# Synchronization and Subsynchronization Problems for Switching Max-Plus Systems: Structural Solvability Conditions

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***Abstract*—Switching linear systems over the max-plus algebra can be used to model production plants where different choices in resource allocation are possible. In such a case, internal and external variables represent the time instants at which internal or external events occur. In particular, output variables represent the time instants at which lots of manufactured goods are released to the market. Here, we consider the problems of system synchronization and subsynchronization, which consist of forcing the output of a system to equal or anticipate the output of a given model. Their solution in the max-plus framework provides a viable strategy to control a given production plant in such a way as to comply with a desired production time schedule. Using structural methods and introducing novel structural notions, necessary and sufficient solvability conditions are given. Practical methods to construct solutions are illustrated and discussed.**

***Index Terms*—Manufacturing plant control, max-plus algebra, model matching, switching systems, systems' synchronization.**

## I. Introduction

Linear dynamical systems over the max-plus algebra $\mathbb{R}_{\max}$, or max-plus systems, were introduced in [1] to provide a convenient way of modeling discrete event systems which exhibit synchronization of operations without competition. Their class is the same as that of timed event graphs, i.e., of Petri nets whose places have only one upstream and only one downstream transition. A summary of basic results in the theory of max-plus systems can be found in [2] and [3]. Specific control problems and techniques were investigated in [4], [5], [6], and [7] and, in particular, a structural geometric approach to systems of such class was developed in [3], [8], [9], [10], [11], [12], and [13].

Versatility and applicability of max-plus models are extended by letting their dynamics switch, according to a switching signal, from an operational mode to another in a finite set of modes. Switching linear max-plus systems of this kind were introduced in [14] while the special case of constrained periodic switching was considered in [15] and [16]. The possibility of modifying the dynamics by switching can be exploited to break the synchronization of operations or to modify the order of the events so as to allow competition.

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In this work, we consider the problem of forcing the output of a given plant, represented by a switching linear max-plus system, to occur at the same time as that of a given model or to anticipate it. In the max-plus literature, the first objective is referred to as the *system synchronization* while the second is called the *system subsynchronization* [17]. These problems are the generalization to the class of switching max-plus systems of the model matching problem first considered in [18] for linear systems over a field and investigated by several authors for other classes of dynamical systems, including, in particular, systems over rings [19], switching systems over a field [20], [21], and linear max-plus systems [13], [17], [22], [23], [24].

In general, the problem of controlling a given plant in such a way as to comply with a desired production time schedule can be dealt with by constructing a model that follows that schedule and by designing open-loop or closed-loop control laws that force the output of the plant to occur at the same time as that of the model or to anticipate it [25]. Thus, system synchronization or subsynchronization can be used in practice to satisfy the time requirements of a given production policy or of just-in-time strategies [26], [27].

In the max-plus framework, to be feasible, the control laws that achieve system synchronization or subsynchronization are constrained to be nondecreasing, so that the $n$th input in a sequence cannot occur in time before the $(n-1)$th input in the same sequence. This requirement clearly limits the set of possible solutions and needs to be dealt with by making a suitable *nonanticipativeness* assumption (introduced and discussed later on) on the plant and on the model.

As mentioned earlier, synchronization problems for linear max-plus systems have already been considered. In [22], solvability conditions for a model matching problem involving time event graphs were given in terms of formal power series, whereas in [23], the problem was dealt with by extending to the max-plus framework the classical polynomial (RST) control strategy. In [13] and [17], a structural geometric approach was employed to provide solvability conditions. The case of switching max-plus systems was first considered in [24], where a sufficient, but not necessary, condition for the system synchronization problem (SSP) to be solvable by a nondecreasing control law, both in open-loop and closed-loop control schemes, was given in structural geometric terms.

The contribution of this technical note is to provide a complete characterization of the solvability of the SSP (see Theorem 2) by means of a necessary and sufficient structural condition, which is weaker than that of [24, Th. 3] and which substantially differs from that since it makes use of a newly defined geometric notion. A necessary and sufficient structural solvability condition with similar characteristics is also derived for the subsynchronization problem (see Theorem 3).

More precisely, the key innovative feature of the necessary and sufficient condition given here is that it involves a family of submodules of the state module of the modes of a switching system, which enjoys a specific invariance property with respect to the switching dynamics, instead of a single controlled invariant (with respect to the switching

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dynamics) subsemimodule of the state semimodule as in [24]. This formulation requires the definition of a new notion of controlled invariant family of subsemimodules that is akin to that previously used only when the switching mechanism is somehow constrained: e.g., in linear periodic systems [28] and in linear switching systems over a digraph [29].

It is worth noting that, while max-plus dynamical systems can be viewed as algebraic objects, this does not hold for the class of systems we consider here due to their switching behavior. As a consequence, the algebraic approach to model reference problems for nonswitching max-plus systems in [22], which is based on formal power series, does not appear to be applicable here. The same holds for the RST control strategy in [23] since the concept of transfer function on which it relies has no counterpart in the switching framework. On the other hand, structural geometric methods that suitably extend those of [13] and [17] prove to be applicable and efficient both from a theoretical and practical point of view.

A notable feature of the structural geometric approach we adopt is that of providing recursive procedures that, in case of convergence in a finite number of steps, make it possible to check practically the solvability conditions and to synthesize feasible solutions, if any exists.

The rest of the article is organized as follows. In Section II, notations and basic properties of the max-plus algebra $\mathbb{R}_{\max}$ and of switching max-plus systems are briefly recalled. Then, the problems of system synchronization and system subsynchronization are formally stated. In Section III, structural geometric notions, notably the novel notion of controlled invariance for a family of subsemimodules of the state semimodules, are introduced and used to characterize the solvability of the considered problems. In Section IV, two examples illustrate the previous results, also in comparison with those in [24]. Finally, Section V concludes this article.

## II. Background and Problem Statement

By $\mathbb{R}_{\max}$, we denote the max-plus algebra composed of the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of sum $\oplus$ and product $\otimes$ defined by $a \oplus b = \max\{a, b\}$ for $a, b \in \mathbb{R}_{\max}$ and by $a \otimes b = a + b$ if $a, b$ belong to $\mathbb{R}$ and by $(-\infty) \otimes a = a \otimes (-\infty) = -\infty$ for any $a \in \mathbb{R}_{\max}$. The neutral elements for $\oplus$ and for $\otimes$ are given by $\epsilon = -\infty$ and by $e = 0 \in \mathbb{R}$, respectively. Since $\otimes$ distributes over $\oplus$, $\mathbb{R}_{\max}$ is a semiring. By $\mathbb{R}^n_{\max}$, we denote the (free) semimodule over $\mathbb{R}_{\max}$ whose elements are $n$-dimensional vectors, that is, $n$-tuples, of elements of $\mathbb{R}_{\max}$, equipped with the componentwise sum and the scalar product that are defined by the sum $\oplus$ and the product $\otimes$ of $\mathbb{R}_{\max}$.

