$$

moreover, for any $q \geq q_1$ since $qp > q_1$ we get

$$\|u_t(\tau, \cdot)\|_{L^q}^p = \|u_t(\tau, \cdot)\|_{L^{qp}}^p \lesssim (1 + \tau)^{-p\gamma} \ell(\tau)^p \|u\|_{X(T)}^p. \quad (34)$$

If, and only if, $p > \bar{p}_{\gamma,1}(\sigma, n)$, then from estimate (33) we may deduce that there exists an exponent $\beta_\gamma(n, \sigma, p) > 1$ such that

$$\|u_t(\tau, \cdot)\|_{L^p}^p \lesssim (1 + \tau)^{-\beta_\gamma(n, \sigma, p)} \|u\|_{X(T)}^p. \quad (35)$$

In particular, for any $q > q_1$ from estimate (34) we see that $\|u_t(\tau, \cdot)\|_{L^{qp}}^p$ also satisfies estimate (35). Thus, for any $q = 1, q_1, \infty$ we get

$$\|\partial_t Nu(t, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^p \int_0^t (1 + t - s)^{-\frac{n}{\sigma}(1 - \frac{1}{q})} \int_0^s (s - \tau)^{-\gamma} (1 + \tau)^{-\beta_\gamma(n, \sigma, p)} d\tau ds.$$

Therefore, for any $p > \bar{p}_{\gamma,1}(n)$ we can apply Proposition 2.2 to get the following desired estimates

$$\|\partial_t Nu(t, \cdot)\|_{L^1} \lesssim (1 + t)^{1-\gamma} \|u\|_{X(T)}^p;$$

moreover, for $n \neq \sigma$ we get

$$\|\partial_t Nu(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{(1 - \frac{n}{\sigma})^+ - \gamma} \|u\|_{X(T)}^p;$$

finally, for $n \geq \sigma$ we find

$$\|\partial_t Nu(t, \cdot)\|_{L^{q_1}} \lesssim (1 + t)^{-\gamma} \ell(t) \|u\|_{X(T)}^p.$$

In a similar way, we estimate $\|Nu(t, \cdot)\|_{L^q}$ for $q = 1, \infty$, and q_0 if $n > 2\sigma$. We apply (26) to get

$$\|Nu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1 + t - s)^{1 - \frac{n}{\sigma}(1 - \frac{1}{q})} \int_0^s (s - \tau)^{-\gamma} \|u_t(\tau, \cdot)\|_{L^1 \cap L^q}^p d\tau ds.$$

Thus, using estimate (35) we find

$$\|Nu(t, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^p \int_0^t (1 + t - s)^{1 - \frac{n}{\sigma}(1 - \frac{1}{q})} \int_0^s (s - \tau)^{-\gamma} (1 + \tau)^{-\beta_\gamma(n, \sigma, p)} d\tau ds.$$

Again, for $p > \bar{p}_{\gamma,1}(n)$, we can apply Proposition 2.2 to get the following desired estimates of $\|Nu\|_{L^q}$:

$$\|Nu(t, \cdot)\|_{L^1} \lesssim (1 + t)^{2-\gamma} \|u\|_{X(T)}^p,$$

moreover, for $n \neq 2\sigma$ we get

$$\|Nu(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{(2 - \frac{n}{\sigma})^+ - \gamma} \|u\|_{X(T)}^p;$$

furthermore, for any $n \geq 2\sigma$ we have

$$\|Nu(t, \cdot)\|_{L^{q_0}} \lesssim (1 + t)^{-\gamma} \ell(t) \|u\|_{X(T)}^p.$$

Finally, we have

$$\|(-\Delta)^{\frac{\sigma}{2}} Nu(t, \cdot)\|_{L^2} \lesssim \|u\|_{X(T)}^p \int_0^t (1 + t - s)^{-\frac{n}{2\sigma}} \int_0^s (s - \tau)^{-\gamma} (1 + \tau)^{-\beta_\gamma(n, \sigma, p)} d\tau ds,$$

which implies

$$\|(-\Delta)^{\frac{\sigma}{2}} Nu(t, \cdot)\|_{L^2} \lesssim (1 + t)^{(1 - \frac{n}{2\sigma})^+ - \gamma} \ell(t)^b \|u\|_{X(T)}^p,$$

where $b = 0$ for any $n \neq 2\sigma$ and $b = 1$ if $n = 2\sigma$.

Collecting the above derived estimates and (31) we have proved

$$\|u^{\text{lin}}\|_{X(T)} + \|Nu\|_{X(T)} \lesssim \|u_1\|_{L^1 \cap L^\infty} + \|u\|_{X(T)}^p,$$

which allows us to get the desired estimate (29).

In order to prove (30) we may rewrite it as

$$\|Nu - Nv\|_{X(T)} \lesssim \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right).$$

Then, we use (3) together with Hölder's inequality to get

$$\|g(u) - g(v)\|_{L^1 \cap L^q} \lesssim \|g(u) - g(v)\|_{L^1 \cap L^q} \left(\|u\|_{L^1 \cap L^q}^{p-1} + \|v\|_{L^1 \cap L^q}^{p-1} \right),$$

and we proceed as in the proof of (29) to get the desired result.

The proof of the decay estimates in (18) follows by straightforward calculations. \square

3. PROOF OF THEOREM 1.2

In order to prove Theorem 1.2, we shall modify the norm of $X(T)$ to take into account of the different profile of the decay rate of the solution, due to the influence of the L^m regularity of the initial datum.

