Existence results for boundary value problems associated with singular strongly nonlinear equations

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Abstract

We consider a strongly nonlinear differential equation of the type

$$(\Phi(a(t,x(t))\,x'(t)))' = f(t,x(t),x'(t)), \qquad \text{a.e. on } [0,T]$$

where f is a Carathédory function, Φ is a strictly increasing homeomorphism (the Φ-Laplacian operator) and the function a is continuous and non-negative. We assume that $a(t,x)$ is bounded from below by a non-negative function $h_1(t)$, independent of x and such that $1/h_1 \in L^p(0,T)$ for some $p>1$, and we require a weak growth condition of Wintner-Nagumo type.

Under these assumptions, we prove existence results for the Dirichlet problem associated to the above equation, as well as for different boundary conditions. Our approach combines fixed point techniques and the upper and lower solutions method.

1 Introduction

Recently many papers have been devoted to the study of boundary value problems (BVPs for short) associated to nonlinear ODEs involving the so-called Φ-*Laplacian operator* (see, e.g., [3, 4, 5, 10]). Namely, equations of the type

$$\big(\Phi(x')\big)' = f(t,x,x'), \tag{1.1}$$

where f is a Carathédory function and $\Phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism such that $\Phi(0) = 0$.

The class of Φ-Laplacian operators includes as a special case the classical r-Laplacian $\Phi(y) := y|y|^{r-2}$, with $r>1$. Such operators arise in some models, e.g. in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. Other models, e.g. reaction-diffusion equations with non-constant diffusivity, and porous media equations, lead to consider mixed differential operators, that is equations of the type

$$\big(a(x)\,\Phi(x')\big)' = f(t,x,x'), \tag{1.2}$$

where a is a continuous positive function (see, e.g., [8]). Furthermore, several papers have been devoted to the case of singular (i.e., with bounded domain) or non-surjective operators (see [1, 6, 9]). Usually, these existence results stem from a combination of fixed point techniques combined with the upper and lower solutions method. In this context, an important tool to get a priori bounds for the derivatives of the solutions is a Nagumo-type growth condition on the function f. Let us observe that, when in the differential operator is present the nonlinear term a, some assumptions are required on the differential operator Φ, which in general is assumed to be homogeneous, or having at most linear growth at infinity.

More recently, in collaboration with Cristina Marcelli, we considered two different generalizations of equation (1.2). In the paper [15], it is investigated the case in which the function a may depend also on t. More precisely, the authors obtain existence results for BVPs associated to the equation

$$\big(a(t,x(t))\,\Phi(x'(t))\big)' = f(t,x(t),x'(t)), \quad \text{a.e. on } I := [0,T]$$

with a continuous and positive, and assuming a weak form of Wintner-Nagumo growth condition. Namely,

$$\big|f(t,x,y)\big| \le \psi\Big(a(t,x)\,|\Phi(y)|\Big)\cdot\Big(\ell(t)+\nu(t)\,|y|^{\frac{s-1}{s}}\Big), \tag{1.3}$$

where $\nu \in L^s(I)$ (for some $s > 1$), $\ell \in L^1(I)$ and $\psi : (0,\infty) \to (0,\infty)$ is a measurable function such that $1/\psi \in L^1_{\mathrm{loc}}(0,\infty)$ and

$$\int_1^\infty \frac{\mathrm{d}s}{\psi(s)} = \infty.$$

This assumption is weaker than other Nagumo-type conditions previously considered, and allows to consider a very general operator Φ, which can be only strictly increasing, not necessarily homogeneous, nor having polynomial growth. Let us also observe that the same equation

$$\big(a(t,x)\,\Phi(x')\big)' = f(t,x,x')$$

was studied in [13, 14] in order to obtain heteroclinic solutions on the real line.

On the other hand, in [7] we considered *possibly singular equations*, that include a non-autonomous differential operator having an explicit dependence on t inside Φ. Namely

$$\Big(\Phi\big(k(t)\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), \text{a.e. on } I \tag{1.4}$$

where the function k is allowed to vanish in a set having null measure, so that equation (1.4) can become singular. In [7] we assumed $1/k \in L^p(I)$ and we look for solutions in the space $W^{1,p}(I)$, rather than $C^1(I,\mathbb{R})$. According our knowledge, very few papers have been devoted to this type of equations, and just for a restricted class of nonlinearities f (see [11, 12]).

In this paper we tackle a further generalization of equation (1.4), allowing also a dependence on x inside Φ. More in detail, we consider the following BVP

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I,\\ x(0)=\nu_1,\ x(T)=\nu_2. \end{cases} \tag{1.5}$$

where $\nu_1,\nu_2 \in \mathbb{R}$, $\Phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism, f is a Carathéodory function and $a : I \times \mathbb{R} \to \mathbb{R}$ is a continuous non-negative function satisfying the following estimate from below:

$$a(t,x) \ge h_1(t) \text{ for every } t \in I \text{ and every } x \in \mathbb{R}, \tag{1.6}$$

where $h_1 \in C(I, \mathbb{R})$ is non-negative and such that $1/h_1 \in L^p(I)$ for some $p > 1$. Notice that, differently to other papers quoted above, here we do not require the positivity of the function a, and thus the equation in (1.5) may be singular. Consequently, as in [7], we look for solutions in the Sobolev space $W^{1,p}(I)$.

For example, estimate (1.6) is verified when $a(t, x)$ has a simpler structure of a product or of a sum, as in the following special cases:

- if $a(t, x) = h(t) \cdot b(x)$, where h is continuous, non-negative and such that $1/h \in L^p(I)$, and b is continuous and such that $\inf_{\mathbb{R}} b > 0$;
- if $a(t, x) = h(t) + b(x)$, where h is continuous, non-negative and such that $1/h \in L^p(I)$, and b is continuous and non-negative.

Our main result, Theorem 3.5 below, yields the existence of a solution of the Dirichlet problem (1.5) assuming a weak Wintner-Nagumo condition, similar to the one in (1.3). Theorem 3.5 extends in a natural way the main result in [7], in the case when $a(t, x) = k(t)$ does not depend on x. The proof is obtained by the method of lower/upper solutions, combined with a fixed point technique applied to an auxiliary functional Dirichlet problem (see Section 2). In Section 3 we provide some illustrating examples in which our main result apply. Finally, in Section 4 we consider different BVPs, such as the periodic problem, Neumann-type problem and Sturm-Liouville-type problem, and with classical techniques we derive existence results.

2 The abstract setting

Let $T > 0$ be fixed and let $I := [0, T]$. Moreover, let $\nu_1, \nu_2 \in \mathbb{R}$. Throughout this section, we shall be concerned with **boundary value problems** of the type

$$\begin{cases} \Big(\Phi\big(A_x(t)\, x'(t)\big)\Big)' = F_x(t), & \text{a.e. on } I, \\ x(0) = \nu_1, \ x(T) = \nu_2, \end{cases} \tag{2.1}$$

where Φ, A and F satisfy the following structural assumptions (for a fixed $p > 1$):

(H1) $\Phi : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing homeomorphism*;

(H2) $A : W^{1,p}(I) \subseteq C(I, \mathbb{R}) \to C(I, \mathbb{R})$ is continuous with respect to the *uniform topology* of $C(I, \mathbb{R})$; moreover, there exist $h_1, h_2 \in C(I, \mathbb{R})$ such that

(H2)$_1$ $h_1, h_2 \geq 0$ on I and $1/h_1, 1/h_2 \in L^p(I)$;
(H2)$_2$ $h_1(t) \leq A_x(t) \leq h_2(t)$ for every $x \in W^{1,p}(I)$ and every $t \in I$;

(H3) $F : W^{1,p}(I) \to L^1(I)$ is continuous (with respect to the usual norms) and there exists a non-negative function $\psi \in L^1(I)$ such that

$$|F_x(t)| \leq \psi(t) \quad \text{for every } x \in W^{1,p}(I) \text{ and a.e. } t \in I. \tag{2.2}$$

Remark 2.1. We point out that, as a consequence of assumptions $(\text{H2})_1$ and $(\text{H2})_2$, for every $x \in W^{1,p}(I)$ we have $A_x \geq h_1 \geq 0$ and

$$\int_0^T \frac{1}{h_2(t)}\, \mathrm{d}t \leq \int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \leq \int_0^T \frac{1}{h_1(t)}\, \mathrm{d}t.$$

Since $1/h_1, 1/h_2 \in L^p(I)$, then the same is true of $1/A_x$ (for any $x \in W^{1,p}(I)$).

In the sequel, we shall indicate by $\mathcal{F}$ the integral operator associated with F, that is, the operator $\mathcal{F} : W^{1,p}(I) \to C(I,\mathbb{R})$ defined by

$$\mathcal{F}_x(t) := \int_0^t F_x(s)\,\mathrm{d}s.$$

Remark 2.2. We observe, for future reference, that $\mathcal{F}$ is *continuous* from $W^{1,p}(I)$ to $C(I,\mathbb{R})$: this follows from the continuity of F and from the estimate

$$\sup_{t\in I} |\mathcal{F}_x(t) - \mathcal{F}_y(t)| \leq \|F_x - F_y\|_{L^1}, \tag{2.3}$$

holding true for every $x, y \in W^{1,p}(I)$. Furthermore, assumption (2.2) gives

$$\sup_{t\in I} |\mathcal{F}_x(t)| \leq \|\psi\|_{L^1}, \quad \text{for every } x \in W^{1,p}(I). \tag{2.4}$$

Definition 2.3. We say that a continuous function $x \in C(I,\mathbb{R})$ is a **solution** of the boundary value problem (2.1) if it satisfies the following properties:

(1) $x \in W^{1,p}(I)$ and $t \mapsto \Phi\big(A_x(t)\,x'(t)\big) \in W^{1,1}(I)$;

(2) $\Big(\Phi\big(A_x(t)\,x'(t)\big)\Big)' = F_x(t)$ for a.e. $t \in I$;

(3) $x(0) = \nu_1$ and $x(T) = \nu_2$.

Remark 2.4. We point out that, if $x \in W^{1,p}(I)$ is a solution of the problem (2.1), by condition (1) in Definition 2.3 (and the fact that Φ is a homeomorphism, see assumption (H1)), there exists a *unique* $\mathcal{A}_x \in C(I,\mathbb{R})$ such that

$$\mathcal{A}_x(t) = A_x(t)\,x'(t) \quad \text{for a.e. } t \in I.$$

We shall use this fact in the next Section 3.

The main result of this section is the following *existence result*.

Theorem 2.5. *Under the structural assumptions* (H1), (H2) *and* (H3)*, the boundary value problem* (2.1) *admits at least one solution* $x \in W^{1,p}(I)$.

The proof of Theorem 2.5 requires some preliminary facts.

Lemma 2.6. *For every fixed* $x \in W^{1,p}(I)$*, there exists a* unique $\xi_x \in \mathbb{R}$ *such that*

$$\int_0^T \frac{1}{A_x(t)}\,\Phi^{-1}\big(\xi_x + \mathcal{F}_x(t)\big)\,\mathrm{d}t = \nu_2 - \nu_1. \tag{2.5}$$

Furthermore, there exists a universal constant $\mathbf{c}_0 > 0$ *such that*

$$|\xi_x| \leq \mathbf{c}_0 \quad \textit{for every } x \in W^{1,p}(I). \tag{2.6}$$

Proof. Let $x \in W^{1,p}(I)$ be fixed and let

$$f_x : \mathbb{R} \longrightarrow \mathbb{R}, \qquad f_x(\xi) := \int_0^T \frac{1}{A_x(t)}\,\Phi^{-1}\big(\xi + \mathcal{F}_x(t)\big)\,\mathrm{d}t.$$

Since $\mathcal{F}_x$ is continuous on I (see Remark 2.2) and since, by assumptions, Φ is continuous on the whole of $\mathbb{R}$, an application of Lebesgue's Dominated Convergence Theorem shows that $f_x \in C(\mathbb{R}, \mathbb{R})$ (see also Remark 2.1); moreover, since Φ is monotone increasing, the same is true of f_x and, by (2.4), we have

$$\begin{aligned} \Phi^{-1}(\xi - \|\psi\|_{L^1}) \cdot \left(\int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \right) &\leq f_x(\xi) \leq \\ &\leq \Phi^{-1}(\xi + \|\psi\|_{L^1}) \cdot \left(\int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \right). \end{aligned} \tag{2.7}$$

From this, we deduce that $f_x(\xi) \to \pm\infty$ as $\xi \to \pm\infty$; thus, by Bolzano's Theorem (and the strict monotonicity of f_x), there exists a unique $\xi_x \in \mathbb{R}$ such that

$$f_x(\xi_x) = \int_0^T \frac{1}{A_x(t)}\, \Phi^{-1}\big(\xi_x + \mathcal{F}_x(t)\big)\, \mathrm{d}t = \nu_2 - \nu_1.$$

We now turn to prove estimate (2.6). To this end we observe that, by (2.5) and the Mean Value Theorem, it is possible to find a suitable $t^* = t^*_x \in I$ such that

$$\Phi^{-1}(\xi_x + \mathcal{F}_x(t^*)) \cdot \left(\int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \right) = \nu_2 - \nu_1;$$

as a consequence, we obtain

$$\xi_x + \mathcal{F}_x(t^*) = \Phi\left((\nu_2 - \nu_1) \cdot \left(\int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \right)^{-1} \right).$$

Now, by crucially exploiting Remark 2.1, we see that

$$\left| (\nu_2 - \nu_1) \cdot \left(\int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \right)^{-1} \right| \leq |\nu_2 - \nu_1| \cdot \left(\int_0^T \frac{1}{h_2(t)}\, \mathrm{d}t \right)^{-1} =: \rho,$$

for every $x \in W^{1,p}(I)$; setting $M := \sup_{[-\rho,\rho]} |\Phi|$, we then get (see also (2.4))

$$\begin{aligned} |\xi_x| &\leq |\xi_x + \mathcal{F}_x(t^*)| + |\mathcal{F}_x(t^*)| \\ &\leq \left| \Phi\left((\nu_2 - \nu_1) \cdot \left(\int_0^T \frac{1}{A_x(t)}\, \mathrm{d}t \right)^{-1} \right) \right| + \sup_{t \in I} |\mathcal{F}_x(t)| \\ &\leq M + \|\psi\|_{L^1} =: \mathbf{c}_0. \end{aligned}$$

Since $\mathbf{c}_0 > 0$ does not depend on x, this gives the desired (2.6). □

We now consider the operator $\mathcal{P} : W^{1,p}(I) \to W^{1,p}(I)$ defined by

$$\mathcal{P}_x(t) := \nu_1 + \int_0^t \frac{1}{A_x(s)}\, \Phi^{-1}\big(\xi_x + \mathcal{F}_x(s)\big)\, \mathrm{d}s, \tag{2.8}$$

where ξ_x is as in Lemma 2.6. We note that $\mathcal{P}$ is well-defined, in the sense that $\mathcal{P}_x \in W^{1,p}(I)$ for every $x \in W^{1,p}(I)$: indeed, assumption $(\mathrm{H2})_2$ and (2.4) give

$$\left| \frac{1}{A_x(s)}\, \Phi^{-1}\big(\xi_x + \mathcal{F}_x(s)\big) \right| \leq \frac{1}{h_1(t)}\, \Phi^{-1}\big(\xi_x + \|\psi\|_{L^1}\big);$$

thus, since $1/h_1 \in L^p(I)$, we conclude that $\mathcal{P}_x \in W^{1,p}(I)$, as claimed. Furthermore, it is easy to see that the solutions of the problem (2.1) (according to Definition 2.3) are precisely **the fixed points (in $W^{1,p}(I)$) of $\mathcal{P}$**.