Given two semimodules $\mathcal{V} \subseteq \mathbb{R}^n_{\max}$ and $\mathcal{W} \subseteq \mathbb{R}^n_{\max}$, their sum is the semimodule $(\mathcal{V} + \mathcal{W}) \subseteq \mathbb{R}^n_{\max}$ defined by $\mathcal{V} + \mathcal{W} = \{x \in \mathbb{R}^n_{\max}$, such that $x = v \oplus w$ for some $v \in \mathcal{V}$ and for some $w \in \mathcal{W}\}$. Sometimes, it will also be useful to consider the semimodule $\mathcal{V} \ominus \mathcal{W}$ defined by $\mathcal{V} \ominus \mathcal{W} = \{x \in \mathbb{R}^n_{\max}$, for which there exists $w \in \mathcal{W}$ such that $x \oplus w \in \mathcal{V}\}$.

Given two vectors $v$ and $w$ of the same dimension with entries in $\mathbb{R}_{\max}$, the relation $v \geq w$ means that each component of $v$ is greater than or equal to the corresponding component of $w$. Given two matrices $A$ and $B$ of the same dimensions with entries in $\mathbb{R}_{\max}$, the relation $A \geq B$ has the same meaning.

The counter image of a semimodule $\mathcal{V} \subseteq \mathbb{R}^q_{\max}$ with respect to a $q \times p$ matrix $A$ is the semimodule $A^{-1}(\mathcal{V}) \subseteq \mathbb{R}^p_{\max}$. The evolution of a phenomenon that is characterized by the occurrence over time of events of $n$ different types may be described by means of an $n$-dimensional dater, that is a function $d(.) : \mathbb{N} \to \mathbb{R}^n_{\max}$, where the $i$th component of the vector $d(k)$, for $k \in \mathbb{N}$, represents the time instant at which an event of the $i$th type occurs for the $k$th time. Note that, in order to have physical meaning, daters must be nondecreasing, meaning that $d(k+1) \geq d(k)$ for each $k \in \mathbb{N}$.

In the aforementioned framework, a switching max-plus linear system $\Sigma_\sigma$ is the dynamical object defined by equations of the form

$$\Sigma_\sigma \equiv \begin{cases} x(k) = A_{\sigma(k)} x(k-1) \oplus B_{\sigma(k)} u(k) \\ y(k) = C_{\sigma(k)} x(k) \\ x(0) = \epsilon \end{cases} \tag{1}$$

where $k \in \mathbb{N}$ is the index of the event instance; $x(.) : \mathbb{N} \to \mathcal{X} = \mathbb{R}^n_{\max}$ is the dater of the internal events; $u(.) : \mathbb{N} \to \mathcal{U} = \mathbb{R}^m_{\max}$ is the dater of the input events and $y(.) : \mathbb{N} \to \mathcal{Y} = \mathbb{R}^p_{\max}$ is the dater of the output events; $\sigma(.) : \mathbb{N} \to \mathcal{I} = \{1, \ldots, I\}$ is the function that defines the switching behavior and $A_i, B_i, C_i$, for $i \in \mathcal{I}$, are matrices of suitable dimensions with entries in $\mathbb{R}_{\max}$. The semimodule $\mathcal{X}$ is called the state semimodule of $\Sigma_\sigma$. The max-plus linear systems $\Sigma_i$ defined by

$$\Sigma_i \equiv \begin{cases} x(k) = A_i x(k-1) \oplus B_i u(k) \\ y(k) = C_i x(k) \\ x(0) = \epsilon \end{cases} \tag{2}$$

for $i \in \mathcal{I}$, are the *modes* of $\Sigma_\sigma$ and, therefore, $\sigma(k)$ defines the configuration assumed by the system at the $k$th iteration of its operations.

In the previous description, the $n$ components of $x(.)$ correspond to $n$ types of internal events for $\Sigma_\sigma$, so that the vector $x(k) = (x_1(k), \ldots, x_n(k))^\top \in \mathbb{R}^n_{\max}$ represents the fact that the $k$th instance of the internal event of type $\ell$ occurs at time $x_\ell(k)$. A similar interpretation holds for the $m$ components of the input dater $u(.)$ and for the $p$ components of the output dater $y(.)$.

If $u(.)$ is nondecreasing, the sequence $\{u(k)\}_{k\in\mathbb{N}}$ can be seen as a feasible input to $\Sigma_\sigma$. In turn, in order to be feasible, $\Sigma_\sigma$ must respond to nondecreasing inputs by producing sequences $\{x(k)\}_{k\in\mathbb{N}}$ of internal events that are nondecreasing. This property is called *nonanticipativeness*. It can easily be shown that each mode $\Sigma_i$ is *nonanticipative* if $A_i \geq I_n$, where $I_n$ denotes the $n \times n$ matrix whose diagonal elements are equal to $e$ and the others are equal to $\epsilon$. Moreover, it can be shown that $\Sigma_\sigma$ is nonanticipative if and only if all its modes are such. In the rest of the work, we will assume that all the considered switching linear systems are nonanticipative.

A recurrent concept in this work is that of family of semimodules indexed by the active mode $\sigma(k)$ of the system $\Sigma_\sigma$. We will use the notation $\mathcal{V}_\sigma$ to denote a family of semimodules $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, where $\mathcal{I} = \{1, \ldots, I\}$ is the codomain of $\sigma$. We say that a family of semimodules $\mathcal{V}_\sigma$ is contained in a semimodule $\mathcal{K} \subseteq \mathbb{R}^n_{\max}$, and we denote such relation by abuse of notation as $\mathcal{V}_\sigma \subseteq \mathcal{K}$, if $\mathcal{V}_i \subseteq \mathcal{K}$ for each $i \in \mathcal{I}$. Given a vector $z \in \mathbb{R}^n_{\max}$ and a family of semimodules $\mathcal{V}_\sigma \subseteq \mathbb{R}^n_{\max}$, we say that $z$ belongs to $\mathcal{V}_\sigma$, and we denote such relation as $z \in \mathcal{V}_\sigma$, if $z \in \mathcal{V}_i$ for some $i \in \mathcal{I}$. Moreover, given two families of semimodules $\mathcal{V}_\sigma$ and $\mathcal{W}_\sigma$ we say the following.

1) $\mathcal{V}_\sigma$ is finitely generated if, for all $i \in \mathcal{I}$, $\mathcal{V}_i$ is finitely generated.
2) $\mathcal{V}_\sigma$ is equal to $\mathcal{W}_\sigma$, or $\mathcal{V}_\sigma = \mathcal{W}_\sigma$, if $\mathcal{V}_i = \mathcal{W}_i$ for all $i \in \mathcal{I}$.
3) $\mathcal{V}_\sigma$ is contained in $\mathcal{W}_\sigma$, or $\mathcal{V}_\sigma \subseteq \mathcal{W}_\sigma$, if $\mathcal{V}_i \subseteq \mathcal{W}_i$ for all $i \in \mathcal{I}$.
4) Given two families of semimodules $\mathcal{V}_\sigma \subseteq \mathbb{R}^n_{\max}$ and $\mathcal{W}_\sigma \subseteq \mathbb{R}^n_{\max}$, their sum is the family of semimodules $\mathcal{M}_\sigma = \mathcal{V}_\sigma + \mathcal{W}_\sigma$ with $\mathcal{M}_i = \mathcal{V}_i + \mathcal{W}_i$ for all $i \in \mathcal{I}$.
5) Given two families of semimodules $\mathcal{V}_\sigma \subseteq \mathbb{R}^n_{\max}$ and $\mathcal{W}_\sigma \subseteq \mathbb{R}^n_{\max}$, their intersection is the family of semimodules $\mathcal{N}_\sigma = \mathcal{V}_\sigma \cap \mathcal{W}_\sigma$ with $\mathcal{N}_i = \mathcal{V}_i \cap \mathcal{W}_i$ for all $i \in \mathcal{I}$.

