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ON A CLASS OF OBSTACLE PROBLEMS WITH (p, q) -GROWTH AND EXPLICIT u -DEPENDENCE

ANDREA GENTILE, TERESA ISERNIA, AND ANTONIA PASSARELLI DI NAPOLI

ABSTRACT. In this paper, we establish the higher differentiability of the gradient of solutions to variational obstacle problems of the type

$$\min \left\{ \int_{\Omega} F(x, u, Du) dx : u \in \mathcal{K}_{\psi}(\Omega) \right\}.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\psi \in W_0^{1,q}(\Omega)$ is a fixed function called obstacle, and $\mathcal{K}_{\psi}(\Omega)$ is the class of admissible functions. The main feature of the energy densities under consideration here is that they satisfy non standard growth conditions with respect to the gradient variable and that they explicitly depend on the pair (x, u) . Assuming that $\psi \in L_{\text{loc}}^{\infty}(\Omega) \cap W_{\text{loc}}^{2,2q-p}(\Omega)$, we are able to prove a second order regularity result for the solution.

1. INTRODUCTION

The aim of this paper is the study of the higher differentiability properties of the gradient of the solutions $u \in W^{1,q}(\Omega)$ to a class of variational obstacle problems of the form

$$\min \left\{ \int_{\Omega} F(x, w, Dw) dx : w \in \mathcal{K}_{\psi}(\Omega) \right\}, \quad (1.1)$$

under the (p, q) -growth conditions, with $2 \leq p \leq q < p + 1$. Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, with $n \geq 2$, and the class of the admissible functions $\mathcal{K}_{\psi}(\Omega)$ is defined as follows

$$\mathcal{K}_{\psi}(\Omega) := \{w \in W_0^{1,q}(\Omega) : w \geq \psi \text{ a.e. in } \Omega\},$$

where $\psi : \Omega \rightarrow [-\infty, +\infty)$, called *obstacle*, fulfills the following regularity assumption

$$\psi \in L_{\text{loc}}^{\infty}(\Omega) \cap W_{\text{loc}}^{2,2q-p}(\Omega). \quad (1.2)$$

To ensure that the discussion remains substantive and avoids trivial cases, we will proceed under the assumption that $\mathcal{K}_{\psi}(\Omega)$ is not empty. We will suppose that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function with respect to (s, ξ) , C^2 with respect to ξ and Lipschitz with respect (x, u) verifying

$$(F_0) \quad \nu(1 + |\xi|^2)^{\frac{q}{2}} \leq F(x, s, \xi) \leq M \left[(1 + |\xi|^2)^{\frac{q}{2}} + |s|^n \right],$$

$$(F_1) \quad \nu(1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle D_{\xi\xi} F(x, s, \xi) \lambda, \lambda \rangle,$$

$$(F_2) \quad |D_{\xi\xi} F(x, s, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{q-2}{2}} + |s|^{\alpha} \right],$$

$$(F_3) \quad |D_{s\xi} F(x, s, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{p+q-4}{4}} + |s|^{\beta} \right],$$

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$$(F_4) \quad |D_{x\xi}F(x, s, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{p+q-2}{4}} + |s|^\gamma \right],$$

$$(F_5) \quad |D_sF(x, s, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{p+q-2}{4}} + |s|^\delta \right]$$

for a.e. $x, y \in \Omega$, for every $s, t \in \mathbb{R}$ and for every $\xi, \eta \in \mathbb{R}^n$, where $2 \leq p < q < p + 1$ and $0 \leq \nu \leq M$ are fixed constants.

One can easily check that the solution to (1.1) solves the following variational inequality

$$\int_{\Omega} \langle A(x, u, Du), D(\varphi - u) \rangle dx \geq \int_{\Omega} b(x, u, Du) (\varphi - u) dx \quad (1.3)$$

for any $\varphi \in \mathcal{K}_\psi$, where the structural functions A and b are given by

$$A(x, s, \xi) = D_\xi F(x, s, \xi) \quad \text{and} \quad b(x, s, \xi) = -D_s F(x, s, \xi).$$

It is well-known that assumptions (F₀)-(F₅) yield that the following

$$(I) \quad \nu (1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle D_\xi A(x, s, \xi) \lambda, \lambda \rangle,$$

$$(II) \quad |A(x, s, \xi) - A(x, s, \eta)| \leq M \left[(1 + |\xi|^2 + |\eta|^2)^{\frac{q-2}{2}} + |s|^\alpha \right] |\xi - \eta|,$$

$$(III) \quad |A(x, s, \xi) - A(x, t, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{p+q-4}{4}} + |s|^\beta \right] |s - t|,$$

$$(IV) \quad |A(x, s, \xi) - A(y, s, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{p+q-2}{4}} + |s|^\gamma \right] |x - y|,$$

$$(V) \quad |b(x, s, \xi)| \leq M \left[(1 + |\xi|^2)^{\frac{p+q-2}{4}} + |s|^\delta \right]$$

hold for a.e. $x, y \in \Omega$, for every $s, t \in \mathbb{R}$ and for every $\xi, \eta \in \mathbb{R}^n$.

The study of regularity theory for obstacle problems is a classical topic in Partial Differential Equations and Calculus of Variations. The formal investigation of obstacle problems began with the works by Fichera [17] and Stampacchia [36], and since then, significant research has been devoted to understanding the regularity of the solutions to these problems. This research interest is motivated by the several applications of obstacle problems in fields such as elastostatics, financial mathematics, and control theory. It is well-known that the solutions to the obstacle problem are generally not of class C^2 , regardless of the regularity of the obstacle, and this led to the development of the concept of weak solutions and to the theory of variational inequalities, after the fundamental work by Lions and Stampacchia [31].

When studying regularity results for solutions to obstacle problems, we are interested in understanding how the regularity properties of the obstacle affect the regularity of the solutions. For linear problems, the solutions typically inherit the regularity of the obstacle; see for instance [4, 6, 14]. In the nonlinear context, this is not always true. Consequently, over the years, there has been an intense research activity aimed at determining the regularity of the solution in relation to the regularity assumed for the obstacle; see [2, 3, 19, 20].

In the last few years, many higher differentiability results of integer or fractional order have been established for solutions to obstacle problems, both in the standard growth (see for instance [9, 15, 23, 24, 32]) and nonstandard growth setting (see, for example [5, 8, 16, 18, 21, 22, 29, 30, 34]). In the quoted papers, the authors obtain some higher differentiability results for solutions, provided a suitable regularity assumption is imposed on the dependence on the x -variable of the coefficients of the variational inequality and on the gradient of the obstacle. However, as far as we know, such regularity hasn't been exploited yet when the energy density in (1.1) explicitly depends on the u -variable.

Actually, we aim to fill this gap by considering here an obstacle problem where the non standard growth of the energy density with respect to the gradient variable combines with the explicit dependence on the u -variable.

It is worth pointing out that the explicit dependence of the integrand on the u -variable in case of (p, q) -growth can not be managed as a simple perturbation. A very clear explanation of this phenomenon can be found in the paper by Marcellini [33], which deals with the local Lipschitz continuity of solutions to equations with p, q -growth and with explicit dependence on the u -variable (see also the very recent papers [10–12]).

In this paper we go further in this direction considering the case of (p, q) -growth conditions with an integrand function that explicitly depends on (x, u) , but now in a constrained minimization problem. The first paper dealing with this case is [13], where the authors proved the local boundedness of solutions to (1.1) by assuming that F is a convex function in (s, ξ) and satisfies the following growth conditions:

$$c_1|\xi|^p \leq F(x, s, \xi) \leq c_2(1 + |s|^\gamma + |\xi|^q),$$

where $\gamma \geq 0$ and $c_1, c_2 > 0$ constants. The crucial aspect in their paper is that they were able to establish the local boundedness under a sharp bound on the gap between the growth exponent, specifically

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1},$$

assuming the boundedness of the obstacle (see also [8]).

Our main result can be stated as follows.

Theorem 1.1. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (1.3) under the assumptions (I)–(V) with exponents $2 \leq p < q < p + 1$ verifying*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1}.$$

If ψ satisfies (1.2), then $V_p(Du) \in W_{\text{loc}}^{1,2}(\Omega)$ and the following estimate

$$\begin{aligned} \int_{\mathcal{B}_{\frac{R}{2}}} |DV_p(Du)|^2 dx &\leq \frac{C}{R^2} \int_{\mathcal{B}_R} (1 + |Du|^2)^{\frac{p}{2}} dx + \frac{c}{R^2} \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx \\ &+ \frac{c}{R^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx + \frac{c}{R^\gamma} |\mathcal{B}_R| \end{aligned} \quad (1.4)$$

holds for any ball $\mathcal{B}_R \Subset \Omega$.

The proof, as it is often the case, relies on the combination of a suitable a priori estimate and an approximation argument. However, the presence of the u variable in the integrand prevents us from approximating the functional from below, that is using functions that behave like t^p , and instead forces us to approximate from above, that is with functions that behave like t^q . For this reason, the Sobolev class $W^{1,q}(\Omega)$ is the natural space in which to search for solutions. Moreover, the presence of the term u also necessitates the use of the local boundedness of the obstacle, and then of the solutions, to exploit the interpolation inequality, which provides the L^{p+2} integrability for the gradient of the solution. This will be a key tool in the proof of our main result.

We conclude this introduction with a brief description of the structure of the paper. In Section 2, we list some notations and classical tools that are useful to the proof of our result,

with a focus on the properties of difference quotients. Section 3 is devoted to some a priori estimates which will be crucial for Section 4, where we provide the proof of the main result.