In view of this fact, we can prove Theorem 2.5 by showing that $\mathcal{P}$ possesses at least one fixed point in $W^{1,p}(I)$; in its turn, the existence of a fixed point of $\mathcal{P}$ follows from Schauder's Fixed Point Theorem if we are able to show that

- $\mathcal{P}$ is bounded in $W^{1,p}(I)$;
- $\mathcal{P}$ is continuous from $W^{1,p}(I)$ into itself;
- $\mathcal{P}$ is *compact*.

These facts are proved in the next lemmas.

Lemma 2.7. *The operator $\mathcal{P}$ defined in* (2.8) *is* ***bounded*** *in $W^{1,p}(I)$, that is, there exists a universal constant $\mathbf{c}_1 > 0$ such that*

$$\|\mathcal{P}_x\|_{W^{1,p}} \leq \mathbf{c}_1 \quad \textit{for every } x \in W^{1,p}(I).$$

Proof. For every $x \in W^{1,p}(I)$, by combining estimates (2.4) and (2.6), we have

$$|\xi_x + \mathcal{F}_x(s)| \leq \mathbf{c}_0 + \|\psi\|_{L^1} =: \eta, \quad \text{for every } s \in I;$$

thus, if we set $\widehat{M} = \max_{[-\eta,\eta]} |\Phi^{-1}|$, we obtain (see assumption $(\mathrm{H2})_2$)

$$\left| \frac{1}{A_x(s)} \, \Phi^{-1}\big(\xi_x + \mathcal{F}_x(s)\big) \right| \leq \frac{M}{h_1(s)} \quad \text{for every } s \in I, \tag{2.9}$$

and the estimate holds *for every $x \in W^{1,p}(I)$*. With such an estimate at hand, we can easily prove the boundedness of $\mathcal{P}$: indeed, by (2.9) we have

$$\|\mathcal{P}_x'\|_{L^p} = \left(\int_0^T \left| \frac{1}{A_x(s)} \, \Phi^{-1}\big(\xi_x + \mathcal{F}_x(s)\big) \right|^p \mathrm{d}s \right)^{1/p} \leq M \, \|1/h_1\|_{L^p}$$

for every $x \in W^{1,p}(I)$; moreover, one has

$$\begin{aligned} \|\mathcal{P}_x\|_{L^p} &\leq \left\{ \int_0^T \left(|\nu_1| + \int_0^t \left| \frac{1}{A_x(s)} \, \Phi^{-1}\big(\xi_x + \mathcal{F}_x(s)\big) \right| \mathrm{d}s \right)^p \mathrm{d}t \right\}^{1/p} \\ &\leq T^{1/p} \left(|\nu_1| + M \, \|1/h_1\|_{L^1} \right), \end{aligned}$$

and again the estimate holds for every $x \in W^{1,p}(I)$. Summing up, if we introduce the constant (which does not depend on x)

$$\mathbf{c}_1 := T^{1/p} \left(|\nu_1| + M \, \|1/h_1\|_{L^1} \right) + M \, \|1/h_1\|_{L^p} > 0,$$

we conclude that, for every $x \in W^{1,p}(I)$, one has

$$\|\mathcal{P}_x\|_{W^{1,p}} = \|\mathcal{P}_x\|_{L^p} + \|\mathcal{P}_x'\|_{L^p} \leq \mathbf{c}_1.$$

This ends the proof. $\square$

Remark 2.8. It is contained in the proof of Lemma 2.7 the following fact, which we shall repeatedly use in the sequel: there exists a constant $M > 0$ such that

$$\max_{t \in I} \left| \Phi^{-1}\big(\xi_x + \mathcal{F}_x(t)\big) \right| \le M, \quad \text{for every } x \in W^{1,p}(I). \tag{2.10}$$

We also highlight that, since the injection $W^{1,p}(I) \subseteq C(I, \mathbb{R})$ is continuous, the boundedness of $\mathcal{P}$ in $W^{1,p}(I)$ implies the boundedness of $\mathcal{P}$ in $C(I, \mathbb{R})$: more precisely, there exists a real $\mathbf{c}_1' > 0$ such that

$$\sup_{t \in I} |\mathcal{P}_x(t)| \le \mathbf{c}_1', \quad \text{for every } x \in W^{1,p}(I). \tag{2.11}$$

We now turn to prove the continuity of $\mathcal{P}$.

Lemma 2.9. *The operator $\mathcal{P}$ defined in (2.8) is **continuous** on $W^{1,p}(I)$.*

Proof. Let $x_0 \in W^{1,p}(I)$ be fixed and let $\{x_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(I)$ be a sequence converging to x_0 as $n \to \infty$. We need to prove that $\mathcal{P}_{x_n} \to \mathcal{P}_{x_0}$ as $n \to \infty$.

To this end, we arbitrarily choose a sub-sequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ and we show that there exists a further sub-sequence $\{x_{n_{k_j}}\}_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} \mathcal{P}_{x_{n_{k_j}}} = \mathcal{P}_{x_0} \quad \text{in } W^{1,p}(I).$$

First of all, by (2.6), the sequence $\{\xi_{x_{n_k}}\}_{k \in \mathbb{N}}$ is bounded in $\mathbb{R}$; thus, there exist an increasing sequence $\{k_j\}_{j \in \mathbb{N}} \subseteq \mathbb{N}$ and a real $\xi_0 \in \mathbb{R}$ such that

$$\xi_j := \xi_{x_{n_{k_j}}} \to \xi_0 \quad \text{as } j \to \infty.$$

Moreover, since $\mathcal{F}$ is continuous from $W^{1,p}(I)$ to $C(I, \mathbb{R})$ (see Remark 2.2), we have $\mathcal{F}_j := \mathcal{F}_{x_{n_{k_j}}} \to \mathcal{F}_{x_0}$ uniformly on I as $j \to \infty$. Finally, since A is continuous *w.r.t. the uniform topology* of $C(I, \mathbb{R})$ (by assumption (H2)), one has

$$A_j := A_{x_{n_{k_j}}} \to A_{x_0} \text{ uniformly on } I \text{ as } j \to \infty.$$

Gathering together all these facts (and reminding that $\Phi \in C(\mathbb{R}, \mathbb{R})$), we get

$$\begin{aligned} \lim_{j \to \infty} \frac{1}{A_j(t)}\, \Phi^{-1}\big(\xi_j + \mathcal{F}_j(t)\big) \\ = \frac{1}{A_{x_0}(t)}\, \Phi^{-1}\big(\xi_0 + \mathcal{F}_{x_0}(t)\big) \quad \text{for a.e. } t \in I. \end{aligned} \tag{2.12}$$

From this, owing to estimate (2.10) and Remark 2.1, we infer that

$$\begin{aligned} \lim_{j \to \infty} \int_0^t \frac{1}{A_j(s)}\, \Phi^{-1}\big(\xi_j + \mathcal{F}_j(s)\big)\, \mathrm{d}s \\ = \int_0^t \frac{1}{A_{x_0}(s)}\, \Phi^{-1}\big(\xi_0 + \mathcal{F}_{x_0}(s)\big)\, \mathrm{d}s \quad \text{for every } t \in I. \end{aligned} \tag{2.13}$$

In particular, since we know from Lemma 2.6 that

$$\int_0^T \frac{1}{A_j(s)}\, \Phi^{-1}\big(\xi_j + \mathcal{F}_j(s)\big)\, \mathrm{d}s = \nu_2 - \nu_1 \quad \text{for every } j \in \mathbb{N},$$

identity (2.13) implies that

$$\int_0^T \frac{1}{A_{x_0}(s)}\,\Phi^{-1}\big(\xi_0 + \mathcal{F}_{x_0}(s)\big)\,\mathrm{d}s = \nu_2 - \nu_1;$$

thus, by the uniqueness property of ξ_x in Lemma 2.6, we get $\xi_0 = \xi_{x_0}$. As a consequence, by exploiting the very definition of $\mathcal{P}$ (see (2.8)), identity (2.13) allows us to conclude that $\mathcal{P}_{x_{n_{k_j}}} \to \mathcal{P}_{x_0}$ point-wise on I as $j \to \infty$.

To complete the proof of the lemma, we need to show that the sequence $\mathcal{P}_{x_{n_{k_j}}}$ actually converges to $\mathcal{P}_{x_0}$ in $W^{1,p}(I)$ as $j \to \infty$. To this end we first observe that, by exploiting estimate (2.10), for almost every $t \in I$ one has

$$\left|\frac{1}{A_j(t)}\,\Phi^{-1}\big(\xi_j + \mathcal{F}_j(t)\big) - \frac{1}{A_{x_0}(t)}\,\Phi^{-1}\big(\xi_{x_0} + \mathcal{F}_{x_0}(t)\big)\right|^p \le 2^p\,\frac{M^p}{h_1^p(t)};$$

as a consequence, since $1/h_1 \in L^p(I)$ (by assumption (H2)), a standard application of Lebesgue's Dominated Convergence Theorem gives (see also (2.12))

$$\begin{aligned}\lim_{j\to\infty} &\|\mathcal{P}'_{x_{n_{k_j}}} - \mathcal{P}'_{x_0}\|_{L^p}^p \\ &= \lim_{j\to\infty}\int_0^T \left|\frac{1}{A_j(t)}\,\Phi^{-1}\big(\xi_j + \mathcal{F}_j(t)\big) - \frac{1}{A_{x_0}(t)}\,\Phi^{-1}\big(\xi_{x_0} + \mathcal{F}_{x_0}(t)\big)\right|^p \mathrm{d}t = 0.\end{aligned}$$

On the other hand, since $\mathcal{P}$ is bounded in $C(I,\mathbb{R})$ (see Remark 2.8), one has

$$|\mathcal{P}_{x_{n_{k_j}}}(t) - \mathcal{P}_{x_0}(t)|^p \le 2^p\,\mathbf{c}_1' \text{ for every } t \in I;$$

thus, again by Lebesgue's Dominated Convergence Theorem, we get

$$\begin{aligned}\lim_{j\to\infty} &\|\mathcal{P}_{x_{n_{k_j}}} - \mathcal{P}_{x_0}\|_{L^p}^p \\ &= \lim_{j\to\infty}\int_0^T |\mathcal{P}_{x_{n_{k_j}}}(t) - \mathcal{P}_{x_0}(t)|^p\,\mathrm{d}t = 0.\end{aligned}$$

Gathering together these facts, we conclude that $\|\mathcal{P}_{x_{n_{k_j}}} - \mathcal{P}_{x_0}\|_{W^{1,p}} \to 0$ as $j \to \infty$, and this finally completes the demonstration of the lemma. □

Finally, we prove that $\mathcal{P}$ is compact.

Lemma 2.10. *The operator $\mathcal{P}$ defined in* (2.8) *is* ***compact*** *on $W^{1,p}(I)$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(I)$ be bounded. We need to prove that the sequence $\{\mathcal{P}_{x_n}\}_{n\in\mathbb{N}}$ possesses a sub-sequence which is convergent to some $y_0 \in W^{1,p}(I)$.

Fist of all, since the sequence $\{\xi_{x_n}\}_{n\in\mathbb{R}}$ is bounded in $\mathbb{R}$ (see (2.6)), there exist a real ξ_0 and a sub-sequence of $\{x_n\}_{n\in\mathbb{N}}$, denoted again by $\{x_n\}_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} \xi_{x_n} = \xi_0 \qquad \text{and} \qquad |\xi_0| \le \mathbf{c}_0. \tag{2.14}$$

Moreover, since $\{x_n\}_{n\in\mathbb{N}}$ is *bounded* in $W^{1,p}(I)$ and $p > 1$, there exist a suitable function $x_0 \in W^{1,p}(I)$ and another sub-sequence of $\{x_n\}_{n\in\mathbb{N}}$, which we still denote by $\{x_n\}_{n\in\mathbb{N}}$, such that $x_n \to x_0$ uniformly on I as $n \to \infty$.

As a consequence, since the operator A is continuous with respect to the uniform topology of $C(I, \mathbb{R})$ (by assumption (H2)), we have

$$A_{x_n} \to A_{x_0} \text{ uniformly on } I \text{ as } j \to \infty. \tag{2.15}$$

We now observe that, by assumption (H3), we have the estimate

$$F_{x_n}(t) \le \psi(t), \text{ holding true for a.e. } t \in I \text{ and every } n \in \mathbb{N};$$

thus, the sequence $\{F_{x_n}\}_{n\in\mathbb{N}}$ is *bounded and equi-integrable* in L^1. Owing to the Dunford-Pettis Theorem, we infer the existence of a function $g \in L^1(I)$ and of another sub-sequence of $\{x_n\}_{n\in\mathbb{N}}$, denoted once again by $\{x_n\}_{n\in\mathbb{N}}$, such that

($\star$) $\lim_{n\to\infty} \int_0^T F_{x_n}(s)\, v(s)\, ds = \int_0^T g(s)\, v(s)\, \mathrm{d}s \quad$ for every $v \in L^\infty(I)$;

($\star$) $\|g\|_{L^1} \le \|\psi\|_{L^1}$.

