ON THE UNIFORM VANISHING PROPERTY AT INFINITY OF $W^{s,p}$-SEQUENCES

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ABSTRACT. We prove that sequences of functions $(u_n) \subset W^{s,p}(\mathbb{R}^N)$, with $s \in (0,1)$ and $p \in (1, \frac{N}{s})$, bounded in $W^{s,p}(\mathbb{R}^N)$, strongly convergent in $L^{\frac{Np}{N-sp}}(\mathbb{R}^N)$ and solving nonlinear fractional p-Laplacian Schrödinger equations in $\mathbb{R}^N$, must vanish at infinity uniformly with respect to $n \in \mathbb{N}$.

1. INTRODUCTION

In this paper, we show the uniform vanishing property at infinity of $W^{s,p}(\mathbb{R}^N)$-sequences, with $s \in (0,1)$ and $p \in (1, \frac{N}{s})$, bounded in $W^{s,p}(\mathbb{R}^N)$, strongly convergent in $L^{p^*_s}(\mathbb{R}^N)$, where $p^*_s := \frac{Np}{N-sp}$, and that are solutions of nonlinear Schrödinger equations in $\mathbb{R}^N$ driven by the fractional p-Laplacian operator $(-\Delta)^s_p$ defined (up to a normalization constant), for $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough, by

$$(-\Delta)^s_p u(x) := 2 \lim_{r\to 0} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N.$$

More precisely, our main result can be stated as follows.

Theorem 1.1. *Let $s \in (0,1)$ and $p \in (1, \frac{N}{s})$. Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|f(x,t)| \leqslant \varepsilon |t|^{p-1} + C_\varepsilon |t|^{q-1} \quad \text{for a.e } x \in \mathbb{R}^N \text{ and for all } t \in \mathbb{R}, \tag{1.1}$$

with $q \in (p, p^*_s]$, and let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous potential such that, for some $V_0 \in (0, \infty)$,

$$V(x) \geqslant V_0 \quad \text{for all } x \in \mathbb{R}^N. \tag{1.2}$$

Let (u_n) be a nonnegative sequence in $W^{s,p}_V(\mathbb{R}^N)$ (see Section 2 for the precise definition of this space) such that $\|u_n\|_{W^{s,p}_V(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$, $u_n \to u$ in $L^{p^*_s}(\mathbb{R}^N)$ and each u_n solves (in weak sense)

$$(-\Delta)^s_p u + V(x)|u|^{p-2}u = f(x,u) \text{ in } \mathbb{R}^N. \tag{1.3}$$

Then, (u_n) satisfies the uniform vanishing property at infinity, i.e.

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0.$$

We recall that nonlocal equations like (1.3) have been extensively investigated in recent years, both for their interesting theoretical structure and in view of concrete real world applications; see [3,4,9,20] and the references therein. Now we give a sketch of the proof of Theorem 1.1. First we use suitable Moser iterations [15] to deduce that $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. The condition $u_n \to u$ in $L^{p^*_s}(\mathbb{R}^N)$ plays a crucial role when $q = p^*_s$ in (1.1), while for $q \in (p, p^*_s)$ it suffices to use the boundedness of (u_n) in $L^{p^*_s}(\mathbb{R}^N)$. The uniform L^∞-bound combined with the strong convergence in $L^{p^*_s}(\mathbb{R}^N)$ and an interpolation argument, implies that $u_n \to u$ in $L^r(\mathbb{R}^N)$ for all $r \in (p, \infty)$. By (1.1), we see that each u_n also satisfies $(-\Delta)^s_p u_n + \sigma u_n^{p-1} \leqslant \tilde{C} u_n^{q-1}$ in $\mathbb{R}^N$, with $\sigma, \tilde{C} > 0$, and in

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light of Browder-Minty theorem we can construct a nonnegative sequence $(z_n) \subset W^{s,p}(\mathbb{R}^N)$ such that each z_n solves $(-\Delta)^s_p z_n + \sigma z_n^{p-1} = \tilde{C} u_n^{q-1}$ in $\mathbb{R}^N$. By comparison, we know that $0 \leqslant u_n \leqslant z_n$ in $\mathbb{R}^N$. Furthermore, thanks to the strong convergence of (u_n) in $L^r(\mathbb{R}^N)$ for all $r \in (p, \infty)$, and the boundedness of (z_n) in $W^{s,p}(\mathbb{R}^N)$, we show that (z_n) strongly converges in $W^{s,p}(\mathbb{R}^N)$. This fact together with a Moser iteration yield $\|z_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. Then we can invoke the interior Hölder regularity result in [11] to infer that $\|z_n\|_{C^{0,\alpha}(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. Accordingly, we apply Lemma 2.1 below to see that $\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |z_n(x)| = 0$. Since $0 \leqslant u_n \leqslant z_n$ in $\mathbb{R}^N$, we conclude that $\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0$. We stress that the hypothesis on the sign of (u_n) is not essential and can be removed; see Remark 2.1.

When $s \to 1$, $(-\Delta)^s_p u$ reduces to the p-Laplacian operator $-\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and the proof of the uniform vanishing property at infinity of sequences $(u_n) \subset W^{1,p}_V(\mathbb{R}^N)$ satisfying $\|u_n\|_{W^{1,p}_V(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$, $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$ and $-\Delta_p u_n + V(x)|u_n|^{p-2}u_n = f(x, u_n)$ in $\mathbb{R}^N$, where f verifies (1.1) with $q \in (p, p^*]$, $p^* := \frac{Np}{N-p}$ and $N > p > 1$, is classically obtained by using appropriate Moser iterations. More precisely, arguing as in [14, Theorem 1.11] (see also [1, Lemma 4.5]), one takes as test function in the weak formulation of equation solved by u_n, the function

$$\varphi^p u_n(\min\{|u_n|, L\})^{p(\gamma-1)},$$

where $L > 0$, $\gamma > 1$, and $\varphi \in C^\infty(\mathbb{R}^N)$ is such that $0 \leqslant \varphi \leqslant 1$ in $\mathbb{R}^N$, $\varphi(x) = 1$ for $|x| \geqslant R$, $\varphi(x) = 0$ for $|x| \leqslant R - r$ and $|\nabla\varphi| \leqslant \frac{2}{r}$, with $R > 0$ and $0 < r \leqslant \frac{R}{2}$. A standard Moser iterative method and $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$ lead to

$$\|u_n\|_{L^\infty(B^c_R(0))} \leqslant C\|u_n\|_{L^{p^*}(B^c_{\frac{R}{2}}(0))} \quad \text{for all } n \in \mathbb{N},$$

from which we derive that $\lim_{|x|\to\infty} |u_n(x)| = 0$ uniformly in $n \in \mathbb{N}$. Alternatively, reasoning as in [7, Theorem 2.3], one can choose as test function

$$u_n(\min\{|u_n|, L\})^{p(\gamma-1)},$$

and by means of a Moser iteration and $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$, we arrive at $(u_n) \subset L^r(\mathbb{R}^N)$ for all $r \in [p, \infty)$ and that there exists a constant $C_r > 0$ such that $\|u_n\|_{L^r(\mathbb{R}^N)} \leqslant C_r$ for all $n \in \mathbb{N}$. These facts combined with the local L^∞-estimate for quasilinear equations due to Serrin [16, Theorem 1] ensure that, fixed $x \in \mathbb{R}^N$ and $t > \frac{N}{p}$,

$$\|u_n\|_{L^\infty(B_1(x))} \leqslant C(\|u_n\|_{L^p(B_2(x))} + \||u_n|^{p^*-1}\|^{\frac{1}{p-1}}_{L^t(B_2(x))}) \quad \text{for all } n \in \mathbb{N},$$

which provides the desired assertion by letting $|x| \to \infty$. However, these techniques do not seem adaptable for equations involving the nonlocal operator $(-\Delta)^s_p$, and thus Theorem 1.1 gives a useful tool to reach the uniform vanishing property at infinity of sequences (u_n) bounded in $W^{s,p}_V(\mathbb{R}^N)$, satisfying $u_n \to u$ in $L^{p^*_s}(\mathbb{R}^N)$ and (1.3). Notice that, when $s \in (0,1)$ and $p = 2$, Theorem 1.1 can be proved by exploiting the properties of the kernel of the resolvent $((-\Delta)^s + \lambda)^{-1}$ on $\mathbb{R}^N$ with $\lambda > 0$ (see [2, Lemma 2.6] and [3, Remark 7.2.10]), or making use of the extension method [8] and applying the L^∞-estimate for subsolutions of degenerate elliptic equations in half-balls (see [3, Lemma 6.3.23]). Nevertheless, these methods are not available for $s \in (0,1)$ and $p \in (1,\infty) \setminus \{2\}$, and new ideas are needed to accomplish the desired result.

