Multiplicity and concentration results for a class of critical fractional Schrödinger-Poisson systems via penalization method

MULTIPLICITY AND CONCENTRATION RESULTS FOR A CLASS OF CRITICAL FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS VIA PENALIZATION METHOD

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Abstract. We deal with the multiplicity and concentration of positive solutions for the following fractional Schrödinger-Poisson type system with critical growth:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2^*_s-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $s \in (\frac{3}{4}, 1)$, $t \in (0,1)$, $(-\Delta)^\alpha$, with $\alpha \in \{s,t\}$, is the fractional Laplacian operator, V is a continuous positive potential and f is a superlinear continuous function with subcritical growth. Using penalization techniques and Ljusternik-Schnirelmann theory, we investigate the relation between the number of positive solutions with the topology of the set where the potential attains its minimum value.

1. Introduction

In this paper we focus our attention on the multiplicity and concentration of positive solutions for the following critical fractional nonlinear Schrödinger-Poisson system:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2^*_s-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $s \in (\frac{3}{4}, 1)$, $t \in (0,1)$, $2^*_s = \frac{6}{3-2s}$ is the critical Sobolev exponent, and $(-\Delta)^\alpha$, with $\alpha \in \{s,t\}$, is the fractional Laplacian operator which may be defined for any $u : \mathbb{R}^3 \to \mathbb{R}$ belonging to the Schwartz class by

$$(-\Delta)^\alpha u(x) = C(3,\alpha) P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2\alpha}} dy \quad (x \in \mathbb{R}^3),$$

where $P.V.$ stands for the Cauchy principal value and $C(3,\alpha)$ is a normalizing constant; see [16]. It is well-known that the study of elliptic equations driven by fractional powers of the Laplacian has received an enormous interest from the mathematical community because nonlocal problems arise in many physical situations in which one has to consider long-range or anomalous diffusions. Moreover, $(-\Delta)^\alpha$ appears as the infinitesimal generator of a Lévy process [9]. For more details and applications we refer the interested reader to [16, 28] and references therein.

Along the paper we will assume that the potential $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying the following hypotheses introduced by del Pino and Felmer in [15]:

(V_1) there exists $V_0 > 0$ such that $V_0 = \inf_{x \in \mathbb{R}^3} V(x)$,

(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 < \min_{\partial\Lambda} V \text{ and } M = \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

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Concerning the nonlinearity $f:\mathbb{R}\to\mathbb{R}$ we assume that it is a continuous function such that $f(t)=0$ for $t\leq 0$ and satisfies the following conditions:
(f_1) $f(t)=o(t^3)$ as $t\to 0$,
(f_2) there exist $q,\sigma\in(4,2^*_s)$, $C_0>0$ such that

$$f(t)\geq C_0 t^{q-1} \quad \forall t>0, \quad \lim_{t\to\infty}\frac{f(t)}{t^{\sigma-1}}=0,$$

(f_3) there exists $\vartheta\in(4,\sigma)$ such that $0<\vartheta F(t)\leq tf(t)$ for all $t>0$,
(f_4) the map $t\mapsto \frac{f(t)}{t^3}$ is increasing in $(0,\infty)$.
We note that when $\phi=0$, then (1.1) reduces to a fractional Schrödinger equation [26] of the type

$$\varepsilon^{2s}(-\Delta)^s u+V(x)u=h(x,u) \text{ in } \mathbb{R}^3, \tag{1.2}$$

which has been widely investigated by many authors in the last two decades; see [5, 17, 19, 25, 34, 36] and references therein. Felmer et al. [19] dealt with the existence, regularity and symmetry of positive solutions to (1.2) when $V=1$ and h is a subcritical nonlinearity satisfying the Ambrosetti-Rabinowitz condition [4]. Secchi [34] studied (1.2) under suitable assumptions on the behavior of the potential V at infinity. Shang et al. [36] established some existence and multiplicity results for a fractional Schrödinger equation with critical growth and requiring the following global assumption on the potential V introduced by Rabinowitz [32]:

$$V_\infty=\liminf_{|x|\to\infty}V(x)>V_0=\inf_{x\in\mathbb{R}^N}V(x)>0. \tag{1.3}$$

Alves and Miyagaki [3], by means of the penalization technique and the extension method [14], focused on the existence and concentration of positive solutions to (1.2) when V verifies (V_1)-(V_2) and h is a subcritical nonlinearity. Subsequently, the multiplicity and concentration of positive solutions to (1.2) with V verifying (V_1)-(V_2), have been considered for critical and supercritical nonlinearities; see [6, 24]. We also mention the papers [11, 35] for some interesting results about critical problems in bounded domains and [7, 8] for critical fractional periodic problems.

We observe that if $s=t=1$, system (1.1) becomes the classical Schrödinger-Poisson system

$$\begin{cases} -\varepsilon^2\,\Delta u+V(x)u+\mu\phi u=g(u) & \text{in } \mathbb{R}^3,\\ -\varepsilon^2\,\Delta\phi=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.4}$$

which describes systems of identical charged particles interacting each other in the case that effects of magnetic field could be ignored and its solution represents, in particular, a standing wave for such a system. For a more detailed physical description of this system we refer to [12]. Concerning some classical existence and multiplicity results for Schrödinger-Poisson systems we quote [10, 22, 23, 33, 40, 43]. For instance, Ruiz [33] gave some existence and nonexistence results to (1.4) when $g(u)=u^p$, $p\in(1,5)$ and $\mu>0$. Azzollini et al. [10] investigated the existence of nontrivial solutions when g verifies Beresticky-Lions type assumptions. Wang et al. [40] considered the existence and concentration of positive solutions to (1.4) involving subcritical nonlinearities. He and Li [23] obtained an existence result for a critical Schrödinger-Poisson system, assuming that the potential V satisfies (V_1)-(V_2).

In the fractional scenario, only few results for fractional Schrödinger-Poisson systems are available in literature. Giammetta [21] studied the local and global well-posedness of a fractional Schrödinger-Poisson system in which the fractional diffusion appears only in the second equation in (1.1). Teng [39] analyzed the existence of ground state solutions for (1.1) with critical Sobolev exponent, by combining Pohozaev-Nehari manifold, arguments of Brezis-Nirenberg type, the monotonicity trick and global compactness Lemma. In [42] Zhang et al. used a perturbation approach to prove the existence of positive solutions to (1.1) with $V(x)=\mu>0$ and g is a

general nonlinearity having subcritical or critical growth. They also investigated the asymptotic behavior of solutions as $\mu \to 0$. Liu and Zhang [27] studied multiplicity and concentration of solutions to (1.1) when the potential V satisfies (1.3). Murcia and Siciliano [30] showed that, for suitably small ε, the number of positive solutions to (1.1) is estimated below by the Ljusternik-Schnirelmann category of the set of minima of the potential.
Motivated by the above papers, in this work we aim to study the multiplicity and concentration of solutions to (1.1) under the local conditions (V_1)-(V_2) on the potential V and assuming (f_1)-(f_4) for the nonlinearity f. In order to state precisely our main result, we recall that if Y is a given closed set of a topological space X, we denote by $cat_Y(Y)$ the Ljusternik-Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y; see [41] for more details. Then we are able to prove the following result:

Theorem 1.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold. Then, for any $\delta > 0$ such that*

$$M_\delta = \{x \in \mathbb{R}^3 : dist(x, M) \leq \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that problem (1.1) admits at least $cat_{M_\delta}(M)$ positive solutions in $\mathcal{H}_\varepsilon \times D^{t,2}(\mathbb{R}^3)$. Moreover, if $(u_\varepsilon, \phi_\varepsilon)$ denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^3$ is a global maximum point of u_ε, then

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0,$$

and there exists $C > 0$ such that

$$0 < u_\varepsilon(x) \leq \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x - x_\varepsilon|^{3+2s}} \quad \text{for all } x \in \mathbb{R}^3.$$

In what follows we give a sketch of the proof of Theorem 1.1 which is obtained applying appropriate variational arguments. Firstly, the lack of information about the behavior of potential V at infinity is overcame considering a modified problem in the spirit of the penalization method introduced by del Pino and Felmer in [15] (see also [1, 2]); see Section 3. Since the nonlinearity f is only continuous, we cannot apply standard Nehari manifold arguments developed, for example, in [2, 27, 22, 36], and for this reason we make use of some variants of critical point theorems due to Szulkin and Weth [38]. We recall that a similar approach has been adopted in [20] to study the multiplicity and concentration behavior of positive solutions for a subcritical Kirchhoff problem. Anyway, the presence of two fractional Laplacian operators and critical Sobolev exponent makes our analysis more complicated and intriguing with respect to the ones in [20] and new arguments will be needed to attack our problem. Moreover, in order to cover some compactness properties for the functional $\mathcal{J}_\varepsilon$ associated to the modified problem, we invoke the Concentration-Compactness Lemma for the fractional Laplacian [17, 31]. Since we are interested in obtaining multiple critical points, we use a technique introduced by Benci and Cerami [13], which consists in making precise comparisons between the category of some sublevel sets of $\mathcal{J}_\varepsilon$ and the category of the set M. Finally, we show that the solutions of the modified problem are also solutions to (1.1), by combining a Moser iteration technique [29] conveniently adapted in the fractional setting and some useful estimates for Bessel operators established in [3, 19].
The paper is organized as follows. In Section 2 we recall some lemmas which we will use along the paper. In Section 3 we introduce the functional setup. In Section 4 we establish an existence result for the modified problem. In Section 5 we deal with the autonomous problem associated with (1.1). In Section 6 we introduce some tools needed to obtain a multiplicity result for the modified problem. In the last section we provide the proof of Theorem 1.1.

2. Preliminaries

We start giving some notations and collecting some useful preliminary results on fractional Sobolev spaces; see [16, 28] for more details.
If $A \subset \mathbb{R}^3$, we denote by $|u|_{L^q(A)}$ the $L^q(A)$-norm of a function $u : \mathbb{R}^3 \to \mathbb{R}$, and by $|u|_q$ its $L^q(\mathbb{R}^3)$-norm. Let us define $D^{s,2}(\mathbb{R}^3)$ as the completion of $C^\infty_c(\mathbb{R}^3)$ with respect to

$$[u]^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dxdy.$$

Then we consider the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : [u] < \infty\}$$

endowed with the norm

$$\|u\|^2 = [u]^2 + |u|_2^2.$$

We recall the following main embeddings for the fractional Sobolev spaces:

Theorem 2.1. [16] *Let $s \in (0, 1)$. Then $H^s(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for any $p \in [2, 2^*_s]$ and compactly in $L^p_{loc}(\mathbb{R}^3)$ for any $p \in [1, 2^*_s)$.*

The following lemma is a version of the well-known concentration-compactness principle:

Lemma 2.1. [19] *If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^s(\mathbb{R}^3)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0$$

where $R > 0$, then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2^*_s)$.

We also recall the following useful technical result.

Lemma 2.2. [31] *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^3)$. Let $\varphi \in C^\infty_c(\mathbb{R}^3)$ and for each $r > 0$ we define $\varphi_r(x) = \varphi(x/r)$. Then, $[u\varphi_r] \to 0$ as $r \to 0$. If in addition $\varphi = 1$ in a neighborhood of the origin, then $[u\varphi_r] \to [u]$ as $r \to \infty$.*

Now, let $s, t \in (0, 1)$ such that $4s + 2t \geq 3$. Using Theorem 2.1 we can see that

$$H^s(\mathbb{R}^3) \subset L^{\frac{12}{3+2t}}(\mathbb{R}^3). \tag{2.1}$$

For any $u \in H^s(\mathbb{R}^3)$, the linear functional $\mathcal{L}_u : D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\mathcal{L}_u(v) = \int_{\mathbb{R}^3} u^2 v \, dx$$

is well defined and continuous in view of Hölder inequality and (2.1). Indeed

$$|\mathcal{L}_u(v)| \leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{2^*_t} dx \right)^{\frac{1}{2^*_t}} \leq C\|u\|^2 \|v\|_{D^{t,2}}, \tag{2.2}$$

where

$$\|v\|^2_{D^{t,2}} = \iint_{\mathbb{R}^6} \frac{|v(x) - v(y)|^2}{|x - y|^{3+2t}} dxdy.$$

Then, by the Lax-Milgram Theorem there exists a unique $\phi^t_u \in D^{t,2}(\mathbb{R}^3)$ such that

$$(-\Delta)^t \phi^t_u = u^2 \text{ in } \mathbb{R}^3.$$

Therefore, we obtain the following t-Riesz formula

$$\phi^t_u(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{3-2t}} \, dy \quad (x \in \mathbb{R}^3), \quad c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3 - 2t)}{\Gamma(t)}. \tag{2.3}$$

In the sequel, we will omit the constant c_t in order to lighten the notation. Finally, we state the following useful properties whose proofs can be found in [27, 39]:

Lemma 2.3. *If $4s+2t\geq 3$, then for all $u\in H^s(\mathbb{R}^3)$ we have:*
(1) $\|\phi^t_u\|_{D^{t,2}}\leq C|u|^2_{\frac{12}{3+2t}}\leq C\|u\|^2_{\varepsilon}$ *and* $\int_{\mathbb{R}^3}\phi^t_u u^2dx\leq C_t|u|^4_{\frac{12}{3+2t}}$. *Moreover $\phi^t_u: H^s(\mathbb{R}^3)\to D^{t,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets;*
(2) $\phi^t_u\geq 0$ *in* $\mathbb{R}^3$;
(3) *if* $y\in\mathbb{R}^3$ *and* $\bar{u}(x)=u(x+y)$ *then* $\phi^t_{\bar{u}}(x)=\phi^t_u(x+y)$ *and* $\int_{\mathbb{R}^3}\phi^t_{\bar{u}}\bar{u}^2dx=\int_{\mathbb{R}^3}\phi^t_u u^2dx$;
(4) $\phi^t_{ru}=r^2\phi^t_u$ *for all* $r\in\mathbb{R}$;
(5) *if* $u_n\rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *then* $\phi^t_{u_n}\rightharpoonup\phi^t_u$ *in* $D^{t,2}(\mathbb{R}^3)$;
(6) *if* $u_n\rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *then* $\int_{\mathbb{R}^3}\phi^t_{u_n}u^2dx=\int_{\mathbb{R}^3}\phi^t_{(u_n-u)}(u_n-u)^2dx+\int_{\mathbb{R}^3}\phi^t_u u^2dx+o_n(1)$.
(7) *if* $u_n\to u$ *in* $H^s(\mathbb{R}^3)$ *then* $\phi^t_{u_n}\to\phi^t_u$ *in* $D^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3}\phi^t_{u_n}u^2dx\to\int_{\mathbb{R}^3}\phi^t_u u^2dx$.

3. Functional Setting

In order to study (1.1), we use the change of variable $x\mapsto\varepsilon x$ and we will look for solutions to

$$\begin{cases}(-\Delta)^s u+V(\varepsilon x)u+\phi^t_u u=f(u)+|u|^{2^*_s-2}u \text{ in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3),\quad u>0 \text{ in } \mathbb{R}^3,\end{cases}\tag{3.1}$$

where ϕ^t_u is given by (2.3). In what follows we introduce a penalization function [15] which will be useful to obtain our results.

Let $K>2$ and $a>0$ such that $f(a)+a^{2^*_s-1}=\frac{V_0}{K}a$ and we define

$$\tilde{f}(t)=\begin{cases}f(t)+(t^+)^{2^*_s-1} & \text{if } t\leq a\\ \frac{V_0}{K}t & \text{if } t>a,\end{cases}$$

and

$$g(x,t)=\chi_\Lambda(x)(f(t)+(t^+)^{2^*_s-1})+(1-\chi_\Lambda(x))\tilde{f}(t).$$

It is easy to check that g satisfies the following properties:
(g_1) $\lim_{t\to0}\frac{g(x,t)}{t^3}=0$ uniformly with respect to $x\in\mathbb{R}^3$,
(g_2) $g(x,t)\leq f(t)+t^{2^*_s-1}$ for all $x\in\mathbb{R}^3$, $t>0$,
(g_3) (i) $0\leq\vartheta G(x,t)<g(x,t)t$ for all $x\in\Lambda$ and $t>0$,
(ii) $0\leq 2G(x,t)<g(x,t)t\leq\frac{V_0}{K}t^2$ for all $x\in\mathbb{R}^3\setminus\Lambda$ and $t>0$,
(g_4) for each $x\in\Lambda$ the function $\frac{g(x,t)}{t^3}$ is increasing in $(0,\infty)$, and for each $x\in\mathbb{R}^3\setminus\Lambda$ the function $\frac{g(x,t)}{t^3}$ is increasing in $(0,a)$.

