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Distributed-Memory Stabilization in a Fractional Cournot–Bertrand Duopoly

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Abstract

This paper generalizes a fractional Cournot–Bertrand duopoly model with delayed self-feedback by replacing discrete memory effects with equal-mean, distributed Gamma memories. The resulting framework preserves the original economic structure while distinguishing the effect of the shape of the memory distribution from that of its average length. We derive the equilibria, the linearized characteristic equations, and real–imaginary crossing conditions for point, weak-Gamma and strong-Gamma memories. For the reference output-feedback configuration, the point-delay benchmark has the reported Hopf threshold $\tau_0 = 3.6355$ at frequency $\psi_0 = 0.5947$, whereas the two Gamma kernels remain separated from an imaginary-axis crossing over the tested equal-mean interval. At the reference crossing frequency, their feedback gains are approximately 0.42 and 0.46, respectively, compared with unit gain for the point delay. Numerical root tracking, stability diagnostics, parameter scans, and solver-convergence checks support the conclusion that distributed aggregation can materially enlarge the practically stable operating region.

Keywords: fractional Cournot–Bertrand duopoly; Caputo derivative; Gamma-distributed delay; Hopf bifurcation; memory effects

MSC: 26A33; 34K18; 91B55



Academic Editor: António Lopes

Received: 5 June 2026

Revised: 29 June 2026

Accepted: 3 July 2026

Published: 6 July 2026

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1. Introduction

Output and price are the two fundamental strategic variables of oligopoly theory. In the Cournot–Bertrand mixed framework, one firm selects the quantity it brings to the market, while the rival selects the price of its differentiated product. The idea of mixed quantity–price competition goes back to Bylka and Komar [1] and was given its modern formulation by Singh and Vives [2], who showed that the choice of the strategic variable is itself an endogenous decision in a differentiated duopoly. Häckner [3] clarified how the degree of product differentiation modifies the classical price–quantity comparisons. In a nonlinear dynamic setting, Ma and Pu [4] and Wang and Ma [5] analyzed boundedly rational Cournot–Bertrand games based on gradient adjustment, showing that the mixed framework can produce a rich bifurcation structure while possessing a larger stability region than the pure Cournot or pure Bertrand counterparts. More recent contributions have studied the role of environmental taxation [6], asymmetric market information [7],

heterogeneous players [8], and demand information sharing [9] within the same family of mixed duopoly models.

A second strand of the literature emphasizes that real firms remember. Decisions on output and price are not based exclusively on current observations: accumulated market shares, past cost trends, and the entire history of the rival's strategies enter the adjustment process with decaying weights. Fractional calculus provides a natural language for this hereditary behavior, since the Caputo derivative weights the whole past of the state variable with a power-law kernel [10–12]. Discrete and continuous fractional-order oligopoly games have been investigated intensively in the last few years: discrete fractional Cournot games with long memory [13], fractional Bertrand games [14], fractional-order discrete Cournot–Bertrand competition [15], differentiated fractional Cournot duopolies [16], fractional Cournot games on networks [17], and conformable Bertrand games with fractional delays [18]. Closely related fractional-order delayed models have been analyzed for duopoly games in the insurance market [19], for Bertrand duopolies with nonidentical delays [20], for financial systems [21–24], for economic growth with environmental pollution [25], and outside economics, for predator–prey and epidemiological systems [26–28], neural networks [29], and chemical reaction models [30,31]. Recent works also document the rich dynamics of fractional models simulated with modern numerical schemes [32,33].

Time delays constitute the third ingredient. Production takes time, information arrives late, and decisions are implemented with lags; consequently, delayed feedback is an unavoidable feature of market adjustment. Within this stream, Wang, Gu, Liu and Xu [34] recently combined the three ingredients above and proposed a fractional-order Cournot–Bertrand duopoly with two discrete self-feedback delays. Using the stability theory of fractional delayed systems initiated by Matignon [35], they derived stability criteria and Hopf bifurcation thresholds in six delay scenarios and validated them numerically.

A point delay is, however, an extreme description of managerial memory: it assumes that the firm reacts to exactly one past observation, the one recorded τ periods earlier. In practice, firms react to weighted averages of past data, moving averages of demand, smoothed inventory reports, processed forecasts. Continuously distributed delays, introduced into economic dynamics through the linear-chain technique by Cushing [36] and MacDonald [37], represent precisely this averaging behavior. Chiarella and Szidarovszky [38] studied dynamic oligopolies with continuously distributed lags; Matsumoto, Chiarella and Szidarovszky analyzed boundedly distributed delay in a dynamic monopoly [39], while Matsumoto and Szidarovszky considered multiple continuously distributed time delays [40]. More recently, Cúlda, Kaslik and Neamțu [41] performed a complete stability and bifurcation analysis of a general Cournot duopoly with four distributed delays, and the same group studied discrete-time mixed oligopolies with multiple delays [42]; delayed two-stage Cournot models with R&D spillovers were investigated in [43]. To the best of our knowledge, the interaction between Caputo fractional dynamics and Gamma-distributed self-feedback memory has not yet been examined in the Cournot–Bertrand mixed framework, and this is the gap filled by the present paper.

Our purpose is deliberately comparative. We keep the economic structure and all parameter values of the reference model [34] unchanged and replace only the concentrated self-feedback delays with weak and strong Gamma kernels normalized to the same mean memory. This normalization is essential: if different parameters or different average lags were used, one could not decide whether a different stability behavior originates from the memory shape or merely from a different market configuration. With the mean fixed, the comparison isolates the effect of the shape of the memory distribution.

The contribution of this paper is fourfold. First, the model is formulated with a general distributed self-feedback memory and specialized to the weak and strong Gamma kernels; the discrete-delay benchmark is recovered as the Dirac limit. Second, all equilibrium, linearization and characteristic-equation computations are carried out explicitly, and the imaginary-axis crossing systems for the three memory mechanisms are derived in a unified real–imaginary framework. Third, we prove rigorous structural results: the kernels’ normalization, mean and Laplace transforms; a uniform bound for the characteristic roots in the closed right half-plane; the non-vanishing of the characteristic function at the origin; and a crossing/transversality criterion that locates the Hopf threshold. Fourth, the numerical section is rebuilt in the style of the reference paper: it contains stable and post-Hopf trajectories with phase portraits, an imaginary-axis crossing diagnostic, a numerical Hopf diagram, a frequency–gain figure that explains the stabilization mechanism, a second diagnostic for the common-delay case, reference-style $\pm 5\%$ perturbation experiments, and sensitivity analyses with respect to the cost and differentiation parameters. The manuscript additionally provides: (a) a transparent computational workflow and mesh-refinement protocol; (b) a two-parameter frequency-domain stability-proximity map; (c) a complex-plane continuation of the dominant characteristic roots for the point-delay benchmark; (d) a compact cross-model comparison table; (e) a near-threshold amplitude-scaling diagnostic consistent with a supercritical point-delay Hopf scenario, without identifying a coordinate-dependent first Lyapunov coefficient; and (f) a precise delimitation of what the present analysis does and does not establish about coexistence of attractors and global dynamics. More specifically, the conceptual contribution is not merely to combine fractional dynamics and distributed delays; it is to put point, weak-Gamma and strong-Gamma self-feedback into one characteristic framework with the same economic benchmark and the same mean memory and then to identify frequency-dependent gain attenuation as the mechanism that separates the distributed cases from the destabilizing point delay.

This paper is organized as follows: Section 2 recalls the reference model and introduces its distributed-memory extension. Section 3 defines the weak and strong Gamma kernels and proves their basic properties. Section 4 computes the equilibria. Section 5 derives the linearized system and the characteristic equation. Section 6 establishes the imaginary-axis crossing equations and the main stability/Hopf criterion. Section 7 presents the numerical comparison with the reference paper, and Section 8 discusses the economic interpretation. Section 9 concludes and outlines future research.

2. The Reference Cournot–Bertrand Model and Its Distributed-Memory Extension

Firm 1 chooses output $q_1(t)$, and firm 2 chooses price $p_2(t)$. Following [34], the inverse demands faced by firm 1 and the demand faced by firm 2 are

$$p_1(t) = 1 - d - q_1(t) + d p_2(t) + d^2 q_1(t), \quad q_2(t) = 1 - d q_1(t) - p_2(t), \quad (1)$$

where $d \in (-1, 1)$ measures product differentiation: the goods are substitutes for $d > 0$ and complements for $d < 0$. With common marginal cost $c \in (0, 1)$, the profit functions are

$$\begin{cases} \Pi_1(t) = q_1(t) [1 - c - d - q_1(t) + d p_2(t) + d^2 q_1(t)], \\ \Pi_2(t) = (p_2(t) - c) [1 - d q_1(t) - p_2(t)]. \end{cases} \quad (2)$$

Since $\Pi_1 = q_1 \{1 - c - d + d p_2 - (1 - d^2) q_1\}$, differentiation with respect to q_1 gives the marginal profit

$$\frac{\partial \Pi_1}{\partial q_1} = 1 - c - d + d p_2 - 2(1 - d^2) q_1, \quad (3)$$

while

$$\frac{\partial \Pi_2}{\partial p_2} = (1 - d q_1 - p_2) - (p_2 - c) = 1 + c - d q_1 - 2 p_2. \quad (4)$$

Both firms are boundedly rational and adjust their strategic variables proportionally to the marginal profit (gradient dynamics). To describe the hereditary character of the profit-realization process, the adjustment is formulated with the Caputo derivative of order $\alpha \in (0, 1)$,

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\sigma)}{(t-\sigma)^\alpha} d\sigma.$$

In [34], the self-feedback terms q_1 and p_2 appearing in the marginal profits in (3) and (4) are evaluated at the delayed times $t - \tau_1$ and $t - \tau_2$. In this paper, those delayed values are replaced by distributed averages. Let $K_1, K_2 : [0, \infty) \rightarrow [0, \infty)$ be probability densities, with $\int_0^\infty K_i(s) ds = 1$, and define

$$Q_1(t) = \int_0^\infty K_1(s) q_1(t-s) ds, \quad P_2(t) = \int_0^\infty K_2(s) p_2(t-s) ds. \quad (5)$$

The distributed-memory fractional Cournot–Bertrand model is

$$\begin{cases} {}^C D_t^\alpha q_1(t) = v_1 q_1(t) \{1 - c - d + d p_2(t) - 2(1 - d^2) Q_1(t)\}, \\ {}^C D_t^\alpha p_2(t) = v_2 p_2(t) \{1 + c - d q_1(t) - 2 P_2(t)\}, \end{cases} \quad (6)$$

with adjustment speeds $v_1, v_2 > 0$. The point-delay model of [34] is recovered by the singular choice $K_i(s) = \delta(s - \tau_i)$, which gives $Q_1(t) = q_1(t - \tau_1)$ and $P_2(t) = p_2(t - \tau_2)$. System (6) therefore contains the reference model as a limiting case, and any difference in the dynamics is attributable solely to the shape of the memory.

3. Weak and Strong Gamma Kernels

Among distributed delays, the Gamma family is the standard choice in economic dynamics [36–38,40,41] because it interpolates between instantaneous feedback and the Dirac point delay while remaining analytically tractable. We use the two classical members, both normalized to the same mean memory $T > 0$.