To our knowledge, our approach is completely new in literature. We emphasize that our arguments can be adapted for the local case $s = 1$; see Remark 2.3. In addition, we observe that Theorem 1.1 and [9, Theorem 7.1] allow us to prove a decay estimate for $u_n(x)$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$; see Corollary 2.1. The remaining cases $N = sp$ and $N < sp$ in Theorem 1.1 are discussed in Section 3. Finally, we demonstrate a Kato's type inequality [12] for $(-\Delta)^s_p$ in Appendix.

2. Proofs of the main results

We start by establishing a very useful lemma that can be seen as a generalization for L^p-sequences uniformly equicontinuous in $\mathbb{R}^N$ of the following well-known fact: if $u : \mathbb{R}^N \to \mathbb{R}$ is uniformly continuous and $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, \infty)$, then $\lim_{|x|\to\infty} |u(x)| = 0$.

Lemma 2.1. *Let $N \geqslant 1$ and $p \in [1, \infty)$. Let $(u_n) \subset L^p(\mathbb{R}^N)$ be a sequence such that:*
(a) $u_n \to u$ in $L^p(\mathbb{R}^N)$,
(b) (u_n) is uniformly equicontinuous in $\mathbb{R}^N$, that is, for all $\eta > 0$ there exists $\delta = \delta_\eta > 0$ such that, if $x, y \in \mathbb{R}^N$ are such that $|x - y| < \delta$, then $|u_n(x) - u_n(y)| < \eta$ for all $n \in \mathbb{N}$.
Then,

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0.$$

Proof. Suppose the assertion of the lemma is false. Then there exist a sequence $(x_k) \subset \mathbb{R}^N$, a subsequence $(n_k) \subset \mathbb{N}$ and $\eta > 0$ such that $|x_k| > k$ and $|u_{n_k}(x_k)| \geqslant \eta$ for all $k \in \mathbb{N}$. Using (b), we can find $\delta = \delta_\eta > 0$ such that, if $|x - y| < \delta$ then $|u_n(x) - u_n(y)| < \frac{\eta}{2}$ for all $n \in \mathbb{N}$. Hence, $|u_{n_k}(x)| > \frac{\eta}{2}$ whenever $|x - x_k| < \delta$. On the other hand, from (a) and [6, Theorem 4.9], we derive that there exists $h \in L^p(\mathbb{R}^N)$ such that, up to a subsequence, $|u_{n_k}(x)| \leqslant h(x)$ for a.e. $x \in \mathbb{R}^N$ and for all $k \in \mathbb{N}$. Consequently,

$$C \geqslant \int_{\mathbb{R}^N} |h|^p\,dx \geqslant \sum_{k=1}^{\infty} \int_{|x-x_k|<\delta} |u_{n_k}(x)|^p\,dx \geqslant \sum_{k=1}^{\infty} \int_{|x-x_k|<\delta} \left(\frac{\eta}{2}\right)^p dx$$
$$= \sum_{k=1}^{\infty} \left(\frac{\eta}{2}\right)^p |B_\delta(0)| = \infty,$$

that is a contradiction. □

Next, in order to prove Theorem 1.1, we collect some notations and definitions. Let $s \in (0, 1)$ and $p \in (1, \frac{N}{s})$. By $\mathcal{D}^{s,p}(\mathbb{R}^N)$ we denote the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to

$$[u]_{s,p} := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\,dxdy \right)^{\frac{1}{p}},$$

or equivalently (see [9, Theorem 2.2])

$$\mathcal{D}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}.$$

The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ given by

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}$$

is equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left([u]_{s,p}^p + \|u\|_{L^p(\mathbb{R}^N)}^p \right)^{\frac{1}{p}}.$$

It is well-known that there exists a constant $S_* = S(N, s, p) > 0$ such that

$$S_* \|u\|_{L^{p_s^*}(\mathbb{R}^N)}^p \leqslant [u]_{s,p}^p \quad \text{for all } u \in \mathcal{D}^{s,p}(\mathbb{R}^N). \tag{2.1}$$

Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $q \in [p, p_s^*]$ and compactly in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, p_s^*)$, and $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$; see [3, 10] for more details.

Let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous potential satisfying (1.2). We define the Banach space $W_V^{s,p}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W_V^{s,p}(\mathbb{R}^N)} := \left([u]_{s,p}^p + \|V^{\frac{1}{p}} u\|_{L^p(\mathbb{R}^N)}^p \right)^{\frac{1}{p}}.$$

Note that (1.2) implies that $W^{s,p}_V(\mathbb{R}^N)\subset W^{s,p}(\mathbb{R}^N)$. For $s=1$, $W^{1,p}_V(\mathbb{R}^N)$ is defined in a similar way by considering $\|\nabla u\|_{L^p(\mathbb{R}^N)}$ in place of $[u]_{s,p}$.

We say that $u\in W^{s,p}_V(\mathbb{R}^N)$ is a weak solution to (1.3) if for all $\psi\in W^{s,p}_V(\mathbb{R}^N)$ it holds

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\psi(x)-\psi(y))\,dxdy+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi\,dx=\int_{\mathbb{R}^N} f(x,u)\psi\,dx.$$

In what follows, when we say that a nonnegative function $u\in W^{s,p}_V(\mathbb{R}^N)$ solves (in weak sense)

$$(-\Delta)^s_p u+V(x)u^{p-1}\leqslant(\geqslant) f(x,u) \text{ in } \mathbb{R}^N,$$

we mean that u fulfills

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\psi(x)-\psi(y))\,dxdy+\int_{\mathbb{R}^N} V(x)u^{p-1}\psi\,dx\leqslant(\geqslant)\int_{\mathbb{R}^N} f(x,u)\psi\,dx$$

for all $\psi\in W^{s,p}_V(\mathbb{R}^N)$ such that $\psi\geqslant 0$ in $\mathbb{R}^N$.

Proof of Theorem 1.1. We split the proof into two main steps.
Step 1 There exists a constant $C>0$ such that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)}\leqslant C \quad \text{for all } n\in\mathbb{N}, \tag{2.2}$$

We perform a Moser iteration argument [15]. For each $L>0$ and $\gamma>1$, we define

$$\phi(t):=tt_L^{p(\gamma-1)} \quad \text{for } t\geqslant 0, \text{ where } t_L:=\min\{t,L\},$$

Notice that ϕ is Lipschitz and nondecreasing in $[0,\infty)$. Let us consider

$$\Phi(t):=\int_0^t (\phi'(\tau))^{\frac{1}{p}}\,d\tau.$$

By [5, Lemma A.2], we know that

$$|a-b|^{p-2}(a-b)(\phi(a)-\phi(b))\geqslant|\Phi(a)-\Phi(b)|^p \quad \text{for all } a,b\geqslant 0. \tag{2.3}$$

Set $u_{L,n}:=\min\{u_n,L\}$ and $w_{L,n}:=u_nu_{L,n}^{\gamma-1}$. Choosing $\phi(u_n)=u_nu_{L,n}^{p(\gamma-1)}\in W^{s,p}_V(\mathbb{R}^N)$ as test function in the weak formulation of (1.3), and exploiting (2.3), we have

$$\begin{aligned}&[\Phi(u_n)]^p_{s,p}+\int_{\mathbb{R}^N}V(x)w^p_{L,n}\,dx\\&\leqslant\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))((u_nu_{L,n}^{p(\gamma-1)})(x)-(u_nu_{L,n}^{p(\gamma-1)})(y))}{|x-y|^{N+sp}}\,dxdy\\&+\int_{\mathbb{R}^N}V(x)u_n^pu_{L,n}^{p(\gamma-1)}\,dx=\int_{\mathbb{R}^N}f(x,u_n)u_nu_{L,n}^{p(\gamma-1)}\,dx.\end{aligned}\tag{2.4}$$

Let us observe that

$$\Phi(t)\geqslant\frac{1}{\gamma}\,tt_L^{\gamma-1} \quad \text{for all } t\geqslant 0,$$

which combined with (2.1) yields

$$[\Phi(u_n)]^p_{s,p}\geqslant S_*\|\Phi(u_n)\|^p_{L^{p^*_s}(\mathbb{R}^N)}\geqslant\frac{1}{\gamma^p}S_*\|w_{L,n}\|^p_{L^{p^*_s}(\mathbb{R}^N)}.$$

Thus, by (2.4),

$$\frac{1}{\gamma^p}S_*\|w_{L,n}\|^p_{L^{p^*_s}(\mathbb{R}^N)}+\int_{\mathbb{R}^N}V(x)w^p_{L,n}\,dx\leqslant\int_{\mathbb{R}^N}f(x,u_n)u_nu_{L,n}^{p(\gamma-1)}\,dx.$$

Using (1.1) (with $\varepsilon = \frac{V_0}{2}$) and (1.2), we see that, for some constant $C_{\frac{V_0}{2}} > 0$,

$$\frac{1}{\gamma^p} S_* \|w_{L,n}\|^p_{L^{p_s^*}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V_0 w^p_{L,n}\, dx \leqslant \int_{\mathbb{R}^N} \frac{V_0}{2} w^p_{L,n}\, dx + C_{\frac{V_0}{2}} \int_{\mathbb{R}^N} u_n^q u_{L,n}^{p(\gamma-1)}\, dx$$

from which we derive

$$\|w_{L,n}\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C_1 \gamma^p \int_{\mathbb{R}^N} u_n^q u_{L,n}^{p(\gamma-1)}\, dx, \tag{2.5}$$

where $C_1 := S_*^{-1} C_{\frac{V_0}{2}} > 0$. Now we distinguish the following cases.