Choosing, in particular, v as the indicator function of $[0, t]$ (with $t \in I$), we get

$$\begin{gathered} \mathcal{F}_{x_n}(t) \to \mathcal{G}(t) := \int_0^t g(s)\, \mathrm{d}s \quad \text{for every } t \in I \\ \text{and} \quad \sup_{t\in I} |\mathcal{G}(t)| \le \|\psi\|_{L^1}. \end{gathered} \tag{2.16}$$

Gathering together (2.14), (2.15) and (2.16), we deduce that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{A_{x_n}(t)}\, \Phi^{-1}\big(\xi_{x_n} + \mathcal{F}_{x_n}(t)\big) \\ &\qquad = \frac{1}{A_{x_0}(t)}\, \Phi^{-1}\big(\xi_0 + \mathcal{G}(t)\big) \quad \text{for a.e. } t \in I. \end{aligned} \tag{2.17}$$

From this, owing to (2.14), (2.16) and Remark 2.1, we conclude that

$$\left| \frac{1}{A_{x_0}(t)}\, \Phi^{-1}\big(\xi_0 + \mathcal{G}(t)\big) \right| \le \frac{M}{h_1(t)} \in L^p(I) \quad \text{for a.e. } t \in I \tag{2.18}$$

and that, for every $t \in I$, one has

$$\begin{aligned} \lim_{n\to\infty} \mathcal{P}_{x_n}(t) &= \lim_{n\to\infty} \left\{ \nu_1 + \int_0^t \frac{1}{A(x_n)(s)}\, \Phi^{-1}\big(\xi_{x_n} + \mathcal{F}_{x_n}(s)\big) \right\} \\ &= \nu_1 + \int_0^t \frac{1}{A_{x_0}(s)}\, \Phi^{-1}\big(\xi_0 + \mathcal{G}(s)\big) =: y_0(t) \quad \text{for every } t \in I. \end{aligned}$$

To complete the proof of the lemma, we need to show that the sequence $\{\mathcal{P}_{x_n}\}_n$ actually converges to y_0 *in* $W^{1,p}(I)$ as $n \to \infty$.

On the one hand, by using estimate (2.18) and by arguing exactly as in the proof of Lemma 2.9, we easily recognize that

$$\begin{aligned} &\lim_{n\to\infty} \|\mathcal{P}'_{x_n} - y'_0\|^p_{L^p} = \\ &= \lim_{n\to\infty} \int_0^T \left| \frac{1}{A_{x_n}(t)}\, \Phi^{-1}\big(\xi_{x_n} + \mathcal{F}_{x_n}(t)\big) - \frac{1}{A_{x_0}(t)}\, \Phi^{-1}\big(\xi_0 + \mathcal{G}(t)\big) \right|^p \mathrm{d}t = 0. \end{aligned}$$

On the other hand, since $\mathcal{P}_{x_n} \to y_0$ point-wise on I, we deduce from (2.11) that

$$|y_0(t)| \leq \mathbf{c}_1' \text{ for every } t \in I;$$

hence, by arguing once again as in the proof of Lemma 2.9, we conclude that

$$\begin{aligned} &\lim_{n\to\infty} \|\mathcal{P}_{x_n} - y_0\|_{L^p}^p \\ &\qquad = \lim_{n\to\infty} \int_0^T |\mathcal{P}_{x_n}(t) - y_0(t)|^p \,\mathrm{d}t = 0. \end{aligned}$$

Summing up, $\mathcal{P}_{x_n} \to y_0$ in $W^{1,p}(I)$ as $n \to \infty$, and the proof is complete. □

Gathering together Lemmas 2.7, 2.9 and 2.10, we can prove Theorem 2.5.

Proof (of Theorem 2.5). We have already pointed out that a function $x \in W^{1,p}(I)$ is a solution of the boundary value problem (2.1) *if and only if* x is a fixed point of the operator $\mathcal{P}$ defined in (2.8). On the other hand, since $\mathcal{P}$ is bounded, continuous and compact on the Banach space $W^{1,p}(I)$, the Schauder Fixed Point Theorem ensures the existence of (at least) one $x \in W^{1,p}(I)$ such that $\mathcal{P}_x = x$, and thus the problem (2.1) possesses at least one solution. This ends the proof. □

3 The Dirichlet problem for singular ODEs

In this section, we exploit the existence result in Theorem 2.5 in order to prove the solvability of boundary value problems of the following type

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), \quad \text{a.e. on } I, \\ x(0) = \nu_1, \ x(T) = \nu_2. \end{cases} \tag{3.1}$$

As in Section 2, $I = [0,T]$ (for some real $T > 0$) and $\nu_1, \nu_2 \in \mathbb{R}$; furthermore, the functions Φ, a and f satisfy the following *structural assumptions*:

(A1) $\Phi : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing homeomorphism*;

(A2) $a \in C(I \times \mathbb{R}, \mathbb{R})$ and there exists $h_1 \in C(I, \mathbb{R})$ such that

- $(\text{A2})_1$ $h_1 \geq 0$ on I and $1/h_1 \in L^p(I)$;
- $(\text{A2})_2$ $a(t,x) \geq h_1(t)$ for every $t \in I$ and every $x \in \mathbb{R}$;

(A3) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a *Carathéodory function*, that is,

- $(*)$ the map $t \mapsto f(t,x,y)$ is measurable on I, for every $(x,y) \in \mathbb{R}^2$;
- $(*)$ the map $(x,y) \mapsto f(t,x,y)$ is continuous on $\mathbb{R}^2$, for almost every $t \in I$.

Remark 3.1. As in Section 2 we point out that, as a consequence of assumptions $(\text{A2})_1$ and $(\text{A2})_2$, for every $(t,x) \in I \times \mathbb{R}$ one has $a(t,x) \geq h_1(t) \geq 0$ and

$$0 \leq \int_0^T \frac{1}{a(t,x(t))}\,\mathrm{d}t \leq \int_0^T \frac{1}{h_1(t)}\,\mathrm{d}t, \quad \text{for every } x \in W^{1,p}(I).$$

In particular, $t \mapsto a(t,x(t)) \in L^p(I)$ *for every* $x \in W^{1,p}(I)$.

We now give the definition of *solution* of the problem (3.1).

Definition 3.2. We say that a continuous function $x \in C(I, \mathbb{R})$ is a **solution** of the Dirichlet problem (3.1) if it satisfies the following properties:

(1) $x \in W^{1,p}(I)$ and $t \mapsto \Phi\big(a(t, x(t))\, x'(t)\big) \in W^{1,1}(I)$;

(2) $\Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big)' = f(t, x(t), x'(t))$ for almost every $t \in I$;

(3) $x(0) = \nu_1$ and $x(T) = \nu_2$.

If x fulfills only (1) and (2), we say that x is a **solution of the ODE**

$$\Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big)' = f(t, x(t), x'(t)). \tag{3.2}$$

In order to clearly state the main result of this section, we also need to introduce the definition of *upper solution* and *lower solution* of the equation in (3.1).

Definition 3.3. We say that a continuous function $\alpha \in C(I, \mathbb{R})$ is a **lower** [resp. **upper**] **solution** of the differential equation (3.2) if

(1) $\alpha \in W^{1,p}(I)$ and $t \mapsto \Phi\big(a(t, \alpha(t))\, \alpha'(t)\big) \in W^{1,1}(I)$;

(2) $\Big(\Phi\big(a(t, \alpha(t))\, \alpha'(t)\big)\Big)' \geq [\leq]\ f(t, \alpha(t), \alpha'(t))$ for almost every $t \in I$.

Remark 3.4. According to Remark 2.4, if $x \in W^{1,p}(I)$ is a solution of the problem (3.1), we denote by $\mathcal{A}_x$ the unique continuous function on I such that

$$\mathcal{A}_x(t) = a(t, x(t))\, x'(t) \quad \text{for a.e. } t \in I.$$

Notice that, as a consequence of condition (1) in Definition 3.2 (and again of the fact that Φ is a homeomorphism, see assumption (H1)), such a function exists.

Analogously, if $\alpha \in W^{1,p}(I)$ is a lower/upper solution of (3.2), we denote by $\mathcal{A}_\alpha$ the unique continuous function on I such that

$$\mathcal{A}_\alpha(t) = a(t, \alpha(t))\, \alpha'(t) \quad \text{for a.e. } t \in I.$$

The existence of such a function follows from (1) in Definition 3.3.

We are ready to state our main existence result.

Theorem 3.5. *Let us assume that, together with the structural assumptions* (A1)-*to*-(A3)*, the following additional hypotheses are satisfied:*

(A1') *there exists a pair of lower and upper solutions $\alpha, \beta \in W^{1,p}(I)$ of the differential equation* (3.2) *such that $\alpha(t) \leq \beta(t)$ for every $t \in I$;*

(A2') *for every $R > 0$ and every non-negative function $\gamma \in L^p(I)$ there exists a non-negative function $h = h_{R,\gamma} \in L^p(I)$ such that*

$$|f(t, x, y(t))| \leq h_{R,\gamma}(t) \tag{3.3}$$

for a.e. $t \in I$, every $|x| \leq R$ and every $y \in L^p(I)$ with $|y(t)| \leq \gamma(t)$ a.e. on I.

(A3') *there exist a constant* $H > 0$*, a non-negative function* $\mu \in L^q(I)$ *(for some* $1 < q \leq \infty$*), a non-negative function* $l \in L^1(I)$ *and a non-negative measurable function* $\psi : (0,\infty) \to (0,\infty)$ *such that*

$$(\star)\ 1/\psi \in L^1_{\mathrm{loc}}(0,\infty) \quad \textit{and} \quad \int_1^\infty \frac{1}{\psi(t)}\,\mathrm{d}t = \infty; \tag{3.4}$$

$$(\star)\ |f(t,x,y)| \leq \psi\Big(|\Phi(a(t,x)\,y)|\Big) \cdot \Big(l(t) + \mu(t)\,|y|^{\frac{q-1}{q}}\Big); \tag{3.5}$$

for a.e. $t \in I$*, every* $x \in [\alpha(t), \beta(t)]$ *and every* $y \in \mathbb{R}$ *with* $|y| \geq H$*.*

Then, for every $\nu_1 \in [\alpha(0), \beta(0)]$ *and every* $\nu_2 \in [\alpha(T), \beta(T)]$*, the (singular) Dirichlet problem* (3.1) *possesses at least one solution* $x \in W^{1,p}(I)$ *satisfying*

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \textit{for every } t \in I. \tag{3.6}$$

Furthermore, if $M > 0$ *is* any *real number such that* $\sup_I |\alpha|, \sup_I |\beta| \leq M$*, it is possible to find a real* $L_0 > 0$*,* only depending on M*, with the following property:* if $L \geq L_0$ is any real number such that $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \leq L$, then

$$\max_{t\in I} \big|x(t)\big| \leq M \quad \text{and} \quad \max_{t\in I} \big|\mathcal{A}_x(t)\big| \leq L. \tag{3.7}$$

The main idea behind the proof of Theorem 3.5 is to think of the Dirichlet problem (3.1) as a *particular case* of an abstract boundary value problem of the form (2.1), and then to apply the existence result contained in Theorem 2.5.

Unfortunately, we *cannot* directly apply our Theorem 2.5 to the problem (3.1): in fact, in general, we cannot expect the (well-defined) functional

$$W^{1,p}(I) \ni x \mapsto F_x := f(t, x(t), x'(t)) \in L^1(I)$$

to satisfy assumption (H3) (or, more precisely, estimate (2.2)).

Thus, following an approach similar to that in [9, 15], we introduce a suitable *truncated version* of problem (3.1), to which Theorem 3.5 can apply.

To this end, to simplify the notation, we first fix some relevant constants we shall need for the proof of Theorem 3.5; henceforth, we suppose that all the assumption in the statement of the cited Theorem 3.5 are satisfied.

Let $M > 0$ be any real number such that $\sup_I |\alpha|, \sup_I |\beta| \leq M$ and let $H > 0$ be the constant appearing in assumption (A3'); moreover, we define

$$a_0 := \max\big\{a(t,x) \,:\, (t,x) \in I \times [-M, M]\big\} \tag{3.8}$$

We choose a real $N > 0$ such that

$$N > \max\left\{H, \frac{2M}{T}\right\} \cdot a_0 \quad \text{and} \quad \Phi(N) \cdot \Phi(-N) < 0 \tag{3.9}$$

and, accordingly, we fix $L_0 > 0$ in such a way that (see (3.4))

$$\begin{aligned} \min\left\{ \int_{\Phi(N)}^{\Phi(L_0)} \frac{1}{\psi(s)}\,\mathrm{d}s, \int_{-\Phi(-N)}^{-\Phi(-L_0)} \frac{1}{\psi(s)}\,\mathrm{d}s \right\} &= 1 + \|l\|_{L^1} + \|\mu\|_{L^q}\,(2M)^{\frac{q-1}{q}} \\ &> \|l\|_{L^1} + \|\mu\|_{L^q}\,(2M)^{\frac{q-1}{q}}. \end{aligned} \tag{3.10}$$

Notice that L_0 depends on the constant M, but not on α, β nor on ν_1 and ν_2.

Finally, we let $L \geq L_0$ be any real number such that

$$L \geq \max\Big\{\sup_{t\in I}|\mathcal{A}_\alpha(t)|, \sup_{t\in I}|\mathcal{A}_\beta(t)|\Big\}. \tag{3.11}$$

Setting $\gamma_0 := L/h_1 \in L^p(I)$, we then consider the following *truncating operators*:

$$\mathcal{T}: W^{1,p}(I) \longrightarrow W^{1,p}(I), \qquad \mathcal{T}(x)(t) := \begin{cases} \alpha(t), & \text{if } x(t) < \alpha(t); \\ x(t), & \text{if } x(t) \in [\alpha(t), \beta(t)]; \\ \beta(t), & \text{if } x(t) > \beta(t); \end{cases}$$

$$\mathcal{D}: L^p(I) \longrightarrow L^p(I), \qquad \mathcal{D}(z)(t) := \begin{cases} -\gamma_0(t), & \text{if } z(t) < -\gamma_0(t); \\ z(t), & \text{if } |z(t)| \leq \gamma_0(t); \\ \gamma_0(t), & \text{if } z(t) > \gamma_0(t). \end{cases}$$

We also consider the truncated function $f^*: I \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f^*(t,x,y) := \begin{cases} f\big(t,\beta(t),\beta'(t)\big) + \arctan\big(x(t)-\beta(t)\big), & \text{if } x(t) > \beta(t); \\ f(t,x,y), & \text{if } x(t) \in [\alpha(t),\beta(t)]; \\ f\big(t,\alpha(t),\alpha'(t)\big) + \arctan\big(x(t)-\alpha(t)\big), & \text{if } x(t) < \alpha(t). \end{cases}$$

By means of the function f^* and of the operators $\mathcal{T}$ and $\mathcal{D}$, we are finally in a position to introduce a "truncated version" of the Dirichlet problem (3.1):

$$\begin{cases} \Big(\Phi\Big(a\big(t,\mathcal{T}(x)(t)\big)\,x'(t)\Big)\Big)' = f^*\Big(t, x(t), \mathcal{D}\big(\mathcal{T}(x)'(t)\big)\Big), & \text{a.e. on } I, \\ x(0) = \nu_1,\ x(T) = \nu_2. \end{cases} \tag{3.12}$$

The next proposition shows that the "abstract" existence result in Theorem 2.5 does apply to the "truncated" Dirichlet problem (3.12).