2. NOTATIONS AND PRELIMINARY RESULTS

In this section we list the notations that we use in this paper and recall some tools that will be useful to prove our results. We denote by either C or c a general constant, which might change in different contexts, possibly even within the same line of estimates. Dependencies on parameters and specific constants will be appropriately highlighted using parentheses or subscripts. For a C^2 -function $f: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$, we write

$$D_\xi f(x, \xi)[\eta] := \left. \frac{d}{dt} \right|_{t=0} f(x, \xi + t\eta) \quad \text{and} \quad D_{\xi\xi} f(x, \xi)[\eta, \eta] := \left. \frac{d^2}{dt^2} \right|_{t=0} f(x, \xi + t\eta)$$

for $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \Omega$.

In what follows, $\mathcal{B}(x, r) = \mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ will denote the ball centered at x of radius r . We shall omit the dependence on the center when it is clear from the context.

Next, we recall some results that will be useful in the following. First we state a well-known iteration lemma whose proof can be found in [28, Lemma 6.1].

Lemma 2.1. *Let $h: [\rho, R] \rightarrow \mathbb{R}$ be a nonnegative bounded function, $0 < \theta < 1$, $A, B \geq 0$ and $\gamma > 0$. Assume that*

$$h(r) \leq \theta h(d) + \frac{A}{(d-r)^\gamma} + B$$

for all $\rho \leq r < d \leq R_0$. Then

$$h(\rho) \leq c \left[\frac{A}{(R_0 - \rho)^\gamma} + B \right],$$

where $c = c(\theta, \gamma) > 0$.

The following Gagliardo-Nirenberg type inequality is stated in [27]. For the proofs, we refer to [7, Appendix A] and [25, Lemma 3.5] (in case $p(x) \equiv p, \forall x$).

Lemma 2.2. *For any $\phi \in C_0^1(\Omega)$ with $\phi \geq 0$, $\mu \in [0, 1]$, and any C^2 map $v: \Omega \rightarrow \mathbb{R}^N$, we have*

$$\begin{aligned} & \int_{\Omega} \phi^2 (\mu^2 + |Dv|^2)^{\frac{p}{2}} |Dv|^2 dx \\ & \leq c \|v\|_{L^\infty(\text{supp}(\phi))}^2 \int_{\Omega} \phi^2 (\mu^2 + |Dv|^2)^{\frac{p-2}{2}} |D^2v|^2 dx \\ & \quad + c \|v\|_{L^\infty(\text{supp}(\phi))}^2 \int_{\Omega} (\phi^2 + |D\phi|^2) (\mu^2 + |Dv|^2)^{\frac{p}{2}} dx, \end{aligned} \tag{2.1}$$

for a constant $c = c(p)$.

In [13] the following local boundedness result has been proved.

Theorem 2.1. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution of (1.1). Assume that*

$$(s, \xi) \mapsto F(x, s, \xi) \quad \text{is convex}$$

and

$$c_1|\xi|^p \leq F(x, s, \xi) \leq c_2(1 + |s|^\gamma + |\xi|^q),$$

for exponents $1 < p \leq q$ and $\gamma > 0$ such that

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1} \quad \text{and} \quad \gamma \leq p^* = \begin{cases} \frac{np}{n-p} & \text{if } p < n \\ \text{any finite exponent} & \text{if } p \geq n. \end{cases}$$

If the obstacle $\psi \in L^\infty_{\text{loc}}(\Omega)$, then $u \in L^\infty_{\text{loc}}(\Omega)$ and the following estimate

$$\|u\|_{L^\infty(\mathcal{B}_{R/2})} \leq [\|\psi\|_{L^\infty(\mathcal{B}_R)} + \|u\|_{W^{1,p}(\mathcal{B}_R)}]^\gamma$$

holds for every ball $\mathcal{B}_R \Subset \Omega$, for $\gamma(n, p) > 0$ and $c = c(\ell, \nu, p, n)$.

Since we are dealing with non linear problems, it will be convenient to use, as usual, the function $V_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined as follows:

$$V_p(\xi) = (\mu^2 + |\xi|^2)^{\frac{p-2}{4}} \xi, \quad \text{for any } \xi \in \mathbb{R}^n.$$

For the auxiliary function V_p , we recall the following estimate, whose proof can be found in [1] for $1 < p < 2$ and in [26] for $p \geq 2$.

Lemma 2.3. *Let $p > 1$. There is a constant $c = c(n, p) > 0$ such that*

$$c^{-1} |\xi - \eta| \leq |V_p(\xi) - V_p(\eta)| (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{2-p}{4}} \leq c |\xi - \eta|,$$

for any $\xi, \eta \in \mathbb{R}^n$.

Remark 2.1. *One can easily check that, for a C^2 -function v , there is a constant $C(p)$ such that*

$$C^{-1} |D^2v|^2 (\mu^2 + |Dv|^2)^{\frac{p-2}{2}} \leq |D[V_p(Dv)]|^2 \leq C |D^2v|^2 (\mu^2 + |Dv|^2)^{\frac{p-2}{2}}. \quad (2.2)$$

In the following, we will always consider $\mu = 1$ both when we apply Lemma 2.2 and when we use the function V_p and its properties. Moreover, for the sake of brevity, we will often use Lemma 2.3 and estimate (2.2) without recalling them explicitly.

2.1. Difference quotients. A key instrument in studying regularity properties of solutions to problems of Calculus of Variations and PDEs is the so called *difference quotients method*. In this section, we recall the definition and some basic results.

Definition 2.1. *Given $h \in \mathbb{R}$ and a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^N$, for any $s = 1, \dots, n$ the finite difference operator in the direction x_s is defined by*

$$\tau_{s,h}F(x) = F(x + he_s) - F(x),$$

where e_s is the unit vector in the direction x_s .

In the following, in order to simplify the notations, we will omit the vector e_s unless it is necessary, denoting

$$\tau_hF(x) = F(x + h) - F(x),$$

where $h \in \mathbb{R}^n$.

We now describe some useful properties of the operator τ_h whose proofs can be found, for example, in [28].

Proposition 2.1. *Let $F \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : d(x, \partial\Omega) > |h|\}.$$

Then

(1) $\tau_h F \in W^{1,p}(\Omega_{|h|})$ and

$$D_i(\tau_h F) = \tau_h(D_i F).$$

(2) If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then

$$\int_{\Omega} F(x)\tau_h G(x)dx = \int_{\Omega} G(x)\tau_{-h}F(x)dx.$$

(3) We have

$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The subsequent result regarding the finite difference operator can be viewed as an integral version of Lagrange's Theorem.

Lemma 2.4. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, and $F, DF \in L^p(\mathcal{B}_R)$ then*

$$\int_{\mathcal{B}_\rho} |\tau_h F(x)|^p dx \leq c(n, p)|h|^p \int_{\mathcal{B}_R} |DF(x)|^p dx.$$

Moreover

$$\int_{\mathcal{B}_\rho} |F(x+h)|^p dx \leq \int_{\mathcal{B}_R} |F(x)|^p dx.$$

The following result is proved in [28].

Lemma 2.5. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^N$, $F \in L^p(\mathcal{B}_R)$ with $1 < p < +\infty$. Suppose that there exist $\rho \in (0, R)$ and $M > 0$ such that*

$$\sum_{s=1}^n \int_{\mathcal{B}_\rho} |\tau_{s,h} F(x)|^p dx \leq M^p |h|^p$$

for $|h| < \frac{R-\rho}{2}$. Then $F \in W^{1,p}(\mathcal{B}_R, \mathbb{R}^N)$. Moreover,

$$\|DF\|_{L^p(\mathcal{B}_\rho)} \leq M$$

and

$$\|F\|_{L^{\frac{np}{n-p}}(\mathcal{B}_\rho)} \leq c(M + \|F\|_{L^p(\mathcal{B}_R)}),$$

with $c = c(n, N, p, \rho, R)$. Furthermore,

$$\frac{\tau_{s,h} F}{|h|} \rightarrow D_s F \quad \text{in } L^p_{\text{loc}}(\Omega), \quad \text{as } h \rightarrow 0,$$

for each $s = 1, \dots, n$.

3. SOME USEFUL A PRIORI ESTIMATES

In this section, we establish some a priori estimates that will be instrumental in proving Theorem 1.1. We require the local boundedness of the local solution u to the variational inequality (1.3), which we will consider as an a priori assumption, unless we are in cases in which it is already granted. Taking into account the local boundedness of u , our set of assumptions can be presented as follows. For every $\Omega' \Subset \Omega$ there exists $K > 0$ such that

$$\|u\|_\infty = \|u\|_{L^\infty(\Omega')} \leq K.$$

Here, we shall assume that there exists a positive constant $M(K)$ such that $A(x, s, \xi)$ and $b(x, s, \xi)$ satisfy the following:

$$\begin{aligned} (\mathcal{A}_1) \quad & \nu (1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle D_\xi A(x, s, \xi) \lambda, \lambda \rangle, \\ (\mathcal{A}_2) \quad & |A(x, s, \xi) - A(x, s, \eta)| \leq M(K) (1 + |\xi|^2 + |\eta|^2)^{\frac{q-2}{2}} |\xi - \eta|, \\ (\mathcal{A}_3) \quad & |A(x, s, \xi) - A(x, t, \xi)| \leq M(K) (1 + |\xi|^2)^{\frac{p+q-4}{4}} |s - t|, \\ (\mathcal{A}_4) \quad & |A(x, s, \xi) - A(y, s, \xi)| \leq M(K) (1 + |\xi|^2)^{\frac{p+q-2}{4}} |x - y|, \\ (\mathcal{A}_5) \quad & |b(x, u, \xi)| \leq M(K) (1 + |\xi|^2)^{\frac{p+q-2}{4}} \end{aligned}$$

for every $x, y \in \Omega'$, for every $s, t \in \mathbb{R}$, for every $\xi, \eta \in \mathbb{R}^n$ and for $|u| \leq K$.

