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Birkhoff–Kellogg type results in product spaces and their application to differential systems

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Abstract. We provide a new version of the well-known Birkhoff–Kellogg invariant-direction Theorem in product spaces. Our results concern operator systems and give the existence of component-wise eigenvalues, instead of scalar eigenvalues as in the classical case, that have corresponding eigenvectors with all components nontrivial and localized by their norm. We also show that, when applied to nonlinear eigenvalue problems for differential equations, this localization property of the eigenvectors provides, in turn, qualitative properties of the solutions. This is illustrated in two contexts of systems of PDEs and ODEs. We show the applicability of our theoretical results with two explicit examples.

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Keywords. Birkhoff-Kellogg type result, nonlinear eigenvalue problem, operator system, nontrivial solution.

1. Introduction

A celebrated result in nonlinear analysis is the Birkhoff–Kellogg invariant-direction Theorem [5]. This theorem has been object of extensive research in the past and finds significant applications in the study of nonlinear eigenvalue problems in infinite-dimensional normed linear spaces, see for example the books [2,13,19], the recent papers [7,16], and references therein. In the version by Krasnosel'skiĭ and Ladyženskiĭ [20], a similar result is set in cones of real Banach spaces, yielding the existence of a pair, constituted by a positive eigenvalue and an eigenvector, the latter localized inside a cone. We stress that a notable, common feature of the two above-mentioned theorems is that they provide a localization of the eigenfunction; this localization, in turn, provides qualitative properties of the solution in the context of applications

to nonlinear eigenvalue problems for differential equations. In the framework of systems the situation is somewhat more delicate. In fact, a direct application of one of the previous two results in product spaces would provide the existence of an eigenvalue with a corresponding (vectorial) eigenfunction that may have trivial components; this issue has been highlighted for example in [16, Definition 1].

In this paper, we present a new version of the Birkhoff–Kellogg Theorem in product spaces, which, instead of a scalar eigenvalue, provides the existence of *component-wise* eigenvalues that have corresponding eigenvectors with all components nontrivial and localized by their norm. In the context of systems of integral equations and their applications, component-wise eigenvalues have been investigated in Chapter 3 of the book [1], where the authors sought constant-sign eigenvectors by means of topological tools such as the Schauder and the Krasnosel'skiĭ-Guo fixed point theorems, and in the book [15] where the authors sought, also via topological fixed point theory, the existence of positive eigenvectors. More in details, we study systems of type

$$\begin{cases} x = \lambda_1 T_1(x, y), \\ y = \lambda_2 T_2(x, y), \end{cases}$$

where $T = (T_1, T_2)$ is a suitable compact operator acting on the Cartesian product of two sets C_1 and C_2, which can be as follows:

(1) C_1 and C_2 are cones;
(2) C_1 is a cone and $C_2 = X_2$ is a infinite dimensional normed space;
(3) both $C_1 = X_1$ and $C_2 = X_2$ are infinite dimensional normed spaces.

In the context of fixed point theory, a component-wise approach has been utilized in the past, see for example [3,4,17,18,27,28]; here we develop a somewhat analogue theory in the framework of nonlinear spectral theory. In particular, we give quite natural conditions yielding the existence of component-wise eigenvalues $\lambda_1, \lambda_2 > 0$ and corresponding eigenfunctions x_0, y_0, both of prescribed non-zero norm. We also provide further results in the settings (2) and (3), which yield the additional existence of negative eigenvalues with corresponding eigenfunctions. Furthermore we present a direct application of the classical result by Krasnosel'skiĭ and Ladyženskiĭ to the context of systems and present a comparison between the two approaches, see Remark 2.8.

Section 3 of the paper is devoted to applications of the theoretical results to differential systems. We stress that these applications are given as a motivation for the theoretical results and are provided for illustrative purposes. In order to keep the paper as readable as possible, we study these problems in the case of continuous nonlinearities.

We firstly focus on the systems of PDEs

$$\begin{cases} -\Delta u = \lambda_1 f(x, u, v), & \text{in } \Omega, \\ -\Delta v = \lambda_2 g(x, u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ denotes the open unit ball in $\mathbb{R}^n$ and f, g are suitable non-negative continuous functions. Problems of this type are well-studied, both

in the classical case of the spectral problem, that is with $\lambda = \lambda_1 = \lambda_2$, see for example [6,9,17,26], and in the case of component-wise eigenvalues with possibly different λ_1, λ_2, see for instance [8,14,24,25] and references therein. Within this framework, we utilize the cone product setting (1); in Theorem 3.1 we provide sufficient conditions for (1.1) that ensure the existence of a pair of positive component-wise eigenvalues with eigenfunctions possessing nontrivial components with localized norms. The result is then illustrated in a specific example.

