Fractional double-phase patterns: concentration and multiplicity of solutions

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FRACTIONAL DOUBLE-PHASE PATTERNS: CONCENTRATION AND MULTIPLICITY OF SOLUTIONS

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Résumé. We consider the following class of fractional problems with unbalanced growth :

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon\,x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1)$, $2 \le p < q < \frac{N}{s}$, $(-\Delta)_t^s$ (with $t \in \{p,q\}$) is the fractional t-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential satisfying local conditions, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity with subcritical growth. Applying suitable variational and topological arguments, we obtain multiple positive solutions for $\varepsilon > 0$ sufficiently small as well as related concentration properties, in relationship with the set where the potential V attains its minimum.

1. Introduction and the main result

1.1. Double-phase problems : an overview.

The present paper was motivated by recent fundamental progress in the mathematical analysis of various nonlinear patterns with unbalanced growth. To the best of our knowledge, the first studies in this field are due to Ball [15, 16] who was interested in models arising in nonlinear elasticity and their qualitative properties (cavitations, discontinuous equilibrium solutions, etc.).

We start by recalling some basic facts concerning double-phase problems. Let $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) be a bounded domain with smooth boundary. Let $u : \Omega \to \mathbb{R}^N$ denote the displacement and assume that Du is the $N \times N$ matrix associated to the deformation gradient. It follows that the total energy is described by an integral of the type

$$I(u) = \int_\Omega f(x, Du(x))dx, \tag{1.1}$$

where the potential $f = f(x,\xi) : \Omega \times \mathbb{R}^{N \times N} \to \mathbb{R}$ is quasiconvex with respect to the second variable.

Ball [15, 16] was interested in potentials given by

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ denotes the determinant of the $N \times N$ matrix ξ. It is assumed that g and h are nonnegative convex functions satisfying the growth hypotheses

$$g(\xi) \ge c_1 |\xi|^p \quad \text{and} \quad \lim_{t \to +\infty} h(t) = +\infty,$$

where $c_1 > 0$ and $1 < p \le N$. We point out that the assumption $p \le N$ was necessary in order to study the existence of cavities for equilibrium solutions, that is, minima of the energy functional (1.1) which are discontinuous at one point where a cavity appears. In fact, every function u with finite energy belongs to the function space $W^{1,p}(\Omega, \mathbb{R}^N)$, hence it is continuous if $p > N$.

Accordingly, Marcellini [53, 54] considered functions $f = f(x, \xi)$ with different growth near the origin and at infinity (unbalanced growth), which satisfy the hypothesis

$$c_1 \, |\xi|^p \le |f(x, \xi)| \le c_2 \, (1 + |\xi|^q) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 \le p \le q$. Regularity and existence of solutions of elliptic equations with (p, q)–growth conditions were studied in [54].

The analysis of non-autonomous energy functionals with energy density changing its ellipticity and growth properties according to the point was developed in several remarkable papers by Mingione *et al.* [17–19, 21, 29, 30]. These contributions are related to the works of Zhikov [73, 74], and they describe the nature of certain phenomena arising in nonlinear elasticity. For instance, Zhikov was interested in providing models for strongly anisotropic materials in the framework of homogenization. The associated functionals also demonstrated their importance in the study of duality theory as well as in the context of the Lavrentiev phenomenon [74]. In relationship with these research directions, Zhikov introduced three different model functionals, mainly in the context of the Lavrentiev phenomenon. These models are the following :

$$
\begin{aligned}
\mathcal{M}(u) \quad &:= \int_\Omega c(x)|Du|^2 dx, \quad 0 < 1/c(\cdot) \in L^t(\Omega), \ t > 1 \\
\mathcal{V}(u) \quad &:= \int_\Omega |Du|^{p(x)} dx, \quad 1 < p(x) < \infty \qquad\qquad (1.2)\\
\mathcal{P}_{p,q}(u) \quad &:= \int_\Omega (|Du|^p + a(x)|Du|^q) \, dx, \quad 0 \le a(x) \le L, \ 1 < p < q.
\end{aligned}
$$

The functional $\mathcal{M}$ is characterized by a loss of ellipticity on the subset of Ω where the potential c vanishes. This functional has been studied in relationship with nonlinear equations involving Muckenhoupt weights. The functional $\mathcal{V}$ is still the object of great interest nowadays and several relevant papers have been developed about it. We refer to Acerbi and Mingione [1] in the context of gradient estimates and contributions to the qualitative analysis of minimizers of nonstandard energy functionals with variable coefficients. The energy functional defined by $\mathcal{V}$ has been used to build consistent models for strongly anisotropic materials : in a material made of different components, the exponent $p(x)$ dictates the geometry of a composite that changes its hardening exponent according to the point. The functional $\mathcal{P}_{p,q}$ defined in (1.2) appears as un upgraded version of $\mathcal{V}$. Again, in this case, the modulating potential $a(x)$ controls the geometry of the composite made by two differential materials, with corresponding hardening exponents p and q.

Following Marcellini's terminology, the functionals defined in (1.2) belong to the realm of energy functionals with nonstandard growth conditions of (p, q)–type. These are functionals of the type defined in relation (1.1), where the energy density satisfies

$$|\xi|^p \le f(x, \xi) \le |\xi|^q + 1, \quad 1 \le p \le q.$$

An alternative relevant example of a functional having (p, q)–growth is given by

$$u \mapsto \int_\Omega |Du|^p \log(1 + |Du|) \, dx, \quad \text{for } p \ge 1,$$

which can be seen as a logarithmic perturbation of the classical p-Dirichlet energy.

The main feature of our paper is the study of a class of *fractional unbalanced double-phase problems*. Such patterns are strictly connected with the analysis of nonlinear problems and stationary waves for models arising in mathematical physics (composite materials, stability of nonlinear damped Kirchhoff systems, fractional quantum mechanics in the study of particles on stochastic fields, fractional superdiffusion, fractional white-noise limit, etc.) ; see, e.g., [64, 65]. For various types of double-phase problems, we refer to the recent papers [14, 32, 55, 56, 62, 63, 72].

1.2. **Statement of the problem and further comments.** In this paper we deal with the existence, multiplicity and concentration behavior of positive solutions for the following class of fractional $p\&q$-Laplacian problems :

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.3)$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1)$, $2 \leq p < q < \frac{N}{s}$, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions. The operator $(-\Delta)_t^s$, with $t \in \{p, q\}$, is the so called fractional t-Laplacian operator which, up to normalization factors, may be defined for every function $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_t^s u(x) = 2 \lim_{r \to 0} \int_{\mathbb{R}^N \setminus \mathcal{B}_r(x)} \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x-y|^{N+st}} dy \quad (x \in \mathbb{R}^N).$$

Problems of this type appear in the case of two different materials that involve power hardening exponents p and q. In this case, the fractional operator $(-\Delta)_t^s$ (with $t \in \{p,q\}$) described the geometry of a composite of two materials.

We point out that in these last years, a considerable attention has been devoted to the study of nonlocal problems driven by fractional operators, both for their interesting theoretical structure and in view of concrete applications, such as, for instance, thin obstacle problem, finance, phase transitions, optimization, anomalous diffusion, conservation laws, image processing, and many others. For more details, we refer the interested reader to [37, 58] for an elementary introduction on this subject.

In the local case $s = 1$, (1.3) becomes a $p\&q$-Laplacian equation of the form :

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x, u) \text{ in } \mathbb{R}^N.$$

The above class of problems comes from a general reaction-diffusion system :

$$u_t = \text{div}(D(u)\nabla u) + c(x, u) \text{ and } D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

which has a wide spectrum of applications in physics and related sciences such as biophysics, plasma physics, solid state physics, and chemical reaction design. In such applications, u represents a concentration, $\text{div}(D(u)\nabla u)$ is the diffusion with diffusion coefficient $D(u)$, and the reaction term $c(x, u)$ relates to source and loss processes. Usually, in chemical and biological applications, the reaction term $c(x, u)$ is a polynomial of u with variable coefficients ; see [27].

Several results for $p\&q$-Laplacian problems set in bounded domains and in the whole of $\mathbb{R}^N$ can be found in [?, 2, 20, 24, 41, 42, 45, 47, 52, 60] and the references therein.

On the other hand, in the nonlocal framework, only few recent works deal with fractional $p\&q$-Laplacian problems. For instance, in [26] the authors studied existence, nonexistence and multiplicity for a nonlocal $p\&q$-subcritical problem. The first author [11] proved an existence result for a critical fractional $p\&q$-problem via mountain pass theorem. The existence of infinitely many nontrivial solutions for a class of fractional $p\&q$-equations involving concave-critical nonlinearities in bounded domains has been investigated in [23]. In [33] the authors establish a Hölder regularity result for nonlocal double phase equations. We also mention [3, 13, 46] for other interesting results.

We stress that, when $p = q = 2$, equation (1.3) boils down a fractional Schrödinger equation (see [51]) of the type

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad (1.4)$$

which has been extensively considered by several authors ; see [40, 44, 57, 67]. In particular way, a great attention has been devoted to the study of solutions of (1.4) which concentrate around critical points of the potential V as $\varepsilon \to 0$; see [6, 9, 10, 31, 39].

When $p = q \neq 2$ and $\varepsilon = 1$ in (1.3), then we obtain a class of fractional p-Laplacian equations :

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) \text{ in } \mathbb{R}^N,$$

for which several existence and multiplicity results have been obtained in this last decade; see for instance [8, 12, 34, 61, 65, 71] and the references therein, and [36, 48] for some interesting regularity results. From of the mathematical point of view, the fractional p-Laplacian has a great attractive since two phenomena are present in it: the nonlinearity of the operator and its nonlocal character. Indeed, some standard tools used to investigate the linear case $p = 2$ seem not to be trivially adaptable in the case $p \neq 2$ due to the lack of Hilbertian structure of $W^{s,p}(\mathbb{R}^N)$ for $p \neq 2$.

1.3. **Multiplicity and concentration of solutions.**

Particularly motivated by [6,9], in this paper we are interested in the multiplicity and concentration behavior of positive solutions to (1.3).

Next, we introduce the assumptions on the potential V and the nonlinearity f. Along the paper, we assume that $V \in \mathcal{C}^0(\mathbb{R}^N, \mathbb{R})$ satisfies the following del Pino-Felmer type conditions [35]:

(V_1) there exists $V_0 > 0$ such that $V_0 := \inf_{x \in \mathbb{R}^N} V(x)$,

(V_2) there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial \Lambda} V \quad \text{and} \quad M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset,$$

while $f \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ fulfills the following hypotheses:

(f_1) $\lim_{|t| \to 0} \dfrac{|f(t)|}{|t|^{p-1}} = 0$;

(f_2) there exists $\nu \in (q, q_s^*)$ such that $\lim_{|t| \to \infty} \dfrac{|f(t)|}{|t|^{\nu-1}} = 0$, where $q_s^* := \dfrac{Nq}{N-sq}$;

(f_3) there exists $\vartheta \in (q, q_s^*)$ such that $0 < \vartheta F(t) := \vartheta \displaystyle\int_0^t f(\tau)\, d\tau \leq t f(t)$ for all $t > 0$;

(f_4) the map $t \mapsto \dfrac{f(t)}{t^{q-1}}$ is increasing for $t > 0$.

Due to the fact that we look for positive solutions to (1.3), we assume that $f(t) = 0$ for $t \leq 0$. To be more precise, in [35] the authors assumed (V_1), and

$$\inf_\Lambda V < \min_{\partial \Lambda} V \tag{1.5}$$

instead of (V_2), and they showed that the following nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x) u = f(u) \text{ in } \mathbb{R}^N, \tag{1.6}$$

admits a single-peak solution which concentrates around the minimum points of V in Λ. Their result can be seen as the localized version of the result of Rabinowitz [66] and Wang [69], who proved the existence of positive solutions to (1.6) for small $\varepsilon > 0$, by assuming the following global condition

$$\liminf_{|x| \to \infty} V(x) > V_0. \tag{1.7}$$

The relevance of (1.5) is that no restriction on the global behavior of V is required other that (V_1), and, in particular, V is not required to be bounded or to belong to a Kato class.

Later, Cingolani and Lazzo [28], assuming (1.7), proved that the multiplicity of solutions to (1.6) is related to topology richness of the set $K := \{x \in \mathbb{R}^N : V(x) = V_0\}$. Subsequently, motivated by [28, 35], in [4] the authors assumed (V_1)-(V_2) and obtained multiple positive solutions for a quasilinear p-Laplacian problem.

In this paper, in order to get a multiplicity result for (1.3), we assume (V_1)-(V_2) as in [4]. Since we aim to relate the number of solutions of (1.3) with the topology of the set M of minima of the potential, it is worth recalling that if Y is a given closed set of a topological space X, we denote by $cat_X(Y)$ the Ljusternik-Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y; see [70] for more details.

Therefore, our main theorem can be stated as follows:

Theorem 1.1. *Assume that* (V_1)-(V_2) *and* (f_1)-(f_4) *hold true. Then, for any* $\delta > 0$ *such that*

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *problem* (1.3) *has at least* $cat_{M_\delta}(M)$ *positive solutions. Moreover, if* u_ε *denotes one of these solutions and* $x_\varepsilon \in \mathbb{R}^N$ *is a global maximum point of* u_ε, *then*

$$\lim_{\varepsilon \to 0} V(\varepsilon\, x_\varepsilon) = V_0.$$

The proof of Theorem 1.1 is obtained by applying suitable variational and topological arguments. Since no informations on the behavior of V at infinity are available, we adapt the penalization method in [35], which consists in making a suitable modification on f, solving an auxiliary problem and then check that, for $\varepsilon > 0$ small enough, the solutions of the new problem are indeed solutions of the original one. To obtain multiple solutions of the modified problem, we make use of some well-known topological techniques proposed in [22], by means of accurate comparisons between the category of some sublevel sets of the modified functional and the category of the set M. Anyway, due to the fact that the nonlinearity is only continuous, one can not apply standard $\mathcal{C}^1$-Nehari manifold arguments due to the lack of differentiability of the associated Nehari manifold. This difficulty will be overcome by using some abstract critical point results obtained in [68]. Compared with the local case $s = 1$, we point out that our result improves Theorem 1.1 in [2] in which the authors investigated the multiplicity of positive solutions for a very general class of quasilinear problems with potentials satisfying (V_1)-(V_2) and involving $\mathcal{C}^1$-subcritical nonlinearities. Indeed, we can not repeat the same arguments developed in [2] but we take inspiration by some ideas developed in [43,68] (see also [9,12]). However, due to the combination of two nonhomogeneous fractional involved operators, our analysis is rather delicate and more fine estimates will be needed to achieve our result. Moreover, in order to show that the solutions of the modified problem are also solutions to (1.3), we can not adapt in our setting the arguments in [2,5,45], due to the nonlocal character of fractional $p\&q$-Laplacian operators, and also fails the strategy used in [6] to study (1.4) based on some useful estimates coming from the good properties of the Bessel kernel established in [40].

In our situation, we develop a suitable Moser iteration scheme [59] to deduce L^∞-estimates and we establish a Hölder regularity result which extends in the fractional $p\&q$-case the interior regularity result proved in [48] (see also [36]) for the fractional p-Laplacian. Indeed, the restriction $p \geq 2$ is related to the use of this regularity result because all variational and topological arguments used to obtain the existence and multiplicity of solutions for the modified problem hold for all $1 < p < q < \frac{N}{s}$. As far as we know, the multiplicity of concentrating solutions to the fractional $p\&q$-Laplacian problem obtained in this paper has not been established in the literature. Moreover, we suspect that our results can be extended to a more general class of anisotropic non-local problems. They should be the local analog of those that in the local case are given by functionals of the type

$$w \mapsto \int_\Omega \varphi_1(|Dw|) + \varphi_2(|Dw|)\, dx,$$

where the conditions satisfied by $\varphi_i(t)$ are typically given by

$$1 < i \leq \frac{\varphi_i'(t)t}{\varphi_i(t)} \leq s. \tag{1.8}$$

In the present case it is

$$\varphi_1(t) = t^p \quad \varphi_2(t) = t^q.$$

The corresponding nonlocal version can be found emulating the nonlinear operators considered in [50]. In this situation, conditions (f_1)-(f_4) on the nonlinearity f have to be modified in terms of the numbers i and s appearing in (1.8); see for instance [2]. Further classes of anisotropic operators that can be considered are described in [21].

The structure of the paper is the following. In Section 2, we introduce some notations and we prove some regularity results for $(-\Delta)_p^s + (-\Delta)_q^s$. Next, we consider the modified problem and we prove a first existence result for it. In Section 3 we study the limiting problem associated with (1.3) and we introduce some tools needed to obtain a multiplicity result for the auxiliary problem. The last section is devoted to the proof of Theorem 1.1.

2. Preliminaries

2.1. **Notations and useful results.** Let $s \in (0,1)$ and $1 < p < \infty$. Let us indicate by $W^{s,p}(\mathbb{R}^N)$ the set of functions $u \in L^p(\mathbb{R}^N)$ such that

$$[u]_{s,p}^p := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dxdy < \infty$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := (|u|_p^p + [u]_{s,p}^p)^{\frac{1}{p}}.$$

For $u, v \in W^{s,p}(\mathbb{R}^N)$, we put

$$\langle u, v \rangle_{s,p} := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, dxdy.$$

When $N > sp$, we also know that, for any $t \in [p, p_s^*]$, there exists $C > 0$ such that

$$|u|_t \leq C\|u\|_{W^{s,p}(\mathbb{R}^N)} \quad \forall u \in W^{s,p}(\mathbb{R}^N), \tag{2.1}$$

and that the embedding $W^{s,p}(\mathbb{R}^N) \subset L^t(K)$ is compact for all $t \in [1, p_s^*]$ and any compact $K \subset \mathbb{R}^N$. In order to deal with fractional $p\&q$-Laplacian problems, we consider the space

$$\mathcal{W} := W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$$

endowed with the norm

$$\|u\|_{\mathcal{W}} := \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}.$$

Since $W^{s,r}(\mathbb{R}^N)$, with $1 < r < \infty$, is a separable reflexive Banach space, we deduce that $\mathcal{W}$ is a separable reflexive Banach space.