Let us consider the following modified problem

$$\begin{cases}(-\Delta)^s u+V(\varepsilon x)u+\phi^t_u u=g(\varepsilon x,u) \text{ in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3),\quad u>0 \text{ in } \mathbb{R}^3.\end{cases}\tag{3.2}$$

It is clear that weak solutions to (3.2) are critical points of the following functional

$$\mathcal{J}_\varepsilon(u)=\frac{1}{2}\|u\|^2_\varepsilon+\frac{1}{4}\int_{\mathbb{R}^3}\phi^t_u u^2dx-\int_{\mathbb{R}^3}G(\varepsilon x,u)dx,$$

defined for all $u\in\mathcal{H}_\varepsilon$ where

$$\mathcal{H}_\varepsilon=\left\{u\in H^s(\mathbb{R}^3):\int_{\mathbb{R}^3}V(\varepsilon x)u^2dx<\infty\right\}$$

is endowed with the norm

$$\|u\|^2_\varepsilon=[u]^2+\int_{\mathbb{R}^3}V(\varepsilon x)u^2dx.$$

Obviously, $\mathcal{H}_\varepsilon$ is a Hilbert space with inner product

$$(u,v)_\varepsilon = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} V(\varepsilon\, x) uv dx.$$

We also note that $\mathcal{J}_\varepsilon \in C^1(\mathcal{H}_\varepsilon, \mathbb{R})$ and its differential is given by

$$\langle \mathcal{J}_\varepsilon'(u), v\rangle = (u,v)_\varepsilon + \int_{\mathbb{R}^3} \phi_u^t uv - \int_{\mathbb{R}^3} g(\varepsilon\, x, u) v dx \quad \forall u, v \in \mathcal{H}_\varepsilon.$$

Let us introduce the Nehari manifold associated to (3.2), that is,

$$\mathcal{N}_\varepsilon = \{u \in \mathcal{H}_\varepsilon \setminus \{0\} : \langle \mathcal{J}_\varepsilon'(u), u\rangle = 0\},$$

and we denote by

$$\mathcal{H}_\varepsilon^+ = \{u \in \mathcal{H}_\varepsilon : |\operatorname{supp}(u^+) \cap \Lambda| > 0\}$$

and $\mathbb{S}_\varepsilon^+ = \mathbb{S}_\varepsilon \cap \mathcal{H}_\varepsilon^+$, where $\mathbb{S}_\varepsilon$ is the unitary sphere in $\mathcal{H}_\varepsilon$. Then $\mathcal{H}_\varepsilon = T_u\mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$ and $T_u\mathbb{S}_\varepsilon^+ = \{v \in \mathcal{H}_\varepsilon : (u,v)_\varepsilon = 0\}$. Let us note that $\mathcal{J}_\varepsilon$ satisfies the following properties:

Lemma 3.1. *The functional $\mathcal{J}_\varepsilon$ has a Mountain-Pass geometry:*
(a) there exist $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$;
(b) there exists $e \in \mathcal{H}_\varepsilon$ such that $\|e\|_\varepsilon > \rho$ and $\mathcal{J}_\varepsilon(e) < 0$.

Proof. (a) Taking into account (g_1), (g_2), (f_2), for any $\xi > 0$ we can find $C_\xi > 0$ such that

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^3} G(\varepsilon\, x, u)\, dx \geq \frac{1}{2}\|u\|_\varepsilon^2 - \xi C\|u\|_\varepsilon^4 - C_\xi C \|u\|_\varepsilon^{2_s^*}.$$

Then there exist $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$.
(b) In view of (g_3)-(i) and Lemma 2.3-(4), we can see that for any $u \in \mathcal{H}_\varepsilon^+$ and $\tau > 0$

$$\begin{aligned}
\mathcal{J}_\varepsilon(\tau u) &\leq \frac{\tau^2}{2}\|u\|_\varepsilon^2 + \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} G(\varepsilon\, x, \tau u)\, dx \\
&\leq \frac{\tau^2}{2}\|u\|_\varepsilon^2 + \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\Lambda_\varepsilon} G(\varepsilon\, x, \tau u)\, dx \\
&\leq \frac{\tau^2}{2}\|u\|_\varepsilon^2 + \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - C_1\tau^\vartheta \int_{\Lambda_\varepsilon} (u^+)^\vartheta\, dx + C_2 |\operatorname{supp}(u^+) \cap \Lambda_\varepsilon|, \qquad (3.3)
\end{aligned}$$

for some positive constants C_1 and C_2. Since $\vartheta \in (4, 2_s^*)$, we get $\mathcal{J}_\varepsilon(\tau u) \to -\infty$ as $\tau \to +\infty$. $\square$

Since f is only continuous, the next results will be crucial to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}_\varepsilon^+$.

Lemma 3.2. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold true. Then,*

(i) For each $u \in \mathcal{H}_\varepsilon^+$, let $h_u : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{J}_\varepsilon(tu)$. Then, there is a unique $t_u > 0$ such that

$$h_u'(t) > 0 \text{ in } (0, t_u)$$
$$h_u'(t) < 0 \text{ in } (t_u, \infty);$$

(ii) there exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}_\varepsilon^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ there is a positive constant $C_{\mathbb{K}}$ such that $t_u \leq C_{\mathbb{K}}$ for any $u \in \mathbb{K}$;

(iii) The map $\hat{m}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\hat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon := \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ is a homeomorphism between $\mathbb{S}_\varepsilon^+$ and $\mathcal{N}_\varepsilon$. Moreover $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$;

(iv) If there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ such that $\operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof. (i) We note that $h_u \in C^1(\mathbb{R}^+, \mathbb{R})$, and in view of Lemma 3.1, we can see that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Then there exists $t_u > 0$ such that $h'_u(t_u) = 0$ and t_u is a global maximum for h_u. This implies that $t_u u \in \mathcal{N}_\varepsilon$. Now, we show the uniqueness of a such t_u. Suppose by contradiction that there exist $t_1 > t_2 > 0$ such that $h'_u(t_1) = h'_u(t_2) = 0$, that is

$$t_1\|u\|_\varepsilon^2 + t_1^3 \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_1 u) u \, dx \tag{3.4}$$

$$t_2\|u\|_\varepsilon^2 + t_2^3 \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_2 u) u \, dx. \tag{3.5}$$

Using (g_4) we can see that

$$\begin{aligned}
\|u\|_\varepsilon^2 \left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) &= \int_{\mathbb{R}^3} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx \\
&= \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx + \int_{\Lambda_\varepsilon} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx \\
&\geq \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx \\
&= \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u > a\}} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx \\
&+ \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u \leq a < t_1 u\}} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx \\
&+ \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_1 u < a\}} \left[\frac{g(\varepsilon\, x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon\, x, t_2 u)}{(t_2 u)^3}\right] u^4 dx =: I + II + III.
\end{aligned}$$

Let us observe that $III \geq 0$ in view of (g_4) and $t_1 > t_2$. Taking into account the definition of g, we have

$$\begin{aligned}
I &\geq \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u > a\}} \left[\frac{V_0}{K}\frac{1}{(t_1 u)^2} - \frac{V_0}{K}\frac{1}{(t_2 u)^2}\right] u^4 dx \\
&= \frac{1}{K}\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u > a\}} V_0 u^2 dx.
\end{aligned}$$

Concerning II, from the definition of g and (g_2), we can infer

$$II \geq \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u \leq a < t_1 u\}} \left[\frac{V_0}{K}\frac{1}{(t_1 u)^2} - \frac{f(t_2 u) + (t_2 u)^{2_s^*-1}}{(t_2 u)^3}\right] u^4 dx.$$

Therefore we get

$$\begin{aligned}
\|u\|_\varepsilon^2 \left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) &\geq \frac{1}{K}\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u > a\}} V_0 u^2 dx \\
&+ \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u \leq a < t_1 u\}} \left[\frac{V_0}{K}\frac{1}{(t_1 u)^2} - \frac{f(t_2 u) + (t_2 u)^{2_s^*-1}}{(t_2 u)^3}\right] u^4 dx.
\end{aligned}$$

Multiplying both sides by $\frac{t_1^2 t_2^2}{t_2^2-t_1^2}<0$ and recalling that $\frac{f(a)}{a}+a^{2_s^*-2}=\frac{V_0}{K}$, we obtain

$$
\begin{aligned}
\|u\|_\varepsilon^2 &\leq \frac{1}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u>a\}} V_0u^2dx+\frac{t_1^2t_2^2}{t_2^2-t_1^2}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\left[\frac{V_0}{K}\frac{1}{(t_1u)^2}-\frac{f(t_2u)+(t_2u)^{2_s^*-1}}{(t_2u)^3}\right]u^4dx\\
&\leq \frac{1}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u>a\}} V_0u^2dx\\
&-\frac{t_2^2}{t_1^2-t_2^2}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\frac{V_0}{K}u^2dx+\frac{t_1^2}{t_1^2-t_2^2}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\frac{f(t_2u)+(t_2u)^{2_s^*-1}}{t_2u}u^2dx\\
&\leq \frac{1}{K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V_0u^2dx\leq \frac{1}{K}\|u\|_\varepsilon^2.
\end{aligned}
$$

Then we can use the facts $u\neq 0$ and $K>2$ to get a contradiction.
(ii) Let $u\in\mathbb{S}_\varepsilon^+$. By (i) there exists $t_u>0$ such that $h_u'(t_u)=0$, or equivalently

$$
t_u+t_u^3\int_{\mathbb{R}^3}\phi_u^tu^2dx=\int_{\mathbb{R}^3}g(\varepsilon\,x,t_uu)\,u\,dx. \tag{3.6}
$$

In the light of (g_1) and (g_2), given $\xi>0$ there exists a positive constant C_ξ such that

$$
|g(x,t)|\leq \xi|t|^3+C_\xi|t|^{2_s^*-1}, \qquad \text{for every } t\in\mathbb{R}.
$$

From (3.6) and applying Theorem 2.1 we can see that

$$
t_u\leq \xi t_u^3C_1+C_\xi t_u^{2_s^*-1}C_2,
$$

which implies that there exists $\tau>0$, independent of u, such that $t_u\geq\tau$. Now, let $\mathbb{K}\subset\mathbb{S}_\varepsilon^+$ be a compact set and we show that t_u can be estimated from above by a constant depending on $\mathbb{K}$. Assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{K}$ such that $t_n:=t_{u_n}\to\infty$. Therefore, there exists $u\in\mathbb{K}$ such that $u_n\to u$ in $\mathcal{H}_\varepsilon$. In view of (3.3), we get

$$
\mathcal{J}_\varepsilon(t_nu_n)\to-\infty. \tag{3.7}
$$

Fix $v\in\mathcal{N}_\varepsilon$. Then, using the fact that $\langle\mathcal{J}_\varepsilon'(v),v\rangle=0$, and assumptions (g_3)-(i) and (g_3)-(ii), we can infer

$$
\begin{aligned}
\mathcal{J}_\varepsilon(v)&=\mathcal{J}_\varepsilon(v)-\frac{1}{\vartheta}\langle\mathcal{J}_\varepsilon'(v),v\rangle\\
&=\left(\frac{\vartheta-2}{2\vartheta}\right)\|v\|_\varepsilon^2+\left(\frac{\vartheta-4}{4\vartheta}\right)\int_{\mathbb{R}^3}\phi_v^tv^2dx+\frac{1}{\vartheta}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}[g(\varepsilon\,x,v)v-\vartheta G(\varepsilon\,x,v)]\,dx\\
&+\frac{1}{\vartheta}\int_{\Lambda_\varepsilon}[g(\varepsilon\,x,v)v-\vartheta G(\varepsilon\,x,v)]\,dx\\
&\geq\left(\frac{\vartheta-2}{2\vartheta}\right)\|v\|_\varepsilon^2+\frac{1}{\vartheta}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}[g(\varepsilon\,x,v)v-\vartheta G(\varepsilon\,x,v)]\,dx\\
&\geq\left(\frac{\vartheta-2}{2\vartheta}\right)\|v\|_\varepsilon^2-\left(\frac{\vartheta-2}{2\vartheta}\right)\frac{1}{K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}V(\varepsilon\,x)v^2dx\\
&\geq\left(\frac{\vartheta-2}{2\vartheta}\right)\left(1-\frac{1}{K}\right)\|v\|_\varepsilon^2
\end{aligned} \tag{3.8}
$$

Taking into account that $\{t_{u_n}u_n\}_{n\in\mathbb{N}}\subset\mathcal{N}_\varepsilon$ and $K>2$, from (3.8) we deduce that (3.7) does not hold, that is an absurd.

(iii) First of all, we observe that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. In fact, by (i), for each $u \in \mathcal{H}_\varepsilon^+$ there exists a unique $\hat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in \mathcal{H}_\varepsilon^+$. Otherwise, if $u \notin \mathcal{H}_\varepsilon^+$, we have

$$|\operatorname{supp}(u^+) \cap \Lambda_\varepsilon| = 0,$$

which together with $(g3)$-(ii) gives

$$\begin{aligned} \|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} \phi_u^t u^2 dx &= \int_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} g(\varepsilon\, x, u^+)\, u^+\, dx \\ &\leq \frac{1}{K} \int_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} V(\varepsilon\, x) u^2 dx \leq \frac{1}{K} \|u\|_\varepsilon^2. \end{aligned} \tag{3.9}$$

Using $\phi_u^t \geq 0$ and (3.9) we get

$$0 < \|u\|_\varepsilon^2 \leq \frac{1}{K} \|u\|_\varepsilon^2$$

and this leads to a contradiction because $K > 2$. Accordingly, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in \mathbb{S}_\varepsilon^+$, m_ε^{-1} is well defined and it is a continuous function. Now, take $u \in \mathbb{S}_\varepsilon^+$ and we have

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_\varepsilon} = \frac{u}{\|u\|_\varepsilon} = u$$

from which we deduce that m_ε is a bijection. Next, we show that $\hat{m}_\varepsilon$ is a continuous function. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\varepsilon^+$ and $u \in \mathcal{H}_\varepsilon^+$ be such that $u_n \to u$ in $\mathcal{H}_\varepsilon$. Since $\hat{m}(tu) = \hat{m}(u)$ for all $t > 0$, we may assume that $\|u_n\|_\varepsilon = \|u\|_\varepsilon = 1$ for all $n \in \mathbb{N}$. Then, in view of (ii), we can find $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Since $t_n u_n \in \mathcal{N}_\varepsilon$, we obtain

$$t_n^2 \|u_n\|_\varepsilon^2 + t_n^4 \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_n u_n)\, t_n u_n\, dx,$$

and taking the limit as $n \to \infty$ we get

$$t_0^2 \|u\|_\varepsilon^2 + t_0^4 \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_0 u)\, t_0 u\, dx$$

which yields $t_0 u \in \mathcal{N}_\varepsilon$ and $t_u = t_0$. Therefore,

$$\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u) \text{ in } \mathcal{H}_\varepsilon,$$

and $\hat{m}_\varepsilon$ and m_ε are continuous functions.
(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be such that $\operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Observing that for each $p \in [2, 2_s^*]$ and $n \in \mathbb{N}$ it holds

$$\begin{aligned} |u_n^+|_{L^p(\Lambda_\varepsilon)} &\leq \inf_{v \in \partial\mathcal{S}_\varepsilon^+} |u_n - v|_{L^p(\Lambda_\varepsilon)} \\ &\leq C_p \inf_{v \in \partial\mathcal{S}_\varepsilon^+} \|u_n - v\|_\varepsilon, \end{aligned}$$

by (g_1), (g_2), and (g_3)-(ii), we can infer that for all $t>0$

$$\begin{aligned}\int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\,dx &= \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} G(\varepsilon x, tu_n)\,dx + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)\,dx\\ &\leq \frac{t^2}{K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V(\varepsilon x)u_n^2 dx + \int_{\Lambda_\varepsilon} F(tu_n) + \frac{t^{2^*_s}}{2^*_s}(u_n^+)^{2^*_s}\,dx\\ &\leq \frac{t^2}{K}\|u_n\|_\varepsilon^2 + C_1 t^4\int_{\Lambda_\varepsilon}(u_n^+)^4 dx + C_2 t^{2^*_s}\int_{\Lambda_\varepsilon}(u_n^+)^{2^*_s} dx\\ &\leq \frac{t^2}{K} + C_1' t^4 \mathrm{dist}(u_n, \partial\mathcal{S}_\varepsilon^+)^4 + C_2' t^{2^*_s}\mathrm{dist}(u_n, \partial\mathcal{S}_\varepsilon^+)^{2^*_s}\end{aligned}$$

from which, for all $t>0$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\,dx \leq \frac{t^2}{K}. \tag{3.10}$$

Recalling the definition of $m_\varepsilon(u_n)$ and using (3.10) we get

$$\begin{aligned}\liminf_{n\to\infty}\mathcal{J}_\varepsilon(tu_n) &\geq \liminf_{n\to\infty}\mathcal{J}_\varepsilon(m_\varepsilon(u_n))\\ &\geq \liminf_{n\to\infty}\left[\frac{t^2}{2}\|u_n\|_\varepsilon^2 + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\,dx\right]\\ &\geq \left(\frac{1}{2}-\frac{1}{K}\right)t^2.\end{aligned}$$

From the arbitrariness of $t>0$ and being $K>2$, we obtain

$$\lim_{n\to\infty}\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) = \infty.$$

Moreover, $\|m_\varepsilon(u_n)\|_\varepsilon\to\infty$ as $n\to\infty$ and this ends the proof of Lemma 3.2. □

Let us introduce the maps

$$\hat{\psi}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}_\varepsilon^+ \to \mathbb{R},$$

by $\hat{\psi}_\varepsilon(u) := \mathcal{J}_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon := \hat{\psi}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$.
In view of Lemma 3.2 and Corollary 2.3 in [38], we obtain the following result.