Our second setting of application is the following BVP for a system of ODEs

$$\begin{cases} u''(t) + \lambda_1 f(t, u(t), v(t)) = 0, \ t \in [0,1], \\ v''(t) + \lambda_2 g(t, u(t), v(t)) = 0, \ t \in [0,1], \\ u'(0) = u(1) + u'(1) = 0, \\ v'(0) = v(1) - \frac{1}{2} v'(1) = 0. \end{cases} \tag{1.2}$$

where f, g are suitable non-negative continuous functions. The BCs that occur in (1.2) have been investigated for a different set of parameters by Lan [21,25]. We rewrite the system (1.2) in terms of a system of Hammerstein integral equations. Since one of the two corresponding kernels changes sign, we use the setting (2), namely the Cartesian product of a conical shell times a ball in the space of continuous functions. In this case our approach yields the existence of two distinct pairs of component-wise eigenvalues with nontrivial eigenfunctions possessing nontrivial components; this is illustrated in a toy model as well.

Overall, the results are new, from both the theoretical and the applied point of view, and complement the ones in [1,2,6,8,9,14–17,24–26,28].

2. Birkhoff–Kellogg type results

We begin this Section by recalling the classical Birkhoff–Kellogg invariant-direction Theorem [5], cf. [12, Theorem 6.1].

Theorem 2.1. *Let U be a bounded open neighbourhood of 0 in an infinite-dimensional normed linear space $(V, \| \|)$, and let $T : \partial U \to V$ be a compact map satisfying $\|T(x)\| \geq \alpha$ for some $\alpha > 0$ for every x in ∂U. Then there exist $x_0 \in \partial U$ and $\lambda_0 \in (0, +\infty)$ such that $x_0 = \lambda_0 T(x_0)$.*

In the following version by Krasnosel'skiĭ and Ladyženskiĭ [20], cf. [19, Theorem 5.5], a similar result is set in cones of real Banach spaces; we recall that a cone K of a normed linear space $(X, \| \|)$ is a closed set with $K + K \subset K$, $\mu K \subset K$ for all $\mu \geq 0$ and $K \cap (-K) = \{0\}$.

Theorem 2.2. *Let X be a real Banach space, $U \subset X$ be an open bounded set with $0 \in U$, $K \subset X$ be a cone, $T : K \cap \overline{U} \longrightarrow K$ be compact and suppose that*

$$\inf_{x \in K \cap \partial U} \|Tx\| > 0.$$

Then there exist $x_0 \in K \cap \partial U$ and $\lambda_0 > 0$ such that $x_0 = \lambda_0 T x_0$.

Before stating our results we fix some notation. Let $(X, \|\ \|)$ be a normed linear space and K a cone in X. Given $r > 0$, by B_r we mean the open ball in X centered at the origin and with radius r, while by $\overline{B}_r$, $\partial\overline{B}_r$ we mean the closed disk and its boundary, respectively. Moreover, we denote by $K_r = B_r \cap K$, and by $\overline{K}_r = \overline{B}_r \cap K$, resp. $\partial\overline{K}_r = \partial\overline{B}_r \cap K$, the closure and boundary of K_r relative to K.

Observe that $\partial\overline{K}_r$ is a retract of $\overline{K}_r$. An explicit example of such a retraction can be found in [10, Example 3] defined as

$$\rho(z) = r\frac{z + (r - \|z\|)^2\, h}{\|z + (r - \|z\|)^2\, h\|}, \quad z \in \overline{K}_r,$$

where $h \in K \setminus \{0\}$ is fixed.

With abuse of notation (the whole space X, when nontrivial, is not a cone) we will still denote $X_r = B_r$, $\overline{X}_r = \overline{B}_r$, $\partial\overline{X}_r = \partial\overline{B}_r$, so that, if X is infinite dimensional, again $\partial\overline{X}_r$ is a retract of $\overline{X}_r$.