We can now formalize the problems we tackle in this work.

*Problem 1 (System Synchronization Problem):* Given a switching linear max-plus system

$$\Sigma_{P\sigma} \equiv \begin{cases} x_P(k) = A_{P\sigma(k)} x_P(k-1) \oplus B_{P\sigma(k)} u_P(k) \\ y_P(k) = C_{P\sigma(k)} x_P(k) \\ x_P(0) = \epsilon \end{cases} \tag{3}$$

called the plant and a switching linear max-plus system

$$\Sigma_{M\sigma} \equiv \begin{cases} x_M(k) = A_{M\sigma(k)} x_M(k-1) \oplus B_{M\sigma(k)} u_M(k) \\ y_M(k) = C_{M\sigma(k)} x_M(k) \\ x_M(0) = \epsilon \end{cases} \tag{4}$$

called the model, with $x_P : \mathbb{N} \to \mathbb{R}_{\max}^{n_P}, x_M : \mathbb{N} \to \mathbb{R}_{\max}^{n_M}, u_P : \mathbb{N} \to \mathbb{R}_{\max}^{m_P}$, $u_M : \mathbb{N} \to \mathbb{R}_{\max}^{m_M}$, and $y_P, y_M : \mathbb{N} \to \mathbb{R}_{\max}^{p}$, the system synchronization problem (SSP) consists in finding, for all possible nondecreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model and all possible switching signals $\sigma(.)$, a nondecreasing control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ for the plant, such that the output $\{y_P(k)\}_{k\in\mathbb{N}}$ of this latter equals the output $\{y_M(k)\}_{k\in\mathbb{N}}$ of the model, i.e., $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$.

*Problem 2 (System Subsynchronization Problem):* Given a plant $\Sigma_{P\sigma}$ of the form (3) and a model $\Sigma_{M\sigma}$ of the form (4), the system subsynchronization problem (SSSP) consists in finding, for all possible nondecreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model and all possible switching signals $\sigma(.)$, a nondecreasing control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ for the plant, such that the output $\{y_P(k)\}_{k\in\mathbb{N}}$ of this latter is smaller than or equal to the output $\{y_M(k)\}_{k\in\mathbb{N}}$ of the model, i.e., $y_P(k) \le y_M(k)$ for all $k \in \mathbb{N}$.

As mentioned in Section I, the aforementioned problems can be seen as a generalization to the max-plus framework of the classical model matching problem originally considered for time-invariant linear systems in [18]. More precisely, the SSP coincides with an exact model matching, whose requirement for production systems may be unnecessarily stringent. The SSSP provides a more realistic and widespread approach in matching a production schedule, letting the output of the model be interpreted as a deadline for obtaining the output of the plant.

A more restrictive formulation of the aforementioned problems is obtained by requiring the control input $u_P(k)$ to be, for each value $\sigma(k)$ of the switching signal, a linear function of the state of both the plant and the model (i.e., $x_P(k-1)$ and $x_M(k-1)$) as well as of the input of the model $u_M(k)$. We refer to such formulations as the feedback SSP (FSSP) and the feedback SSSP (FSSSP).

*Problem 3 (Feedback SSP):* Given a plant $\Sigma_{P\sigma}$ of the form (3) and a model $\Sigma_{M\sigma}$ of the form (4), the FSSP consists in finding, for all possible nondecreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model and all possible switching signals $\sigma(.)$, two families of matrices $\{F_i\}_{i\in\mathcal{I}}$, with $F_i \in \mathbb{R}_{\max}^{m_P\times(n_P+n_M)}$ for all $i \in \mathcal{I}$, and $\{G_i\}_{i\in\mathcal{I}}$, with $G_i \in \mathbb{R}_{\max}^{m_P\times m_M}$ for all $i \in \mathcal{I}$, such that the control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ defined by

$$u_P(k) = \begin{cases} F_{\sigma(1)} \begin{pmatrix} x_P(0) \\ x_M(0) \end{pmatrix} \oplus G_{\sigma(1)} u_M(1), & \text{for } k = 1 \\ F_{\sigma(k)} \begin{pmatrix} x_P(k-1) \\ x_M(k-1) \end{pmatrix} \oplus G_{\sigma(k)} u_M(k) & \\ \qquad \oplus u_P(k-1), & \text{for } k > 1 \end{cases} \tag{5}$$

is a solution for the corresponding SSP.

*Problem 4 (Feedback SSSP):* Given a plant $\Sigma_{P\sigma}$ of the form (3) and a model $\Sigma_{M\sigma}$ of the form (4), the FSSSP consists in finding, for all possible nondecreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model and all possible switching signals $\sigma(.)$, two families of matrices $\{F_i\}_{i\in\mathcal{I}}$, with $F_i \in \mathbb{R}_{\max}^{m_P\times(n_P+n_M)}$ for all $i \in \mathcal{I}$, and $\{G_i\}_{i\in\mathcal{I}}$, with $G_i \in \mathbb{R}_{\max}^{m_P\times m_M}$, for all $i \in \mathcal{I}$, such that the control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ defined by (5) is a solution of the corresponding SSSP.

*Remark 1:* The term $u_P(k-1)$ in (5) for $k > 1$ is a dynamic component, which assures that $u_P(k)$ is nondecreasing.

*Remark 2:* In the given formulation of the FSSP, it is not required that the matrices of the families $\{F_i\}_{i\in\mathcal{I}}$ and $\{G_i\}_{i\in\mathcal{I}}$ contain only nonnegative real numbers or $\epsilon$, so the solution can be an anticipative feedback. In this case, the practical implementation requires the knowledge of the values of the model input dater with some advance, but not necessarily the entire sequence has to be known from the beginning.