In what follows, we establish some useful regularity results for fractional $p\&q$-Laplace problems. To do this we follow the approach in [48] used to study the regularity for the fractional p-Laplacian.

From now on, we will assume that $2 \leq p < q < \frac{N}{s}$. We define

$$W^{s,t}(\Omega) := \{u \in L^t(\Omega) : \|u\|_{W^{s,t}(\Omega)} < \infty\},$$

where

$$\|u\|_{W^{s,t}(\Omega)}^t := \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dxdy + \int_{\Omega} |u|^t \, dx,$$

and

$$W_0^{s,t}(\Omega) := \{u \in W^{s,t}(\Omega) : u = 0 \text{ in } \Omega^c\}.$$

If $\Omega \subset \mathbb{R}^N$ is bounded we set

$$\widetilde{W}^{s,t}(\Omega) := \left\{u \in L_{loc}^t(\mathbb{R}^N) : \exists U \supseteq \Omega : \|u\|_{W^{s,t}(\Omega)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{t-1}}{(1+|x|)^{N+st}} \, dx < \infty\right\},$$

and if Ω is unbounded, we set

$$\widetilde{W}_{loc}^{s,t}(\Omega) := \{u \in L_{loc}^t(\mathbb{R}^N) : u \in \widetilde{W}^{s,t}(\Omega') \text{ for any bounded } \Omega' \subseteq \Omega\}.$$

Let $u : \mathbb{R}^N \to \mathbb{R}$ be any measurable function. We recall that the non-local tail centered at $x \in \mathbb{R}^N$ with radius $R > 0$ is defined (see [36]) as :

$$\text{Tail}_t(u; x; R) := \left(R^{st} \int_{\mathcal{B}_R^c(x)} \frac{|u(y)|^{t-1}}{|x - y|^{N+st}} \, dy \right)^{\frac{1}{t-1}}.$$

When $x_0 = 0$, we write $\text{Tail}_t(u; R) := \text{Tail}_t(u; 0; R)$.

Along this section, in order to give advantages in readability, we use the following notation : for all $a \in \mathbb{R}$ and $t > 0$, we set $a^t := |a|^{t-1}a$.

Now we give the following definitions :

Definition 2.1. *Let Ω be bounded, $u \in \widetilde{W}^{s,p}(\Omega) \cap \widetilde{W}^{s,q}(\Omega)$ and $f \in (W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega))^*$. We say that u is a weak solution of $(-\Delta)_p^s u + (-\Delta)_q^s u = f$ in Ω if, for all $\varphi \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$,*

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dxdy + \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{q-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sq}} \, dxdy = \langle f, \varphi \rangle$$

If Ω is unbounded, we say that $u \in \widetilde{W}_{loc}^{s,p}(\Omega) \cap \widetilde{W}_{loc}^{s,q}(\Omega)$ solves $(-\Delta)_p^s u + (-\Delta)_q^s u = f$ in Ω if it does so in any bounded open set $\Omega' \subseteq \Omega$.

The inequality $(-\Delta)_p^s u + (-\Delta)_q^s u \leq f$ weakly in f will mean that

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dxdy + \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{q-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sq}} \, dxdy \leq \langle f, \varphi \rangle$$

for all $\varphi \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$, $\varphi \geq 0$, and similarly for $(-\Delta)_p^s u + (-\Delta)_q^s u \geq f$. Moreover, given $K > 0$ and Ω bounded, we say that $|(-\Delta)_p^s u + (-\Delta)_q^s u| \geq K$ weakly in Ω, if $-K \leq (-\Delta)_p^s u + (-\Delta)_q^s u \leq K$ weakly in Ω. Arguing as in the proof of Lemma 2.3 in [48], we can prove that the above definitions make sense.

The next result can be obtained following the same lines of the proof of Lemma 2.8 in [48].

Lemma 2.1. *Suppose $u \in \widetilde{W}_{loc}^{s,p}(\Omega) \cap \widetilde{W}_{loc}^{s,q}(\Omega)$ solves $(-\Delta)_p^s u + (-\Delta)_q^s u = f$ weakly in Ω, for some $f \in L_{loc}^1(\Omega)$. Let $v \in L_{loc}^1(\Omega)$ be such that*

$$\text{dist}(supp(v), \Omega) > 0, \quad \int_\Omega \frac{|v(x)|^{t-1}}{(1 + |x|^{N+st})} \, dx < \infty \quad \forall t \in \{p, q\},$$

and define for a.e. Lebesgue point $x \in \Omega$ of u

$$h_t(x) = 2 \int_{supp(v)} \frac{(u(x) - u(y) - v(y))^{t-1} - (u(x) - u(y))^{t-1}}{|x - y|^{N+st}} \, dy \quad \forall t \in \{p, q\}.$$

Then, $u + v \in \widetilde{W}_{loc}^{s,p}(\Omega) \cap \widetilde{W}_{loc}^{s,q}(\Omega)$ and it solves $(-\Delta)_p^s(u + v) + (-\Delta)_q^s(u + v) = f + h_p + h_q$ weakly in Ω.

We also have the following comparison principle whose proof is similar to the one in Proposition 2.10 in [48].

Lemma 2.2. *Let Ω be bounded, $u, v \in \widetilde{W}^{s,p}(\Omega) \cap \widetilde{W}^{s,q}(\Omega)$ such that $u \leq v$ in Ω^c and, for all $\varphi \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$, $\varphi \geq 0$ in Ω,*

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dxdy + \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{q-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sq}} \, dxdy$$

$$\leq \iint_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))^{p-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dxdy + \iint_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))^{q-1}(\varphi(x) - \varphi(y))}{|x - y|^{N+sq}} \, dxdy.$$

Then $u \leq v$ in Ω.

Next we prove a weak Harnack-type inequality for non-negative supersolutions.

Theorem 2.1. *There exists universal $\sigma \in (0,1)$, $\tilde{C} > 0$ with the following property : if $u \in \widetilde{W}^{s,p}(\mathcal{B}_{R/3}) \cap \widetilde{W}^{s,q}(\mathcal{B}_{R/3})$ satisfies weakly*

$$\begin{cases} (-\Delta)^s_p u + (-\Delta)^s_q u \geq -K & \text{in } \mathcal{B}_{R/3}, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

for some $K \geq 0$, then

$$\inf_{\mathcal{B}_{R/4}} u \geq \sigma \left(\fint_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} u^{p-1} \, dx \right)^{\frac{1}{p-1}} - \tilde{C}(KR^{sq})^{\frac{1}{q-1}}.$$

Démonstration. The proof follows the same lines of the proof of Theorem 5.2 in [48]. Note that by Lemma 2.1 and the following elementary inequality

$$a^t - (a-b)^t \geq 2^{1-t}b^t \quad \forall a \in \mathbb{R}, b \geq 0, t \geq 1,$$

we have weakly in $\mathcal{B}_{R/3}$

$$\begin{aligned}
(-\Delta)^s_p w + (-\Delta)^s_q w &= (-\Delta)^s_p(\sigma L \varphi_R) + (-\Delta)^s_q(\sigma L \varphi_R) + h_p + h_q \\
&\leq \frac{C_1(\sigma L)^{p-1}}{R^{sp}} + \frac{C_1(\sigma L)^{q-1}}{R^{sq}} - \frac{C_2 L^{p-1}}{R^{sp}} - \frac{C_3 L^{q-1}}{R^{sq}} \\
&\leq -\frac{C_2 L^{p-1}}{2R^{sp}} - \frac{C_3 L^{q-1}}{2R^{sq}} \\
&\leq -\frac{C_3 L^{q-1}}{2R^{sq}} =: -\tilde{C}^{-(q-1)} \frac{L^{q-1}}{R^{sq}},
\end{aligned}$$

provided that

$$\sigma < \min \left\{ 1, \left(\frac{C_2}{2C_1} \right)^{\frac{1}{p-1}}, \left(\frac{C_3}{2C_1} \right)^{\frac{1}{q-1}} \right\}.$$

Here, $L = \left(\fint_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} u^{p-1} \, dx \right)^{\frac{1}{p-1}}$ and $w = \sigma L \varphi_R + \chi_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} u$, where φ_R is as in Theorem 5.2 in [48]. Finally, one uses Lemma 2.2 instead of Proposition 2.10 in [48] to study the case $L > \tilde{C}(KR^{sq})^{\frac{1}{q-1}}$. $\qquad \square$

As in [48], we extend Theorem 2.1 to supersolutions which are only non-negative in a ball. To do this, we introduce a tail term (see [36]).

Lemma 2.3. *There exist $\sigma \in (0,1)$, $\tilde{C} > 0$, and for all $\varepsilon > 0$ a constant $C_\varepsilon > 0$ with the following property : if $u \in \widetilde{W}^{s,p}(\mathcal{B}_{R/3}) \cap \widetilde{W}^{s,q}(\mathcal{B}_{R/3}) \cap L^\infty(\mathbb{R}^N)$ satisfies weakly*

$$\begin{cases} (-\Delta)^s_p u + (-\Delta)^s_q u \geq -K & \text{in } \mathcal{B}_{R/3}, \\ u \geq 0 & \text{in } \mathcal{B}_R, \end{cases}$$

for some $K \geq 0$, then there exists $M > 0$ such that

$$\inf_{\mathcal{B}_{R/4}} u \geq \sigma \left(\fint_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} u^{p-1} \, dx \right)^{\frac{1}{p-1}} - \tilde{C}(KR^{sq})^{\frac{1}{q-1}} - C_\varepsilon \mathrm{Tail}_q(u_-; R) - \varepsilon \sup_{\mathcal{B}_R} u - R^{\frac{(q-p)s}{q-1}} M(\varepsilon + C_\varepsilon).$$

Démonstration. The proof follows the same lines of the proof of Lemma 5.3 in [48]. For this reason, we only point out the differences. Applying Lemma 2.1 to functions u and $v = u_-$ so that $u + v = u_+$ and $\Omega = \mathcal{B}_{R/3}$, we have weakly in $\mathcal{B}_{R/3}$

$$(-\Delta)_p^s u_+ + (-\Delta)_q^s u_+ = (-\Delta)_p^s u + (-\Delta)_q^s u + h_p + h_q$$

$$= -K + 2 \int_{\mathcal{B}_{R/3}^c} \frac{(u(x) - u(y) - u_-(y))^{p-1} - (u(x) - u(y))^{p-1}}{|x-y|^{N+sp}} \, dy$$

$$+ 2 \int_{\mathcal{B}_{R/3}^c} \frac{(u(x) - u(y) - u_-(y))^{q-1} - (u(x) - u(y))^{q-1}}{|x-y|^{N+sq}} \, dy$$

$$\geq -K + 2 \int_{\{u<0\}} \frac{(u(x))^{p-1} - (u(x) - u(y))^{p-1}}{|x-y|^{N+sp}} \, dy$$

$$+ 2 \int_{\{u<0\}} \frac{(u(x))^{q-1} - (u(x) - u(y))^{q-1}}{|x-y|^{N+sq}} \, dy$$

$$\geq -K + C \int_{\{u<0\}} \frac{(u(x))^{p-1} - (u(x) - u(y))^{p-1}}{|y|^{N+sp}} \, dy$$

$$+ C \int_{\{u<0\}} \frac{(u(x))^{q-1} - (u(x) - u(y))^{q-1}}{|y|^{N+sq}} \, dy,$$

where in the last inequality we used that $|x - y| \geq |y| - |x| \geq \frac{2}{3}|y|$ for all $x \in \mathcal{B}_{R/3}$ and $y \in \{u < 0\} \subset \mathcal{B}_R^c$. Now, using

$$(a+b)^t - a^t \leq \theta a^t + C_\theta b^t \quad \forall a, b \geq 0, t \geq 1, C_\theta \to \infty \text{ as } \theta \to 0,$$

we can see that for all $\theta > 0$ there exists $C_\theta > 0$ such that weakly in $\mathcal{B}_{R/3}$

$$(-\Delta)_p^s u_+ + (-\Delta)_q^s u_+ \geq -K - \theta (\sup_{\mathcal{B}_R} u)^{p-1} \int_{\mathcal{B}_R^c} \frac{dy}{|y|^{N+sp}} - \frac{C_\theta}{R^{sp}} \text{Tail}_p(u_-; R)^{p-1}$$

$$- \theta (\sup_{\mathcal{B}_R} u)^{q-1} \int_{\mathcal{B}_R^c} \frac{dy}{|y|^{N+sq}} - \frac{C_\theta}{R^{sq}} \text{Tail}_q(u_-; R)^{q-1}$$

$$\geq -K - \frac{C\theta}{R^{sp}} (\sup_{\mathcal{B}_R} u)^{p-1} - \frac{C_\theta}{R^{sp}} \text{Tail}_p(u_-; R)^{p-1}$$

$$- \frac{C\theta}{R^{sq}} (\sup_{\mathcal{B}_R} u)^{q-1} - \frac{C_\theta}{R^{sq}} \text{Tail}_q(u_-; R)^{q-1} =: -\tilde{K}.$$

Hence, applying Theorem 2.1 to u_+ we have

$$\inf_{\mathcal{B}_{R/4}} u \geq \sigma \left(\fint_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} u^{p-1} \, dx \right)^{\frac{1}{p-1}} - \tilde{C}(\tilde{K} R^{sq})^{\frac{1}{q-1}}. \tag{2.2}$$

On the other hand, recalling that $\|u\|_{L^\infty(\mathbb{R}^N)} \leq C_0$, it follows that

$$(\sup_{\mathcal{B}_R} u)^{\frac{p-1}{q-1}} \leq C_0^{\frac{p-1}{q-1}} \text{ and } (\text{Tail}_p(u_-; R))^{\frac{p-1}{q-1}} \leq \left(\frac{\omega_{N-1}}{sp} \right)^{\frac{1}{q-1}} C_0^{\frac{p-1}{q-1}}.$$

Take $M = \max \left\{ 1, \left(\frac{\omega_{N-1}}{sp} \right)^{\frac{1}{q-1}} \right\} C_0^{\frac{p-1}{q-1}}$. Therefore, using $0 < \frac{1}{q-1} \leq 1$ ($q > p \geq 2$) and that

$$(a+b)^t \leq a^t + b^t \quad \forall a, b \geq 0, 0 < t \leq 1,$$

we obtain

$$
\begin{aligned}
(\tilde{K}R^{sq})^{\frac{1}{q-1}} &\leq (KR^{sq})^{\frac{1}{q-1}} + (C\theta)^{\frac{1}{q-1}}(\sup_{\mathcal{B}_R} u) + C_\theta^{\frac{1}{q-1}}\mathrm{Tail}_q(u_-;R) \\
&\quad + (C\theta R^{s(q-p)})^{\frac{1}{q-1}}(\sup_{\mathcal{B}_R} u)^{\frac{p-1}{q-1}} + (C_\theta R^{s(q-p)})^{\frac{1}{q-1}}(\mathrm{Tail}_p(u_-;R))^{\frac{p-1}{q-1}} \\
&\leq (KR^{sq})^{\frac{1}{q-1}} + (C\theta)^{\frac{1}{q-1}}(\sup_{\mathcal{B}_R} u) + C_\theta^{\frac{1}{q-1}}\mathrm{Tail}_q(u_-;R) \\
&\quad + R^{\frac{(q-p)s}{q-1}}M((C\theta)^{\frac{1}{q-1}} + C_\theta^{\frac{1}{q-1}}).
\end{aligned}
\tag{2.3}
$$

Then, for any $\varepsilon > 0$ and $\theta > 0$ such that $\tilde{C}(C\theta)^{\frac{1}{q-1}} < \varepsilon$, it follows from (2.2) and (2.3) that

$$
\begin{aligned}
\inf_{\mathcal{B}_{R/4}} u &\geq \sigma \left(\int_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} u^{p-1}\, dx \right)^{\frac{1}{p-1}} - \tilde{C}(KR^{sq})^{\frac{1}{q-1}} \\
&\quad - C_\varepsilon \mathrm{Tail}_q(u_-;R) - \varepsilon \sup_{\mathcal{B}_R} u - R^{\frac{(q-p)s}{q-1}}M(\varepsilon + C_\varepsilon).
\end{aligned}
$$

$\square$

Remark 2.1. *Note that differently from Lemma 5.3 in [48], the presence of fractional p&q-Laplacians forces to require that $u \in L^\infty(\mathbb{R}^N)$ in order to estimate the "additional terms" $(\sup_{\mathcal{B}_R} u)^{\frac{p-1}{q-1}}$ and $(\mathrm{Tail}_p(u_-;R))^{\frac{p-1}{q-1}}$.*

In view of the above results, we can deduce an estimate of the oscillation of a bounded function u such that $(-\Delta)_p^s u + (-\Delta)_q^s u$ is locally bounded. For $R > 0$ and $x_0 \in \mathbb{R}^N$, we define

$$
Q(u;x_0;R) := \|u\|_{L^\infty(B_R(x_0))} + \mathrm{Tail}_q(u;x_0;R)
$$

and if $x_0 = 0$ we use the notation $Q(u;R) := Q(u;0;R)$. In what follows, for a universal constant, we mean a constant $C = C(N,s,p,q)$, that is depends only on N,s,p,q.