Let X_1 and X_2 be normed linear spaces and $C_1 \subset X_1$, $C_2 \subset X_2$ such that for each $i \in \{1, 2\}$ either

(a) $C_i = K_i$ is a cone; or
(b) $C_i = X_i$ is an infinite dimensional normed space.

In the following we will consider the product space $X_1 \times X_2$ endowed with the maximum norm. With abuse of notation, the norm in any space will be still denoted by $\|\ \|$. The next result is a version of the Birkhoff–Kellogg Theorem in product spaces.

Theorem 2.3. *Let r_1, r_2 be positive constants and suppose that*

$$T = (T_1, T_2) : \overline{C}_{1,r_1} \times \overline{C}_{2,r_2} \longrightarrow C_1 \times C_2$$

is a compact map satisfying that

$$\inf_{\|x\|=r_1,\ \|y\|=r_2} \|T_1(x, y)\| > 0 \quad \text{and} \quad \inf_{\|x\|=r_1,\ \|y\|=r_2} \|T_2(x, y)\| > 0. \tag{2.1}$$

Then there exist $\lambda_1, \lambda_2 > 0$ and $(x_0, y_0) \in \overline{C}_{1,r_1} \times \overline{C}_{2,r_2}$ with $\|x_0\| = r_1$ and $\|y_0\| = r_2$ such that

$$\begin{cases} x_0 = \lambda_1\, T_1(x_0, y_0), \\ y_0 = \lambda_2\, T_2(x_0, y_0). \end{cases} \tag{2.2}$$

Proof. For each $i = 1, 2$, let us consider a retraction $\rho_i : \overline{C}_{i,r_i} \to \partial\overline{C}_{i,r_i}$. Now, define the auxiliary map $N = (N_1, N_2) : \overline{C}_{1,r_1} \times \overline{C}_{2,r_2} \to \overline{C}_{1,r_1} \times \overline{C}_{2,r_2}$ as

$$N(x, y) = \left(r_1 \frac{T_1\left(\rho_1(x), \rho_2(y)\right)}{\|T_1\left(\rho_1(x), \rho_2(y)\right)\|}, r_2 \frac{T_2\left(\rho_1(x), \rho_2(y)\right)}{\|T_2\left(\rho_1(x), \rho_2(y)\right)\|} \right),$$

and observe that, by (2.1), N is well-defined. Since N is a compact map, Schauder fixed point theorem (see e.g. [12, §6, Theorem 3.2]) ensures that N has at least one fixed point $(x_0, y_0) \in \overline{C}_{1,r_1} \times \overline{C}_{2,r_2}$. Observe that

$N\left(\overline{C}_{1,r_1}\times\overline{C}_{2,r_2}\right)\subset\partial\overline{C}_{1,r_1}\times\partial\overline{C}_{2,r_2}$ and so it follows that $(x_0,y_0)\in\partial\overline{C}_{1,r_1}\times\partial\overline{C}_{2,r_2}$, that is,

$$x_0=r_1\frac{T_1(x_0,y_0)}{\|T_1(x_0,y_0)\|},\quad y_0=r_2\frac{T_2(x_0,y_0)}{\|T_2(x_0,y_0)\|}.$$

Taking $\lambda_i=r_i/\|T_i(x_0,y_0)\|$, $i=1,2$, the proof is finished. □

Remark 2.4. It should be noted that, under the assumptions of Theorem 2.3, the existence of $\lambda_0>0$ and $(x_0,y_0)\in\overline{C}_{1,r_1}\times\overline{C}_{2,r_2}$ with $\|x_0\|=r_1$ and $\|y_0\|=r_2$ solving the equation

$$\begin{cases}x_0=\lambda_0\,T_1(x_0,y_0),\\ y_0=\lambda_0\,T_2(x_0,y_0)\end{cases}$$

cannot be guaranteed.

Indeed, consider as Banach spaces $X_1=X_2=\mathbb{R}$, the cones $K_1=K_2=[0,+\infty)$, $r_1=r_2=1$ and the continuous function $T:[0,1]\times[0,1]\to[0,+\infty)\times[0,+\infty)$ given by

$$T(x,y)=(T_1(x,y),T_2(x,y))=(2x+y,x+3y).$$

Note that $\inf\limits_{\|x\|=r_1,\ \|y\|=r_2}|T_1(x,y)|=3>0$ and $\inf\limits_{\|x\|=r_1,\ \|y\|=r_2}|T_2(x,y)|=4>0$, but there is no $\lambda\in(0,+\infty)$ such that $(1,1)=\lambda T(1,1)=\lambda\,(3,4)$.