Definition 1. The weak Gamma kernel (exponential kernel) with mean $T > 0$ is

$$K_w(s; T) = \frac{1}{T} e^{-s/T}, \quad s \geq 0, \quad (7)$$

and the strong Gamma kernel (Erlang kernel of order two) with the same mean T is

$$K_s(s; T) = \frac{4s}{T^2} e^{-2s/T}, \quad s \geq 0. \quad (8)$$

Proposition 1. For every $T > 0$, kernels (7) and (8) are probability densities on $[0, \infty)$ with mean T , and their Laplace transforms $\hat{K}(\lambda; T) = \int_0^\infty e^{-\lambda s} K(s; T) ds$, $\text{Re } \lambda > -1/T$ resp. $\text{Re } \lambda > -2/T$, are

$$\hat{K}_w(\lambda; T) = \frac{1}{1 + \lambda T}, \quad \hat{K}_s(\lambda; T) = \frac{1}{(1 + \lambda T/2)^2}. \quad (9)$$

Proof. Using $\int_0^\infty s^n e^{-as} ds = n!/a^{n+1}$ for $a > 0$ and $n \in \mathbb{N}_0$, we compute for the weak kernel as

$$\int_0^\infty K_w ds = \frac{1}{T} \cdot \frac{1}{1/T} = 1, \quad \int_0^\infty s K_w ds = \frac{1}{T} \cdot \frac{1}{(1/T)^2} = T,$$

and, for $\text{Re}(\lambda + 1/T) > 0$,

$$\hat{K}_w(\lambda; T) = \frac{1}{T} \int_0^\infty e^{-(\lambda+1/T)s} ds = \frac{1}{T} \cdot \frac{1}{\lambda + 1/T} = \frac{1}{1 + \lambda T}.$$

For the strong kernel,

$$\int_0^\infty K_s ds = \frac{4}{T^2} \cdot \frac{1!}{(2/T)^2} = 1, \quad \int_0^\infty s K_s ds = \frac{4}{T^2} \cdot \frac{2!}{(2/T)^3} = T,$$

and, for $\text{Re}(\lambda + 2/T) > 0$,

$$\hat{K}_s(\lambda; T) = \frac{4}{T^2} \int_0^\infty s e^{-(\lambda+2/T)s} ds = \frac{4}{T^2} \cdot \frac{1}{(\lambda + 2/T)^2} = \frac{1}{(1 + \lambda T/2)^2}. \quad \square$$

The equality of the means is the key normalization: comparing the weak kernel, the strong kernel and the Dirac delay $\delta(s - T)$ at the same T is a comparison of memory shapes not of different average lags. It does not make the three mechanisms dynamically identical: the Dirac kernel concentrates all feedback weight at one lag, whereas the Gamma kernels spread it over a continuum of shorter and longer lags. Thus “equal mean” is a controlled comparison of average memory length, not a claim of equal effective delay at every frequency. Figure 1 displays the two kernels at the mean $T = 3.6355$, the first Hopf threshold reported in [34] for the output-feedback case. The weak kernel is monotonically decreasing and weights the most recent past most heavily; the strong kernel vanishes at $s = 0$, peaks at the positive lag $s = T/2$, and then decays, so that it concentrates information around a positive lag without concentrating it at a single lag. Economically, this does not mean that a firm ignores current market information; the contemporaneous variables $q_1(t)$ and $p_2(t)$ remain in the adjustment equations. The zero value at $s = 0$ concerns only the delayed self-feedback aggregate; it models a two-stage reporting/processing pipeline in which a current observation is not instantaneously fed back into the self-correction channel but becomes influential after collection and processing.

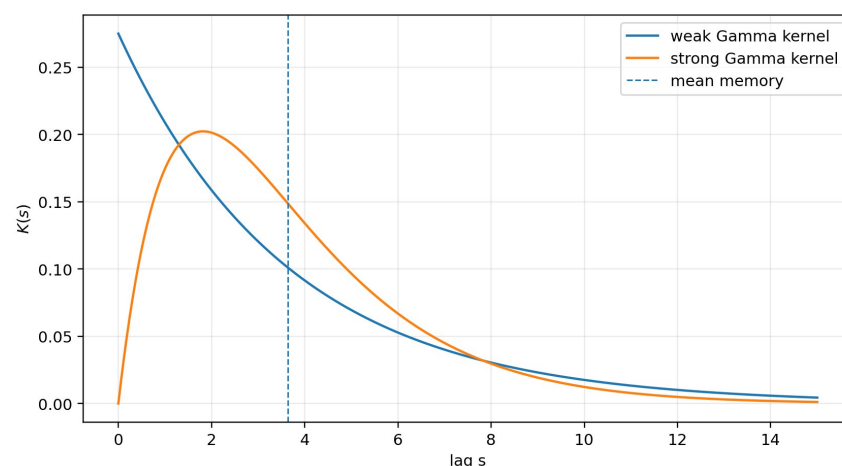


Figure 1. Weak and strong Gamma kernels with the same mean memory $T = 3.6355$; the first Hopf threshold is reported for the reference point-delay output-feedback case. The weak kernel weights the recent past most strongly; the strong kernel peaks at the positive lag $s = T/2$ and assigns zero weight to the present instant. The vertical dashed line marks the common mean.

Remark 1 (Linear-chain reduction). The Gamma kernels admit the classical linear-chain representation [36,37]. For the weak kernel, if

$$w(t) = \int_0^\infty K_w(s; T) q_1(t-s) ds,$$

then differentiating under the integral sign gives the exact first-order equation

$$\dot{w}(t) = \frac{1}{T}(q_1(t) - w(t)).$$

For the strong Gamma kernel, the equivalent chain has two stages with rate $2/T$,

$$\dot{z}_1(t) = \frac{2}{T}(q_1(t) - z_1(t)), \quad \dot{z}_2(t) = \frac{2}{T}(z_1(t) - z_2(t)),$$

and the distributed memory is $Q_1(t) = z_2(t)$. The same construction is used for the price-memory variable. Hence the distributed model in (6) is equivalent to a finite-dimensional system of mixed fractional/integer order, which is the structure integrated in the numerical experiments in Section 7.

4. Equilibria

Because the kernels have unit mass, constant functions are reproduced by the convolution: if $(q_1, p_2) \equiv (\bar{q}, \bar{p})$ then $Q_1 \equiv \bar{q}$ and $P_2 \equiv \bar{p}$. Consequently the equilibria of (6) coincide with those of the point-delay model and satisfy

$$\bar{q} \{1 - c - d + d\bar{p} - 2(1 - d^2)\bar{q}\} = 0, \quad \bar{p} \{1 + c - d\bar{q} - 2\bar{p}\} = 0.$$

Proposition 2. System (6) has the algebraic boundary equilibria

$$E_0 = (0, 0), \quad E_p = \left(0, \frac{1+c}{2}\right), \quad E_q = \left(\frac{1-c-d}{2(1-d^2)}, 0\right),$$

where E_q is economically admissible only when $1 - c - d > 0$, and the unique interior equilibrium

$$E_* = (q_1^*, p_2^*), \quad q_1^* = \frac{2 + cd - 2c - d}{4 - 3d^2}, \quad p_2^* = \frac{2 + 2c + cd - d - d^2 - 2cd^2}{4 - 3d^2}, \quad (10)$$

which is positive whenever

$$4 - 3d^2 > 0, \quad 2 + cd - 2c - d > 0, \quad 2 + 2c + cd - d - d^2 - 2cd^2 > 0. \quad (11)$$

Proof. At an interior equilibrium both braces vanish:

$$1 - c - d + d\bar{p} - 2(1 - d^2)\bar{q} = 0, \quad 1 + c - d\bar{q} - 2\bar{p} = 0.$$

The second equation gives $\bar{p} = (1 + c - d\bar{q})/2$. By substituting it into the first, we have

$$1 - c - d + \frac{d}{2}(1 + c - d\bar{q}) - 2(1 - d^2)\bar{q} = 0 \iff 2(1 - d^2)\bar{q} + \frac{d^2}{2}\bar{q} = 1 - c - d + \frac{d}{2}(1 + c).$$

By multiplying by 2 and collecting terms, we have

$$(4 - 4d^2 + d^2)\bar{q} = (4 - 3d^2)\bar{q} = 2 - 2c - 2d + d + cd = 2 + cd - 2c - d,$$

which yields q_1^* in (10). Back-substitution gives

$$p_2^* = \frac{1+c-dq_1^*}{2} = \frac{(1+c)(4-3d^2)-d(2+cd-2c-d)}{2(4-3d^2)} = \frac{2+2c+cd-d-d^2-2cd^2}{4-3d^2},$$

after expanding $(1+c)(4-3d^2)-d(2+cd-2c-d) = 4+4c-3d^2-3cd^2-2d-cd^2+2cd+d^2 = 4+4c+2cd-2d-2d^2-4cd^2$ and dividing numerator and denominator by 2. Positivity of q_1^* and p_2^* is equivalent to (11). $\square$

The conditions in (11) are exactly the assumption (κ_1) of the reference paper [34], so all parameter sets used there remain admissible here.

5. Linearization and Characteristic Equations

Let $u(t) = q_1(t) - q_1^*$ and $v(t) = p_2(t) - p_2^*$. By linearity of the convolution,

$$Q_1(t) = q_1^* + \int_0^\infty K_1(s) u(t-s) ds, \quad P_2(t) = p_2^* + \int_0^\infty K_2(s) v(t-s) ds.$$

Write the first equation of (6) as ${}^C D_t^\alpha q_1 = v_1 q_1 G_1(p_2, Q_1)$, with $G_1(p, Q) = 1 - c - d + dp - 2(1 - d^2)Q$. Since $G_1(p_2^*, q_1^*) = 0$ at E_* , the Jacobian contribution of factor q_1 vanishes, and the first-order expansion retains only $v_1 q_1^* dG_1$. The same argument applies to the second equation. The linearization of (6) at E_* is therefore

$$\begin{cases} {}^C D_t^\alpha u(t) = e_1 v(t) + e_3 \int_0^\infty K_1(s) u(t-s) ds, \\ {}^C D_t^\alpha v(t) = e_2 u(t) + e_4 \int_0^\infty K_2(s) v(t-s) ds, \end{cases} \quad (12)$$

with

$$e_1 = d v_1 q_1^*, \quad e_2 = -d v_2 p_2^*, \quad e_3 = -2(1 - d^2) v_1 q_1^*, \quad e_4 = -2v_2 p_2^*. \quad (13)$$

For $d \in (0, 1)$ and $q_1^*, p_2^* > 0$, one has $e_1 > 0$, $e_2 < 0$, $e_3 < 0$, and $e_4 < 0$. By taking Laplace transforms in (12) and using $\mathcal{L}\{{}^C D_t^\alpha u\}(\lambda) = \lambda^\alpha U(\lambda) - \lambda^{\alpha-1}u(0)$ together with $\mathcal{L}\{\int_0^\infty K(s)u(\cdot-s)ds\}(\lambda) = \widehat{K}(\lambda)U(\lambda) + (\text{terms from the initial history})$, the homogeneous part of the transformed system reads. Equivalently, the characteristic equation is obtained by inserting the exponential-mode ansatz $(u(t), v(t)) = (U, V)e^{\lambda t}$ into the homogeneous linearized system; then $\int_0^\infty K(s)e^{\lambda(t-s)}ds = \widehat{K}(\lambda)e^{\lambda t}$. The prescribed initial values and prehistory affect only the inhomogeneous transform terms and not the modal determinant.

$$\begin{pmatrix} \lambda^\alpha - e_3 \widehat{K}_1(\lambda) & -e_1 \\ -e_2 & \lambda^\alpha - e_4 \widehat{K}_2(\lambda) \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix},$$

where λ^α denotes the principal branch. Nontrivial modes correspond to the vanishing of the determinant, i.e., to the characteristic equation

$$\Delta(\lambda) = (\lambda^\alpha - e_3 \widehat{K}_1(\lambda))(\lambda^\alpha - e_4 \widehat{K}_2(\lambda)) - e_1 e_2 = 0. \quad (14)$$

The three memory mechanisms are encoded by the transform alone:

$$\widehat{K}_i(\lambda) = e^{-\lambda \tau_i} \text{ (point delay), } \widehat{K}_i(\lambda) = \frac{1}{1 + \lambda T_i} \text{ (weak), } \widehat{K}_i(\lambda) = \frac{1}{(1 + \lambda T_i/2)^2} \text{ (strong).}$$

If there is no self-feedback memory, $\hat{K}_1 = \hat{K}_2 \equiv 1$, and (14) reduces to the quadratic

$$z^2 - (e_3 + e_4)z + e_3e_4 - e_1e_2 = 0, \quad z = \lambda^\alpha. \quad (15)$$

By Matignon's theorem [35], the equilibrium of the memoryless fractional system is locally asymptotically stable if both roots z_j of (15) satisfy the sector condition

$$|\arg z_j| > \frac{\alpha\pi}{2}. \quad (16)$$

With $e_3 + e_4 < 0$ and $e_3e_4 - e_1e_2 > 0$ (assumption (κ_2) of [34]), both roots are real and negative, so (16) holds for every $\alpha \in (0, 1)$.