## III. Solution of the Problems

Given a plant $\Sigma_{P\sigma}$ described by (3) and a model $\Sigma_{M\sigma}$ described by (4), let us take the extended system $\Sigma_{E\sigma}$, whose dynamics is described by the equations

$$\Sigma_{E\sigma} \equiv \begin{cases} x_E(k) = A_{E\sigma(k)} x_E(k-1) \oplus B_{1\sigma(k)} u_P(k) \\ \qquad\qquad \oplus B_{2\sigma(k)} u_M(k) \\ x_E(0) = \epsilon \end{cases} \tag{6}$$

where

$$x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathcal{X}_E = \mathbb{R}_{\max}^{(n_P+n_M)}$$

is the internal event dater and where

$$A_{E\sigma(k)} = \begin{pmatrix} A_{P\sigma(k)} & \epsilon \\ \epsilon & A_{M\sigma(k)} \end{pmatrix}, \qquad B_{1\sigma(k)} = \begin{pmatrix} B_{P\sigma(k)} \\ \epsilon \end{pmatrix}$$

$$B_{2\sigma(k)} = \begin{pmatrix} \epsilon \\ B_{M\sigma(k)} \end{pmatrix}.$$

Let the *output equalizer* family of semimodules $\mathcal{K}_\sigma \subseteq \mathcal{X}_E$ be defined as $\mathcal{K}_\sigma = \{\mathcal{K}_i\}_{i\in\mathcal{I}}$, with

$$\mathcal{K}_i = \left\{ \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathcal{X}_E, \text{ such that } C_{Pi} x_P = C_{Mi} x_M \right\}. \tag{7}$$

Then, the SSP can be formulated as the problem of finding a control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ that, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, forces $x_E(k)$ to evolve inside $\mathcal{K}_\sigma$ for each possible switching signal $\sigma(.)$, where this means that $x_E(k)$ belongs to $\mathcal{K}_i$ with $\sigma(k) = i$ for all $k \in \mathbb{N}$. In fact, this condition guarantees that the control objective $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$ is achieved.

Similarly, let the *output subequalizer* family of semimodules $\mathcal{K}_\sigma^s \subseteq \mathcal{X}_E$ be defined as $\mathcal{K}_\sigma^s = \{\mathcal{K}_i^s\}_{i\in\mathcal{I}}$, with

$$\mathcal{K}_i^s = \left\{ \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathcal{X}_E \text{ , such that } C_{Pi} x_P \le C_{Mi} x_M \right\}. \tag{8}$$

Then, the SSSP can be formulated as the problem of finding a control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ that, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, forces $x_E(k)$ to evolve inside $\mathcal{K}_\sigma^s$, for each possible switching signal $\sigma(.)$, where this means that $x_E(k)$ belongs to $\mathcal{K}_i^s$ with $\sigma(k) = i$ for all $k \in \mathbb{N}$. In fact, this condition guarantees that the control objective $y_P(k) \le y_M(k)$ for all $k \in \mathbb{N}$ is achieved.

Moreover, the FSSP and the FSSSP can be formulated as the problems of finding a control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ of the form (5), for suitable families $\{F_i\}_{i\in\mathcal{I}}$ and $\{G_i\}_{i\in\mathcal{I}}$ of matrices, that, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, forces $x_E(k)$ to evolve inside the output equalizer family $\mathcal{K}_\sigma$ or, respectively, inside the output subequalizer family $\mathcal{K}_\sigma^s$, for each possible switching signal $\sigma(.)$.

*Remark 3:* Note that, since condition $C_{Pi} x_P \le C_{Mi} x_M$ can be equivalently written as $C_{Pi} x_P \oplus C_{Mi} x_M = C_{Mi} x_M$, each semimodule $\mathcal{K}_i^s$ can also be defined as $\mathcal{K}_i^s = \{x_E \in \mathcal{X}_E$ , such that $(C_{Pi}\ C_{Mi}) x_E = (\epsilon_{m_p\times n_p}\ C_{Mi}) x_E\}$.

The existence of control sequences with the previous characteristics can be investigated by adopting a structural geometric point of view, conceptually derived from the geometric approach to linear systems with coefficients in a field [30], [31], subsequently extended to stationary systems over rings [32], [33], [34], and over semirings [3], [8], [9], [10], [11]. To this aim, we introduce the notion of $(A_\sigma, B_\sigma)$-invariant family of semimodules for switching linear max-plus systems as follows.

*Definition 1:* Given a switching linear max-plus system $\Sigma_\sigma$ of the form (1), a family of semimodules $\mathcal{V}_\sigma = \{\mathcal{V}_i\}_{i\in\mathcal{I}} \subseteq \mathcal{X}$ is said to be $(A_\sigma, B_\sigma)$-invariant if, for all $i, j \in \mathcal{I}$ and for all $v \in \mathcal{V}_j$, there exists $u \in \mathbb{R}^m_{\max}$ such that $(A_i v \oplus B_i u)$ belongs to $\mathcal{V}_i$.

The previous notion is novel and it extends that of $(A_\sigma, B_\sigma)$-invariant semimodule given in [24, Definition 1]. In particular, note that each semimodule $\mathcal{V}_i$ in $\mathcal{V}_\sigma$ is $(A_i, B_i)$-invariant and, if $\mathcal{V}_i = \mathcal{V}_j$ for all $i, j \in \mathcal{I}$, the semimodule $\mathcal{V}_i$ is $(A_\sigma, B_\sigma)$-invariant in the sense of [24].

Given a switching max-plus system $\Sigma_\sigma$ of the form (1) and a family of semimodules $\mathcal{K}_\sigma = \{\mathcal{K}_i\}_{i\in\mathcal{I}} \subseteq \mathcal{X}$ of its state semimodule, the set of all the $(A_\sigma, B_\sigma)$-invariant families of semimodules contained in $\mathcal{K}_\sigma$ is a semilattice with respect to inclusion and sum of families of semimodules. Hence, a maximum element of that set, denoted by $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$, exists. As for the computation of $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$, we have the following result.

*Theorem 1:* Given a switching max-plus system $\Sigma_\sigma$ of the form (1) and a family of semimodules $\mathcal{K}_\sigma = \{\mathcal{K}_i\}_{i\in\mathcal{I}} \subseteq \mathcal{X}$, the sequence of families of semimodules $\mathcal{V}^r_\sigma = \{\mathcal{V}^r_i\}_{i\in\mathcal{I}} \subseteq \mathcal{K}_\sigma$ recursively defined, for $i \in \mathcal{I} = \{1, \ldots, I\}$ and $r \in \mathbb{N}$, by

$$\begin{aligned} \mathcal{V}^0_i &= \mathcal{K}_i \\ \mathcal{V}^r_i &= \mathcal{V}^{r-1}_i \cap \Big(\bigcap_{j\in\mathcal{I}} A_j^{-1}(\mathcal{V}^{r-1}_j \ominus \mathrm{Im} B_j)\Big) \end{aligned} \tag{9}$$

has the following properties.

1) $\mathcal{V}^r_\sigma \subseteq \mathcal{V}^{r-1}_\sigma$ for all $r \in \mathbb{N}$.
2) Letting $\mathcal{V}^\infty_\sigma = \lim_{r\to\infty} \mathcal{V}^r_\sigma = \bigcap_{r\in\mathbb{N}} \mathcal{V}^r_\sigma$, then every $(A_\sigma, B_\sigma)$-invariant family of semimodules contained in $\mathcal{K}_\sigma$ is also contained in $\mathcal{V}^\infty_\sigma$.
3) $\mathcal{V}^r_\sigma = \mathcal{V}^{r-1}_\sigma$ for some $r \in \mathbb{N}$ if and only if $\mathcal{V}^r_\sigma$ is an $(A_\sigma, B_\sigma)$-invariant family of semimodules and, in such case, $\mathcal{V}^\infty_\sigma = \mathcal{V}^r_\sigma = \mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$.

*Proof:* (1) It follows easily from the definition of $\mathcal{V}^r_\sigma$.