Proposition 3.1. *Assume that hypotheses (V_1)-(V_2) and (f_1)-(f_4) hold true. Then,*

(a) $\hat{\psi}_\varepsilon \in C^1(\mathcal{H}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}_\varepsilon'(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon}\langle \mathcal{J}_\varepsilon'(\hat{m}_\varepsilon(u)), v\rangle$$

for every $u\in\mathcal{H}_\varepsilon^+$ and $v\in\mathcal{H}_\varepsilon$;

(b) $\psi_\varepsilon\in C^1(\mathbb{S}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \psi_\varepsilon'(u), v\rangle = \|m_\varepsilon(u)\|_\varepsilon\langle \mathcal{J}_\varepsilon'(m_\varepsilon(u)), v\rangle,$$

for every

$$v\in T_u\mathbb{S}_\varepsilon^+ := \{v\in\mathcal{H}_\varepsilon : (u,v)_\varepsilon = 0\};$$

(c) If $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for ψ_ε, then $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{J}_\varepsilon$. If $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{N}_\varepsilon$ is a bounded $(PS)_d$ sequence for $\mathcal{J}_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for the functional ψ_ε;

(d) u is a critical point of ψ_ε if, and only if, $m_\varepsilon(u)$ is a nontrivial critical point for $\mathcal{J}_\varepsilon$. Moreover, the corresponding critical values coincide and

$$\inf_{u\in\mathbb{S}_\varepsilon^+}\psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon}\mathcal{J}_\varepsilon(u).$$

Remark 3.1. *As in [38], we can see that the following equalities hold*

$$c_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u) = \inf_{u\in\mathcal{H}^+_\varepsilon} \max_{t>0} \mathcal{J}_\varepsilon(tu) = \inf_{u\in\mathbb{S}^+_\varepsilon} \max_{t>0} \mathcal{J}_\varepsilon(tu).$$

Remark 3.2. *Let us note that if $u \in \mathcal{N}_\varepsilon$, using (g_1), (g_2) and taking $\xi \in (0, \frac{1}{2})$ we can see that*

$$\begin{aligned} 0 &= \|u\|^2_\varepsilon + \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} g(\varepsilon\, x, u)u dx \\ &\geq \frac{1}{2}\|u\|^2_\varepsilon - C\|u\|^{2^*_s}_\varepsilon \end{aligned}$$

which implies that $\|u\|_\varepsilon \geq \alpha > 0$ for some α independent of u.

4. Existence result for the modified problem

In this section we focus our attention on the existence of positive solutions to (3.2) for small $\varepsilon > 0$. We begin showing that the functional $\mathcal{J}_\varepsilon$ satisfies the Palais-Smale condition at any level $d < \frac{s}{3}S_*^{\frac{3}{2s}}$ ($(PS)_d$ in short), where S_* is the best constant of the Sobolev embedding $H^s(\mathbb{R}^3)$ into $L^{2^*_s}(\mathbb{R}^3)$. We recall that the existence of Palais-Smale sequences of $\mathcal{J}_\varepsilon$ is justified by Lemma 3.1 and Mountain-Pass Lemma [41]. Firstly, we note that any Palais-Smale sequence is bounded.

Lemma 4.1. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(PS)_d$ sequence for $\mathcal{J}_\varepsilon$. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS) sequence at the level d, that is

$$\mathcal{J}_\varepsilon(u_n) \to d \text{ and } \mathcal{J}'_\varepsilon(u_n) \to 0 \text{ in } \mathcal{H}^{-1}_\varepsilon.$$

Arguing as in the proof of Lemma 3.2-(ii) (see formula (3.8) there), we can deduce that

$$\begin{aligned} C + \|u_n\|_\varepsilon &\geq \mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle \\ &\geq \left(\frac{\vartheta-2}{2\vartheta}\right)\left(1 - \frac{1}{K}\right)\|u_n\|^2_\varepsilon. \end{aligned}$$

Since $\vartheta > 4$ and $K > 2$, we can conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$. □

The next result will be fundamental to obtain compactness of bounded Palais-Smale sequences.

Lemma 4.2. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(PS)_d$ sequence for $\mathcal{J}_\varepsilon$. Then, for each $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that*

$$\limsup_{n\to\infty}\left[\int_{\mathbb{R}^3\setminus B_R} dx \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}}\, dy + \int_{\mathbb{R}^3\setminus B_R} V(\varepsilon\, x)u_n^2\, dx\right] < \zeta.$$

Proof. For any $R > 0$, let $\eta_R \in C^\infty(\mathbb{R}^3)$ be such that $\eta_R = 0$ in B_R and $\eta_R = 1$ in B^c_{2R}, with $0 \leq \eta_R \leq 1$ and $|\nabla\eta_R| \leq \frac{C}{R}$, where C is a constant independent of R. Since $\{\eta_R u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, it follows that $\langle \mathcal{J}'_\varepsilon(u_n), \eta_R u_n\rangle = o_n(1)$, that is

$$\begin{aligned} &\left[\iint_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}}\eta_R(x)\, dxdy + \int_{\mathbb{R}^3} V(\varepsilon\, x)u_n^2\eta_R\, dx\right] + \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 \eta_R dx \\ &= o_n(1) + \int_{\mathbb{R}^3} g(\varepsilon\, x, u_n)u_n\eta_R\, dx - \iint_{\mathbb{R}^6} \frac{(\eta_R(x) - \eta_R(y))(u_n(x) - u_n(y))}{|x-y|^{3+2s}} u_n(y)\, dxdy. \end{aligned}$$

Take $R>0$ such that $\Lambda_\varepsilon \subset B_R$. Then, using (g_3)-(ii) we get

$$\begin{aligned}&\left[\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\eta_R(x)\,dxdy + \int_{\mathbb{R}^3} V(\varepsilon\, x)u_n^2\eta_R\,dx\right] \\ &\leq \int_{\mathbb{R}^3} \frac{1}{K} V(\varepsilon\, x)u_n^2\eta_R\,dx - \iint_{\mathbb{R}^6} \frac{(\eta_R(x)-\eta_R(y))(u_n(x)-u_n(y))}{|x-y|^{3+2s}} u_n(y)\,dxdy + o_n(1)\end{aligned}$$

which implies that

$$\begin{aligned}&\left(1-\frac{1}{K}\right)\left[\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\eta_R(x)\,dxdy + \int_{\mathbb{R}^3} V(\varepsilon\, x)u_n^2\eta_R\,dx\right] \\ &\qquad \leq -\iint_{\mathbb{R}^6} \frac{(\eta_R(x)-\eta_R(y))(u_n(x)-u_n(y))}{|x-y|^{3+2s}} u_n(y)\,dxdy + o_n(1). \end{aligned} \tag{4.1}$$

Now, we aim to show that

$$\lim_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^6} \frac{(\eta_R(x)-\eta_R(y))(u_n(x)-u_n(y))}{|x-y|^{3+2s}} u_n(y)\,dxdy = 0. \tag{4.2}$$

Applying Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ we can see that

$$\begin{aligned}&\left|\iint_{\mathbb{R}^6} \frac{(\eta_R(x)-\eta_R(y))(u_n(x)-u_n(y))}{|x-y|^{3+2s}} u_n(y)\,dxdy\right| \\ &\leq \left(\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^6} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(y)|^2\,dxdy\right)^{\frac{1}{2}} \\ &\leq C\left(\iint_{\mathbb{R}^6} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(y)|^2\,dxdy\right)^{\frac{1}{2}}\end{aligned}$$

so it is enough to prove

$$\lim_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^6} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(y)|^2\,dxdy = 0$$

to infer that (4.2) holds true.

Firstly, we note that $\mathbb{R}^6$ can be written as

$$\mathbb{R}^6 = ((\mathbb{R}^3\setminus B_{2R})\times(\mathbb{R}^3\setminus B_{2R})) \cup ((\mathbb{R}^3\setminus B_{2R})\times B_{2R}) \cup (B_{2R}\times\mathbb{R}^3) =: \mathbb{X}^1_R \cup \mathbb{X}^2_R \cup \mathbb{X}^3_R.$$

Then

$$\begin{aligned}\iint_{\mathbb{R}^6} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2 dxdy &= \iint_{\mathbb{X}^1_R} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2 dxdy \\ &+ \iint_{\mathbb{X}^2_R} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2 dxdy + \iint_{\mathbb{X}^3_R} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2 dxdy.\end{aligned} \tag{4.3}$$

In what follows, we estimate each integrals in (4.3). Since $\eta_R = 1$ in $\mathbb{R}^3\setminus B_{2R}$, we have

$$\iint_{\mathbb{X}^1_R} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}} dxdy = 0. \tag{4.4}$$

Let $k>4$. Clearly, we have

$$\mathbb{X}^2_R = (\mathbb{R}^3\setminus B_{2R})\times B_{2R} \subset ((\mathbb{R}^3\setminus B_{kR})\times B_{2R}) \cup ((B_{kR}\setminus B_{2R})\times B_{2R}).$$

Let us observe that, if $(x,y) \in (\mathbb{R}^3 \setminus B_{kR}) \times B_{2R}$, then

$$|x-y| \geq |x| - |y| \geq |x| - 2R > \frac{|x|}{2}.$$

Then, taking into account that $0 \leq \eta_R \leq 1$, $|\nabla \eta_R| \leq \frac{C}{R}$ and applying Hölder inequality, we can see

$$\begin{aligned}
&\iint_{\mathbb{X}^2_R} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}} dxdy \\
&= \int_{\mathbb{R}^3 \setminus B_{kR}} dx \int_{B_{2R}} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}} dy + \int_{B_{kR} \setminus B_{2R}} dx \int_{B_{2R}} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}} dy \\
&\leq 2^{5+2s} \int_{\mathbb{R}^3 \setminus B_{kR}} dx \int_{B_{2R}} \frac{|u_n(x)|^2}{|x|^{3+2s}}\, dy + \frac{C}{R^2} \int_{B_{kR} \setminus B_{2R}} dx \int_{B_{2R}} \frac{|u_n(x)|^2}{|x-y|^{3+2(s-1)}}\, dy \\
&\leq C R^3 \int_{\mathbb{R}^3 \setminus B_{kR}} \frac{|u_n(x)|^2}{|x|^{3+2s}}\, dx + \frac{C}{R^2} (kR)^{2(1-s)} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^2 dx \\
&\leq C R^3 \left(\int_{\mathbb{R}^3 \setminus B_{kR}} |u_n(x)|^{2^*_s} dx \right)^{\frac{2}{2^*_s}} \left(\int_{\mathbb{R}^3 \setminus B_{kR}} \frac{1}{|x|^{\frac{9}{2s}+3}}\, dx \right)^{\frac{2s}{3}} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^2 dx \\
&\leq \frac{C}{k^3} \left(\int_{\mathbb{R}^3 \setminus B_{kR}} |u_n(x)|^{2^*_s} dx \right)^{\frac{2}{2^*_s}} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^2 dx \\
&\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^2 dx. \qquad (4.5)
\end{aligned}$$

Now, we fix $\delta \in (0,1)$, and we note that

$$\begin{aligned}
&\iint_{\mathbb{X}^3_R} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}}\, dxdy \\
&\leq \int_{B_{2R} \setminus B_{\delta R}} dx \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}}\, dy + \int_{B_{\delta R}} dx \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}}\, dy. \quad (4.6)
\end{aligned}$$

Let us estimate the first integral in (4.6). Thus,

$$\int_{B_{2R} \setminus B_{\delta R}} dx \int_{\mathbb{R}^3 \cap \{y : |x-y| < R\}} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}}\, dy \leq \frac{C}{R^{2s}} \int_{B_{2R} \setminus B_{\delta R}} |u_n(x)|^2 dx$$

and

$$\int_{B_{2R} \setminus B_{\delta R}} dx \int_{\mathbb{R}^3 \cap \{y : |x-y| \geq R\}} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}}\, dy \leq \frac{C}{R^{2s}} \int_{B_{2R} \setminus B_{\delta R}} |u_n(x)|^2 dx$$

from which we obtain

$$\int_{B_{2R} \setminus B_{\delta R}} dx \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x-y|^{3+2s}}\, dy \leq \frac{C}{R^{2s}} \int_{B_{2R} \setminus B_{\delta R}} |u_n(x)|^2 dx. \qquad (4.7)$$

Now, using the definition of η_R, $\delta \in (0,1)$, and $0 \leq \eta_R \leq 1$, we have

$$\begin{aligned}
\int_{B_{\delta R}} dx \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\, dy &= \int_{B_{\delta R}} dx \int_{\mathbb{R}^3 \setminus B_R} \frac{|u_n(x)|^2 |\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\, dy \\
&\leq 4 \int_{B_{\delta R}} dx \int_{\mathbb{R}^3 \setminus B_R} \frac{|u_n(x)|^2}{|x-y|^{3+2s}}\, dy \\
&\leq C \int_{B_{\delta R}} |u_n|^2 dx \int_{(1-\delta)R}^{\infty} \frac{1}{r^{1+2s}} dr \\
&= \frac{C}{[(1-\delta)R]^{2s}} \int_{B_{\delta R}} |u_n|^2 dx \qquad (4.8)
\end{aligned}$$

where we used the fact that if $(x,y) \in B_{\delta R} \times (\mathbb{R}^3 \setminus B_R)$, then $|x-y| > (1-\delta)R$.
Taking into account (4.6), (4.7) and (4.8) we deduce

$$\begin{aligned}
&\iint_{\mathbb{X}^3_R} \frac{|u_n(x)|^2 |\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\, dxdy \\
&\qquad \leq \frac{C}{R^{2s}} \int_{B_{2R} \setminus B_{\delta R}} |u_n(x)|^2 dx + \frac{C}{[(1-\delta)R]^{2s}} \int_{B_{\delta R}} |u_n(x)|^2 dx. \qquad (4.9)
\end{aligned}$$