Theorem 2.2. *There exist universal $\alpha \in (0,1)$, $C > 0$ with the following property : if if $u \in \widetilde{W}^{s,p}(\mathcal{B}_{R_0}) \cap \widetilde{W}^{s,q}(\mathcal{B}_{R_0}) \cap L^\infty(\mathbb{R}^N)$ satisfies $|(-\Delta)_p^s u + (-\Delta)_q^s u| \leq K$ weakly in $\mathcal{B}_{R_0}$ for some $R_0 > 0$, then for all $r \in (0, R_0)$,*

$$
\mathrm{osc}_{\mathcal{B}_r} u \leq C \left[(KR_0^{sq})\frac{1}{q-1} + Q(u;R_0) + R_0^{\frac{(q-p)s}{q-1}} \right] \frac{r^\alpha}{R_0^\alpha}.
$$

Démonstration. We argue as in the proof of Theorem 5.4 in [48]. For any $j \geq 0$, we denote by $R_j := \frac{R_0}{4^j}$, $\mathcal{B}_j := \mathcal{B}_{R_j}$. As in [48], we verify that there exists a universal $\alpha \in (0,1)$ and a number $\lambda > 0$ (depending on all the data), a nondecreasing sequence $\{m_j\}_{j\in\mathbb{N}}$, a nonincreasing sequence $\{M_j\}_{j\in\mathbb{N}}$ such that

$$
m_j \leq \inf_{\mathcal{B}_j} u \leq \sup_{\mathcal{B}_j} u \leq M_j, \quad M_j - m_j = \lambda R_j^\alpha.
$$

Note that, Step 0 (that is $j = 0$) is similar to the one in [48]. Concerning Inductive step in [48], we proceed as follows. Fix $\sigma \in (0,1)$. Then, using Lemma 2.3, it follows that formula (5.5) in [48]

becomes

$$
\begin{aligned}
\sigma(M_j - m_j) \le{} & \inf_{\mathcal{B}_{j+1}}(M_j - u) + \inf_{\mathcal{B}_{j+1}}(u - m_j) + 2\tilde{C}(KR_j^{sq})^{\frac{1}{q-1}} \\
& + C_\varepsilon[\mathrm{Tail}_q((M_j - u)_-; R_j) + \mathrm{Tail}_q((u - m_j)_-; R_j)] \\
& + \varepsilon[\sup_{\mathcal{B}_j}(M_j - u) + \sup_{\mathcal{B}_j}(u - m_j)] + 2M(C_\varepsilon + \varepsilon)R_j^{\frac{s(q-p)}{q-1}}.
\end{aligned}
$$

Take $\varepsilon = \frac{\sigma}{4}$ and set $C := \max\{2\tilde{C}, C_\varepsilon, 2M(C_\varepsilon + \varepsilon)\}$. Then, formula (5.6) in [48] becomes

$$
\begin{aligned}
\mathrm{osc}_{\mathcal{B}_{j+1}} u \le{} & \left(1 - \frac{\sigma}{2}\right)(M_j - m_j) \\
& + C[(KR_j^{sq})^{\frac{1}{q-1}} + \mathrm{Tail}_q((M_j - u)_-; R_j) + \mathrm{Tail}_q((u - m_j)_-; R_j) + R_j^{\frac{s(q-p)}{q-1}}].
\end{aligned}
$$

Choosing $\alpha < \frac{s(q-p)}{q-1} < \frac{sq}{q-1}$, as in [48], we can see that

$$
\mathrm{Tail}_q((u - m_j)_-; R_j) \le C\left[\lambda S(\alpha)^{\frac{1}{q-1}} + \frac{Q(u; R_0)}{R_0^\alpha}\right]R_j^\alpha
$$

where $S(\alpha) := \sum_{h=1}^\infty \frac{(4^{\alpha h} - 1)^{q-1}}{4^{sqh}} \to 0$ as $\alpha \to 0^+$. A similar estimate holds for $\mathrm{Tail}_q((M_j - u)_-; R_j)$. Consequently, noting that $R_j = R_0/4^j$ and $R_{j+1} = R_j/4$, we get

$$
\begin{aligned}
\mathrm{osc}_{\mathcal{B}_{j+1}} u \le{} & \left(1 - \frac{\sigma}{2}\right)\lambda R_j^\alpha + C\left[(KR_j^{sq})^{\frac{1}{q-1}} + \lambda S(\alpha)^{\frac{1}{q-1}}R_j^\alpha + \frac{Q(u; R_0)}{R_0^\alpha}R_j^\alpha + R_j^{\frac{s(q-p)}{q-1}}\right] \\
\le{} & 4^\alpha\left[\left(1 - \frac{\sigma}{2}\right) + CS(\alpha)^{\frac{1}{q-1}}\right]\lambda R_{j+1}^\alpha + 4^\alpha C\left[R_0^{\frac{sq}{q-1} - \alpha}K^{\frac{1}{q-1}} + \frac{Q(u; R_0)}{R_0^\alpha} + R_0^{\frac{s(q-p)}{q-1} - \alpha}\right]R_{j+1}^\alpha.
\end{aligned}
$$

Choose $\alpha \in (0, \frac{s(q-p)}{q-1})$ such that

$$
4^\alpha\left[\left(1 - \frac{\sigma}{2}\right) + CS(\alpha)^{\frac{1}{q-1}}\right] \le 1 - \frac{\sigma}{4}
$$

and we set

$$
\lambda := \frac{4^{\alpha+1}}{\sigma}C\left[R_0^{\frac{sq}{q-1} - \alpha}K^{\frac{1}{q-1}} + \frac{Q(u; R_0)}{R_0^\alpha} + R_0^{\frac{s(q-p)}{q-1} - \alpha}\right], \tag{2.4}
$$

so that

$$
\mathrm{osc}_{\mathcal{B}_{j+1}} u \le \lambda R_{j+1}^\alpha.
$$

Then, we may pick m_{j+1} and M_{j+1} such that

$$
m_j \le m_{j+1} \le \inf_{\mathcal{B}_{j+1}} u \le \sup_{\mathcal{B}_{j+1}} u \le M_{j+1} \le M_j, \quad M_{j+1} - m_{j+1} = \lambda R_{j+1}^\alpha.
$$

Now, fix $r \in (0, R_0)$ and find an integer $j \ge 0$ such that $R_{j+1} \le r < R_j$, thus $R_j \le 4r$. Hence, by the above claim and (2.4) we can deduce that

$$
\mathrm{osc}_{\mathcal{B}_r} u \le \mathrm{osc}_{\mathcal{B}_j} u \le \lambda R_j^\alpha \le C\left[R_0^{\frac{sq}{q-1}}K^{\frac{1}{q-1}} + Q(u; R_0) + R_0^{\frac{s(q-p)}{q-1}}\right]\frac{r^\alpha}{R_0^\alpha}.
$$

$\square$

In the light of Theorem 2.2, we can argue as in the proof of Corollary 5.5 in [48] to obtain the following result :

Corollary 2.1. *There exist universal* $\alpha \in (0,1)$, $C > 0$ *with the following property : if if* $u \in \widetilde{W}^{s,p}(\mathcal{B}_{2R_0}(x_0)) \cap \widetilde{W}^{s,q}(\mathcal{B}_{2R_0}(x_0)) \cap L^{\infty}(\mathbb{R}^N)$ *satisfies* $|(-\Delta)_p^s u + (-\Delta)_q^s u| \leq K$ *weakly in* $\mathcal{B}_{2R_0}(x_0)$,

$$[u]_{\mathcal{C}^\alpha(\mathcal{B}_{R_0}(x_0))} \leq C \left[(KR_0^{sq})^{\frac{1}{q-1}} + Q(u; x_0; 2R_0) + R_0^{\frac{(q-p)s}{q-1}} \right] R_0^{-\alpha},$$

where $[u]_{\mathcal{C}^\alpha(\Omega)} = \sup_{x,y \in \Omega, x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

2.2. **The modified problem.** Now, we adapt the penalization method introduced in [35] to study our problem. First of all, without loss of generality, we will assume that

$$0 \in \Lambda \text{ and } V(0) = V_0.$$

We note that $\frac{f(t)}{t^{p-1}+t^{q-1}}$ is increasing for $t > 0$. Indeed,

$$\frac{f(t)}{t^{p-1} + t^{q-1}} = \frac{f(t)}{t^{q-1}} \frac{t^{q-1}}{t^{p-1} + t^{q-1}},$$

$\frac{f(t)}{t^{q-1}}$ is increasing for $t > 0$ by (f_4), and $\frac{t^{q-1}}{t^{p-1}+t^{q-1}}$ is increasing for $t > 0$ since $q > p$.
    Let

$$K > \frac{q}{p} \left(\frac{\vartheta - p}{\vartheta - q} \right)$$

and $a > 0$ be such that

$$f(a) = \frac{V_0}{K}(a^{p-1} + a^{q-1})$$

and we define

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \leq a, \\ \frac{V_0}{K}(t^{p-1} + t^{q-1}) & \text{if } t > a, \end{cases}$$

and

$$g(x,t) := \begin{cases} \chi_\Lambda(x)f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

It is easy to check that g satisfies the following properties :

(g_1) $\displaystyle\lim_{t\to 0^+} \frac{g(x,t)}{t^{p-1}} = 0$ uniformly with respect to $x \in \mathbb{R}^N$,

(g_2) $g(x,t) \leq f(t)$ for all $x \in \mathbb{R}^N$, $t > 0$,

(g_3) (i) $0 < \vartheta G(x,t) < g(x,t)t$ for all $x \in \Lambda$ and $t > 0$,
        (ii) $0 \leq pG(x,t) < g(x,t)t \leq \frac{V_0}{K}(t^p + t^q)$ for all $x \in \Lambda^c$ and $t > 0$,

(g_4) for each $x \in \Lambda$ the function $t \mapsto \dfrac{g(x,t)}{(t^{p-1} + t^{q-1})}$ is increasing in $(0,\infty)$, and for each $x \in \Lambda^c$ the

    function $t \mapsto \dfrac{g(x,t)}{(t^{p-1} + t^{q-1})}$ is increasing in $(0, a)$.

We stress that if u_ε is a solution to

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = g(\varepsilon x, u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.5}$$

having the property that $u_\varepsilon(x) \leq a$ for all $x \in \Lambda_\varepsilon^c$, where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$, then $g(\varepsilon x, u_\varepsilon) = f(u_\varepsilon)$, and thus u_ε is also a solution to (1.3).
    For any $\varepsilon > 0$, we define the space

$$\mathbb{X}_\varepsilon := \left\{ u \in \mathcal{W} : \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^p + |u|^q)\, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{\mathbb{X}_\varepsilon} := \|u\|_{V_\varepsilon,p} + \|u\|_{V_\varepsilon,q},$$

where

$$\|u\|_{V_\varepsilon,t} := \left([u]_{s,t}^t + \int_{\mathbb{R}^N} V(\varepsilon\,x)|u|^t\,dx \right)^{\frac{1}{t}} \qquad \text{for all } t > 1.$$

In order to study (2.5), we look for critical points of the functional $\mathcal{I}_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{R}$ defined as

$$\mathcal{I}_\varepsilon(u) := \frac{1}{p}\|u\|_{V_\varepsilon,p}^p + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q - \int_{\mathbb{R}^N} G(\varepsilon\,x,u)\,dx.$$

It is standard to verify that $\mathcal{I}_\varepsilon \in \mathcal{C}^1(\mathbb{X}_\varepsilon,\mathbb{R})$ and its differential is given by

$$\langle \mathcal{I}_\varepsilon'(u),\varphi \rangle = \langle u,\varphi \rangle_{s,p} + \int_{\mathbb{R}^N} V(\varepsilon\,x)|u|^{p-2}u\,\varphi\,dx + \langle u,\varphi \rangle_{s,q}$$

$$+ \int_{\mathbb{R}^N} V(\varepsilon\,x)|u|^{q-2}u\,\varphi\,dx - \int_{\mathbb{R}^N} g(\varepsilon\,x,u)\varphi\,dx$$

for any $u,\varphi \in \mathbb{X}_\varepsilon$.

Next, we show that $\mathcal{I}_\varepsilon$ possesses a mountain pass geometry [7].

Lemma 2.4. *The functional $\mathcal{I}_\varepsilon$ satisfies the following conditions :*
(i) there exist $\alpha,\rho > 0$ such that $\mathcal{I}_\varepsilon(u) \geq \alpha$ for $\|u\|_{\mathbb{X}_\varepsilon} = \rho$;
(ii) there exists $e \in \mathbb{X}_\varepsilon$ with $\|e\|_{\mathbb{X}_\varepsilon} > \rho$ and $\mathcal{I}_\varepsilon(e) < 0$.

Démonstration. (i) By (g_1), (g_2), (f_1) and (f_2), for any given $\zeta > 0$ there exists $C_\zeta > 0$ such that

$$|g(x,t)| \leq \zeta|t|^{p-1} + C_\zeta|t|^{\nu-1} \quad \forall(x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Take $\zeta \in (0,V_0)$. Then we have

$$\mathcal{I}_\varepsilon(u) \geq \frac{1}{p}\|u\|_{V_\varepsilon,p}^p + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q - \frac{\zeta}{p}|u|_p^p - \frac{C_\zeta}{\nu}|u|_\nu^\nu$$

$$\geq C_1\|u\|_{V_\varepsilon,p}^p + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q - \frac{C_\zeta}{\nu}|u|_\nu^\nu.$$

Choosing $\|u\|_{\mathbb{X}_\varepsilon} = \rho \in (0,1)$ and using $1 < p < q$, we have $\|u\|_{V_\varepsilon,p} < 1$ and then $\|u\|_{V_\varepsilon,p}^p \geq \|u\|_{V_\varepsilon,p}^q$. This fact combined with

$$a^t + b^t \geq C_t(a+b)^t \quad \forall a,b \geq 0 \quad \forall t > 1,$$

and Sobolev embeddings yield

$$\mathcal{I}_\varepsilon(u) \geq C_2\|u\|_{\mathbb{X}_\varepsilon}^q - \frac{C_\zeta}{\nu}|u|_\nu^\nu \geq C_2\|u\|_{\mathbb{X}_\varepsilon}^q - C_3\|u\|_{\mathbb{X}_\varepsilon}^\nu.$$

Since $\nu > q$, we can find $\alpha > 0$ such that $\mathcal{I}_\varepsilon(u) \geq \alpha$ for $\|u\|_{\mathbb{X}_\varepsilon} = \rho$.

(ii) Take $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $u \geq 0$, $u \not\equiv 0$ and $\text{supp}(u) \subset \Lambda_\varepsilon$. Using (f_3) we know that

$$F(t) \geq At^\vartheta - B \quad \forall t > 0.$$

Hence,

$$\mathcal{I}_\varepsilon(tu) \leq \frac{t^p}{p}\|u\|_{\varepsilon,p}^p + \frac{t^q}{q}\|u\|_{\varepsilon,q}^q - At^\vartheta \int_{\Lambda_\varepsilon} u^\vartheta\,dx + B_1 \to -\infty \quad \text{as } t \to \infty$$

where we used $\vartheta > q > p$. $\qquad\square$

By Lemma 2.4, we can define the minimax level

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \mathcal{I}_\varepsilon(\gamma(t)) \quad \text{where} \quad \Gamma_\varepsilon := \left\{ \gamma \in \mathcal{C}^0([0,1], \mathbb{X}_\varepsilon) : \gamma(0) = 0, \mathcal{I}_\varepsilon(\gamma(1)) < 0 \right\}.$$

Next, we show that the modified functional satisfies the Palais-Smale condition.

Lemma 2.5. *$\mathcal{I}_\varepsilon$ verifies the Palais-Smale condition at any level $c \in \mathbb{R}$.*

Démonstration. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence at the level c, that is

$$\mathcal{I}_\varepsilon(u_n) = c + o_n(1) \quad \text{and} \quad \mathcal{I}'_\varepsilon(u_n) = o_n(1).$$

Let us prove that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$. Using (g_3) and $q > p$ we can see that

$$
\begin{aligned}
C_0(1 + \|u_n\|_{\mathbb{X}_\varepsilon}) &\geq \mathcal{I}_\varepsilon(u_n) - \frac{1}{\vartheta} \langle \mathcal{I}'_\varepsilon(u_n), u_n \rangle \\
&= \left(\frac{1}{p} - \frac{1}{\vartheta} \right) \|u_n\|^p_{V_\varepsilon, p} + \left(\frac{1}{q} - \frac{1}{\vartheta} \right) \|u_n\|^q_{V_\varepsilon, q} + \frac{1}{\vartheta} \int_{\Lambda_\varepsilon^c} [g(\varepsilon x, u_n)u_n - \vartheta G(\varepsilon x, u_n)]\, dx \\
&\quad + \frac{1}{\vartheta} \int_{\Lambda_\varepsilon} [g(\varepsilon x, u_n)u_n - \vartheta G(\varepsilon x, u_n)]\, dx \\
&\geq \left(\frac{1}{q} - \frac{1}{\vartheta} \right) [\|u_n\|^p_{V_\varepsilon, p} + \|u_n\|^q_{V_\varepsilon, q}] - \left(\frac{1}{p} - \frac{1}{\vartheta} \right) \frac{1}{K} \int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u_n|^p + |u_n|^q)\, dx \\
&\geq \left[\left(\frac{1}{q} - \frac{1}{\vartheta} \right) - \left(\frac{1}{p} - \frac{1}{\vartheta} \right) \frac{1}{K} \right] (\|u_n\|^p_{V_\varepsilon, p} + \|u_n\|^q_{V_\varepsilon, q}) =: \tilde{C}(\|u_n\|^p_{V_\varepsilon, p} + \|u_n\|^q_{V_\varepsilon, q})
\end{aligned}
$$

where $\tilde{C} > 0$ because of $K > \left(\frac{\vartheta - p}{\vartheta - q} \right) \frac{q}{p}$.