Case 1 When $q \in (p, p_s^*)$.

Because

$$u_n^q u_{L,n}^{p(\gamma-1)} = u_n^{q-p} w^p_{L,n},$$

the Hölder inequality and (2.5) lead to

$$\|w_{L,n}\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C_1 \gamma^p \|u_n\|^{q-p}_{L^{p_s^*}(\mathbb{R}^N)} \|w_{L,n}\|^p_{L^{\alpha_s^*}(\mathbb{R}^N)},$$

where

$$\alpha_s^* := \frac{p p_s^*}{p_s^* - (q-p)} \in (p, p_s^*).$$

Exploiting the fact that

$$\|u_n\|_{L^{p_s^*}(\mathbb{R}^N)} \leqslant K \quad \text{for all } n \in \mathbb{N}, \tag{2.6}$$

we have

$$\|w_{L,n}\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C_2 \gamma^p \|w_{L,n}\|^p_{L^{\alpha_s^*}(\mathbb{R}^N)},$$

where $C_2 := C_1 K^{q-p} > 0$. Note that, if $u_n \in L^{\gamma \alpha_s^*}(\mathbb{R}^N)$, then, due to $u_{L,n} \leqslant u_n$, we find

$$\|w_{L,n}\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C_2 \gamma^p \|u_n\|^{p\gamma}_{L^{\gamma\alpha_s^*}(\mathbb{R}^N)} < \infty,$$

and applying Fatou's lemma, as $L \to \infty$, we get

$$\|u_n\|_{L^{\gamma p_s^*}(\mathbb{R}^N)} \leqslant C_2^{\frac{1}{p\gamma}} \gamma^{\frac{1}{\gamma}} \|u_n\|_{L^{\gamma\alpha_s^*}(\mathbb{R}^N)}, \tag{2.7}$$

and so $u_n \in L^{\gamma p_s^*}(\mathbb{R}^N)$. Now we set $\gamma := \frac{p_s^*}{\alpha_s^*} > 1$, and observe that, since $u_n \in L^{p_s^*}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, (2.7) holds for this choice of γ. Thus, owing to $\gamma^2 \alpha_s^* = \gamma p_s^*$, we see that (2.7) is true with γ replaced by γ^2. Hence,

$$\begin{aligned}
\|u_n\|_{L^{\gamma^2 p_s^*}(\mathbb{R}^N)} &\leqslant C_2^{\frac{1}{p\gamma^2}} \gamma^{\frac{2}{\gamma^2}} \|u_n\|_{L^{\gamma^2\alpha_s^*}(\mathbb{R}^N)} \\
&\leqslant C_2^{\frac{1}{p}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)} \gamma^{\frac{1}{\gamma}+\frac{2}{\gamma^2}} \|u_n\|_{L^{\gamma\alpha_s^*}(\mathbb{R}^N)} \\
&= C_2^{\frac{1}{p}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)} \gamma^{\frac{1}{\gamma}+\frac{2}{\gamma^2}} \|u_n\|_{L^{p_s^*}(\mathbb{R}^N)},
\end{aligned}$$

where we have used the fact that $\gamma\alpha_s^* = p_s^*$. Iterating this process, we obtain

$$\|u_n\|_{L^{\gamma^m p_s^*}(\mathbb{R}^N)} \leqslant C_2^{\sum_{j=1}^m \frac{1}{p\gamma^j}} \gamma^{\sum_{j=1}^m \frac{j}{\gamma^j}} \|u_n\|_{L^{p_s^*}(\mathbb{R}^N)} \quad \text{for all } n, m \in \mathbb{N}.$$

In view of (2.6), we arrive at

$$\|u_n\|_{L^{\gamma^m p_s^*}(\mathbb{R}^N)} \leqslant C_2^{\sum_{j=1}^m \frac{1}{p\gamma^j}} \gamma^{\sum_{j=1}^m \frac{j}{\gamma^j}} K \quad \text{for all } n, m \in \mathbb{N}. \tag{2.8}$$

Letting $m \to \infty$ in (2.8), we deduce that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C_2^{\sigma_1} \gamma^{\sigma_2} K =: C_3 \quad \text{for all } n \in \mathbb{N},$$

where

$$\sigma_1 := \sum_{j=1}^{\infty} \frac{1}{p\gamma^j} < \infty \quad \text{and} \quad \sigma_2 := \sum_{j=1}^{\infty} \frac{j}{\gamma^j} < \infty.$$

Case 2 When $q = p^*_s$.

Suppose $(u_n) \subset L^{p\gamma}(\mathbb{R}^N)$ and prove that $(u_n) \subset L^{p^*_s\gamma}(\mathbb{R}^N)$. Since $u_n^{p^*_s} u_{L,n}^{p(\gamma-1)} = h_n w_{L,n}^p$, where $h_n := u_n^{p^*_s - p} \in L^{\frac{N}{sp}}(\mathbb{R}^N)$, and recalling (2.5), we have

$$\|w_{L,n}\|^p_{L^{p^*_s}(\mathbb{R}^N)} \leqslant C_1 \gamma^p \int_{\mathbb{R}^N} h_n w_{L,n}^p \, dx. \tag{2.9}$$

Let $M > 0$ to be fixed later and set

$$A_{n,M} := \{x \in \mathbb{R}^N : h_n(x) \leqslant M\} \text{ and } B_{n,M} := \{x \in \mathbb{R}^N : h_n(x) > M\}.$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^N} h_n w_{L,n}^p \, dx &= \int_{A_{n,M}} h_n w_{L,n}^p \, dx + \int_{B_{n,M}} h_n w_{L,n}^p \, dx \\ &\leqslant M \|w_{L,n}\|^p_{L^p(\mathbb{R}^N)} + \left(\int_{B_{n,M}} h_n^{\frac{N}{sp}} \, dx \right)^{\frac{sp}{N}} \|w_{L,n}\|^p_{L^{p^*_s}(\mathbb{R}^N)}. \end{aligned} \tag{2.10}$$

Thanks to $u_n \to u$ in $L^{p^*_s}(\mathbb{R}^N)$, we can take $M \equiv M_\gamma > 0$ sufficiently large such that

$$\left(\int_{B_{n,M}} h_n^{\frac{N}{sp}} \, dx \right)^{\frac{p}{sN}} \leqslant \frac{1}{2C_1 \gamma^p} \quad \text{for all } n \in \mathbb{N}. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), and using $w_{L,n} \leqslant u_n^\gamma$, we get

$$\|w_{L,n}\|^q_{L^{p^*_s}(\mathbb{R}^N)} \leqslant 2C_1 \gamma^p M_\beta \|u_n\|^{p\gamma}_{L^{p\gamma}(\mathbb{R}^N)},$$

and invoking Fatou's lemma, as $L \to \infty$, we reach

$$\|u_n\|^{p\gamma}_{L^{p^*_s\gamma}(\mathbb{R}^N)} \leqslant 2C_1 M_\gamma \gamma \|u_n\|^{p\gamma}_{L^{p\gamma}(\mathbb{R}^N)}. \tag{2.12}$$

Therefore a bootstrap argument can start: because $(u_n) \subset L^{p^*_s}(\mathbb{R}^N)$, we can apply (2.12) with $\gamma = \frac{p^*_s}{p}$ to obtain that $(u_n) \subset L^{\left(\frac{p^*_s}{p}\right)p^*_s}(\mathbb{R}^N)$. We can then apply again (2.12) and, after k iterations, we find $(u_n) \subset L^{\left(\frac{p^*_s}{p}\right)^k p^*_s}(\mathbb{R}^N)$ and thus $(u_n) \subset L^r(\mathbb{R}^N)$ for all $r \in [p^*_s, \infty)$. We now go back to inequality (2.5). By $0 \leqslant u_{L,n} \leqslant u_n$ and letting $L \to \infty$ in (2.5), we have

$$\left(\int_{\mathbb{R}^N} u_n^{p^*_s\gamma} \, dx \right)^{\frac{p}{p^*_s}} \leqslant C_1 \gamma^p \int_{\mathbb{R}^N} u_n^{p^*_s + p(\gamma-1)} \, dx,$$

and so

$$\left(\int_{\mathbb{R}^N} u_n^{p^*_s\gamma} \, dx \right)^{\frac{1}{p^*_s(\gamma-1)}} \leqslant (\sqrt[p]{C_1}\gamma)^{\frac{1}{\gamma-1}} \left(\int_{\mathbb{R}^N} u_n^{p^*_s + p(\gamma-1)} \, dx \right)^{\frac{1}{p(\gamma-1)}}. \tag{2.13}$$