6. Imaginary-Axis Crossing and the Hopf Criterion

A loss of stability through an oscillatory mode requires a pair of characteristic roots on the imaginary axis. By substituting $\lambda = i\omega$, $\omega > 0$, into (14) and using the principal branch

$$(i\omega)^\alpha = \omega^\alpha \left(\cos \frac{\alpha\pi}{2} + i \sin \frac{\alpha\pi}{2} \right) = A + iB, \quad A = \omega^\alpha \cos \frac{\alpha\pi}{2}, \quad B = \omega^\alpha \sin \frac{\alpha\pi}{2},$$

we obtain explicit crossing systems for each configuration analyzed in the reference paper. If only the output feedback is distributed, $K_2(s) = \delta(s)$, the characteristic function becomes

$$\Delta_q(\lambda, T) = (\lambda^\alpha - e_3\hat{K}(\lambda; T))(\lambda^\alpha - e_4) - e_1e_2. \quad (17)$$

By writing $\hat{K}(i\omega; T) = H_R(\omega, T) + iH_I(\omega, T)$ and expanding

$$\Delta_q(i\omega, T) = [(A - e_3H_R) + i(B - e_3H_I)] [(A - e_4) + iB] - e_1e_2,$$

the real and imaginary parts give the crossing system

$$(A - e_3H_R)(A - e_4) - (B - e_3H_I)B - e_1e_2 = 0, \quad (18)$$

$$(A - e_3H_R)B + (B - e_3H_I)(A - e_4) = 0. \quad (19)$$

From (9),

$$\frac{1}{1 + i\omega T} = \frac{1 - i\omega T}{1 + \omega^2 T^2} \implies H_R = \frac{1}{1 + \omega^2 T^2}, \quad H_I = -\frac{\omega T}{1 + \omega^2 T^2} \quad (\text{weak}), \quad (20)$$

and by setting $x = \omega T/2$,

$$\frac{1}{(1 + ix)^2} = \frac{(1 - ix)^2}{(1 + x^2)^2} \implies H_R = \frac{1 - x^2}{(1 + x^2)^2}, \quad H_I = -\frac{2x}{(1 + x^2)^2} \quad (\text{strong}), \quad (21)$$

while for the point-delay benchmark, $e^{-i\omega\tau} = \cos \omega\tau - i \sin \omega\tau$, i.e.,

$$H_R = \cos(\omega\tau), \quad H_I = -\sin(\omega\tau). \quad (22)$$

If both feedback variables use the same kernel with the same parameter, the characteristic function is

$$\Delta_b(\lambda, T) = (\lambda^\alpha - e_3\hat{K}(\lambda; T))(\lambda^\alpha - e_4\hat{K}(\lambda; T)) - e_1e_2, \quad (23)$$

and the same expansion yields

$$(A - e_3 H_R)(A - e_4 H_R) - (B - e_3 H_I)(B - e_4 H_I) - e_1 e_2 = 0, \quad (24)$$

$$(A - e_3 H_R)(B - e_4 H_I) + (B - e_3 H_I)(A - e_4 H_R) = 0. \quad (25)$$

Before stating the main criterion, we prove two structural facts: the characteristic roots that may compromise stability are confined to a bounded set, and the origin is never a root.

Lemma 1. Let $\widehat{K}_1, \widehat{K}_2$ be Laplace transforms of probability densities on $[0, \infty)$ (including the Dirac case). Then every root λ of (14) with $\operatorname{Re} \lambda \geq 0$ satisfies

$$|\lambda|^\alpha \leq \rho^+ := \frac{|e_3| + |e_4| + \sqrt{(|e_3| + |e_4|)^2 + 4(|e_3 e_4| + |e_1 e_2|)}}{2}. \quad (26)$$

Proof. For $\operatorname{Re} \lambda \geq 0$,

$$|\widehat{K}_i(\lambda)| = \left| \int_0^\infty e^{-\lambda s} K_i(s) ds \right| \leq \int_0^\infty e^{-s \operatorname{Re} \lambda} K_i(s) ds \leq \int_0^\infty K_i(s) ds = 1,$$

and for the principal branch with $\lambda = r e^{i\theta}$, $|\theta| \leq \pi/2$, one has $|\lambda^\alpha| = r^\alpha = |\lambda|^\alpha$. Expanding (14),

$$\lambda^{2\alpha} = (e_3 \widehat{K}_1 + e_4 \widehat{K}_2) \lambda^\alpha - e_3 e_4 \widehat{K}_1 \widehat{K}_2 + e_1 e_2,$$

so by the triangle inequality

$$|\lambda|^{2\alpha} \leq (|e_3| + |e_4|) |\lambda|^\alpha + |e_3 e_4| + |e_1 e_2|.$$

Thus $x = |\lambda|^\alpha$ satisfies $x^2 - (|e_3| + |e_4|)x - (|e_3 e_4| + |e_1 e_2|) \leq 0$, which forces $x \leq \rho^+$. $\square$

Proposition 3. Assume that $e_3 e_4 - e_1 e_2 \neq 0$ (in particular, under assumption (κ_2) of [34], $e_3 e_4 - e_1 e_2 > 0$). Then $\lambda = 0$ is not a root of (14) for any kernel and any memory parameter.

Proof. At $\lambda = 0$, $\lambda^\alpha = 0$ and $\widehat{K}_i(0) = \int_0^\infty K_i = 1$; hence $\Delta(0) = (-e_3)(-e_4) - e_1 e_2 = e_3 e_4 - e_1 e_2 \neq 0$. $\square$

Theorem 1. Assume (11), $e_3 + e_4 < 0$, and $e_3 e_4 - e_1 e_2 > 0$, so that memoryless system (15) satisfies Matignon's condition (16). Fix one of the kernel families above with the memory parameter $T \in (0, T_{\max}]$, and let $\Delta(\lambda, T)$ denote the associated characteristic function (17) or (23). Then we have the following:

- (i) If the corresponding crossing system, (18) and (19) or (24) and (25), has no solution (ω, T) with $\omega > 0$ and $T \in (0, T_{\max}]$, then $\Delta(\cdot, T)$ has no roots in the closed right half-plane for any $T \in (0, T_{\max}]$, and the equilibrium E_* remains locally asymptotically stable on this interval.
- (ii) If the crossing system has a solution (ω_0, T_0) at which $\partial_\lambda \Delta(i\omega_0, T_0) \neq 0$ and the transversality condition

$$\operatorname{Re} \left(\frac{d\lambda}{dT} \right) \Big|_{(i\omega_0, T_0)} = \operatorname{Re} \left(- \frac{\partial_T \Delta}{\partial_\lambda \Delta} \right) \Big|_{(i\omega_0, T_0)} \neq 0 \quad (27)$$

holds, then a simple conjugate pair of characteristic roots crosses the imaginary axis transversally at $T = T_0$. Therefore T_0 is a Hopf crossing of the linearized fractional distributed-delay system. If no earlier crossing occurs for $0 < T < T_0$, then T_0 is the first Hopf threshold.

Proof. (i) For each fixed T , the function $\lambda \mapsto \Delta(\lambda, T)$ is analytic in the open right half-plane and continuous up to the imaginary axis; λ^α (principal branch) is analytic on $\mathbb{C} \setminus (-\infty, 0]$, the Gamma transforms are rational with poles at $\lambda = -1/T$ or $\lambda = -2/T$ in the open left half-plane, and $e^{-\lambda\tau}$ is entire. Moreover, for every compact subset of the closed right half-plane that avoids the origin, Δ is jointly continuous in (λ, T) , and as $T \rightarrow 0^+$, the Gamma transforms converge to 1 uniformly on compact subsets. Hence the characteristic function converges locally uniformly to the memoryless characteristic function, which has no roots with $\operatorname{Re} \lambda \geq 0$ by (16). By Lemma 1, all roots that could enter the closed right half-plane lie in the compact set $\mathcal{K} = \{\lambda : \operatorname{Re} \lambda \geq 0, |\lambda| \leq (\rho^+)^{1/\alpha}\}$, uniformly in T . Proposition 3 excludes the origin from the root set. Therefore, by the argument principle applied on a large right half-disk with a small indentation around the origin and then passing to the limit in the indentation radius, the number of roots in $\operatorname{Re} \lambda > 0$ can change only if a root crosses the imaginary axis at some $\lambda = i\omega$, $\omega > 0$. Such a crossing is exactly equivalent to one of the real-imaginary systems, (18) and (19) or (24) and (25). If no solution of the relevant crossing system exists for $0 < T \leq T_{\max}$, the number of right-half-plane roots remains equal to its memoryless value, namely, zero. Local asymptotic stability of E_* then follows from the stability theory of fractional functional differential equations, for which the absence of characteristic roots in the closed right half-plane (with the principal branch) is the stability criterion [10,35].

(ii) Since $\partial_\lambda \Delta(i\omega_0, T_0) \neq 0$, the implicit-function theorem applied to the analytic equation $\Delta(\lambda, T) = 0$ yields a unique C^1 root branch $\lambda(T)$, with $\lambda(T_0) = i\omega_0$ and $\lambda'(T) = -\partial_T \Delta / \partial_\lambda \Delta$. Condition (27) states that the real part of this branch has nonzero derivative at T_0 ; hence the conjugate pair crosses the imaginary axis transversally. If T_0 is the first such crossing, the equilibrium changes stability there; otherwise the statement identifies a transverse Hopf crossing of the linearized dynamics. $\square$

Remark 2. Following the established practice for fractional delayed systems [19,20,27,28,34], the transversal imaginary-axis crossing of Theorem 1(ii) is referred to as a Hopf bifurcation of the fractional model, and the simulations of Section 7 indeed display the onset of sustained oscillations beyond T_0 . We emphasize that a complete nonlinear normal-form analysis (direction and stability of the bifurcating periodic solutions) for Caputo systems with distributed delay is, to our knowledge, not yet available in the literature; for this reason we confine the rigorous statement to the characteristic-root crossing and treat the post-threshold periodic regime as a numerically documented phenomenon.