Now, assume by contradiction that $\|u_n\|_{\mathbb{X}_\varepsilon} \to \infty$. Then we have the following cases :
Case 1 : $\|u_n\|_{V_\varepsilon, p} \to \infty$ and $\|u_n\|_{V_\varepsilon, q} \to \infty$.
Then, for n large, we have $\|u_n\|^{q-p}_{V_\varepsilon, q} \geq 1$, that is $\|u_n\|^q_{V_\varepsilon, q} \geq \|u_n\|^p_{V_\varepsilon, q}$ which implies that

$$C_0(1 + \|u_n\|_{\mathbb{X}_\varepsilon}) \geq \tilde{C}(\|u_n\|^p_{V_\varepsilon, p} + \|u_n\|^p_{V_\varepsilon, q}) \geq C_1(\|u_n\|_{V_\varepsilon, p} + \|u_n\|_{V_\varepsilon, q})^p = C_1 \|u_n\|^p_{\mathbb{X}_\varepsilon}$$

and this gives a contradiction.
Case 2 : $\|u_n\|_{V_\varepsilon, p} \to \infty$ and $\|u_n\|_{V_\varepsilon, q}$ is bounded.
Therefore,

$$C_0(1 + \|u_n\|_{V_\varepsilon, p} + \|u_n\|_{V_\varepsilon, q}) = C_0(1 + \|u_n\|_{\mathbb{X}_\varepsilon}) \geq \tilde{C} \|u_n\|^p_{V_\varepsilon, p}$$

and thus

$$C_0 \left(\frac{1}{\|u_n\|^p_{V_\varepsilon, p}} + \frac{1}{\|u_n\|^{p-1}_{V_\varepsilon, p}} + \frac{\|u_n\|_{V_\varepsilon, q}}{\|u_n\|^p_{V_\varepsilon, p}} \right) \geq \tilde{C}.$$

Since $p > 1$ and passing to the limit as $n \to \infty$ we deduce that $0 < \tilde{C} \leq 0$ and this is a contradiction.
Case 3 : $\|u_n\|_{V_\varepsilon, q} \to \infty$ and $\|u_n\|_{V_\varepsilon, p}$ is bounded.
We can proceed as in Case 2.

Consequently, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$, and we may assume that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$ and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q_s^*)$. Next, we show that the weak limit u is a critical point of $\mathcal{I}_\varepsilon$.

Consider the sequence

$$h_n(x, y) := \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{\frac{N+sp}{p'}}},$$

and let
$$h(x,y) := \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\frac{N+sp}{p'}}},$$

where $p' = \frac{p}{p-1}$. It is easy to check that $\{h_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^{p'}(\mathbb{R}^{2N})$ with $h_n \to h$ a.e. in $\mathbb{R}^{2N}$. Since $L^{p'}(\mathbb{R}^{2N})$ is a reflexive space, there exists a subsequence, still denoted by $\{h_n\}_{n \in \mathbb{N}}$, such that $h_n \rightharpoonup h$ in $L^{p'}(\mathbb{R}^{2N})$, that is

$$\iint_{\mathbb{R}^{2N}} h_n(x,y)k(x,y)dxdy \to \iint_{\mathbb{R}^{2N}} h(x,y)k(x,y)dxdy \quad \forall k \in L^p(\mathbb{R}^{2N}).$$

Then, for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, taking

$$k(x,y) = \frac{(\phi(x) - \phi(y))}{|x - y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^{2N}),$$

and we can see that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dxdy$$
$$\to \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dxdy.$$

In a similar way we can prove that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+sq}} dxdy$$
$$\to \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+sq}} dxdy.$$

Taking into account that

$$\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{p-2}u_n\phi\, dx \to \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}u\phi\, dx,$$

$$\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{q-2}u_n\phi\, dx \to \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{q-2}u\phi\, dx,$$

$$\int_{\mathbb{R}^N} g(\varepsilon x, u_n)\phi\, dx \to \int_{\mathbb{R}^N} g(\varepsilon x, u)\phi\, dx,$$

and that $\langle \mathcal{I}_\varepsilon'(u_n), \phi \rangle = o_n(1)$, we can deduce that $\langle \mathcal{I}_\varepsilon'(u), \phi \rangle = 0$ for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, which together with the density of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ in $\mathbb{X}_\varepsilon$ implies that u is a critical point of $\mathcal{I}_\varepsilon$. In particular, $\langle \mathcal{I}_\varepsilon'(u), u \rangle = 0$. In order to prove the strong convergence, we show the next claim :
Claim : For any $\eta > 0$ there exists $R = R(\eta) > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathcal{B}_R^c} \left(\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} + \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+sq}}\, dy + V(\varepsilon x)(|u_n|^p + |u_n|^q) \right) dx < \eta. \quad (2.6)$$

For any $R > 0$, let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $0 \leq \psi_R \leq 1$, $\psi_R = 0$ in $\mathcal{B}_{R/2}$, $\psi_R = 1$ in $\mathcal{B}_R^c$, and $|\nabla \psi_R| \leq \frac{C}{R}$, for some constant $C > 0$ independent of R.

Since $\{\psi_R u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$, it follows that $\langle \mathcal{I}'_\varepsilon(u_n), \psi_R u_n \rangle = o_n(1)$, that is

$$
\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \psi_R(x)\, dxdy + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+sq}} \psi_R(x)\, dxdy
$$
$$
+ \int_{\mathbb{R}^N} V(\varepsilon\, x)|u_n|^p \psi_R\, dx + \int_{\mathbb{R}^N} V(\varepsilon\, x)|u_n|^q \psi_R\, dx
$$
$$
= o_n(1) + \int_{\mathbb{R}^N} g(\varepsilon\, x, u_n)\psi_R u_n\, dx
$$
$$
- \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x - y|^{N+sp}} u_n(y)\, dxdy
$$
$$
- \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x - y|^{N+sq}} u_n(y)\, dxdy
$$

Take $R > 0$ such that $\Lambda_\varepsilon \subset \mathcal{B}_{R/2}$. By the definition of ψ_R and (g_3)-(ii) we get

$$
\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \psi_R(x)\, dxdy + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+sq}} \psi_R(x)\, dxdy
$$
$$
+ \left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^N} V(\varepsilon\, x)(|u_n|^p + |u_n|^q)\psi_R\, dx
$$
$$
\leq o_n(1) - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x - y|^{N+sp}} u_n(y)\, dxdy
$$
$$
- \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x - y|^{N+sq}} u_n(y)\, dxdy. \tag{2.7}
$$

Now, using the Hölder inequality and the boundedness of $\{u_n\}_{n \in \mathbb{N}}$ in $\mathbb{X}_\varepsilon$ we have

$$
\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x - y|^{N+sp}} u_n(y)\, dxdy \right|
$$
$$
\leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x - y|^{N+sp}} |u_n(y)|^p\, dxdy \right)^{\frac{1}{p}}. \tag{2.8}
$$

On the other hand, using the definition of ψ_R, polar coordinates and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$, we can see that

$$\iint_{\mathbb{R}^{2N}} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{N+sp}} |u_n(x)|^p dxdy$$

$$= \int_{\mathbb{R}^N} \int_{|y-x|>R} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{N+sp}} |u_n(x)|^p dxdy + \int_{\mathbb{R}^N} \int_{|y-x|\leq R} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{N+sp}} |u_n(x)|^p dxdy$$

$$\leq C \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|y-x|>R} \frac{dy}{|x-y|^{N+sp}} \right) dx + \frac{C}{R^p} \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|y-x|\leq R} \frac{dy}{|x-y|^{N+sp-p}} \right) dx$$

$$\leq C \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|z|>R} \frac{dz}{|z|^{N+sp}} \right) dx + \frac{C}{R^p} \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|z|\leq R} \frac{dz}{|z|^{N+sp-p}} \right) dx$$

$$\leq C \int_{\mathbb{R}^N} |u_n(x)|^p dx \left(\int_R^\infty \frac{d\rho}{\rho^{sp+1}} \right) + \frac{C}{R^p} \int_{\mathbb{R}^N} |u_n(x)|^p dx \left(\int_0^R \frac{d\rho}{\rho^{sp-p+1}} \right)$$

$$\leq \frac{C}{R^{sp}} \int_{\mathbb{R}^N} |u_n(x)|^p dx + \frac{C}{R^p} R^{-sp+p} \int_{\mathbb{R}^N} |u_n(x)|^p dx$$

$$\leq \frac{C}{R^{sp}} \int_{\mathbb{R}^N} |u_n(x)|^p dx \leq \frac{C}{R^{sp}},$$

which together with (2.8) implies that

$$\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{N+sp}} u_n(y)\, dxdy \right| \leq \frac{C}{R^s}. \tag{2.9}$$

In a similar way, we can prove that

$$\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{N+sq}} u_n(y)\, dxdy \right| \leq \frac{C}{R^s}. \tag{2.10}$$

Putting together (2.7), (2.9) and (2.10) we can infer that (2.6) is verified.

From (2.6), we can also deduce that $u_n \to u$ in $L^p(\mathbb{R}^N)$. Indeed, for any $\eta > 0$ there exits $R = R(\eta) > 0$ for which (2.6) holds true, and using the locally compact embedding $\mathbb{X}_\varepsilon \Subset L^p_{loc}(\mathbb{R}^N)$ we get for n large

$$|u_n - u|_p^p = |u_n - u|_{L^p(\mathcal{B}_R)}^p + |u_n - u|_{L^p(\mathcal{B}_R^c)}^p$$

$$\leq \eta + |u_n - u|_{L^p(\mathcal{B}_R^c)}^p$$

$$\leq \eta + 2^{p-1}(|u_n|_{L^p(\mathcal{B}_R^c)}^p + |u|_{L^p(\mathcal{B}_R^c)}^p)$$

$$\leq \eta + \frac{2^{p-1}}{V_0} \left[\int_{\mathcal{B}_R^c} \left(\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \, dy + V(\varepsilon\, x)|u_n|^p \right) dx \right.$$

$$\left. + \int_{\mathcal{B}_R^c} \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dy + V(\varepsilon\, x)|u|^p \right) dx \right] < \eta + \frac{2^p}{V_0}\eta =: \kappa\eta.$$

By the arbitrariness of η, we get the assertion. Moreover, by interpolation, it holds that $u_n \to u$ in $L^\sigma(\mathbb{R}^N)$ for any $\sigma \in [p, q_s^*)$. Consequently, from (f_1), (f_2) and (g_2) we have

$$\int_{\mathbb{R}^N} g(\varepsilon\, x, u_n) u_n \, dx \to \int_{\mathbb{R}^N} g(\varepsilon\, x, u) u \, dx. \tag{2.11}$$

On the other hand, using $\langle \mathcal{I}'_\varepsilon(u_n), u_n \rangle = o_n(1)$ and $\langle \mathcal{I}'_\varepsilon(u), u \rangle = 0$, we know that

$$\|u_n\|^p_{V_\varepsilon, p} + \|u_n\|^q_{V_\varepsilon, q} = \int_{\mathbb{R}^N} g(\varepsilon\, x, u_n) u_n \, dx + o_n(1)$$

and

$$\|u\|^p_{V_\varepsilon, p} + \|u\|^q_{V_\varepsilon, q} = \int_{\mathbb{R}^N} g(\varepsilon\, x, u) u \, dx$$

which combined with (2.11) yield

$$\|u_n\|^p_{V_\varepsilon, p} + \|u_n\|^q_{V_\varepsilon, q} = \|u\|^p_{V_\varepsilon, p} + \|u\|^q_{V_\varepsilon, q} + o_n(1).$$

In view of the Brezis-Lieb Lemma [25], we know that

$$\|u_n - u\|^p_{V_\varepsilon, p} = \|u_n\|^p_{V_\varepsilon, p} - \|u\|^p_{V_\varepsilon, p} + o_n(1) \text{ and } \|u_n - u\|^q_{V_\varepsilon, q} = \|u_n\|^q_{V_\varepsilon, q} - \|u\|^q_{V_\varepsilon, q} + o_n(1),$$

so we can deduce that

$$\|u_n - u\|^p_{V_\varepsilon, p} + \|u_n - u\|^q_{V_\varepsilon, q} = o_n(1)$$

which gives $\|u_n - u\|_{\mathbb{X}_\varepsilon} = o_n(1)$ as $n \to \infty$. This ends the proof of lemma. $\qquad\square$

Theorem 2.3. *Suppose that (V_1)-(V_2) and (f_1)-(f_4) hold. Then, for any $\varepsilon > 0$, (2.5) has a nontrivial nonnegative solution.*

Démonstration. In view of Lemma 2.4 and Lemma 2.5, we can apply the mountain pass theorem [7] to deduce that, for all $\varepsilon > 0$, there exists $u_\varepsilon \in \mathbb{X}_\varepsilon$ such that $\mathcal{I}_\varepsilon(u_\varepsilon) = c_\varepsilon$ and $\mathcal{I}'_\varepsilon(u_\varepsilon) = 0$. Moreover, $u_\varepsilon \geq 0$ in $\mathbb{R}^N$. Indeed, using $\langle \mathcal{I}'_\varepsilon(u_\varepsilon), u_\varepsilon^- \rangle = 0$, where $u_\varepsilon^- = \min\{u_\varepsilon, 0\}$, $g(\varepsilon\, \cdot, t) = 0$ for $t \leq 0$, and the following elementary inequality,

$$|x - y|^{t-2}(x - y)(x^- - y^-) \geq |x^- - y^-|^t \quad \forall x, y \in \mathbb{R} \quad \forall t > 1,$$

we can see that

$$\|u^-\|^p_{V_\varepsilon, p} + \|u^-\|^q_{V_\varepsilon, q} \leq 0$$

which implies that $u_\varepsilon^- = 0$, that is $u_\varepsilon \geq 0$ in $\mathbb{R}^N$ and $u \not\equiv 0$. $\qquad\square$

Since we are interested in multiple critical points of the functional $\mathcal{I}_\varepsilon$, we introduce the Nehari manifold associated with (2.5), namely

$$\mathcal{N}_\varepsilon := \{ u \in \mathbb{X}_\varepsilon : \langle \mathcal{I}'_\varepsilon(u), u \rangle = 0 \}.$$

Let us define

$$\mathbb{X}^+_\varepsilon := \{ u \in \mathbb{X}_\varepsilon : |\mathrm{supp}(u^+) \cap \Lambda_\varepsilon| > 0 \}$$

and $\mathbb{S}^+_\varepsilon := \mathbb{S}_\varepsilon \cap \mathbb{X}^+_\varepsilon$, where $\mathbb{S}_\varepsilon := \{ u \in \mathbb{X}_\varepsilon : \|u\|_{\mathbb{X}_\varepsilon} = 1 \}$ is the unit sphere in $\mathbb{X}_\varepsilon$. Note that $\mathbb{S}^+_\varepsilon$ is a incomplete $\mathcal{C}^{1,1}$-manifold of codimension one, hence $\mathbb{X}_\varepsilon = T_u \mathbb{S}^+_\varepsilon \oplus \mathbb{R}u$ for all $u \in \mathbb{S}^+_\varepsilon$, where

$$T_u \mathbb{S}^+_\varepsilon := \left\{ v \in \mathbb{X}_\varepsilon : \langle u, v \rangle_{s,p} + \langle u, v \rangle_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon\, x)(|u|^{p-2}uv + |u|^{q-2}uv) \, dx = 0 \right\}.$$

Since f is only continuous, the next two results will be fundamental to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}^+_\varepsilon$.