Let $\gamma_1 := \frac{p^*_s}{p}$ and define γ_m inductively so that $p^*_s + p(\gamma_{m+1} - 1) = p^*_s \gamma_m$ for $m \in \mathbb{N}$. Hence,

$$\gamma_m = \gamma_1^{m-1}(\gamma_1 - 1) + 1 \text{ for } m \in \mathbb{N}, \text{ and } \lim_{m \to \infty} \gamma_m = \infty.$$

Put

$$\Psi_{m,n} := \left(\int_{\mathbb{R}^N} u_n^{p^*_s \gamma_m} \, dx \right)^{\frac{1}{p^*_s(\gamma_m - 1)}} \quad \text{for all } m, n \in \mathbb{N}.$$

Consequently, (2.13) can be written as

$$\Psi_{m+1,n} \leqslant C_{m+1}^{\frac{1}{\gamma_{m+1}-1}} \Psi_{m,n} \quad \text{for all } m, n \in \mathbb{N},$$

where $C_{m+1} := \sqrt[q]{C_1}\gamma_{m+1}$. Iterating the above relation, we find

$$\Psi_{m+1,n} \leqslant \left(\prod_{j=2}^{m+1} C_j^{\frac{1}{\gamma_j - 1}} \right) \Psi_{1,n} \quad \text{for all } m \in \mathbb{N} \setminus \{1\}, n \in \mathbb{N}. \tag{2.14}$$

In view of (2.6), it follows from (2.12) with $\gamma = \gamma_1 = \frac{p_s^*}{p}$ that, for some $C' > 0$,

$$\Psi_{1,n} \leqslant \left(2C_1 M_{\frac{p_s^*}{p}} \frac{p_s^*}{p} \right)^{\frac{(p_s^*)^2}{p_s^* - p}} \|u_n\|_{L^{p_s^*}(\mathbb{R}^N)}^{\frac{(p_s^*)^2}{p_s^*-p}} \leqslant C' \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, standard calculations show that there exists $C'' > 0$ such that

$$\prod_{j=2}^{m+1} C_j^{\frac{1}{\gamma_j - 1}} = (\sqrt[q]{C_1})^{\sum_{j=2}^{m+1} \frac{1}{\gamma_1^{j-1}(\gamma_1 - 1)}} \prod_{j=2}^{m+1} \left(\gamma_1^{j-1}(\gamma_1 - 1) + 1 \right)^{\frac{1}{\gamma_1^{j-1}(\gamma_1-1)}} \leqslant C'' \quad \text{for all } m \in \mathbb{N} \setminus \{1\}.$$

Then, passing to the limit as $m \to \infty$ in (2.14), we can infer that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C'C'' \quad \text{for all } n \in \mathbb{N}.$$

Step 2 Conclusion.
From (2.2) and [11, Corollary 5.5], we know that $u_n \in C^0(\mathbb{R}^N)$ for all $n \in \mathbb{N}$ (indeed, $u_n \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$). Using (2.2), (u_n) is bounded in $L^p(\mathbb{R}^N)$, $u_n \to u$ in $L^{p_s^*}(\mathbb{R}^N)$ and the interpolation inequality for L^r-spaces, we get

$$\|u_n\|_{L^r(\mathbb{R}^N)} \leqslant C \quad \text{for all } n \in \mathbb{N} \text{ and } r \in [p, \infty], \tag{2.15}$$

and

$$u_n \to u \text{ in } L^r(\mathbb{R}^N) \quad \text{for all } r \in (p, \infty). \tag{2.16}$$

Now, we fix $\varepsilon \in (0, V_0)$. Exploiting (1.1) and (1.2), we have that u_n solves (in weak sense)

$$(-\Delta)_p^s u_n + \sigma u_n^{p-1} \leqslant \tilde{C} u_n^{q-1} \text{ in } \mathbb{R}^N,$$

where $\sigma := V_0 - \varepsilon > 0$ and $\tilde{C} := C_\varepsilon > 0$.

Let us introduce the operator $\mathcal{A} : W^{s,p}(\mathbb{R}^N) \to (W^{s,p}(\mathbb{R}^N))^*$, where $(W^{s,p}(\mathbb{R}^N))^*$ denotes the dual of $W^{s,p}(\mathbb{R}^N)$, given by

$$\langle \mathcal{A}(u), v \rangle := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dxdy$$
$$+ \sigma \int_{\mathbb{R}^N} |u|^{p-2}uv \, dx \quad \text{for } u, v \in W^{s,p}(\mathbb{R}^N).$$

We claim that $\mathcal{A}$ satisfies all assumptions of Browder-Minty theorem (see [19, Theorem 26.A]), and as a result $\mathcal{A}$ is surjective. Recall that $W^{s,p}(\mathbb{R}^N)$ is a (real) reflexive Banach space[1]. Next we verify that $\mathcal{A}$ is hemicontinuous, i.e. the real function $t \mapsto \langle \mathcal{A}(u + tv), w \rangle$ is continuous on $[0, 1]$ for all $u, v, w \in W^{s,p}(\mathbb{R}^N)$. Let $(t_n) \subset [0, 1]$ be such that $t_n \to t$. We have,

$\langle \mathcal{A}(u + t_n v), w \rangle$

[1]Let us consider the linear isometry $T : W^{s,p}(\mathbb{R}^N) \to E := L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^{2N})$ defined as $T(u) := \left(u, \frac{u(x)-u(y)}{|x-y|^{\frac{N+sp}{p}}} \right)$. Since $W^{s,p}(\mathbb{R}^N)$ is a Banach space, $T(W^{s,p}(\mathbb{R}^N))$ is a closed subspace of E. Using this fact and that $L^r(\mathbb{R}^k)$ is a reflexive, separable and uniformly convex Banach space for all $r \in (1, \infty)$ and $k \in \mathbb{N}$ (see [6, Chapter 4]), we deduce that $W^{s,p}(\mathbb{R}^N)$ is a reflexive, separable and uniformly convex space for all $s \in (0, 1)$ and $p \in (1, \infty)$.

$$= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)+t_n(v(x)-v(y))|^{p-2}(u(x)-u(y)+t_n(v(x)-v(y)))(w(x)-w(y))}{|x-y|^{N+sp}}\,dxdy$$
$$+ \int_{\mathbb{R}^N} |u+t_nv|^{p-2}(u+t_nv)w\,dx,$$

and

$$\left|\frac{|u(x)-u(y)+t_n(v(x)-v(y))|^{p-2}(u(x)-u(y)+t_n(v(x)-v(y)))(w(x)-w(y))}{|x-y|^{N+sp}}\right|$$
$$\leqslant \frac{(|u(x)-u(y)|+|v(x)-v(y)|)^{p-1}|w(x)-w(y)|}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^{2N}),$$
$$||u+t_nv|^{p-2}(u+t_nv)w| \leqslant (|u|+|v|)^{p-1}|w| \in L^1(\mathbb{R}^N).$$

Then, applying the dominated convergence theorem, we deduce the assertion. Bearing in mind the following Simon's inequalities [17, formula (2.2)],

$$(|x|^{p-2}x-|y|^{p-2}y)\cdot(x-y) \geqslant \begin{cases} c_p|x-y|^p & \text{if } p\in[2,\infty), \\ c_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p\in(1,2), \end{cases} \qquad \text{for all } x,y\in\mathbb{R}^N,$$

we see that $\mathcal{A}$ is monotone, i.e. $\langle \mathcal{A}(u)-\mathcal{A}(v), u-v\rangle \geqslant 0$ for all $u,v\in W^{s,p}(\mathbb{R}^N)$. Finally, $\mathcal{A}$ is coercive (due to $\langle \mathcal{A}(u),u\rangle/\|u\|_{W^{s,p}(\mathbb{R}^N)} \geqslant \min\{1,\sigma\}\|u\|^{p-1}_{W^{s,p}(\mathbb{R}^N)} \to \infty$ as $\|u\|_{W^{s,p}(\mathbb{R}^N)}\to\infty$). Therefore, Browder-Minty theorem implies that $\mathcal{A}$ is surjective[2], that is, for all $v\in(W^{s,p}(\mathbb{R}^N))^*$ there exists $u\in W^{s,p}(\mathbb{R}^N)$ such that $\mathcal{A}(u)=v$. Thus, since $\tilde{C}u_n^{q-1}\in L^{\frac{p}{p-1}}(\mathbb{R}^N)\subset(W^{s,p}(\mathbb{R}^N))^*$, there exists $z_n\in W^{s,p}(\mathbb{R}^N)$ solving (in weak sense)

$$(-\Delta)^s_p z_n + \sigma|z_n|^{p-2}z_n = \tilde{C}u_n^{q-1} \text{ in } \mathbb{R}^N. \tag{2.17}$$