Here the exponents p and q are such that

$$2 \leq p \leq q < p + 1. \quad (3.1)$$

Let us point out that assumption (\mathcal{A}_1) implies in particular

$$\langle A(x, s, \xi) - A(x, s, \eta), \xi - \eta \rangle \geq \nu (1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}} |\xi - \eta|^2.$$

Along this section we will suppose that $A(x, s, \xi)$ satisfies (\mathcal{A}_1) – (\mathcal{A}_4) , $b(x, s, \xi)$ obeys (\mathcal{A}_5) , ψ verifies (1.2) and (p, q) fulfills (3.1).

The first result of this section is an a priori estimate for locally bounded solutions $u \in \mathcal{K}_\psi(\Omega)$ to the variational inequality (1.3) such that $Du \in L^{2q-p}(\Omega)$.

Theorem 3.1. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (1.3) such that $Du \in L_{\text{loc}}^{2q-p}(\Omega)$. If $\psi \in W_{\text{loc}}^{2, 2q-p}(\Omega)$ then*

$$V_p(Du) \in W_{\text{loc}}^{1, 2}(\Omega),$$

and the following estimate holds

$$\begin{aligned} & \int_{\mathcal{B}_\rho} |DV_p(Du)|^2 dx \\ & \leq \frac{c}{(R - \rho)^2} \int_{\mathcal{B}_R} (1 + |Du|^{2q-p} + |D\psi|^{2q-p}) dx + c \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx \end{aligned} \quad (3.2)$$

for every concentric balls $\mathcal{B}_\rho \subset \mathcal{B}_R \Subset \Omega$ and for positive constant $c = c(n, p, q, \nu, L)$.

Proof. Let us fix a direction $s = 1, \dots, n$ and $h \in \mathbb{R}$ such that $|h|$ is sufficiently small. Let e_s be the unit vector in the s -direction and let us denote the vector he_s with h in order to simplify the notation. Consider \mathcal{B}_R such that $\mathcal{B}_{4R} \Subset \Omega$, with $R < \frac{1}{4}$, and arbitrary radii

$$0 < \rho \leq r < \tilde{s} < t < \tilde{t} < \tilde{r} < R.$$

Let us consider a cut-off function $\eta \in C_c^\infty(\mathcal{B}_t)$ such that

$$\begin{cases} \eta \equiv 1 & \text{on } \mathcal{B}_{\bar{s}}, \\ 0 \leq \eta \leq 1, \\ |D\eta| \leq \frac{c}{t-\bar{s}}. \end{cases}$$

Setting

$$v_1(x) = \eta^2(x)[(u - \psi)(x + h) - (u - \psi)(x)]$$

and

$$v_2(x) = \eta^2(x - h)[(u - \psi)(x - h) - (u - \psi)(x)],$$

one can easily check that the functions

$$\varphi_1(x) = u + \lambda v_1$$

and

$$\varphi_2(x) = u + \lambda v_2$$

are admissible test functions for any $\lambda \in [0, 1)$. Inserting φ_1 and φ_2 in inequality (1.3), we get

$$\int_{\Omega} \langle A(x, u, Du), D(\eta^2 \tau_h(u - \psi)) \rangle dx \geq \int_{\Omega} b(x, u, Du) \eta^2 \tau_h(u - \psi) dx$$

and

$$\int_{\Omega} \langle A(x, u, Du), D(\eta^2(x - h) \tau_{-h}(u - \psi)) \rangle dx \geq \int_{\Omega} b(x, u, Du) \eta^2(x - h) \tau_{-h}(u - \psi) dx.$$

By using the change of variable $y = x - h$ in the integrals of the previous estimate, we get

$$\begin{aligned} & - \int_{\Omega} \langle A(x + h, u(x + h), Du(x + h)), D(\eta^2 \tau_h(u - \psi)) \rangle dx \\ & \geq - \int_{\Omega} b(x + h, u(x + h), Du(x + h)) \eta^2(x) \tau_h(u - \psi) dx \end{aligned}$$

and so, summing up,

$$\begin{aligned} & \int_{\Omega} \langle A(x + h, u(x + h), Du(x + h)) - A(x, u, Du), D(\eta^2 \tau_h(u - \psi)) \rangle dx \\ & \leq \int_{\Omega} \tau_h b(x, u, Du) \eta^2 \tau_h(u - \psi) dx \\ & = - \int_{\Omega} b(x, u, Du) \tau_{-h}(\eta^2 \tau_h(u - \psi)) dx. \end{aligned}$$

The previous inequality can be rewritten as

$$\begin{aligned} I_0 & := \int_{\Omega} \langle A(x + h, u(x + h), Du(x + h)) - A(x + h, u(x + h), Du(x)), \eta^2 \tau_h Du \rangle dx \\ & \leq \int_{\Omega} \langle A(x + h, u(x + h), Du(x + h)) - A(x + h, u(x + h), Du(x)), \eta^2 \tau_h D\psi \rangle dx \\ & \quad + \int_{\Omega} \langle A(x + h, u(x + h), Du(x)) - A(x + h, u(x), Du(x)), \eta^2 \tau_h D(\psi - u) \rangle dx \\ & \quad + \int_{\Omega} \langle A(x + h, u(x), Du(x)) - A(x, u(x), Du(x)), \eta^2 \tau_h D(\psi - u) \rangle dx \end{aligned}$$

$$\begin{aligned}
 & + 2 \int_{\Omega} \langle A(x+h, u(x+h), Du(x+h)) - A(x+h, u(x+h), Du(x)), \eta D\eta \tau_h(\psi - u) \rangle dx \\
 & + 2 \int_{\Omega} \langle A(x+h, u(x+h), Du(x)) - A(x+h, u(x), Du(x)), \eta D\eta \tau_h(\psi - u) \rangle dx \\
 & + 2 \int_{\Omega} \langle A(x+h, u(x), Du(x)) - A(x, u(x), Du(x)), \eta D\eta \tau_h(\psi - u) \rangle dx \\
 & + \int_{\Omega} b(x, u, Du) \tau_{-h}(\eta^2 \tau_h(\psi - u)) dx \\
 & =: \sum_{i=1}^6 I_i + J
 \end{aligned}$$

that yields

$$I_0 \leq \sum_{i=1}^6 |I_i| + |J|. \quad (3.3)$$

At this point, we proceed to estimate all terms within (3.3). Using assumption (\mathcal{A}_1) we can see that

$$\begin{aligned}
 I_0 & \geq \nu \int_{\Omega} \eta^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
 & \geq c(\nu) \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx,
 \end{aligned} \quad (3.4)$$

where we also applied Lemma 2.3.

Next, from (\mathcal{A}_2) and Young's inequality we obtain

$$\begin{aligned}
 |I_1| & \leq M(K) \int_{\Omega} \eta^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{q-2}{2}} |\tau_h Du| |\tau_h D\psi| dx \\
 & \leq \varepsilon \int_{\Omega} \eta^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
 & \quad + C_{\varepsilon} \int_{\Omega} \eta^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{2q-p-2}{2}} |\tau_h D\psi|^2 dx \\
 & \leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon} \tilde{I}_1.
 \end{aligned}$$

In order to estimate \tilde{I}_1 , we first apply Hölder's inequality with exponents $\frac{2q-p}{2}$ and $\frac{2q-p}{2q-p-2}$ and then we use Lemma 2.4 and Young's inequality, to deduce that

$$\begin{aligned}
 \tilde{I}_1 & \leq \left(\int_{\mathcal{B}_t} (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{2q-p}{2}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_t} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \\
 & \leq |h|^2 \left(\int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_{\tilde{t}}} |D^2\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \\
 & \leq c|h|^2 \int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + c|h|^2 \int_{\mathcal{B}_{\tilde{t}}} |D^2\psi|^{2q-p} dx,
 \end{aligned}$$

where we used the a priori assumption $Du \in L_{\text{loc}}^{2q-p}(\Omega)$, the assumption $D\psi \in L_{\text{loc}}^{2q-p}(\Omega)$ and the properties of η . Combining the previous estimates we get

$$|I_1| \leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon} |h|^2 \int_{\mathcal{B}_{\bar{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + C_{\varepsilon} |h|^2 \int_{\mathcal{B}_{\bar{t}}} |D^2\psi|^{2q-p} dx. \quad (3.5)$$

By virtue of assumption (\mathcal{A}_3) , we have

$$\begin{aligned} |I_2| &\leq M(K) \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p+q-4}{4}} |\tau_h u| |\tau_h D(\psi - u)| dx \\ &\leq M(K) \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p+q-4}{4}} |\tau_h u| |\tau_h D\psi| dx \\ &\quad + M(K) \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p+q-4}{4}} |\tau_h u| |\tau_h Du| dx \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

By the properties of η , using Lemma 2.4, the Hölder inequality with the triple of exponents $\frac{2q-p}{2q-p-2}$, $2q-p$ and $2q-p$, Young's inequality, and exploiting that $\frac{(p+q-4)(2q-p)}{2(2q-p-2)} \leq 2q-p$ (since $p \leq q$), we can estimate $I_{2,1}$ as follows

$$\begin{aligned} I_{2,1} &\leq M \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{(p+q-4)(2q-p)}{4(2q-p-2)}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_t} |\tau_h u|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left(\int_{\mathcal{B}_t} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\ &\leq M |h|^2 \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{(p+q-4)(2q-p)}{4(2q-p-2)}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_{\bar{t}}} |Du|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left(\int_{\mathcal{B}_{\bar{t}}} |D^2\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\ &\leq c |h|^2 \int_{\mathcal{B}_{\bar{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + c |h|^2 \int_{\mathcal{B}_{\bar{t}}} |D^2\psi|^{2q-p} dx. \end{aligned}$$