Lemma 2.6. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold true. Then, we have the following results :*
  (i) For each $u \in \mathbb{X}^+_\varepsilon$, let $h_u : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{I}_\varepsilon(tu)$. Then, there is a unique $t_u > 0$ such that

$$h'_u(t) > 0 \text{ for all } t \in (0, t_u)$$
$$h'_u(t) < 0 \text{ for all } t \in (t_u, \infty);$$

(ii) *there exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}_\varepsilon^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$, there is a positive constant $C_\mathbb{K}$ such that $t_u \leq C_\mathbb{K}$ for any $u \in \mathbb{K}$;*

(iii) *The map $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\hat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon := \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ is a homeomorphism between $\mathbb{S}_\varepsilon^+$ and $\mathcal{N}_\varepsilon$. Moreover $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}}$;*

(iv) *If there is a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ such that $\operatorname{dist}(u_n, \partial \mathbb{S}_\varepsilon^+) \to 0$ then $\|m_\varepsilon(u_n)\|_{\mathbb{X}_\varepsilon} \to \infty$ and $\mathcal{I}_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

Démonstration. (i) Arguing as in the proof of Lemma 2.4, we can see that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Then there exists a global maximum point $t_u > 0$ for h_u such that $h_u'(t_u) = 0$, that is $t_u u \in \mathcal{N}_\varepsilon$. Next we show the uniqueness of a such t_u. Assume by contradiction that there exist $t_1 > t_2 > 0$ such that $h_u'(t_1) = h_u'(t_2) = 0$, or equivalently

$$t_1^{p-1}\|u\|_{V_\varepsilon,p}^p + t_1^{q-1}\|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} g(\varepsilon x, t_1 u) u \, dx \tag{2.12}$$

$$t_2^{p-1}\|u\|_{V_\varepsilon,p}^p + t_2^{q-1}\|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} g(\varepsilon x, t_2 u) u \, dx. \tag{2.13}$$

Dividing (2.12) by t_1^{q-1} and (2.13) by t_2^{q-1} respectively, we get

$$\frac{\|u\|_{V_\varepsilon,p}^p}{t_1^{q-p}} + \|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{q-1}} u^q dx,$$

and

$$\frac{\|u\|_{V_\varepsilon,p}^p}{t_2^{q-p}} + \|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{q-1}} u^q dx.$$

Subtracting the above identities and using (g_4) and (f_4) we obtain

$$\begin{aligned}
\left(\frac{1}{t_1^{q-p}} - \frac{1}{t_2^{q-p}}\right) \|u\|_{V_\varepsilon,p}^p &= \int_{\mathbb{R}^N} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx \\
&\geq \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx \\
&\quad + \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx \\
&\quad + \int_{\Lambda_\varepsilon^c \cap \{t_1 u < a\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx \\
&\geq \frac{V_0}{K}\left(\frac{1}{t_1^{q-p}} - \frac{1}{t_2^{q-p}}\right) \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a\}} u^p dx \\
&\quad + \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \left[\frac{V_0}{K}\left(\frac{1}{(t_1 u)^{q-p}} + 1\right) - \frac{f(t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx.
\end{aligned}$$

In view of $t_1 > t_2$ and $q > p$ we have

$$
\begin{aligned}
\|u\|_{\varepsilon,p}^p &\leq \frac{V_0}{K} \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a\}} u^p dx + \frac{(t_1 t_2)^{q-p}}{t_2^{q-p} - t_1^{q-p}} \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \left[\frac{V_0}{K} \left(\frac{1}{(t_1 u)^{q-p}} + 1 \right) - \frac{f(t_2 u)}{(t_2 u)^{q-1}} \right] u^q dx \\
&= \frac{1}{K} \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a\}} V_0 u^p dx \\
&\quad - \frac{t_2^{q-p}}{t_1^{q-p} - t_2^{q-p}} \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \frac{V_0}{K} u^p dx + \frac{(t_1 t_2)^{q-p}}{t_2^{q-p} - t_q^{q-p}} \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \left[\frac{V_0}{K} - \frac{f(t_2 u)}{(t_2 u)^{q-1}} \right] u^q dx \\
&\leq \frac{1}{K} \int_{\Lambda_\varepsilon^c} V_0 u^p dx + \frac{t_1^{q-p}}{t_2^{q-p} - t_1^{q-p}} \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \frac{V_0}{K} u^p dx \\
&\leq \frac{1}{K} \int_{\Lambda_\varepsilon^c} V_0 u^p dx \leq \frac{1}{K} \|u\|_{\varepsilon,p}^p
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
\frac{f(t_2 u)}{(t_2 u)^{q-1}} &= \frac{f(t_2 u)}{(t_2 u)^{p-1} + (t_2 u)^{q-1}} \frac{(t_2 u)^{p-1} + (t_2 u)^{q-1}}{(t_2 u)^{q-1}} \\
&\leq \frac{V_0}{K} \left(\frac{1}{(t_2 u)^{q-p}} + 1 \right) \quad \text{in } \Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}.
\end{aligned}
$$

Since $u \neq 0$ and $K > 1$, we get a contradiction.

(ii) Let $u \in \mathbb{S}_\varepsilon^+$. By (i) there exists $t_u > 0$ such that $h_u'(t_u) = 0$, or equivalently

$$
t_u^{p-1} \|u\|_{V_\varepsilon,p}^p + t_u^{q-1} \|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} g(\varepsilon x, t_u u)\, u\, dx.
$$

From (g_1)-(g_2) and (2.1), for all $\xi > 0$ we obtain

$$
t_u^{p-1} \|u\|_{V_\varepsilon,p}^p + t_u^{q-1} \|u\|_{V_\varepsilon,q}^q \leq \int_{\mathbb{R}^3} g(\varepsilon x, t_u u)\, u\, dx \leq \xi t_u^{p-1} \|u\|_{V_\varepsilon,p}^p + C_\xi t_u^{\nu-1} \|u\|_{V_\varepsilon,q}^\nu,
$$

and choosing ξ sufficiently small, we have

$$
C t_u^{p-1} \|u\|_{V_\varepsilon,p}^p + t_u^{q-1} \|u\|_{V_\varepsilon,q}^q \leq C t_u^{\nu-1} \|u\|_{V_\varepsilon,q}^\nu \leq C t_u^{\nu-1}.
$$

Now, if $t_u \leq 1$, then $t_u^{q-1} \leq t_u^{p-1}$, and using the facts that $1 = \|u\|_{\mathbb{X}_\varepsilon} \geq \|u\|_{V_\varepsilon,p}$ and $q > p$ imply that $\|u\|_{V_\varepsilon,p}^p \geq \|u\|_{V_\varepsilon,p}^q$, we deduce that

$$
C t_u^{q-1} = C t_u^{q-1} \|u\|_{\mathbb{X}_\varepsilon}^q \leq t_u^{q-1} (C \|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \leq t_u^{q-1} (C \|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \leq C t_u^{\nu-1}.
$$

Since $\nu > q$, we can find $\tau > 0$, independent of u, such that $t_u \geq \tau$.

When $t_u > 1$, then $t_u^{q-1} > t_u^{p-1}$, and using again the fact that $1 = \|u\|_{\mathbb{X}_\varepsilon} \geq \|u\|_{V_\varepsilon,p}$ and $q > p$ imply that $\|u\|_{V_\varepsilon,p}^p \geq \|u\|_{V_\varepsilon,p}^q$, we get

$$
C t_u^{p-1} = C t_u^{p-1} \|u\|_{\mathbb{X}_\varepsilon}^q \leq t_u^{p-1} (C \|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \leq t_u^{p-1} (C \|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \leq C t_u^{\nu-1}.
$$

Thanks to $\nu > q > p$, there exists $\tau > 0$, independent of u, such that $t_u \geq \tau$.

Now, let $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ be a compact set, and assume by contradiction that there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ such that $t_n := t_{u_n} \to \infty$. Therefore, there exists $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. From the proof of (ii) in Lemma 2.4, we know that

$$
\mathcal{I}_\varepsilon(t_n u_n) \to -\infty. \tag{2.14}
$$

On the other hand, fixed $v \in \mathcal{N}_\varepsilon$, by $\langle \mathcal{I}'_\varepsilon(v), v \rangle = 0$ and (g_3) we can see that

$$\mathcal{I}_\varepsilon(v) = \mathcal{I}_\varepsilon(v) - \frac{1}{\vartheta} \langle \mathcal{I}'_\varepsilon(v), v \rangle$$
$$\geq \tilde{C}(\|v\|^p_{V_\varepsilon, p} + \|v\|^q_{V_\varepsilon, q}).$$

Taking $v_n = t_{u_n} u_n \in \mathcal{N}_\varepsilon$ in the above inequality we obtain

$$\mathcal{I}_\varepsilon(t_n u_n) \geq \tilde{C}(\|v_n\|^p_{V_\varepsilon, p} + \|v_n\|^q_{V_\varepsilon, q}).$$

Since $\|v_n\|_{\mathbb{X}_\varepsilon} = t_n \to \infty$ and $\|v_n\|_{\mathbb{X}_\varepsilon} = \|v_n\|_{\varepsilon, p} + \|v_n\|_{\varepsilon, q}$, we can use (2.14) to obtain a contradiction.
(iii) Firstly, we note that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (i), for each $u \in \mathbb{X}_\varepsilon^+$ there exists a unique $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in \mathbb{X}_\varepsilon^+$. Otherwise, if $u \notin \mathbb{X}_\varepsilon^+$, we get

$$|\operatorname{supp}(u^+) \cap \Lambda_\varepsilon| = 0,$$

which together with $(g3)$-(ii) yields

$$\|u\|^p_{V_\varepsilon, p} + \|u\|^q_{V_\varepsilon, q} = \int_{\mathbb{R}^N} g(\varepsilon x, u) \, u \, dx = \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u) \, u \, dx + \int_{\Lambda_\varepsilon} g(\varepsilon x, u) \, u \, dx$$
$$= \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u^+) \, u^+ \, dx$$
$$\leq \frac{1}{K} \int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u|^p + |u|^q) dx$$
$$\leq \frac{1}{K}(\|u\|^p_{V_\varepsilon, p} + \|u\|^q_{V_\varepsilon, q}) \tag{2.15}$$

and this leads to a contradiction because $K > 1$. Consequently, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} \in \mathbb{S}_\varepsilon^+$, m_ε^{-1} is well defined and continuous. From $u \in \mathbb{S}_\varepsilon^+$, we can see that

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{\mathbb{X}_\varepsilon}} = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} = u$$

which implies that m_ε is a bijection.
Next, we prove that $\hat{m}_\varepsilon$ is a continuous function. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{X}_\varepsilon^+$ and $u \in \mathbb{X}_\varepsilon^+$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. Since $\hat{m}(tu) = \hat{m}(u)$ for all $t > 0$, we may assume that $\|u_n\|_{\mathbb{X}_\varepsilon} = \|u\|_{\mathbb{X}_\varepsilon} = 1$ for all $n \in \mathbb{N}$. In view of (ii), there exists $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Since $t_n u_n \in \mathcal{N}_\varepsilon$, we have

$$t_n^p \|u_n\|^p_{V_\varepsilon, p} + t_n^q \|u_n\|^q_{V_\varepsilon, q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_n u_n) \, t_n u_n \, dx,$$

and letting $n \to \infty$ we obtain

$$t_0^p \|u\|^p_{V_\varepsilon, p} + t_0^q \|u\|^q_{V_\varepsilon, q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_0 u) \, t_0 u \, dx,$$

which implies that $t_0 u \in \mathcal{N}_\varepsilon$. By (i), we deduce that $t_u = t_0$ and this shows that $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in $\mathbb{X}_\varepsilon^+$. Therefore, $\hat{m}_\varepsilon$ and m_ε are continuous functions.
(iv) Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be such that $\operatorname{dist}(u_n, \partial \mathbb{S}_\varepsilon^+) \to 0$. Observing that for each $r \in [p, q_s^*]$ and $n \in \mathbb{N}$ it holds

$$|u_n^+|_{L^r(\Lambda_\varepsilon)} \leq \inf_{v \in \partial \mathbb{S}_\varepsilon^+} |u_n - v|_{L^r(\Lambda_\varepsilon)}$$
$$\leq C_r \inf_{v \in \partial \mathbb{S}_\varepsilon^+} \|u_n - v\|_{\mathbb{X}_\varepsilon},$$

and using (g_1), (g_2), and (g_3)-(ii), we can see that for all $t > 0$

$$\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\, dx = \int_{\Lambda_\varepsilon^c} G(\varepsilon x, tu_n)\, dx + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)\, dx$$

$$\leq \frac{V_0}{Kp} \int_{\Lambda_\varepsilon^c} (t^p|u_n|^p + t^q|u_n|^q)dx + \int_{\Lambda_\varepsilon} F(tu_n)\, dx$$

$$\leq \frac{t^p}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx + \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx$$

$$+ C_1 t^p \int_{\Lambda_\varepsilon} (u_n^+)^p dx + C_2 t^\nu \int_{\Lambda_\varepsilon} (u_n^+)^\nu dx$$

$$\leq \frac{t^p}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx + \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx$$

$$+ C_p' t^p \mathrm{dist}(u_n, \partial \mathbb{S}_\varepsilon^+)^p + C_\nu' t^\nu \mathrm{dist}(u_n, \partial \mathbb{S}_\varepsilon^+)^\nu.$$

Accordingly,

$$\int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\, dx \leq \frac{t^p}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx + \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx + o_n(1). \qquad (2.16)$$

Now, we recall that $K > \frac{q}{p} > 1$, and that $1 = \|u_n\|_{\mathbb{X}_\varepsilon} \geq \|u_n\|_{V_\varepsilon, p}$ implies that $\|u_n\|_{V_\varepsilon, p}^p \geq \|u_n\|_{V_\varepsilon, p}^q$. Then, for all $t > 1$, we deduce that

$$\frac{t^p}{p}\|u_n\|_{\varepsilon,p}^p + \frac{t^q}{q}\|u_n\|_{\varepsilon,q}^q - \frac{t^p}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx - \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx$$

$$= \frac{t^p}{p}[u_n]_{s,p}^p + t^p \left(\frac{1}{p} - \frac{1}{Kp}\right) \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx + \frac{t^q}{q}[u_n]_{s,q}^q + t^q \left(\frac{1}{q} - \frac{1}{Kp}\right) \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx$$

$$\geq C_1 t^p \|u_n\|_{V_\varepsilon,p}^p + C_2 t^q \|u_n\|_{V_\varepsilon,q}^q$$

$$\geq C_1 t^p \|u_n\|_{V_\varepsilon,p}^q + C_2 t^q \|u_n\|_{V_\varepsilon,q}^q$$

$$\geq C_1 t^p \|u_n\|_{V_\varepsilon,p}^q + C_2 t^p \|u_n\|_{V_\varepsilon,q}^q$$

$$\geq C_3 t^p (\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^q = C_3 t^p.$$

$$(2.17)$$

Taking in mind the definition of $m_\varepsilon(u_n)$ and using (2.16), (2.17) we have

$$\liminf_{n\to\infty} \mathcal{I}_\varepsilon(m_\varepsilon(u_n)) \geq \liminf_{n\to\infty} \mathcal{I}_\varepsilon(tu_n) \geq C_3 t^p \quad \forall t > 1$$

which implies that

$$\lim_{n\to\infty} \mathcal{I}_\varepsilon(m_\varepsilon(u_n)) = \infty.$$

On the other hand, by the definition of $\mathcal{I}_\varepsilon$,

$$\liminf_{n\to\infty} \left\{ \frac{1}{p}\|m_\varepsilon(u_n)\|_{V_\varepsilon,p}^p + \frac{1}{q}\|m_\varepsilon(u_n)\|_{V_\varepsilon,q}^q \right\} \geq \liminf_{n\to\infty} \mathcal{I}_\varepsilon(m_\varepsilon(u_n)) = \infty$$

so that $\|m_\varepsilon(u_n)\|_{\mathbb{X}_\varepsilon} \to \infty$ as $n \to \infty$. This ends the proof of lemma. □

Let us define the maps

$$\hat{\psi}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}_\varepsilon^+ \to \mathbb{R},$$

by $\hat{\psi}_\varepsilon(u) := \mathcal{I}_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon := \hat{\psi}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$. The next result is a consequence of Lemma 2.6 and Corollary 2.3 in [68].

Proposition 2.1. *Assume that* (V_1)-(V_2) *and* (f_1)-(f_4) *hold true. Then,*

(a) $\hat{\psi}_\varepsilon \in \mathcal{C}^1(\mathbb{X}_\varepsilon^+, \mathbb{R})$ and

$$\langle \hat{\psi}_\varepsilon'(u), v \rangle = \frac{\|\hat{m}_\varepsilon(u)\|_{\mathbb{X}_\varepsilon}}{\|u\|_{\mathbb{X}_\varepsilon}} \langle \mathcal{I}_\varepsilon'(\hat{m}_\varepsilon(u)), v \rangle \quad \forall u \in \mathbb{X}_\varepsilon^+ \quad \forall v \in \mathbb{X}_\varepsilon;$$

(b) $\psi_\varepsilon \in \mathcal{C}^1(\mathbb{S}_\varepsilon^+, \mathbb{R})$ and

$$\langle \psi_\varepsilon'(u), v \rangle = \|m_\varepsilon(u)\|_{\mathbb{X}_\varepsilon} \langle \mathcal{I}_\varepsilon'(m_\varepsilon(u)), v \rangle, \quad \forall v \in T_u\mathbb{S}_\varepsilon^+;$$

(c) if $\{u_n\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for ψ_ε, then $\{m_\varepsilon(u_n)\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{I}_\varepsilon$. If $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_d$ sequence for $\mathcal{I}_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for the functional ψ_ε;

(d) u is a critical point of ψ_ε if, and only if, $m_\varepsilon(u)$ is a critical point for $\mathcal{I}_\varepsilon$. Moreover, the corresponding critical values coincide and

$$\inf_{u \in \mathbb{S}_\varepsilon^+} \psi_\varepsilon(u) = \inf_{u \in \mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u).$$

Remark 2.2. As in [68], c_ε has the following variational characterization :

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u) = \inf_{u \in \mathbb{X}_\varepsilon^+} \max_{t > 0} \mathcal{I}_\varepsilon(tu) = \inf_{u \in \mathbb{S}_\varepsilon^+} \max_{t > 0} \mathcal{I}_\varepsilon(tu).$$

We conclude this section by showing a result which will be used later.

Corollary 2.2. The functional ψ_ε verifies the $(PS)_d$ condition on $\mathbb{S}_\varepsilon^+$.