Let us focus now on operators defined in the product of a cone times an infinite dimensional normed space. In this case, an additional solution can be obtained.

Theorem 2.5. *Let K_1 be a cone in the normed linear space X_1, and X_2 be an infinite dimensional normed space. Let r_1,r_2 be positive constants and suppose that $T=(T_1,T_2):\overline{K}_{1,r_1}\times\overline{B}_{2,r_2}\longrightarrow K_1\times X_2$ is a compact map satisfying that*

$$\inf_{\|x\|=r_1,\ \|y\|=r_2}\|T_1(x,y)\|>0\quad\text{and}\quad\inf_{\|x\|=r_1,\ \|y\|=r_2}\|T_2(x,y)\|>0.$$

Then there exist $\lambda_{1,1},\lambda_{2,1},\lambda_{1,2}>0$, $\lambda_{2,2}<0$ and $(x_{0,j},y_{0,j})\in\overline{K}_{1,r_1}\times\overline{B}_{2,r_2}$, $j=1,2$, with $\|x_{0,j}\|=r_1$ and $\|y_{0,j}\|=r_2$ such that

$$\begin{cases}x_{0,j}=\lambda_{1,j}\,T_1(x_{0,j},y_{0,j}),\\ y_{0,j}=\lambda_{2,j}\,T_2(x_{0,j},y_{0,j}),\end{cases}\qquad(j=1,2).$$

Proof. The first solution is ensured by Theorem 2.3. In order to obtain the second one, just consider the map $\tilde{N}:\overline{K}_{1,r_1}\times\overline{B}_{2,r_2}\to\overline{K}_{1,r_1}\times\overline{B}_{2,r_2}$ given by

$$\tilde{N}(x,y)=\left(r_1\frac{T_1\left(\rho_1(x),\rho_2(y)\right)}{\|T_1\left(\rho_1(x),\rho_2(y)\right)\|},-r_2\frac{T_2\left(\rho_1(x),\rho_2(y)\right)}{\|T_2\left(\rho_1(x),\rho_2(y)\right)\|}\right).$$

As a consequence of Schauder fixed point theorem, $\tilde{N}$ has a fixed point $(x_{0,2},y_{0,2})$ located in $\partial\overline{K}_{1,r_1}\times\partial\overline{B}_{2,r_2}$, that is,

$$\begin{cases}x_{0,2}=\lambda_{1,2}\,T_1(x_{0,2},y_{0,2}),\\ y_{0,2}=\lambda_{2,2}\,T_2(x_{0,2},y_{0,2}),\end{cases}$$

where $\lambda_{1,2}=r_1/\|T_1\left(x_{0,2},y_{0,2}\right)\|>0$ and $\lambda_{2,2}=-r_2/\|T_2\left(x_{0,2},y_{0,2}\right)\|<0$. □

Remark 2.6. Under the assumptions of Theorem 2.3, if both C_1 and C_2 are infinite dimensional normed spaces, then there exist four couples of numbers λ_1, λ_2 and associated points $(x_0, y_0) \in \overline{B}_{1,r_1} \times \overline{B}_{2,r_2}$ with $\|x_0\| = r_1$ and $\|y_0\| = r_2$ such that (2.2) holds. Indeed, it suffices to apply the Schauder theorem to each auxiliary map

$$N_{j,k}(x,y) = \left((-1)^j r_1 \frac{T_1(\rho_1(x), \rho_2(y))}{\|T_1(\rho_1(x), \rho_2(y))\|}, (-1)^k r_2 \frac{T_2(\rho_1(x), \rho_2(y))}{\|T_2(\rho_1(x), \rho_2(y))\|} \right), \quad j,k \in \{1,2\}.$$

Let us now stress, once again, that a key feature of our results is the localization of the solutions. In order to compare our results with the classical ones we provide now a direct application of Theorem 2.2 to the situation in which, for each $i \in \{1,2\}$, $C_i = K_i$ is a cone. An analogous statement holds when C_i, $i \in \{1,2\}$, is either a cone or an infinite dimensional normed space.