Proof. [Theorem 1.2] Now, for a given $T > 0$, we fix

$$X(T) = \mathcal{C}^1([0, T], L^m \cap L^\infty) \cap \mathcal{C}([0, T], H^\sigma),$$

if $m \leq 2$, and

$$X(T) = \mathcal{C}^1([0, T], L^m \cap L^\infty),$$

if $m > 2$.

Assume first that $n < \sigma$. Then we set

$$\|u\|_{X_0(T)} := \sup_{s \in [0, T]} \left\{ (1+s)^{-1} \|u(s, \cdot)\|_{L^m} + (1+s)^{-1+\frac{n}{m\sigma}} \|u(s, \cdot)\|_{L^\infty} + (1+s)^{\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{2})} \|(-\Delta)^{\frac{\sigma}{2}} u(s, \cdot)\|_{L^2} \right\},$$

if $m \leq 2$ and

$$\|u\|_{X_0(T)} := \sup_{s \in [0, T]} \left\{ (1+s)^{-1} \|u(s, \cdot)\|_{L^m} + (1+s)^{-1+\frac{n}{m\sigma}} \|u(s, \cdot)\|_{L^\infty} \right\},$$

if $m > 2$. Then we define

$$\|u\|_{X(T)} := \|u\|_{X_0(T)} + \sup_{s \in [0, T]} \left\{ \|u_t(s, \cdot)\|_{L^m} + (1+s)^{\frac{n}{m\sigma}} \|u_t(s, \cdot)\|_{L^\infty} \right\}.$$

If $n \geq \sigma$, we modify the norm into

$$\begin{aligned} \|u\|_{X(T)} := & \|u\|_{X_0(T)} + \sup_{s \in [0, T]} \left\{ \|u_t(s, \cdot)\|_{L^m} + (1+s)^{\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{\tilde{p}_m(\sigma, n)})} \|u_t(s, \cdot)\|_{L^{\tilde{p}_m(\sigma, n)}} \right. \\ & \left. + (1+s)^{1-\frac{n}{\sigma}(1-\frac{1}{m})} (\ell(s)^{-1} \|u_t(s, \cdot)\|_{L^{q_1}} + \|u_t(s, \cdot)\|_{L^\infty}) \right\}, \end{aligned}$$

where $q_1 = n/(n-\sigma)$ is the solution to

$$\frac{n}{\sigma} \left(1 - \frac{1}{q} \right) = 1,$$

if $n > \sigma$, and $q_1 = \infty$ if $n = \sigma$, as in the proof of Theorem 1.1. Moreover, if $n \geq 2\sigma$, we replace $(1+s)^{-1+\frac{n}{m\sigma}} \|u(s, \cdot)\|_{L^\infty}$ in $\|u\|_{X_0(T)}$ by

$$(1+s)^{1-\frac{n}{\sigma}(1-\frac{1}{m})} (\ell(s)^{-1} \|u(s, \cdot)\|_{L^{q_0}} + \|u(s, \cdot)\|_{L^\infty}),$$

where $q_0 = n/(n-2\sigma)$ is the solution to

$$\frac{n}{\sigma} \left(1 - \frac{1}{q_0} \right) = 2,$$

if $n > 2\sigma$, and $q_0 = \infty$ if $n = 2\sigma$, as in the proof of Theorem 1.1. We proceed similarly to modify $(1+s)^{\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{2})} \|(-\Delta)^{\frac{\sigma}{2}} u(s, \cdot)\|_{L^2}$ in $\|u\|_{X_0(T)}$.

Applying Proposition 2.1, it follows immediately that

$$\|u^{\text{lin}}\|_{X(T)} \lesssim \|u_1\|_{L^m \cap L^\infty}, \quad (36)$$

and so we conclude $u^{\text{lin}} \in X(T)$. It remains to estimate Nu to prove (29)-(30).

If $u \in X(T)$, then for any $p \geq \tilde{p}_m(\sigma, n)$, we may estimate

$$\|u_t(t, \cdot)\|_{L^p} \leq \|u\|_{X(T)} (1+t)^{-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{\tilde{p}_m(\sigma, n)})}. \quad (37)$$

By using the derived $(L^1 \cap L^q) - L^q$ estimates stated in (26), applying the Minkowski's integral inequality, we get for $q \in [1, \infty]$:

$$\|\partial_t N u(t, \cdot)\|_{L^q} \lesssim \int_0^t (1+t-s)^{-\frac{n}{\sigma}(1-\frac{1}{q})} \int_0^s (s-\tau)^{-\gamma} \| |u_t(\tau, \cdot)|^p \|_{L^1 \cap L^p} d\tau ds.$$

Thus, employing estimates (37), due to $p \geq \tilde{p}_m(\sigma, n)$, we get

$$\|\partial_t N u(t, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^p \int_0^t (1+t-s)^{-\frac{n}{\sigma}(1-\frac{1}{q})} \int_0^s (s-\tau)^{-\gamma} (1+\tau)^{\frac{n}{\sigma}(1-\frac{1}{m})+\gamma-2} d\tau ds.$$

Indeed, for $p \geq \tilde{p}_m(\sigma, n)$, we get

$$(1+\tau)^{-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{\tilde{p}_m(\sigma, n)})p} \leq (1+\tau)^{-\frac{n}{\sigma}(\frac{\tilde{p}_m(\sigma, n)}{m}-1)} = (1+\tau)^{\frac{n}{\sigma}(1-\frac{1}{m})+\gamma-2}.$$

In particular,

$$2 - \gamma - \frac{n}{\sigma} \left(1 - \frac{1}{m}\right) < 1,$$

due to $\gamma > \bar{\gamma} = 1 - n(1 - 1/m)/\sigma$.