Proposition 3.6. *Let the above assumptions and notation apply. Then, there exists (at least) one solution $x \in W^{1,p}(I)$ of the Dirichlet problem* (3.12).

Proof. We consider the operators A and F defined as follows:

$$\begin{aligned} &A: W^{1,p}(I) \longrightarrow C(I,\mathbb{R}), \qquad A_x(t) := a\big(t, \mathcal{T}(x)(t)\big), \\ &F: W^{1,p}(I) \longrightarrow L^1(I), \qquad F_x(t) := f^*\Big(t, x(t), \mathcal{D}\big(\mathcal{T}(x)'(t)\big)\Big) \end{aligned}$$

By means of these operators, the problem (3.12) can be re-written as

$$\begin{cases} \Big(\Phi\big(A_x(t)\,x'(t)\big)\Big)' = F_x(t), & \text{a.e. on } I, \\ x(0) = \nu_1,\ x(T) = \nu_2. \end{cases}$$

We claim that A and F satisfy assumptions (H2) and (H3) in Theorem 2.5.

First of all, since $\mathcal{T}$ is continuous with respect to the uniform topology of $C(I,\mathbb{R})$ (as is very easy to see) and since, by the choice of M, we have

$$-M \leq \alpha(t) \leq \mathcal{T}(x)(t) \leq \beta(t) \leq M \quad \text{for every } t \in I,$$

the uniform continuity of a on $I \times [-M, M]$ implies that A is continuous from $W^{1,p}(I)$ (as a subspace of $C(I,\mathbb{R})$) to $C(I,\mathbb{R})$. Moreover, by (3.8) one has

$$A_x(t) = a\big(t, \mathcal{T}(x)(t)\big) \leq a_0 \quad \text{for every } x \in W^{1,p}(I) \text{ and every } t \in I.$$

Finally, by assumption (A2), there exists $h_1 \in C(I,\mathbb{R})$ such that

($\star$) $h_1 \geq 0$ and $1/h_1 \in L^p(I)$;

($\star$) $A_x(t) \geq h_1(t)$ for every $x \in W^{1,p}(I)$ and every $t \in I$.

Thus, the operator A satisfies (H2) in Theorem 2.5 (with $h_2(t) \equiv a_0$).

As for the functional F, by arguing exactly as in [7, Theorem 3.1], one can recognize that it is continuous from $W^{1,p}(I)$ to $L^1(I)$ and that

$$|F_x(t)| \leq \Theta(t) := \max\Big\{h_{M,\gamma_0}(t), |f(t,\alpha(t),\alpha'(t)|, |f(t,\beta(t),\beta'(t)|\Big\} + \frac{\pi}{2}$$

for every $x \in W^{1,p}(I)$ and almost every $t \in I$ (here, γ_{M,γ_0} is the function appearing in assumption (A2') and corresponding to M and $\gamma_0 = L/h_1$). Since, obviously, $\Theta \in L^1(I)$, we conclude that F satisfies assumption (H3) in Theorem 2.5.

Gathering together all these facts, we are allowed to apply Theorem 2.5 to problem (3.12), which therefore admits (at least) one solution $x \in W^{1,p}(I)$. $\square$

We now turn to prove that any solution of (3.12) actually solves (3.1).

Proposition 3.7. *Let the above assumptions and notation apply. If $x \in W^{1,p}(I)$ is any solution of the truncated problem* (3.12)*, the following facts hold:*

(i) $\alpha(t) \leq x(t) \leq \beta(t)$ *for every* $t \in I$*;*

(ii) $\sup_I |x| \leq M$*;*

(iii) $\big|\mathcal{A}_x(t)\big| \leq L$ *for every* $t \in I$*;*

(iv) $|x'(t)| \leq L/h_1(t) = \gamma_0(t)$ *for a.e.* $t \in I$*.*

Proof. Let $x \in W^{1,p}(I)$ be any solution of the truncated problem (3.12). According to Remark 2.4, we denote by $\mathcal{A}_x$ the unique continuous function on I such that

$$\mathcal{A}_x(t) = A(x(t))\, x'(t) = a\big(x, \mathcal{T}(x)(t)\big)\, x'(t) \quad \text{for a.e. } t \in I.$$

Once we have proved that $x(t) \in [\alpha(t), \beta(t)]$ for all $t \in I$, we shall obtain

$$\mathcal{A}_x(t) = a(t, x(t))\, x'(t) \quad \text{for a.e. } t \in I.$$

Let us then turn to prove statements (i)-to-(iii).

(i) Let us assume, by contradiction, that $x(\bar{t}) \notin [\alpha(\bar{t}), \beta(\bar{t})]$ for some $\bar{t} \in I$; moreover, to fix ideas, let us suppose that $x(\bar{t}) < \alpha(\bar{t})$.

Since, by assumptions, $\nu_1 = x(0) \geq \alpha(0)$ and $\nu_2 = x(T) \geq \alpha(T)$, it is possible to find suitable points $t_1, t_2, \theta \in I$, with $t_1 < \theta < t_2$, such that

(a) $x(\theta) - \alpha(\theta) = \min_{t \in I} \big(x(t) - \alpha(t)\big) < 0$;

(b) $x(t_1) - \alpha(t_1) = x(t_2) - \alpha(t_2) = 0$ and $x(t) < \alpha(t)$ for every $t \in (t_1, t_2)$.

In particular, from (b) we infer that $\mathcal{T}(x) \equiv \alpha$ on (t_1, t_2) and that

$$\begin{aligned} f^*\Big(t, x(t), \mathcal{D}\big(\mathcal{T}(x)'(t)\big)\Big) &= f(t, \alpha(t), \alpha'(t)) + \arctan\big(x(t) - \alpha(t)\big) \\ &< f(t, \alpha(t), \alpha'(t)), \qquad \text{for a.e. } t \in (t_1, t_2). \end{aligned}$$

As a consequence, since x solves the Dirichlet problem (3.12) and α is a lower solution of the ODE (3.2), for almost every $t \in (t_1, t_2)$ we obtain

$$\begin{aligned} \big(\Phi(\mathcal{A}_x(t))\big)' &= \Big(\Phi\Big(a\big(t, \mathcal{T}(x)(t)\big)\, x'(t)\Big)\Big)' \\ &= f^*\Big(t, x(t), \mathcal{D}\big(\mathcal{T}(x)'(t)\big)\Big) < f(t, \alpha(t), \alpha'(t)) \\ &\le \Big(\Phi\Big(a\big(t, \alpha(t)\big)\, \alpha'(t)\Big)\Big)' = \big(\Phi(\mathcal{A}_\alpha(t))\big)'. \end{aligned} \tag{3.13}$$

We now introduce the subsets I_1, I_2 of I defined as follows:

$$I_1 := \{t \in (t_1, \theta) : x'(t) < \alpha'(t)\} \quad \text{and} \quad I_2 := \{t \in (\theta, t_2) : x'(t) > \alpha'(t)\}.$$

Since $x < \alpha$ on (t_1, t_2), it is readily seen that both I_1 and I_2 must have positive measure; thus, it is possible to find $\tau_1 \in I_1$ and $\tau_2 \in I_2$ such that

($\star$) $0 < h_1(\tau_i) \le a(\tau_i, \alpha(\tau_i))$ for $i = 1, 2$;

($\star$) $\mathcal{A}_\alpha(\tau_i) = a(\tau_i, \alpha(\tau_i))\, \alpha'(\tau_i)$ for $i = 1, 2$ (see Remark 3.4);

($\star$) $\mathcal{A}_x(\tau_i) = a(\tau_i, \mathcal{T}(x)(\tau_i))\, x'(\tau_i) = a(\tau_i, \alpha(\tau_i))\, x'(\tau_i)$ for $i = 1, 2$.

From this, by integrating both sides of inequality (3.13) on $[\tau_1, \theta]$, we get

$$\Phi\big(\mathcal{A}_x(\theta)\big) - \Phi\Big(a\big(\tau_1, \alpha(\tau_1)\big)\, x'(\tau_1)\Big) \le \Phi\big(\mathcal{A}_\alpha(\theta)\big) - \Phi\Big(a\big(\tau_1, \alpha(\tau_1)\big)\, \alpha'(\tau_1)\Big);$$

hence, by the choice of τ_1 and the fact that Φ is strictly increasing, one has

$$\Phi\big(\mathcal{A}_x(\theta)\big) - \Phi\big(\mathcal{A}_\alpha(\theta)\big) < 0. \tag{3.14}$$

On the other hand, if we integrate both sides of (3.13) on $[\theta, \tau_2]$ we get

$$\Phi\Big(a\big(\tau_2, \alpha(\tau_2)\big)\, x'(\tau_2)\Big) - \Phi\big(\mathcal{A}_x(\theta)\big) \le \Phi\Big(a\big(\tau_2, \alpha(\tau_2)\big)\, \alpha'(\tau_2)\Big) - \Phi\big(\mathcal{A}_\alpha(\theta)\big)$$

and thus, by the choice of τ_2 and again the strict monotonicity of Φ, we obtain

$$\Phi\big(\mathcal{A}_x(\theta)\big) - \Phi\big(\mathcal{A}_\alpha(\theta)\big) > 0,$$

This is clearly in contradiction with (3.14), hence $x \ge \alpha$ on I. By arguing analogously one can also prove that $x \le \beta$ on I, and statement (i) is established.

(ii) By statement (i) and the choice of M, we immediately get

$$-M \le \alpha(t) \le x(t) \le \beta(t) \le M \quad \text{for every } t \in I.$$

(iii) We split the proof of this statement into two steps.

Step I: We begin by showing that, if $N > 0$ is as in (3.9), then

$$\min_{t\in I} \big|\mathcal{A}_x(t)\big| \leq N. \tag{3.15}$$

We argue again by contradiction and, to fix ideas, we assume that

$$\mathcal{A}_x(t) = a\big(t, x(t)\big)\, x'(t) > N \quad \text{for a.e. } t \in I.$$

By integrating both sides of this inequality on $[0, T]$, we get

$$\int_0^T \mathcal{A}_x(t)\,\mathrm{d}t = \int_0^T a(t, x(t))\, x'(t)\,\mathrm{d}t > NT;$$

from this, by statement (ii), (3.8) and the choice of N (see (3.9)), we obtain

$$\begin{aligned} NT &< \int_0^T a(t, x(t))\, x'(t)\,\mathrm{d}t \leq a_0 \cdot \int_0^T x'(t)\,\mathrm{d}t \\ &= (\nu_2 - \nu_1)\cdot a_0 = |\nu_2 - \nu_1| \cdot a_0 \leq (2M)\cdot a_0 < NT. \end{aligned}$$

This is clearly a contradiction, hence $\mathcal{A}_x \leq N$ on I. By arguing analogously one can also prove that $\mathcal{A}_x \geq -N$ on I, and (3.15) is established.

Step II: We now turn to prove statement (iii). To this end, arguing once again by contradiction, we assume that there exists $\bar{t} \in I$ such that

$$\big|\mathcal{A}_x(\bar{t})\big| > L;$$

moreover, to fix ideas, we suppose that $\mathcal{A}_x(\bar{t}) > L$.

Since, by definition, $L > N$, from Step I and Remark 3.4 we infer the existence of two points $t_1, t_2 \in I$, with $t_1 < t_2$, such that (for example)

(∗) $\mathcal{A}_x(t_1) = N$ and $\mathcal{A}_x(t_2) = L$;

(∗∗) $0 < N < \mathcal{A}_x(t) < L$ for every $t \in (t_1, t_2)$;

from this, by statement (ii), (3.8) and the choice of N (see (3.9)), we obtain

$$0 < H < \frac{N}{a_0} < x'(t) < \frac{L}{h_1(t)} = \gamma_0(t) \quad \text{for a.e. } t \in (t_1, t_2). \tag{3.16}$$

Now, by definition of $\mathcal{D}$, we deduce from (3.16) that $\mathcal{D}(x') = x'$ a.e. on (t_1, t_2); moreover, by statement (i) and the definition of f^*, we have

$$f^*\Big(t, x(t), \mathcal{D}\big(\mathcal{T}(x)'(t)\big)\Big) = f(t, x(t), x'(t)) \quad \text{for a.e. } t \in (t_1, t_2).$$

As a consequence, since $x(t) \in [\alpha(t), \beta(t)]$ for every $t \in I$ (by statement (i)) and since $x'(t) > H$ for a.e. $t \in (t_1, t_2)$ (again by (3.16)), we are entitled to apply estimate (3.5), which gives (remind that x solves (3.12) and see (∗∗))

$$\begin{aligned} \left|\Big(\Phi\big(\mathcal{A}_x(t)\big)\Big)'\right| &= \left|\Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big)'\right| = \big|f(t, x(t), x'(t))\big| \\ &\leq \psi\Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big) \cdot \Big(l(t) + \mu(t)\,(x'(t))^{\frac{q-1}{q}}\Big) \\ &= \psi\Big(\Phi\big(\mathcal{A}_x(t)\big)\Big) \cdot \Big(l(t) + \mu(t)\,(x'(t))^{\frac{q-1}{q}}\Big) \qquad \big(\text{a.e. on } (t_1, t_2)\big). \end{aligned}$$

In particular, by exploiting this inequality, we obtain (remind that $L \geq L_0$)

$$\int_{\Phi(N)}^{\Phi(L_0)} \frac{1}{\psi(s)}\, ds \leq \int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(s)}\, \mathrm{d}s = \int_{\Phi(\mathcal{A}_x(t_1))}^{\Phi(\mathcal{A}_x(t_2))} \frac{1}{\psi(s)}\, \mathrm{d}s$$

$$\leq \int_{t_0}^{t_1} \frac{\big(\Phi\big(\mathcal{A}_x(t)\big)\big)'}{\psi\big(\Phi\big(\mathcal{A}_x(t)\big)\big)}\, \mathrm{d}t \leq \int_{t_0}^{t_1} \Big(l(t) + \mu(t)\,(x'(t))^{\frac{q-1}{q}}\Big)\, \mathrm{d}t$$

$\big($by Hölder's inequality$\big)$

$$\leq \|l\|_{L^1} + \|\mu\|_{L^q} \cdot \left(\int_{t_0}^{t_1} x'(t)\, \mathrm{d}t\right)^{\frac{q-1}{q}}$$

$$\leq \|l\|_{L^1} + \|\mu\|_{L^q} \cdot \big(x(t_1) - x(t_0)\big)^{\frac{q-1}{q}}$$

$\big($by statement (ii)$\big)$

$$\leq \|l\|_{L^1} + \|\mu\|_{L^q} \cdot (2M)^{\frac{q-1}{q}}.$$

This is in contradiction with the choice of L_0 (see (3.10)), hence $\mathcal{A}_x \leq L$ on I. Analogously, one can show that $\mathcal{A}_x \geq -L$ on I and statement (iii) is proved.