Theorem 2.7. *Let X_1 and X_2 be real Banach spaces and let $K_1 \subset X_1$, $K_2 \subset X_2$ cones. Let r_1, r_2 be positive constants and suppose that*

$$T = (T_1, T_2) : \overline{K}_{1,r_1} \times \overline{K}_{2,r_2} \longrightarrow K_1 \times K_2,$$

is a compact map satisfying that

$$\inf_{\|x\|=r_1,\ \|y\|\le r_2} \|T_1(x,y)\| > 0 \quad \text{and} \quad \inf_{\|x\|\le r_1,\ \|y\|=r_2} \|T_2(x,y)\| > 0. \tag{2.3}$$

Then there exists $\lambda_0 > 0$ and $(x_0, y_0) \in \overline{K}_{1,r_1} \times \overline{K}_{2,r_2}$ with $\|x_0\| = r_1$ or $\|y_0\| = r_2$ such that

$$\begin{cases} x_0 = \lambda_0 T_1(x_0, y_0), \\ y_0 = \lambda_0 T_2(x_0, y_0). \end{cases} \tag{2.4}$$

Proof. First of all note that $K = K_1 \times K_2$ is a cone in the space $X = X_1 \times X_2$. Let $U = B_{1,r_1} \times B_{2,r_2}$ and take $(x,y) \in K \cap \partial U$. Now either $(x,y) \in \partial B_{1,r_1} \times \overline{B}_{2,r_2}$ or $(x,y) \in \overline{B}_{1,r_1} \times \partial B_{2,r_2}$. In the first case we have

$$\|T(x,y)\| \ge \|T_1(x,y)\|$$

while in the second case we obtain

$$\|T(x,y)\| \ge \|T_2(x,y)\|.$$

In both cases, by condition (2.3), we have

$$\begin{aligned} \|T(x,y)\| &\ge \min\{\|T_1(x,y)\|, \|T_2(x,y)\|\} \\ &\ge \min\left\{ \inf_{\|x\|=r_1,\ \|y\|\le r_2} \|T_1(x,y)\|, \inf_{\|x\|\le r_1,\ \|y\|=r_2} \|T_2(x,y)\| \right\} > 0. \end{aligned}$$

This implies the inequality

$$\inf_{(x,y)\in K\cap\partial U} \|T(x,y)\| > 0.$$

Thus, by Theorem 2.2, there exist $\lambda_0 > 0$ and $(x_0, y_0) \in K \cap \partial U$ such that (2.4) holds. □

Remark 2.8. Note that condition (2.3) is stronger than (2.1); this situation is particularly meaningful in the applications, see the hypothesis *b*) in Theorem 3.3. Furthermore, note that Theorem 2.3 provides a better localization of the eigenfunctions (both of them are nontrivial), since Theorem 2.7 provides couples (x_0, y_0) in which one of the components can be trivial.

3. Some applications to differential systems

In this Section we present some applications of the above results to the context of systems of ODEs and PDEs.

3.1. Eigenvalues for a system of elliptic PDEs

We begin by illustrating the applicability of Theorem 2.3 in the context of PDEs. In particular, we discuss the existence of eigenvalues and eigenfunctions of quasilinear elliptic systems subject to homogeneous Dirichlet boundary conditions of the form

$$\begin{cases} -\Delta u = \lambda_1 f(x,u,v), & \text{in } \Omega, \\ -\Delta v = \lambda_2 g(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ denotes the open unit ball in $\mathbb{R}^n$, $f : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions.

It is folklore (see for example [19, Section 7.2] or [11, Section 4.2]), that if h is sufficiently smooth, the unique solution of the linear PDE

$$-\Delta w = h(x) \ \text{ in } \Omega, \quad w = 0 \ \text{ on } \partial\Omega, \tag{3.2}$$

can be represented in the integral form

$$w(x) = \int_\Omega k(x,y)\, h(y)\, dy =: L(h)(x), \tag{3.3}$$

where k is the Green's function associated to the PDE. The explicit formula for the Green's function for the ball depends on the dimension of the space and can be found for example in [11, Section 2]. It is also well known that the Green's function is non-negative and continuous for $x \neq y$ and singular on the diagonal. The recent paper [29] provides the expression of the Green's function and studies its properties in more general domains. Note that if the function h that occurs in (3.2) is continuous one may still study the solutions of the corresponding Hammerstein integral equation (3.3) and may interpret those as *weak* solutions of (3.2), see [19, Section 7.2]. Now, denote by

$$P := \{u \in \mathcal{C}(\overline{\Omega}) : u \geq 0\},$$

where $\mathcal{C}(\overline{\Omega})$ is the space of continuous functions endowed with the usual supremum norm $\|u\|_\infty = \max_{x\in\overline{\Omega}} |u(x)|$. Note that P is a cone in $\mathcal{C}(\overline{\Omega})$ and the linear operator L leaves the cone P invariant and is compact, see for example [19, Section 7.2] and [29, Proposition 5.1].