Furthermore, using Young's and Hölder's Inequality properly, Lemma 2.4, and taking into account that we have $\frac{(q-2)(2q-p)}{2q-p-2} \leq 2q-p$ thanks to $p \leq q$, we get

$$\begin{aligned} I_{2,2} &\leq \varepsilon \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx + C_{\varepsilon} \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{q-2}{2}} |\tau_h u|^2 dx \\ &\leq \varepsilon \int_{\Omega} \eta^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\ &\quad + C_{\varepsilon} \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{(q-2)(2q-p)}{2(2q-p-2)}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_t} |\tau_h u|^{2q-p} dx \right)^{\frac{2}{2q-p}} \\ &\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon} |h|^2 \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{(q-2)(2q-p)}{2(2q-p-2)}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_{\bar{t}}} |Du|^{2q-p} dx \right)^{\frac{2}{2q-p}} \\ &\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon} |h|^2 \int_{\mathcal{B}_{\bar{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx. \end{aligned}$$

Hence,

$$|I_2| \leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon} |h|^2 \int_{\mathcal{B}_{\bar{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + c |h|^2 \int_{\mathcal{B}_{\bar{t}}} |D^2\psi|^{2q-p} dx. \quad (3.6)$$

Next using assumption (\mathcal{A}_4) we estimate I_3 as follows

$$\begin{aligned} |I_3| &\leq M(K)|h| \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p+q-2}{4}} |\tau_h D(\psi - u)| dx \\ &\leq M(K)|h| \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p+q-2}{4}} |\tau_h D\psi| dx + M(K)|h| \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p+q-2}{4}} |\tau_h Du| dx \\ &=: I_{3,1} + I_{3,2}. \end{aligned}$$

We focus on $I_{3,1}$. Exploiting Hölder's inequality with exponents $\frac{2q-p}{2q-p-1}$ and $2q-p$, the properties of η , Lemma 2.4 and the fact that $p+q-2 \leq 2(2q-p-1)$ in view of $p \leq q$ we obtain

$$\begin{aligned} I_{3,1} &\leq c|h| \left(\int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{(p+q-2)(2q-p)}{4(2q-p-1)}} dx \right)^{\frac{2q-p-1}{2q-p}} \left(\int_{\Omega} \eta^2 |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\ &\leq c|h|^2 \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{(p+q-2)(2q-p)}{4(2q-p-1)}} dx \right)^{\frac{2q-p-1}{2q-p}} \left(\int_{\mathcal{B}_{\tilde{t}}} |D^2\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\ &\leq c|h|^2 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + c|h|^2 \int_{\mathcal{B}_{\tilde{t}}} |D^2\psi|^{2q-p} dx. \end{aligned}$$

Moreover, for what concerns $I_{3,2}$ we get

$$\begin{aligned} I_{3,2} &\leq \varepsilon \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx + C_{\varepsilon}|h|^2 \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{q}{2}} dx \\ &\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon}|h|^2 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q}{2}} dx. \end{aligned}$$

Combining the estimates for $I_{3,1}$ and $I_{3,2}$ we can see that

$$|I_3| \leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + C_{\varepsilon}|h|^2 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + c|h|^2 \int_{\mathcal{B}_{\tilde{t}}} |D^2\psi|^{2q-p} dx. \quad (3.7)$$

In order to estimate I_4 we use assumption (\mathcal{A}_2) , Young's and Hölder's inequalities and Lemma 2.4 as follows

$$\begin{aligned} |I_4| &\leq 2M(K) \int_{\Omega} \eta (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{q-2}{2}} |\tau_h Du| |D\eta| |\tau_h(\psi - u)| dx \\ &\leq \varepsilon \int_{\Omega} \eta^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\ &\quad + C_{\varepsilon} \int_{\mathcal{B}_t} |D\eta|^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{2q-p-2}{2}} |\tau_h(\psi - u)|^2 dx \\ &\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + \frac{C_{\varepsilon}}{|t - \tilde{s}|^2} \int_{\mathcal{B}_t} (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{2q-p-2}{2}} |\tau_h(\psi - u)|^2 dx \\ &\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx \\ &\quad + \frac{C_{\varepsilon}}{|t - \tilde{s}|^2} \left(\int_{\mathcal{B}_t} (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{2q-p}{2}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_t} |\tau_h(\psi - u)|^{2q-p} dx \right)^{\frac{2}{2q-p}} \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx \\
&\quad + \frac{C_\varepsilon |h|^2}{|t - \tilde{s}|^2} \left(\int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{\mathcal{B}_{\tilde{t}}} |D(\psi - u)|^{2q-p} dx \right)^{\frac{2}{2q-p}} \\
&\leq \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx + \frac{C_\varepsilon |h|^2}{|t - \tilde{s}|^2} \int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx + \frac{C_\varepsilon |h|^2}{|t - \tilde{s}|^2} \int_{\mathcal{B}_{\tilde{t}}} |D\psi|^{2q-p} dx.
\end{aligned} \tag{3.8}$$

Combining assumption (\mathcal{A}_3) with Hölder's inequality with the triplet of exponents $\frac{q}{q-2}$, q , q , Lemma 2.4 and Young's inequality, we get

$$\begin{aligned}
|I_5| &\leq 2M(K) \int_{\Omega} \eta (1 + |Du|^2)^{\frac{p+q-4}{4}} |\tau_h u| |D\eta| |\tau_h(\psi - u)| dx \\
&\leq \frac{2M(K)}{t - \tilde{s}} \left(\int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{q(p+q-4)}{4(q-2)}} dx \right)^{\frac{q-2}{q}} \left(\int_{\mathcal{B}_{\tilde{t}}} |\tau_h u|^q dx \right)^{\frac{1}{q}} \left(\int_{\mathcal{B}_{\tilde{t}}} |\tau_h(\psi - u)|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{c|h|^2}{t - \tilde{s}} \int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{q}{2}} dx + \frac{c|h|^2}{t - \tilde{s}} \int_{\mathcal{B}_{\tilde{t}}} |D\psi|^q dx,
\end{aligned} \tag{3.9}$$

where we also used the fact that, since $2 \leq p < q$, we have $\frac{q(p+q-4)}{2(q-2)} \leq q$.

In order to estimate I_6 , we combine (\mathcal{A}_4) together with Hölder's and Young's inequality, the definition of η and Lemma 2.4

$$\begin{aligned}
|I_6| &\leq 2M(K)|h| \int_{\Omega} (1 + |Du|^2)^{\frac{p+q-2}{4}} \eta |D\eta| |\tau_h(\psi - u)| dx \\
&\leq \frac{c|h|}{t - \tilde{s}} \left(\int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{q(p+q-2)}{4(q-1)}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathcal{B}_{\tilde{t}}} |\tau_h(\psi - u)|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{c|h|^2}{t - \tilde{s}} \int_{\mathcal{B}_{\tilde{t}}} (1 + |Du|^2)^{\frac{q}{2}} dx + \frac{c|h|}{t - \tilde{s}} \int_{\mathcal{B}_{\tilde{t}}} |D\psi|^q dx,
\end{aligned} \tag{3.10}$$

where we also used the fact that, since $p \leq q$, we have $\frac{q(p+q-2)}{2(q-1)} \leq q$.

Now we focus on J . First, we represent $\tau_{-h}(\eta^2 \tau_h(\psi - u))$ as

$$\begin{aligned}
\tau_{-h}(\eta^2 \tau_h(\psi - u)) &= \eta^2(x-h) \tau_h((\psi - u)(x-h)) - \eta^2(x) \tau_h((\psi - u)(x)) \\
&= \int_0^1 \frac{d}{d\theta} \left[\eta^2(x - \theta h) \tau_h((\psi - u)(x - \theta h)) \right] d\theta \\
&= -h \int_0^1 \left[2\eta(x - \theta h) D\eta(x - \theta h) \tau_h((\psi - u)(x - \theta h)) \right. \\
&\quad \left. + \eta^2(x - \theta h) \tau_h(D(\psi - u)(x - \theta h)) \right] d\theta.
\end{aligned}$$

Then, thanks to (\mathcal{A}_5) , we have

$$|J| \leq |h| M(K) \int_{\Omega} (1 + |Du|^2)^{\frac{p+q-2}{4}} \int_0^1 2\eta(x - \theta h) |D\eta(x - \theta h)| |\tau_h((\psi - u)(x - \theta h))| d\theta dx$$

$$\begin{aligned}
& + |h|M(K) \int_{\Omega} (1 + |Du|^2)^{\frac{p+q-2}{4}} \int_0^1 \eta^2(x - \theta h) |\tau_h(D((\psi - u)(x - \theta h)))| d\theta dx \\
& \leq |h|M(K) \int_{\Omega} (1 + |Du|^2)^{\frac{p+q-2}{4}} \int_0^1 2\eta(x - \theta h) |D\eta(x - \theta h)| |\tau_h(\psi - u)(x - \theta h)| d\theta dx \\
& \quad + |h|M(K) \int_{\Omega} (1 + |Du|^2)^{\frac{p+q-2}{4}} \int_0^1 \eta^2(x - \theta h) |\tau_h(D\psi(x - \theta h))| d\theta dx \\
& \quad + |h|M(K) \int_{\Omega} (1 + |Du|^2)^{\frac{p+q-2}{4}} \int_0^1 \eta^2(x - \theta h) |\tau_h(Du(x - \theta h))| d\theta dx \\
& =: J_1 + J_2 + J_3.
\end{aligned}$$

Using the properties of η and Hölder's inequality, we get

$$\begin{aligned}
J_1 & \leq \frac{c(K)|h|}{t - \tilde{s}} \int_0^1 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{p+q-2}{4}} |\tau_h(\psi - u)(x - \theta h)| dx d\theta \\
& \leq \frac{c(K)|h|}{t - \tilde{s}} \int_0^1 \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q(p+q-2)}{4(q-1)}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathcal{B}_t} |\tau_h(\psi - u)(x - \theta h)|^q dx \right)^{\frac{1}{q}} d\theta.
\end{aligned}$$