Putting together (4.3), (4.4), (4.5) and (4.9), we can infer

$$\begin{aligned}
\iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\, dxdy &\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^2 dx \\
&\quad + \frac{C}{R^{2s}} \int_{B_{2R} \setminus B_{\delta R}} |u_n(x)|^2 dx + \frac{C}{[(1-\delta)R]^{2s}} \int_{B_{\delta R}} |u_n(x)|^2 dx. \qquad (4.10)
\end{aligned}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^3)$, by Theorem 2.1 we may assume that $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$ for some $u \in H^s(\mathbb{R}^3)$. Taking the limit as $n \to \infty$ in (4.10) we have

$$\begin{aligned}
&\limsup_{n\to\infty} \iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\, dxdy \\
&\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR} \setminus B_{2R}} |u(x)|^2 dx + \frac{C}{R^{2s}} \int_{B_{2R} \setminus B_{\delta R}} |u(x)|^2 dx + \frac{C}{[(1-\delta)R]^{2s}} \int_{B_{\delta R}} |u(x)|^2 dx \\
&\leq \frac{C}{k^3} + Ck^2 \left(\int_{B_{kR} \setminus B_{2R}} |u(x)|^{2^*_s} dx \right)^{\frac{2}{2^*_s}} + C \left(\int_{B_{2R} \setminus B_{\delta R}} |u(x)|^{2^*_s} dx \right)^{\frac{2}{2^*_s}} + C \left(\frac{\delta}{1-\delta} \right)^{2s} \left(\int_{B_{\delta R}} |u(x)|^{2^*_s} dx \right)^{\frac{2}{2^*_s}},
\end{aligned}$$

where in the last passage we used the Hölder inequality.
Since $u \in L^{2^*_s}(\mathbb{R}^3)$, $k > 4$ and $\delta \in (0,1)$, we obtain

$$\limsup_{R\to\infty} \int_{B_{kR} \setminus B_{2R}} |u(x)|^{2^*_s} dx = \limsup_{R\to\infty} \int_{B_{2R} \setminus B_{\delta R}} |u(x)|^{2^*_s} dx = 0.$$

Choosing $\delta = \frac{1}{k}$, we get

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&\quad\le \lim_{k\to\infty}\limsup_{R\to\infty}\Big[\frac{C}{k^3}+Ck^2\left(\int_{B_{kR}\setminus B_{2R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}+C\left(\int_{B_{2R}\setminus B_{\frac{1}{k}R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}} \\
&\qquad+C\left(\frac{1}{k-1}\right)^{2s}\left(\int_{B_{\frac{1}{k}R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}\Big] \\
&\quad\le \lim_{k\to\infty}\frac{C}{k^3}+C\left(\frac{1}{k-1}\right)^{2s}\left(\int_{\mathbb{R}^3}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}=0.
\end{aligned}$$

Putting together (4.1), (4.2) and using the definition of η_R, we deduce that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus B_R}dx\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dy+\int_{\mathbb{R}^3\setminus B_R}V(\varepsilon\, x)u_n^2\,dx=0.$$

This ends the proof of Lemma 4.2. □

Proposition 4.1. *The functional $\mathcal{J}_\varepsilon$ verifies the $(PS)_d$ condition in $\mathcal{H}_\varepsilon$ at any level $d<\frac{s}{3}S_*^{\frac{3}{2s}}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS) sequence for $\mathcal{J}_\varepsilon$ at the level d. By Lemma 4.1 we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, and, up to a subsequence, we may assume that

$$u_n \rightharpoonup u \text{ in } \mathcal{H}_\varepsilon. \tag{4.11}$$

In view of Lemma 4.2, for each $\zeta>0$ there exists $R=R(\zeta)>0$ such that

$$\limsup_{n\to\infty}\left[\int_{\mathbb{R}^3\setminus B_R}dx\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dy+\int_{\mathbb{R}^3\setminus B_R}V(\varepsilon\, x)u_n^2\,dx\right]<\zeta. \tag{4.12}$$

Using (4.12) and $\mathcal{H}_\varepsilon \Subset L^r_{loc}(\mathbb{R}^3)$ for all $r\in[2,2^*_s)$, it is easy to deduce that $u_n\to u$ in $L^r(\mathbb{R}^3)$ for all $r\in[2,2^*_s)$. In particular

$$u_n\to u \text{ in } L^{\frac{12}{3+2t}}(\mathbb{R}^3). \tag{4.13}$$

Then, in view of (4.11) and (4.13), we can apply (1) and (6) of Lemma 2.3 to infer that

$$\phi^t_{u_n}\to\phi^t_u \text{ in } D^{t,2}(\mathbb{R}^3) \tag{4.14}$$

and

$$\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx\to\int_{\mathbb{R}^3}\phi^t_u u^2dx. \tag{4.15}$$

Putting together (4.11), (4.13) and (4.14) it is easy to check that as $n\to\infty$

$$\int_{\mathbb{R}^3}\phi^t_{u_n}u_n\psi\,dx=\int_{\mathbb{R}^3}\phi^t_u u\psi\,dx+o_n(1)$$

for all $\psi\in C^\infty_c(\mathbb{R}^3)$. Since it is clear that (4.11) and (f_1)-(f_2) yield

$$(u_n,\psi)_\varepsilon\to(u,\psi)_\varepsilon \quad\text{and}\quad \int_{\mathbb{R}^3}g(\varepsilon\, x,u_n)\psi\,dx\to\int_{\mathbb{R}^3}g(\varepsilon\, x,u)\psi\,dx,$$

for all $\psi \in C_c^\infty(\mathbb{R}^3)$, we can infer that u is a critical point of $\mathcal{J}_\varepsilon$. In particular

$$\|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \int_{\mathbb{R}^3} g(\varepsilon x, u)u\, dx. \tag{4.16}$$

In what follows, we show that

$$\int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\, dx \to \int_{\mathbb{R}^3} g(\varepsilon x, u)u\, dx. \tag{4.17}$$

Using (4.12), (f_1), (f_2), (g_2) and Theorem 2.1 we can see that

$$\int_{\mathbb{R}^3\setminus B_R} g(\varepsilon x, u_n)u_n\, dx \leq C(\zeta + \zeta^{\frac{\sigma}{2}} + \zeta^{\frac{2_s^*}{2}}), \tag{4.18}$$

for any n big enough. On the other hand, choosing R large enough, we may assume that

$$\int_{\mathbb{R}^3\setminus B_R} g(\varepsilon x, u)u\, dx \leq \zeta. \tag{4.19}$$

From the arbitrariness of $\zeta > 0$, we can see that (4.18) and (4.19) yield

$$\int_{\mathbb{R}^3\setminus B_R} g(\varepsilon x, u_n)u_n\, dx \to \int_{\mathbb{R}^3\setminus B_R} g(\varepsilon x, u)u\, dx \tag{4.20}$$

as $n \to \infty$. Now, we note that by the definition of g we know that

$$g(\varepsilon x, u_n)u_n \leq f(u_n)u_n + a^{2_s^*} + \frac{V_0}{K}u_n^2 \text{ in } \mathbb{R}^3 \setminus \Lambda_\varepsilon.$$

Since $B_R \cap (\mathbb{R}^3 \setminus \Lambda_\varepsilon)$ is bounded, we can use (f_1)-(f_2), the Dominated Convergence Theorem and the strong convergence in $L^r_{loc}(\mathbb{R}^3)$ for all $r \in [1, 2_s^*)$, to see that

$$\int_{B_R\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)} g(\varepsilon x, u_n)u_n\, dx \to \int_{B_R\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)} g(\varepsilon x, u)u\, dx \tag{4.21}$$

as $n \to \infty$.
At this point, we aim to show that

$$\lim_{n\to\infty} \int_{\Lambda_\varepsilon} |u_n|^{2_s^*}\, dx = \int_{\Lambda_\varepsilon} |u|^{2_s^*}\, dx. \tag{4.22}$$

Indeed, if we assume that (4.22) is true, from Theorem 2.1, (g_2), (f_1)-(f_2), (4.11) and the Dominated Convergence Theorem, we can see that

$$\int_{B_R\cap\Lambda_\varepsilon} g(\varepsilon x, u_n)u_n\, dx \to \int_{B_R\cap\Lambda_\varepsilon} g(\varepsilon x, u)u\, dx. \tag{4.23}$$

Putting together (4.20), (4.21) and (4.23), we can conclude that (4.17) holds. Hence, in view of $\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle = o_n(1)$, we can see that (4.15), (4.16) and (4.17) imply that $\|u_n\|_\varepsilon \to \|u\|_\varepsilon$, and then $u_n \to u$ in $\mathcal{H}_\varepsilon$ (since $\mathcal{H}_\varepsilon$ is a Hilbert space).

It remains to prove that (4.22) is satisfied. Invoking the Concentration-Compactness Lemma for the fractional Laplacian [17, 31], we can find an at most countable index set I, sequences $\{x_i\}_{i\in I} \subset \mathbb{R}^3$, $\{\mu_i\}_{i\in I}, \{\nu_i\}_{i\in I} \subset (0,\infty)$ such that

$$\begin{aligned} &\mu \geq |(-\Delta)^{\frac{s}{2}} u|^2 + \sum_{i\in I} \mu_i \delta_{x_i}, \\ &\nu = |u|^{2_s^*} + \sum_{i\in I} \nu_i \delta_{x_i} \quad \text{and } S_* \nu_i^{\frac{2}{2_s^*}} \leq \mu_i \end{aligned} \tag{4.24}$$

for any $i \in I$, where δ_{x_i} is the Dirac mass at the point x_i. Let us show that $\{x_i\}_{i\in I} \cap \Lambda_\varepsilon = \emptyset$. Assume by contradiction that $x_i \in \Lambda_\varepsilon$ for some $i \in I$. For any $\rho > 0$, we define $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$ where $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ is such that $\psi = 1$ in B_1, $\psi = 0$ in $\mathbb{R}^3 \setminus B_2$ and $|\nabla\psi|_\infty \leq 2$. We suppose that $\rho > 0$ is such that $supp(\psi_\rho) \subset \Lambda_\varepsilon$. Since $\{\psi_\rho u_n\}_{n\in\mathbb{N}}$ is bounded, we have $\langle \mathcal{J}'_\varepsilon(u_n), \psi_\rho u_n\rangle = o_n(1)$, from which

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \psi_\rho(y)\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&\leq \iint_{\mathbb{R}^6} \psi_\rho(y)\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dxdy + \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2\psi_\rho dx + \int_{\mathbb{R}^3} V(\varepsilon\, x)u_n^2\psi_\rho dx \\
&\leq -\iint_{\mathbb{R}^6} u_n(x)\frac{(u_n(x)-u_n(y)(\psi_\rho(x)-\psi_\rho(y)))}{|x-y|^{3+2s}}\,dxdy \\
&+ \int_{\mathbb{R}^3} u_n\psi_\rho f(u_n)\,dx + \int_{\mathbb{R}^3} \psi_\rho|u_n|^{2^*_s}\,dx + o_n(1).
\end{aligned} \tag{4.25}$$

Due to the fact that f has subcritical growth and ψ_ρ has compact support, we obtain that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)u_n\psi_\rho\,dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}\psi_\rho f(u)u\,dx = 0. \tag{4.26}$$

Now, we show that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^6} u_n(x)\frac{(u_n(x)-u_n(y)(\psi_\rho(x)-\psi_\rho(y)))}{|x-y|^{3+2s}}\,dxdy = 0. \tag{4.27}$$

Using Hölder inequality and the fact that $\{u_n\}$ is bounded in $\mathcal{H}_\varepsilon$, we can see that

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^6} u_n(x)\frac{(u_n(x)-u_n(y)(\psi_\rho(x)-\psi_\rho(y)))}{|x-y|^{3+2s}}\,dxdy\right| \\
&\leq \left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}}\left(\iint_{\mathbb{R}^6}|u_n(x)|^2\frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}} \\
&\leq C\left(\iint_{\mathbb{R}^6}|u_n(x)|^2\frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, if we prove that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^6}|u_n(x)|^2\frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy = 0, \tag{4.28}$$

then (4.27) is satisfied. In what follows, we modify suitably the arguments in Lemma 4.2. Let us note that $\mathbb{R}^6$ can be written as

$$\mathbb{R}^6 = ((\mathbb{R}^3\setminus B_{2\rho}(x_i))\times(\mathbb{R}^3\setminus B_{2\rho}(x_i)))\cup(B_{2\rho}(x_i)\times\mathbb{R}^3)\cup((\mathbb{R}^3\setminus B_{2\rho}(x_i))\times B_{2\rho}(x_i)) =: X^1_\rho\cup X^2_\rho\cup X^3_\rho.$$

Hence

$$\begin{aligned}
&\iint_{\mathbb{R}^6}|u_n(x)|^2\frac{(\psi_\rho(x)-\psi_\rho(y))^2}{|x-y|^{3+2s}}\,dxdy \\
&= \iint_{X^1_\rho}|u_n(x)|^2\frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy + \iint_{X^2_\rho}|u_n(x)|^2\frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&+ \iint_{X^3_\rho}|u_n(x)|^2\frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy.
\end{aligned} \tag{4.29}$$

Since $\psi=0$ in $\mathbb{R}^3\setminus B_2$, we have

$$\iint_{X^1_\rho} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy=0. \tag{4.30}$$

Using $0\leq \psi\leq 1$ and the Mean Value Theorem, we can see that

$$\begin{aligned}
&\iint_{X^2_\rho} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&=\int_{B_{2\rho}(x_i)} dx \int_{\{y\in\mathbb{R}^3:|x-y|\leq \rho\}} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dy+\int_{B_{2\rho}(x_i)} dx \int_{\{y\in\mathbb{R}^3:|x-y|> \rho\}} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dy \\
&\leq \rho^{-2}|\nabla\psi|^2_\infty \int_{B_{2\rho}(x_i)} dx \int_{\{y\in\mathbb{R}^3:|x-y|\leq \rho\}} \frac{|u_n(x)|^2}{|x-y|^{3+2s-2}}\,dy+4\int_{B_{2\rho}(x_i)} dx \int_{\{y\in\mathbb{R}^3:|x-y|> \rho\}} \frac{|u_n(x)|^2}{|x-y|^{3+2s}}\,dy \\
&\leq C\rho^{-2s}\int_{B_{2\rho}(x_i)} |u_n(x)|^2\,dx + C\rho^{-2s}\int_{B_{2\rho}(x_i)} |u_n(x)|^2\,dx \leq C\rho^{-2s}\int_{B_{2\rho}(x_i)} |u_n(x)|^2\,dx,
\end{aligned} \tag{4.31}$$

for some $C>0$ independent of ρ and n. On the other hand

$$\begin{aligned}
&\iint_{X^3_\rho} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&=\int_{\mathbb{R}^3\setminus B_{2\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|\leq \rho\}} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dy \\
&+\int_{\mathbb{R}^3\setminus B_{2\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|> \rho\}} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dy =: A_{\rho,n}+B_{\rho,n}.
\end{aligned} \tag{4.32}$$

Now, we note that $|x-y|<\rho$ and $|y-x_i|<2\rho$ imply $|x-x_i|<3\rho$, so we get

$$\begin{aligned}
A_{\rho,n} &\leq \rho^{-2}|\nabla\psi|^2_\infty \int_{B_{3\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|\leq \rho\}} \frac{|u_n(x)|^2}{|x-y|^{1+2s}}\,dy \\
&\leq C\rho^{-2}\int_{B_{3\rho}(x_i)} |u_n(x)|^2\,dx \int_0^\rho \frac{1}{r^{2s-1}}\,dr \\
&\leq C\rho^{-2s}\int_{B_{3\rho}(x_i)} |u_n(x)|^2\,dx,
\end{aligned} \tag{4.33}$$

for some $C>0$ independent of ρ and n. Let us observe, that for all $k>4$ it holds

$$(\mathbb{R}^3\setminus B_{2\rho}(x_i))\times B_{2\rho}(x_i)\subset (B_{k\rho}(x_i)\times B_{2\rho}(x_i))\cup ((\mathbb{R}^3\setminus B_{k\rho}(x_i))\times B_{2\rho}(x_i)).$$