With these ingredients in mind, we say that a pair $(u,v) \in \mathcal{C}(\overline{\Omega}) \times \mathcal{C}(\overline{\Omega})$ is a (*weak*) *solution* of problem (3.1) if (u,v) is a solution of the integral system

$$\begin{cases} u(x) = \lambda_1 \displaystyle\int_\Omega k(x,y)\, f(y,u(y),v(y))\, dy, \\ v(x) = \lambda_2 \displaystyle\int_\Omega k(x,y)\, g(y,u(y),v(y))\, dy. \end{cases}$$

In other words, (u, v) is a solution of (3.1) if and only if

$$\begin{cases} u = \lambda_1 T_1(u, v), \\ v = \lambda_2 T_2(u, v), \end{cases} \tag{3.4}$$

where

$$T_1(u, v)(x) = \int_\Omega k(x, y)\, f(y, u(y), v(y))\, dy,$$
$$T_2(u, v)(x) = \int_\Omega k(x, y)\, g(y, u(y), v(y))\, dy.$$

We are now in position to state the following Theorem.

Theorem 3.1. *Let r_1, r_2 be positive constants and suppose that there exist two continuous functions $\underline{f}, \underline{g} : \overline{\Omega} \to \mathbb{R}_+$ such that the following conditions hold:*

a) $f(x, u, v) \geq \underline{f}(x)$ on $\overline{\Omega} \times [0, r_1] \times [0, r_2]$ and

$$\sup_{x \in \overline{\Omega}} \int_\Omega k(x, y)\, \underline{f}(y)\, dy > 0;$$

b) $g(x, u, v) \geq \underline{g}(x)$ on $\overline{\Omega} \times [0, r_1] \times [0, r_2]$ and

$$\sup_{x \in \overline{\Omega}} \int_\Omega k(x, y)\, \underline{g}(y)\, dy > 0.$$

Then there exist $\lambda_1, \lambda_2 > 0$ and $(u_0, v_0) \in P \times P$ with $\|u_0\|_\infty = r_1$ and $\|v_0\|_\infty = r_2$ that satisfy the system (3.4).

Proof. Let us consider the Banach spaces $X_1 = X_2 = \mathcal{C}(\overline{\Omega})$ and the cones $K_1 = K_2 = P$. Due to the compactness and invariance properties of the operator L defined in (3.3), combined with the continuity of f, g, the operator

$$T = (T_1, T_2) : \overline{P}_{r_1} \times \overline{P}_{r_2} \longrightarrow P \times P,$$

defined in (3.1), is compact. Now let us consider $(u, v) \in P \times P$ such that $\|u\|_\infty = r_1, \|v\|_\infty = r_2$. For every $x \in \overline{\Omega}$ we have

$$\|T_1(u, v)\|_\infty \geq T_1(u, v)(x) = \int_\Omega k(x, y)\, f(y, u(y), v(y))\, dy \geq \int_\Omega k(x, y)\, \underline{f}(y)\, dy.$$

Then we get

$$\|T_1(u, v)\|_\infty \geq \sup_{x \in \overline{\Omega}} \int_\Omega k(x, y)\, \underline{f}(y)\, dy. \tag{3.5}$$

Note that the RHS of (3.5) does not depend on the particular (u, v) chosen. Therefore we obtain

$$\inf_{\|u\|_\infty = r_1,\ \|v\|_\infty = r_2} \|T_1(u, v)\|_\infty \geq \sup_{x \in \overline{\Omega}} \int_\Omega k(x, y)\, \underline{f}(y)\, dy > 0.$$

A similar argument applies in the case of the component T_2. Then a direct application of Theorem 2.3 yields the result. □

In the following example we show the applicability of Theorem 3.1.