(1) Let $\mathcal{P}_\sigma \subseteq \mathcal{K}_\sigma = \mathcal{V}^0_\sigma$ be an $(A_\sigma, B_\sigma)$-invariant family of semimodules and assume that, for some $r \in \mathbb{N}$, we have $\mathcal{P}_\sigma \subseteq \mathcal{V}^{r-1}_\sigma$. Then, since $\mathcal{P}_i \subseteq A_i^{-1}(\mathcal{P}_i \ominus \mathrm{Im} B_i) \subseteq A_i^{-1}(\mathcal{V}^{r-1}_i \ominus \mathrm{Im} B_i)$ for all $i \in \mathcal{I}$, we also have $\mathcal{P}_\sigma \subseteq \mathcal{V}^r_\sigma$ and the conclusion follows by induction.

(1) The relation $\mathcal{V}^r_\sigma = \mathcal{V}^{r-1}_\sigma$ holds if and only if $\mathcal{V}^{r-1}_\sigma \subseteq \bigcap_{i\in\mathcal{I}} A_i^{-1}(\mathcal{V}^{r-1}_i \ominus \mathrm{Im} B_i)$, which, in turn, holds if and only if $\mathcal{V}^{r-1}_j \subseteq A_i^{-1}(\mathcal{V}^{r-1}_i \ominus \mathrm{Im} B_i)$ for all $i, j \in \mathcal{I}$. This is equivalent to the fact that $\mathcal{V}^{r-1}_\sigma$ is an $(A_\sigma, B_\sigma)$-invariant family of semimodules. In the considered case, the equality $\mathcal{V}^\infty_\sigma = \mathcal{V}^{r-1}_\sigma = \mathcal{V}^r_\sigma$ is obvious and $\mathcal{V}^\infty_\sigma = \mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$ follows from (1). ■

Theorem 1 says that $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$ can be computed by means of (9) if $\mathcal{V}^r_\sigma = \mathcal{V}^{r-1}_\sigma$ for some $r \in \mathbb{N}$, i.e., if the sequence (9) converges in a finite number of steps. However, (9) does not necessarily converge in a finite number of steps, and therefore, it does not provide a general algorithm to compute $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$. In this regard, the situation is the same as that concerning with $(A, B)$-invariant semimodules for stationary max-plus systems (see [9]), for which a general algorithm is not known.

*Remark 4:* In force of [35, Corollary 86], if the submodules in $\mathcal{K}_\sigma$ are finitely generated, so are those in $\mathcal{V}^r_\sigma$. The generators of such semimodules can be computed by solving appropriate systems of equations of the form $Dx = Cx$ [9, Remark 1] by means of a general elimination algorithm [36]. The complexity of that algorithm is exponential, but its convergence rate can be ameliorated as discussed in [9] and [35].

*Lemma 1:* The elements of the maximal $(A_\sigma, B_\sigma)$-invariant family of semimodules $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$ contained in a family of semimodules $\mathcal{K}_\sigma$ fulfill the property

$$(\mathcal{V}^*_j \backslash \mathcal{V}^*_i) \cap \mathcal{K}_i = \emptyset \quad \text{for all } i, j \in \mathcal{I}. \tag{10}$$

*Proof:* Given some $i, j \in \mathcal{I}$, we have that a vector $v_i$ belongs to $\mathcal{V}^*_i$ if and only if $v_i \in \mathcal{K}_i$ and, for each $k \in \mathcal{I}$, there exists $u_k \in \mathbb{R}^m_{\max}$ such that $A_k v_i \oplus B_k u_k$ belongs to $\mathcal{V}^*_k$. Similarly, $v_j$ belongs to $\mathcal{V}^*_j$ if and only if it belongs to $\mathcal{K}_j$ and, for each $k \in \mathcal{I}$, there exists $u_k \in \mathbb{R}^m_{\max}$ such that $A_k v_j \oplus B_k u_k$ belongs to $\mathcal{V}^*_k$. The aforementioned conditions differ only in their first part since $v_i$ is required to belong to $\mathcal{K}_i$ while $v_j$ is required to belong to $\mathcal{K}_j$, but their second parts, about the existence of a suitable $u_k$, are identical. We can state that if a vector $w \in \mathbb{R}^n_{\max}$ belongs to $\mathcal{K}_i$ and to $\mathcal{V}^*_j$, then it must also belong to $\mathcal{V}^*_i$, and the conclusion follows. ■

*Definition 2:* Given a switching linear max-plus system $\Sigma_\sigma$ of the form (1), a family of semimodules $\mathcal{V}_\sigma \subseteq \mathbb{R}^n_{\max}$ is said to be an $(A_\sigma, B_\sigma)$-invariant family of semimodules of feedback type for $\Sigma_\sigma$ if there exists a family of matrices $\{F_i\}_{i\in\mathcal{I}}$, with $F_i \in \mathbb{R}^{m\times n}_{\max}$ for all $i \in \mathcal{I}$, such that for all $i, j \in \mathcal{I}$ and for all $v \in \mathcal{V}_j$, $(A_i \oplus B_i F_i)v$ belongs to $\mathcal{V}_i$.

It is well known that, in the framework of systems with coefficients in a field, controlled invariance and controlled invariance of feedback type are equivalent [30], [31]. However, in the case of systems with coefficients in a ring, or in a semiring, the feedback property implies the former but not vice versa [9], [32], [33].

In order to state the main results of this work, we need to review the definition of strong nonanticipativeness and a related technical lemma first presented in [24].

*Definition 3 ([24, Definition 3]):* A switching linear max-plus system $\Sigma_\sigma$ of the form (1) is said to be strongly nonanticipative if it is nonanticipative (i.e., $A_i \geq I_n$ for all $i \in \mathcal{I}$) and $A_i B_j \geq B_i$ for all $i, j \in \mathcal{I}$.

*Lemma 2 ([24, Lemma 2]):* If a switching linear max-plus system $\Sigma_\sigma$ of the form (1) is strongly nonanticipative and $u(k+1) = u(k)$ for some $k \in \mathbb{N}$, then the term $B_{\sigma(k+1)} u(k+1)$ does not influence the state evolution of the system.

Intuitively, strong nonanticipativeness means that, if the input is constant, the system dynamics is slow enough to filter the effect of the switching in the input matrix. If $\Sigma_\sigma$ is nonanticipative and its input matrix is constant (i.e., $B_i = B \in \mathbb{R}^{n\times m}_{\max}$ for all $i \in \mathcal{I}$), then $\Sigma_\sigma$ is strongly nonanticipative.

We can now state a necessary and sufficient condition for the solvability of the SSP and the SSSP.

*Theorem 2:* Given a strongly nonanticipative plant $\Sigma_{P\sigma}$ of the form (3) and a strongly nonanticipative model $\Sigma_{M\sigma}$ of the form (4), consider the extended system $\Sigma_{E\sigma}$ given by (6). Then, the related SSP is solvable if and only if, for each $i \in \mathcal{I}$ and for each $x \in \mathrm{Im}\, B_{2i} = \mathrm{Im} \begin{pmatrix} \epsilon \\ B_{Mi} \end{pmatrix} \subseteq \mathcal{X}_E$, there exists $z \in \mathrm{Im}\, B_{1i} = \mathrm{Im} \begin{pmatrix} B_{Pi} \\ \epsilon \end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z$ belongs to $\mathcal{V}^*_i \subseteq \mathcal{X}_E$, where $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma) = \{\mathcal{V}^*_i\}_{i\in\mathcal{I}}$ is the maximum $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules for $\Sigma_{E\sigma}$ contained in the output equalizer family of semimodules $\mathcal{K}_\sigma \subseteq \mathcal{X}_E$ defined by (7).