We shall now distinguish two cases. If $n < \sigma$ or $q < q_1$, that is,

$$\frac{n}{\sigma} \left(1 - \frac{1}{q}\right) < 1,$$

by applying Proposition 2.3, due to

$$2 - \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) - \gamma - \left(2 - \gamma - \frac{n}{\sigma} \left(1 - \frac{1}{m}\right)\right) = \frac{n}{\sigma} \left(\frac{1}{m} - \frac{1}{q}\right),$$

we obtain:

$$\|\partial_t N u(t, \cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{q})} \|u\|_{X(T)}^p.$$

If $n \geq \sigma$ and $q = q_1$, or, respectively, $q > q_1$, then, by applying Proposition 2.3, due to

$$1 - \gamma - \left(2 - \gamma - \frac{n}{\sigma} \left(1 - \frac{1}{m}\right)\right) = -1 + \frac{n}{\sigma} \left(1 - \frac{1}{m}\right),$$

we obtain:

$$\|\partial_t N u(t, \cdot)\|_{L^q} \lesssim (1+t)^{-1+\frac{n}{\sigma}(1-\frac{1}{m})} \ell(t) \|u\|_{X(T)}^p,$$

or, respectively,

$$\|\partial_t N u(t, \cdot)\|_{L^q} \lesssim (1+t)^{-1+\frac{n}{\sigma}(1-\frac{1}{m})} \|u\|_{X(T)}^p.$$

We proceed similarly to estimate $\|N u(t, \cdot)\|_{L^q}$, and $\|(-\Delta)^{\frac{\sigma}{2}} N u(t, \cdot)\|_{L^2}$ if $m \leq 2$.

We proved estimate (29). With the same approach used in the proof of Theorem 1.1 we get estimate (30), and we conclude the proof. \square

4. PROOF OF THEOREM 1.3

In order to prove our non-existence result, we employ the test functions method. In particular, to treat the non-local differential operators $(-\Delta)^\sigma$ and $(-\Delta)^{\sigma/2}$ we apply the following fundamental lemma:

Lemma 4.1 ([14]). *Let $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $q > n$. Fixed $\theta \notin \mathbb{N}$, we define $m_\theta = [\theta]$, the greatest integer which is smaller than θ , and $s_\theta = \theta - m_\theta$. Then, there exists a positive constant $A_{n,q,\theta}$ depending only on n, q and θ such that for any $x \in \mathbb{R}^n$,*

$$|((-\Delta)^\theta \langle \cdot \rangle^{-q})(x)| \leq \langle x \rangle^{-n-2s_\theta}. \quad (38)$$

Proof. If $\theta = s_\theta \in (0, 1)$, then the proof follows by straightforward calculations as in Lemma 1.5 in [14]. In the case $\theta = m_\theta + s_\theta$ with $m_\theta \geq 1$ we use the representation

$$((-\Delta)^\theta \langle \cdot \rangle^{-q})(x) = ((-\Delta)^{s_\theta} ((-\Delta)^{m_\theta} \langle \cdot \rangle^{-q}))(x),$$

and we remark that $((-\Delta)^{m_\theta} \langle \cdot \rangle^{-q})(x)$ can be written as a linear combination of functions $\varphi_i(x) := \langle x \rangle^{-q_i}$, with $q_i > n$. Then, we apply the already known result to $((-\Delta)^{s_\theta} \langle \cdot \rangle^{-q_i})(x)$ to conclude the proof. \square

Remark 4.1. If $\theta \in \mathbb{N}$ then the following estimate holds:

$$|(-\Delta)^\theta \langle x \rangle^{-q}| \lesssim \langle x \rangle^{-q-2\theta},$$

for any $q > n$.

4.1. The general case $\sigma > 0$ and $p < p_\gamma(\sigma, n)$, $m \geq 1$. For any $\alpha \in (0, 1)$ and for a fixed $T > 0$ we denote by $J_{0|t}^\alpha, J_{t|T}^\alpha$ the fractional integral operators defined by

$$J_{0|t}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-(1-\alpha)} f(\tau) d\tau,$$

$$J_{t|T}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{-(1-\alpha)} f(\tau) d\tau,$$

and the following fractional differential operators:

$$D_{0|t}^\alpha := \partial_t J_{0|t}^{1-\alpha}, \quad D_{t|T}^\alpha := \partial_t J_{t|T}^{1-\alpha}.$$

The following properties are satisfied:

$$\int_0^T (D_{0|t}^\alpha f)(t)g(t)dt = \int_0^T f(t)(D_{t|T}^\alpha g)(t)dt, \quad (39)$$

$$D_{0|t}^\alpha J_{0|t}^\alpha f(t) = f(t). \quad (40)$$

Let us introduce the function $\omega := \omega(t) \in C_c([0, \infty))$ defined by

$$\omega(t) = \begin{cases} (1-t/T) & \text{if } t \in [0, T], \\ 0 & \text{if } t > T. \end{cases} \quad (41)$$

Then, it holds $\text{supp } \omega = [0, T]$ and $\omega(t)^\beta \in C_c^k([0, \infty))$ for any $\beta > k \geq 0$. Moreover, the following useful lemma holds.