Taking $z_n^- := \min\{z_n,0\}$ in the weak formulation of (2.17), and using $u_n\geqslant 0$ and the elementary inequality

$$|x-y|^{p-2}(x-y)(x^- - y^-) \geqslant |x^- - y^-|^p \quad \text{for all } x,y\in\mathbb{R},$$

it is easy to check that $z_n\geqslant 0$ in $\mathbb{R}^N$. By comparison (see [9, Theorem 7.1]), we obtain that $0\leqslant u_n\leqslant z_n$ in $\mathbb{R}^N$ and for all $n\in\mathbb{N}$. Now, testing (2.17) with z_n, we derive that

$$[z_n]^p_{s,p} + \sigma\|z_n\|^p_{L^p(\mathbb{R}^N)} = \tilde{C}\int_{\mathbb{R}^N} u_n^{q-1}z_n\,dx, \tag{2.18}$$

and exploiting the Young's inequality with $\eta\in(0,\sigma)$, i.e. $ab\leqslant \eta a^p + C_\eta b^{\frac{p}{p-1}}$ for all $a,b\geqslant 0$, on the right hand side of (2.18), we have

$$[z_n]^p_{s,p} + \sigma\|z_n\|^p_{L^p(\mathbb{R}^N)} \leqslant \eta\tilde{C}\|z_n\|^p_{L^p(\mathbb{R}^N)} + C_\eta\tilde{C}\int_{\mathbb{R}^N}|u_n|^{\frac{(q-1)p}{p-1}}\,dx.$$

In view of $\frac{(q-1)p}{p-1}\in(p,\infty)$ and (2.15), we know that

$$\int_{\mathbb{R}^N}|u_n|^{\frac{(q-1)p}{p-1}}\,dx \leqslant C \quad \text{for all } n\in\mathbb{N},$$

and so $\|z_n\|_{W^{s,p}(\mathbb{R}^N)}\leqslant C$ for all $n\in\mathbb{N}$. Hence, up to a subsequence, we may assume that

$$z_n \rightharpoonup z \quad \text{in } W^{s,p}(\mathbb{R}^N). \tag{2.19}$$

From (2.19) and $u_n\to u$ in $L^{\frac{(q-1)p}{p-1}}(\mathbb{R}^N)$ (thanks to (2.16)), it follows that z solves (in weak sense)

$$(-\Delta)^s_p z + \sigma|z|^{p-2}z = \tilde{C}u^{q-1} \text{ in } \mathbb{R}^N.$$

[2]Indeed, $\mathcal{A}$ is invertible because $\mathcal{A}$ is strictly monotone.

In particular,

$$[z]_{s,p}^p + \sigma \|z\|_{L^p(\mathbb{R}^N)}^p = \tilde{C} \int_{\mathbb{R}^N} u^{q-1} z \, dx, \tag{2.20}$$

Next we show that

$$\int_{\mathbb{R}^N} u_n^{q-1} z_n \, dx \to \int_{\mathbb{R}^N} u^{q-1} z \, dx. \tag{2.21}$$

Let us observe that

$$\begin{aligned} &\int_{\mathbb{R}^N} u_n^{q-1} z_n \, dx - \int_{\mathbb{R}^N} u^{q-1} z \, dx \\ &= \int_{\mathbb{R}^N} (u_n^{q-1} - u^{q-1}) z_n \, dx + \int_{\mathbb{R}^N} (z_n - z) u^{q-1} \, dx =: X_n + Y_n. \end{aligned}$$

In light of (2.19), it is clear that $Y_n \to 0$. On the other hand, employing the Hölder inequality, (2.19), (2.16) with $r = \frac{(q-1)p}{p-1}$, and the dominated convergence theorem, we have

$$\begin{aligned} |X_n| &\leqslant \left(\int_{\mathbb{R}^N} |u_n^{q-1} - u^{q-1}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \|z_n\|_{L^p(\mathbb{R}^N)} \\ &\leqslant C \left(\int_{\mathbb{R}^N} |u_n^{q-1} - u^{q-1}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \to 0. \end{aligned}$$

Therefore, (2.21) is satisfied. Combining (2.18), (2.20) and (2.21), we deduce that

$$[z_n]_{s,p}^p + \sigma \|z_n\|_{L^p(\mathbb{R}^N)}^p = [z]_{s,p}^p + \sigma \|z\|_{L^p(\mathbb{R}^N)}^p + o_n(1),$$

and recalling that $W^{s,p}(\mathbb{R}^N)$ is a uniformly convex space, we infer that $z_n \to z$ in $W^{s,p}(\mathbb{R}^N)$. In particular, $z_n \to z$ in $L^t(\mathbb{R}^N)$ for all $t \in [p, p_s^*]$ and condition (a) of Lemma 2.1 is true.

Testing (2.17) with $z_n z_{L,n}^{p(\gamma-1)}$ and exploiting $\tilde{C} u_n^{q-1} \leqslant \tilde{C} z_n^{q-1}$, we can proceed as in Step 1 to obtain the following estimate

$$\|z_n z_{L,n}^{\gamma-1}\|_{L^{p_s^*}(\mathbb{R}^N)}^p \leqslant \tilde{C}_1 \gamma^p \int_{\mathbb{R}^N} z_n^q z_{L,n}^{p(\gamma-1)} \, dx,$$

where $\tilde{C}_1 := S_*^{-1} \tilde{C} > 0$, instead of (2.5). On account of $z_n \to z$ in $L^{p_s^*}(\mathbb{R}^N)$, we can perform the same iteration arguments given in Step 1 to arrive at

$$\|z_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C \quad \text{for all } n \in \mathbb{N}. \tag{2.22}$$

Invoking [11, Corollary 5.5] and using (2.22), we see that, fixed $x_0 \in \mathbb{R}^N$, $(z_n) \subset C^{0,\alpha}(B_{1/2}(x_0))$ for some $\alpha \in (0,1)$ depending only on N, s, p and independent of $n \in \mathbb{N}$ and x_0, and it holds

$$[z_n]_{C^{0,\alpha}(B_{1/2}(x_0))} := \sup_{x,y \in B_{1/2}(x_0),\, x \neq y} \frac{|z_n(x) - z_n(y)|}{|x-y|^\alpha} \leqslant C_{N,s,p} \quad \text{for all } n \in \mathbb{N}, \tag{2.23}$$

where $C_{N,s,p} > 0$ is independent of $x_0 \in \mathbb{R}^N$. Combining (2.22) with (2.23), we get

$$\|z_n\|_{C^{0,\alpha}(\mathbb{R}^N)} := \|z_n\|_{L^\infty(\mathbb{R}^N)} + [z_n]_{C^{0,\alpha}(\mathbb{R}^N)} \leqslant C' \quad \text{for all } n \in \mathbb{N}. \tag{2.24}$$

In fact, fixed $x, y \in \mathbb{R}^N$, if $|x-y| \geqslant 1$, then (2.22) implies

$$|z_n(x) - z_n(y)| \leqslant 2\|z_n\|_{L^\infty(\mathbb{R}^N)} \leqslant 2C \leqslant 2C|x-y|^\alpha,$$

whereas, if $|x-y| < 1$, then $|x - \frac{x+y}{2}| = |y - \frac{x+y}{2}| = \frac{|x-y|}{2} < \frac{1}{2}$ and from (2.23) we have

$$|z_n(x) - z_n(y)| \leqslant \left| z_n(x) - z_n\left(\frac{x+y}{2}\right) \right| + \left| z_n(y) - z_n\left(\frac{x+y}{2}\right) \right| \leqslant C|x-y|^\alpha.$$

In particular, (2.24) implies that condition (b) of Lemma 2.1 holds. Indeed, fixed $\eta > 0$ and setting $\delta := \left(\frac{\eta}{2C'}\right)^{\frac{1}{\alpha}}$, we obtain that, for all $x, y \in \mathbb{R}^N$ such that $|x-y| < \delta$,

$$|z_n(x) - z_n(y)| \leqslant C'|x-y|^\alpha < \eta \quad \text{ for all } n \in \mathbb{N},$$

and this proves the assertion.