Remark 3 (Why distributed memory is harder to destabilize). The crossing equations clarify the mechanism. On the imaginary axis the point delay has constant unit modulus, $|e^{-i\omega\tau}| = 1$, for all ω ; the delayed channel transmits every frequency at full strength, only rotating its phase, and a suitable τ can always align the phase so as to push the system toward resonance. The Gamma kernels, in contrast, are low-pass filters,

$$|\widehat{K}_w(i\omega; T)| = \frac{1}{\sqrt{1 + \omega^2 T^2}}, \quad |\widehat{K}_s(i\omega; T)| = \frac{1}{1 + \omega^2 T^2/4},$$

both strictly decreasing in ω and vanishing as $\omega \rightarrow \infty$. At the frequencies relevant for the crossing, the feedback strength $|e_3| |\widehat{K}|$ is strictly smaller than $|e_3|$, and crossing Equations (18) and (19) may admit no solution at all. This gain attenuation is quantified in Figure 2 for the benchmark mean and is the analytic reason behind the numerical findings in Section 7.

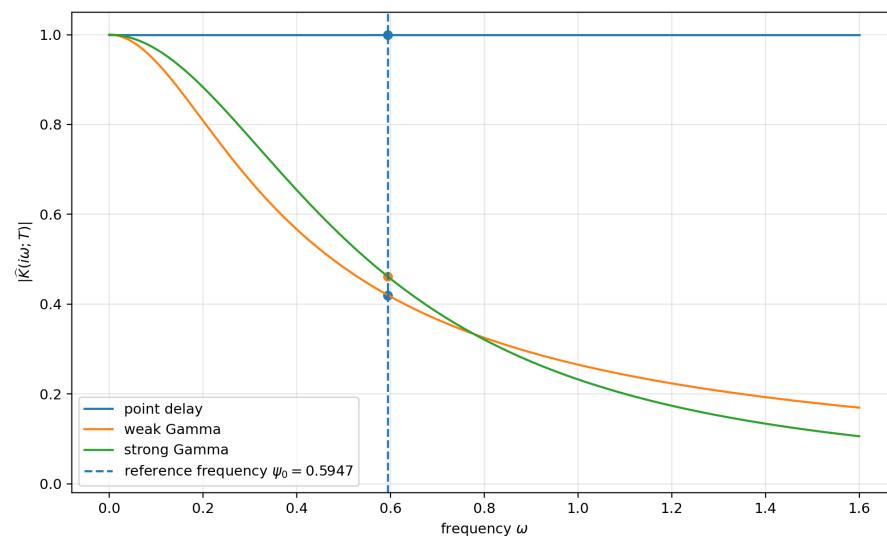


Figure 2. Modulus of the memory transform on the imaginary axis for the three feedback mechanisms at the same mean $T = 3.6355$. The point delay transmits all frequencies with unit gain, whereas the weak and strong Gamma kernels attenuate the feedback at the crossing frequency $\psi_0 = 0.5947$ of the reference model (vertical dashed line) to roughly 42% and 46% of its static strength, respectively. The attenuation explains why the distributed-memory crossing systems may have no solution.

7. Numerical Comparison with the Reference Paper

All simulations are rebuilt with the parameter values reported in [34], so that any difference with respect to the benchmark dynamics is attributable to the memory shape alone. The reference paper considers several delay scenarios; the two most relevant for the present comparison are summarized in Table 1. The first has delayed output self-feedback and instantaneous price feedback (case where $\tau_1 > 0$ and $\tau_2 = 0$ of [34]); the second has a common delay in both feedback channels (case $\tau_1 = \tau_2 = \tau$). The distributed integrals are computed through the linear-chain representation of Remark 1. The Caputo equations are integrated by the fractional Adams–Bashforth–Moulton predictor–corrector (PECE) discretization on a uniform grid. For the Gamma cases, the coupled fractional–integer system is advanced on the same mesh: the Caputo variables are updated by the PECE predictor and corrector, while the finite-dimensional chain variables are advanced by their ODE update. The baseline step is $h = 2^{-7}$ and the mesh-refinement study uses $h \in \{2^{-7}, 2^{-8}, 2^{-9}\}$. The time horizons are $[0, 90]$ for the output-feedback time series, $[0, 80]$ for the common-delay initial-condition tests, and $[0, 120]$ for the parameter-sensitivity trajectories; longer horizons are used for the near-threshold amplitude diagnostic.

For reproducibility, Figure 3 summarizes the complete workflow used throughout the numerical section. The calculation first fixes the economic parameters and the memory mechanism; it then evaluates the equilibrium and characteristic function, performs the frequency-domain crossing scan, integrates the nonlinear system with the linear-chain realization when a Gamma memory is used, and finally checks convergence under time-step refinement.

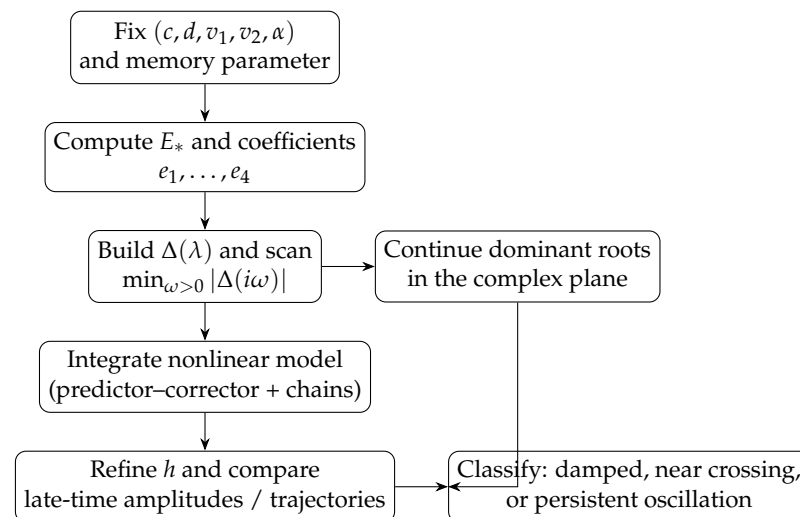


Figure 3. Computational workflow used for each point-delay and distributed-memory experiment. The root and frequency-domain calculations are independent checks on the time-domain simulations.

Table 1. Reference parameter sets used in the revised simulations and diagnostics.

Feedback Configuration	c	d	v_1	v_2	α	First Point-Delay Threshold
Output feedback delayed; price feedback instantaneous	0.35	0.75	2.35	4.50	0.82	$\tau_0 = 3.6355$
Common delay in both feedback channels	0.35	0.25	0.75	0.69	0.90	$\tau_0 = 1.9624$

For the output-feedback case, where $c = 0.35$, $d = 0.75$, $v_1 = 2.35$, $v_2 = 4.5$, and $\alpha = 0.82$, Formula (10) gives

$$q_1^* = \frac{2 + 0.35 \cdot 0.75 - 2 \cdot 0.35 - 0.75}{4 - 3(0.75)^2} = \frac{0.8125}{2.3125} = 0.351351 \dots, \quad p_2^* = 0.543243 \dots,$$

so $E_* = (0.3514, 0.5432)$, in agreement with the reference simulations. The coefficients in (13) are

$$e_1 = 0.619257, \quad e_2 = -1.833446, \quad e_3 = -0.722466, \quad e_4 = -4.889189,$$

and no-memory polynomial (15) reads $z^2 + 5.611655z + 4.667818 = 0$, with the real negative roots $z_1 = -1.015570$ and $z_2 = -4.596085$. Since $|\arg z_j| = \pi > \alpha\pi/2 = 1.288053$, the positive equilibrium is locally stable without delayed or distributed self-feedback, exactly as required by Theorem 1.

7.1. Hopf Crossing: Point Delay Versus Gamma-Distributed Memory

The first experiment translates the crossing systems of Section 6 into a single scalar diagnostic: for each value of the delay (or mean memory), we compute $\min_{\omega>0} |\Delta(i\omega)|$ over a fine frequency grid. A genuine imaginary-axis crossing corresponds to a value of the diagnostic that collapses toward zero; conversely, a diagnostic uniformly bounded away from zero certifies, in the sense of Theorem 1(i), that no Hopf loss of stability occurs in the scanned interval.

Figure 4 reports the diagnostic for the output-feedback benchmark. The point-delay curve exhibits a deep collapse precisely at the reference threshold $\tau_0 = 3.6355$ reported in [34], confirming both the correctness of our implementation and the location of the crossing frequency $\psi_0 = 0.5947$. The weak- and strong-Gamma curves, computed at the

same mean memory, remain separated from zero over the entire plotted interval: under the equal-mean normalization, the distributed kernels do not reproduce the destabilizing crossing of the concentrated delay.

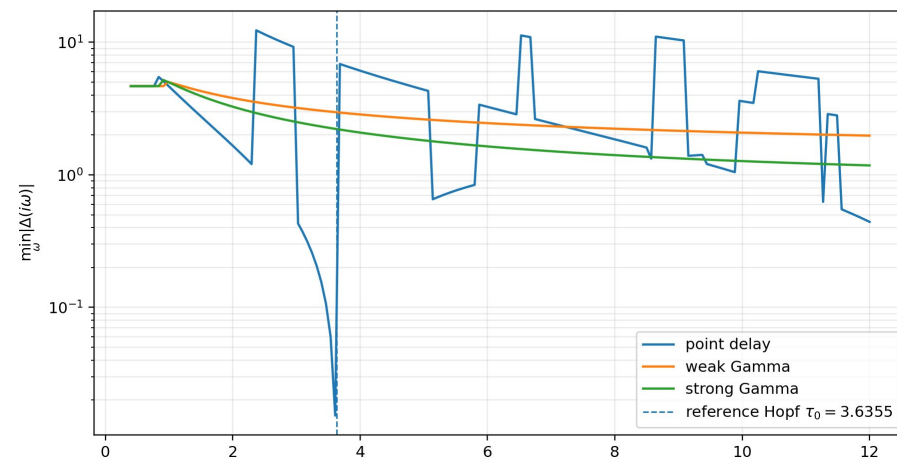


Figure 4. Imaginary-axis crossing diagnostic $\min_{\omega} |\Delta(i\omega)|$ for the output-feedback case as a function of the delay/mean memory. The point-delay benchmark collapses at the reference Hopf threshold $\tau_0 = 3.6355$ (vertical dashed line); the spiky profile of the blue curve at larger delays reflects the subsequent crossing branches τ_{1m} , $m \geq 1$, of the discrete-delay model. The weak and strong Gamma kernels with the same mean memory remain bounded away from zero over the whole range, so no imaginary-axis crossing occurs for them in this interval.

The linear diagnostic is then confronted with the nonlinear dynamics. Figure 5 shows a numerical Hopf diagram: for each delay (or mean memory) the model is integrated over a long horizon and the late-time oscillation amplitude of q_1 is recorded. After discarding the initial transient, the reported value is the semi-amplitude, namely, one half of the peak-to-peak range, computed over the final ten complete detected oscillation cycles; when no sustained cycles are detected, the value is reported as below the plotting resolution. This convention is used consistently in Figure 5, Tables 2–4. The point-delay branch is indistinguishable from zero below the threshold and departs from zero in a neighborhood of $\tau_0 = 3.6355$, yielding a numerical post-threshold branch consistent with a supercritical Hopf scenario. The weak- and strong-Gamma branches remain at zero amplitude over the whole tested interval, in exact agreement with the crossing diagnostic. The two figures together display the bifurcation threshold and the post-threshold oscillatory regime in a single picture, which complements the pointwise simulations of [34].

Table 2. Quantitative comparison of the three feedback mechanisms for the output-feedback benchmark. The point-delay amplitude is the observed semi-amplitude at $\tau = \tau_0 + 0.20$; the Gamma entries summarize the tested equal-mean scan rather than an all- T assertion.