Now, we make use of a change of variable, taking $|h| < \frac{\tilde{t}-t}{2}$, setting for each $\theta \in (0, 1)$ $y = x + \theta h$, so that $y \in \mathcal{B}_{t+\theta|h|} \subset \mathcal{B}_{\frac{t+\tilde{t}}{2}}$. Then we exploit Lemma 2.4 and Young's inequality, thus obtaining

$$\begin{aligned}
J_1 & \leq \frac{c(K)|h|}{t - \tilde{s}} \int_0^1 \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q(p+q-2)}{4(q-1)}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathcal{B}_{t+\theta|h|}} |\tau_h(\psi - u)(y)|^q dy \right)^{\frac{1}{q}} d\theta \\
& \leq \frac{c(K)|h|^2}{t - \tilde{s}} \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q(p+q-2)}{4(q-1)}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} |D(\psi - u)|^q dx \right)^{\frac{1}{q}} \\
& \leq \frac{c(K)|h|^2}{t - \tilde{s}} \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} (1 + |Du|^2)^{\frac{q}{2}} dx + \frac{c|h|^2}{t - \tilde{s}} \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} |D\psi|^q dx,
\end{aligned}$$

where we used that $p + q - 2 \leq 2(q - 1)$. Now we consider J_2 . Again, we first apply Hölder's inequality, then the same change of variable we used to get the estimate of J_1 , and then Young's inequality and Lemma 2.4, thus obtaining

$$\begin{aligned}
J_2 & \leq c|h| \int_0^1 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{p+q-2}{4}} |\tau_h(D\psi(x - \theta h))| dx d\theta \\
& \leq c|h| \int_0^1 \left(\int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q(p+q-2)}{4(q-1)}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathcal{B}_t} |\tau_h(D\psi(x - \theta h))|^q dx \right)^{\frac{1}{q}} \\
& \leq c|h|^2 \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} (1 + |Du|^2)^{\frac{q}{2}} dx + c|h|^2 \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} |D^2\psi|^q dx.
\end{aligned}$$

Finally, we focus on J_3 . By Young's inequality and the definition of η we get

$$J_3 \leq c|h| \int_0^1 \int_{\Omega} \eta^2(x - \theta h) (1 + |Du|^2)^{\frac{p+q-2}{4}} |\tau_h(Du(x - \theta h))| dx d\theta$$

$$\begin{aligned}
&\leq c\sigma \int_0^1 \int_{\Omega} \eta^2(x - \theta h) (1 + |Du|^2)^{\frac{p-2}{2}} |\tau_h(Du(x - \theta h))|^2 dx d\theta \\
&\quad + c_{\sigma} |h|^2 \int_0^1 \int_{\Omega} \eta^2(x - \theta h) (1 + |Du|^2)^{\frac{q}{2}} dx d\theta \\
&\leq c\sigma \int_0^1 \int_{\mathcal{B}_t} \eta^2(x - \theta h) |\tau_h(V_p(Du)(x - \theta h))|^2 dx d\theta + c_{\sigma} |h|^2 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q}{2}} dx \\
&\leq c\sigma \int_0^1 \int_{\mathcal{B}_{t+\theta|h|}} \eta^2(y) |\tau_h(V_p(Du)(y))|^2 dy d\theta + c_{\sigma} |h|^2 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q}{2}} dx \\
&\leq c\sigma \int_{\mathcal{B}_{\tilde{t}}} |\tau_h(V_p(Du))|^2 dx + \frac{c}{\sigma} |h|^2 \int_{\mathcal{B}_t} (1 + |Du|^2)^{\frac{q}{2}} dx
\end{aligned}$$

where we made the change of variable $y = x + \theta h$ and we chose $|h| < \frac{\tilde{t}-t}{2}$ so that $y \in \mathcal{B}_{t+\theta|h|} \subset \mathcal{B}_{\frac{t+\tilde{t}}{2}}$. Here $\sigma > 0$ is a parameter that will be chosen later. Hence,

$$\begin{aligned}
|J| &\leq c\sigma \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} |\tau_h(V_p(Du))|^2 dx + \frac{c}{\sigma} |h|^2 \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \\
&\quad + \frac{c|h|^2}{t-\tilde{s}} \int_{\mathcal{B}_{\tilde{r}}} |D\psi|^q dx + c|h|^2 \int_{\mathcal{B}_{\tilde{r}}} |D^2\psi|^q dx.
\end{aligned} \tag{3.11}$$

Inserting (3.4)–(3.11) in (3.3), and choosing ε sufficiently small, we have

$$\begin{aligned}
\int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx &\leq c\sigma \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} |\tau_h V_p(Du)|^2 dx + \frac{c|h|^2}{\sigma|t-\tilde{s}|^2} \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \\
&\quad + c|h|^2 \int_{\mathcal{B}_{\tilde{r}}} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{|t-\tilde{s}|^2} \int_{\mathcal{B}_{\tilde{r}}} |D\psi|^{2q-p} dx.
\end{aligned}$$

Now, exploiting the fact that $\eta \equiv 1$ on $\mathcal{B}_{\tilde{s}}$, and choosing $\sigma > 0$ such that $c\sigma = \kappa$, with $\kappa \in (0, 1)$, we get

$$\begin{aligned}
\int_{\mathcal{B}_{\tilde{s}}} |\tau_h V_p(Du)|^2 dx &\leq \kappa \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} |\tau_h V_p(Du)|^2 dx + \frac{c|h|^2}{\kappa|t-\tilde{s}|^2} \int_{\mathcal{B}_{\frac{t+\tilde{t}}{2}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \\
&\quad + c|h|^2 \int_{\mathcal{B}_{\tilde{r}}} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{|t-\tilde{s}|^2} \int_{\mathcal{B}_{\tilde{r}}} |D\psi|^{2q-p} dx.
\end{aligned}$$

We point out that this estimate holds for any $\rho < r < \tilde{s} < t < \tilde{t} < \tilde{r} < R$, with constants independent of the radii, and so, we may choose

$$\tilde{s} = r + \frac{\tilde{r} - r}{4} \quad t = r + \frac{\tilde{r} - r}{2} \quad \tilde{t} = r + \frac{3(\tilde{r} - r)}{4} \tag{3.12}$$

so that

$$t - \tilde{s} = \frac{\tilde{r} - r}{4} = \frac{1}{2}(\tilde{t} - \tilde{s}).$$

With these choices, the previous estimate can be written as

$$\begin{aligned} \int_{\mathcal{B}_{\tilde{s}}} |\tau_h V_p(Du)|^2 dx &\leq \kappa \int_{\mathcal{B}_{\tilde{t}}} |\tau_h V_p(Du)|^2 dx + \frac{c|h|^2}{\kappa(\tilde{t} - \tilde{s})^2} \int_{\mathcal{B}_{r + \frac{5}{8}(\tilde{r}-r)}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \\ &\quad + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{(\tilde{t} - \tilde{s})^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx. \end{aligned} \quad (3.13)$$

Now, applying the iteration Lemma 2.1, we get

$$\begin{aligned} \int_{\mathcal{B}_r} |\tau_h V_p(Du)|^2 dx &\leq \frac{c|h|^2}{(\tilde{r} - r)^2} \int_{\mathcal{B}_{\tilde{r}}} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \\ &\quad + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{(\tilde{r} - r)^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx \end{aligned}$$

and then by Lemma 2.5 we obtain

$$\begin{aligned} \int_{\mathcal{B}_\rho} |DV_p(Du)|^2 dx &\leq \frac{c}{(R - \rho)^2} \int_{\mathcal{B}_R} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \\ &\quad + c \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c}{(R - \rho)^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx. \end{aligned}$$

This completes the proof of Theorem 3.1. \square

At this point, we observe for further needs that Theorem 3.1, in the particular case $p = q$, yields that if $u \in \mathcal{K}_\psi(\Omega)$ is a solution to the obstacle problem (1.3), then $V_p(Du) \in W_{\text{loc}}^{1,2}(\Omega)$ together with estimate (3.2). More precisely we have the following.

Theorem 3.2. *Suppose that (\mathcal{A}_1) – (\mathcal{A}_5) hold true with $p = q$. Let $\psi \in W_{\text{loc}}^{2,q}(\Omega)$ and let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (1.3). Then $V_q(Du) \in W_{\text{loc}}^{1,2}(\Omega)$ and the following estimate*

$$\begin{aligned} \int_{\mathcal{B}_\rho} |DV_q(Du)|^2 dx &\leq \frac{c}{(R - \rho)^2} \int_{\mathcal{B}_R} (1 + |Du|)^q dx \\ &\quad + \frac{c}{(R - \rho)^2} \int_{\mathcal{B}_R} |D\psi|^q dx + c \int_{\mathcal{B}_R} |D^2\psi|^q dx, \end{aligned} \quad (3.14)$$

holds for all concentric balls $\mathcal{B}_\rho \subset \mathcal{B}_R \Subset \Omega$ and for positive constant $c = c(n, q, \nu, L, \|u\|_\infty)$.

Proof. Let us observe that in this case, the assumption $Du \in L^{2q-p}(\Omega)$ reduces to $Du \in L^q(\Omega)$, since $p = q$. Therefore we do not need to require any additional integrability for the Du .

Following the lines of the proof of Theorem 3.1, we arrive at (3.13) which, in this case, can be written as follows

$$\begin{aligned} \int_{\mathcal{B}_{\tilde{s}}} |\tau_h V_q(Du)|^2 dx &\leq \frac{1}{2} \int_{\mathcal{B}_{\tilde{t}}} |\tau_h V_q(Du)|^2 dx + \frac{c|h|^2}{|t - \tilde{s}|^2} \int_{\mathcal{B}_{r + \frac{5}{8}(\tilde{r}-r)}} (1 + |Du|^q) dx \\ &\quad + \frac{c|h|^2}{|t - \tilde{s}|^2} \int_{\mathcal{B}_R} |D\psi|^q dx + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^q dx. \end{aligned}$$

Arguing exactly as we did at the end of the proof of Theorem 3.1, using Lemmas 2.1 and 2.5, we get the conclusion, with estimate (3.14). \square

Finally, we go back to the case $p \leq q < p + 1$ and we obtain the following a priori estimate for solutions $u \in \mathcal{K}_\psi(\Omega)$ to (1.3).

