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NON-DEFECTIVITY OF SEGRE–VERONESE VARIETIES

HIROTACHI ABO, MARIA CHIARA BRAMBILLA, FRANCESCO GALUPPI, AND ALESSANDRO ONETO

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Abstract. We prove that Segre–Veronese varieties are never secant defective if each degree is at least three. The proof is by induction on the number of factors, degree, and dimension. As a corollary, we give an almost optimal non-defectivity result for Segre–Veronese varieties with one degree equal to one and all the others at least three.

1. Introduction

A *Segre–Veronese variety* is the embedding of a multi-projective space by a very ample line bundle. It parameterizes the rank-one partially symmetric tensors, and the compactification of the space parameterizing those with partially symmetric rank at most m is called the mth secant variety of the Segre–Veronese variety. This paper concerns the problem of classifying the so-called *defective* secant varieties of Segre–Veronese varieties, the ones with dimensions smaller than expected. This problem is very classical and has its roots in XIX century algebraic geometry, see [BCC+18]. It is also closely related to partially symmetric tensor rank, partially symmetric tensor border rank, simultaneous rank, and partially symmetric tensor decompositions, as well as their uniqueness, which are relevant topics to many branches of modern applied sciences, see [Lan12]. Hence, it has the potential to impact a variety of areas, including mathematics, computer science, and statistics.

Our goal is to establish non-defectivity for a large family of Segre–Veronese varieties. The simplest examples of Segre–Veronese varieties are Veronese varieties, whose defectivity is completely understood due to the celebrated theorem by

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Alexander and Hirschowitz [AH95]. Beyond Veronese varieties, this classification problem, however, is still far from complete.

Some cases are better understood than the others. For example, the conjecturally complete list of defective secant varieties for Segre–Veronese varieties with two factors was suggested by Abo and Brambilla in [AB13]. Significant progress towards this conjecture was made by Galuppi and Oneto in [GO22]: they proved that if the bi-projective space is embedded by a linear system of degree at least three in both factors, then its secant varieties are all non-defective. In this paper, we extend this result to an arbitrary number of factors.

Catalisano, Geramita, and Gimigliano carried out the first systematic study of the secant varieties of Segre–Veronese varieties in [CGG05, CGG08]. In these papers, they discovered many defective cases, including *unbalanced cases* (where one of the factors of the multi-projective space has a much larger dimension than the rest). Several of these defective cases were later generalized by Abo and Brambilla [AB12], as well as Laface, Massaranti, and Richter [LMR22].

Regarding the secant non-defectivity, Laface and Postinghel in [LP13] employed toric approaches to show that the secant varieties of Segre–Veronese varieties of an arbitrary number of copies of the projective line are never defective. Ballico [Bal23] and Ballico, Bernardi, and Mańdziuk [BBM24] proved non-defectivity for more families of Segre–Veronese varieties, with some assumptions on the dimensions.

Araujo, Massarenti, and Rischter [AMR19] developed a new approach using osculating projections and obtained an asymptotic bound under which the secant varieties of Segre–Veronese varieties always have the expected dimensions. Their bound was improved by Laface, Massarenti, and Richter [LMR22].

Very recently, Taveira Blomenhofer and Casarotti [BC23] significantly improved the bound from [LMR22], showing that most secant varieties of Segre–Veronese varieties are not defective. However, there is still a range of values of m for which the non-defectivity of the mth secant variety of a Segre–Veronese variety is not known. As the longevity of the classification problem of the defective cases suggests, making this final stretch is the most difficult part. The primary goal of this paper is to fill this gap for Segre–Veronese varieties embedded with degree at least three in all factors. In the remaining part of this introduction, we introduce basic notation and state our main results.

For given k-tuples $\mathbf{n} = (n_1, n_2, \dots, n_k)$ and $\mathbf{d} = (d_1, d_2, \dots, d_k)$ of positive integers, we write $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$ and $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ for the Segre–Veronese variety obtained by embedding $\mathbb{P}^{\mathbf{n}}$ by the morphism associated with its complete linear system $|\mathscr{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d})|$. The closure of the union of secant $(m-1)$-planes to $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ is called the mth secant variety of $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ and denoted by $\sigma_m(\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}})$. We say that it is non-defective if it is not *m-defective* for any positive integer m, that is, if the dimension of $\sigma_m(\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}})$ equals the expected one, defined by a naïve parameter count. See Section 2 for explicit definitions.