Therefore, we can exploit Lemma 2.1 to conclude that

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |z_n(x)| = 0.$$

Recalling that $0 \leqslant u_n \leqslant z_n$ in $\mathbb{R}^N$ and for all $n \in \mathbb{N}$, the above limit gives

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0.$$

The proof of the theorem is now complete. □

Remark 2.1. *The conclusion of Theorem 1.1 remains true without requiring any hypothesis of sign on the sequence (u_n). In fact, if we use a Kato's type inequality for $(-\Delta)^s_p$ (see Theorem 4.1 in Appendix), we deduce that $|u_n|$ satisfies $(-\Delta)^s_p|u_n| + V(x)|u_n|^{p-1} \leqslant |f(x,u_n)|$ in $\mathbb{R}^N$ which combined with (1.1) gives $(-\Delta)^s_p|u_n| + \sigma|u_n|^{p-1} \leqslant \tilde{C}|u_n|^{q-1}$ in $\mathbb{R}^N$. Then we can repeat the same reasonings developed in Steps 1 and 2 by replacing u_n by $|u_n|$.*

Remark 2.2. *If $V \in L^\infty(\mathbb{R}^N)$, then it is not needed to introduce the sequence (z_n) given in the proof of Theorem 1.1. In fact, in this situation, we have that $(-\Delta)^s_p u_n = h_n$ in $\mathbb{R}^N$, where*

$$h_n(x) := -V(x)u_n^{p-1}(x) + f(x, u_n(x)).$$

Now, $h_n \in L^\infty(\mathbb{R}^N)$ (by $V \in L^\infty(\mathbb{R}^N)$ and (2.2)), and using the same argument to prove (2.24), we infer that $\|u_n\|_{C^{0,\alpha}(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. Since $u_n \to u$ in $L^{p^*_s}(\mathbb{R}^N)$, we can apply Lemma 2.1 to get the desired result.

Remark 2.3. *If we replace the fractional p-Laplacian operator $(-\Delta)^s_p$ by the p-Laplacian operator $-\Delta_p$, the same conclusion of Theorem 1.1 is valid for nonnegative sequence $(u_n) \subset W^{1,p}_V(\mathbb{R}^N)$ such that $\|u_n\|_{W^{1,p}(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$, $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$ and each u_n solves $-\Delta_p u_n + V(x)u_n^{p-1} = f(x,u_n)$ in $\mathbb{R}^N$, where f verifies (1.1) with $q \in (p, p^*]$. Henceforth, we just point out some differences. If we test the equation with $u_n u_{L,n}^{p(\gamma-1)}$, then it appears the term*

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n u_{L,n}^{p(\gamma-1)})\, dx$$

instead of the double integral in (2.4). Observing that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n u_{L,n}^{p(\gamma-1)})\, dx \\
&= \int_{\mathbb{R}^N} u_{L,n}^{p(\gamma-1)} |\nabla u_n|^p\, dx + p(\gamma-1)\int_{\{u_n \leqslant L\}} u_{L,n}^{p(\gamma-1)}|\nabla u_{L,n}|^p\, dx \\
&\geqslant \int_{\mathbb{R}^N} u_{L,n}^{p(\gamma-1)} |\nabla u_n|^p\, dx,
\end{aligned}$$

and that, by standard calculations,

$$\|w_{L,n}\|^p_{L^{p^*}(\mathbb{R}^N)} \leqslant 2^p \mathcal{S}_*^{-1}\gamma^p \int_{\mathbb{R}^N} u_{L,n}^{p(\gamma-1)}|\nabla u_n|^p\, dx,$$

we have, similarly to the proof of Theorem 1.1, the following inequality in place of (2.5)

$$\|w_{L,n}\|^p_{L^{p^*}(\mathbb{R}^N)} \leqslant C_1\gamma^p \int_{\mathbb{R}^N} u_n^q u_{L,n}^{p(\gamma-1)}\, dx,$$

where $C_1 := 2^p \mathcal{S}_*^{-1} C_{\frac{V_0}{2}}$ (here $\mathcal{S}_*$ is the best constant related to the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$). Then, the same iteration arguments used in Step 1 in Theorem 1.1 show that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C \quad \text{for all } n \in \mathbb{N}.$$

Moreover, instead of the result in [11], we invoke the standard regularity theory for quasilinear elliptic equations (see for instance [13, 14, 16, 18]) to deduce that $\|z_n\|_{C^{0,\alpha}(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. Repeating the reasonings made in Step 2 of Theorem 1.1 (with the appropriate modifications), we achieve $\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0$.

Corollary 2.1. *Under the same assumptions of Theorem 1.1, with $f \in C^0(\mathbb{R}^N \times \mathbb{R})$, there exists a constant $C > 0$ independent on $n \in \mathbb{N}$ such that*

$$0 \leqslant u_n(x) \leqslant \frac{C}{|x|^{\frac{N+sp}{p-1}}} \quad \text{for all } |x| >> 1 \text{ and } n \in \mathbb{N}.$$

Proof. Using Theorem 1.1, $\frac{|f(x,t)|}{t^{p-1}} \to 0$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$ (by (1.1)), and (1.2), we can find $k_0 > 0$ such that

$$\begin{aligned}(-\Delta)_p^s u_n + \frac{V_0}{2} u_n^{p-1} &= f(x, u_n) - \left(V(x) - \frac{V_0}{2}\right) u_n^{p-1} \\ &\leqslant f(x, u_n) - \frac{V_0}{2} u_n^{p-1} \leqslant 0 \text{ in } \overline{B}_{k_0}^c(0). \end{aligned} \tag{2.25}$$

On the other hand, by [9, Lemma 7.1], taking $\alpha := \frac{N+sp}{p-1}$ and a positive function $\Upsilon \in C^2(\mathbb{R}^N)$ such that Υ is radially symmetric, decreasing and $\Upsilon(x) = \frac{1}{|x|^\alpha}$ for all $|x| > 1$, there exists $k_1 >> 1$ such that

$$(-\Delta)_p^s \Upsilon + c_1 \Upsilon^{p-1} \geqslant 0 \text{ in } \overline{B}_{k_1}^c(0), \tag{2.26}$$

for some $c_1 > 0$. Define $\phi(x) := m\Upsilon(rx)$, where $m, r > 0$ will be chosen later in a suitable way. By means of (2.26), we see that

$$(-\Delta)_p^s \phi = \frac{m}{r^{sp}} (-\Delta)_p^s \Upsilon(r\,\cdot) \geqslant -\frac{c_1}{r^{sp}} \phi^{p-1} \text{ in } \overline{B}_{\frac{k}{r}}^c(0),$$

and setting

$$r := \left(\frac{2c_1}{V_0}\right)^{\frac{1}{sp}} \text{ and } k_2 := \frac{k_1}{r},$$

we obtain that

$$(-\Delta)_p^s \phi + \frac{V_0}{2} \phi^{p-1} \geqslant 0 \text{ in } \overline{B}_{k_2}^c(0). \tag{2.27}$$

Put $\kappa := \max\{k_0, k_2\}$. Combining (2.25) and (2.27), we get

$$(-\Delta)_p^s u_n + \frac{V_0}{2} u_n^{p-1} \leqslant 0 \leqslant (-\Delta)_p^s \phi + \frac{V_0}{2} \phi^{p-1} \text{ in } \overline{B}_\kappa^c(0).$$

Using (2.2) and that Υ is a positive and continuous function in $\mathbb{R}^N$, we can choose

$$m := \frac{C_3}{\min_{\overline{B}_{\kappa r}(0)} \Upsilon} > 0$$

in order to have $u_n \leqslant \phi$ in $\overline{B}_\kappa(0)$. Then, by comparison (see [9, Theorem 7.1]), we derive that $u_n \leqslant \phi$ in $\mathbb{R}^N$. Therefore,

$$0 \leqslant u_n(x) \leqslant \frac{m}{r^{\frac{N+sp}{p-1}} |x|^{\frac{N+sp}{p-1}}} \quad \text{for all } |x| > \frac{1}{r} \text{ and } n \in \mathbb{N}.$$

The proof of the corollary is now complete. ◻

3. Some final comments

In this section, we discuss the remaining cases $N < sp$ and $N = sp$, with $s \in (0,1)$ and $p \in (1, \infty)$.

Let us first assume $N < sp$. If $q \in (p, \infty)$ in (1.1), $(u_n) \subset W^{s,p}_V(\mathbb{R}^N)$ is such that $\|u_n\|_{W^{s,p}_V(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$, and $u_n \to u$ in $L^r(\mathbb{R}^N)$ for some $r \in [p, \infty)$, then the conclusion of Theorem 1.1 still remains true. In fact, thanks to the continuous embedding $W^{s,p}(\mathbb{R}^N) \subset C^{0, s-\frac{N}{p}}(\mathbb{R}^N)$, we deduce that (u_n) is a uniformly equicontinuous sequence and thus we can apply Lemma 2.1.