(iv) From statement (iii) and assumption (A2) we immediately infer that

$$|x'(t)| \leq \frac{L}{a(t, x(t))} \leq \frac{L}{h_1(t)} \quad \text{for almost every } t \in I,$$

and the proof is finally complete. □

By combining Propositions 3.6 and 3.7, we can finally prove Theorem 3.5.

Proof (of Theorem 3.5). First of all, by Proposition 3.6, there exists (at least) one solution $x \in W^{1,p}(I)$ of the "truncated" Dirichlet problem (3.12); moreover, by statements (i) and (iii) of Proposition 3.7 (and the very definitions of the operators $\mathcal{T}$ and $\mathcal{D}$), for almost every $t \in I$ we obtain

$$\Big(\Phi\Big(a\big(t, x(t)\big)\, x'(t)\Big)\Big)' = \Big(\Phi\Big(a\big(t, \mathcal{T}(x)(t)\big)\, x'(t)\Big)\Big)'$$

$$= f^*\Big(t, x(t), \mathcal{D}\big(\mathcal{T}(x)'(t)\big)\Big) = f(t, x(t), x'(t)).$$

Thus, x is actually a solution of the Dirichlet problem (3.1). To complete the demonstration of the theorem, we show that x satisfies (3.6) and (3.7).

As for (3.6), it is precisely statement (i) of Proposition 3.7; estimate (3.7), instead, follows from statements (ii) and (iii) of the same proposition. □

Some examples. We close the section with a few illustrating examples, in which we consider a generic function $a(t, x)$ satisfying assumption (A2). We explicitly point out that (A2) is verified, for example, in the following special cases:

(1.) when $a(t,x)$ has a product structure

$$a(t,x) = h(t) \cdot b(x),$$

where $h : I \to \mathbb{R}$ is a continuous non-negative function on I such that $1/h \in L^p(I)$ (for a certain $p > 1$) and $b \in C(\mathbb{R})$ is such that $\inf_{\mathbb{R}} b > 0$;

(2.) when $a(t,x)$ is a sum

$$a(t,x) = h(t) + b(x),$$

where $h : I \to \mathbb{R}$ is a continuous non-negative function on I such that $1/h \in L^p(I)$ (for a certain $p > 1$) and $b \in C(\mathbb{R},\mathbb{R})$ is non-negative on $\mathbb{R}$.

In the next Example 3.8, the growth of the right-hand side f with respect the variable y is linear, and this allows the choice $\psi \equiv 1$ in the Wintner-Nagumo condition (3.5). Thus, condition (3.5) does not require any relation among the differential operator Φ, the function a appearing inside Φ and the function f.

Example 3.8. Let us consider the Dirichlet problem

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = \sigma(t)(x(t)+\rho(t)) + g(x(t))\,x'(t) \\ x(0) = \nu_1,\ x(T) = \nu_2, \end{cases} \tag{3.17}$$

where φ, a, σ, ρ and g satisfy the following assumptions:

($\star$) $\Phi : \mathbb{R} \to \mathbb{R}$ is a generic strictly increasing homeomorphism;

($\star$) $a \in C(I \times \mathbb{R}, \mathbb{R})$ satisfies assumption (A2);

($\star$) $\sigma \in L^1(I)$ and $\sigma \geq 0$ a.e. on I;

($\star$) $\rho \in C(I)$ and $g \in C(\mathbb{R},\mathbb{R})$ are generic.

We aim to show that our Theorem 3.5 can be applied to problem (3.17). To this end, we consider the function f defined as follows:

$$f : I \times \mathbb{R}^2 \to \mathbb{R}, \qquad f(t,x,y) := \sigma(t)(x+\rho(t)) + g(x)y.$$

Obviously, f is a Carathéodory function; moreover, it is very easy to recognize that f satisfies assumption (A2)'. Indeed, let $R > 0$ be arbitrarily fixed and let γ be a non-negative function in $L^1(I)$; setting $M_R := \max_{[-R,R]} |g|$, we then have

$$\big|f(t,x,y(t))\big| \leq \sigma(t)\,\big(R + |\rho(t)|\big) + M_R \cdot \gamma(t) =: h_{R,\gamma}(t)$$

for every $|x| \leq R$ and every $y \in L^1(I)$ such that $|y(t)| \leq \gamma(t)$ for a.e. $t \in I$. Since the function $h_{R,\gamma}$ is non-negative and belongs to $\in L^1(I)$ (by the assumptions on σ, ρ and γ), we conclude that f fulfills (3.3) in assumption (A2)', as claimed.

We now observe that, setting $N := \max_I |\rho|$, the constant functions

$$\alpha(t) := -N \qquad \beta(t) := N \qquad \big(\text{for } t \in I$$

are, respectively, a lower and a upper solution of (3.2) such that $\alpha \leq \beta$ on I; hence, assumption (A1)' is satisfied. Furthermore, since we have

$$|f(t,x,y)| \;\leq\; (2N)\,\sigma(t) + \Big(\max_{x\in[-N,N]} |g(x)|\Big) \cdot |y|$$

for every $t \in I$, every $|x| \leq N$ and every $y \in \mathbb{R}$, we conclude that f also satisfies assumption (A3)' with the choice (here, $M_N := \max_{[-N,N]} |g|$)

$$H := 1, \quad \psi \equiv 1, \quad l(t) := 2N\,\sigma(t), \quad \mu(t) := M_N \quad \text{and} \quad q = \infty.$$

We are then entitled to apply Theorem 3.5, which ensures that there exists (at least) one solution of problem (3.17) for every fixed $\nu_1, \nu_2 \in [-N, N]$.

In the next Example 3.9 we provide an application of Theorem 3.5 for a rather general right-hand side, with possible superlinear growth with respect to u'.

Example 3.9. Let us consider the following Dirichlet problem

$$\begin{cases} \Big(\Phi_r\big(a(t,x(t))\,x'(t)\big)\Big)' = \sigma(t) \cdot g(x(t)) \cdot |x'(t)|^\delta \\ u(0) = \nu_1, \ u(T) = \nu_2, \end{cases} \tag{3.18}$$

where Φ_r, a, σ, g and the exponent δ satisfy the following assumptions:

($\star$) $\Phi_r : \mathbb{R} \to \mathbb{R}$ is the standard r-Laplacian, that is,

$$\Phi_r(\xi) := |\xi|^{r-2} \cdot \xi \qquad \big(\text{for a suitable } r > 1\big);$$

($\star$) $a \in C(I \times \mathbb{R}, \mathbb{R})$ satisfies assumption (A2), that is,

- $h_1 \geq 0$ on I and $1/h_1 \in L^p(I)$ (for some $p > 1$);
- $a(t,x) \geq h_1(t)$ for every $t \in I$ and every $x \in \mathbb{R}$.

($\star$) $\sigma \in L^\tau(I)$ for a suitable $\tau > 1$ satisfying the relation

$$\frac{1}{\tau} + \frac{r-1}{p} < 1; \tag{3.19}$$

($\star$) $g \in C(\mathbb{R}, \mathbb{R})$ is a generic function;

($\star$) δ is a positive real constant satisfying the relation

$$\delta \leq 1 - \frac{1}{\tau} + (r-1)\left(1 - \frac{1}{p}\right). \tag{3.20}$$

We aim to show that our Theorem 3.5 can be applied to problem (3.9). To this end, we consider the function f defined as follows:

$$f : I \times \mathbb{R}^2 \to \mathbb{R}, \qquad f(t,x,y) := \sigma(t) \cdot g(x) \cdot |y|^\delta.$$

Obviously, f is a Carathéodory function; moreover, it is not difficult to recognize that f satisfies assumption (A2)'. Indeed, let $R > 0$ be arbitrarily fixed and let γ be a non-negative function in $L^1(I)$; setting $M_R := \max_{[-R,R]} |g|$, we then have

$$\big|f(t,x,y(t))\big| \leq M_R \cdot |\sigma(t)| \cdot (\gamma(t))^\delta =: h_{R,\gamma}(t)$$

for every $|x| \leq R$ and every $y \in L^1(I)$ such that $|y(t)| \leq \gamma(t)$ a.e. on I. Now, by combining (3.19) with (3.20) we readily see that

$$\delta < \left(1 - \frac{1}{\tau}\right) p; \tag{3.21}$$

from this, by Hölder's inequality (and the assumptions on σ and γ), we infer that $h_{R,\gamma}$ (which is non-negative on I) belongs to $L^1(I)$, whence f satisfies (A2)'.

In order to prove that also assumptions (A1)' and (A3)' are satisfied we first notice that, if $N > 0$ is *arbitrary*, the constant functions

$$\alpha(t) := -N \qquad \beta(t) := N \qquad \big(\text{for } t \in I$$

are, respectively, a lower and a upper solution of (3.2) such that $\alpha \le \beta$ on I; hence, assumption (A1)' is fulfilled. Moreover, by (3.20) we have

$$\delta \le (r-1) + \frac{q-1}{q}, \quad \text{where } q := \frac{\tau\, p}{p + \tau\,(r-1)} > 1.$$

From this, setting $M_N := \max_{[-N,N]} |g|$, we obtain

$$\begin{aligned}
\big|f(t,x,y)\big| &\le M_N \cdot |\sigma(t)| \cdot |y|^{\delta} \le M_N \cdot |\sigma(t)| \cdot |y|^{r-1} \cdot |y|^{\frac{q-1}{q}} \\
&= \big|\Phi(a(t,x)y)\big| \cdot \left(\frac{M_N \cdot |\sigma(t)|}{(a(t,x))^{r-1}}\right) |y|^{\frac{q-1}{q}} \\
&\quad \big(\text{since } a(t,x) \ge h_1(t) \text{ for every } (t,x) \in I \times \mathbb{R}\big) \\
&\le \big|\Phi(a(t,x)y)\big| \cdot \left(\frac{M_N \cdot |\sigma(t)|}{(h_1(t))^{r-1}}\right) |y|^{\frac{q-1}{q}}
\end{aligned}$$

for a.e. $t \in I$, every $|x| \le N$ and every $|y| \ge 1$. Thus, if we are able to prove that

$$t \mapsto \frac{|\sigma(t)|}{(h_1(t))^{r-1}} \in L^q(I), \tag{3.22}$$

we can conclude that f satisfies assumption (A3)' with the choice

$$H := 1, \quad \psi(s) := s, \quad l(t) := 0, \quad \mu(t) := \frac{M_N \cdot |\sigma(t)|}{(h_1(t))^{r-1}}$$

and q as above. On the other hand, the needed (3.22) is an easy consequence of Hölder's inequality, assumption (3.19) and of the fact that $1/h_1 \in L^p(I)$.

We are then entitled to apply Theorem 3.5, which ensures the existence of (at least) one solution of the Dirichlet problem (3.18) *for every* $\nu_1\, \nu_2 \in \mathbb{R}$.

4 General nonlinear boundary conditions

The main aim of this last section is to prove the solvability of *general boundary value problems* associated with the (possibly singular) differential equation

$$\Big(\Phi\big(a(t,x(t))\, x'(t)\big)\Big)' = f(t,x(t),x'(t)), \qquad \text{a.e. on } I. \tag{4.1}$$

(here, Φ, a and f satisfy the assumptions (A1)-to-(A3) introduced in Section 3). As a particular case, we shall obtain existence results for *periodic boundary value problems* and for *Sturm-Liouville-type problems* (associated with (4.1)).

To be more precise, taking for fixed all the notation introduced so far, we aim to study the following general boundary value problems (associated with (4.1)):

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ g(x(0),x(T),\mathcal{A}_x(0),\mathcal{A}_x(T)) = 0, \\ x(T) = h(x(0)). \end{cases} \tag{4.2}$$

Here, $h:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}^4\to\mathbb{R}$ satisfy the following general assumptions:

(G1) $h\in C(\mathbb{R},\mathbb{R})$ and is increasing on $\mathbb{R}$;

(G2) $g\in C(\mathbb{R}^4,\mathbb{R})$ and, for every fixed $u,v\in\mathbb{R}$, it holds that

(G2)$_1$ $g(u,v,\cdot,z)$ is increasing for every fixed $z\in\mathbb{R}$;

(G2)$_2$ $g(u,v,w,\cdot)$ is decreasing for every fixed $w\in\mathbb{R}$.

We now state one of the main existence results of this section.

Theorem 4.1. *Let us assume that* all *the hypotheses of Theorem 3.5 are satisfied and that the functions g and h fulfill the assumptions* (G1)-(G2) *introduced above. Moreover, if $\alpha,\beta\in W^{1,p}(I)$ are as in assumption* (A1'), *we suppose that*

$$\begin{cases} g(\alpha(0),\alpha(T),\mathcal{A}_\alpha(0),\mathcal{A}_\alpha(T)) \geq 0, \\ \alpha(T) = h(\alpha(0)) \end{cases} \qquad \begin{cases} g(\beta(0),\beta(T),\mathcal{A}_\beta(0),\mathcal{A}_\beta(T)) \leq 0, \\ \beta(T) = h(\beta(0)). \end{cases} \tag{4.3}$$

Finally, let us assume that the function a satisfies the following condition:

$$a(0,x)\neq 0 \quad \text{and} \quad a(T,x)\neq 0 \qquad \text{for every } x\in\mathbb{R}.$$

Then the problem (4.2) *possesses one solution $x\in W^{1,p}(I)$ such that*

$$\alpha(t)\leq x(t)\leq\beta(t) \qquad \text{for every } t\in I. \tag{4.4}$$

Furthermore, if $M>0$ is any real number such that $\sup_I|\alpha|, \sup_I|\beta|\leq M$ and $L_0>0$ is as in Theorem 3.5 (see (3.10))*, the following fact holds true:* for every real number $L\geq L_0$ such that $\sup_I|\mathcal{A}_\alpha|, \sup_I|\mathcal{A}_\beta|\leq L$, we have

$$\max_{t\in I}\big|x(t)\big|\leq M \quad \text{and} \quad \max_{t\in I}\big|\mathcal{A}_x(t)\big|\leq L. \tag{4.5}$$

The basic idea behind the proof of Theorem 4.1, inspired by the work of Cabada and Pouso [3] and already exploited in [15], is to think of the boundary value problem (4.2) as a "superposition" of Dirichlet problems to which our existence result in Theorem 3.5 apply. Following this approach, we first establish a compactness-type result for the solutions of the differential equation in (3.1).