Lemma 4.2. *For any $\beta > 0$ it holds*

$$\int_t^\infty \omega(\tau)^\beta d\tau = \frac{T}{\beta+1} \omega(t)^{\beta+1}. \quad (42)$$

Furthermore, for any $\alpha \in (0, 1)$ and $\beta > \alpha$ it holds

$$D_{t|T}^\alpha \omega(t)^\beta = C(\alpha, \beta) T^{-\alpha} \omega(t)^{\beta-\alpha}, \quad (43)$$

where

$$C(\alpha, \beta) = \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}.$$

Proof. The identity (42) can be easily obtained by applying the definition of ω stated in (41). For the proof of (43) see for instance Lemma 4.1 in [3]. \square

We set $\alpha := 1 - \gamma$ and we fix β such that

$$\beta > (\alpha+1)p', \quad (44)$$

where $p' = p/(p-1)$ is the Hölder conjugate of p . Let $\Psi = \Psi(x) \in C^\infty(\mathbb{R}^n)$ be defined by

$$\Psi(x) = \langle x \rangle^{-q}, \text{ for some } q > n, \quad (45)$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Then, for any $R > 0$ we define $\Psi_R(x) = \Psi(x/R)$ and

$$\Phi_R(t, x) = \omega(t)^\beta \Psi_R(x).$$

Finally we introduce the functions

$$\varphi(t, x) := D_t^\alpha \Phi_R(t, x), \quad \tilde{\varphi}(t, x) := \int_t^\infty \varphi(\tau, x) d\tau,$$

defined for any $t \geq 0$ and $x \in \mathbb{R}^n$. It is clear that we have $\partial_t \tilde{\varphi} = -\varphi$; moreover, by Lemma 4.2 we can conclude that $\text{supp } \varphi, \text{supp } \tilde{\varphi} \subset [0, T] \times \mathbb{R}^n$; explicitly:

$$\varphi(t, x) = C(\alpha, \beta) T^{-\alpha} \omega(t)^{\beta-\alpha} \Psi_R(x); \quad (46)$$

$$\tilde{\varphi}(t, x) = \frac{C(\alpha, \beta)}{\beta + 1 - \alpha} T^{1-\alpha} \omega(t)^{\beta-\alpha+1} \Psi_R(x). \quad (47)$$

Let us suppose by contradiction that there exists u a global (in time) weak solution to (1). Then, for any $R > 0$ we can define the integral term

$$I_R := \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} J_{0|t}^\alpha (|u_t|^p) \varphi \, dx \, dt.$$

By properties (39) and (40) we find on one hand the identity

$$I_R = \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} |u_t|^p \Phi_R \, dx \, dt. \quad (48)$$

On the other hand, since u is a weak solution to the Cauchy problem (1), we get

$$I_R + \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) \, dx = - \int_0^T \int_{\mathbb{R}^n} u_t \varphi_t \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} u_t (\mu(-\Delta)^{\frac{\sigma}{2}} \varphi + (-\Delta)^\sigma \tilde{\varphi}) \, dx \, dt. \quad (49)$$

By using Young inequality we can estimate

$$\int_0^T \int_{\mathbb{R}^n} |u_t| |\varphi_t| \, dx \, dt \leq \epsilon \int_0^T \int_{\mathbb{R}^n} |u_t|^p \Phi_R \, dx \, dt + C_\epsilon \int_0^T \int_{\mathbb{R}^n} |\varphi_t|^{p'} \Phi_R^{-\frac{1}{p-1}} \, dx \, dt$$

By (46), since $\omega'(t) = -1/T$ for any $t \in (0, T)$, we find

$$\varphi_t(t, x) = -C(\alpha, \beta)(\beta - \alpha) T^{-1-\alpha} \omega(t)^{\beta-\alpha-1} \Psi_R(x).$$

Thus, there exists $C > 0$ such that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |\varphi_t|^{p'} \Phi_R^{-\frac{1}{p-1}} \, dx \, dt &= C T^{-(\alpha+1)p'} \int_0^T \int_{\mathbb{R}^n} \omega(t)^{\beta-(\alpha+1)p'} \Psi(x/R) \, dx \, dt \\ &\lesssim T^{-(\alpha+1)p'+1} R^n; \end{aligned}$$

in the last estimate we used that the exponent $\beta - (\alpha + 1)p'$ is positive and the function $\Psi = \Psi(x)$ is integrable on \mathbb{R}^n . In order to treat the remaining integral terms we use Lemma 4.1; in particular, since $q > n$, there exists $\delta > 0$ such that

$$|(-\Delta)^\sigma \Psi_R(x)| \lesssim \langle x \rangle^{-n-\delta}, \quad |(-\Delta)^{\sigma/2} \Psi_R(x)| \lesssim \langle x \rangle^{-n-\delta};$$

moreover, we recall that the following scaling identity holds for any $\theta > 0$:

$$(-\Delta)^\theta \Psi_R(x) = R^{-2\theta} ((-\Delta)^\theta \Psi)(x/R). \quad (50)$$

Thus, applying again Young inequality we find

$$\int_0^T \int_{\mathbb{R}^n} |u_t| |(-\Delta)^{\frac{\sigma}{2}} \varphi| \, dx \, dt \leq \epsilon \int_0^T \int_{\mathbb{R}^n} |u_t|^p \Phi_R \, dx \, dt + C_\epsilon \int_0^T \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\sigma}{2}} \varphi|^{p'} \Phi_R^{-\frac{1}{p-1}} \, dx \, dt.$$