Now we suppose $N = sp$. We replace (1.1) by the following condition: there exists $\alpha_0 \geqslant 0$ such that, for all $\varepsilon > 0$, $q > p$ and $\alpha > \alpha_0$ there exists $C_{\varepsilon,q,\alpha} > 0$ such that

$$|f(x,t)| \leqslant \varepsilon\, |t|^{p-1} + C_{\varepsilon,q,\alpha} |t|^{q-1} \Theta_{N,s}(\alpha |t|^{\frac{p}{p-1}}),$$

where

$$\Theta_{N,s}(t) := e^t - \sum_{j=0}^{j_p - 2} \frac{t^j}{j!} \quad \text{ with } j_p := \min\{k \in \mathbb{N} : k \geqslant p\},$$

and assume that $(u_n) \subset W^{s,p}_V(\mathbb{R}^N)$ is such that $\|u_n\|_{W^{s,p}_V(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$ if $\alpha_0 = 0$, and $\limsup_{n\to\infty} \|u_n\|^{\frac{p}{p-1}}_{W^{s,p}_V(\mathbb{R}^N)} < \frac{\alpha_*}{\alpha_0}$ if $\alpha_0 > 0$, for some suitable $\alpha_* > 0$, and $u_n \to u$ in $L^r(\mathbb{R}^N)$ for some $r \in [p, \infty)$. Then the conclusion of Theorem 1.1 still holds. In this case, we invoke the following version of the Trudinger-Moser inequality [20, Theorem 1.3]:

$$\sup_{u \in W^{s,p}(\mathbb{R}^N),\, \|u\|_{W^{s,p}_{V_0}(\mathbb{R}^N)} \leqslant 1} \int_{\mathbb{R}^N} \Theta_{N,s}(\alpha |u|^{\frac{p}{p-1}})\, dx < \infty \quad \text{ for all } \alpha \in [0, \alpha_*). \tag{3.1}$$

The main differences concern the estimates below to handle (2.4) in Step 1. Using the growth assumption on f, we see that

$$\int_{\mathbb{R}^N} f(x,u_n) u_n u_{L,n}^{p(\gamma-1)}\, dx \leqslant \int_{\mathbb{R}^N} \frac{V_0}{2} w_{L,n}^p\, dx + C \int_{\mathbb{R}^N} \Theta_{N,s}(\alpha |u_n|^{\frac{p}{p-1}}) u_n^q u_{L,n}^{p(\gamma-1)}\, dx.$$

Select $\sigma > 1$ such that $\sigma(q-p) \geqslant p$. Choose $t > 1$ and $\alpha > 0$ small if $\alpha_0 = 0$, while $t > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 if $\alpha_0 > 0$, such that

$$\alpha t \|u_n\|^{\frac{p}{p-1}}_{W^{s,p}_{V_0}(\mathbb{R}^N)} < \alpha_* \quad \text{ for all sufficiently large } n \in \mathbb{N}.$$

Let $\mu > 1$ be such that $\frac{1}{t} + \frac{1}{\sigma} + \frac{1}{\mu} = 1$. From the generalized Hölder inequality, $\Theta_{N,s}(a)^\rho \leqslant \Theta_{N,s}(\rho a)$ for all $a \geqslant 0$ and $\rho \geqslant 1$, the boundedness of (u_n) in $W^{s,p}(\mathbb{R}^N)$, the Sobolev embedding $W^{s,p}(\mathbb{R}^N) \subset L^\nu(\mathbb{R}^N)$ for all $\nu \in [p, \infty)$, and (3.1), we have, for all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Theta_{N,s}(\alpha |u_n|^{\frac{p}{p-1}}) u_n^q u_{L,n}^{p(\gamma-1)}\, dx \\
&\leqslant \|u_n\|^{q-p}_{L^{\sigma(q-p)}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \Theta_{N,s}\left(\alpha t \|u_n\|^{\frac{p}{p-1}}_{W^{s,p}_{V_0}(\mathbb{R}^N)} \left(\frac{|u_n|}{\|u_n\|_{W^{s,p}_{V_0}(\mathbb{R}^N)}} \right)^{\frac{p}{p-1}} \right) dx \right)^{\frac{1}{t}} \|w_{L,n}\|^p_{L^{\mu p}(\mathbb{R}^N)} \\
&\leqslant C \|w_{L,n}\|^p_{L^{\mu p}(\mathbb{R}^N)},
\end{aligned}$$

where $C > 0$ is independent of L and n. On the other hand, fixed $\tau > \mu p$, and exploiting $w_{L,n}^p \geqslant \Phi(u_n)^p$ (by (2.3)), the Sobolev embedding $W^{s,p}_{\frac{V_0}{2}}(\mathbb{R}^N) \subset L^\tau(\mathbb{R}^N)$, and $\Phi(u_n) \geqslant \frac{1}{\gamma} w_{L,n}$, we obtain

$$[\Phi(u_n)]^p_{s,p} + \int_{\mathbb{R}^N} \frac{V_0}{2} w_{L,n}^p\, dx \geqslant \|\Phi(u_n)\|^p_{W^{s,p}_{\frac{V_0}{2}}(\mathbb{R}^N)} \geqslant C \|\Phi(u_n)\|^p_{L^\tau(\mathbb{R}^N)} \geqslant \frac{C}{\gamma^p} \|w_{L,n}\|^p_{L^\tau(\mathbb{R}^N)}.$$

Consequently,

$$\|w_{L,n}\|^p_{L^\tau(\mathbb{R}^N)} \leqslant C\gamma^p \|w_{L,n}\|^p_{L^{\mu p}(\mathbb{R}^N)},$$

and letting $L \to \infty$ we find

$$\|u_n\|_{L^{\gamma\tau}(\mathbb{R}^N)} \leqslant C^{\frac{1}{p\gamma}} \gamma^{\frac{1}{\gamma}} \|u_n\|_{L^{\mu p\gamma}(\mathbb{R}^N)}.$$

Therefore, taking $\kappa := \frac{\tau}{\mu p} > 1$, we get, for all $m \in \mathbb{N}$,

$$\|u_n\|_{L^{\tau\kappa^m}(\mathbb{R}^N)} \leqslant C^{\sum_{i=1}^m p^{-1}\kappa^{-i}} \kappa^{\sum_{i=1}^m i\kappa^{-i}} \|u_n\|_{L^\tau(\mathbb{R}^N)},$$

from which $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all sufficiently large $n \in \mathbb{N}$.

4. Appendix

We establish a Kato's type inequality [12] for the fractional p-Laplacian operator.

Theorem 4.1. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let $V \in C^0(\mathbb{R}^N)$ with $V(x) \geqslant V_0 > 0$ in $\mathbb{R}^N$, and $g \in L^1_{loc}(\mathbb{R}^N)$ such that $g\xi \in L^1(\mathbb{R}^N)$ for all $\xi \in W^{s,p}_V(\mathbb{R}^N)$. Let $u \in W^{s,p}_V(\mathbb{R}^N)$ be such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\psi(x)-\psi(y))\,dxdy + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi\,dx = \int_{\mathbb{R}^N} g\psi\,dx, \tag{4.1}$$

for all $\psi \in W^{s,p}_V(\mathbb{R}^N)$. Then it holds

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{||u(x)|-|u(y)||^{p-2}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V(x)|u|^{p-1}\varphi\,dx \\ &\leqslant \int_{\mathbb{R}^N} \operatorname{sign}(u) g\varphi\,dx,\end{aligned} \tag{4.2}$$

for all $\varphi \in W^{s,p}_V(\mathbb{R}^N)$ such that $\varphi \geqslant 0$ in $\mathbb{R}^N$, where

$$\operatorname{sign}(u)(x) := \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Proof. The proof is inspired by some arguments found in [3, Theorem 17.3.5] and [5, Theorem 3.1]. We start by recalling the following useful inequality (see [5, Lemma A.1]): if $\zeta : \mathbb{R} \to \mathbb{R}$ is a C^1 convex function, $\varepsilon \geqslant 0$, $J_p(t) := |t|^{p-2}t$ and $J_{p,\varepsilon}(t) := (\varepsilon + |t|^2)^{\frac{p-2}{2}} t$ with $t \in \mathbb{R}$, then

$$\begin{aligned}&J_p(a-b)[AJ_{p,\varepsilon}(\zeta'(a)) - BJ_{p,\varepsilon}(\zeta'(b))] \\ &\geqslant \left(\varepsilon(a-b)^2 + (\zeta(a)-\zeta(b))^2\right)^{\frac{p-2}{2}} (\zeta(a)-\zeta(b))(A-B) \quad \text{for all } a,b \in \mathbb{R},\ A,B \geqslant 0.\end{aligned} \tag{4.3}$$