Proposition 4.2. *For every natural n, let $x_n\in W^{1,p}$ be a solution of the equation*

$$\Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), \qquad \text{a.e. on } I. \tag{4.6}$$

We assume that, together with (A1)*-to-*(A3)*, f satisfies assumption* (A2$'$) *of Theorem 3.5; moreover, we suppose that there exist $M,L>0$ such that*

$$\sup_I|x_n|\leq M \quad \text{and} \quad \sup_I|\mathcal{A}_{x_n}|\leq L \qquad \text{for every } n\in\mathbb{N}. \tag{4.7}$$

Then, there exist a sub-sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and a solution $x_0\in W^{1,p}(I)$ of the equation (4.1) *with the following properties:*

(1) $x_{n_k}(t) \to x_0(t)$ *for every* $t \in I$ *as* $n \to \infty$;

(2) $\mathcal{A}_{x_{n_k}}(t) \to \mathcal{A}_{x_0}(t)$ *for every* $t \in I$ *as* $n \to \infty$.

Proof. For every natural n, we set $z_n := \big(\Phi(\mathcal{A}_{x_n})\big)'$. Since x_n is a solution of (4.6), by (4.7) and the fact that f satisfies (A2′), we have (see also (3.3))

$$\big|z_n(t)\big| = \big|f(t, x_n(t), x_n'(t))\big| \le h_{M,\gamma_0}(t) \quad \text{for a.e.}\, t \in I, \tag{4.8}$$

where, as usual, $h_{M,\gamma_0} \in L^p(I)$ is the function appearing in assumption (A2′) and corresponding to M and $\gamma_0 = L/h_1$. Moreover, again by (4.7), one has

$$|x_n'(t)| \le \frac{L}{h_1(t)} = \gamma_0(t) \quad \text{for a.e.}\, t \in I. \tag{4.9}$$

As a consequence, both $\{z_n\}_n$ and $\{x_n'\}_n$ are *uniformly integrable* in $L^1(I)$. By Dunford-Pettis Theorem (see, e.g., [2]), we then infer the existence of $g, h \in L^1(I)$ such that, up to a sub-sequence, $x_n' \rightharpoonup g$ and $z_n \rightharpoonup h$ in $L^1(I)$ as $n \to \infty$.

Now, since $\{x_n\}_n$ is bounded in $\mathbb{R}$ (again by (4.7)), we can assume that $x_n(0)$ converges to some $\nu_0 \in \mathbb{R}$ as $n \to \infty$; from this, reminding that $x_n' \rightharpoonup g$, we get

$$x_n(t) = x_n(0) + \int_0^t x_n'(s)\,\mathrm{d}s \underset{n\to\infty}{\longrightarrow} \nu_0 + \int_0^t g(s)\,\mathrm{d}s =: x_0(t) \quad \forall\, t \in I. \tag{4.10}$$

Notice that, by its very definition, x_0 satisfies the following properties:

(a) x_0 is absolutely continuous on I and $x_0' = g \in L^1(I)$;

(b) $\sup_I |x_0| \le M$ (this follows also from (4.7)) .

Thus, to complete the demonstration, we need to prove that x_0 is a solution of the equation (4.6) and that $\mathcal{A}_{x_n} \to \mathcal{A}_{x_0}$ point-wise on I as $n \to \infty$.

First of all, since also $\{\mathcal{A}_{x_n}(0)\}_n$ is bounded in $\mathbb{R}$ (by (4.7)), we can suppose that $\mathcal{A}_{x_n}(0) \to \nu_0'$ as $n \to \infty$ for some $\nu_0' \in \mathbb{R}$; thus, since $z_n \rightharpoonup h$, we have

$$\Phi\big(\mathcal{A}_{x_n}(t)\big) = \Phi\big(\mathcal{A}_{x_n}(0)\big) + \int_0^t z_n'(s)\,\mathrm{d}s \underset{n\to\infty}{\longrightarrow} \Phi(\nu_0') + \int_0^t h(s)\,\mathrm{d}s.$$

As a consequence, by the continuity of Φ^{-1}, we obtain

$$\mathcal{A}_{x_n}(t) \underset{n\to\infty}{\longrightarrow} \Phi^{-1}\left(\Phi(\nu_0') + \int_0^t h(s)\,\mathrm{d}s\right) =: \mathcal{U}(t) \quad \text{for every } t \in I. \tag{4.11}$$

Notice that, by definition, $\mathcal{U} \in C(I, \mathbb{R})$ and it satisfies the following properties:

$(\mathrm{a})_1$ $\Phi \circ \mathcal{U}$ is absolutely continuous on I and $(\Phi \circ \mathcal{U})' = h \in L^1(I)$;

$(\mathrm{b})_1$ $\sup_I |\mathcal{U}| \le L$ (this follows also from (4.7)) .

Now, since a is continuous on $I \times \mathbb{R}$, we derive from (4.10) that $a(t, x_n(t))$ converges to $a(t, x_0(t))$ for every $t \in I$ as $n \to \infty$; thus, the above (4.11) gives

$$x_n'(t) \underset{n\to\infty}{\longrightarrow} \frac{1}{a(t, x_0(t))}\,\mathcal{U}(t) \quad \text{for a.e.}\, t \in I. \tag{4.12}$$

Taking into account (4.9) and the fact that $1/h_1(t) \in L^p(I)$ (see assumption (A2)), it is easy to recognize that $x_n' \to \mathcal{U}/a(\cdot, x_0(\cdot))$ also in $L^1(I)$; on the other hand, since we already know that $x_n' \rightharpoonup g$ in $L^1(I)$ as $n \to \infty$, we necessarily have

$$g(t) = \frac{1}{a(t, x_0(t))}\,\mathcal{U}(t) \quad \text{a.e. on } I. \tag{4.13}$$

From this, by reminding that $g = x_0'$ (see (a)), we infer that

($\star$) $x_0' = g \in L^p(I)$, whence $x_0 \in W^{1,p}(I)$;

($\star$) $a(t, x_0)\, x_0' = \mathcal{U}$ a.e. on I;

($\star$) $\Phi \circ \big(a(t, x_0)\, x_0'\big) = \Phi \circ \mathcal{U} \in W^{1,1}(I)$ and $\big(\Phi(a(t, x_0)\, x_0')\big)' = h$ (see $(\mathrm{a})_1$).

We now turn to prove that x_0 solves the ODE (4.6). To this end we observe that, by (4.12) and (4.13), we have $x_n'(t) \to g(t) = x_0'(t)$ for a.e. $t \in I$; as a consequence, since x_n is a solution of (4.6) for every n and f is a Carathéodory function (see assumption (A3)), we obtain (remind that $x_n \to x_0$ point-wise on I)

$$z_n = \Big(\Phi\big(\mathcal{A}_{x_n}\big)\Big)' = f(t, x_n(t), x_n'(t)) \underset{n\to\infty}{\longrightarrow} f(t, x_0(t), x_0'(t)) \quad \text{for a.e. } t \in I.$$

On the other hand, by (4.8), we have that $z_n \to f(t, x_0(t), x_0'(t)$ also in $L^1(I)$; since we already know that $z_n \rightharpoonup h$ in $L^1(I)$, we conclude that

$$\big(\Phi(a(t, x_0(t))\, x_0'(t))\big)' = h = f(t, x_0(t), x_0'(t)) \quad \text{for a.e. } t \in I,$$

that is, x_0 is a solution of (4.6). Finally, since $\mathcal{U}$ is a continuous function on I such that $\mathcal{U} = a(t, x_0)\, x_0'$ a.e. on I, we have $\mathcal{U} = \mathcal{A}_x$ on I and, by (4.11),

$$\mathcal{A}_{x_n}(t) \underset{n\to\infty}{\longrightarrow} \mathcal{A}_x(t) \quad \text{for every } t \in I.$$

This ends the proof. □

We also need the following technical lemma.

Lemma 4.3. *Let α, $\beta \in W^{1,p}(I)$ be, respectively, a lower and a upper solution of the equation* (4.1) *such that $\alpha \le \beta$. Moreover, let us assume that*

$$a(0, x) \neq 0 \quad \textit{and} \quad a(T, x) \neq 0 \qquad \textit{for every } x \in \mathbb{R}.$$

Then the following facts hold true:

(i) *if $\alpha(0) = \beta(0)$, then $\mathcal{A}_\alpha(0) \le \mathcal{A}_\beta(0)$;*

(ii) *if $\alpha(T) = \beta(T)$, then $\mathcal{A}_\alpha(T) \ge \mathcal{A}_\beta(T)$.*

Proof. We only prove statement (i), since (ii) can be demonstrated analogously.

First of all, since both $a(0, \alpha(0))$ and $a(0, \beta(0))$ are different from 0 (by assumption), it is possible to find $\delta > 0$ such that, for a.e. $t \in [0, \delta]$, we have

$$\alpha'(t) = \frac{\mathcal{A}_\alpha(t)}{a(t, \alpha(t))} =: u_1(t) \quad \text{and} \quad \beta'(t) = \frac{\mathcal{A}_\beta(t)}{a(t, \beta(t))} =: u_2(t)$$

moreover, both u_1 and u_2 are continuous on $[0,\delta]$. Let us now assume, by contradiction, that $\mathcal{A}_\alpha(0) > \mathcal{A}_\beta(0)$. Since, by assumption, $\alpha(0) = \beta(0)$ (and a is non-negative on $I \times \mathbb{R}$), there exists $\delta' < \delta$ such that

$$\alpha'(t) = u_1(t) > u_2(t) = \beta'(t) \quad \text{for a.e. } t \in [0,\delta'];$$

thus, by integrating this inequality on $[0,\delta']$, we get

$$\begin{aligned}\alpha(t) &= \alpha(0) + \int_0^t \alpha'(s)\,\mathrm{d}s = \beta(0) + \int_0^t \alpha'(s)\,\mathrm{d}s \\ &> \beta(0) + \int_0^t \beta'(s)\,\mathrm{d}s = \beta(t) \qquad \text{(for every } t \in [0,\delta'],\end{aligned}$$

which contradicts the fact that $\alpha \le \beta$ on I. This ends the proof. □

With Proposition 4.2 and Lemma 4.3 at hand, we can prove Theorem 4.1.

Proof (of Theorem 4.1). Let $\nu \in [\alpha(0), \beta(0)]$ be fixed. Since, by assumption, h is increasing on $\mathbb{R}$ and α, β satisfy (4.3), we have $h(\nu) \in [\alpha(T), \beta(T)]$; as a consequence, by the existence result in Theorem 3.5, the Dirichlet problem

$$(\mathrm{D}_\nu) \qquad \begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ x(0) = \nu, \ x(T) = h(\nu) \end{cases}$$

admits one solution x_ν such that $\alpha \le x_\nu \le \beta$ on I. Moreover, if $M > 0$ is such that $\sup_I |\alpha|, \sup_I |\beta| \le M$ and $L_0 > 0$ is as in Theorem 3.5, we can choose any real number $L \ge L_0$ such that $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \le L$, obtaining

$$(*) \qquad \sup_{t\in I} |x_\nu(t)| \le M \quad \text{and} \quad \sup_{t \in I} |\mathcal{A}_{x_\nu}(t)| \le L.$$

With this choice of L, we consider the following set:

$$V := \Big\{\nu \in [\alpha(0),\beta(0)] : \exists \text{ a solution } x_\nu \in W^{1,p}(I) \text{ of } (\mathrm{D}_\nu) \text{ s.t. } \alpha \le x_\nu \le \beta, \\ x_\nu \text{ satisfies } (*) \text{ and } g(x_\nu(0), x_\nu(T), \mathcal{A}_{x_\nu}(0), \mathcal{A}_{x_\nu}(T)) \ge 0\Big\}.$$

Claim I: We have $\nu := \alpha(0) \in V$. In fact, by Theorem 3.5, there exists a solution $x_\nu \in W^{1,p}(I)$ of (D_ν) such that $\alpha \le x_\nu \le \beta$ on I and satisfying $(*)$; in particular, we have $x_\nu(T) = h(\alpha(0)) = \alpha(T)$ and $x(t) \ge \alpha(t)$ for every $t \in I$. From this, by applying Lemma 4.3 (with x_ν in place of β), we get $\mathcal{A}_x(0) \ge \mathcal{A}_\alpha(0)$.

Analogously, since $x_\nu(T) = \alpha(T)$ and $x_\nu \le \beta$, again by Lemma 4.3 we have $\mathcal{A}_{x_\nu}(T) \le \mathcal{A}_\alpha(T)$; thus, by assumptions $(\mathrm{G2})_1, (\mathrm{G2})_2$ and (4.3) we get

$$g(x_\nu(0), x_\nu(T), \mathcal{A}_{x_\nu}(0), \mathcal{A}_{x_\nu}(T)) \ge g(\alpha(0), \alpha(T), \mathcal{A}_\alpha(0), \mathcal{A}_\alpha(T)) \ge 0.$$

This proves that $\nu \in V$, as claimed. In particular, $V \ne \varnothing$.

Claim II: If $\overline{\nu} := \sup V$, we have $\overline{\nu} \in V$. In fact, if $\overline{\nu} = \alpha(0)$, then we have already proved in Claim I that $\overline{\nu} \in V$; if, instead, $\overline{\nu} > \alpha(0)$, we choose a sequence $\{\nu_n\}_n \subseteq V$ such that $\nu_n \to \overline{\nu}$ as $n \to \infty$ and $\nu_n \le \overline{\nu}$ for every n. Since, for every n, we have $\nu_n \in V$, there exists a solution x_n of (D_{ν_n}) such that

(a) $\alpha \le x_n \le \beta$ on I;

(b) x_n satisfies $(*)$;

(c) $g(x_n(0), x_n(T), \mathcal{A}_{x_n}(0), \mathcal{A}_{x_n}(T)) \ge 0$.