Theorem 3.3. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (1.3) under the assumptions (\mathcal{A}_1) – (\mathcal{A}_5) with $p \leq q < p + 1$. Assume that $u \in L^\infty_{\text{loc}}(\Omega)$ and $V_p(Du) \in W^{1,2}_{\text{loc}}(\Omega)$. Then the following estimates hold*

$$\begin{aligned} \int_{\mathcal{B}_\rho} |DV_p(Du)|^2 dx &\leq (1 + \|u\|_{L^\infty(\mathcal{B}_R)})^\gamma \left(\frac{c}{(R-\rho)^2} + \frac{c}{(R-\rho)^{\frac{2}{p-q+1}}} \right) \int_{\mathcal{B}_R} (1 + |Du|^2)^{\frac{p}{2}} dx \\ &\quad + \left(c + \frac{c}{(R-\rho)^2} \right) \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx, \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \int_{\mathcal{B}_\rho} (1 + |Du|)^{p+2} dx &\leq c (1 + \|u\|_{L^\infty(\mathcal{B}_R)})^\gamma \left(\frac{c}{(R-\rho)^2} + \frac{c}{(R-\rho)^{\frac{2}{p-q+1}}} \right) \int_{\mathcal{B}_R} (1 + |Du|^2)^{\frac{p}{2}} dx \\ &\quad + \left(c + \frac{c}{(R-\rho)^2} \right) \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx, \end{aligned} \quad (3.16)$$

for all ball $\mathcal{B}_\rho \subset \mathcal{B}_R \Subset \Omega$, for a positive constant $c = c(n, p, q, \nu, L)$ and a positive exponent $\gamma = \gamma(p, q)$.

Proof. Note that the a priori assumptions $u \in L^\infty_{\text{loc}}(\Omega)$ and $V_p(Du) \in W^{1,2}_{\text{loc}}(\Omega)$ imply, through Lemma 2.2, that $Du \in L^{p+2}_{\text{loc}}(\Omega)$. Furthermore, the assumption $q < p + 1$ implies that $p \leq 2q - p < p + 2$, and so there exists $\vartheta \in [0, 1)$ such that

$$1 = \frac{\vartheta(2q-p)}{p} + \frac{(1-\vartheta)(2q-p)}{p+2} \quad \text{i.e.} \quad \vartheta = \frac{p(p-q+1)}{2q-p}.$$

Hence, by means of the interpolation inequality, we get

$$\int_{\mathcal{B}_\rho} (1 + |Du|^2)^{\frac{2q-p}{2}} dx \leq \left(\int_{\mathcal{B}_\rho} (1 + |Du|^2)^{\frac{p+2}{2}} dx \right)^{\frac{(1-\vartheta)(2q-p)}{p+2}} \left(\int_{\mathcal{B}_\rho} (1 + |Du|^2)^{\frac{p}{2}} dx \right)^{\frac{\vartheta(2q-p)}{p}},$$

for every ball $\mathcal{B}_\rho \Subset \Omega$. Inserting the previous estimate in (3.13), using Young's inequality and exploiting the definition of ϑ we get

$$\begin{aligned} \int_{\mathcal{B}_{\tilde{s}}} |\tau_h V_p(Du)|^2 dx &\leq \kappa \int_{\mathcal{B}_{\tilde{t}}} |\tau_h V_p(Du)|^2 dx \\ &\quad + \frac{c|h|^2}{\kappa(\tilde{t}-\tilde{s})^2} \left(\int_{\mathcal{B}_{r+\frac{5}{8}(\tilde{r}-r)}} (1 + |Du|^2)^{\frac{p+2}{2}} dx \right)^{\frac{(1-\vartheta)(2q-p)}{p+2}} \left(\int_{\mathcal{B}_R} (1 + |Du|^2)^{\frac{p}{2}} dx \right)^{\frac{\vartheta(2q-p)}{p}} \\ &\quad + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{(\tilde{t}-\tilde{s})^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx \\ &\leq \kappa \int_{\mathcal{B}_{\tilde{t}}} |\tau_h V_p(Du)|^2 dx + \kappa|h|^2 \int_{\mathcal{B}_{r+\frac{5}{8}(\tilde{r}-r)}} (1 + |Du|^2)^{\frac{p+2}{2}} dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{c|h|^2}{\kappa^{\frac{1}{p-q+1}}(\tilde{t}-\tilde{s})^{\frac{2}{p-q+1}}} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
 & + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{(\tilde{t}-\tilde{s})^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx,
 \end{aligned} \tag{3.17}$$

with a constant c independent of all the radii. Using the interpolation inequality at (2.1) of Lemma 2.2 with $\mu = 1$ and a cut off function ϕ between the balls $\mathcal{B}_{r+\frac{5}{8}(\tilde{r}-r)}$ and $\mathcal{B}_{\tilde{r}}$, we have that

$$\begin{aligned}
 & \int_{\mathcal{B}_{r+\frac{5}{8}(\tilde{r}-r)}} (1+|Du|)^{p+2} dx \\
 & \leq c\|u\|_{L^\infty(\mathcal{B}_R)}^2 \int_{\mathcal{B}_{\tilde{r}}} |DV_p(Du)|^2 dx + \frac{c\|u\|_{L^\infty(\mathcal{B}_R)}^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx.
 \end{aligned} \tag{3.18}$$

Inserting (3.18) in (3.17), we obtain

$$\begin{aligned}
 \int_{\mathcal{B}_{\tilde{s}}} |\tau_h V_p(Du)|^2 dx & \leq \kappa \int_{\mathcal{B}_{\tilde{t}}} |\tau_h V_p(Du)|^2 dx \\
 & + c\kappa|h|^2\|u\|_{L^\infty(\mathcal{B}_R)}^2 \int_{\mathcal{B}_{\tilde{r}}} |DV_p(Du)|^2 dx \\
 & + \frac{c\kappa|h|^2\|u\|_{L^\infty(\mathcal{B}_R)}^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
 & + \frac{c|h|^2}{\kappa^{\frac{1}{p-q+1}}(\tilde{t}-\tilde{s})^{\frac{2}{p-q+1}}} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
 & + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{(\tilde{t}-\tilde{s})^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx.
 \end{aligned}$$

With the same choice of the radii as in (3.12) and by the use of Lemma 2.4 together with the assumption $V_p(Du) \in W_{\text{loc}}^{1,2}(\Omega)$, previous estimate yields

$$\begin{aligned}
 \int_{\mathcal{B}_r} |\tau_h V_p(Du)|^2 dx & \leq c\kappa|h|^2 \int_{\mathcal{B}_{\tilde{r}}} |DV_p(Du)|^2 dx \\
 & + c\kappa|h|^2\|u\|_{L^\infty(\mathcal{B}_R)}^2 \int_{\mathcal{B}_{\tilde{r}}} |DV_p(Du)|^2 dx \\
 & + \frac{c|h|^2\|u\|_{L^\infty(\mathcal{B}_R)}^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
 & + \frac{c\kappa|h|^2}{\kappa^{\frac{1}{p-q+1}}(\tilde{r}-r)^{\frac{2}{p-q+1}}} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
 & + c|h|^2 \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c|h|^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx.
 \end{aligned}$$

Hence, by virtue Lemma 2.5, it holds that

$$\int_{\mathcal{B}_r} |DV_p(Du)|^2 dx \leq c\kappa(1+\|u\|_{L^\infty(\mathcal{B}_R)}^2) \int_{\mathcal{B}_{\tilde{r}}} |DV_p(Du)|^2 dx$$

$$\begin{aligned}
& + \frac{c\kappa\|u\|_{L^\infty(\mathcal{B}_R)}^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
& + \frac{c}{\kappa^{\frac{1}{p-q+1}}(\tilde{r}-r)^{\frac{2}{p-q+1}}} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
& + c \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx.
\end{aligned}$$

Choosing κ such that

$$c\kappa(1+\|u\|_{L^\infty(\mathcal{B}_R)}^2) = \frac{1}{2} \iff \kappa = \frac{1}{2c(1+\|u\|_{L^\infty(\mathcal{B}_R)})}$$

by the iteration Lemma 2.1, it follows

$$\begin{aligned}
\int_{\mathcal{B}_\rho} |DV_p(Du)|^2 dx & \leq c(1+\|u\|_{L^\infty(\mathcal{B}_R)})^\gamma \left(\frac{1}{(R-\rho)^2} + \frac{1}{(R-\rho)^{\frac{2}{p-q+1}}} \right) \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
& + c \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \frac{c}{(R-\rho)^2} \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx, \tag{3.19}
\end{aligned}$$

with $\gamma = \gamma(p, q) > 0$.