Démonstration. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be a (PS) sequence for ψ_ε at the level d. Then

$$\psi_\varepsilon(u_n) \to d \quad \text{and} \quad \psi_\varepsilon'(u_n) \to 0 \text{ in } (T_{u_n}\mathbb{S}_\varepsilon^+)'.$$

By Proposition 2.1-(c), it follows that $\{m_\varepsilon(u_n)\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{I}_\varepsilon$ in $\mathbb{X}_\varepsilon$. Then, by Lemma 2.5, we can see that $\mathcal{I}_\varepsilon$ satisfies the $(PS)_d$ condition in $\mathbb{X}_\varepsilon$, so there exists $u \in \mathbb{S}_\varepsilon^+$ such that, up to a subsequence,

$$m_\varepsilon(u_n) \to m_\varepsilon(u) \text{ in } \mathbb{X}_\varepsilon.$$

Applying Lemma 2.6-(iii), we conclude that $u_n \to u$ in $\mathbb{S}_\varepsilon^+$. $\qquad \square$

3. A MULTIPLICITY PROPERTY FOR THE MODIFIED PROBLEM

3.1. The limiting problem.
Let us consider the limiting problem associated with (1.3), that is

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V_0(|u|^{p-2}u + |u|^{q-2}u) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

The corresponding energy functional is given by

$$\mathcal{E}_{V_0}(u) := \frac{1}{p}[u]_{s,p}^p + \frac{1}{q}[u]_{s,q}^q + V_0\left[\frac{1}{p}|u|_p^p + \frac{1}{q}|u|_q^q\right] - \int_{\mathbb{R}^N} F(u)\, dx$$

which is well-defined on the space $\mathbb{Y}_{V_0} = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{\mathbb{Y}_{V_0}} := \|u\|_{s,p} + \|u\|_{s,q},$$

where

$$\|u\|_{s,t} := \left([u]_{s,t}^t + V_0|u|_t^t\right)^{\frac{1}{t}} \quad \text{for all } t \geq 1.$$

It is easy to check that $\mathcal{E}_{V_0} \in \mathcal{C}^1(\mathbb{Y}_{V_0}, \mathbb{R})$ and its differential is given by

$$\langle \mathcal{E}_{V_0}'(u), \varphi \rangle = \langle u, \varphi \rangle_{s,p} + \langle u, \varphi \rangle_{s,q} + V_0\left[\int_{\mathbb{R}^N} |u|^{p-2}u\, \varphi\, dx + \int_{\mathbb{R}^N} |u|^{q-2}u\, \varphi\, dx\right] - \int_{\mathbb{R}^N} f(u)\varphi\, dx$$

for any $u, \varphi \in \mathbb{Y}_{V_0}$. Let us define the Nehari manifold associated with $\mathcal{E}_{V_0}$

$$\mathcal{M}_{V_0} := \{u \in \mathbb{Y}_{V_0} \setminus \{0\} : \langle \mathcal{E}_{V_0}'(u), u \rangle = 0\},$$

and $d_{V_0} := \inf_{u \in \mathcal{M}_{V_0}} \mathcal{E}_{V_0}(u)$.

We denote by $\mathbb{Y}_{V_0}^+$ the open subset of $\mathbb{Y}_{V_0}$ defined as

$$\mathbb{Y}_{V_0}^+ := \{u \in \mathbb{Y}_{V_0} : |\operatorname{supp}(u^+)| > 0\},$$

and $\mathbb{S}_{V_0}^+ := \mathbb{S}_{V_0} \cap \mathbb{Y}_{V_0}^+$, where $\mathbb{S}_{V_0}$ is the unit sphere of $\mathbb{Y}_{V_0}$. We note that $\mathbb{S}_{V_0}^+$ is a incomplete $\mathcal{C}^{1,1}$-manifold of codimension 1 modeled on $\mathbb{Y}_{V_0}$ and contained in $\mathbb{Y}_{V_0}^+$. Thus, $\mathbb{Y}_{V_0} = T_u\mathbb{S}_{V_0}^+ \oplus \mathbb{R}u$ for each $u \in \mathbb{S}_{V_0}^+$, where

$$T_u\mathbb{S}_{V_0}^+ := \left\{v \in \mathbb{Y}_{V_0} : \langle u, v \rangle_{s,p} + \langle u, v \rangle_{s,q} + V_0 \int_{\mathbb{R}^N} (|u|^{p-2}u + |u|^{q-2}u)v\, dx = 0\right\}.$$

As in Section 2, we can see that the following results hold.

Lemma 3.1. *Assume that (f_1)-(f_4) hold true. Then,*
(i) *For each $u \in \mathbb{Y}_{V_0}^+$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{E}_{V_0}(tu)$. Then, there is a unique $t_u > 0$ such that*

$$h_u'(t) > 0 \text{ for all } t \in (0, t_u)$$
$$h_u'(t) < 0 \text{ for all } t \in (t_u, \infty);$$

(ii) *there exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}_{V_0}^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_{V_0}^+$ there is a positive constant $C_{\mathbb{K}}$ such that $t_u \leq C_{\mathbb{K}}$ for any $u \in \mathbb{K}$;*
(iii) *The map $\hat{m}_\mu : \mathbb{Y}_{V_0}^+ \to \mathcal{M}_{V_0}$ given by $\hat{m}_{V_0}(u) = t_u u$ is continuous and $m_{V_0} := \hat{m}_{V_0}|_{\mathbb{S}_{V_0}^+}$ is a homeomorphism between $\mathbb{S}_{V_0}^+$ and $\mathcal{M}_{V_0}$. Moreover $m_{V_0}^{-1}(u) = \frac{u}{\|u\|_{\mathbb{Y}_{V_0}}}$;*
(iv) *If there is a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{S}_{V_0}^+$ such that $\operatorname{dist}(u_n, \partial\mathbb{S}_{V_0}^+) \to 0$ then $\|m_{V_0}(u_n)\|_{\mathbb{Y}_{V_0}} \to \infty$ and $\mathcal{E}_{V_0}(m_{V_0}(u_n)) \to \infty$.*

Let us define the maps

$$\hat{\psi}_\mu : \mathbb{Y}_{V_0}^+ \to \mathbb{R} \quad \text{and} \quad \psi_\mu : \mathbb{S}_{V_0}^+ \to \mathbb{R},$$

by $\hat{\psi}_{V_0}(u) := \mathcal{E}_{V_0}(\hat{m}_{V_0}(u))$ and $\psi_{V_0} := \hat{\psi}_{V_0}|_{\mathbb{S}_{V_0}^+}$.

Proposition 3.1. *Assume that (f_1)-(f_4) hold true. Then,*
(a) $\hat{\psi}_{V_0} \in \mathcal{C}^1(\mathbb{Y}_{V_0}^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}_{V_0}'(u), v \rangle = \frac{\|\hat{m}_{V_0}(u)\|_{\mathbb{Y}_{V_0}}}{\|u\|_{Y_{V_0}}} \langle \mathcal{E}_{V_0}'(\hat{m}_{V_0}(u)), v \rangle \quad \forall u \in \mathbb{Y}_{V_0}^+ \quad \forall v \in \mathbb{Y}_{V_0};$$

(b) $\psi_{V_0} \in \mathcal{C}^1(\mathbb{S}_{V_0}^+, \mathbb{R})$ *and*

$$\langle \psi_{V_0}'(u), v \rangle = \|m_{V_0}(u)\|_{\mathbb{Y}_{V_0}} \langle \mathcal{E}_{V_0}'(m_{V_0}(u)), v \rangle, \quad \forall v \in T_u\mathbb{S}_{V_0}^+;$$

(c) *If $\{u_n\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for ψ_{V_0}, then $\{m_{V_0}(u_n)\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{E}_{V_0}$. If $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$ is a bounded $(PS)_d$ sequence for $\mathcal{E}_{V_0}$, then $\{m_{V_0}^{-1}(u_n)\}_{n \in \mathbb{N}}$ is a $(PS)_d$ sequence for the functional ψ_{V_0};*
(d) *u is a critical point of ψ_{V_0} if, and only if, $m_{V_0}(u)$ is a nontrivial critical point for $\mathcal{E}_{V_0}$. Moreover, the corresponding critical values coincide and*

$$\inf_{u \in \mathbb{S}_{V_0}^+} \psi_{V_0}(u) = \inf_{u \in \mathcal{M}_{V_0}} \mathcal{E}_{V_0}(u).$$

Remark 3.1. *We have the following variational characterization for d_{V_0} :*

$$d_{V_0} = \inf_{u \in \mathcal{M}_{V_0}} \mathcal{E}_{V_0}(u) = \inf_{u \in \mathbb{Y}_{V_0}^+} \max_{t > 0} \mathcal{E}_{V_0}(tu) = \inf_{u \in \mathbb{S}_{V_0}^+} \max_{t > 0} \mathcal{E}_{V_0}(tu) > 0.$$

The next lemma allows us to assume that the weak limit of a $(PS)_{d_{V_0}}$ sequence of $\mathcal{E}_{V_0}$ is nontrivial.

Lemma 3.2. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ be a $(PS)_{d_{V_0}}$ sequence for $\mathcal{E}_{V_0}$ such that $u_n \rightharpoonup 0$ in $\mathbb{Y}_{V_0}$. Then, one and only one of the following alternatives occurs :*
(a) $u_n \to 0$ in $\mathbb{Y}_{V_0}$, or
(b) there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q\, dx \geq \beta.$$

Démonstration. Assume by contradiction that (b) is not true. Then, by Lemma 2.2 in [3], it follows that

$$u_n \to 0 \quad \text{in } L^\sigma(\mathbb{R}^N) \text{ for all } \sigma \in (p, q_s^*). \tag{3.2}$$

Using (3.2) and (f_1)-(f_2), we have

$$\int_{\mathbb{R}^N} f(u_n)u_n\, dx = o_n(1) \quad \text{as } n \to \infty.$$

On the other hand, arguing as in Lemma 2.5, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, and we may assume that $u_n \rightharpoonup u$ in $\mathbb{Y}_{V_0}$. Taking into account the above facts and that $\langle \mathcal{E}'_{V_0}(u_n), u_n \rangle = o_n(1)$, we get

$$\|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q = \int_{\mathbb{R}^N} f(u_n)u_n\, dx = o_n(1),$$

which implies that $\|u_n\|_\mu \to 0$ as $n \to \infty$, and this is a contradiction because of $\mathcal{E}_{V_0}(u_n) \to d_{V_0} > 0$. Consequently, (a) holds true. $\qquad\square$

Now, we prove an existence result for (3.1).

Theorem 3.1. *Problem (3.1) admits a positive ground state solution.*

Démonstration. Using a variant of the mountain-pass theorem without the (PS)-condition (see [70]), there exists a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ for $\mathcal{E}_{V_0}$ at the level d_{V_0}. Arguing as in Lemma 2.5, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, so we may assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } \mathbb{Y}_{V_0}, \\ u_n &\to u \quad \text{in } L_{loc}^\sigma(\mathbb{R}^N) \text{ for all } \sigma \in [1, p_s^*). \end{aligned}$$

Proceeding as in the proof of Lemma 2.5, we can show that $\mathcal{E}'_{V_0}(u) = 0$.

Now, if $u \neq 0$, then u is a nontrivial solution to (3.1). Assume that $u = 0$. Then $\|u_n\|_{\mathbb{Y}_{V_0}} \not\to 0$ in $\mathbb{Y}_{V_0}$. By Lemma 3.2, we can find a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q dx \geq \beta > 0. \tag{3.3}$$

Let us define

$$\tilde{v}_n(x) := u_n(x + y_n).$$

From the invariance by translations of $\mathbb{R}^N$, it is clear that $\|\tilde{v}_n\|_{\mathbb{Y}_{V_0}} = \|u_n\|_{\mathbb{Y}_{V_0}}$, so $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$ and there exists $\tilde{v}$ such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{Y}_{V_0}$, $\tilde{v}_n \to \tilde{v}$ in $L_{loc}^m(\mathbb{R}^N)$ for any $m \in [1, q_s^*)$ and $\tilde{v} \neq 0$ in view of (3.3). Moreover, $\mathcal{E}_{V_0}(\tilde{v}_n) = \mathcal{E}_{V_0}(u_n)$ and $\mathcal{E}'_{V_0}(\tilde{v}_n) = o_n(1)$, and arguing as before it is easy to check that $\mathcal{E}'_{V_0}(\tilde{v}) = 0$.

Now, let u be the solution obtained before and we prove that u is a ground state solution. It is clear that $d_{V_0} \leq \mathcal{E}_{V_0}(u)$. On the other hand, by Fatou's Lemma and (f_3)-(f_4), we can see that

$$\mathcal{E}_\mu(u) = \mathcal{E}_{V_0}(u) - \frac{1}{q}\langle \mathcal{E}'_{V_0}(u), u \rangle \leq \liminf_{n \to \infty} \left[\mathcal{E}_{V_0}(u_n) - \frac{1}{q}\langle \mathcal{E}'_{V_0}(u_n), u_n \rangle \right] = d_{V_0}$$

which yields $d_{V_0} = \mathcal{E}_{V_0}(u)$.

Finally, we prove that the ground state is positive. Let $u^- := \min\{u, 0\}$. Using $\langle \mathcal{E}'_{V_0}(u), u^- \rangle = 0$, $f(t) = 0$ for $t \leq 0$, and

$$|x - y|^{t-2}(x - y)(x^- - y^-) \geq |x^- - y^-|^t \quad \forall x, y \in \mathbb{R} \quad \forall t > 1,$$

we have

$$\|u^-\|_{s,p}^p + \|u^-\|_{s,q}^q \leq 0$$

which gives $u^- = 0$, that is $u \geq 0$ in $\mathbb{R}^N$. Therefore, $u \geq 0$ and $u \not\equiv 0$ in $\mathbb{R}^N$. Arguing as in Lemma 4.1, we can deduce that $u \in L^\infty(\mathbb{R}^N)$. Moreover, by Corollary 2.1, we obtain that $u \in \mathcal{C}^0(\mathbb{R}^N)$. Arguing as in the proof of Theorem 1.1-(ii) in [49], we conclude that $u > 0$ in $\mathbb{R}^N$. $\square$

Next we give a compactness result for the autonomous problem which will be used in the sequel.

Lemma 3.3. *Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$ be a sequence such that $\mathcal{E}_{V_0}(u_n) \to d_{V_0}$. Then, $\{u_n\}_{n \in \mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}_{V_0}$.*

Démonstration. Since $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$ and $\mathcal{E}_\mu(u_n) \to d_{V_0}$, we apply Lemma 3.1-$(iii)$, Proposition 3.1-$(d)$ and use the definition of d_{V_0} to infer that

$$v_n := m_{V_0}^{-1}(u_n) = \frac{u_n}{\|u_n\|_{\mathbb{Y}_{V_0}}} \in \mathbb{S}_{V_0}^+ \quad \forall n \in \mathbb{N}$$

and

$$\psi_{V_0}(v_n) = \mathcal{E}_\mu(u_n) \to d_{V_0} = \inf_{v \in \mathbb{S}_{V_0}^+} \psi_{V_0}(v).$$

Let us define $\mathcal{F} : \overline{\mathbb{S}}_{V_0}^+ \to \mathbb{R} \cup \{\infty\}$ as

$$\mathcal{F}(u) := \begin{cases} \psi_\mu(u) & \text{if } u \in \mathbb{S}_{V_0}^+ \\ \infty & \text{if } u \in \partial \mathbb{S}_{V_0}^+. \end{cases}$$

We note that

— $(\overline{\mathbb{S}}_{V_0}^+, \delta_{V_0})$, where $\delta_{V_0}(u, v) := \|u - v\|_{\mathbb{Y}_{V_0}}$, is a complete metric space;

— $\mathcal{F} \in \mathcal{C}(\overline{\mathbb{S}}_{V_0}^+, \mathbb{R} \cup \{\infty\})$, by Lemma 3.1-$(iv)$;

— $\mathcal{F}$ is bounded below, by Proposition 3.1-(d).

Applying the Ekeland variational principle [38] to $\mathcal{F}$, we can find $\{\hat{v}_n\}_{n \in \mathbb{N}} \subset \mathbb{S}_{V_0}^+$ such that $\{\hat{v}_n\}_{n \in \mathbb{N}}$ is a $(PS)_{d_{V_0}}$ sequence for ψ_{V_0} on $\mathbb{S}_{V_0}^+$ and $\|\hat{v}_n - v_n\|_{\mathbb{Y}_{V_0}} = o_n(1)$. Then, using Proposition 3.1, Theorem 3.1 and arguing as in the proof of Corollary 2.2, we obtain the thesis. $\square$

3.2. **The barycenter map.** In this subsection, we establish a relation between the topology of M and the number of positive solutions to (2.5). For this reason, we take $\delta > 0$ such that

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\} \subset \Lambda,$$

and consider $\eta \in \mathcal{C}^\infty([0, \infty), [0, 1])$ non increasing such that $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$, $\eta(t) = 0$ if $t \geq \delta$ and $|\eta'(t)| \leq c$ for some $c > 0$.