Example 3.2. Take the open set $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and consider the system

$$\begin{cases} -\Delta u = \lambda_1(1+x^2)e^u(2+\cos v), & \text{in } \Omega, \\ -\Delta v = \lambda_2(1+y^2)(1+v^2)(2+\sin u), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Now, note that the function

$$w(x,y) := \frac{1}{4}(1-x^2-y^2),$$

is the unique solution of the "torsion problem"

$$-\Delta w = 1 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Now fix $r_1, r_2 > 0$ and observe that the conditions $a)$ and $b)$ of Theorem 3.1 are satisfied with the choice of $\underline{f}(x,y) = \underline{g}(x,y) \equiv 1$, since

$$\sup_{(x,y)\in\overline{\Omega}} \int_\Omega k((x,y),(\xi,\eta))\, d(\xi,\eta) = \sup_{(x,y)\in\overline{\Omega}} \frac{1}{4}(1-x^2-y^2) = \frac{1}{4}.$$

Note that (r_1, r_2) can be chosen arbitrarily in $(0,+\infty)\times(0,+\infty)$, thus we obtain the existence of infinitely many couples of type (λ_1,λ_2), with $\lambda_1,\lambda_2 > 0$, and associated couples of nonnegative functions (u_0, v_0) with prescribed norm that satisfy the system (3.6).

3.2. Eigenvalues for a system of ODEs

Here we apply Theorem 2.5 to the study of eigenvalues and eigenfunctions for the following class of BVPs for systems of ODEs:

$$\begin{cases} u''(t) + \lambda_1 f(t,u(t),v(t)) = 0,\ t \in [0,1], \\ v''(t) + \lambda_2 g(t,u(t),v(t)) = 0,\ t \in [0,1], \\ u'(0) = u(1) + u'(1) = 0, \\ v'(0) = v(1) - \frac{1}{2}v'(1) = 0, \end{cases} \tag{3.7}$$

where $f : [0,1]\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}_+$ and $g : [0,1]\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}_+$ are continuous functions.

Note that the system (3.7) can be rewritten as a system of Hammerstein integral equations, namely

$$\begin{cases} u(t) = \lambda_1 \displaystyle\int_0^1 k_1(t,s)\, f(s,u(s),v(s))\, ds, \\ v(t) = \lambda_2 \displaystyle\int_0^1 k_2(t,s)\, g(s,u(s),v(s))\, ds, \end{cases} \tag{3.8}$$

where k_1 and k_2 are the corresponding Green's functions, which are given by

$$k_1(t,s) = \begin{cases} 2-t, & s \le t, \\ 2-s, & s > t, \end{cases}$$

and

$$k_2(t,s) = \frac{1}{2}\begin{cases} 1-2t, & s \le t, \\ 1-2s, & s > t. \end{cases}$$

We point that the kernel k_1 is continuous, non-negative, and has been studied in details in [21,23] while the kernel k_2 is continuous, changes sign, and

can be obtained via a direct calculation. In this case we utilize the space $C[0,1]$, endowed with the usual supremum norm $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$. We construct, following the computations in [23], the cone

$$K = \left\{ u \in C[0,1] : \min_{t\in[0,1]} u(t) \geq \left(1 - \frac{t}{2}\right) \|u\|_\infty \right\}, \tag{3.9}$$

and consider its product with the space itself.

With these ingredients we can state the following Theorem; the construction of the minorant $\underline{f}$ follows the argument in [22].

Theorem 3.3. *Let r_1, r_2 be positive constants and suppose that there exist two measurable functions $\underline{f} : [0,1] \to \mathbb{R}_+$ and $\underline{g} : [0,1] \to \mathbb{R}_+$ such that the following conditions hold:*

a) $f(t,u,v) \geq \underline{f}(t)$ on $[0,1] \times [r_1/2, r_1] \times [-r_2, r_2]$ and

$$\sup_{t\in[0,1]} \int_0^1 k_1(t,s)\, \underline{f}(s)\, ds > 0;$$

b) $g(t,u,v) \geq \underline{g}(t)$ on $[0,1] \times [r_1/2, r_1] \times [-r_2, r_2]$ and

$$\int_0^1 \underline{g}(s)\, ds > 0.$$

Then there exist $\lambda_{1,1}, \lambda_{2,1}, \lambda_{1,2} > 0$, $\lambda_{2,2} < 0$ and $(u_{0,j}, v_{0,j}) \in \overline{K}_{r_1} \times \overline{B}_{r_2}$, $j = 1,2$, with $\|u_{0,j}\| = r_1$ and $\|v_{0,j}\| = r_2$ that satisfy the system (3.8).