Then, we have the following estimates

$$\begin{aligned}
&\int_{B_{k\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|> \rho\}} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\,dy \\
&\leq 4\int_{B_{k\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|> \rho\}} |u_n(x)|^2 \frac{1}{|x-y|^{3+2s}}\,dy \\
&\leq 4\int_{B_{k\rho}(x_i)} |u_n(x)|^2\,dx \int_{\{z\in\mathbb{R}^3:|z|> \rho\}} \frac{1}{|z|^{3+2s}}\,dz \\
&\leq C\rho^{-2s}\int_{B_{k\rho}(x_i)} |u_n(x)|^2\,dx,
\end{aligned} \tag{4.34}$$

for some $C>0$ independent of ρ and n. On the other hand, $|x-x_i|\geq k\rho$ and $|y-x_i|<2\rho$ imply

$$|x-y|\geq |x-x_i|-|y-x_i|\geq \frac{|x-x_i|}{2}+\frac{k\rho}{2}-2\rho>\frac{|x-x_i|}{2},$$

which together with $0\leq \psi\leq 1$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^3\setminus B_{k\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|>\rho\}} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\, dy \\
&\leq C\int_{\mathbb{R}^3\setminus B_{k\rho}(x_i)} dx \int_{\{y\in B_{2\rho}(x_i):|x-y|>\rho\}} \frac{|u_n(x)|^2}{|x-x_i|^{3+2s}}\, dy \\
&\leq C\rho^3 \int_{\mathbb{R}^3\setminus B_{k\rho}(x_i)} \frac{|u_n(x)|^2}{|x-x_i|^{3+2s}}\, dx \\
&\leq C\rho^3 \left(\int_{\mathbb{R}^3\setminus B_{k\rho}(x_i)} |u_n(x)|^{2^*_s}\, dx\right)^{\frac{2}{2^*_s}} \left(\int_{\mathbb{R}^3\setminus B_{k\rho}(x_i)} |x-x_i|^{-(3+2s)\frac{2^*_s}{2^*_s-2}}\, dx\right)^{\frac{2^*_s-2}{2^*_s}} \\
&\leq Ck^{-3}\left(\int_{\mathbb{R}^3\setminus B_{k\rho}(x_i)} |u_n(x)|^{2^*_s}\, dx\right)^{\frac{2}{2^*_s}}, \qquad (4.35)
\end{aligned}$$

for some $C>0$ independent of ρ and n. Taking into account (4.34) and (4.35), and the fact that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{2^*_s}(\mathbb{R}^3)$, we can find $C>0$ independent of ρ and n such that

$$B_{\rho,n}\leq C\rho^{-2s}\int_{B_{k\rho}(x_i)} |u_n(x)|^2\, dx + Ck^{-3}. \qquad (4.36)$$

Putting together (4.29)-(4.33) and (4.36), we have

$$\iint_{\mathbb{R}^6} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\, dxdy \leq C\rho^{-2s}\int_{B_{k\rho}(x_i)} |u_n(x)|^2\, dx + Ck^{-3}, \qquad (4.37)$$

for some $C>0$ independent of ρ and n. Since $u_n\to u$ strongly in $L^2_{loc}(\mathbb{R}^3)$, we can deduce that

$$\lim_{n\to\infty} C\rho^{-2s}\int_{B_{k\rho}(x_i)} |u_n(x)|^2\, dx + Ck^{-3} = C\rho^{-2s}\int_{B_{k\rho}(x_i)} u^2(x)\, dx + Ck^{-3}.$$

Moreover, using the Hölder inequality, we get

$$\begin{aligned}
C\rho^{-2s}\int_{B_{k\rho}(x_i)} |u(x)|^2\, dx + Ck^{-3} &\leq C\rho^{-2s}\left(\int_{B_{k\rho}(x_i)} |u(x)|^{2^*_s}\, dx\right)^{\frac{2}{2^*_s}} |B_{k\rho}(x_i)|^{1-\frac{2}{2^*_s}} + Ck^{-3} \\
&\leq Ck^{2s}\left(\int_{B_{k\rho}(x_i)} |u(x)|^{2^*_s}\, dx\right)^{\frac{2}{2^*_s}} + Ck^{-3} \to Ck^{-3} \text{ as } \rho\to 0.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\lim_{\rho\to 0}\limsup_{n\to\infty} \iint_{\mathbb{R}^6} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\, dxdy \\
&= \lim_{k\to\infty}\lim_{\rho\to 0}\limsup_{n\to\infty} \iint_{\mathbb{R}^6} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\, dxdy = 0,
\end{aligned}$$

that is (4.28) holds true.

Now, by (4.24) and taking the limit as $\rho\to 0$ and $n\to\infty$ in (4.25), we can deduce that $\nu_i\geq \mu_i$. In view of the last statement in (4.24), we have $\nu_i\geq S^{\frac{3}{2s}}$, and using (g_3), (V_1) and $K>2$ we can deduce that

$$
\begin{aligned}
d&=\mathcal{J}_\varepsilon(u_n)-\frac{1}{4}\langle \mathcal{J}'_\varepsilon(u_n),u_n\rangle+o_n(1)\\
&\geq \frac{1}{4}\|u_n\|^2_\varepsilon+\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}\left[\frac{1}{4}u_n g(\varepsilon x,u_n)-G(\varepsilon x,u_n)\right]dx+\frac{4s-3}{12}\int_{\Lambda_\varepsilon}|u_n|^{2^*_s}\,dx+o_n(1)\\
&\geq \frac{1}{4}\left[\int_{\Lambda_\varepsilon}\psi_\rho|(-\Delta)^{\frac{s}{2}}u_n|^2dx+\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}V(\varepsilon x)u_n^2dx\right]-\frac{1}{4K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}V_0u_n^2dx+\frac{4s-3}{12}\int_{\Lambda_\varepsilon}|u_n|^{2^*_s}\,dx+o_n(1)\\
&\geq \frac{1}{4}\int_{\Lambda_\varepsilon}\psi_\rho|(-\Delta)^{\frac{s}{2}}u_n|^2dx+\left(\frac{1}{4}-\frac{1}{4K}\right)\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}V(\varepsilon x)u_n^2dx+\frac{4s-3}{12}\int_{\Lambda_\varepsilon}|u_n|^{2^*_s}\,dx+o_n(1)\\
&\geq \frac{1}{4}\int_{\Lambda_\varepsilon}\psi_\rho|(-\Delta)^{\frac{s}{2}}u_n|^2dx+\frac{4s-3}{12}\int_{\Lambda_\varepsilon}\psi_\rho|u_n|^{2^*_s}\,dx+o_n(1).
\end{aligned}
$$

Then, in view of (4.24), $\nu_i\geq S^{\frac{3}{2s}}$ and taking the limit as $n\to\infty$, we find

$$
\begin{aligned}
d&\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\mu_i+\frac{4s-3}{12}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\nu_i\\
&\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)S_*\nu_i^{2/2^*_s}+\frac{4s-3}{12}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\nu_i\\
&\geq \frac{1}{4}S_*^{\frac{3}{2s}}+\frac{4s-3}{12}S_*^{\frac{3}{2s}}=\frac{s}{3}S_*^{\frac{3}{2s}},
\end{aligned}
$$

which gives a contradiction. This means that (4.22) holds and we can conclude the proof. □

Corollary 4.1. *The functional ψ_ε verifies the $(PS)_d$ condition on $\mathbb{S}^+_\varepsilon$ for any $d<\frac{s}{3}S_*^{\frac{3}{2s}}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{S}^+_\varepsilon$ be a (PS) sequence for ψ_ε at the level d, that is

$$\psi_\varepsilon(u_n)\to d \quad\text{and}\quad \psi'_\varepsilon(u_n)\to 0 \text{ in } (T_{u_n}\mathbb{S}^+_\varepsilon)'.$$

Using Proposition 3.1-(c) we can see that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{J}_\varepsilon$ in $\mathcal{H}_\varepsilon$. Then, from Proposition 4.1 we can see that $\mathcal{J}_\varepsilon$ verifies the $(PS)_d$ condition in $\mathcal{H}_\varepsilon$, so there exists $u\in\mathbb{S}^+_\varepsilon$ such that, up to a subsequence,

$$m_\varepsilon(u_n)\to m_\varepsilon(u) \text{ in } \mathcal{H}_\varepsilon.$$

By Lemma 3.2-(iii), we can infer that $u_n\to u$ in $\mathbb{S}^+_\varepsilon$. □

Now, we conclude this section giving the proof of the main result of this section:

Theorem 4.1. *Assume that (V_1)-(V_2) and $(f1)$-$(f4)$ hold. Then, problem (3.2) admits a positive ground state for all $\varepsilon>0$.*

Proof. Arguing as in the proof of Lemma 3.1 in [27], we can prove that $c_\varepsilon<\frac{s}{3}S_*^{\frac{3}{2s}}$ for all $\varepsilon>0$. Then, taking into account Lemma 3.1, Lemma 4.1, Proposition 4.1, and applying mountain pass theorem [4], we can see that $\mathcal{J}_\varepsilon$ admits a nontrivial critical point $u\in\mathcal{H}_\varepsilon$. Since $\langle \mathcal{J}'_\varepsilon(u),u^-\rangle=0$, where $u^-=\min\{u,0\}$, it is easy to check that $u\geq 0$ in $\mathbb{R}^3$. Moreover, proceeding as in the proof of Lemma 7.1 below, we can see that $u\in L^\infty(\mathbb{R}^3)$. From Proposition 2.9 in [37], we deduce that $u\in C^{1,\alpha}(\mathbb{R}^3)$ and by the maximum principle [37] we can conclude that $u>0$ in $\mathbb{R}^3$. Finally, we show that u is a ground state solution. Indeed, in view of (g_3) and applying Fatou's Lemma we obtain

$$\begin{aligned}
c_\varepsilon &\leq \mathcal{J}_\varepsilon(u) - \frac{1}{\vartheta}\langle \mathcal{J}'_\varepsilon(u), u\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u\|_\varepsilon^2 + \left(\frac{1}{4} - \frac{1}{\vartheta}\right)\int_{\mathbb{R}^3} \phi_u^t u^2 dx + \int_{\mathbb{R}^3} \frac{1}{\vartheta} g(\varepsilon\, x, u)u - G(\varepsilon\, x, u)dx \\
&\leq \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u_n\|_\varepsilon^2 + \left(\frac{1}{4} - \frac{1}{\vartheta}\right)\int_{\mathbb{R}^3} \phi_u^t u_n^2 dx + \int_{\mathbb{R}^3} \frac{1}{\vartheta} g(\varepsilon\, x, u_n)u_n - G(\varepsilon\, x, u_n)dx\right] \\
&= \liminf_{n\to\infty}\left[\mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle\right] \\
&= c_\varepsilon
\end{aligned}$$

which implies that $\mathcal{J}_\varepsilon(u) = c_\varepsilon$. □

5. The autonomous problem

In this section we consider the limit problem associated to (3.2). More precisely, we deal with the following autonomous problem

$$\begin{cases} (-\Delta)^s u + V_0 u + \phi_u^t u = f(u) + |u|^{2_s^*-2}u \text{ in } \mathbb{R}^3, \\ u \in H^s(\mathbb{R}^3), \quad u > 0 \text{ in } \mathbb{R}^3. \end{cases} \tag{5.1}$$

The Euler-Lagrange functional associated to (5.1) is given by

$$\mathcal{J}_0(u) = \frac{1}{2}\left(\iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} V_0 u^2 dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F(u)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} (u^+)^{2_s^*} dx$$

which is well defined on the Hilbert space $\mathcal{H}_0 := H^s(\mathbb{R}^3)$ endowed with the inner product

$$(u, \varphi)_0 = \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} V_0 u(x)\,\varphi(x)dx.$$

The norm induced by this inner product is

$$\|u\|_0^2 = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} V_0 u^2 dx.$$

The Nehari manifold associated to $\mathcal{J}_0$ is given by

$$\mathcal{N}_0 = \{u \in \mathcal{H}_0 \setminus \{0\} : \langle \mathcal{J}'_0(u), u\rangle = 0\}.$$

We denote by $\mathcal{H}_0^+$ the open subset of $\mathcal{H}_0$ defined as

$$\mathcal{H}_0^+ = \{u \in \mathcal{H}_0 : |\operatorname{supp}(u^+)| > 0\},$$

and $\mathbb{S}_0^+ = \mathbb{S}_0 \cap \mathcal{H}_0^+$, where $\mathbb{S}_0$ is the unit sphere of $\mathcal{H}_0$. We note that $\mathbb{S}_0^+$ is a incomplete $C^{1,1}$-manifold of codimension 1 modeled on $\mathcal{H}_0$ and contained in $\mathcal{H}_0^+$. Thus $\mathcal{H}_0 = T_u\mathbb{S}_0^+ \oplus \mathbb{R}u$ for each $u \in \mathbb{S}_0^+$, where $T_u\mathbb{S}_0^+ = \{u \in \mathcal{H}_0 : (u, v)_0 = 0\}$.
As in Section 3, we can see that the following results hold.

Lemma 5.1. *Assume that (f_1)-(f_4) hold true. Then,*

(i) For each $u \in \mathcal{H}_0^+$, let $h_u : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{J}_0(tu)$. Then, there is a unique $t_u > 0$ such that

$$h'_u(t) > 0 \ in \ (0, t_u)$$
$$h'_u(t) < 0 \ in \ (t_u, \infty);$$

(ii) *there exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}_0^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_0^+$ there is a positive constant $C_{\mathbb{K}}$ such that $t_u \leq C_{\mathbb{K}}$ for any $u \in \mathbb{K}$;*

(iii) *The map $\hat{m}_0 : \mathcal{H}_0^+ \to \mathcal{N}_0$ given by $\hat{m}_0(u) = t_u u$ is continuous and $m_0 := \hat{m}_0|_{\mathbb{S}_0^+}$ is a homeomorphism between $\mathbb{S}_0^+$ and $\mathcal{N}_0$. Moreover $m_0^{-1}(u) = \frac{u}{\|u\|_0}$;*

(iv) *If there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_0^+$ such that $\mathrm{dist}(u_n, \partial\mathbb{S}_0^+) \to 0$, then $\|m_0(u_n)\|_0 \to \infty$ and $\mathcal{J}_0(m_0(u_n)) \to \infty$.*

Let us define the maps

$$\hat{\psi}_0 : \mathcal{H}_0^+ \to \mathbb{R} \quad \text{and} \quad \psi_0 : \mathbb{S}_0^+ \to \mathbb{R},$$

by $\hat{\psi}_0(u) := \mathcal{J}_0(\hat{m}_0(u))$ and $\psi_0 := \hat{\psi}_0|_{\mathbb{S}_0^+}$.

Proposition 5.1. *Assume that assumptions (f_1)-(f_4) hold true. Then,*

(a) *$\hat{\psi}_0 \in C^1(\mathcal{H}_0^+, \mathbb{R})$ and*

$$\langle \hat{\psi}_0'(u), v\rangle = \frac{\|\hat{m}_0(u)\|_0}{\|u\|_0} \langle \mathcal{J}_0'(\hat{m}_0(u)), v\rangle$$

for every $u \in \mathcal{H}_0^+$ and $v \in \mathcal{H}_0$;

(b) *$\psi_0 \in C^1(\mathbb{S}_0^+, \mathbb{R})$ and*

$$\langle \psi_0'(u), v\rangle = \|m_0(u)\|_0 \langle \mathcal{J}_0'(m_0(u)), v\rangle,$$

for every

$$v \in T_u\mathbb{S}_0^+ := \{v \in \mathcal{H}_0 : (u, v)_0 = 0\}\,;$$

(c) *If $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for ψ_0, then $\{m_0(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{J}_0$. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ is a bounded $(PS)_d$ sequence for $\mathcal{J}_0$, then $\{m_0^{-1}(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for the functional ψ_0;*

(d) *u is a critical point of ψ_0 if, and only if, $m_0(u)$ is a nontrivial critical point for $\mathcal{J}_0$. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_0^+} \psi_0(u) = \inf_{u\in\mathcal{N}_0} \mathcal{J}_0(u).$$

Remark 5.1. *As in Section 3, we have the following equalities*

$$c_0 := \inf_{u\in\mathcal{N}_0} \mathcal{J}_0(u) = \inf_{u\in\mathcal{H}_0^+} \max_{t>0} \mathcal{J}_0(tu) = \inf_{u\in\mathbb{S}_0^+} \max_{t>0} \mathcal{J}_0(tu).$$

We recall the following lemma whose proof can be found in [27] (see Lemma 3.3 there):

Lemma 5.2. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_0$ be a $(PS)_d$ sequence for $\mathcal{J}_0$ with $d < \frac{s}{3} S_*^{\frac{3}{2s}}$ and $u_n \rightharpoonup 0$. Then, only one of the alternative below holds:*

(a) *$u_n \to 0$ in $\mathcal{H}_0$;*

(b) *there exist a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx \geq \beta > 0.$$

Remark 5.2. *Let us observe that, if $\{u_n\}_{n\in\mathbb{N}}$ is a (PS) sequence at the level c_0 for the functional $\mathcal{J}_0$ such that $u_n \rightharpoonup u$, then we may assume $u \neq 0$. Otherwise, if $u_n \rightharpoonup 0$ and, once it does not occur $u_n \to 0$ in $\mathcal{H}_0$, in view of Lemma 5.2 we can find $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx \geq \beta > 0.$$

Set $v_n(x) = u_n(x + y_n)$, and making a change of variable, we can see that $\{v_n\}_{n\in\mathbb{N}}$ is a $(PS)_{c_0}$ sequence for $\mathcal{J}_0$, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_0$ and there exists $v \in \mathcal{H}_0$ such that $v_n \rightharpoonup v$ with $v \neq 0$.