For any $y \in M$, we define

$$\Psi_{\varepsilon, y}(x) := \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

where $w \in \mathbb{Y}_{V_0}$ is a positive ground state solution to the autonomous problem (3.1) (whose existence is guaranteed by Theorem 3.1). Let $t_\varepsilon > 0$ be the unique number such that

$$\max_{t \geq 0} \mathcal{I}_\varepsilon(t \Psi_{\varepsilon,y}) = \mathcal{I}_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

and consider $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ defined as

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y}.$$

Lemma 3.4. *The functional Φ_ε satisfies*

$$\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \ uniformly \ in \ y \in M.$$

Démonstration. Assume by contradiction that there exist $\delta_0 > 0$, $\{y_n\}_{n \in \mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - d_{V_0}| \geq \delta_0. \tag{3.4}$$

Note that, for each $n \in \mathbb{N}$ and for all $z \in \mathcal{B}_{\frac{\delta}{\varepsilon_n}}$, we have $\varepsilon_n z \in \mathcal{B}_\delta$. Then, $\varepsilon_n z + y_n \in \mathcal{B}_\delta(y_n) \subset M_\delta \subset \Lambda$. Using the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, and recalling that $G = F$ in Λ and $\eta(t) = 0$ for $t \geq \delta$, we can write

$$\begin{aligned}
\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p} \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \frac{t_{\varepsilon_n}^q}{q} \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_{\varepsilon_n} \Psi_{\varepsilon_n,y_n}) \, dx \\
&= \frac{t_{\varepsilon_n}^p}{p} \left([\eta(|\varepsilon_n \cdot|)w]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^p \, dz \right) \\
&\quad + \frac{t_{\varepsilon_n}^q}{q} \left([\eta(|\varepsilon_n \cdot|)w]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^q \, dz \right) \\
&\quad - \int_{\mathbb{R}^N} F(t_{\varepsilon_n} \eta(|\varepsilon_n z|)w(z)) \, dz. \tag{3.5}
\end{aligned}$$

In what follows, we show that the sequence $\{t_{\varepsilon_n}\}_{n \in \mathbb{N}}$ satisfies $t_{\varepsilon_n} \to 1$ as $\varepsilon_n \to 0$. Firstly, we show that $t_{\varepsilon_n} \to t_0 \in [0, \infty)$. By the definition of t_{ε_n}, it follows that $\langle \mathcal{I}'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n) \rangle = 0$, which gives

$$\frac{1}{t_{\varepsilon_n}^p} \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q = \int_{\mathbb{R}^N} \left[\frac{f(t_{\varepsilon_n} \eta(|\varepsilon_n z|)w(z))}{(t_{\varepsilon_n} \eta(|\varepsilon_n z|)w(z))^{q-1}} \right] (\eta(|\varepsilon_n z|)w(z))^q \, dz, \tag{3.6}$$

where we used the fact that $g = f$ on Λ. Since $\eta(|x|) = 1$ for $x \in \mathcal{B}_{\frac{\delta}{2}}$ and $\mathcal{B}_{\frac{\delta}{2}} \subset \mathcal{B}_{\frac{\delta}{\varepsilon_n}}$ for n large enough, from (3.6) it follows that

$$t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \geq \int_{\mathcal{B}_{\frac{\delta}{2}}} \left[\frac{f(t_{\varepsilon_n} w(z))}{(t_{\varepsilon_n} w(z))^{q-1}} \right] |w(z)|^q \, dz.$$

Since w is continuous and positive in $\mathbb{R}^N$, we can find a vector $\hat{z} \in \mathbb{R}^N$ such that

$$w(\hat{z}) = \min_{z \in \overline{\mathcal{B}_{\frac{\delta}{2}}}} w(z) > 0.$$

Then, by (f_4), we deduce that

$$t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \geq \left[\frac{f(t_{\varepsilon_n} w(\hat{z}))}{(t_{\varepsilon_n} w(\hat{z}))^{q-1}} \right] |w(\hat{z})|^q |\mathcal{B}_{\frac{\delta}{2}}|. \tag{3.7}$$

Suppose by contradiction that $t_{\varepsilon_n} \to \infty$. Let us observe that Lemma 2.3 in [3] yields

$$\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},r} \to \|w\|_{s,r} \in (0, \infty) \quad \forall r \in \{p, q\}. \tag{3.8}$$

On the other hand, from $t_{\varepsilon_n} \to \infty$, $q > p$ and (3.8), it follows that

$$t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \to \|w\|_{s,q}^q. \tag{3.9}$$

and by (f_3) we have

$$\lim_{n\to\infty} \frac{f(t_{\varepsilon_n} w(\hat{z}))}{(t_{\varepsilon_n} w(\hat{z}))^{q-1}} = \infty. \tag{3.10}$$

Putting together (3.7), (3.9) and (3.10) we reach a contradiction. Therefore, $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, we may assume that $t_{\varepsilon_n} \to t_0$ for some $t_0 \geq 0$. Indeed, using (3.6), (3.8), (f_1)-(f_2), we can verify that $t_0 > 0$.

Hence, letting $n \to \infty$ in (3.6), and using (3.8) and the dominated convergence theorem, we get

$$t_0^{p-q} \|w\|_{s,p}^p + \|w\|_{s,q}^q = \int_{\mathbb{R}^N} \frac{f(t_0 w)}{(t_0 w)^{q-1}} w^q \, dx. \tag{3.11}$$

Since $w \in \mathcal{N}_{V_0}$, we can see that

$$\|w\|_{s,p}^p + \|w\|_{s,q}^q = \int_{\mathbb{R}^N} f(w) w \, dx. \tag{3.12}$$

Putting together (3.11) and (3.12) we can deduce that

$$(t_0^{p-q} - 1)\|w\|_{s,p}^p = \int_{\mathbb{R}^N} \left[\frac{f(t_0 w)}{(t_0 w)^{q-1}} - \frac{f(w)}{w^{q-1}} \right] w^q \, dx$$

which combined with (f_4) yields $t_0 = 1$. Accordingly, taking the limit as $n \to \infty$ in (3.5), we obtain

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \mathcal{E}_{V_0}(w) = d_{V_0},$$

that is a contradiction thanks to (3.4). $\qquad\square$

Next we prove a compactness result which will be fundamental to prove that the solutions of (2.5) are indeed solutions to (1.3).

Lemma 3.5. *Let $\varepsilon_n \to 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that the translated sequence*

$$\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$$

has a subsequence which converges in $\mathbb{Y}_{V_0}$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}} := \{\varepsilon_n \tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n \to y_0 \in M$.

Démonstration. Since $\langle \mathcal{I}_{\varepsilon_n}'(u_n), u_n \rangle = 0$ and $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$, we can argue as in the proof of Lemma 2.5 to show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_{\varepsilon_n}$. Let us observe that $\|u_n\|_{\mathbb{X}_{\varepsilon_n}} \nrightarrow 0$ since $d_{V_0} > 0$. Therefore, proceeding as in Lemma 3.2, we can find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(\tilde{y}_n)} |u_n|^q dx \geq \beta.$$

Set $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$. Clearly, $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } \mathbb{Y}_{V_0},$$

for some $\tilde{u} \neq 0$. Let $\{t_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ be such that $\tilde{v}_n := t_n \tilde{u}_n \in \mathcal{M}_{V_0}$ (see Lemma 3.1-(i)), and set $y_n := \varepsilon_n \tilde{y}_n$. Then, by $u_n \in \mathcal{N}_{\varepsilon_n}$ and (g_2), it follows that

$$d_{V_0} \leq \mathcal{E}_{V_0}(\tilde{v}_n) \leq \frac{1}{p}[\tilde{v}_n]_{s,p}^p + \frac{1}{q}[\tilde{v}_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q \right) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n) \, dx$$

$$\leq \frac{t_n^p}{p}[u_n]_{s,p}^p + \frac{t_n^q}{q}[u_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x) \left(\frac{t_n^p}{p}|u_n|^p + \frac{t_n^q}{q}|u_n|^q \right) dx - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_n u_n) \, dx$$

$$= \mathcal{I}_{\varepsilon_n}(t_n u_n) \leq \mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1),$$

which gives

$$\mathcal{E}_{V_0}(\tilde{v}_n) \to d_{V_0} \text{ and } \{\tilde{v}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}. \tag{3.13}$$

Moreover, (3.13) implies that $\{\tilde{v}_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, so we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}$. Obviously, $\{t_n\}_{n \in \mathbb{N}}$ is bounded and it holds $t_n \to t_0 \geq 0$. If $t_0 = 0$, from the boundedness of $\{\tilde{u}_n\}_{n \in \mathbb{N}}$, we get $\|\tilde{v}_n\|_{\mathbb{Y}_{V_0}} = t_n \|\tilde{u}_n\|_{\mathbb{Y}_{V_0}} \to 0$, that is $\mathcal{E}_{V_0}(\tilde{v}_n) \to 0$ in contrast with the fact $d_{V_0} > 0$. Then, $t_0 > 0$. From the uniqueness of the weak limit, we have $\tilde{v} = t_0 \tilde{u}$ and $\tilde{u} \neq 0$. By Lemma 3.3, we infer that

$$\tilde{v}_n \to \tilde{v} \text{ in } \mathbb{Y}_{V_0}, \tag{3.14}$$

and consequently $\tilde{u}_n \to \tilde{u}$ in $\mathbb{Y}_{V_0}$. Furthermore,

$$\mathcal{E}_{V_0}(\tilde{v}) = d_{V_0} \text{ and } \langle \mathcal{E}'_{V_0}(\tilde{v}), \tilde{v} \rangle = 0.$$

Next, we show that $\{y_n\}_{n \in \mathbb{N}}$ has a subsequence, still denoted by itself, such that $y_n \to y_0 \in M$. We begin by proving the boundedness of $\{y_n\}_{n \in \mathbb{N}}$. Suppose by contradiction that $\{y_n\}_{n \in \mathbb{N}}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}_{n \in \mathbb{N}}$, such that $|y_n| \to \infty$. Take $R > 0$ such that $\Lambda \subset \mathcal{B}_R$. We may suppose that $|y_n| > 2R$ for n large enough, so, for any $x \in \mathcal{B}_{R/\varepsilon_n}$ we get

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R.$$

Then,

$$\|\tilde{u}_n\|_{s,p}^p + \|\tilde{u}_n\|_{s,q}^q \leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, \tilde{u}_n) \tilde{u}_n \, dx$$

$$\leq \int_{\mathcal{B}_{R/\varepsilon_n}} \tilde{f}(\tilde{u}_n) \tilde{u}_n \, dx + \int_{\mathcal{B}_{R/\varepsilon_n}^c} f(\tilde{u}_n) \tilde{u}_n \, dx.$$

Since $\tilde{u}_n \to \tilde{u}$ in $\mathbb{Y}_{V_0}$, it follows from the dominated convergence theorem that

$$\int_{\mathcal{B}_{R/\varepsilon_n}^c} f(\tilde{u}_n) \tilde{u}_n \, dx = o_n(1).$$

On the other hand, noticing that $\tilde{f}(\tilde{u}_n) \tilde{u}_n \leq \frac{V_0}{K}(|\tilde{u}_n|^p + |\tilde{u}_n|^q)$, we get

$$\|\tilde{u}_n\|_{s,p}^p + \|\tilde{u}_n\|_{s,q}^q \leq \frac{1}{K} \int_{\mathcal{B}_{R/\varepsilon_n}} V_0(|\tilde{u}_n|^p + |\tilde{u}_n|^q) \, dx + o_n(1).$$

Therefore,

$$\left(1 - \frac{1}{K} \right) (\|\tilde{u}_n\|_{s,p}^p + \|\tilde{u}_n\|_{s,q}^q) \leq o_n(1),$$

and recalling that $\tilde{u}_n \to \tilde{u} \neq 0$, we obtain a contradiction.

Thus, $\{y_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}^N$ and, up to a subsequence, we may assume that $y_n \to y_0$. If $y_0 \notin \overline{\Lambda}$, then there exists $r > 0$ such that $y_n \in \mathcal{B}_{r/2}(y_0) \subset \overline{\Lambda}^c$ for any n large enough. Reasoning as before, we get a contradiction. Hence, $y \in \overline{\Lambda}$. In order to prove that $V(y_0) = V_0$, we assume by

contradiction that $V(y_0) > V_0$. Taking into account (3.14), Fatou's Lemma and the invariance of $\mathbb{R}^N$ by translations, we have

$$d_{V_0} = \mathcal{E}_{V_0}(\tilde{v}) < \liminf_{n\to\infty} \left[\frac{1}{p}[\tilde{v}_n]_{s,p}^p + \frac{1}{q}[\tilde{v}_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q \right) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n)\, dx \right]$$
$$\leq \liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0}$$

which does not make sense. By (V_2) we conclude that $y_0 \in M$. $\square$

Now, we introduce the following subset of $\mathcal{N}_\varepsilon$:

$$\widetilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : \mathcal{I}_\varepsilon(u) \leq d_{V_0} + h_1(\varepsilon)\},$$

where $h_1(\varepsilon) := \sup_{y \in M} |\mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) - d_{V_0}|$. By Lemma 3.4, it follows that $h_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. By the definition of $h_1(\varepsilon)$, we know that, for all $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and then $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$.

For any $\delta > 0$ given by Lemma 3.4, we take $\rho = \rho(\delta) > 0$ such that $M_\delta \subset \mathcal{B}_\rho$, and we consider $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\Upsilon(x) := \begin{cases} x & \text{if } |x| < \rho \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)(|u(x)|^p + |u(x)|^q)\, dx}{\displaystyle\int_{\mathbb{R}^N} (|u(x)|^p + |u(x)|^q)\, dx}.$$

Let us note that β_ε has the following property.

Lemma 3.6. *The function β_ε verifies the following limit*

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in M.$$

Démonstration. Suppose by contradiction that there exist $\delta_0 > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{3.15}$$

Using the definitions of $\Phi_{\varepsilon_n}(y_n)$, β_{ε_n}, ψ and the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N}[\Upsilon(\varepsilon_n z + y_n) - y_n](|\eta(|\varepsilon_n z|)\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^q)\, dz}{\int_{\mathbb{R}^N}(|\eta(|\varepsilon_n z|)\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^q)\, dz}.$$

Taking into account $\{y_n\}_{n\in\mathbb{N}} \subset M \subset \mathcal{B}_\rho$ and applying the dominated convergence theorem, we infer that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1)$$

which contradicts (3.15).
                                                                                    $\square$

Lemma 3.7. *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Démonstration. Let $\varepsilon_n \to 0$ as $n \to \infty$. Then there exists $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Our claim is to show that there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0. \tag{3.16}$$

Recalling that $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we deduce that

$$d_{V_0} \leq c_{\varepsilon_n} \leq \mathcal{I}_{\varepsilon_n}(u_n) \leq d_{V_0} + h(\varepsilon_n)$$

which implies that $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. By Lemma 3.5, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n\,\tilde{y}_n \in M_\delta$ for n sufficiently large. Thus

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\displaystyle\int_{\mathbb{R}^N}[\Upsilon(\varepsilon_n\,z + y_n) - y_n](|u_n(z + \tilde{y}_n)|^p + |u_n(z + \tilde{y}_n)|^q)\,dz}{\displaystyle\int_{\mathbb{R}^N}(|u_n(z + \tilde{y}_n)|^p + |u_n(z + \tilde{y}_n)|^q)\,dz}.$$

Since $u_n(\cdot + \tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $\varepsilon_n\,z + y_n \to y \in M$, we deduce that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$, that is (3.16) holds. $\qquad\square$

3.3. Multiple solutions for (2.5).

In this subsection we give a relation between the topology of M and the number of solutions to (2.5). Since $\mathbb{S}_\varepsilon^+$ is not complete, we can not apply directly standard Ljusternik-Schnirelmann theory, but we overcome this difficulty using the abstract results in [68].

Theorem 3.2. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold true. Then, given $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\bar{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_\delta)$, problem (2.5) has at least $cat_{M_\delta}(M)$ positive solutions.*

Démonstration. For any $\varepsilon > 0$, we consider the map $\alpha_\varepsilon : M \to \mathbb{S}_\varepsilon^+$ defined as $\alpha_\varepsilon(y) := m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. By Lemma 3.4, it holds that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y \in M. \tag{3.17}$$

Set

$$\widetilde{\mathcal{S}}_\varepsilon^+ := \{w \in \mathbb{S}_\varepsilon^+ : \psi_\varepsilon(w) \leq d_{V_0} + h_1(\varepsilon)\},$$

where $h_1(\varepsilon) := \sup_{y\in M} |\psi_\varepsilon(\alpha_\varepsilon(y)) - d_{V_0}| \to 0$ as $\varepsilon \to 0$ in view of (3.17). Since $\psi_\varepsilon(\alpha_\varepsilon(y)) \in \widetilde{\mathcal{S}}_\varepsilon^+$ we deduce that $\widetilde{\mathcal{S}}_\varepsilon^+ \neq \emptyset$.

In the light of Lemma 3.4, Lemma 2.6-(iii), Lemma 3.7 and Lemma 3.6, we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the following diagram is well defined for any $\varepsilon \in (0, \bar{\varepsilon})$:

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \alpha_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta.$$

By Lemma 3.6, and decreasing $\bar{\varepsilon}$ if necessary, we obtain that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for all $y \in M$, for some function $\theta(\varepsilon, y)$ satisfying $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$ and for all $\varepsilon \in (0, \bar{\varepsilon})$. Then, $H(t, y) = y + (1-t)\theta(\varepsilon, y)$ with $(t, y) \in [0, 1] \times M$, is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $id : M \to M_\delta$. Consequently,

$$cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M) \geq cat_{M_\delta}(M). \tag{3.18}$$

Applying Corollary 2.2 and Theorem 27 in [68], with $c = c_\varepsilon \leq d_{V_0} + h_1(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, we obtain that Ψ_ε has at least $cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Combining Proposition 2.1-(d) with (3.18), we can infer that $\mathcal{I}_\varepsilon$ admits at least $cat_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. $\qquad\square$

4. PROOF OF THEOREM 1.1

This last section is devoted to the main result of this work. The idea is to show that the solutions obtained in Theorem 3.2 satisfy, for $\varepsilon > 0$ small enough, the estimate $u_\varepsilon(x) \leq a$ for all $x \in \Lambda_\varepsilon^c$. This fact implies that these solutions are indeed solutions of the original problem (1.3). To achieve our purpose, we first provide the following result which plays a fundamental role to study the behavior of the maximum points of the solutions.