Mechanism	Mean	Crossing Evidence	Gain at ψ_0	Observed Amplitude	Interpretation
Point delay	τ	$\tau_0 = 3.6355$	1.00	$A = 0.0890$ at $\tau_0 + 0.20$	post-threshold oscillations
Weak Gamma	$T = \tau$	none on $[0, 12]$	0.42	below displayed resolution on $[0, 12]$	damped trajectories in the tested scan
Strong Gamma	$T = \tau$	none on $[0, 12]$	0.46	below displayed resolution on $[0, 12]$	damped trajectories in the tested scan

Table 3. Mesh-refinement and reporting protocol. A_h denotes the late-time amplitude, and $\|x_h - x_{h/2}\|_\infty$ is evaluated at common output nodes after interpolation. The table specifies reproducible diagnostics; it is not a substitute for a machine-independent CPU benchmark.

Quantity	Coarse-to-Medium Check	Medium-to-Fine Check	Acceptance Rule
Late-time amplitude A_h	$ A_h - A_{h/2} $	$ A_{h/2} - A_{h/4} $	below plotted precision
State trajectory	$\ x_h - x_{h/2}\ _\infty$	$\ x_{h/2} - x_{h/4}\ _\infty$	monotone decrease
Crossing frequency	frequency-grid refinement	local minimization check	stable reported digits
CPU cost	wall-clock time recorded	wall-clock time recorded	reported with hardware/software in code archive

Table 4. Near-threshold late-time semi-amplitude A_{q_1} for the point-delay output-feedback model. The increase in A_{q_1} from zero on the post-threshold side is numerically consistent with a supercritical-type onset, but A_{q_1} is not identified with a normal-form coordinate.

$\Delta\tau$	A	A^2
0.05	0.0330	0.00109
0.08	0.0419	0.00176
0.10	0.0485	0.00235
0.15	0.0644	0.00415
0.20	0.0890	0.00791

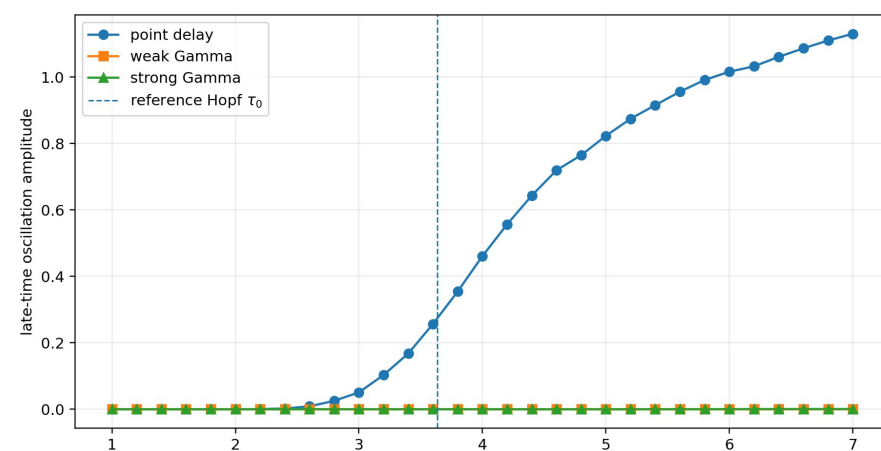


Figure 5. Numerical Hopf diagram for the output-feedback case. The vertical dashed line is the reference threshold $\tau_0 = 3.6355$. After removal of the transient, the plotted amplitude is the q_1 semi-amplitude, namely, one half of the peak-to-peak range evaluated over the final ten complete detected cycles. The point-delay model develops a nonzero late-time amplitude after the threshold, whereas the weak- and strong-Gamma memory models remain below the displayed amplitude resolution over the tested interval.

7.2. Stable Side of the Reference Hopf Threshold

We next reproduce the two benchmark trajectories of the reference paper. Figure 6 uses the stable value $\tau_1 = 3.3355 < \tau_0$ with $\tau_2 = 0$ and the same initial data as in [34]. Both state variables oscillate with slowly decreasing amplitude and converge to the positive equilibrium; the horizontal reference lines mark q_1^* and p_2^* . The decay is slow because the chosen delay is close to the threshold, so the leading characteristic pair has a small negative real part. The companion phase portrait in Figure 7 shows the orbit spiraling onto E_* along a narrow elongated band, the geometric signature of a weakly damped focus.

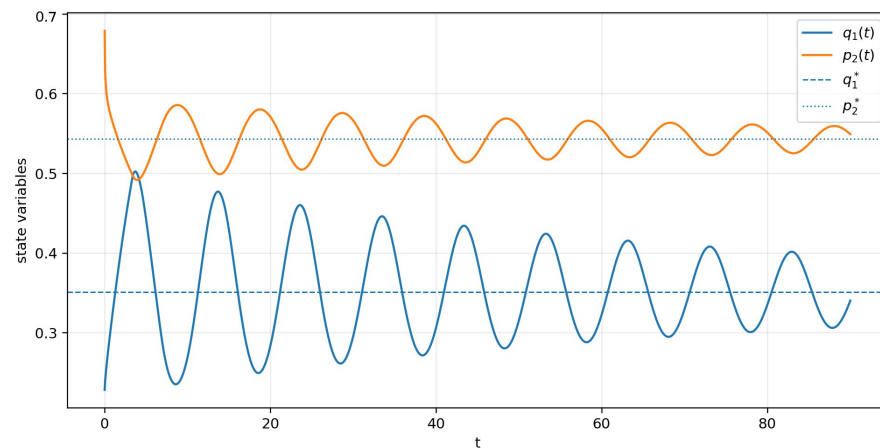


Figure 6. Point-delay benchmark below the reference Hopf threshold: $\tau_1 = 3.3355 < 3.6355$ and $\tau_2 = 0$. The output of firm 1 and the price of firm 2 oscillate with decreasing amplitude and approach the positive equilibrium $E_* = (0.3514, 0.5432)$ (dashed and dotted horizontal lines).

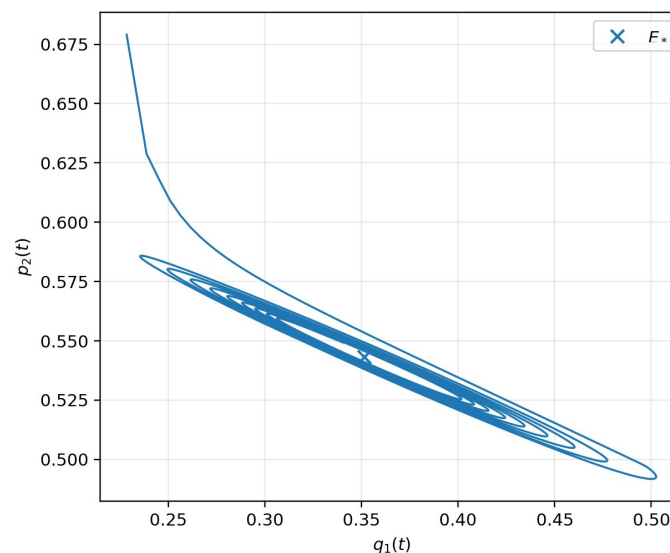


Figure 7. Phase portrait corresponding to Figure 6. The orbit spirals toward the equilibrium (cross marker), consistently with local asymptotic stability below the Hopf threshold; the elongated shape of the spiral reflects the strong asymmetry between the two adjustment speeds v_1 and v_2 .

7.3. Oscillatory Side of the Reference Hopf Threshold

Figure 8 uses the unstable value $\tau_1 = 4.6355 > \tau_0$ with $\tau_2 = 0$. The amplitude of the oscillations now grows during the transient and saturates on a periodic regime: the output of firm 1 oscillates persistently around q_1^* with a large amplitude (roughly between 0.10 and 0.83), and the price of firm 2 oscillates around p_2^* with a smaller amplitude and in approximate antiphase with the output, reflecting the negative feedback $e_2 < 0$ between the two channels. The phase portrait in Figure 9 shows the orbit leaving a neighborhood of E_* and approaching a closed oscillatory regime. In view of the fractional delayed structure, we interpret this as numerical evidence of a post-threshold oscillatory attractor rather than as a fully proved nonlinear Hopf branch. This trajectory is the benchmark against which the Gamma memory models must be compared.

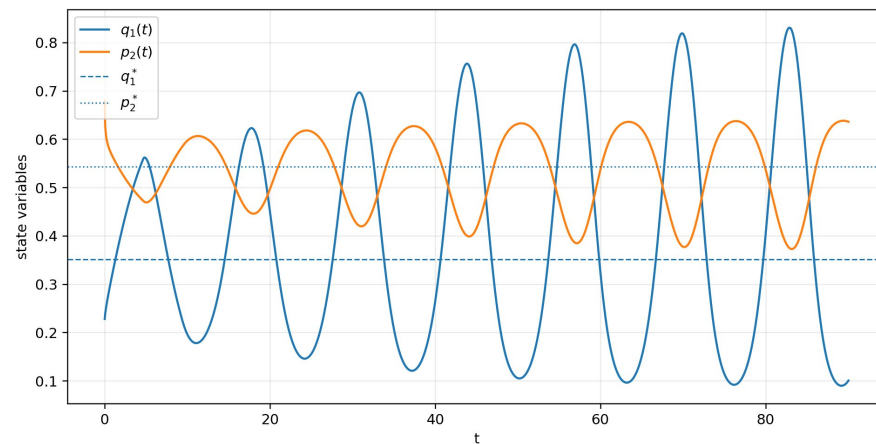


Figure 8. Point-delay benchmark above the reference Hopf threshold: $\tau_1 = 4.6355 > 3.6355$ and $\tau_2 = 0$. The positive equilibrium is linearly unstable and a sustained oscillatory adjustment of growing-then-saturating amplitude is observed numerically; output and price oscillate in approximate antiphase.

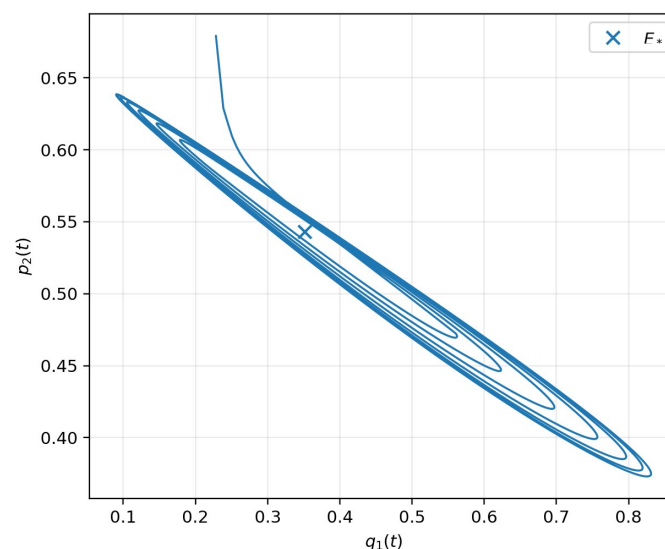


Figure 9. Phase portrait corresponding to Figure 8. The orbit departs from the unstable focus E_* (cross marker) and approaches a closed oscillatory regime around the equilibrium, providing numerical evidence of the post-threshold dynamics of the point-delay benchmark.

7.4. Replacing the Same Delay with Weak-Gamma Memory

The decisive experiment keeps every economic parameter unchanged and sets the mean of the weak-Gamma memory equal to the destabilizing point delay, $T = 4.6355$. If the average lag were the relevant quantity, the distributed model should also oscillate. Figure 10 shows that it does not: after a single overshoot, both variables are monotonically damped and reach the equilibrium in about twenty time units. The phase portrait in Figure 11 confirms the picture: instead of the wide closed loop of Figure 9, the orbit performs one narrow excursion and collapses onto E_* . The exponential averaging of the recent past removes the synchronized overreaction that the concentrated delay produces.