By (38), (46) and (50) with $\theta = \sigma/2$ we find

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\sigma}{2}} \varphi|^{p'} \Phi_R^{-\frac{1}{p-1}} \, dx \, dt &\lesssim T^{-\alpha p'} R^{-\sigma p'} \int_0^T \int_{\mathbb{R}^n} \omega(t)^{\beta-\alpha p'} \langle x/R \rangle^{-n-\delta} \, dx \, dt \\ &\lesssim T^{-\alpha p'+1} R^{n-\sigma p'}; \end{aligned}$$

in the last estimate we used that the exponent $\beta - \alpha p'$ is positive due to assumption (44) and the function $\langle x \rangle^{-n-\sigma}$ is integrable. In a similar way we estimate the last integral term:

$$\int_0^T \int_{\mathbb{R}^n} |u_t| |(-\Delta)^\sigma \tilde{\varphi}| dt dx \leq \epsilon \int_0^T \int_{\mathbb{R}^n} |u_t|^p \Phi_R dx dt + C_\epsilon \int_0^T \int_{\mathbb{R}^n} |(-\Delta)^\sigma \tilde{\varphi}|^{p'} \Phi_R^{-\frac{1}{p-1}} dx dt.$$

By (38), (47) and (50) with $\theta = \sigma$ we get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |(-\Delta)^\sigma \tilde{\varphi}|^{p'} \Phi_R^{-\frac{1}{p-1}} dx dt &\lesssim T^{(1-\alpha)p'} R^{-2\sigma p'} \int_0^T \int_{\mathbb{R}^n} \omega(t)^{\beta - (\alpha-1)p'} \langle x/R \rangle^{-n-2\delta} dx dt \\ &\lesssim T^{(1-\alpha)p'+1} R^{n-2\sigma p'}. \end{aligned} \quad (51)$$

Collecting the previous estimates we obtain that there exists $C > 0$ such that

$$(1 - 3\epsilon)I_R + \int_{\mathbb{R}^n} u_1(x)\varphi(0, x) dx \leq CT^{1-\alpha p'} R^n (T^{-p'} + R^{-\sigma p'} + T^{p'} R^{-2\sigma p'}). \quad (52)$$

Being $\varphi \geq 0$, since u_1 is non-negative, we find

$$\int_{\mathbb{R}^n} u_1(x)\varphi(0, x) dx \geq 0.$$

Thus, fixing $\epsilon \in (0, 1/3)$, by (52), we get

$$I_R \lesssim T^{1-\alpha p'} R^n (T^{-p'} + R^{-\sigma p'} + T^{p'} R^{-2\sigma p'}). \quad (53)$$

Therefore, if $p < p_\gamma(\sigma, n)$ it is sufficient to set $R = T^{\frac{1}{\sigma}}$ to get

$$I_R \lesssim T^{1+n/\sigma - (\alpha+1)p'}.$$

In fact, it holds $1 + n/\sigma - (\alpha + 1)p' < 0$ if, and only if, $p < p_\gamma(\sigma, n)$, or $n < \alpha\sigma$. In this case, by Beppo Levi's theorem on monotone convergence, since $\Phi_{T^{1/\sigma}} \nearrow 1$ as $T \rightarrow \infty$, we derive

$$\lim_{T \rightarrow \infty} I_{T^{\frac{1}{\sigma}}} = \int_0^\infty \int_{\mathbb{R}^n} |u_t|^p dx dt = 0;$$

hence, $u_t \equiv 0$. As a consequence, $u(t, x) = u(0, x) = 0$, for any $x \in \mathbb{R}^n$. In order to complete the proof of Theorem 1.3 it remains to treat the critical case $p = p_\gamma(\sigma, n)$ in the case $\sigma/2 \in \mathbb{N}$.

4.2. The critical exponent $p_\gamma(\sigma, n)$ in the case $\sigma/2 \in \mathbb{N}$. In the critical case $\bar{p} := p_\gamma(\sigma, n)$, we set $R = T^{\frac{1}{\sigma}} K^{-\frac{1}{\sigma}}$ for some fixed $K > 1$. Then, from estimate (52) we get the existence of $C > 0$ such that

$$(1 - 3\epsilon) \int_0^T \int_{\mathbb{R}^n} |u_t|^p \Phi_{(T/K)^{1/\sigma}} dx dt \leq CK^{-\frac{n}{\sigma}} (1 + K^{\bar{p}'} + K^{2\bar{p}'});$$

thus, taking $T \rightarrow \infty$ we find

$$\int_0^\infty \int_{\mathbb{R}^n} |u_t|^p dx dt \leq C(K), \quad (54)$$

for some $C(K) > 0$, i.e. $u_t \in L^p$.

In order to prove our result in the critical case $p = p_\gamma(\sigma, n)$ when $\sigma/2 \in \mathbb{N}$, it is sufficient to use a classical test function; we will use the same notations as in Section 4.1. Let $\Psi \in C_c^\infty(\mathbb{R}^n)$ be a radial test function such that

- $\text{supp } \Psi = B_1$;
- $\Psi(x) = 1$, for any $x \in B_{1/2}$;
- Ψ is decreasing.

Then, for any integer $\theta \in \mathbb{N}$, it holds $(-\Delta)^\theta \Psi_R(x) = 0$ for any $|x| \leq R/2$.