In order to obtain (3.16) we combine estimates (3.18) and (3.19) to have

$$\begin{aligned}
& \int_{\mathcal{B}_{r+\frac{5}{8}(\tilde{r}-r)}} (1+|Du|)^{p+2} dx \\
& \leq c\|u\|_{L^\infty(\mathcal{B}_R)}^2 \int_{\mathcal{B}_{\tilde{r}}} |DV_p(Du)|^2 dx + \frac{c\|u\|_{L^\infty(\mathcal{B}_R)}^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \\
& \leq c\|u\|_{L^\infty(\mathcal{B}_R)}^\gamma \left[\left(\frac{c}{(R-\tilde{r})^2} + \frac{c}{(R-\tilde{r})^{\frac{2}{p-q+1}}} \right) \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx \right. \\
& \quad \left. + \frac{c}{(R-\tilde{r})^2} \int_{\mathcal{B}_R} |D^2\psi|^{2q-p} dx + \int_{\mathcal{B}_R} |D\psi|^{2q-p} dx \right] \\
& \quad + \frac{c\|u\|_{L^\infty(\mathcal{B}_R)}^2}{(\tilde{r}-r)^2} \int_{\mathcal{B}_R} (1+|Du|^2)^{\frac{p}{2}} dx,
\end{aligned}$$

where we used (3.15) on the concentric balls $\mathcal{B}_{\tilde{r}}$ and \mathcal{B}_R . At this point, the conclusion follows, choosing r, \tilde{r} as follows

$$\tilde{r} = R - \frac{R-\rho}{4} \quad r = R - \frac{R-\rho}{2}.$$

□

4. PROOF OF THEOREM 1.1

We rewrite the assumptions of Section 3 under the notation $\alpha = \frac{p+q}{2}$, and we compare conditions (\mathcal{A}_1) – (\mathcal{A}_5) with the following ones in terms of (p, α) :

$$(\mathbb{A}_1) \langle A(x, s, \xi) - A(x, s, \eta), \xi - \eta \rangle \geq \nu (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2,$$

$$(\mathbb{A}_2) |A(x, s, \xi) - A(x, s, \eta)| \leq M(K) (1 + |\xi|^2 + |\eta|^2)^{\frac{q-2}{2}} |\xi - \eta|,$$

$$\begin{aligned}
 (\mathbb{A}_3) \quad & |A(x, s, \xi) - A(x, t, \xi)| \leq M(K) (1 + |\xi|^2)^{\frac{\alpha-2}{2}} |s - t|, \\
 (\mathbb{A}_4) \quad & |A(x, s, \xi) - A(y, s, \xi)| \leq M(K) (1 + |\xi|^2)^{\frac{\alpha-1}{2}} |x - y|, \\
 (\mathbb{A}_5) \quad & |b(x, u, \xi)| \leq M(K) (1 + |\xi|^2)^{\frac{\alpha-1}{2}}.
 \end{aligned}$$

Since $\alpha \leq q$ we have

$$(1 + |\xi|^2)^{\frac{\alpha-2}{2}} \leq (1 + |\xi|^2)^{\frac{q-2}{2}} \quad \text{and} \quad (1 + |\xi|^2)^{\frac{\alpha-1}{2}} \leq (1 + |\xi|^2)^{\frac{q-1}{2}}.$$

Therefore assumptions (\mathbb{A}_1) – (\mathbb{A}_5) with the choice $\alpha = \frac{p+q}{2}$ imply the familiar (p, q) -growth, see also [33, Section 6.2]. Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let $u \in W_{\text{loc}}^{1,q}(\Omega) \cap \mathcal{K}_\psi(\Omega)$ be a solution to (1.1). Take a radius $\tilde{R} < 1$ such that $\mathcal{B}_{\tilde{R}} \Subset \Omega$. For each $\varepsilon \in (0, 1)$, define

$$F_\varepsilon(x, s, \xi) := F(x, s, \xi) + \varepsilon |\xi|^q.$$

Note that, thanks to (\mathbb{A}_1) – (\mathbb{A}_2) , there exist two positive constants c_1 and c_2 such that F_ε fullfills the following growth condition

$$\varepsilon |\xi|^q + c_1 (1 + |\xi|^2)^{\frac{p}{2}} \leq F_\varepsilon(x, s, \xi) \leq c_2 (1 + |\xi|^2)^{\frac{q}{2}}, \quad (4.1)$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. We stress that c_1 and c_2 are independent of ε . Now we set

$$A_\varepsilon(x, s, \xi) := D_\xi F_\varepsilon(x, s, \xi) \quad \text{and} \quad b_\varepsilon(x, s, \xi) := -D_s F_\varepsilon(x, s, \xi).$$

From the definition of F_ε it follows that

$$b_\varepsilon(x, s, \xi) = -D_s F(x, s, \xi) = b(x, s, \xi).$$

Moreover, thanks to some algebraic inequalities that can be found in [35, Section 2], assumptions (I) – (V) imply the following conditions

$$\begin{aligned}
 (I_\varepsilon) \quad & \langle A_\varepsilon(x, s, \xi) - A_\varepsilon(x, s, \eta), \xi - \eta \rangle \geq \tilde{\nu} (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2 + \varepsilon \cdot \tilde{c} |\xi - \eta|^q, \\
 (II_\varepsilon) \quad & |A_\varepsilon(x, s, \xi) - A_\varepsilon(x, s, \eta)| \leq \widetilde{M} (1 + |\xi|^2 + |\eta|^2)^{\frac{q-2}{2}} |\xi - \eta|, \\
 (III_\varepsilon) \quad & |A_\varepsilon(x, s, \xi) - A_\varepsilon(x, t, \xi)| \leq \widetilde{M} (1 + |\xi|^2)^{\frac{\alpha-2}{2}} |s - t|, \\
 (IV_\varepsilon) \quad & |A_\varepsilon(x, s, \xi) - A_\varepsilon(y, s, \xi)| \leq \widetilde{M} (1 + |\xi|^2)^{\frac{\alpha-1}{2}} |x - y|, \\
 (V_\varepsilon) \quad & |b_\varepsilon(x, s, \xi)| \leq \widetilde{M} (1 + |\xi|^2)^{\frac{\alpha-1}{2}},
 \end{aligned}$$

where the constants $\tilde{\nu}, \widetilde{M}, \tilde{c}$ are positive and independent of ε .

Consider the obstacle problem

$$\min \left\{ \int_{\mathcal{B}_{\tilde{R}}} F_\varepsilon(x, w, Dw) \, dx : w \in u + W_0^{1,q}(\mathcal{B}_{\tilde{R}}) \cap \mathcal{K}_\psi(\Omega) \right\}. \quad (4.2)$$

Let $u_\varepsilon \in u + W_0^{1,q}(\mathcal{B}_{\tilde{R}}) \cap \mathcal{K}_\psi(\Omega)$ be a solution to (4.2). Thus u_ε satisfies the variational inequality

$$\int_{\Omega} \langle A_\varepsilon(x, u_\varepsilon, Du_\varepsilon), D(\varphi - u_\varepsilon) \rangle \, dx \geq \int_{\Omega} b(x, u_\varepsilon, Du_\varepsilon) (\varphi - u_\varepsilon) \, dx$$

for any $\varphi \in \mathcal{K}_\psi$.

Since $A_\varepsilon(x, u, \xi)$ and $b(x, u, \xi)$ satisfy assumptions (\mathcal{A}_1) – (\mathcal{A}_5) with $p = q$, we may apply Theorem 3.2 to each function u_ε , thus obtaining that $V_q(Du_\varepsilon) \in W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$. We point out that, since $p < q$, we trivially have that $V_q(Du_\varepsilon) \in W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$ implies $V_p(Du_\varepsilon) \in W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$.

Since (4.1) yields

$$c_1 (1 + |\xi|^2)^{\frac{p}{2}} \leq F_\varepsilon(x, s, \xi) \leq c_2 (1 + |\xi|^2)^{\frac{q}{2}}, \quad (4.3)$$

and (I_ε) implies

$$\langle A_\varepsilon(x, s, \xi) - A_\varepsilon(x, s, \eta), \xi - \eta \rangle \geq \tilde{\nu} (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$, we can apply Theorems 2.1 and 3.3 to each function u_ε to deduce the estimate

$$\begin{aligned} \int_{\mathcal{B}_{\frac{\rho}{2}}} |DV_p(Du_\varepsilon)|^2 dx &\leq c(\rho) (1 + \|u\|_{L^\infty(\mathcal{B}_\rho)})^\gamma \left[\int_{\mathcal{B}_\rho} (1 + |Du_\varepsilon|^2)^{\frac{p}{2}} dx \right. \\ &\quad \left. + \int_{\mathcal{B}_\rho} (|D^2\psi|^{2q-p} + |D\psi|^{2q-p}) dx \right] \end{aligned} \quad (4.4)$$

for any ball $\mathcal{B}_\rho \Subset \mathcal{B}_{\tilde{R}}$ and with a constant $c(\rho)$ independent of ε . It follows from (4.3) and the minimality of u_ε that

$$\begin{aligned} \int_{\mathcal{B}_{\tilde{R}}} |Du_\varepsilon|^p dx &\leq \int_{\mathcal{B}_{\tilde{R}}} (1 + |Du_\varepsilon|^2)^{\frac{p}{2}} dx \leq c \int_{\mathcal{B}_{\tilde{R}}} F_\varepsilon(x, u_\varepsilon, Du_\varepsilon) dx \\ &\leq c \int_{\mathcal{B}_{\tilde{R}}} F_\varepsilon(x, u, Du) dx \leq c \int_{\mathcal{B}_{\tilde{R}}} (1 + |Du|^2)^{\frac{q}{2}} dx, \end{aligned} \quad (4.5)$$

which implies that the sequence $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,p}(\mathcal{B}_{\tilde{R}})$. Then there exists a function $\tilde{u} \in u + W_0^{1,p}(\mathcal{B}_{\tilde{R}})$ such that, up to a subsequence, we have

$$u_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,p}(\mathcal{B}_{\tilde{R}}), \quad (4.6)$$

as $\varepsilon \rightarrow 0$. Then, up to a subsequence, we may assume that

$$u_\varepsilon \rightarrow \tilde{u} \quad \text{strongly in } L^p(\mathcal{B}_{\tilde{R}}),$$

and

$$u_\varepsilon \rightarrow \tilde{u} \quad \text{a.e. in } \mathcal{B}_{\tilde{R}} \quad (4.7)$$

as $\varepsilon \rightarrow 0$. On the other hand, the energy densities F_ε satisfy the assumptions of Theorem 2.1 and so