Next we prove that (4.2) is valid for all $\varphi \in C^\infty_c(\mathbb{R}^N)$ such that $\varphi \geqslant 0$ in $\mathbb{R}^N$. Fix $\varphi \in C^\infty_c(\mathbb{R}^N)$ such that $\varphi \geqslant 0$ in $\mathbb{R}^N$. For $\delta > 0$ sufficiently small, we define the smooth convex Lipschitz function $\zeta_\delta(t) := \sqrt{t^2+\delta^2}$ for $t \in \mathbb{R}$. Observe that, for all $t, \tau \in \mathbb{R}$,

$$|\zeta'_\delta(t)| \leqslant 1, \quad |\zeta'_\delta(t) - \zeta'_\delta(\tau)| \leqslant \frac{1}{\delta}|t-\tau|. \tag{4.4}$$

We first consider the case $p \geqslant 2$. Let $\psi_\delta := \varphi|\zeta'_\delta(u)|^{p-2}\zeta'_\delta(u)$. Notice that $\psi_\delta \in W^{s,p}_V(\mathbb{R}^N)$ because $\varphi \in C^\infty_c(\mathbb{R}^N)$, $u \in W^{s,p}_V(\mathbb{R}^N)$, $|\zeta'_\delta(u)|^{p-2}\zeta'_\delta(u)$ is bounded (by (4.4)) and Lipschitz (thanks to $||a|^{p-2}a - |b|^{p-2}b| \leqslant C_p(|a|+|b|)^{p-2}|a-b|$ for all $a,b \in \mathbb{R}$, (4.4) and $[u]_{s,p} < \infty$), and thus the assertion follows from [3, Lemma 1.1.4]. Then, inserting $\psi = \psi_\delta$ into (4.1), and applying (4.3) with

$$\zeta := \zeta_\delta,\ \varepsilon := 0,\ a := u(x),\ b := u(y),\ A := \varphi(x),\ B := \varphi(y),$$

we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|\zeta_\delta(u(x))-\zeta_\delta(u(y))|^{p-2}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dxdy+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi_\delta\,dx\\ &\leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\psi_\delta(x)-\psi_\delta(y))\,dxdy+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi_\delta\,dx\\ &=\int_{\mathbb{R}^N} g\psi_\delta\,dx\end{aligned}$$

and so

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|\zeta_\delta(u(x))-\zeta_\delta(u(y))|^{p-2}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dxdy\\ &+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi_\delta\,dx \leqslant \int_{\mathbb{R}^N} g\psi_\delta\,dx.\end{aligned}$$

Next we show that we can pass to the limit as $\delta\to 0$ in the above relation to deduce (4.2). By virtue of $\zeta_\delta(t)\to|t|$ as $\delta\to 0$ and

$$\begin{aligned}&\left|\frac{|\zeta_\delta(u(x))-\zeta_\delta(u(y))|^{p-2}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\right|\\ &\leqslant \frac{|\zeta_\delta(u(x))-\zeta_\delta(u(y))|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\\ &\leqslant \frac{|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\in L^1(\mathbb{R}^{2N}),\end{aligned}$$

where we have used the fact that $\zeta_\delta(t)$ has Lipschitz constant equals to one, we can apply the dominated convergence theorem to see that

$$\begin{aligned}&\lim_{\delta\to 0}\iint_{\mathbb{R}^{2N}} \frac{|\zeta_\delta(u(x))-\zeta_\delta(u(y))|^{p-2}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dxdy\\ &=\iint_{\mathbb{R}^{2N}} \frac{||u(x)|-|u(y)||^{p-2}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy.\end{aligned}$$

Since $\zeta'_\delta(t)\to \mathrm{sign}(t)$ as $\delta\to 0$ and $|V(x)|u|^{p-2}u\psi_\delta|\leqslant V(x)|u|^{p-1}\varphi\in L^1(\mathbb{R}^N)$ (in light of (4.4), $u\in W^{s,p}_V(\mathbb{R}^N)$ and $\varphi\in C^\infty_c(\mathbb{R}^N)$), we can again invoke the dominated convergence theorem to obtain

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi_\delta\,dx=\int_{\mathbb{R}^N} V(x)|u|^{p-1}\varphi\,dx.$$

Finally, thanks to $|\zeta'_\delta(t)|^{p-2}\zeta'_\delta(t)\to\mathrm{sign}(t)$ as $\delta\to 0$ and $|g\psi_\delta|\leqslant |g|\varphi\in L^1(\mathbb{R}^N)$ (owing to (4.4) and the assumption on g), the dominated convergence theorem yields

$$\int_{\mathbb{R}^N} g\psi_\delta\,dx\to\int_{\mathbb{R}^N}\mathrm{sign}(u)g\varphi\,dx.$$

When $p\in(1,2)$, ψ_δ is no more a legitimate test function and we consider a slight modification of it. More precisely, we pick $\psi=\tilde{\psi}_\delta:=\varphi(\delta+|\zeta'_\delta(u)|^2)^{\frac{p-2}{2}}\zeta'_\delta(u)$ in (4.1). Observe that $\tilde{\psi}_\delta\in W^{s,p}_V(\mathbb{R}^N)$ because $\varphi\in C^\infty_c(\mathbb{R}^N)$, $u\in W^{s,p}_V(\mathbb{R}^N)$, $(\delta+|\zeta'_\delta(u)|^2)^{\frac{p-2}{2}}\zeta'_\delta(u)$ is bounded (in view of $|(\delta+t^2)^{\frac{p-2}{2}}t|\leqslant |t|^{p-1}$ for all $t\in\mathbb{R}$ and (4.4)) and Lipschitz (due to $|(\delta+t^2)^{\frac{p-2}{2}}t-(\delta+\tau^2)^{\frac{p-2}{2}}\tau|\leqslant(p-1)\delta^{\frac{p-2}{2}}|t-\tau|$ for all $t,\tau\in\mathbb{R}$, (4.4) and $[u]_{s,p}<\infty$), and hence the assertion follows from [3, Lemma 1.1.4]. Using (4.3) with

$$\zeta:=\zeta_\delta,\ \varepsilon:=\delta,\ a:=u(x),\ b:=u(y),\ A:=\varphi(x),\ B:=\varphi(y),$$

we get

$$\iint_{\mathbb{R}^{2N}} \frac{[\delta(u(x)-u(y))^2+(\zeta_\delta(u(x))-\zeta_\delta(u(y)))^2]^{\frac{p-2}{2}}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dxdy$$
$$+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\tilde{\psi}_\delta\,dx$$
$$\leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\tilde{\psi}_\delta(x)-\tilde{\psi}_\delta(y))\,dxdy+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\tilde{\psi}_\delta\,dx$$
$$=\int_{\mathbb{R}^N} g\tilde{\psi}_\delta\,dx$$

and thus

$$\iint_{\mathbb{R}^{2N}} \frac{[\delta(u(x)-u(y))^2+(\zeta_\delta(u(x))-\zeta_\delta(u(y)))^2]^{\frac{p-2}{2}}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dxdy$$
$$+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\tilde{\psi}_\delta\,dx \leqslant \int_{\mathbb{R}^N} g\tilde{\psi}_\delta\,dx.$$

In what follows we justify the passage to the limit as $\delta \to 0$ in the above relation. Since $p \in (1,2)$, we see that

$$\left|\frac{[\delta(u(x)-u(y))^2+(\zeta_\delta(u(x))-\zeta_\delta(u(y)))^2]^{\frac{p-2}{2}}(\zeta_\delta(u(x))-\zeta_\delta(u(y)))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\right|$$
$$\leqslant \frac{|\zeta_\delta(u(x))-\zeta_\delta(u(y))|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}$$
$$\leqslant \frac{|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^{2N}),$$

and using $\zeta_\delta(t) \to |t|$ as $\delta \to 0$, $(\delta+|\zeta_\delta'(t)|^2)^{\frac{p-2}{2}}\zeta_\delta'(t) \to \mathrm{sign}(t)$ as $\delta \to 0$, $|\tilde{\psi}_\delta| \leqslant |\zeta_\delta'(t)|^{p-1}\varphi \leqslant \varphi$, $V(x)|u|^{p-1}\varphi \in L^1(\mathbb{R}^N)$ and $|g|\varphi \in L^1(\mathbb{R}^N)$, as before we can apply the dominated convergence theorem to achieve our aim. Hence we have proved that (4.2) holds for all $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \geqslant 0$ in $\mathbb{R}^N$. Because $C_c^\infty(\mathbb{R}^N)$ is dense in $W_V^{s,p}(\mathbb{R}^N)$, we get the assertion. □

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