Applying the same ideas used in Section 4.1 we can estimate

$$\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} J_{0|T}^\alpha (|u_t|^{\bar{p}}) \varphi dx dt \leq - \int_0^T \int_{\mathbb{R}^n} u_t \varphi_t dx dt + \int_0^T \int_{\mathbb{R}^n} u_t ((-\Delta)^{\frac{\sigma}{2}} \varphi + (-\Delta)^\sigma \tilde{\varphi}) dt dx. \quad (55)$$

Then, taking $R = T^{1/\sigma} K^{-1/\sigma}$, we find

$$\int_0^T \int_{\mathbb{R}^n} |u_t| |\varphi_t| dx dt \leq \epsilon \int_0^T \int_{\mathbb{R}^n} |u_t|^{\bar{p}} \Phi_R dx dt + C_\epsilon K^{-\frac{n}{\sigma}}.$$

and

$$\int_0^T \int_{\mathbb{R}^n} |u_t| |(-\Delta)^{\frac{\sigma}{2}} \varphi| dt dx \leq \epsilon \int_0^T \int_{\mathbb{R}^n} |u_t|^{\bar{p}} \Phi_R dx dt + C_\epsilon K^{-\frac{n}{\sigma} + \bar{p}'}$$

In the critical case we can improve the estimate for the last integral term: we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |u_t| |(-\Delta)^\sigma \tilde{\varphi}| dx dt &= \int_0^T \int_{|x| > (T/K)^{1/\sigma}} |u_t| |(-\Delta)^\sigma \tilde{\varphi}| dx dt \\ &\leq \left(\int_0^T \int_{|x| > (T/K)^{1/\sigma}} |(-\Delta)^\sigma \tilde{\varphi}|^{\bar{p}'} \Phi_R^{-\frac{1}{\bar{p}-1}} dx dt \right)^{\frac{1}{\bar{p}'}} \\ &\quad \times \left(\int_0^T \int_{|x| > (T/K)^{1/\sigma}} |u_t|^{\bar{p}} \Phi_R dx dt \right)^{\frac{1}{\bar{p}}}. \end{aligned}$$

In analogous way as in (51) we get

$$\int_0^T \int_{\mathbb{R}^n} |(-\Delta)^\sigma \tilde{\varphi}|^{\bar{p}'} \Phi_R^{-\frac{1}{\bar{p}-1}} dx dt \lesssim K^{-\frac{n}{\sigma} + 2\bar{p}'},$$

and so by (55), taking account of (48), we obtain

$$\begin{aligned} (1 - 2\epsilon) \int_0^T \int_{\mathbb{R}^n} |u_t|^{\bar{p}} \Phi_R dx dt &\leq \bar{C} K^{-\frac{n}{\sigma}} (1 + K^{\bar{p}'}) \\ &\quad + \tilde{C} K^{-\frac{n}{\sigma\bar{p}'} + 2} \left(\int_0^T \int_{|x| > (T/K)^{1/\sigma}} |u_t|^{\bar{p}} \Phi_R dx dt \right)^{\frac{1}{\bar{p}}}, \end{aligned}$$

for some constants $\bar{C}, \tilde{C} > 0$. Since $u_t \in L^p$ by (54), for any fixed $K > 0$ it holds,

$$\lim_{T \rightarrow \infty} \int_0^T \int_{|x| > (T/K)^{1/\sigma}} |u_t|^{\bar{p}} \Phi_R dx dt = 0;$$

thus, we finally get

$$\int_0^\infty \int_{\mathbb{R}^n} |u_t|^{\bar{p}} dx dt \leq \bar{C} K^{-\frac{n}{\sigma}} (1 + K^{\bar{p}'}).$$

The right hand side is arbitrarily small since it holds

$$\bar{p}' = \frac{n + \sigma}{(\alpha + 1)\sigma} < \frac{n}{\sigma},$$

for any $\gamma > (n - \sigma)/n$. Thus, we conclude $u_t \equiv 0$, and so for any $t \geq 0$ we have $u(t, x) = u(0, x) = 0$. This concludes the proof of Theorem 1.3.

4.3. The case $u_1 \in L^m$, $m > 1$. By our assumption on u_1 in Theorem 1.4, we have

$$u_1(x) \gtrsim |x|^{-\frac{n}{m}} (\log(|x|))^{-1};$$

thus, we find

$$\begin{aligned} \int_{\mathbb{R}^n} u_1(x) \langle x/R \rangle^{-q} dx &\geq \int_{|x| \leq R} u_1(x) \langle x/R \rangle^{-q} dx \geq 2^{-\frac{q}{2}} \int_{|x| \leq R} u_1(x) dx \\ &\gtrsim \int_{|x| \leq R} |x|^{-\frac{n}{m}} (\log(|x|))^{-1} dx \gtrsim \int_0^R r^{\frac{n(m-1)}{m} - 1} (\log(r))^{-1} dr \\ &\gtrsim (\log(R))^{-1} R^{\frac{n(m-1)}{m}}, \end{aligned} \tag{56}$$

where we used

$$\langle x/R \rangle^{-q} = (1 + |x|^2/R^2)^{-\frac{q}{2}} \geq 2^{-\frac{q}{2}} \quad \text{for any } |x| \leq R.$$

Recalling that

$$\varphi(0, x) = C(\alpha, \beta) T^{-\alpha} \langle x/R \rangle^{-q},$$

the nonnegativity of u_1 , together with estimates (52) and (56) implies:

$$\begin{aligned} C(\alpha, \beta) T^{-\alpha} (\log(R))^{-1} R^{\frac{n(m-1)}{m}} &\lesssim C(\alpha, \beta) T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \langle x/R \rangle^{-q} dx \\ &= \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) dx \lesssim T^{1-\alpha p'} R^n (T^{-p'} + R^{-\sigma p'} + T^{p'} R^{-2\sigma p'}). \end{aligned}$$

Setting again $R = T^{\frac{1}{\sigma}}$ we get

$$(\log(R))^{-1} T^{-\alpha + \frac{n(m-1)}{\sigma m}} \lesssim T^{1-(\alpha+1)p' + \frac{n}{\sigma}},$$

which leads to a contradiction for $T \rightarrow \infty$ if $p < \tilde{p}_m(\sigma, n)$. This completes the proof of Theorem 1.4.