Proof. Let us consider the Banach spaces $X_1 = X_2 = C[0,1]$ and the cone K as in (3.9). Note that the operator

$$T = (T_1, T_2) : \overline{K}_{r_1} \times \overline{B}_{r_2} \longrightarrow K \times C[0,1],$$

defined by

$$T_1(u,v)(t) := \int_0^1 k_1(t,s)\, f(s, u(s), v(s))\, ds,$$
$$T_2(u,v)(t) := \int_0^1 k_2(t,s)\, g(s, u(s), v(s))\, ds,$$

is compact, as a consequence of the classical Arzelà–Ascoli Theorem.

Firstly, let us take $(u,v) \in K \times C[0,1]$ such that $\|u\|_\infty = r_1, \|v\|_\infty = r_2$. Note that for every $t \in [0,1]$ we have

$$u(t) \geq \left(1 - \frac{t}{2}\right) r_1 \geq \frac{r_1}{2}.$$

Therefore, we have

$$\|T_1(u,v)\|_\infty \geq T_1(u,v)(t) = \int_0^1 k_1(t,s)\, f(s, u(s), v(s))\, ds \geq \int_0^1 k_1(t,s)\, \underline{f}(s)\, ds,$$

for every $t \in [0,1]$. Thus we obtain

$$\|T_1(u,v)\|_\infty \geq \sup_{t\in[0,1]} \int_0^1 k_1(t,s)\, \underline{f}(s)\, ds. \tag{3.10}$$

Note that the RHS of (3.10) does not depend on the particular (u, v) chosen. Therefore we obtain

$$\inf_{\|u\|_\infty = r_1,\ \|v\|_\infty = r_2} \|T_1(u,v)\|_\infty \geq \sup_{t\in[0,1]} \int_0^1 k_1(t,s)\, \underline{f}(s)\, ds > 0.$$

Secondly, let us take $(u, v) \in K \times C[0, 1]$ such that $\|u\|_\infty = r_1, \|v\|_\infty = r_2$. Note that

$$\begin{aligned} \|T_2(u,v)\|_\infty &\geq |T_2(u,v)(1)| \\ &= \int_0^1 -k_2(1,s)\, g(s,u(s),v(s))\, ds \geq \int_0^1 -k_2(1,s)\, \underline{g}(s)\, ds. \end{aligned} \tag{3.11}$$

Note that the RHS of (3.11) does not depend on the particular (u, v) chosen. Therefore we obtain

$$\inf_{\|u\|_\infty = r_1,\ \|v\|_\infty = r_2} \|T_2(u,v)\|_\infty \geq \int_0^1 -k_2(1,s)\, \underline{g}(s)\, ds = \frac{1}{2}\int_0^1 \underline{g}(s)\, ds > 0.$$

Then a direct application of Theorem 2.5 yields the result. □

In the following example we show the applicability of Theorem 3.3.

Example 3.4. Consider the system

$$\begin{cases} u''(t) + \lambda_1 t(u^2 + v^2) = 0, \ t \in [0,1], \\ v''(t) + \lambda_2 t e^{uv} = 0, \ t \in [0,1], \\ u'(0) = u(1) + u'(1) = 0 = v'(0) = v(1) - \frac{1}{2}v'(1). \end{cases}$$

Now fix $r_1, r_2 > 0$, then the choice of $\underline{f}(t) = \dfrac{t\, r_1^2}{4}$ gives, by direct calculation,

$$\sup_{t\in[0,1]} \int_0^1 k_1(t,s) \frac{t\, r_1^2}{4}\, ds = \frac{r_1^2}{4} > 0,$$

while choosing $\underline{g}(t) = te^{-r_1 r_2}$ yields

$$\int_0^1 te^{-r_1 r_2}\, dt = \frac{e^{-r_1 r_2}}{2} > 0.$$

Then the conditions a) and b) of Theorem 3.3 are satisfied. Furthermore since the pair (r_1, r_2) can be chosen arbitrarily in $(0, +\infty) \times (0, +\infty)$, we obtain the existence of two distinct families of uncountably many pairs: both $(\lambda_{1,1}, \lambda_{2,1})$, with $\lambda_{i,1} > 0$, $i = 1, 2$, and $(\lambda_{1,2}, \lambda_{2,2})$, with $\lambda_{1,2} > 0$ and $\lambda_{2,2} < 0$, each of them with the associated eigenfunctions of prescribed norms.

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Declarations

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