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty(\mathcal{B}_\rho)} &\leq c [\|\psi\|_{L^\infty(\mathcal{B}_{\tilde{R}})} + \|u_\varepsilon\|_{W^{1,p}(\mathcal{B}_{\tilde{R}})}] \\ &\leq c [\|\psi\|_{L^\infty(\mathcal{B}_{\tilde{R}})} + \|u_\varepsilon - u\|_{W^{1,p}(\mathcal{B}_{\tilde{R}})} + \|u\|_{W^{1,p}(\mathcal{B}_{\tilde{R}})}] \\ &\leq c [\|\psi\|_{L^\infty(\mathcal{B}_{\tilde{R}})} + \|Du_\varepsilon - Du\|_{L^p(\mathcal{B}_{\tilde{R}})} + \|u\|_{W^{1,p}(\mathcal{B}_{\tilde{R}})}] \\ &\leq c [\|\psi\|_{L^\infty(\mathcal{B}_{\tilde{R}})} + \|Du\|_{L^q(\mathcal{B}_{\tilde{R}})} + \|u\|_{W^{1,p}(\mathcal{B}_{\tilde{R}})}] \\ &\leq c [\|\psi\|_{L^\infty(\mathcal{B}_{\tilde{R}})} + \|u\|_{W^{1,q}(\mathcal{B}_{\tilde{R}})}] \end{aligned} \quad (4.8)$$

where we used (4.5) and the constant c is independent of ε . Therefore inserting (4.5), (4.8) in (4.4) we obtain

$$\begin{aligned} \int_{\mathcal{B}_{\frac{\rho}{2}}} |DV_p(Du_\varepsilon)|^2 dx &\leq c(\rho) (\|\psi\|_{L^\infty(\mathcal{B}_\rho)} + \|u\|_{W^{1,q}(\mathcal{B}_\rho)})^\gamma \\ &\times \left[\int_{\mathcal{B}_\rho} (1 + |Du|^2)^{\frac{p}{2}} dx + \int_{\mathcal{B}_\rho} (|D^2\psi|^{2q-p} + |D\psi|^{2q-p}) dx \right] \end{aligned}$$

and so $\{V_p(Du_\varepsilon)\}_\varepsilon$ is bounded in $W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$. As a consequence of the previous estimate and Lemma 2.2 we have that $\{Du_\varepsilon\}_\varepsilon$ is bounded in $L_{\text{loc}}^{p+2}(\mathcal{B}_{\tilde{R}})$ uniformly with respect to ε . So there exists a function $\tilde{w} \in W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$ such that, up to a subsequence,

$$V_p(Du_\varepsilon) \rightharpoonup \tilde{w} \quad \text{in } W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}}),$$

which implies that, up to a subsequence,

$$V_p(Du_\varepsilon) \rightarrow \tilde{w} \quad \text{in } L_{\text{loc}}^2(\mathcal{B}_{\tilde{R}}), \quad (4.9)$$

and so

$$V_p(Du_\varepsilon) \rightarrow \tilde{w} \quad \text{a.e. in } \mathcal{B}_{\tilde{R}} \quad (4.10)$$

as $\varepsilon \rightarrow 0$. Thanks to (4.6), we know that, as $\varepsilon \rightarrow 0$,

$$Du_\varepsilon \rightharpoonup D\tilde{u} \quad \text{in } L^p(\mathcal{B}_{\tilde{R}}), \quad (4.11)$$

and since the function $\xi \mapsto V_p(\xi)$ is continuous and bijective, (4.10) combined with (4.9) and (4.11) implies

$$V_p(D\tilde{u}) = \tilde{w} \quad \text{a.e. in } \mathcal{B}_{\tilde{R}}. \quad (4.12)$$

Hence,

$$Du_\varepsilon \rightarrow D\tilde{u} \quad \text{a.e. in } \mathcal{B}_{\tilde{R}}, \text{ as } \varepsilon \rightarrow 0. \quad (4.13)$$

Therefore, using (4.5), the Dominated Convergence Theorem, (4.12), and (4.13), we can pass to the limit in (4.4) to obtain

$$\begin{aligned} \int_{\mathcal{B}_{\frac{\rho}{2}}} |DV_p(D\tilde{u})|^2 dx &\leq c(\rho) [\|\psi\|_{L^\infty(\mathcal{B}_{\tilde{R}})} + \|u\|_{W^{1,q}(\mathcal{B}_{\tilde{R}})}]^\gamma \\ &\times \left[\int_{\mathcal{B}_\rho} (1 + |D\tilde{u}|^2)^{\frac{p}{2}} dx + \int_{\mathcal{B}_\rho} (|D^2\psi|^{2q-p} + |D\psi|^{2q-p}) dx \right], \end{aligned} \quad (4.14)$$

and so $V_p(D\tilde{u}) \in W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$.

By (4.8), we get that the family of functions $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(\mathcal{B}_R)$, therefore there exists a function $\tilde{v} \in L_{\text{loc}}^\infty(\mathcal{B}_R)$ such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{*} \tilde{v} \quad \text{in } L^\infty(\mathcal{B}_{\tilde{R}}) \quad (4.15)$$

as $\varepsilon \rightarrow 0$. Putting together (4.15) and (4.7), we derive

$$\tilde{v} = \tilde{u} \quad \text{a.e. in } \mathcal{B}_{\tilde{R}},$$

which implies $\tilde{u} \in L_{\text{loc}}^\infty(\mathcal{B}_R)$. Now, exploiting that $V_p(D\tilde{u}) \in W_{\text{loc}}^{1,2}(\mathcal{B}_{\tilde{R}})$ and applying Lemma 2.2, we get

$$D\tilde{u} \in L_{\text{loc}}^{p+2}(\mathcal{B}_{\tilde{R}}),$$

and so, since $q < p + 1$, a covering argument implies $\tilde{u} \in W^{1,q}(\mathcal{B}_{\tilde{R}})$. So we have $\tilde{u} \in u + W_0^{1,q}(\mathcal{B}_{\tilde{R}})$, and since the set $\mathcal{K}_\psi(\Omega)$ is closed, we also have $\tilde{u} \in \mathcal{K}_\psi(\Omega)$. Taking into account the minimality of u and u_ε , the semicontinuity of $F(x, s, \xi)$, and recalling the definition of F_ε , we get

$$\begin{aligned}
\int_{\mathcal{B}_{\tilde{R}}} F(x, u, Du) dx &\leq \int_{\mathcal{B}_{\tilde{R}}} F(x, \tilde{u}, D\tilde{u}) dx \leq \int_{\mathcal{B}_{\tilde{R}}} [F(x, \tilde{u}, D\tilde{u}) + \varepsilon |D\tilde{u}|^q] dx \\
&\leq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{B}_{\tilde{R}}} [F(x, u_\varepsilon, Du_\varepsilon) + \varepsilon |D\tilde{u}|^q] dx \\
&\leq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{B}_{\tilde{R}}} [F_\varepsilon(x, u_\varepsilon, Du_\varepsilon) + \varepsilon |D\tilde{u}|^q - \varepsilon |Du_\varepsilon|^q] dx \\
&\leq \liminf_{\varepsilon \rightarrow 0} \left(\int_{\mathcal{B}_{\tilde{R}}} F_\varepsilon(x, u, Du) dx + \varepsilon \int_{\mathcal{B}_{\tilde{R}}} (|D\tilde{u}|^q - |Du_\varepsilon|^q) dx \right) \\
&= \liminf_{\varepsilon \rightarrow 0} \left(\int_{\mathcal{B}_{\tilde{R}}} F(x, u, Du) dx + \varepsilon \int_{\mathcal{B}_{\tilde{R}}} (|Du|^q + |D\tilde{u}|^q - |Du_\varepsilon|^q) dx \right) \\
&= \int_{\mathcal{B}_{\tilde{R}}} F(x, u, Du) dx,
\end{aligned}$$

where we used that $\{Du_\varepsilon\}_\varepsilon$ is bounded in $L_{\text{loc}}^{p+2}(\mathcal{B}_{\tilde{R}})$ and therefore also in $L_{\text{loc}}^q(\mathcal{B}_{\tilde{R}})$. By virtue of the strict convexity of F , last inequality implies $u = \tilde{u}$ on $\mathcal{B}_{\tilde{R}}$. Finally, recalling (4.14), we get

$$\begin{aligned}
\int_{\mathcal{B}_{\frac{\rho}{2}}} |DV_p(Du)|^2 dx &\leq \frac{C}{\rho^2} \int_{\mathcal{B}_\rho} (1 + |Du|^2)^{\frac{p}{2}} dx + \frac{c}{\rho^2} \int_{\mathcal{B}_\rho} |D^2\psi|^{2q-p} dx \\
&\quad + \frac{c}{\rho^2} \int_{\mathcal{B}_\rho} |D\psi|^{2q-p} dx + \frac{c}{\rho^\gamma} |\mathcal{B}_\rho|,
\end{aligned}$$

for any ball $\mathcal{B}_\rho \Subset \mathcal{B}_{\tilde{R}}$. A standard covering argument yields the conclusion with estimate (1.4). \square

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ANDREA GENTILE

DIPARTIMENTO DI INGEGNERIA CIVILE, EDILE E ARCHITETTURA

UNIVERSITÀ POLITECNICA DELLE MARCHE

VIA BRECCE BIANCHE 12, 60131 ANCONA, ITALY

Email address: a.gentile@staff.univpm.it

TERESA ISERNIA

DIPARTIMENTO DI INGEGNERIA INDUSTRIALE E SCIENZE MATEMATICHE

UNIVERSITÀ POLITECNICA DELLE MARCHE

VIA BRECCE BIANCHE 12, 60131 ANCONA, ITALY

Email address: t.isernia@staff.univpm.it

ANTONIA PASSARELLI DI NAPOLI

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI “R. CACCIOPPOLI”

UNIVERSITÀ DEGLI STUDI DI NAPOLI “FEDERICO II”

VIA CINTIA, 80126 NAPLES, ITALY

Email address: antpassa@unina.it