5. CONCLUDING REMARKS

The assumption $u(0, x) = 0$ in (1) may be easily replaced in Theorems 1.1 and 1.2 by the smallness of the initial data $u_0(x) = u(0, x)$ in an appropriate space. For instance, we may supplement (16) in the statement of Theorem 1.1 with

$$u_0 \in W^{2\kappa, 1} \cap W^{2\kappa, \infty} \quad \text{with} \quad \|u_0\|_{W^{2\kappa, 1} \cap W^{2\kappa, \infty}} \leq \epsilon, \quad (57)$$

with κ positive integer such that $\sigma \leq 2\kappa$. Here $W^{2\kappa, 1}$ and $W^{2\kappa, \infty}$ are the Sobolev spaces of order 2κ . By the equivalence of the norm in Sobolev spaces and in Bessel potential spaces of positive even order, the smallness assumption in (57) implies, in particular, that

$$u_0, (-\Delta)^{\frac{\sigma}{2}} u_0 \in L^1 \cap L^\infty, \quad \|u_0\|_{L^1 \cap L^\infty} + \|(-\Delta)^{\frac{\sigma}{2}} u_0\|_{L^1 \cap L^\infty} \leq \epsilon.$$

This is sufficient to replace (31) in the proof of Theorem 1.1, by

$$\|u^{\text{lin}}\|_{X(T)} \lesssim \|u_0\|_{W^{2\kappa, 1} \cap W^{2\kappa, \infty}} + \|u_1\|_{L^1 \cap L^\infty}, \quad (58)$$

where now

$$u^{\text{lin}} = K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x), \quad \text{with } K_0 = \partial_t K_1 + \mu (-\Delta)^{\frac{\sigma}{2}} K_1.$$

Similarly, we may supplement (19) in the statement of Theorem 1.2 with

$$u_0 \in W^{2\kappa, m} \cap W^{2\kappa, \infty} \quad \text{with} \quad \|u_0\|_{W^{2\kappa, m} \cap W^{2\kappa, \infty}} \leq \epsilon. \quad (59)$$

On the other hand, the assumption $u(0, x) = 0$ plays a different role in the proof of Theorems 1.3 and 1.4. In particular, if we assume that $u(0, x) = u_0(x)$, for some $u_0 \in L^1$, positive, in Theorem 1.4, then (49) becomes

$$I_R + \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) dx - \int_{\mathbb{R}^n} u_0(x) (-\Delta)^\sigma \tilde{\varphi}(0, x) dx = \int_0^T \int_{\mathbb{R}^n} u_t (-\varphi_t + \mu (-\Delta)^{\frac{\sigma}{2}} \varphi + (-\Delta)^\sigma \tilde{\varphi}) dt dx. \quad (60)$$

Due to the fact that

$$\int_{\mathbb{R}^n} u_0(x) (-\Delta)^\sigma \tilde{\varphi}(0, x) dx \quad (61)$$

is uniformly bounded with respect to T , the presence of the first initial datum u_0 does not invalidate the argument in the proof of Theorem 1.4.

However, by following the proof of Theorem 1.4 one may only deduce that $u(t, x) = u_0(x)$ if $p < \tilde{p}_m(\sigma, n)$. Then we shall exclude weak solutions which are constant in time, to keep the statement of Theorem 1.4 true.

Removing the initial datum assumption $u_0 \equiv 0$ in Theorem 1.3 is even more problematic since the information that (61) is uniformly bounded with respect to T is no longer sufficient to conclude the proof of Theorem 1.3.

We mention that another open problem is if the nonexistence of global in time weak solutions for $p \in (1, \gamma^{-1})$ is true when $n > \sigma$ and $\gamma \in (0, (n - \sigma)/n)$. In the analogous case of the Cauchy problem where the nonlinear memory term is $G(t, u)$, this can be proved setting $R = \log T$ (see [3] for the proof when $\sigma = 1$, i.e. for (13); the proof can be easily generalized to general powers σ). With this choice the inequality

$$I_R \lesssim T^{1-\alpha p'} R^n (T^{-2p'} + T^{-p'} R^{-\sigma p'} + R^{-2\sigma p'}),$$

leads to $I_{\log T} \rightarrow 0$ as $T \rightarrow \infty$, provided that $\alpha p' > 1$. Namely, the intent is to take a very small growth of R with respect to T so that the nonexistence result follows from the scaling of the term u_{tt} alone.

However, this approach does not seem to work for model (1). Indeed, writing (53) as

$$I_R \lesssim T^{1-(\alpha-1)p'} R^n (T^{-2p'} + T^{-p'} R^{-\sigma p'} + R^{-2\sigma p'}),$$

we see that for a given increasing function $R = R(T)$, we may determine that $I_{R(T)} \rightarrow 0$ as $T \rightarrow \infty$ if, and only if

$$\lim_{T \rightarrow \infty} T^{1-(\alpha-1)p'} R(T)^n (T^{-2p'} + R(T)^{-2\sigma p'}) = 0.$$

Both $1 - (\alpha - 1)p' = 1 + \gamma p'$ and n are positive powers, so that the previous strategy, in which we privilege one variable, does not work.