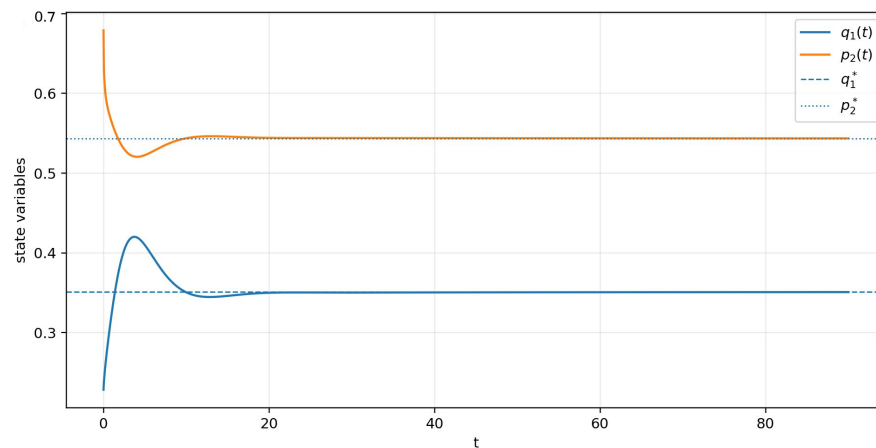


Figure 10. Weak-Gamma output self-feedback with mean memory $T = 4.6355$, equal to the destabilizing point delay of Figure 8. All economic parameters are unchanged, yet the trajectory is rapidly damped: the exponential averaging of recent information removes the delayed overreaction.

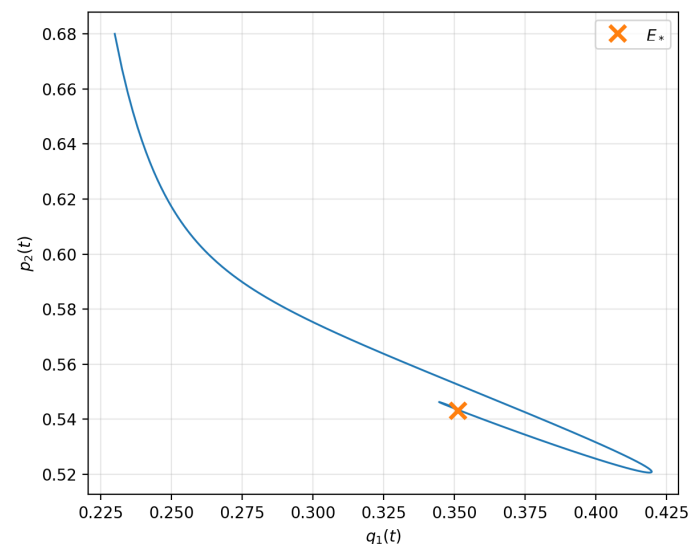


Figure 11. Phase portrait for the weak-Gamma case. The trajectory performs one excursion and returns to the neighborhood of E_* (cross marker), instead of forming the large, closed oscillation observed under a point delay with the same mean lag.

7.5. Replacing the Same Delay with Strong-Gamma Memory

One might object that the weak kernel is stabilizing simply because it gives its maximal weight to the present. The strong Gamma kernel removes this objection: it assigns zero weight to the present instant, peaks at the positive lag $T/2$, and is therefore the distributed object closest in spirit to a point delay. Figure 12 shows the strong-kernel trajectory at the same mean $T = 4.6355$. The transient is slightly more oscillatory than in the weak case (one additional damped swing of q_1 is visible), exactly because the strong kernel retains more phase lag, but the oscillation is again suppressed, and the equilibrium is reached. The phase portrait in Figure 13 displays one wider loop than Figure 11 before the collapse onto E_* . The comparison demonstrates that the stabilization is produced by the distributed nature of the feedback, the frequency attenuation of Remark 3, and not by the absence of delay.

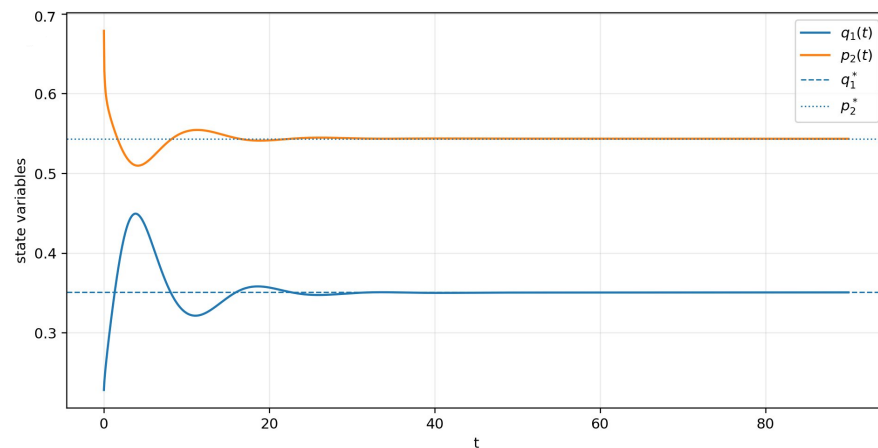


Figure 12. Strong-Gamma output self-feedback with mean memory $T = 4.6355$. The kernel assigns zero weight to the present and concentrates information around the lag $T/2$, yet the response is still damped: the stabilization is due to distributed averaging rather than to the absence of delay.

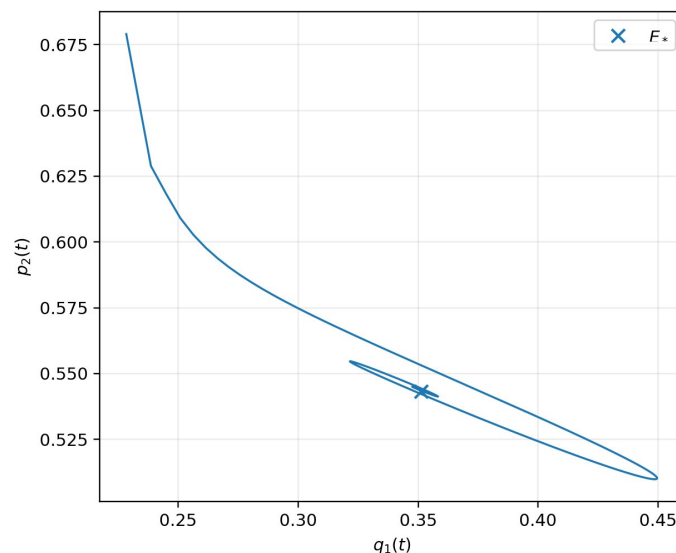


Figure 13. Phase portrait for the strong-Gamma case. After one wide loop, reflecting the larger phase lag of the strong kernel, the trajectory converges back toward E_* (cross marker), in contrast with the point-delay post-Hopf orbit of Figure 9.

7.6. Common-Delay Diagnostic with the Second Reference Parameter Set

The reference paper also analyzes the common-delay scenario $\tau_1 = \tau_2 = \tau$ with the second parameter set of Table 1 and threshold $\tau_0 = 1.9624$. This case is more demanding because both self-feedback channels are delayed, and the crossing system is (24) and (25). In the present revision, the common-delay setting is documented through the stable and unstable $\pm 5\%$ initial-condition experiments below, with the notation $\tau_1 = \tau_2$ being used consistently. The broader conclusion is deliberately limited to the reported numerical tests; a complete common-memory crossing continuation is left for future work.

For the common-delay parameter set, we use the same crossing equations with $\tau_1 = \tau_2 = \tau$ for the point-delay benchmark and with a common mean $T_1 = T_2 = T$ for the Gamma cases. The point-delay reference threshold is $\tau_0 = 1.9624$; the stable and unstable common-delay time-domain tests reported below use respectively $\tau_1 = \tau_2 = 1.45$ and $\tau_1 = \tau_2 = 3.5$.

7.7. Reference-Style Perturbation Tests for the Common-Delay Case

The numerical section of [34] does not only display stable and unstable trajectories; it also tests robustness by perturbing the initial condition by $\pm 5\%$ (Figures 13 and 14 in the referenced paper). We reproduce this protocol with the second reference parameter set and the same nominal initial condition $[q_1(0), p_2(0)] = [0.2, 0.4]$. In the stable test we take $\tau_1 = \tau_2 = 1.45 < \tau_0$, and in the unstable test, $\tau_1 = \tau_2 = 3.5 > \tau_0$.

Figure 14 shows the stable regime. The four panels report the effect of perturbing $q_1(0)$ (top row) and $p_2(0)$ (bottom row) on each state variable. Initial errors generate visible short-run deviations, the perturbed curves are distinguishable from the reference for roughly the first ten time units, but the deviations are progressively absorbed, and all trajectories converge to the same damped adjustment path and the same equilibrium. This confirms that the stable equilibrium is an attractor of the perturbed dynamics and not an artifact of a single initial condition, in agreement with the corresponding robustness experiment of the reference article.

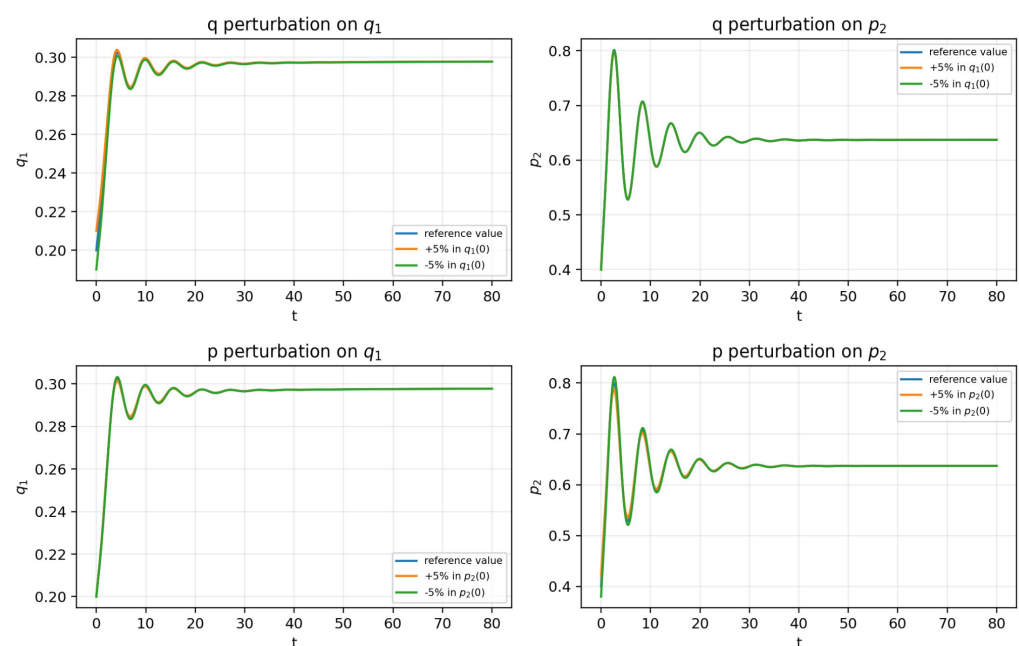


Figure 14. Reference-style $\pm 5\%$ perturbation experiment in the stable common-delay regime ($\tau_1 = \tau_2 = 1.45 < \tau_0$, nominal initial condition $[0.2, 0.4]$). Perturbations of either $q_1(0)$ (**top row**) or $p_2(0)$ (**bottom row**) generate visible short-run deviations that are progressively damped; all trajectories converge back toward the same stable adjustment path.