On account of (b) we are entitled to apply Proposition 4.2, which provides us with a solution $x_0 \in W^{1,p}(I)$ of (4.1) such that (up to a sub-sequence)

$$x_n(t) \to x_0(t) \quad \text{and} \quad \mathcal{A}_{x_n}(t) \to \mathcal{A}_{x_0}(t) \qquad \text{for every } t \in I.$$

Now, since $\nu_n \to \overline{\nu}$ and h is continuous, it is readily seen that x_0 is a solution of $(\mathrm{D}_{\overline{\nu}})$; moreover, since x_n satisfies $(*)$ and $\alpha \le x_n \le \beta$ on I for every $n \in \mathbb{N}$, then the same is true of x_0. Finally, by (c) and the continuity of g we conclude that

$$\begin{aligned} &g(x_0(0), x_0(T), \mathcal{A}_{x_0}(0), \mathcal{A}_{x_0}(T)) \\ &\qquad = \lim_{n\to\infty} g(x_n(0), x_n(T), \mathcal{A}_{x_n}(0), \mathcal{A}_{x_n}(T)) \ge 0, \end{aligned} \tag{4.14}$$

and this proves that $\overline{\nu} \in V$.

With Claims I and II at hand, we can finally prove the existence of a solution for the problem (4.2). In fact, let $\overline{\nu} = \sup V$ and let x_0 be as in Claim II.

If $\overline{\nu} = \beta(0)$, we have $x_0(0) = \beta(0)$ and $x_0(T) = h(\beta(0)) = \beta(T)$ (by (4.3)); since, in particular, we also have $x_0 \le \beta$ on I, from Lemma 4.3 we infer that

$$\mathcal{A}_{x_0}(0) \le \mathcal{A}_\beta(0) \text{ and } \mathcal{A}_{x_0}(T) \ge \mathcal{A}_\beta(T).$$

Hence, by the monotonicity of g (see (G2)), by (4.3) and (4.14) we obtain

$$\begin{aligned} 0 \le g(x_0(0), x_0(T), \mathcal{A}_{x_0}(0), \mathcal{A}_{x_0}(T)) &= g(\beta(0), \beta(T), \mathcal{A}_{x_0}(0), \mathcal{A}_{x_0}(T)) \\ &\le g(\beta(0), \beta(T), \mathcal{A}_\beta(0), \mathcal{A}_\beta(T)) \le 0, \end{aligned}$$

and this proves that x_0 is a solution of (4.2) satisfying (4.4) and (4.5).

If, instead $\overline{\nu} < \beta(0)$, we choose a sequence $\{\mu_m\}_m \subseteq [\alpha(0), \beta(0)]$ such that $\mu_m \to \overline{\nu}$ as $m \to \infty$ and $\mu_m > \overline{\nu}$ for every m. Since, by Claim II, x_0 is a solution of $(\mathrm{D}_{\overline{\nu}})$ such that $x_0 \le \beta$ and satisfying $(*)$, we can think of x_0 and β as, respectively, a lower and a upper solution of (4.1) satisfying (A1') in Theorem 3.5; moreover, by $(*)$ and the choice of M and $L \ge L_0$ (see (3.10)), we have

$$\sup_{t\in I} |x_0(t)|,\ \sup_{t\in I} |\beta(t)| \le M \quad \text{and} \quad \sup_{t\in I} |\mathcal{A}_{x_0}(t)|,\ \sup_{t\in I} |\mathcal{A}_\beta(t)| \le L.$$

Hence, for every m there exists a solution u_m of $(\mathrm{D}_{\mu_\mathrm{m}})$ such that

- $\alpha \le x_0 \le u_m \le \beta$ on I;
- $\sup_I |u_m| \le M$ and $\sup_I |\mathcal{A}_{u_m}| \le L$.

In particular, u_m satisfies $(*)$ for any m. We can then apply Proposition 4.2, which provides us with a solution u_0 of (4.1) such that (up to a sub-sequence)

$$u_m(t) \to u_0(t) \quad \text{and} \quad \mathcal{A}_{u_m}(t) \to \mathcal{A}_{u_0}(t) \qquad \text{for every } t \in I.$$

In particular, since $\mu_m \to \overline{\nu}$ and h is continuous, u_0 solves $(\mathrm{D}_{\overline{\nu}})$; hence

$$u_0(T) = h(u_0(0)).$$

We now observe that, since $\mu_m > \overline{\nu} = \sup V$, then $\mu_m \notin V$; as a consequence, since $\alpha \le u_m \le \beta$ on I and u_m satisfies $(*)$ for every m, we necessarily have

$$g(u_m(0), u_m(T), \mathcal{A}_{u_m}(0), \mathcal{A}_{u_m}(T)) < 0.$$

From this, by the continuity of g (see assumption (G1)), we get

$$g(u_0(0), u_0(T), \mathcal{A}_{u_0}(0), \mathcal{A}_{u_0}(T)) \le 0. \tag{4.15}$$

On the other hand, since both x_0 and u_0 solve $(\mathrm{D}_{\overline{\nu}})$, we have $u_0(0) = x_0(0) = \overline{\nu}$ and $u_0(T) = x_0(T) = h(\overline{\nu})$; moreover, since $u_m \ge x_0$ for every m, then the same is true of u_0. From this, by exploiting once again Lemma 4.3 we infer that

$$\mathcal{A}_{u_0}(0) \ge \mathcal{A}_{x_0}(0) \quad \text{and} \quad \mathcal{A}_{u_0}(T) \le \mathcal{A}_{x_0}(T).$$

By (4.15), (4.14) and the by monotonicity of g we then obtain

$$\begin{aligned} 0 &\ge g(u_0(0), u_0(T), \mathcal{A}_{u_0}(0), \mathcal{A}_{u_0}(T)) = g(x_0(0), x_0(T), \mathcal{A}_{u_0}(0), \mathcal{A}_{u_0}(T)) \\ &\ge g(x_0(0), x_0(T), \mathcal{A}_{x_0}(0), \mathcal{A}_{x_0}(T)) \ge 0, \end{aligned}$$

and this shows that u_0 is a solution of (4.2). Finally, since $\alpha \le u_m \le \beta$ on I and u_m satisfies $(*)$ for every m, we conclude that u_0 satisfies both (4.4) and (4.5). □

As a particular case of Theorem 4.1, we have the following result.

Corollary 4.4. *Let us assume that* all *the hypotheses of Theorem 3.5 are satisfied; moreover, if* $\alpha, \beta \in W^{1,p}(I)$ *are as in assumption* (A1'), *we suppose that*

$$\begin{cases} \mathcal{A}_\alpha(0) \ge \mathcal{A}_\alpha(T), \\ \alpha(T) = \alpha(0), \end{cases} \qquad \begin{cases} \mathcal{A}_\beta(0) \le \mathcal{A}_\beta(T), \\ \beta(T) = \beta(0). \end{cases}$$

Finally, let us assume that the function a *satisfies the following condition:*

$$a(0,x) \ne 0 \quad \textit{and} \quad a(T,x) \ne 0 \qquad \textit{for every } x \in \mathbb{R}.$$

Then, there exists (at least) one solution $x \in W^{1,p}(I)$ *of the periodic problem*

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \textit{a.e. on } I, \\ \mathcal{A}_x(0) = \mathcal{A}_x(T), \\ x(0) = x(T). \end{cases}$$

Proof. It is a straightforward consequence of Theorem 4.1 applied to the functions

$$h(r) = r \quad \text{and} \quad g(u,v,w,z) = w - z$$

(which trivially satisfy assumptions (G1) and (G2)). This ends the proof. □

We conclude the present section by briefly turning our attention to Sturm-Liouville-type and Neumann-type problems associated with the ODE (4.1).
To be more precise, we consider the following boundary value problems:

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ p(x(0), \mathcal{A}_x(0)) = 0, \;\; q(x(T), \mathcal{A}_x(T)) = 0. \end{cases} \tag{4.16}$$

Here, the functions $p, q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following general assumptions:

(S1) $p \in C(\mathbb{R}^2, \mathbb{R})$ and, for every $s \in \mathbb{R}$, the map $p(s, \cdot)$ is increasing on $\mathbb{R}$;

(S2) $q \in C(\mathbb{R}^2, \mathbb{R})$ and, for every $s \in \mathbb{R}$, the map $q(s, \cdot)$ is decreasing on $\mathbb{R}$.

The following theorem is the second main result of this section.

Theorem 4.5. *Let us assume that* all *the hypotheses of Theorem 3.5 are satisfied and that the functions p and q fulfill the assumptions* (S1)-(S2) *introduced above. Moreover, if $\alpha, \beta \in W^{1,p}(I)$ are as in assumption* (A1'), *we suppose that*

$$\begin{cases} p(\alpha(0), \mathcal{A}_\alpha(0)) \geq 0, \\ q(\alpha(T), \mathcal{A}_\alpha(T)) \geq 0; \end{cases} \qquad \begin{cases} p(\beta(0), \mathcal{A}_\beta(0)) \leq 0, \\ q(\beta(T), \mathcal{A}_\beta(T)) \leq 0. \end{cases} \tag{4.17}$$

Finally, let us assume that a satisfies the following "compatibility" condition:

$$a(0, x) \neq 0 \quad \text{and} \quad a(T, x) \neq 0 \qquad \text{for every } x \in \mathbb{R}.$$

Then the problem (4.16) *possesses one solution $x \in W^{1,p}(I)$ such that*

$$\alpha(t) \leq x(t) \leq \beta(t) \qquad \text{for every } t \in I. \tag{4.18}$$

Furthermore, if $M > 0$ is any real number such that $\sup_I |\alpha|, \sup_I |\beta| \leq M$ and $L_0 > 0$ is as in Theorem 3.5 (see (3.10))*, the following fact holds true:* for every real number $L \geq L_0$ such that $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \leq L$, we have

$$\max_{t \in I} \big|x(t)\big| \leq M \quad \text{and} \quad \max_{t \in I} \big|\mathcal{A}_x(t)\big| \leq L. \tag{4.19}$$

The proof of Theorem 4.5 relies on the following lemma.

Lemma 4.6. *Let the assumptions and the notation of Theorem 4.5 apply.*

Then, for every fixed $\nu \in [\alpha(T), \beta(T)]$, the boundary value problem

$$(\mathrm{D}_\nu) \qquad \begin{cases} \Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big)' = f(t, x(t), x'(t)), & \text{a.e. on } I, \\ p(x(0), \mathcal{A}_x(0)) = 0, \\ x(T) = \nu. \end{cases}$$

possesses at least one solution $x \in W^{1,p}(I)$ such that $\alpha \leq x \leq \beta$ on I.

Furthermore, if M and L_0 are as in the statement of Theorem 4.5, then for every $L \geq L_0$ such that $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \leq L$ we have

$$\sup_{t \in I} |x(t)| \leq M \quad \text{and} \quad \sup_{t \in I} |\mathcal{A}_x(t)| \leq L. \tag{4.20}$$

Proof. We fix $\nu \in [\alpha(T), \beta(T)]$ and we consider the following functions:

($\star$) $h : \mathbb{R} \to \mathbb{R}, \quad h(r) := \nu$;

($\star$) $g : \mathbb{R}^4 \to \mathbb{R}, \quad g(u, v, w, z) := p(u, w)$.

Then, by means of these functions, we can re-write the problem (D_ν) as

$$\begin{cases} \Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big)' = f(t, x(t), x'(t)), & \text{a.e. on } I, \\ g(x(0), x(T), \mathcal{A}_x(0), \mathcal{A}_x(T)) = 0, \\ x(T) = h(x(0)). \end{cases}$$

Now, taking into account (S1), it is readily seen that h and g satisfy, respectively, assumptions (G1) and (G2) in the statement of Theorem 4.1; thus, to prove the lemma, it suffices to show the existence of a lower and a upper solution for (4.1) satisfying (A1') and (4.3) (with the above choice of h and g).

To this end we first observe that, by assumption, α and β are, respectively, a lower and a upper solution for (4.1) satisfying (A1') (that is, $\alpha \le \beta$ on I); as a consequence, by Theorem 3.5, the Dirichlet problem

$$(\mathrm{D})_1 \qquad \begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I,\\ x(0)=\alpha(0),\ x(T)=\nu \end{cases}$$

possesses (at least) one solution $x_1 \in W^{1,p}(I)$ such that $\alpha \le x_1 \le \beta$ on I. Moreover, if M and L_0 are as in the statement of Theorem 4.1, we can fix any real number $L \ge L_0$ such that $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \le L$, obtaining

$$\sup_{t\in I} |x_1(t)| \le M \quad \text{and} \quad \sup_{t\in I} |\mathcal{A}_{x_1}(t)| \le L. \tag{4.21}$$

We claim that the function x_1, which is obviously a lower solution of (4.1), satisfies the first assumption in (4.3) (with g as above). In fact, since $x_1(0) = \alpha(0)$ and $x_1 \ge \alpha$ on I, from Lemma 4.3 (with x_1 in place of β) we infer that

$$\mathcal{A}_{x_1}(0) \ge \mathcal{A}_\alpha(0);$$

as a consequence, by assumption (S1) and by (4.17), we obtain

$$\begin{aligned} g(x_1(0), x_1(T), \mathcal{A}_{x_1}(0), \mathcal{A}_{x_1}(T)) &= p(x_1(0), \mathcal{A}_{x_1}(0)) \\ &= p(\alpha(0), \mathcal{A}_{x_1}(0)) \ge p(\alpha(0), \mathcal{A}_\alpha(0)) \ge 0. \end{aligned}$$

Furthermore, since x_1 solves $(\mathrm{D})_1$, we have $x_1(T) = h(x_1(0)) = \nu$, and this proves that x_1 satisfies the first assumption in (4.3).

We now turn to prove the existence of a upper solution x_2 of (4.1) such that $x_2 \ge x_1$ on I and satisfying the second assumption in (4.3).