Moreover, arguing as in the proof of Proposition 3.4 in [27], we have the following existence result:

Theorem 5.1. *Problem* (5.1) *admits a positive ground state solution.*

Now we give a compactness result for the autonomous problem which we will use later.

Lemma 5.3. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ *be a sequence such that* $\mathcal{J}_0(u_n) \to c_0$. *Then* $\{u_n\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* $H^s(\mathbb{R}^3)$.

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ and $\mathcal{J}_0(u_n) \to c_0$, we can apply Lemma 5.1-(iii), Proposition 5.1-(d) and the definition of c_0 to infer that

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in \mathbb{S}_0^+$$

and

$$\psi_0(v_n) = \mathcal{J}_0(u_n) \to c_0 = \inf_{v\in\mathbb{S}_0^+} \psi_0(v).$$

Let us introduce the following map $\mathcal{F} : \overline{\mathbb{S}}_0^+ \to \mathbb{R}\cup\{\infty\}$ defined by setting

$$\mathcal{F}(u) := \begin{cases} \psi_0(u) & \text{if } u \in \mathbb{S}_0^+ \\ \infty & \text{if } u \in \partial\mathbb{S}_0^+. \end{cases}$$

We note that

- $(\overline{\mathbb{S}}_0^+, d_0)$, where $d(u,v) = \|u-v\|_0$, is a complete metric space;
- $\mathcal{F} \in C(\overline{\mathbb{S}}_0^+, \mathbb{R}\cup\{\infty\})$, by Lemma 5.1-$(iv)$;
- $\mathcal{F}$ is bounded below, by Proposition 5.1-(d).

Hence, applying the Ekeland's variational principle [18] to $\mathcal{F}$, we can find $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_0^+$ such that $\{\hat{v}_n\}_{n\in\mathbb{N}}$ is a $(PS)_{c_0}$ sequence for ψ_0 on $\mathbb{S}_0^+$ and $\|\hat{v}_n - v_n\|_0 = o_n(1)$. Then, using Proposition 5.1, Theorem 5.1 and arguing as in the proof of Corollary 4.1 we obtain the thesis. □

Finally, we prove the following useful relation between c_ε and c_0:

Lemma 5.4. *It holds* $\lim_{\varepsilon\to 0} c_\varepsilon = c_0$.

Proof. For any $R > 0$ we set $u_R(x) = \psi_R(x)u_0(x)$, where u_0 is positive ground state of (5.1) which is given by Theorem 5.1, and $\psi_R(x) = \psi(x/R)$ with $\psi \in C_c^\infty(\mathbb{R}^3)$, $\psi \in [0,1]$, $\psi = 1$ if $|x| \leq \frac{1}{2}$ and $\psi = 0$ if $|x| \geq 1$. For simplicity, we assume that $supp(\psi) \subset B_1 \subset \Lambda$. By Lemma 2.2 and the Dominated Convergence Theorem we can see that

$$u_R \to u_0 \text{ in } H^s(\mathbb{R}^3) \quad \text{ as } R \to \infty. \tag{5.2}$$

For each $\varepsilon, R > 0$ there exists $t_{\varepsilon,R} > 0$ such that

$$\mathcal{J}_\varepsilon(t_{\varepsilon,R}u_R) = \max_{t\geq 0} \mathcal{J}_\varepsilon(tu_R).$$

Then $\mathcal{J}_\varepsilon'(t_{\varepsilon,R}u_R) = 0$ and this implies that

$$\frac{1}{t_{\varepsilon,R}^2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_R|^2 + V(\varepsilon x)u_R^2 dx + \int_{B_R} \phi_{u_R}^t u_R^2 dx = \int_{B_R} \frac{f(t_{\varepsilon,R}u_R)}{(t_{\varepsilon,R}u_R)^3}u_R^4 dx + t_{\varepsilon,R}^{2_s^*-4}\int_{B_R} |u_R|^{2_s^*}dx. \tag{5.3}$$

From the last equality, we can deduce that for any $R > 0$ we have

$$0 < \lim_{\varepsilon\to 0} t_{\varepsilon,R} = t_R < \infty.$$

Taking the limit as $\varepsilon \to 0$ in (5.3) we get

$$\frac{1}{t_R^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_R|^2+V_0u_R^2dx+\int_{B_R}\phi_{u_R}^t u_R^2dx=\int_{B_R}\frac{f(t_Ru_R)}{(t_Ru_R)^3}u_R^4dx+t_R^{2_s^*-4}\int_{B_R}|u_R|^{2_s^*}dx. \quad (5.4)$$

Putting together (5.2) and (5.4) we deduce that $t_R=1$ and $\mathcal{J}_0(t_Ru_R)=\max_{t\geq 0}\mathcal{J}_0(tu_R)$.

Accordingly, by (5.4), we have

$$c_\varepsilon \leq \max_{t\geq 0}\mathcal{J}_\varepsilon(tu_R)=\mathcal{J}_\varepsilon(t_{\varepsilon,R}u_R)$$

which implies that

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq \mathcal{J}_0(t_Ru_R).$$

Taking the limit as $R\to\infty$ and using (5.2) we get

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq c_0.$$

On the other hand, in view of (V_1), we know that $c_\varepsilon \geq c_0$ for all $\varepsilon>0$. Then we can conclude that $c_\varepsilon \to c_0$ as $\varepsilon \to 0$. □

6. Barycenter map and multiplicity of solutions to (1.1)

In this section, our main purpose is to apply the Ljusternik-Schnirelmann category theory to prove a multiplicity result for problem (3.2). We begin proving the following technical result.

Lemma 6.1. *Let $\varepsilon_n \to 0^+$ and $\{u_n\}_{n\in\mathbb{N}}\subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n)\to c_0$. Then there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ such that the translated sequence*

$$\tilde{u}_n(x):=u_n(x+\tilde{y}_n)$$

has a subsequence which converges in $H^s(\mathbb{R}^3)$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}}:=\{\varepsilon_n\tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n\to y_0\in M$.

Proof. Since $\langle \mathcal{J}'_{\varepsilon_n}(u_n),u_n\rangle=0$ and $\mathcal{J}_{\varepsilon_n}(u_n)\to c_0$, it is easy to see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\varepsilon_n}$. Let us observe that $\|u_n\|_{\varepsilon_n}\not\to 0$ since $c_0>0$. Therefore, arguing as in Lemma 5 in [1], we can find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ and constants $R,\alpha>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(\tilde{y}_n)}|u_n|^2dx\geq\alpha.$$

Set $\tilde{u}_n(x):=u_n(x+\tilde{y}_n)$. Then $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^3)$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } H^s(\mathbb{R}^3),$$

for some $\tilde{u}\neq 0$. Let $\{t_n\}_{n\in\mathbb{N}}\subset(0,+\infty)$ be such that $\tilde{v}_n:=t_n\tilde{u}_n\in\mathcal{N}_0$ (see Lemma 5.1-(i)), and set $y_n:=\varepsilon_n\tilde{y}_n$. Then, using (g_2) and Lemma 2.3-(4), we can see that

$$\begin{aligned}
c_0\leq \mathcal{J}_0(\tilde{v}_n)&\leq\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2+V(\varepsilon_n x+y_n)\tilde{v}_n^2dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{\tilde{v}_n}^t\tilde{v}_n^2dx-\int_{\mathbb{R}^3}\left(F(\tilde{v}_n)+\frac{1}{2_s^*}|\tilde{v}_n|^{2_s^*}\right)dx\\
&\leq\frac{t_n^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2+V(\varepsilon_n z)u_n^2dx+\frac{t_n^4}{4}\int_{\mathbb{R}^3}\phi_{u_n}^tu_n^2dx-\int_{\mathbb{R}^3}G(\varepsilon_n z,t_nu_n)dx\\
&=\mathcal{J}_{\varepsilon_n}(t_nu_n)\leq\mathcal{J}_{\varepsilon_n}(u_n)=c_0+o_n(1),
\end{aligned}$$

which gives

$$\mathcal{J}_0(\tilde{v}_n)\to c_0 \text{ and } \{\tilde{v}_n\}_{n\in\mathbb{N}}\subset\mathcal{N}_0. \quad (6.1)$$

In particular, (6.1) yields that $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^3)$, so we may assume that $\tilde{v}_n\rightharpoonup\tilde{v}$. Obviously, $\{t_n\}_{n\in\mathbb{N}}$ is bounded and we may assume that $t_n\to t_0\geq 0$. If $t_0=0$, from the boundedness of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, we get $\|\tilde{v}_n\|_0 = t_n\|\tilde{u}_n\|_0 \to 0$, that is $\mathcal{J}_0(\tilde{v}_n) \to 0$ in contrast with the fact $c_0 > 0$. Then, $t_0 > 0$. From the uniqueness of the weak limit we have $\tilde{v} = t_0\tilde{u}$ and $\tilde{v} \neq 0$. Using Lemma 5.3 we deduce that

$$\tilde{v}_n \to \tilde{v} \text{ in } H^s(\mathbb{R}^3), \tag{6.2}$$

which implies that $\tilde{u}_n \to \tilde{u}$ in $H^s(\mathbb{R}^3)$ and

$$\mathcal{J}_0(\tilde{v}) = c_0 \text{ and } \langle \mathcal{J}_0'(\tilde{v}), \tilde{v}\rangle = 0.$$

Now, we show that $\{y_n\}_{n\in\mathbb{N}}$ admits a subsequence, still denoted by $\{y_n\}_{n\in\mathbb{N}}$, such that $y_n \to y_0 \in M$. Assume by contradiction that $\{y_n\}_{n\in\mathbb{N}}$ is not bounded. Then there exists a subsequence, still denoted by $\{y_n\}_{n\in\mathbb{N}}$, verifying $|y_n| \to +\infty$. Since $u_n \in \mathcal{N}_{\varepsilon_n}$, we can see that

$$\|\tilde{u}_n\|_0^2 \leq [\tilde{u}_n]^2 + \int_{\mathbb{R}^3} V(\varepsilon_n\, x + y_n)\tilde{u}_n^2 dx + \int_{\mathbb{R}^3} \phi^t_{\tilde{u}_n}\tilde{u}_n^2 dx = \int_{\mathbb{R}^3} g(\varepsilon_n\, x + y_n, \tilde{u}_n)\tilde{u}_n\, dx.$$

Take $R > 0$ such that $\Lambda \subset B_R(0)$, and assume that $|y_n| > 2R$ for n large. Thus, for any $x \in B_{R/\varepsilon_n}(0)$ we get $|\,\varepsilon_n\, x + y_n| \geq |y_n| - |\,\varepsilon_n\, x| > R$ for all n large enough.
Hence, from the definition of g, we deduce that

$$\|v_n\|_0^2 \leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(\tilde{u}_n)\tilde{u}_n\, dx + \int_{\mathbb{R}^3\setminus B_{R/\varepsilon_n}(0)} f(\tilde{u}_n)\tilde{u}_n + \tilde{u}_n^{2^*_s}\, dx.$$

Since $\tilde{u}_n \to \tilde{u}$ in $H^s(\mathbb{R}^3)$, we can apply the Dominated Convergence Theorem to see that

$$\int_{\mathbb{R}^3\setminus B_{R/\varepsilon_n}(0)} f(\tilde{u}_n)\tilde{u}_n\, dx = o_n(1).$$

Therefore

$$\|\tilde{u}_n\|_0^2 \leq \frac{1}{K}\int_{B_{R/\varepsilon_n}(0)} V_0\tilde{u}_n^2\, dx + o_n(1),$$

which yields

$$\left(1 - \frac{1}{K}\right)\|\tilde{u}_n\|_0^2 \leq o_n(1).$$

But $\tilde{u}_n \to \tilde{u} \neq 0$ and $K > 2$, so we get a contradiction. Thus $\{y_n\}_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, we may assume that $y_n \to y_0$. If $y_0 \notin \overline{\Lambda}$, then there exists $r > 0$ such that $y_n \in B_{r/2}(y_0) \subset \mathbb{R}^3 \setminus \overline{\Lambda}$ for any n large enough. Reasoning as before, we get a contradiction. Hence $y_0 \in \overline{\Lambda}$. Now, we show that $V(y_0) = V_0$. Assume by contradiction that $V(y_0) > V_0$. Taking into account (6.2), Fatou's Lemma and the invariance of $\mathbb{R}^3$ by translations, we have

$$\begin{aligned} c_0 &< \liminf_{n\to\infty}\Big[\frac{1}{2}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2 + V(\varepsilon_n\, z + y_n)\tilde{v}_n^2\right) + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{\tilde{v}_n}\tilde{v}_n^2 dx - \int_{\mathbb{R}^3}\left(F(\tilde{v}_n) + \frac{1}{2^*_s}|\tilde{v}_n|^2\right) dx\Big] \\ &\leq \liminf_{n\to\infty}\mathcal{J}_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty}\mathcal{J}_{\varepsilon_n}(u_n) = c_0 \end{aligned}$$

which is impossible. Therefore, in view of (V_2), we can conclude that $y_0 \in M$. □

Now, we aim to relate the number of positive solutions of (3.2) to the topology of the set Λ. For this reason, we take $\delta > 0$ such that

$$M_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \leq \delta\} \subset \Lambda,$$

and we consider $\eta \in C_0^\infty(\mathbb{R}_+, [0,1])$ be such that $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \geq \delta$.
For any $y \in \Lambda$, we define

$$\Psi_{\varepsilon,y}(x) = \eta(|\,\varepsilon\, x - y|)w\left(\frac{\varepsilon\, x - y}{\varepsilon}\right)$$

where $w \in H^s(\mathbb{R}^3)$ is a positive ground state solution to problem (5.1) (see Theorem 5.1).