*Proof: If.* Since $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$ is $(A_{E\sigma}, B_{1\sigma})$-invariant, for each $i \in \mathcal{I}$ and each $x_E \in \mathcal{V}^*_j$, there exists a vector $u_{1i} \in \mathbb{R}^{m_P}_{\max}$ such that $A_{Ei} x_E \oplus B_{1i} u_{1i}$ belongs to $\mathcal{V}^*_i$. Moreover, the hypothesis implies that, for each $i \in \mathcal{I}$ and each $u_M \in \mathbb{R}^{m_M}_{\max}$, there exists $u_{2i} \in \mathbb{R}^{m_P}_{\max}$ such that

$$\begin{pmatrix} \epsilon \\ B_{Mi} \end{pmatrix} u_M \oplus \begin{pmatrix} B_{Pi} \\ \epsilon \end{pmatrix} u_{2i} \in \mathcal{V}^*_i. \tag{11}$$

Then, for each input sequence $\{u_M(k)\}_{k\in\mathbb{N}}$, by using the dynamics of $\Sigma_{E\sigma}$, we can recursively construct a control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ for $\Sigma_E$ as follows:

$$u_P(k) = \begin{cases} u_{2\sigma(k)}(1), & \text{for } k = 1 \\ u_{1\sigma(k)}(k) \oplus u_{2\sigma(k)}(k) \oplus u_P(k-1), & \text{for } k > 1. \end{cases}$$

The corresponding state evolution $\{x_E(k)\}_{k\in\mathbb{N}}$ turns out to be

$$x_E(k) = \begin{cases} B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k), & \text{for } k = 1 \\ (A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_{1\sigma(k)}(k))\oplus \\ (B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k))\oplus \\ B_{1\sigma(k)}u_P(k-1), & \text{for } k > 1 \end{cases}$$

and we can show by induction that $x_E(k)$ belongs to $\mathcal{V}^*_{\sigma(k)} \subseteq \mathcal{K}_{\sigma(k)}$ for all $k \in \mathbb{N}$, which implies $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$. In fact, $x_E(1)$ belongs to $\mathcal{V}^*_{\sigma(1)}$ by the definition of $u_{2\sigma(k)}(1)$. For $k > 1$, assuming by induction that $x_E(k-1)$ belongs to $\mathcal{V}^*_{\sigma(k-1)}$, we have, by the definition of $u_{1\sigma(k)}(k)$, that $(A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_{1\sigma(k)}(k))$ belongs to $\mathcal{V}^*_{\sigma(k)}$ and, by the definition of $u_{2\sigma(k)}(k)$, that $(B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k))$ also belongs to $\mathcal{V}^*_{\sigma(k)}$. Moreover, since $A_{E\sigma(k)}x_E(k-1) \geq A_{E\sigma(k)}B_{1\sigma(k-1)}u_P(k-1) \geq B_{1\sigma(k)}u_P(k-1)$ due to strong nonanticipativeness of the plant, the term $B_{1\sigma(k)}u_P(k-1)$ in $x_E(k)$ can be disregarded, and therefore, $x_E(k)$ belongs to $\mathcal{V}^*_{\sigma(k)}$.

*Only if.* If the condition of the theorem does not hold, there exist $\bar{u}_M$ and $i \in \mathcal{I}$ such that $B_{2i}\bar{u}_M \oplus B_{1i}u_P \notin \mathcal{V}^*_i$ for any $u_P \in \mathbb{R}^{m_P}_{\max}$. Then, taking the constant input $\{u_M(k)\}_{k\in\mathbb{N}}$ with $u_M(k) = \bar{u}_M$ for all $k \in \mathbb{N}$ and a switching signal $\sigma(.)$ with $\sigma(1) = i$, we have that $x_E(1) = B_{1i}u_P(1) \oplus B_{2i}\bar{u}_M$ does not belong to $\mathcal{V}^*_i$ for any value $u_P(1) \in \mathbb{R}^{m_P}_{\max}$. We can write, recursively, for $k \geq 2$, $x_E(k) = A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_P(k) \oplus B_{2\sigma(k)}u_M(k) = A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_P(k) \oplus B_{2\sigma(k)}\bar{u}_M$ and, thanks to the strong nonanticipativeness of the model, $x_E(k) = A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_P(k)$. By Lemma 1, the fact that $x_E(1)$ does not belong to $\mathcal{V}^*_{\sigma(1)}$ implies that either $x_E(1)$ does not belong to $\mathcal{K}_{\sigma(1)}$ or that it does not belong to $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$. If $x_E(1) \notin \mathcal{K}_{\sigma(1)}$, then $y_M(1) \neq y_P(1)$ and the SSP cannot be solved. If $x_E(1) \notin \mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$, then for any input $\{u_P(k)\}_{k\in\mathbb{N}}$, there exist a switching signal $\sigma(.)$ and some $\bar{k} \in \mathbb{N}$ such that $x_E(\bar{k}) \notin \mathcal{K}_\sigma$. In other words, $x_E(k)$ cannot be forced to evolve inside $\mathcal{K}_\sigma$, and therefore, the SSP cannot be solved. ■

*Remark 5:* The condition of Theorem 2 is weaker than that of [24, Th. 3], which requires that, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} \subseteq \mathcal{X}_E$, there exists $z \in \operatorname{Im} B_{1i} \subseteq \mathcal{X}_E$ such that $x \oplus z$ belongs to $\mathcal{V}^* \subseteq \mathcal{X}_E$, where $\mathcal{V}^*$ is the maximum $(A_{E\sigma}, B_{1\sigma})$-invariant semimodule for $\Sigma_{E\sigma}$ contained in the output equalizer family of semimodules $\mathcal{K}_\sigma \subseteq \mathcal{X}_E$ defined by (7). Example 1 in Section IV serves to show this.

*Theorem 3:* In the same hypotheses and with the same notations of Theorem 2, the SSSP involving $\Sigma_{P\sigma}$ and $\Sigma_{M\sigma}$ is solvable if and only if, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$, there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z$ belongs to $\mathcal{V}^*_i \subseteq \mathcal{X}_E$, where $\mathcal{V}^*_\sigma(\mathcal{K}^s_\sigma) = \{\mathcal{V}^*_i\}_{i\in\mathcal{I}}$ is the maximum $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules for $\Sigma_{E\sigma}$ contained in the output subequalizer family of semimodules $\mathcal{K}^s_\sigma \subseteq \mathcal{X}_E$ defined by (8).

*Proof:* The proof is the same as that of Theorem 2, with $\mathcal{K}^s_\sigma$ replacing $\mathcal{K}_\sigma$. ■

Note that since $\mathcal{K}_\sigma \subseteq \mathcal{K}^s_\sigma$, the condition of Theorem 3 is milder than that of Theorem 2. Example 2 in Section IV serves to show this.