This approach also does not work if one consider the L^m theory, trying to generalize the proof of Theorem 1.4. On the other hand, we notice that in Theorem 1.4, the restriction $m < n^2/(n^2 - \sigma^2)$ does not appear. As a consequence, the range for nonexistence of global in time weak solutions under the assumption (23) also covers some values of $\gamma < (n - \sigma)/\sigma$. More precisely,

$$\gamma^{-1} > \tilde{p}_m(\sigma, n) = 1 + \frac{(2 - \gamma)\sigma m}{n} \iff 0 < \gamma < \frac{n + 2\sigma m - \sqrt{n^2 + (2\sigma m)^2}}{2\sigma m}.$$

REFERENCES

- [1] Cazenave T., Dickstein F., Weissler F.B., An equation whose Fujita critical exponent is not given by scaling, *Nonlinear Anal.* **68**, 862–874 (2008).
- [2] Cui, S.: Local and global existence of solutions to semilinear parabolic initial value problems. *Nonlinear Anal.* **43**, 293–323 (2001)
- [3] D’Abbicco M., A wave equation with structural damping and nonlinear memory, *Nonlinear Differential Equations and Applications*, **21** (5), 751–773 (2014) ISSN: 1021-9722; <http://dx.doi.org/10.1007/s00030-014-0265-2>
- [4] D’Abbicco, M.: The influence of a nonlinear memory on the damped wave equation. *Nonlinear Anal.* **95**, 130–145 (2014). doi:10.1016/j.na.2013.09.006
- [5] D’Abbicco, M., A new critical exponent for the heat and damped wave equations with nonlinear memory and not integrable data. To appear in *Anomalies in PDE’s*, Cicognani, M., Del Santo, D., Parmeggiani, A., Reissig, M. (eds), 2020. In: Springer INdAM Series.
- [6] D’Abbicco M., Ebert M.R., A classification of structural dissipations for evolution operators, *Math. Methods Appl. Sci.* **39**, 2558–2582 (2016). <http://dx.doi.org/10.1002/mma.3713>.
- [7] D’Abbicco M., Ebert M., A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations, *Nonlinear Anal.* **149**, 1–40 (2017).
- [8] D’Abbicco M., Ebert M.R., The critical exponent for nonlinear damped σ -evolution equations, <https://arxiv.org/abs/2005.10946>.
- [9] D’Abbicco, M., Lucente, S.: The beam equation with nonlinear memory. *Z. Angew. Math. Phys.* **67**, 60 (2016). <https://doi.org/10.1007/s00033-016-0655-x>
- [10] D’Abbicco M., Reissig M., Semilinear Structural Damped Waves, *Mathematical Methods in Applied Sciences*, **37** 1570–1592 (2014), <http://dx.doi.org/10.1002/mma.2913>
- [11] Dannawi I., Kirane M. and Fino A. Z., Finite time blow-up for damped wave equations with space-time dependent potential and nonlinear memory, *Nonlinear Differ. Equ. Appl.*, 25–38 (2018).
- [12] Fino, A.: Critical exponent for damped wave equations with nonlinear memory. *Nonlinear Anal.* **74**, 5495–5505 (2011).
- [13] Fino, A.Z., Kirane, M.: Qualitative properties of solutions to a time-space fractional evolution equation. *Quart. Appl. Math.* **70**(1), 133–157 (2012). <https://doi.org/10.1090/S0033-569X-2011-01246-9>
- [14] Fujiwara K., A note for the global nonexistence of semirelativistic equations with nongauge invariant power type nonlinearity, *Math. Methods Appl. Sci.* **41** no. 13, 4955–4966 (2018).
- [15] Ikehata R., Asymptotic profiles for wave equations with strong damping, *J. Differential Equations* **257**, 2159–2177 (2014). <http://dx.doi.org/10.1016/j.jde.2014.05.031>
- [16] P. Marcati, K. Nishihara, The $L^p - L^q$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, *J. Differential Equations* **191**, 445–469 (2003).

- [17] T. Narazaki, $L^p - L^q$ estimates for damped wave equations and their applications to semilinear problem, *J. Math. Soc. Japan* **56**, 586–626 (2004).
- [18] Nishihara K., $L^p - L^q$ estimates for solutions to the damped wave equations in 3-dimensional space and their applications, *Math. Z.* **244**, 631–649 (2003).
- [19] Duong P. T. , Kainane M., Reissig M., Global existence for semi-linear structurally damped σ -evolution models, *J. Math. Anal. Appl.* **431**, 569–596 (2015). <http://dx.doi.org/10.1016/j.jmaa.2015.06.001>.
- [20] Zhang, Q., Li, Y.: The critical exponent for a time fractional diffusion equation with nonlinear memory. *Math Methods Appl. Sci.* 41(16), 6443-6456 (2018). <https://doi.org/10.1002/mma.5169>

MARCELLO D'ABBICCO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BARI, VIA E. ORABONA 4, 70125 BARI, ITALY

Email address: `marcello.dabbicco@uniba.it`

GIOVANNI GIRARDI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BARI, VIA E. ORABONA 4, 70125 BARI, ITALY

Email address: `giovanni.girardi@uniba.it`