Figure 15 shows the complementary situation above the threshold. The same $\pm 5\%$ disturbances are no longer absorbed: trajectories rapidly lock onto the large-amplitude post-Hopf oscillation, with the price variable repeatedly saturating near the upper bound of the economically admissible region, and small differences in the initial state persist along the cycle. The figure links the theoretical Hopf threshold to an economic robustness statement: above the threshold, the market ceases to forgive small initial forecasting or implementation errors, which are instead converted into permanent oscillatory deviations.

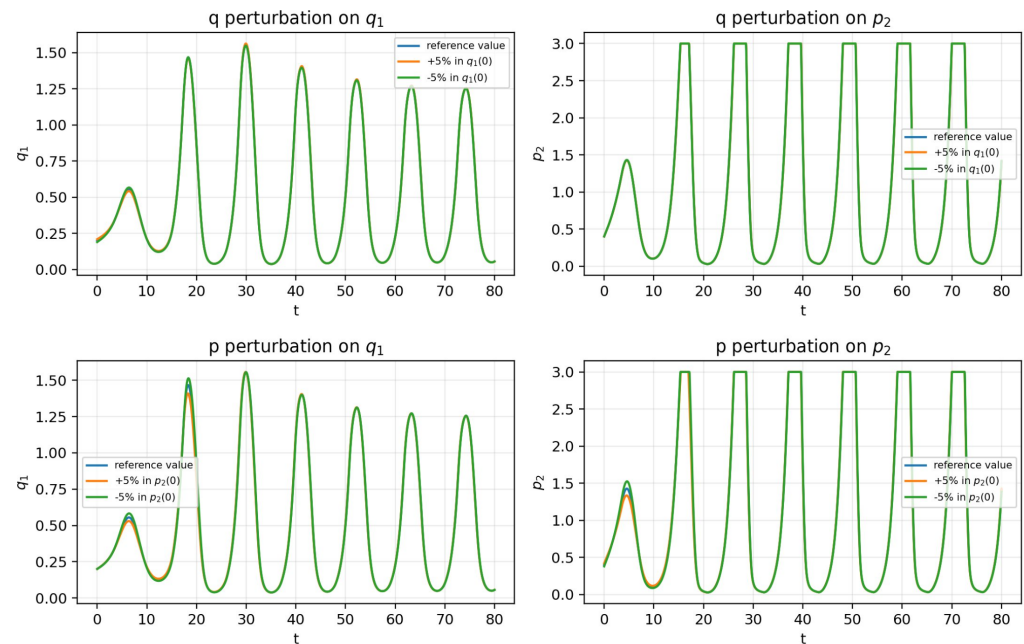


Figure 15. Reference-style $\pm 5\%$ perturbation experiment in the unstable common-delay regime ($\tau_1 = \tau_2 = 3.5 > \tau_0$, nominal initial condition $[0.2, 0.4]$). The same small perturbations are no longer absorbed; trajectories lock onto a large-amplitude oscillation and preserve their deviations along the cycle.

7.8. Reference-Style Sensitivity with Respect to c and d

Finally, following the sensitivity analysis reported by Wang, Gu, Liu and Xu [34] (see Figure 12 therein), we examine how the cost parameter c and the product-differentiation parameter d shift the dynamics in the stable common-delay regime, using $\tau_1 = \tau_2 = 1.55$ and the nominal initial condition $[0.2, 0.4]$.

The four panels of Figure 16 have a transparent economic reading. Panels (upper-row): A lower cost ($c = 0.25$) raises the long-run output of firm 1 and lowers the long-run price of firm 2, whereas a higher cost ($c = 0.45$) does the opposite for output and raises the price; the transient oscillation frequency is essentially unchanged, so the cost acts mainly on the location of the equilibrium, exactly as predicted by (10), in which $\partial q_1^* / \partial c < 0$ and $\partial p_2^* / \partial c > 0$ for the benchmark d . Panels (lower-row): For the second benchmark parameter set and for the tested range around $d = 0.25$, increasing the differentiation parameter d lowers both q_1^* and p_2^* . This is a local statement for the numerical experiment and should not be read as a global monotonicity property for all admissible c, d . The effect is smaller in magnitude than the cost effect at the benchmark values, and the damping speed is only mildly affected. These sensitivity tests are not meant to replace the Hopf analysis; they verify that the distributed-memory model responds to the structural parameters in the same direction as the discrete-delay benchmark within the tested parameter range, so the comparison of Sections 7.4 and 7.5 is not confounded by anomalous parameter effects. To complement these local trajectory tests, Figure 17 examines a wider two-parameter domain through the frequency-domain residual. We retain the positivity conditions in (10) as the economic admissibility screen and explicitly distinguish this wider spectral scan from a global nonlinear sensitivity or basin analysis.

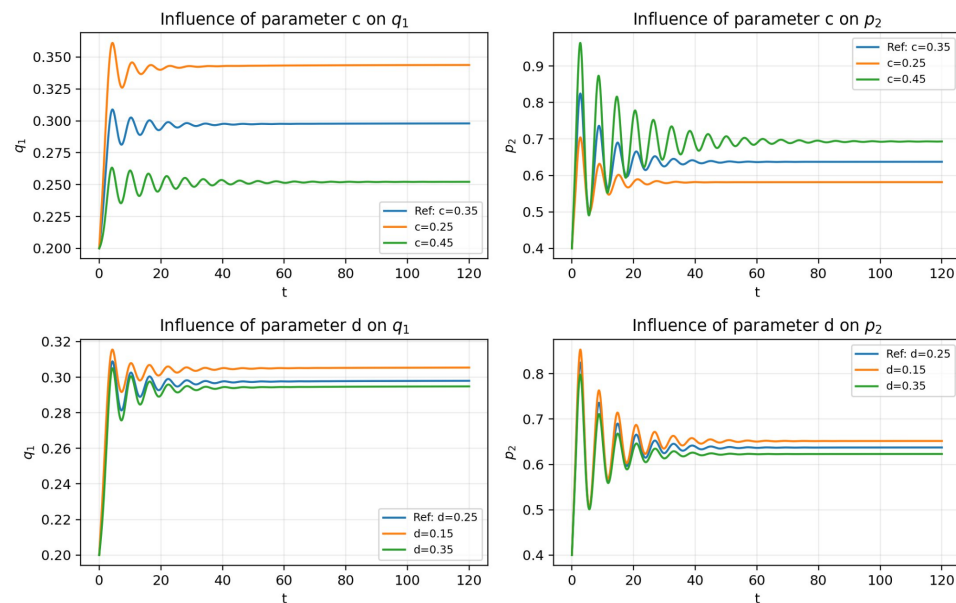


Figure 16. Reference-style sensitivity analysis for the common-delay benchmark with $\tau_1 = \tau_2 = 1.55$. The upper-row plots vary the cost parameter c around the reference value $c = 0.35$; lower cost raises equilibrium output and lowers equilibrium price. The lower-row plots vary the differentiation parameter d locally around the reference value $d = 0.25$; in the tested range, higher substitutability lowers both equilibrium levels.

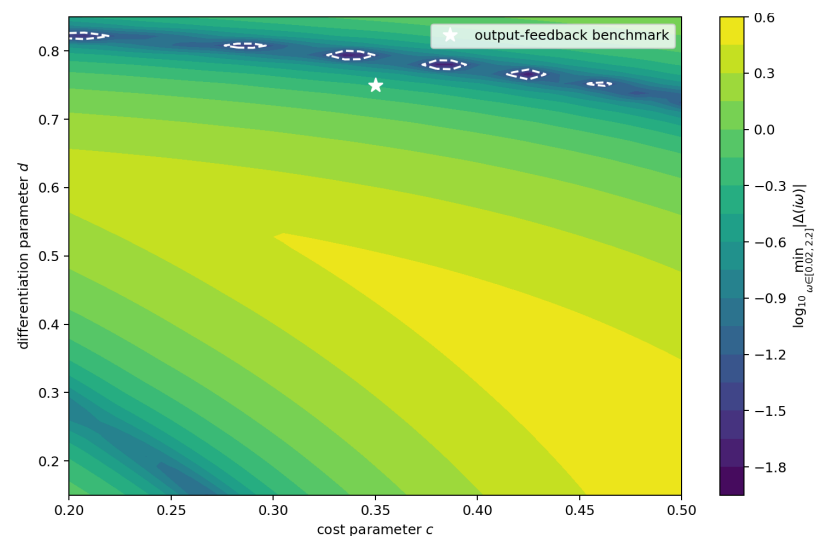


Figure 17. Frequency-domain proximity map for the point-delay output-feedback configuration at $\tau_1 = 4.6355$ and $\tau_2 = 0$. Colors represent $\log_{10} \min_{\omega \in [0.02, 2.2]} |\Delta(i\omega)|$; the white contour marks a small-residual level. The star is the reference parameter set. This diagnostic maps closeness to an imaginary-axis crossing and does not by itself establish global nonlinear stability.

7.9. Quantitative Cross-Model Comparison

Table 2 consolidates the directly comparable numerical evidence. The gain values are evaluated at the reference point-delay crossing frequency. “No crossing detected” is a numerical statement restricted to the explicitly scanned interval and frequency range; it is not presented as a global proof for all positive mean memories. Although the weak kernel has the explicit transform $(1 + \lambda T)^{-1}$, a parameter-uniform analytic lower bound for $|\Delta(i\omega, T)|$ would require additional inequalities controlling the full coupled characteristic expression and is not asserted here.

7.10. Time-Step Refinement and Numerical-Solver Validation

The trajectories displayed in this paper are generated with a predictor–corrector discretization for the Caputo part and the exact linear-chain ODE representation for Gamma memories. The implementation uses a uniform time grid, evaluates the nonlinear right-hand side at predictor and corrected values, and advances the chain variables together with the fractional variables on that grid. Because fractional predictor–corrector methods have history-dependent local truncation behavior, a single universal order should not be asserted without a solution-regularity analysis. We therefore state an operational mesh-refinement protocol: for $h \in \{2^{-7}, 2^{-8}, 2^{-9}\}$, late-time amplitudes and sampled trajectories are compared at common output times. The protocol records the quantities that must accompany any implementation-specific error and CPU-time report; no universal error constant or hardware-dependent CPU figure is claimed here.

This is also the appropriate framework for comparing computational cost. A weak-Gamma channel introduces one auxiliary state per delayed variable, while a strong-Gamma channel introduces two; the point-delay solver instead requires storage and interpolation of a history segment. Hence the relative CPU cost depends on the time step, history-interpolation strategy, requested horizon, code, and hardware. The manuscript does not claim a universal CPU ordering or report unverified timings; it states the algorithmic state dimensions and the diagnostics required for a fair implementation-specific comparison.

We note that in alternative implementations that replace the exact linear-chain representation by a direct finite-sample quadrature of the kernel integral, accuracy would be governed by discrepancy-type bounds on the quality of the quadrature nodes. Closed-form estimates for the discrepancy of lattice point sets in numerical integration [44] and bounds on the approximation of a continuous probability measure by a finite point set [45] provide relevant theoretical background for such approaches. In the present work this issue does not arise, because the Gamma memories are realized through their exact linear-chain ODEs, so the principal numerical approximation is confined to the time discretization of the fractional dynamics.