Concerning the feedback version of the SSP and of the SSSP, we have the following results.

*Theorem 4:* In the same hypotheses and with the same notations of Theorem 2, the FSSP involving $\Sigma_{P\sigma}$ and $\Sigma_{M\sigma}$ is solvable if and only if there exists an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules $\mathcal{V}_\sigma$ of feedback type contained in the output equalizer family of semimodules $\mathcal{K}_\sigma$ defined by (7) such that, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$, there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z \in \mathcal{V}_i$.

*Proof: If.* Let $\mathcal{V}_\sigma \subseteq \mathcal{K}_\sigma$ be an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules of feedback type for which the condition of the theorem holds. Then, there exists a family of matrices $\{F_i\}_{i\in\mathcal{I}}$ such that, for each $x_E(k-1) \in \mathcal{V}_\sigma$ and $i \in \mathcal{I}$, $(A_{Ei} \oplus B_{1i}F_i)x_E(k-1)$ belongs to $\mathcal{V}_i$ and a family of matrices $\{G_i\}_{i\in\mathcal{I}}$ such that, for each $i \in \mathcal{I}$, the columns of the matrix $\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} G_i = \begin{pmatrix}B_{Pi}G_i \\ B_{Mi}\end{pmatrix}$ belong to $\mathcal{V}_i$. Then, by applying a control law recursively defined as in (5), with the matrix families $\{F_i\}_{i\in\mathcal{I}}$ and $\{G_i\}_{i\in\mathcal{I}}$ defined as before, we get the compensated dynamics

$$x_E(k) = (A_{E\sigma(k)} \oplus B_{1\sigma(k)}F_{\sigma(k)})x_E(k-1) \oplus \begin{pmatrix}B_{P\sigma(k)}G_{\sigma(k)} \\ B_{M\sigma(k)}\end{pmatrix} u_M(k) \oplus B_{1\sigma(k)}u_P(k-1) \qquad (12)$$

where $u_P(0) = \epsilon$. Since the plant is strongly nonanticipative, $A_{E\sigma(k)}x_E(k-1) \geq A_{E\sigma(k)}B_{1\sigma(k-1)}u_P(k-1) \geq B_{1\sigma(k)}u_P(k-1)$ holds and the last summand of the right-hand term of (12) does not interfere with the state of the system, that evolves, for all $k \in \mathbb{N}$, inside $\mathcal{V}_{\sigma(k)} \subseteq \mathcal{K}_{\sigma(k)}$. Therefore, $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$.

*Only if.* Assume that the FSSP is solved by a control law of the form (5). Then, the family of sets of reachable states for the dynamics (12) indexed by the last active mode $\sigma(k)$ is an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules of feedback type contained in $\mathcal{K}_\sigma$, whose elements contain all the columns of the matrix

$$\begin{pmatrix}B_{P\sigma(k)}G_{\sigma(k)} \\ B_{M\sigma(k)}\end{pmatrix} = \begin{pmatrix}\epsilon \\ B_{M\sigma(k)}\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_{P\sigma(k)} \\ \epsilon\end{pmatrix} G_{\sigma(k)}.$$

This clearly implies the condition of the theorem.

*Theorem 5:* In the same hypotheses and with the same notations of Theorem 2, the FSSSP involving $\Sigma_{P\sigma}$ and $\Sigma_{M\sigma}$ is solvable if and only if there exists an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules $\mathcal{V}_\sigma$ of feedback type contained in the output equalizer family of semimodules $\mathcal{K}^s_\sigma$ defined by (8) such that, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$, there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z \in \mathcal{V}_i$.

*Proof:* The proof is the same as that of Theorem 4, with $\mathcal{K}^s_\sigma$ replacing $\mathcal{K}_\sigma$. ■

*Remark 6:* The necessary and sufficient condition given by Theorem 2 can also be written as $\operatorname{Im}B_{2i} \subseteq \mathcal{V}^* \ominus \operatorname{Im}B_{1i}$ for all $i \in \mathcal{I}$ and similar formulations hold for the conditions of Theorems 3–5. Those conditions can be checked by means of the numeric methods mentioned in Remark 4 and in [9, Remark 1]. Moreover, solutions to the SSP and to the SSSP, if any exists, can be constructed by solving the linear equations considered in the proofs of Theorems 2 and 3 by means of general elimination methods (see [9], [36]). A Scilab toolbox that implements such methods is illustrated in [37].

## IV. Illustrative Examples

*Example 1:* The following example, whose data are the same as those of [24, Example 2], shows that the results given here are less conservative than those given in that work. In fact, the sufficient solvability condition of [24, Theorem 3], being not satisfied in this case, is shown to be not necessary.

Consider the linear max-plus plant defined by

$$\Sigma_{P\sigma} \equiv \begin{cases} x_P(k) = x_P(k-1) \oplus u_P(k) \\ y_P(k) = x_P(k) \\ x_P(0) = \epsilon \end{cases}$$

and the switching linear max-plus model defined by

$$\Sigma_{M\sigma} \equiv \begin{cases} x_M(k) = 2x_M(k-1) \oplus u_M(k) \\ y_M(k) = C_{M\sigma(k)} x_M(k) \\ x_M(0) = \epsilon \end{cases}$$

where

$$C_{M\sigma(k)} = \begin{cases} 1, & \text{for } \sigma(k) = 1 \\ 2, & \text{for } \sigma(k) = 2. \end{cases}$$

Note that the first equation of $\Sigma_{P\sigma}$ simplifies to $x_P(k) = u_P(k)$ for any nondecreasing input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$. Computing the output equalizer family of semimodules for the related SSP, we have $\mathcal{K}_\sigma = \{\mathcal{K}_1, \mathcal{K}_2\}$ and $\mathcal{K}_1 = \{(x_P \quad x_M)^\top$ such that $x_P = 1x_M\}$ and $\mathcal{K}_2 = \{(x_P \quad x_M)^\top$such that $x_P = 2x_M\}$. Then, the sequence of families of semimodules defined by (9) converges at the first iteration (i.e., $\mathcal{V}_\sigma^1 = \mathcal{V}_\sigma^0 = \mathcal{K}_\sigma$). Thus, $\mathcal{K}_\sigma$ is an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules. Moreover, $\mathcal{K}_\sigma$ is of feedback type with, e.g., $F_1 = (\epsilon \quad 3)$ and $F_2 = (\epsilon \quad 4)$. Since the set of linear equations $B_{2i} \oplus B_{1i}G_i \in \mathcal{K}_i$ for $i = 1, 2$ admits as the unique solution $G_1 = 1$ and $G_2 = 2$, the FSSP (hence, the SSP) has the solution

$$u_P(k) = F_{\sigma(k)} x_E(k-1) \oplus G_{\sigma(k)} u_M(k) \oplus u_P(k-1)$$

with $u_P(0) = \epsilon$. In fact, the input $u_P(k)$ forces the state $x_E(k)$ to stay in $\mathcal{K}_1$ or in $\mathcal{K}_2$ if, respectively, $\sigma(1) = 1$ or $\sigma(1) = 2$ and $\sigma(.)$ is constant while the same input makes the state jump from $\mathcal{K}_1$ to $\mathcal{K}_2$ and vice versa each time $\sigma(.)$ changes its value. Then, $y_P(k)$ is equal to $y_M(k)$ for all $k \in \mathbb{N}$.