Lemma 4.1. *Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution to (2.5). Then $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$, and there exists $\{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ such that $\tilde{u}_n := u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N)$ and $|\tilde{u}_n|_\infty \leq C$ for all $n \in \mathbb{N}$, for some $C > 0$. Moreover,*

$$\tilde{u}_n(x) \to 0 \ as \ |x| \to \infty \ uniformly \ in \ n \in \mathbb{N}. \tag{4.1}$$

Démonstration. Since $\{u_n\}_{n \in \mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$, we can argue as in the proof of Lemma 3.7 to see that $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. By Lemma 3.5, we can find $\{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ such that $\tilde{u}_n := u_n(\cdot + \tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $y_n := \varepsilon_n \tilde{y}_n \to y_0 \in M$. Now we develop a Moser iteration argument [59]. For any $L > 0$ and $\beta > 1$, we take

$$\gamma(\tilde{u}_n) := \tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)} \in \mathbb{X}_\varepsilon,$$

where $\tilde{u}_{n,L} := \min\{\tilde{u}_n, L\}$, as test function in the problem solved by $\tilde{u}_n$ and we have

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sp}} \, dxdy$$

$$+ \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{q-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sq}} \, dxdy$$

$$+ \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|\tilde{u}_n|^p \tilde{u}_{n,L}^{q(\beta-1)} \, dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|\tilde{u}_n|^q \tilde{u}_{n,L}^{q(\beta-1)} \, dx$$

$$= \int_{\mathbb{R}^N} g_n(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n)\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)} \, dx.$$

From the growth assumptions on g, we know that for all $\xi > 0$ there exists $C_\xi > 0$ such that

$$|g(x,t)| \leq \xi|t|^{p-1} + C_\xi|t|^{q_s^*-1} \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Using the above inequality, (V_1) and choosing $\xi \in (0, V_0)$, we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sp}} \, dxdy$$

$$+ \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{q-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n \tilde{u}_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sq}} \, dxdy$$

$$\leq C \int_{\mathbb{R}^N} |\tilde{u}_n|^{q_s^*} \tilde{u}_{n,L}^{q(\beta-1)} \, dx. \tag{4.2}$$

Let us define

$$\lambda(t) := \frac{|t|^q}{q} \quad \text{and} \quad \Gamma(t) := \int_0^t (\gamma'(\tau))^{\frac{1}{q}} d\tau.$$

Since γ is an increasing function, we can infer

$$(a - b)(\gamma(a) - \gamma(b)) \geq 0 \quad \text{for any } a, b \in \mathbb{R}.$$

Combining the above inequality with the Jensen inequality we have

$$\lambda'(a - b)(\gamma(a) - \gamma(b)) \geq |\Gamma(a) - \Gamma(b)|^q \quad \text{for any } a, b \in \mathbb{R},$$

from which

$$|\Gamma(\tilde{u}_n)(x) - \Gamma(\tilde{u}_n)(y)|^q \leq |\tilde{u}_n(x) - \tilde{u}_n(y)|^{q-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n\tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n\tilde{u}_{n,L}^{q(\beta-1)})(y)).$$

We can also note that $\Gamma(\tilde{u}_n) \geq \frac{1}{\beta}\tilde{u}_n\tilde{u}_{n,L}^{\beta-1}$. Thus, by the Sobolev inequality we deduce that

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{q-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n\tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n\tilde{u}_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sq}}\,dxdy$$

$$\geq [\Gamma(\tilde{u}_n)]_{s,q}^q \geq S_*|\Gamma(\tilde{u}_n)|_{q_s^*}^q \geq \frac{S_*}{\beta^q}|\tilde{u}_n\tilde{u}_{n,L}^{\beta-1}|_{q_s^*}^q, \qquad (4.3)$$

where S_* denotes the Sobolev constant of the embedding $W^{s,q}(\mathbb{R}^N) \subset L^{q_s^*}(\mathbb{R}^N)$.

On the other hand, we can see that

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))((\tilde{u}_n\tilde{u}_{n,L}^{q(\beta-1)})(x) - (\tilde{u}_n\tilde{u}_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sp}}\,dxdy$$

$$= \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))[(\tilde{u}_n(x) - \tilde{u}_n(y))\tilde{u}_{n,L}^{q(\beta-1)}(x) + \tilde{u}_n(y)(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))]}{|x-y|^{N+sp}}\,dxdy$$

$$= \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^p}{|x-y|^{N+sp}}\tilde{u}_{n,L}^{q(\beta-1)}(x)\,dxdy$$

$$+ \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))\,\tilde{u}_n(y)\,(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))}{|x-y|^{N+sp}}\,dxdy \geq 0. \qquad (4.4)$$

Indeed,

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))\,\tilde{u}_n(y)\,(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))}{|x-y|^{N+sp}}\,dxdy$$

$$= \int_{\{\tilde{u}_n(x)\geq L\}} \int_{\{\tilde{u}_n(y)\leq L\}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))\,\tilde{u}_n(y)\,(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))}{|x-y|^{N+sp}}\,dxdy$$

$$+ \int_{\{\tilde{u}_n(x)\leq L\}} \int_{\{\tilde{u}_n(y)\leq L\}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))\,\tilde{u}_n(y)\,(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))}{|x-y|^{N+sp}}\,dxdy$$

$$+ \int_{\{\tilde{u}_n(x)\geq L\}} \int_{\{\tilde{u}_n(y)\geq L\}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))\,\tilde{u}_n(y)\,(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))}{|x-y|^{N+sp}}\,dxdy$$

$$+ \int_{\{\tilde{u}_n(x)\leq L\}} \int_{\{\tilde{u}_n(y)\geq L\}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))\,\tilde{u}_n(y)\,(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y))}{|x-y|^{N+sp}}\,dxdy$$

$$=: I + II + III + IV.$$

Note that $III = 0$, and that $I \geq 0$. Indeed, when $\tilde{u}_n(x) \geq L$ and $\tilde{u}_n(y) \leq L$ we have

$$\tilde{u}_n(x) - \tilde{u}_n(y) \geq \tilde{u}_n(x) - L \geq 0 \quad \text{and} \quad \tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y) = L^{q(\beta-1)} - \tilde{u}_n^{q(\beta-1)}(y) \geq 0.$$

On the other hand, when $\tilde{u}_n(x) \leq L$ and $\tilde{u}_n(y) \leq L$, we can see that

$$(\tilde{u}_n(x) - \tilde{u}_n(y))(\tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y)) = (\tilde{u}_n(x) - \tilde{u}_n(y))(\tilde{u}_n^{q(\beta-1)}(x) - \tilde{u}_n^{q(\beta-1)}(y)) \geq 0,$$

then $II \geq 0$. Finally, when $\tilde{u}_n(x) \leq L$ and $\tilde{u}_n(y) \geq L$, we have

$$\tilde{u}_n(x) - \tilde{u}_n(y) \leq L - \tilde{u}_n(y) \leq 0 \quad \text{and} \quad \tilde{u}_{n,L}^{q(\beta-1)}(x) - \tilde{u}_{n,L}^{q(\beta-1)}(y) = \tilde{u}_n^{q(\beta-1)}(x) - L^{q(\beta-1)} \leq 0,$$

which gives $IV \geq 0$. Putting together (4.2), (4.3) and (4.4), we conclude that

$$|\tilde{u}_n \tilde{u}_{n,L}^{\beta-1}|_{q_s^*}^q \leq C\beta^q \int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*} \tilde{u}_{n,L}^{q(\beta-1)} dx. \tag{4.5}$$

Take $\beta = \frac{q_s^*}{q}$ and fix $R > 0$. Then,

$$\int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*} \tilde{u}_{n,L}^{q_s^*-q} dx = \int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*-q} \left(\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q}} \right)^q dx$$

$$= \int_{\{\tilde{u}_n \leq R\}} \tilde{u}_n^{q_s^*-q} \left(\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q}} \right)^q dx + \int_{\{\tilde{u}_n \geq R\}} \tilde{u}_n^{q_s^*-q} \left(\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q}} \right)^q dx$$

$$=: I_1 + I_2.$$

Since $0 \leq \tilde{u}_{n,L} \leq \tilde{u}_n$, we can see that

$$I_1 \leq \int_{\{\tilde{u}_n \leq R\}} R^{q_s^*-q} \tilde{u}_n^{q_s^*} dx,$$

and applying Hölder's inequality with $\frac{q_s^*}{q_s^*-q}$ and $\frac{q_s^*}{q}$, we have

$$I_2 \leq \left(\int_{\{\tilde{u}_n \geq R\}} \tilde{u}_n^{q_s^*} dx \right)^{\frac{q_s^*-q}{q_s^*}} \left(\int_{\mathbb{R}^N} (\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q_s^*}})^{q_s^*} dx \right)^{\frac{q}{q_s^*}}.$$

Since $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ strongly converges in $\mathbb{Y}_{V_0}$, we can take R sufficiently large such that

$$\left(\int_{\{\tilde{u}_n \geq R\}} \tilde{u}_n^{q_s^*} dx \right)^{\frac{q_s^*-q}{q_s^*}} \leq \epsilon \beta^{-q},$$

and consequently

$$I_2 \leq \epsilon \beta^{-q} \left(\int_{\mathbb{R}^N} (\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q_s^*}})^{q_s^*} dx \right)^{\frac{q}{q_s^*}}.$$

Summing up, by the estimates for I_1 and I_2, we obtain

$$\int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*} \tilde{u}_{n,L}^{q_s^*-q} dx \leq \int_{\mathbb{R}^N} R^{q_s^*-q} \tilde{u}_n^{q_s^*} dx + \epsilon \beta^{-q} \left(\int_{\mathbb{R}^N} (\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q_s^*}})^{q_s^*} dx \right)^{\frac{q}{q_s^*}}. \tag{4.6}$$

Combining (4.5) with (4.6) we can infer that

$$\left(\int_{\mathbb{R}^N} (\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q}})^{q_s^*} dx \right)^{\frac{q}{q_s^*}} \leq C\beta^q \int_{\mathbb{R}^N} R^{q_s^*-q} \tilde{u}_n^{q_s^*} dx + C\epsilon \left(\int_{\mathbb{R}^N} (\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q_s^*}})^{q_s^*} dx \right)^{\frac{q}{q_s^*}}.$$

Note that, choosing $0 < \epsilon < \frac{1}{C}$, we have

$$\left(\int_{\mathbb{R}^N} (\tilde{u}_n \tilde{u}_{n,L}^{\frac{q_s^*-q}{q}})^{q_s^*} dx \right)^{\frac{q}{q_s^*}} \leq \bar{C}\beta^q \int_{\mathbb{R}^N} R^{q_s^*-q} \tilde{u}_n^{q_s^*} dx < \infty,$$

and taking the limit as $L \to \infty$ we deduce that $\tilde{u}_n \in L^{\frac{(q_s^*)^2}{q}}(\mathbb{R}^N)$.
Using $0 \leq \tilde{u}_{n,L} \leq \tilde{u}_n$ and letting $L \to \infty$ in (4.5), we get

$$|\tilde{u}_n|_{\beta q_s^*}^{\beta q} \leq C\beta^q \int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*+q(\beta-1)} dx,$$

which yields

$$\left(\int_{\mathbb{R}^N} \tilde{u}_n^{\beta q_s^*} \, dx\right)^{\frac{1}{q(\beta-1)}} \leq (\bar{C}\beta)^{\frac{1}{\beta-1}} \left(\int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*+q(\beta-1)} \, dx\right)^{\frac{1}{q(\beta-1)}}. \tag{4.7}$$

For $m \geq 1$, we set

$$q_s^* + q(\beta_{m+1} - 1) = \beta_m \, q_s^* \quad \text{and} \quad \beta_1 = \frac{q_s^*}{q}.$$

In particular

$$\beta_{m+1} = \beta_1^m(\beta_1 - 1) + 1,$$

and $\lim_{m\to\infty} \beta_m = \infty$. Let us define

$$K_m := \left(\int_{\mathbb{R}^N} \tilde{u}_n^{q_s^*\beta_m} \, dx\right)^{\frac{1}{q_s^*(\beta_m-1)}}.$$

Then (4.7) reads as

$$K_{m+1} \leq (\bar{C}\beta_{m+1})^{\frac{1}{\beta_{m+1}-1}} K_m.$$

A standard iteration argument shows that there exists $C_0 > 0$, independent of m, such that

$$K_{m+1} \leq \prod_{k=1}^m (\bar{C}\beta_{k+1})^{\frac{1}{\beta_{k+1}-1}} K_1 \leq C_0 K_1.$$

Taking the limit as $m \to \infty$ we conclude that $|\tilde{u}_n|_\infty \leq C$ uniformly in $n \in \mathbb{N}$. Now, we note that $\tilde{u}_n$ is such that

$$(-\Delta)_p^s \tilde{u}_n + (-\Delta)_q^s \tilde{u}_n = h_n \quad \text{in } \mathbb{R}^N,$$

where

$$h_n := -V(\varepsilon_n \, x + \varepsilon_n \, \tilde{y}_n)(\tilde{u}_n^{p-1} + \tilde{u}_n^{q-1}) + g(\varepsilon_n \, x + \varepsilon_n \, \tilde{y}_n, \tilde{u}_n).$$

Using the growth assumptions on g, Corollary 2.1 and that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap \mathbb{Y}_{V_0}$, we can deduce that $\tilde{u}_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. $\qquad\square$

Finally, we give the proof of the main result of this work.

Proof of Theorem 1.1. Fix $\delta > 0$ sufficiently small such that $M_\delta \subset \Lambda$.
Claim : There exists $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (2.5), it holds

$$|u_\varepsilon|_{L^\infty(\Lambda_\varepsilon^c)} < a. \tag{4.8}$$

We argue by contradiction. Thus, there exists a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, $u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $\mathcal{I}'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_{\varepsilon_n}|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geq a. \tag{4.9}$$

It is clear that $\mathcal{I}_{\varepsilon_n}(u_{\varepsilon_n}) \to d_{V_0}$ as in the first part of the proof of Lemma 3.5. Then, by Lemma 3.5, we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\tilde{u}_n := u_{\varepsilon_n}(\cdot + \tilde{y}_n) \to \tilde{u}$ in $\mathbb{Y}_{V_0}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

Now, taking $r > 0$ such that $\mathcal{B}_r(y_0) \subset \mathcal{B}_{2r}(y_0) \subset \Lambda$, we get $\mathcal{B}_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Moreover, for any $y \in \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, it holds

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \text{ for } n \text{ sufficiently large.}$$

For these values of n, we have $\Lambda_{\varepsilon_n}^c \subset \mathcal{B}_{\frac{r}{\varepsilon_n}}^c(\tilde{y}_n)$. On the other hand, by (4.1), we know that

$$\tilde{u}_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N},$$

so we can find $R > 0$ such that

$$\tilde{u}_n(x) < a \quad \text{for any } |x| \geq R, n \in \mathbb{N}.$$

Accordingly,

$$u_{\varepsilon_n}(x) < a \text{ for any } x \in \mathcal{B}_R^c(\tilde{y}_n), n \in \mathbb{N}.$$

On the other hand, there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ it holds

$$\Lambda_{\varepsilon_n}^c \subset \mathcal{B}_{\frac{r}{\varepsilon_n}}^c(\tilde{y}_n) \subset \mathcal{B}_R^c(\tilde{y}_n).$$

Therefore, we deduce that $u_{\varepsilon_n}(x) < a$ for any $x \in \Lambda_{\varepsilon_n}^c$ and $n \geq \nu$, which is impossible due to (4.9). This ends the proof of the claim.

Let $\bar{\varepsilon}_\delta > 0$ given by Theorem 3.2, and fix $\varepsilon \in (0, \varepsilon_\delta)$, where $\varepsilon_\delta := \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. In the light of Theorem 3.2, we know that (2.5) admits at least $cat_{M_\delta}(M)$ nontrivial solutions. Let us denote by u_ε one of these solutions. Since $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ satisfies (4.8), by the definition of g it follows that u_ε is also a solution of (1.3). Hence, (1.3) has at least $cat_{M_\delta}(M)$ nontrivial solutions.

Next, we study the behavior of the maximum points of solutions of (1.3). Take $\varepsilon_n \to 0$ and consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{X}_{\varepsilon_n}$ of solutions to (2.5) as above. Let us observe that (g_1) implies that we can find $\gamma \in (0, a)$ such that

$$g(\varepsilon\, x, t)t \leq \frac{V_0}{K}(t^p + t^q) \quad \text{for any } x \in \mathbb{R}^N, 0 \leq t \leq \gamma. \tag{4.10}$$

Arguing as before, we can find $R > 0$ such that

$$|u_n|_{L^\infty(\mathcal{B}_R^c(\tilde{y}_n))} < \gamma. \tag{4.11}$$

Moreover, up to extract a subsequence, we may assume that

$$|u_n|_{L^\infty(\mathcal{B}_R(\tilde{y}_n))} \geq \gamma. \tag{4.12}$$

Indeed, if (4.12) does not hold, it follows from (4.11) that $|u_n|_\infty < \gamma$. Then, combining $\langle \mathcal{I}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ with (4.10), we obtain

$$\|u_n\|_{V_{\varepsilon_n}, p}^p + \|u_n\|_{V_{\varepsilon_n}, q}^q \leq \int_{\mathbb{R}^N} g(\varepsilon_n\, x, u_n)u_n\, dx \leq \frac{V_0}{K} \int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q)\, dx$$

that is $\|u_n\|_{\mathbb{X}_{\varepsilon_n}} = 0$, which does not make sense. Accordingly, (4.12) holds true.

Taking into account (4.11) and (4.12), we can deduce that if p_n is a global maximum point of u_n then $p_n = \tilde{y}_n + q_n$, for some $q_n \in \mathcal{B}_R$. Consequently, $\varepsilon_n\, p_n \to y_0 \in M$ and using the continuity of V we obtain that $V(\varepsilon_n\, p_n) \to V(y_0) = V_0$ as $n \to \infty$. This ends the proof of Theorem 1.1. $\qquad \square$

Remark 4.1. *We suspect that it is possible to prove that the solutions u of (1.3) have a polynomial decay at infinity of the type $0 < u(x) \leq \frac{C}{|x|^\sigma}$ for $|x| \gg 1$, where $\sigma = \sigma(N, s, p, q) > 0$.*

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