Let $t_\varepsilon > 0$ be the unique number such that

$$\max_{t\geq 0} \mathcal{J}_\varepsilon(t\Psi_{\varepsilon,y}) = \mathcal{J}_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

Finally, we introduce $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}.$$

Lemma 6.2. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = c_0 \textit{ uniformly in } y \in M.$$

Proof. Assume by contradiction that there exist $\delta_0 > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \delta_0. \tag{6.3}$$

Let us observe that using the change of variable $z = \dfrac{\varepsilon_n\, x - y_n}{\varepsilon_n}$, if $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$, it follows that $\varepsilon_n\, z \in B_\delta(0)$ and $\varepsilon_n\, x + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Then, recalling that $G(x,t) = F(t) + \frac{1}{2^*_s} t^{2^*_s}$ for $(x,t) \in \Lambda \times \mathbb{R}_+$ we have

$$\begin{aligned}
\mathcal{J}_\varepsilon(\Phi_{\varepsilon_n}(y_n)) = \frac{t^2_{\varepsilon_n}}{2} &\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(\eta(|\,\varepsilon_n\, z|)w(z))|^2\, dz + \int_{\mathbb{R}^3} V(\varepsilon_n\, z + y_n)(\eta(|\,\varepsilon_n\, z|)w(z))^2\, dz \right) \\
&+ \frac{t^4_{\varepsilon_n}}{4} \int_{\mathbb{R}^3} \phi^t_{\eta(|\,\varepsilon_n\, z|)}(\eta(|\,\varepsilon_n\, z|)w(z))^2 dz - \int_{\mathbb{R}^3} F(t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z))\, dz \\
&- \frac{t^{2^*_s}_{\varepsilon_n}}{2^*_s} \int_{\mathbb{R}^3} (\eta(|\,\varepsilon_n\, z|)w(z))^{2^*_s} dz.
\end{aligned} \tag{6.4}$$

Now, we aim to show that the sequence $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ verifies $t_{\varepsilon_n} \to 1$ as $\varepsilon_n \to 0$. From the definition of t_{ε_n}, it follows that $\langle \mathcal{J}'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$ which gives

$$\begin{aligned}
&t^2_{\varepsilon_n} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(\eta(|\,\varepsilon_n\, z|)w(z))|^2 + V(\varepsilon_n\, z + y_n)(\eta(|\,\varepsilon_n\, z|)w(z))^2\, dz \right) + t^4_{\varepsilon_n} \int_{\mathbb{R}^3} \phi^t_{\eta(|\,\varepsilon_n\, z|)}(\eta(|\,\varepsilon_n\, z|)w(z))^2 dz \\
&= \int_{\mathbb{R}^3} g(\varepsilon_n\, z + y_n, t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)) t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z) dz.
\end{aligned} \tag{6.5}$$

Since $\eta = 1$ in $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{\varepsilon_n}}(0)$ for all n sufficiently large, (6.5) yields

$$\begin{aligned}
&\frac{1}{t^2_{\varepsilon_n}} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,y_n}|^2 + V(\varepsilon_n\, x)\Psi^2_{\varepsilon_n,y_n} dx + \int_{\mathbb{R}^3} \phi^t_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} dx \\
&= \int_{\mathbb{R}^3} \frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) + (t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^{2^*_s-1}}{(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^3} \Psi^4_{\varepsilon_n,y_n} dx \\
&\geq t^{2^*_s-4}_{\varepsilon_n} \int_{B_{\frac{\delta}{2}}(0)} |w(z)|^{2^*_s}\, dz.
\end{aligned}$$

From the continuity of w we can find a vector $\hat{z} \in \mathbb{R}^3$ such that

$$w(\hat{z}) = \min_{z\in B_{\frac{\delta}{2}}} w(z) > 0,$$

which implies that

$$\begin{aligned}&\frac{1}{t_{\varepsilon_n}^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n\,x)\Psi_{\varepsilon_n,y_n}^2dx+\int_{\mathbb{R}^3}\phi^t_{\Psi_{\varepsilon_n,y_n}}\Psi_{\varepsilon_n,y_n}^2dx\\&\geq t_{\varepsilon_n}^{2^*_s-4}w^{2^*_s}(\hat{z})|B_{\frac{\delta}{2}}(0)|.\end{aligned}\tag{6.6}$$

Now, assume by contradiction that $t_{\varepsilon_n}\to\infty$. Let us observe that Lemma 2.2, Lemma 2.3-(7) and the Dominated Convergence Theorem yield

$$\begin{aligned}&\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n}\to\|w\|^2_0\in(0,\infty)\ \text{ and }\int_{\mathbb{R}^3}\phi^t_{\Psi_{\varepsilon_n,y_n}}\Psi^2_{\varepsilon_n,y_n}dx\to\int_{\mathbb{R}^3}\phi^t_w w^2dx\\&|\Psi_{\varepsilon_n,y_n}|_{2^*_s}\to|w|_{2^*_s}\ \text{ and }\int_{\mathbb{R}^3}\frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^3}\Psi^4_{\varepsilon_n,y_n}dx\to\int_{\mathbb{R}^3}\frac{f(t_0w)}{(t_0w)^3}w^4dx.\end{aligned}\tag{6.7}$$

Hence, using $t_{\varepsilon_n}\to\infty$, (6.6) and (6.7) we obtain

$$\int_{\mathbb{R}^3}\phi^t_w w^2dx=\infty,$$

that is a contradiction. Therefore $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and, up to subsequence, we may assume that $t_{\varepsilon_n}\to t_0$ for some $t_0\geq 0$. Let us prove that $t_0>0$. Suppose by contradiction that $t_0=0$. Then, taking into account (6.7) and the growth assumptions on g, we can see that (6.5) gives

$$\|t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n}\to 0$$

which is impossible in view of $t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\in\mathcal{N}_{\varepsilon_n}$ and Remark 3.2. Hence $t_0>0$. Thus, taking the limit as $n\to\infty$ in (6.5), we deduce from (6.7) and the Dominated Convergence Theorem that

$$\frac{1}{t_0^2}\|w\|^2_0+\int_{\mathbb{R}^3}\phi^t_w w^2dx=\int_{\mathbb{R}^3}\frac{f(t_0w)+(t_0w)^{2^*_s-1}}{(t_0w)^3}\,w^4\,dx.$$

In the light of $w\in\mathcal{N}_0$ and (f_5) we can infer that $t_0=1$. Then, passing to the limit as $n\to\infty$ in (6.4), by $t_{\varepsilon_n}\to 1$ and (6.7) we obtain

$$\lim_{n\to\infty}\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))=\mathcal{J}_0(w)=c_0,$$

which contradicts (6.3). □

At this point, we are in the position to define the barycenter map. For any $\delta>0$, we take $\rho=\rho(\delta)>0$ such that $M_\delta\subset B_\rho$, and we consider $\Upsilon:\mathbb{R}^3\to\mathbb{R}^3$ given by

$$\Upsilon(x)=\begin{cases}x & \text{if } |x|<\rho\\ \frac{\rho x}{|x|} & \text{if } |x|\geq\rho.\end{cases}$$

We define the barycenter map $\beta_\varepsilon:\mathcal{N}_\varepsilon\to\mathbb{R}^3$ as follows

$$\beta_\varepsilon(u)=\frac{\displaystyle\int_{\mathbb{R}^3}\Upsilon(\varepsilon\,x)u^2(x)\,dx}{\displaystyle\int_{\mathbb{R}^3}u^2(x)\,dx}.$$

Arguing as Lemma 5.4 in [27], we can see that the function β_ε verifies the following limit:

Lemma 6.3.

$$\lim_{\varepsilon\to0}\beta_\varepsilon(\Phi_\varepsilon(y))=y\ \textit{uniformly in}\ y\in M.$$

Next, we introduce a subset $\widetilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ taking a function $h_1:\mathbb{R}^+\to\mathbb{R}^+$ such that $h_1(\varepsilon)\to 0$ as $\varepsilon\to 0$, and setting

$$\widetilde{\mathcal{N}}_\varepsilon=\{u\in\mathcal{N}_\varepsilon:\mathcal{J}_\varepsilon(u)\leq c_0+h_1(\varepsilon)\}.$$

Fixed $y\in M$, from Lemma 6.2 it follows that $h_1(\varepsilon)=|\mathcal{J}_\varepsilon(\Phi_\varepsilon(y))-c_0|\to 0$ as $\varepsilon\to 0$. Therefore $\Phi_\varepsilon(y)\in\widetilde{\mathcal{N}}_\varepsilon$, and $\widetilde{\mathcal{N}}_\varepsilon\neq\emptyset$ for any $\varepsilon>0$. Moreover, proceeding as in Lemma 5.5 in [27], we have:

Lemma 6.4. *For any $\delta>0$, there holds that*

$$\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}}_\varepsilon}\operatorname{dist}(\beta_\varepsilon(u),M_\delta)=0.$$

In order to prove that (3.2) admits at least $cat_{M_\delta}(M)$ positive solutions, we recall the following useful abstract result whose proof can be found in [13].

Lemma 6.5. *Let I, I_1 and I_2 be closed sets with $I_1\subset I_2$, and let $\pi:I\to I_2$ and $\psi:I_1\to I$ be two continuous maps such that $\pi\circ\psi$ is homotopically equivalent to the embedding $j:I_1\to I_2$. Then $cat_I(I)\geq cat_{I_2}(I_1)$.*

Since $\mathbb{S}^+_\varepsilon$ is a not complete metric space, we cannot apply directly standard Ljusternik-Schnirelmann theory. Anyway, we will make use of some abstract category results contained in [38].

Theorem 6.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold true. Then, given $\delta>0$ there exists $\bar{\varepsilon}_\delta>0$ such that, for any $\varepsilon\in(0,\bar{\varepsilon}_\delta)$, problem (3.2) has at least $cat_{M_\delta}(M)$ positive solutions.*

Proof. For any $\varepsilon>0$, we consider the map $\alpha_\varepsilon:M\to\mathbb{S}^+_\varepsilon$ defined by $\alpha_\varepsilon(y)=m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. Using Lemma 6.2, we can see that

$$\lim_{\varepsilon\to 0}\psi_\varepsilon(\alpha_\varepsilon(y))=\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(\Phi_\varepsilon(y))=c_0 \text{ uniformly in } y\in M. \tag{6.8}$$

Set

$$\widetilde{\mathcal{S}}^+_\varepsilon=\{w\in\mathbb{S}^+_\varepsilon:\psi_\varepsilon(w)\leq c_0+h_1(\varepsilon)\},$$

where $h_1(\varepsilon)\to 0$ as $\varepsilon\to 0^+$. It follows from (6.8) that $h_1(\varepsilon)=|\psi_\varepsilon(\alpha_\varepsilon(y))-c_0|\to 0$ as $\varepsilon\to 0^+$ uniformly in $y\in M$, so there exists $\bar{\varepsilon}>0$ such that $\psi_\varepsilon(\alpha_\varepsilon(y))\in\widetilde{\mathcal{S}}^+_\varepsilon$ and $\widetilde{\mathcal{S}}^+_\varepsilon\neq\emptyset$ for all $\varepsilon\in(0,\bar{\varepsilon})$. In the light of Lemma 3.2-(ii), Lemma 6.2, Lemma 6.3 and Lemma 6.4, we can find $\bar{\varepsilon}=\bar{\varepsilon}_\delta>0$ such that the following diagram

$$M\xrightarrow{\Phi_\varepsilon}\Phi_\varepsilon(M)\xrightarrow{m_\varepsilon^{-1}}\alpha_\varepsilon(M)\xrightarrow{m_\varepsilon}\Phi_\varepsilon(M)\xrightarrow{\beta_\varepsilon}M_\delta$$

is well defined for any $\varepsilon\in(0,\bar{\varepsilon})$.

Thanks to Lemma 6.3, and decreasing $\bar{\varepsilon}$ if necessary, we can see that $\beta_\varepsilon(\Phi_\varepsilon(y))=y+\theta(\varepsilon,y)$ for all $y\in M$, for some function $\theta(\varepsilon,y)$ verifying $|\theta(\varepsilon,y)|<\frac{\delta}{2}$ uniformly in $y\in M$ and for all $\varepsilon\in(0,\bar{\varepsilon})$. Then, it is easy to check that $H(t,y)=y+(1-t)\theta(\varepsilon,y)$ with $(t,y)\in[0,1]\times M$ is a homotopy between $\beta_\varepsilon\circ\Phi_\varepsilon=(\beta_\varepsilon\circ m_\varepsilon)\circ(m_\varepsilon^{-1}\circ\Phi_\varepsilon)$ and the inclusion map $id:M\to M_\delta$. This fact together with Lemma 6.5 implies that

$$cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)\geq cat_{M_\delta}(M). \tag{6.9}$$

Applying Corollary 4.1, Lemma 5.4 and Corollary 28 in [38] with $c=c_\varepsilon\leq c_0+h_1(\varepsilon)=d$ and $K=\alpha_\varepsilon(M)$, we can deduce that ψ_ε has at least $cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}^+_\varepsilon$. Taking into account Proposition 3.1-(d) and (6.9), we can infer that (3.2) has at least $cat_{M_\delta}(M)$ solutions. □

7. PROOF OF THEOREM 1.1

This last section is devoted to the proof of Theorem 1.1 in which we prove that the solutions obtained in Section 6 are indeed solutions of the original problem (1.1).

Firstly, we use a Moser iteration argument [29] to prove the following useful L^∞-estimate for the solutions of the modified problem (3.2).

Lemma 7.1. *Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution to (3.2). Then, up to a subsequence, $v_n := u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N)$, and there exists $C > 0$ such that*

$$|v_n|_\infty \leq C \text{ for all } n \in \mathbb{N}.$$

Proof. For any $L > 0$ and $\beta > 1$, let us define the function

$$\gamma(v_n) = \gamma_{L,\beta}(v_n) = v_n v_{L,n}^{2(\beta-1)} \in \mathcal{H}_\varepsilon$$

where $v_{L,n} = \min\{v_n, L\}$. Since γ is an increasing function, we have

$$(a-b)(\gamma(a) - \gamma(b)) \geq 0 \quad \text{ for any } a, b \in \mathbb{R}.$$

Let us consider

$$\mathcal{E}(t) = \frac{|t|^2}{2} \quad \text{ and } \quad \Gamma(t) = \int_0^t (\gamma'(\tau))^{\frac{1}{2}} d\tau.$$

Then, applying Jensen inequality, we get for all $a, b \in \mathbb{R}$ such that $a > b$,

$$\begin{aligned}
\mathcal{E}'(a-b)(\gamma(a)-\gamma(b)) = (a-b)(\gamma(a)-\gamma(b)) &= (a-b)\int_b^a \gamma'(t)dt \\
&= (a-b)\int_b^a (\Gamma'(t))^2 dt \geq \left(\int_b^a (\Gamma'(t))dt\right)^2.
\end{aligned}$$

The same argument works when $a \leq b$. Therefore

$$\mathcal{E}'(a-b)(\gamma(a)-\gamma(b)) \geq |\Gamma(a) - \Gamma(b)|^2 \text{ for any } a, b \in \mathbb{R}. \tag{7.1}$$

By (7.1), we can see that

$$|\Gamma(v_n)(x) - \Gamma(v_n)(y)|^2 \leq (v_n(x) - v_n(y))((v_n v_{L,n}^{2(\beta-1)})(x) - (v_n v_{L,n}^{2(\beta-1)})(y)). \tag{7.2}$$

Choosing $\gamma(v_n) = v_n v_{L,n}^{2(\beta-1)}$ as test function in (3.2), and using (7.2) and $\phi_{u_n}^t \geq 0$, we obtain

$$\begin{aligned}
&[\Gamma(v_n)]^2 + \int_{\mathbb{R}^3} V_n(x)|v_n|^2 v_{L,n}^{2(\beta-1)} dx \\
&\leq \iint_{\mathbb{R}^6} \frac{(v_n(x) - v_n(y))}{|x-y|^{N+2s}}((v_n v_{L,n}^{2(\beta-1)})(x) - (v_n v_{L,n}^{2(\beta-1)})(y))\, dxdy + \int_{\mathbb{R}^3} V_n(x)|v_n|^2 v_{L,n}^{2(\beta-1)} dx \\
&\leq \int_{\mathbb{R}^3} g_n(v_n) v_n v_{L,n}^{2(\beta-1)} dx,
\end{aligned} \tag{7.3}$$

where we used the notations $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $g_n(v_n) = g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n)$.
Since

$$\Gamma(v_n) \geq \frac{1}{\beta} v_n v_{L,n}^{\beta-1},$$

and applying Theorem 2.1, we have

$$[\Gamma(v_n)]^2 \geq S_* |\Gamma(v_n)|_{2_s^*}^2 \geq \left(\frac{1}{\beta}\right)^2 S_* |v_n v_{L,n}^{\beta-1}|_{2_s^*}^2. \tag{7.4}$$

From assumptions (g_1) and (g_2), for any $\xi>0$ there exists $C_\xi>0$ such that

$$|g_n(v_n)|\leq \xi|v_n|+C_\xi|v_n|^{2^*_s-1}. \tag{7.5}$$

Taking $\xi\in(0,V_0)$, and using (7.4) and (7.5), we can see that (7.3) yields

$$|w_{L,n}|^2_{L^{2^*_s}(\mathbb{R}^3)}\leq C\beta^2\int_{\mathbb{R}^3}|v_n|^{2^*_s}v_{L,n}^{2(\beta-1)}dx. \tag{7.6}$$

where $w_{L,n}:=v_nv_{L,n}^{\beta-1}$. Now, we take $\beta=\frac{2^*_s}{2}$ and fix $R>0$. Recalling that $0\leq v_{L,n}\leq v_n$, we have

$$\begin{aligned}\int_{\mathbb{R}^3}v_n^{2^*_s}v_{L,n}^{2(\beta-1)}dx&=\int_{\mathbb{R}^3}v_n^{2^*_s-2}v_n^2v_{L,n}^{2^*_s-2}dx\\&=\int_{\mathbb{R}^3}v_n^{2^*_s-2}(v_nv_{L,n}^{\frac{2^*_s-2}{2}})^2dx\\&\leq\int_{\{v_n<R\}}R^{2^*_s-2}v_n^{2^*_s}dx+\int_{\{v_n>R\}}v_n^{2^*_s-2}(v_nv_{L,n}^{\frac{2^*_s-2}{2}})^2dx\\&\leq\int_{\{v_n<R\}}R^{2^*_s-2}v_n^{2^*_s}dx+\left(\int_{\{v_n>R\}}v_n^{2^*_s}dx\right)^{\frac{2^*_s-2}{2^*_s}}\left(\int_{\mathbb{R}^3}(v_nv_{L,n}^{\frac{2^*_s-2}{2}})^{2^*_s}dx\right)^{\frac{p}{2^*_s}}.\end{aligned} \tag{7.7}$$

Since v_n is bounded in $\in L^{2^*_s}(\mathbb{R}^3)$, we can see that for any R sufficiently large

$$\left(\int_{\{v_n>R\}}v_n^{2^*_s}dx\right)^{\frac{2^*_s-2}{2^*_s}}\leq\varepsilon\,\beta^{-2}. \tag{7.8}$$

Putting together (7.6), (7.7) and (7.8) we get

$$\left(\int_{\mathbb{R}^3}(v_nv_{L,n}^{\frac{2^*_s-2}{2}})^{2^*_s}\right)^{\frac{2}{2^*_s}}\leq C\beta^2\int_{\mathbb{R}^3}R^{2^*_s-2}v_n^{2^*_s}dx<\infty$$

and taking the limit as $L\to\infty$, we obtain $v_n\in L^{\frac{(2^*_s)^2}{2}}(\mathbb{R}^3)$.