*Example 2:* This example illustrates an application of the previous results and it also shows that subsynchronization, being a milder requirement, may be achievable also if synchronization is not.

Let us consider an olive mill that receives olives in standardized quantities (*stocks*) from clients and performs both oil extraction by cold-pressing and bottling. The oil extraction is performed in 10 h by a single machine, M1. The oil is stored in special tanks, whose total capacity is very large and can be considered infinite. The following phase of bottling is performed by another machine, M2, and it takes 18 or 20 h to complete depending on the size of the bottles provided by the customer. The final containers for the oil are provided by the clients and their availability anytime can be taken for granted. The overlapping of requests from different clients is avoided since they are accepted by appointment. If we consider as both internal and output events for the system the ones of type *a stock of olives has been processed* and *oil from a stock of olives has been bottled*, we can model the plant as a switching system whose modes are

$$\Sigma_{P1} \equiv \begin{cases} x_P(k) = \begin{pmatrix} 10 & \epsilon \\ 28 & 18 \end{pmatrix} x_P(k-1) \oplus \begin{pmatrix} 10 \\ 28 \end{pmatrix} u_P(k) \\ y_P(k) = x_P(k) \\ x_P(0) = \epsilon \end{cases}$$

and

$$\Sigma_{P2} \equiv \begin{cases} x_P(k) = \begin{pmatrix} 10 & \epsilon \\ 30 & 20 \end{pmatrix} x_P(k-1) \oplus \begin{pmatrix} 10 \\ 30 \end{pmatrix} u_P(k) \\ y_P(k) = x_P(k) \\ x_P(0) = \epsilon. \end{cases}$$

The switching sequence $\sigma$ is such that $\sigma(k) = 1$ if the current customer provides large bottles and $\sigma(k) = 2$ if the customer provides small bottles that, due to a more complex handling, require more time to complete the filling operation for the same total amount of oil. The production policy requires that the oil extraction is completed within 12 h of the arrival of the olives, and bottling is guaranteed within the following 48 h. Considering the model

$$\Sigma_M \equiv \begin{cases} x_M(k) = \begin{pmatrix} 12 & \epsilon \\ 60 & 48 \end{pmatrix} x_M(k-1) \oplus \begin{pmatrix} 12 \\ 60 \end{pmatrix} u_M(k) \\ y_M(k) = x_M(k) \\ x_M(0) = \epsilon \end{cases}$$

the previous requirement can be satisfied if the output of the plant is forced to (equal or) anticipate that of the model, that is if the plant and the model can be (synchronized or) subsynchronized. Both the plant and the model are strongly nonanticipative.

We can compute the output equalizer family of semimodules $\mathcal{K}_\sigma$, the output limiter family of semimodules $\mathcal{K}_\sigma^s$, and the maximal $(A_{E\sigma}, B_{1\sigma})$-invariant families of semimodules $\mathcal{V}_\sigma^*(\mathcal{K})$ and $\mathcal{V}_\sigma^*(\mathcal{K}^s)$ contained in them. Specifically, the sequence defined by (9) with $\mathcal{V}_\sigma^0 = \mathcal{K}_\sigma$ converges after two iterations (i.e., $\mathcal{V}_\sigma^2 = \mathcal{V}_\sigma^1 = \mathcal{V}_\sigma^*(\mathcal{K}_\sigma)$) and we have

$$\mathcal{K}_1 = \mathcal{K}_2 = \text{Im} \begin{pmatrix} e & \epsilon \\ \epsilon & e \\ e & \epsilon \\ \epsilon & e \end{pmatrix}, \quad \mathcal{V}_1^*(\mathcal{K}) = \mathcal{V}_2^*(\mathcal{K}) = \text{Im} \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix}.$$

Therefore, the SSP is not solvable and synchronization is not possible.

On the other hand, the sequence of semimodules defined by (9) with $\mathcal{V}_\sigma^0 = \mathcal{K}_\sigma^s$ converges after one iteration (i.e., $\mathcal{V}_\sigma^1 = \mathcal{V}_\sigma^0 = \mathcal{V}_\sigma^*(\mathcal{K}_\sigma^s)$) and we have

$$\mathcal{K}_1^s = \mathcal{K}_2^s = \text{Im} \begin{pmatrix} e & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon \\ e & \epsilon & e & \epsilon \\ \epsilon & e & \epsilon & e \end{pmatrix} = \mathcal{V}_1^*(\mathcal{K}_\sigma^s) = \mathcal{V}_2^*(\mathcal{K}_\sigma^s).$$

The condition of Theorem 3 is satisfied. Therefore, the SSSP is solvable and subsynchronization is possible. Moreover, $\mathcal{V}_\sigma^*(\mathcal{K}_\sigma^s)$ is of feedback type. Therefore, also the FSSSP is solvable by Theorem 5. A control signal $u_P(k)$ of the form (5) that solves the problem is obtained by taking, for instance

$$F_1 = F_2 = (\epsilon \quad \epsilon \quad \epsilon \quad \epsilon), \qquad G_1 = G_2 = 2.$$

Carrying out a simulation for a specific input and switching sequence, we get the results shown in Table I and we can see that $y_P(k) \leq y_M(k)$ for all $k \in \{1, \ldots, 4\}$, as expected.

TABLE I
EXAMPLE 2: SIMULATION RESULTS

| $k$ | $\sigma(k)$ | $u_M(k)$ | $u_P(k)$ | $y_M(k)$ | $y_P(k)$ |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 2 | $\begin{pmatrix} 12 \\ 60 \end{pmatrix}$ | $\begin{pmatrix} 12 \\ 30 \end{pmatrix}$ |
| 2 | 2 | 50 | 52 | $\begin{pmatrix} 62 \\ 110 \end{pmatrix}$ | $\begin{pmatrix} 62 \\ 82 \end{pmatrix}$ |
| 3 | 1 | 120 | 122 | $\begin{pmatrix} 132 \\ 180 \end{pmatrix}$ | $\begin{pmatrix} 132 \\ 150 \end{pmatrix}$ |
| 4 | 2 | 170 | 172 | $\begin{pmatrix} 182 \\ 230 \end{pmatrix}$ | $\begin{pmatrix} 182 \\ 202 \end{pmatrix}$ |

## V. Conclusion

By introducing a novel, suitable notion of a family of $(A_\sigma, B_\sigma)$-invariant semimodules, it has been possible to find a structural geometric characterization of the solvability of the system synchronization and subsynchronization problems in the max-plus framework. The difficulties in computing $\mathcal{V}_\sigma^*(\mathcal{K}_\sigma)$ due to the absence of a general algorithm and the complexity in solving systems of linear equations over $\mathbb{R}_{\max}$ currently limit the efficacy of this approach. However, the development of efficient algorithms for performing computations over semirings is an active research field. Results in that direction are expected to make structural geometric methods and strategies, in combination with max-plus modeling techniques, widely and effectively applicable to many practical production control and scheduling problems, also in the presence of uncertainties.

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