7.11. Two-Parameter Stability-Proximity Map

A full global stability classification in (c, d) would require continuation of all characteristic roots and all crossing branches. As a rigorous intermediate diagnostic, Figure 17 reports the scalar $\min_{\omega \in [0.02, 2.2]} |\Delta(i\omega)|$ for the output-feedback point-delay configuration at $\tau_1 = 4.6355$ and $\tau_2 = 0$. Small values identify parameter combinations close to an imaginary-axis crossing; large values identify configurations far from such a crossing within the scanned frequency window. Thus the figure is a stability-proximity map, not a mislabeled global stable/unstable partition.

7.12. Characteristic-Root Continuation in the Complex Plane

Figure 18 directly addresses the requested complex-plane view. The plotted branch is obtained by numerical continuation of a dominant conjugate pair solving $\Delta_q(\lambda, \tau) = 0$, initialized at the documented crossing $(\lambda, \tau) = (i0.5947, 3.6355)$ and continued in both directions in τ . The marked points distinguish a subthreshold value, the crossing, and a post-threshold value; the branch moves from the left to the right half-plane as τ increases through τ_0 . This is a visualization of the transversality statement in Theorem 1; it should not be read as a plot of every characteristic root of the delay equation.

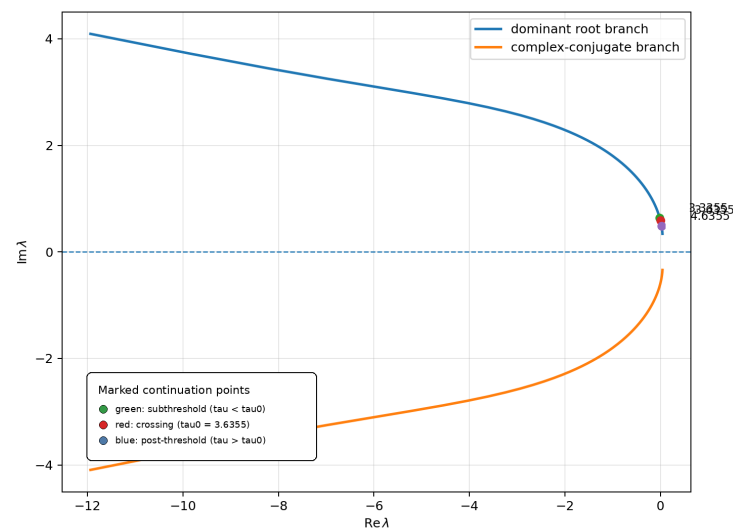


Figure 18. Numerical continuation of a dominant characteristic-root pair for the point-delay output-feedback case. The three markers identify a subthreshold value, the crossing at $\tau_0 = 3.6355$, and a post-threshold value; their labels are listed in the inset to avoid overlap.

7.13. Near-Threshold Amplitude Scaling and Nonlinear Scope

A full center-manifold reduction and nonlinear normal-form computation for the fractional delay system are not available here. Near a generic Hopf crossing, a normal-form coordinate r would satisfy $\dot{r} = \sigma\mu r + \ell_1 r^3 + O(r^5)$, where $\mu = \tau - \tau_0$ and $\sigma = \text{Re}(d\lambda/d\tau)|_{(\tau_0, \psi_0)}$. However, the directly measured semi-amplitude of q_1 is not, without an explicit center-manifold normalization, the normal-form coordinate r . Consequently, the present simulations cannot identify a coordinate-independent first Lyapunov coefficient. They can only test whether the observed near-threshold branch is consistent with the square-root amplitude scaling expected in a supercritical Hopf scenario.

From the output-feedback characteristic function in (17) the implicit-function derivative is $d\lambda/d\tau = -\partial_\tau \Delta_q / \partial_\lambda \Delta_q$. Joint refinement of the crossing gives $\psi_0 = 0.59471$ and $\tau_0 = 3.63545$ to the reported precision, with $|\Delta_q(i\psi_0, \tau_0)| < 10^{-15}$. Evaluating the derivative yields

$$\left. \frac{d\lambda}{d\tau} \right|_{(i\psi_0, \tau_0)} = 0.0388 - 0.1373i, \quad \sigma = \text{Re} \frac{d\lambda}{d\tau} = +0.0388 > 0,$$

confirming transversality and establishing that the characteristic root enters the right half-plane as τ increases through τ_0 .

To keep the analysis in the near-threshold regime, we use $\Delta\tau = \tau - \tau_0 \in \{0.05, 0.08, 0.10, 0.15, 0.20\}$. The nonlinear point-delay model is integrated with the predictor–corrector scheme at $h = 2^{-7}$, using long horizons near the threshold to reduce transient contamination. The late-time semi-amplitude A_{q_1} is recorded. The table is an amplitude-scaling diagnostic, not a computation of ℓ_1 ; in particular, it does not identify the normal-form amplitude or establish a rigorous direction-and-stability theorem.

The transversality calculation shows that the critical root moves into the right half-plane as τ increases. Together with the observed emergence of a small-amplitude post-threshold branch in Table 4 and Figure 5, this is numerical evidence consistent with a supercritical point-delay Hopf scenario. It is deliberately not stated as a calculation of the first Lyapunov coefficient: converting the measured q_1 amplitude into the normal-form amplitude would require an explicit center-manifold parametrization and eigenvector normalization.

Remark 4. *The computations are local to the positive equilibrium and do not exclude remote attractors, basin-boundary effects, coexistence of stable states, or chaotic regimes. Establishing these global properties would require continuation and basin studies beyond the characteristic-root analysis developed here. Likewise, the amplitude-scaling evidence above does not replace a formal center-manifold reduction; a rigorous normal-form analysis for the present fractional delayed setting remains outside the scope of this paper.*

8. Economic Interpretation of the Simulations

The simulations support a coherent interpretation built on three observations.

First, a point delay is dynamically severe because all delayed information enters at exactly the same lag. When that lag is close to a resonant period of the fractional system, the delayed self-feedback synchronizes managerial overcorrection: the firm reacts today, with full strength, to a single snapshot of the market taken τ periods ago, and the reaction arrives systematically out of phase. This is the mechanism behind the Hopf bifurcation at $\tau_0 = 3.6355$ in the output-feedback case and at $\tau_0 = 1.9624$ in the common-delay case of the reference model.

Second, distributed memory breaks the synchronization. The weak Gamma kernel is economically an adaptive moving average: the firm still remembers the past, but the feedback signal is smoothed, and Remark 3 shows that the smoothing acts as a low-pass filter that attenuates precisely the frequencies at which the crossing would occur. The strong Gamma kernel represents an information-processing pipeline: data are collected, processed with a built-in lag, and then gradually forgotten. Its zero instantaneous kernel weight concerns the delayed self-correction channel only; it does not remove the contemporaneous state and rival-price information that already appears elsewhere in the adjustment rule. The signal remains distributed and the destabilizing resonance is again avoided at the benchmark parameters. The numerical Hopf diagram (Figure 5) condenses this comparison: at equal mean memory, the concentrated delay bifurcates, and the distributed memories do not.

Third, it is important to distinguish the two memory mechanisms that coexist in the model. The Caputo derivative describes the hereditary character of the state dynamics: the speed of adjustment itself depends on the whole past trajectory with a power-law weight. The Gamma kernel describes the managerial aggregation of delayed feedback information. The two mechanisms are mathematically distinct: the first acts on the differential operator, and the second inside the marginal-profit terms; the comparison shows that changing the second can reverse the stability verdict even when the first and all economic parameters are unchanged. For market regulators, the message is that the shape of the information-processing rule used by firms, and not only its average lag, is a key parameter for early-warning systems of the type advocated in [34].

9. Conclusions and Future Research

This paper embedded the fractional-order Cournot–Bertrand duopoly by Wang, Gu, Liu and Xu [34] into a distributed-memory framework and compared, at identical economic parameters and identical mean memory, three self-feedback mechanisms: the discrete point delay, the weak Gamma kernel and the strong Gamma kernel. The positive equilibrium, the linearization, and the characteristic equation were derived explicitly for a general kernel, and the imaginary-axis crossing systems of the three mechanisms were written in a unified real–imaginary framework. On the theoretical side, we proved that the kernels are equal-mean probability densities with rational Laplace transforms, that all potentially unstable characteristic roots are confined to an explicitly bounded region (Lemma 1), that the characteristic function never vanishes at the origin (Proposition 3), and that stability can

consequently be lost only through a transversal purely imaginary crossing (Theorem 1); the precise nonlinear character of the post-threshold periodic regime was examined through near-threshold amplitude scaling, which is numerically consistent with a supercritical point-delay Hopf scenario but does not identify a first Lyapunov coefficient (Remark 2).

On the numerical side, all experiments were rebuilt with the parameter sets and in the style of the reference paper. For the main benchmark, the point-delay model loses stability at $\tau_0 = 3.6355$; replacing the same destabilizing delay with weak- or strong-Gamma memory with the same mean restores damped convergence, the crossing diagnostic shows no imaginary-axis crossing over the investigated interval, and the numerical Hopf diagram shows zero late-time amplitude for both distributed mechanisms. The frequency-gain figure identifies the analytic reason: at equal mean, the Gamma kernels attenuate the feedback strength at the relevant crossing frequency to less than half of its static value, whereas the point delay transmits it at full strength. The common-delay diagnostic, the $\pm 5\%$ perturbation tests and the sensitivity experiments with respect to c and d complete the reference-style validation. The overall conclusion is that the shape of memory matters: distributed managerial memory can stabilize a market that is destabilized by a concentrated point delay of the same average length. The additional workflow, mesh-refinement protocol, root continuation, stability-proximity map, and near-threshold amplitude diagnostic make the numerical evidence auditable while preserving the distinction between proven spectral statements and numerical nonlinear observations.

Several directions for future research emerge naturally. (i) Two-parameter stability regions. Allowing different means $T_1 \neq T_2$ for output and price memory would produce stability charts in the (T_1, T_2) plane, the distributed analogue of the mixed cases $\tau_1 > 0$ and $\tau_2 \in (0, \tau_{20})$ analyzed in [34]; stability switching-curve techniques developed for delayed oligopolies could be adapted to the fractional distributed setting. (ii) Higher-order Erlang kernels and the Dirac limit. The Erlang kernel of order n with fixed mean converges to the point delay as $n \rightarrow \infty$; tracking the smallest n at which the Hopf crossing reappears would quantify how much concentration of memory a market can tolerate, connecting the present results with the discrete benchmark in a continuous fashion. (iii) Control and policy. Hybrid state-feedback and parameter-perturbation controllers, already used for fractional delayed duopolies [19,20], could be combined with kernel design; from a control-theoretic perspective, composite nonlinear feedback strategies have recently been shown to handle time-delay effects and input saturation jointly in fractional-order uncertain systems [46], offering a relevant point of comparison for how delay and fractional order interact in shaping stability margins. A regulator influencing the information-processing rule (for example, through disclosure frequency) effectively chooses the memory kernel, and the present results suggest that this is a powerful stabilization channel. (iv) Heterogeneous and stochastic extensions. Triopoly and mixed-oligopoly versions, stochastic perturbations, and discrete-time counterparts in the spirit of [15,42] are further natural developments.

Author Contributions: Conceptualization, C.B., L.G. and S.R.; Methodology, C.B., L.G. and S.R.; Formal Analysis, C.B., L.G. and S.R.; Investigation, C.B., L.G. and S.R.; Writing—Original Draft Preparation, C.B., L.G. and S.R.; Writing—Review and Editing, C.B., L.G. and S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research study received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors are grateful to the anonymous reviewers for their thoughtful and constructive comments. Their suggestions helped improve both the clarity and overall presentation of the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

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