First of all, we notice that x_1 and β are, respectively, a lower and a upper solution for (4.1) such that $x_1 \le \beta$ on I and $\nu = x_1(T) \in [x_1(T), \beta(T)]$; moreover, by (4.21) and the choice of M and $L \ge L_0$, we have

$$\sup_{t\in I} |x_1(t)|,\ \sup_{t\in I} |\beta(t)| \le M \quad \text{and} \quad \sup_{t\in I} |\mathcal{A}_{x_1}(t)|,\ \sup_{t\in I} |\mathcal{A}_\beta(t)| \le L.$$

As a consequence, by Theorem 3.5, the Dirichlet problem

$$(\mathrm{D})_2 \qquad \begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I,\\ x(0)=\beta(0),\ x(T)=\nu \end{cases}$$

possesses a solution $x_2 \in W^{1,p}(I)$ such that $x_1 \le x_2 \le \beta$ on I, further satisfying

$$\sup_{t\in I} |x_2(t)| \le M \quad \text{and} \quad \sup_{t\in I} |\mathcal{A}_{x_2}(t)| \le L. \tag{4.22}$$

for the same $M, L > 0$ fixed at beginning. We claim that x_2, which is obviously a upper solution of (4.1), satisfies the second assumption in (4.3). In fact, since

$x_2(0) = \beta(0)$ and $x_2 \le \beta$ on I, by exploiting Lemma 4.3 we get $\mathcal{A}_{x_2}(0) \le \mathcal{A}_\beta(0)$; thus, by assumption (G2) and again by (4.17) we have

$$
\begin{aligned}
g(x_2(0), x_2(T), \mathcal{A}_{x_2}(0), \mathcal{A}_{x_2}(T)) &= p(x_2(0), \mathcal{A}_{x_2}(0)) \\
&= p(\beta(0), \mathcal{A}_{x_2}(0)) \le p(\beta(0), \mathcal{A}_\beta(0)) \le 0.
\end{aligned}
$$

Furthermore, since x_2 solves (D_2), one has $x_2(T) = h(x_2(0)) = \nu$, and this proves that $x_2 \ge x_1$ satisfies the second assumption in (4.17).

Gathering together all these facts, we can conclude that all the assumptions in Theorem 4.1 are fulfilled (with the above choice of g and h); as a consequence, there exists (at least) one solution $x \in W^{1,p}(I)$ of (D_ν) such that

$$\alpha \le x_1 \le x \le x_2 \le \beta \quad \text{on } I.$$

In particular, by the choice of $L \ge L_0$ (according to (3.10)) and the fact that x_1, x_2 fulfill, respectively, (4.21) and (4.22), we deduce that x satisfies (4.20) with *the very same* $M, L > 0$ *fixed at the beginning.* This ends the proof. □

Remark 4.7. Let the assumptions and the notation of Lemma 4.6 apply.

By giving a closer inspection to the proof of this lemma, we see that the only property of α and β we have used is the following (see (4.17)):

$$(\bigstar) \qquad p(\alpha(0), \mathcal{A}_\beta(0)) \ge 0 \qquad \text{and} \qquad p(\beta(0), \mathcal{A}_\beta(0)) \le 0.$$

Hence, Lemma 4.6 still holds if we replace (4.17) with the weaker $(\bigstar)$.

Thanks to Lemma 4.6, we are able to prove Theorem 4.5.

Proof (of Theorem 4.5). Let $\nu \in [\alpha(T), \beta(T)]$ be fixed. By Lemma 4.6, there exists (at least) one solution $x_\nu \in W^{1,p}(I)$ of the boundary value problem

$$
(\mathrm{D}_\nu) \qquad
\begin{cases}
\Big(\Phi\big(a(t, x(t))\, x'(t)\big)\Big)' = f(t, x(t), x'(t)), & \text{a.e. on } I, \\
p(x(0), \mathcal{A}_x(0)) = 0, \\
x(T) = \nu,
\end{cases}
$$

such that $\alpha \le x_\nu \le \beta$ on I; moreover, if $M, L_0 > 0$ are as in the statement of the theorem (that is, L_0 is as in (3.10)), we can choose any real number $L \ge L_0$ such that $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \le L$, obtaining (see Lemma 4.6)

$$(*) \qquad \sup_{t \in I} |x_\nu(t)| \le M \quad \text{and} \quad \sup_{t \in I} |\mathcal{A}_{x_\nu}(t)| \le L.$$

With such a choice of $L \ge L_0$, we consider the following set

$$
\begin{aligned}
V := \Big\{ \nu \in [\alpha(T), \beta(T)] : \exists \text{ a solution } x_\nu \in W^{1,p}(I) \text{ of } (\mathrm{D}_\nu) \text{ s.t. } \alpha \le x_\nu \le \beta, \\
x_\nu \text{ satisfies } (*) \text{ and } q(x_\nu(T), \mathcal{A}_{x_\nu}(T)) \ge 0 \Big\}.
\end{aligned}
$$

Step I: We have $\nu = \alpha(T) \in V$. In fact, by Lemma 4.6, there exists at least one solution $x_\nu \in W^{1,p}(I)$ of (D_ν) such that $\alpha \le x_\nu \le \beta$ and satisfying $(*)$; in particular, we have $x_\nu(T) = \alpha(T)$ and $x_\nu \ge \alpha$ on I. Hence, by Lemma 4.3 we get

$$\mathcal{A}_{x_\nu}(T) \le \mathcal{A}_\alpha(T).$$

From this, by assumption (S2) and (4.17), we obtain

$$q(x_\nu(T), \mathcal{A}_{x_\nu}(T)) = q(\alpha(T), \mathcal{A}_{x_\nu}(T)) \geq q(\alpha(T), \mathcal{A}_\alpha(T)) \geq 0.$$

This proves that $\nu \in V$, as claimed. In particular, $V \neq \varnothing$.

Step II: Setting $\overline{\nu} := \sup V$, we have $\overline{\nu} \in V$. In fact, if $\overline{\nu} = \alpha(T)$, we have already proved in Step I that $\overline{\nu} \in V$; if, instead, $\overline{\nu} > \alpha(T)$, we can choose a sequence $\{\nu_n\}_n \subseteq V$ such that $\nu_n \to \overline{\nu}$ as $n \to \infty$ and $\nu_n \leq \overline{\nu}$ for every n. Since, for every n, we have $\nu_n \in V$, there exists a solution x_n of (D_{ν_n}) such that

(a) $\alpha \leq x_n \leq \beta$ on I;

(b) x_n satisfies $(*)$;

(c) $q(x_n(T), \mathcal{A}_{x_n}(T)) \geq 0$.

On account of (b) we are entitled to apply Proposition 4.2, which provides us with a solution $x_0 \in W^{1,p}(I)$ of (4.1) such that, up to a sub-sequence,

$$x_n(t) \to x_0(t) \quad \text{and} \quad \mathcal{A}_{x_n}(t) \to \mathcal{A}_{x_0}(t) \qquad \text{for every } t \in I.$$

Now, since $\nu_n \to \overline{\nu}$ and p is continuous, we see that x_0 solves $(\mathrm{D}_{\overline{\nu}})$; hence

$$p(x_0(0), \mathcal{A}_{x_0}(0)) = 0.$$

Moreover, since x_n satisfies $(*)$ and $\alpha \leq x_n \leq \beta$ on I for every natural n, then the same is true of x_0. Finally, by (c) and the continuity of q, we obtain

$$q(x_0(T), \mathcal{A}_{x_0}(T)) = \lim_{n\to\infty} q(x_n(T), \mathcal{A}_{x_n}(T)) \geq 0. \tag{4.23}$$

This proves that $\overline{\nu} \in V$, as claimed.

Now we have established Claims I and II, we can finally prove the existence of a solution to (4.16). In fact, let $\overline{\nu} = \sup V \in V$ and let x_0 be as in Claim II.

If $\overline{\nu} = \beta(T)$, we have $x_0(T) = \overline{\nu} = \beta(T)$ and $p(x_0(0), \mathcal{A}_{x_0}(0)) = 0$; since, in particular, $x_0 \leq \beta$ on I, Lemma 4.3 implies that $\mathcal{A}_{x_0}(T) \geq \mathcal{A}_\beta(T)$; from this, by (4.23), the monotonicity of q (see assumption (S2)) and (4.17), we obtain

$$0 \leq q(x_0(T), \mathcal{A}_{x_0}(T)) = q(\beta(T), \mathcal{A}_{x_0}(T) \leq q(\beta(T), \mathcal{A}_\beta(T)) \leq 0,$$

and this proves that x_0 is a solution of (4.16) satisfying (4.18) and (4.19).

If, instead, $\overline{\nu} < \beta(T)$, we choose a sequence $\{\mu_m\}_m \subseteq [\alpha(T), \beta(T)]$ such that $\mu_m \to \overline{\nu}$ as $m \to \infty$ and $\mu_m > \overline{\nu}$ for any m. Since, by Claim II, x_0 solves $(\mathrm{D}_{\overline{\nu}})$ and $x_0 \leq \beta$ on I, we can think of x_0 and β as, respectively, a lower and a upper solution of (4.1) satisfying (A1') in Theorem 3.5 and property $(\bigstar)$ in Remark 4.7 (see (4.17)); moreover, by $(*)$ and the choice of M and $L \geq L_0$ we have

$$\sup_{t\in I} |x_0(t)|, \ \sup_{t\in I} |\beta(t)| \leq M \quad \text{and} \quad \sup_{t\in I} |\mathcal{A}_{x_0}(t)|, \ \sup_{t\in I} |\mathcal{A}_\beta(t)| \leq L.$$

Hence, by Remark 4.7, for any m there exists a solution u_m of (D_{μ_m}) such that

- $\alpha \leq x_0 \leq u_m \leq \beta$ on I;
- $\sup_I |u_m| \leq M$ and $\sup_I |\mathcal{A}_{u_m}| \leq L$.

In particular, u_m satisfies $(*)$ for every m. We can then apply Proposition 4.2, which provides us with a solution u_0 of (4.1) such that, up to a sub-sequence,

$$u_m(t) \to u_0(t) \quad \text{and} \quad \mathcal{A}_{u_m}(t) \to \mathcal{A}_{u_0}(t) \qquad \text{for every } t \in I.$$

Thus, since $\mu_m \to \overline{\nu}$ and p is continuous, we see that u_0 solves $(\mathrm{D}_{\overline{\nu}})$; hence

$$p(u_0(0), \mathcal{A}_{u_0}(0)) = 0.$$

We now observe that, since $\mu_m > \overline{\nu} = \sup V$, then $\mu_m \notin V$; as a consequence, since $\alpha \le u_m \le \beta$ on I and u_m satisfies $(*)$ for every m, we necessarily have

$$q(u_m(T), \mathcal{A}_{u_m}(T)) < 0 \quad \text{for every } m \in \mathbb{N}$$

From this, by the continuity of q (see assumption (S2)), we get

$$q(u_0(T), \mathcal{A}_{u_0}(T)) = \lim_{m\to\infty} q(u_m(T), \mathcal{A}_{u_m}(T)) \le 0. \tag{4.24}$$

On the other hand, since both x_0 and u_0 solve $(\mathrm{D}_{\overline{\nu}})$, we have $x_0(T) = \overline{\nu} = u_0(T)$; moreover, since $u_m \ge x_0$ for every natural m (by the construction of u_m), then the same is true of u_0. From this, by using Lemma 4.3 we infer that

$$\mathcal{A}_{u_0}(T) \le \mathcal{A}_{x_0}(T).$$

By (4.24), (4.23) and the monotonicity of q (see (S2)), we then get

$$0 \ge q(u_0(T), \mathcal{A}_{u_0}(T)) = q(x_0(T), \mathcal{A}_{u_0}(T)) \ge q(x_0(T), \mathcal{A}_{x_0}(T)) \ge 0,$$

and this shows that u_0 is a solution of (4.19). Finally, since $\alpha \le u_m \le \beta$ and u_m satisfies $(*)$ for every m, we conclude that u_0 fulfills both (4.18) and (4.19). □

From Theorem 4.5 we easily deduce the following results.

Corollary 4.8. *Let us assume that* all *the hypotheses of Theorem 3.5 are satisfied. Moreover, let $\ell_1, \ell_2, \nu_1, \nu_2 \in \mathbb{R}$ and let $m_1, m_2 \in [0,\infty)$. If α and β are as in assumption* (A1)'*, we suppose that the following conditions are satisfied:*

$$\begin{cases} \ell_1\,\alpha(0) + m_1\,\mathcal{A}_\alpha(0) \ge \nu_1, \\ \ell_2\,\alpha(T) - m_2\,\mathcal{A}_\alpha(T) \ge \nu_2; \end{cases} \qquad \begin{cases} \ell_1\,\beta(0) + m_1\,\mathcal{A}_\beta(0) \le \nu_1, \\ \ell_2\,\beta(T) - m_2\,\mathcal{A}_\beta(T) \le \nu_2 \end{cases}$$

Finally, we assume that the function a fulfills the the following assumption:

$$a(0,x) \ne 0 \quad \textit{and} \quad a(T,x) \ne 0 \qquad \textit{for every } x \in \mathbb{R}.$$

*Then, there exists a solution $x \in W^{1,p}(I)$ of the **Sturm-Liouville problem***

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \textit{a.e. on } I, \\ \ell_1\,x(0) + m_1\,\mathcal{A}_x(0) = \nu_1, \\ \ell_2\,x(T) - m_2\,\mathcal{A}_x(T) = \nu_2. \end{cases}$$

Proof. It is a direct consequence of Theorem 4.5 applied to the functions

$$p(s,t) := \ell_1\,s + m_1\,t - \nu_1 \qquad \text{and} \qquad q(s,t) := \ell_2\,s - m_2\,t - \nu_2,$$

which satisfy assumptions (S1)-(S2) (since $m_1\,m_2 \ge 0$). This ends the proof. □

Corollary 4.9. *Let us assume that* all *the hypotheses of Theorem 3.5 are satisfied. Moreover, let* $\nu_1,\ \nu_2 \in \mathbb{R}$ *be arbitrarily fixed. If* α *and* β *are as in assumption* (A1)', *we suppose that the following conditions are satisfied:*

$$\begin{cases} \mathcal{A}_\alpha(0) \geq \nu_1, \\ \mathcal{A}_\alpha(T) \leq \nu_2; \end{cases} \qquad \begin{cases} \mathcal{A}_\beta(0) \leq \nu_1, \\ \mathcal{A}_\beta(T) \geq \nu_2 \end{cases}$$

Finally, we assume that the function a fulfills the following usual assumption:

$$a(0,x) \neq 0 \quad \text{and} \quad a(T,x) \neq 0 \qquad \text{for every } x \in \mathbb{R}.$$

Then, there exists (at least) one solution $x \in W^{1,p}(I)$ *of the* ***Neumann problem***

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ \mathcal{A}_x(0) = \nu_1, \\ \mathcal{A}_x(T) = \nu_2. \end{cases}$$

Proof. It is another direct consequence of Theorem 4.5 applied to the functions

$$p(s,t) := t - \nu_1 \qquad \text{and} \qquad q(s,t) := \nu_2 - t,$$

which obviously satisfy assumptions (S1)-(S2). This ends the proof. □

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