Now, using $0\leq v_{L,n}\leq v_n$ and passing to the limit as $L\to\infty$ in (7.6), we have

$$|v_n|^{\beta 2}_{L^{\beta 2^*_s}(\mathbb{R}^3)}\leq C\beta^2\int_{\mathbb{R}^3}v_n^{2^*_s+2(\beta-1)},$$

from which we deduce that

$$\left(\int_{\mathbb{R}^3}v_n^{\beta 2^*_s}dx\right)^{\frac{1}{(\beta-1)2^*_s}}\leq(C\beta)^{\frac{1}{\beta-1}}\left(\int_{\mathbb{R}^3}v_n^{2^*_s+2(\beta-1)}\right)^{\frac{1}{2(\beta-1)}}.$$

For $m\geq1$ we define β_{m+1} inductively so that $2^*_s+2(\beta_{m+1}-1)=2^*_s\beta_m$ and $\beta_1=\frac{2^*_s}{2}$. Then we have

$$\left(\int_{\mathbb{R}^3}v_n^{\beta_{m+1}2^*_s}dx\right)^{\frac{1}{(\beta_{m+1}-1)2^*_s}}\leq(C\beta_{m+1})^{\frac{1}{\beta_{m+1}-1}}\left(\int_{\mathbb{R}^3}v_n^{2^*_s\beta_m}\right)^{\frac{1}{2^*_s(\beta_m-1)}}.$$

Let us define

$$D_m=\left(\int_{\mathbb{R}^3}v_n^{2^*_s\beta_m}\right)^{\frac{1}{2^*_s(\beta_m-1)}}.$$

Using an iteration argument, we can find $C_0 > 0$ independent of m such that

$$D_{m+1} \leq \prod_{k=1}^{m} (C\beta_{k+1})^{\frac{1}{\beta_{k+1}-1}} D_1 \leq C_0 D_1.$$

Taking the limit as $m \to \infty$ we get $|v_n|_\infty \leq K$ for all $n \in \mathbb{N}$. □

Now, we are ready to give the proof of our main result.

Proof of Theorem 1.1. Take $\delta > 0$ such that $M_\delta \subset \Lambda$. We begin proving that there exists $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (3.2), it results

$$|u_\varepsilon|_{L^\infty(\mathbb{R}^3 \setminus \Lambda_\varepsilon)} < a. \tag{7.9}$$

Assume by contradiction that for some subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, we can find $u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $\mathcal{J}'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_{\varepsilon_n}|_{L^\infty(\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n})} \geq a. \tag{7.10}$$

Since $\mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) \leq c_0 + h_1(\varepsilon_n)$ and $h_1(\varepsilon_n) \to 0$, we can argue as in the first part of the proof of Lemma 6.1, to deduce that $\mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) \to c_0$. In view of Lemma 6.1, we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $\tilde{u}_n = u_{\varepsilon_n}(\cdot + \tilde{y}_n) \to \tilde{u}$ in $H^s(\mathbb{R}^3)$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$. Now, if we choose $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$, we can see that $B_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Then, for any $y \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$ it holds

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \text{ for } n \text{ sufficiently large.}$$

Therefore

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \tag{7.11}$$

for any n big enough.
Now, we observe that $\tilde{u}_n$ is a solution to

$$(-\Delta)^s \tilde{u}_n + \tilde{u}_n = \xi_n \text{ in } \mathbb{R}^3,$$

where

$$\xi_n(x) := g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) - V_n(x)\tilde{u}_n + \tilde{u}_n - \phi^t_{\tilde{u}_n}\tilde{u}_n$$

and

$$V_n(x) := V(\varepsilon_n x + \varepsilon_n \tilde{y}_n).$$

Put $\xi(x) := f(\tilde{u}) - V_0\tilde{u} + \tilde{u} - \phi^t_{\tilde{u}}\tilde{u}$. Using Lemma 7.1, the interpolation in the L^p spaces, $\tilde{u}_n \to \tilde{u}$ in $H^s(\mathbb{R}^3)$, the growth assumptions on g, $\varepsilon_n \tilde{y}_n \to y_0 \in M$ and Lemma 2.3-(7) we can see that

$$\xi_n \to \xi \text{ in } L^p(\mathbb{R}^3) \quad \forall p \in [2, \infty),$$

so there exists $C > 0$ such that

$$|\xi_n|_\infty \leq C \text{ for any } n \in \mathbb{N}.$$

Consequently, $\tilde{u}_n(x) = (\mathcal{K} * \xi_n)(x) = \int_{\mathbb{R}^3} \mathcal{K}(x-z)\xi_n(z)\, dz$, where $\mathcal{K}$ is the Bessel kernel and satisfies the following properties [19]:

(i) $\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^3 \setminus \{0\}$,

(ii) there is $C > 0$ such that $\mathcal{K}(x) \leq \dfrac{C}{|x|^{3+2s}}$ for any $x \in \mathbb{R}^3 \setminus \{0\}$,

(iii) $\mathcal{K} \in L^r(\mathbb{R}^3)$ for any $r \in [1, \frac{3}{3-2s})$.

Then, arguing as in Lemma 2.6 in [3], we can see that

$$\tilde{u}_n(x) \to 0 \text{ as } |x| \to \infty \tag{7.12}$$

uniformly in $n \in \mathbb{N}$. Therefore there exists $R > 0$ such that

$$\tilde{u}_n(x) < a \text{ for } |x| \geq R, n \in \mathbb{N}.$$

Hence $u_{\varepsilon_n}(x) < a$ for any $x \in \mathbb{R}^3 \setminus B_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. This fact and (7.11) show that there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ and $r/\varepsilon_n > R$ we have

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^3 \setminus B_R(\tilde{y}_n),$$

which implies that $u_{\varepsilon_n}(x) < a$ for any $x \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n}$ and $n \geq \nu$. This is impossible in view of (7.10). Let $\bar{\varepsilon}_\delta > 0$ be given by Theorem 6.1, and we fix $\varepsilon \in (0, \varepsilon_\delta)$ where $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. In the light of Theorem 6.1, we know that problem (3.2) admits at least $cat_{M_\delta}(M)$ nontrivial solutions. Let us denote by u_ε one of these solutions. Since $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ satisfies (7.9), from the definition of g it follows that u_ε is a solution of (3.1). Then $\hat{u}(x) = u(x/\varepsilon)$ is a solution to (1.1), and we can conclude that (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions.

Finally, we study the behavior of the maximum points of solutions to problem (1.1). Take $\varepsilon_n \to 0$ and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\varepsilon_n}$ of solutions to (3.1) as above. Let us observe that (g_1) implies that we can find $\gamma > 0$ such that

$$g(\varepsilon\, x, t)t \leq \frac{V_0}{K} t^2 \text{ for any } x \in \mathbb{R}^3, t \leq \gamma. \tag{7.13}$$

Arguing as before, we can find $R > 0$ such that

$$|u_n|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{7.14}$$

Moreover, up to extract a subsequence, we may assume that

$$|u_n|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{7.15}$$

Indeed, if (7.15) does not hold, in view of (7.14) we can see that $|u_n|_{L^\infty(\mathbb{R}^3)} < \gamma$. Then, using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and (7.13) we can infer

$$\|u_n\|^2_{\varepsilon_n} \leq \|u_n\|^2_{\varepsilon_n} + \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} g(\varepsilon_n\, x, u_n) u_n\, dx \leq \frac{V_0}{K} \int_{\mathbb{R}^3} u_n^2\, dx$$

which yields $\|u_n\|_{\varepsilon_n} = 0$, and this is impossible. Hence (7.15) holds true. Taking into account (7.14) and (7.15) we can deduce that the maximum points $p_n \in \mathbb{R}^3$ of u_n belong to $B_R(\tilde{y}_n)$. Therefore $p_n = \tilde{y}_n + q_n$, for some $q_n \in B_R(0)$. Consequently, $\eta_{\varepsilon_n} = \varepsilon_n\, \tilde{y}_n + \varepsilon_n\, q_n$ is the maximum point of $\hat{u}_n(x) = u_n(x/\varepsilon_n)$. Since $|q_n| < R$ for any $n \in \mathbb{N}$ and $\varepsilon_n\, \tilde{y}_n \to y_0 \in M$ (in view of Lemma 6.1), from the continuity of V we can infer that

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0.$$

Let us conclude the proof of Theorem 1.1 by giving an estimate of the decay of solutions to (1.1). According to Lemma 4.3 in [19], we know that there exists a positive function w such that

$$0 < w(x) \leq \frac{C}{1 + |x|^{3+2s}}, \tag{7.16}$$

and

$$(-\Delta)^s w + \frac{V_0}{2} w \geq 0 \text{ in } \mathbb{R}^3 \setminus B_{R_1}, \tag{7.17}$$

for some suitable $R_1 > 0$. By (f_1), the definition of g and (7.12), we can find $R_2 > 0$ sufficiently large such that

$$(-\Delta)^s \tilde{u}_{\varepsilon_n} + \frac{V_0}{2}\tilde{u}_{\varepsilon_n} = g(\varepsilon_n x + \varepsilon_n \tilde{y}_{\varepsilon_n}, \tilde{u}_{\varepsilon_n}) - \left(V_n - \frac{V_0}{2}\right)\tilde{u}_{\varepsilon_n} - \phi^t_{\tilde{u}_{\varepsilon_n}}\tilde{u}_{\varepsilon_n}$$
$$\leq g(\varepsilon_n x + \varepsilon_n \tilde{y}_{\varepsilon_n}, \tilde{u}_{\varepsilon_n}) - \frac{V_0}{2}\tilde{u}_{\varepsilon_n} \leq 0 \text{ in } \mathbb{R}^3 \setminus B_{R_2}. \tag{7.18}$$

Choose $R_3 = \max\{R_1, R_2\}$, and we set

$$a = \inf_{B_{R_3}} w > 0 \text{ and } \tilde{w}_{\varepsilon_n} = (b+1)w - a\tilde{u}_{\varepsilon_n}, \tag{7.19}$$

where $b = \sup_{n\in\mathbb{N}} |\tilde{u}_{\varepsilon_n}|_\infty < \infty$. Now we prove that

$$\tilde{w}_{\varepsilon_n} \geq 0 \text{ in } \mathbb{R}^3. \tag{7.20}$$

We first note that (7.17), (7.18) and (7.19) yield

$$\tilde{w}_{\varepsilon_n} \geq ba + w - ba > 0 \text{ in } B_{R_3}, \tag{7.21}$$

$$(-\Delta)^s \tilde{w}_{\varepsilon_n} + \frac{V_0}{2}\tilde{w}_{\varepsilon_n} \geq 0 \text{ in } \mathbb{R}^3 \setminus B_{R_3}. \tag{7.22}$$

Now, we argue by contradiction and we assume that there exists a sequence $\{\bar{x}_{n,k}\} \subset \mathbb{R}^3$ such that

$$\inf_{x\in\mathbb{R}^3} \tilde{w}_{\varepsilon_n}(x) = \lim_{k\to\infty} \tilde{w}_{\varepsilon_n}(\bar{x}_{n,k}) < 0. \tag{7.23}$$

By (7.12), (7.16) and the definition of $\tilde{w}_{\varepsilon_n}$, it is clear that $|\tilde{w}_{\varepsilon_n}(x)| \to 0$ as $|x| \to \infty$, uniformly in $n \in \mathbb{N}$. Thus we can deduce that $\{\bar{x}_{n,k}\}$ is bounded, and, up to subsequence, we may assume that there exists $\bar{x}_n \in \mathbb{R}^3$ such that $\bar{x}_{n,k} \to \bar{x}_n$ as $k \to \infty$. Thus, from (7.23), we get

$$\inf_{x\in\mathbb{R}^3} \tilde{w}_{\varepsilon_n}(x) = \tilde{w}_{\varepsilon_n}(\bar{x}_n) < 0. \tag{7.24}$$

In the light of the minimality of $\bar{x}_n$ and the representation formula for the fractional Laplacian [16], we can see that

$$(-\Delta)^s \tilde{w}_{\varepsilon_n}(\bar{x}_n) = C(3,s)\int_{\mathbb{R}^3} \frac{2\tilde{w}_{\varepsilon_n}(\bar{x}_n) - \tilde{w}_{\varepsilon_n}(\bar{x}_n+\xi) - \tilde{w}_{\varepsilon_n}(\bar{x}_n-\xi)}{|\xi|^{3+2s}} d\xi \leq 0. \tag{7.25}$$

Taking into account (7.21) and (7.23) we can infer that $\bar{x}_n \in \mathbb{R}^3 \setminus B_{R_3}$. This together with (7.24) and (7.25) yields

$$(-\Delta)^s \tilde{w}_{\varepsilon_n}(\bar{x}_n) + \frac{V_0}{2}\tilde{w}_{\varepsilon_n}(\bar{x}_n) < 0,$$

which contradicts (7.22). Thus, (7.20) holds true and using (7.16) we get

$$\tilde{u}_{\varepsilon_n}(x) \leq \frac{\tilde{C}}{1+|x|^{3+2s}} \text{ for all } x \in \mathbb{R}^3, n \in \mathbb{N}, \tag{7.26}$$

for some $\tilde{C} > 0$. Since $\hat{u}_{\varepsilon_n}(x) = u_{\varepsilon_n}(\frac{x}{\varepsilon_n}) = \tilde{u}_{\varepsilon_n}(\frac{x}{\varepsilon_n} - \tilde{y}_{\varepsilon_n})$ and $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_{\varepsilon_n} + \varepsilon_n q_{\varepsilon_n}$, from (7.26) we obtain

$$\begin{aligned}
0 < \hat{u}_{\varepsilon_n}(x) = u_{\varepsilon_n}\left(\frac{x}{\varepsilon_n}\right) &= \tilde{u}_{\varepsilon_n}\left(\frac{x}{\varepsilon_n} - \tilde{y}_{\varepsilon_n}\right) \\
&\leq \frac{\tilde{C}}{1 + |\frac{x}{\varepsilon_n} - \tilde{y}_{\varepsilon_n}|^{3+2s}} \\
&= \frac{\tilde{C}\,\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \varepsilon_n \tilde{y}_{\varepsilon_n}|^{3+2s}} \\
&\leq \frac{\tilde{C}\,\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \eta_{\varepsilon_n}|^{3+2s}} \quad \forall x \in \mathbb{R}^3.
\end{aligned}$$

This ends the proof of Theorem 1.1. □

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