Theorem 1.1. *Let $k \geq 2$. If $d_1, d_2, \dots, d_k \geq 3$, then $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ is not defective.*

The proof of this theorem, presented in Section 3, is an application of the *differential Horace method*, which enables us to show the secant non-defectivity of a Segre–Veronese variety by induction on dimension and degree. This type of approach often leads to a complicated nested induction. The significance of this paper is to overcome this complication and to give a clean proof.

As a consequence of Theorem 1.1, we deduce an almost optimal non-defectivity result for the Segre–Veronese variety having one factor embedded in degree 1 and all the others at least three.

Theorem 1.2. *Let $k \geq 2$, let $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_k)$ be k-tuples of positive integers, let $|\mathbf{n}| = n_1 + n_2 + \cdots + n_k$, and let $N_{\mathbf{n},\mathbf{d}} = \prod_{i=1}^{k} \binom{n_i+d_i}{n_i}$. If $d_1, d_2, \ldots, d_k \geq 3$ and if*

$$m \leq (n_0+1)\left\lfloor \frac{N_{\mathbf{n},\mathbf{d}}}{n_0+|\mathbf{n}|+1} \right\rfloor \qquad or \qquad m \geq (n_0+1)\left\lceil \frac{N_{\mathbf{n},\mathbf{d}}}{n_0+|\mathbf{n}|+1} \right\rceil,$$

then $\mathrm{SV}^{(1,\mathbf{d})}_{(n_0,\mathbf{n})}$ is not m-defective.

The proof, presented in Section 4, is based on an inductive method which allows to deduce non-defectivity results for a Segre product $\mathbb{P}^n \times X$ from the non-defectivity of the algebraic variety X, see Proposition 4.1. It is worth noting that Theorem 1.2 is stronger than [BC23, Theorem 4.8] for these specific multidegrees, see Remark 4.6 for more details.

While the rank of a tensor tells us about the length of a minimal decomposition as a sum of rank-one elements, identifiability is the uniqueness of such decomposition. For applied purposes, it is very important to know when the Segre–Veronese variety is identifiable, namely when the general partially symmetric tensor has a unique decomposition. Thanks to [MM22, Theorem 1.5], the non-defectivity of a variety has direct consequences on its identifiability, so we immediately get the following corollaries of Theorems 1.1 and 1.2.

Corollary 1.3. *Let $k \geq 2$, let $\mathbf{n}, \mathbf{d} \in \mathbb{N}^k$ be tuples of positive integers with $d_1, d_2, \ldots, d_k \geq 3$. Let $|\mathbf{n}| = n_1 + n_2 + \cdots + n_k$ and let $N_{\mathbf{n},\mathbf{d}} = \prod_{i=1}^{k} \binom{n_i+d_i}{n_i}$.*

(1) *If $m(|\mathbf{n}|+1) \leq N_{\mathbf{n},\mathbf{d}}$, then $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ is $(m-1)$-identifiable.*
(2) *If $m \leq (n_0+1)\left\lfloor \frac{N_{\mathbf{n},\mathbf{d}}}{n_0+|\mathbf{n}|+1} \right\rfloor$, then $\mathrm{SV}^{(1,\mathbf{d})}_{(n_0,\mathbf{n})}$ is $(m-1)$-identifiable.*

In order to complete the classification of Segre–Veronese varieties, it remains to solve the cases in which one of the degrees is 1 or 2. A major difficulty here is that, besides the unbalanced ones, several defective cases are known, and it is complicated to shape a general inductive strategy that avoids them. We underline that all known balanced defective cases appear when the number of factors is four or less. For this reason, we want to explicitly draw attention to the following question.

Question. Is it true that the only defective cases for Segre–Veronese varieties with at least five factors are the unbalanced cases?

During the final part of the preparation of the present manuscript, Ballico privately informed us that he indipendently obtained a result similar to Theorem 1.1, which eventually appeared in [Bal24].

2. Tools and background

We work over an algebraically closed field $\Bbbk$ of characteristic zero.

Given an algebraic variety $X \subset \mathbb{P}^N$, the *mth secant variety*

$$\sigma_m(X) = \overline{\bigcup_{x_1,x_2,\ldots,x_m \in X} \langle x_1, x_2, \ldots, x_m \rangle} \subset \mathbb{P}^N$$

of X is the Zariski-closure of the union of all linear spaces spanned by m points of X.

The notion of *expected dimension* of $\sigma_m(X) \subset \mathbb{P}^N$ follows from a straightforward parameter count:

$$\exp . \dim \sigma_m(X) = \min\{N, m \dim(X) + m - 1\}.$$

It is immediate to see that this is always an upper bound for the actual dimension: we say that X is m-*defective* if $\dim \sigma_m(X) < \exp . \dim \sigma_m(X)$.

Let us fix some notation we will use throughout the paper.

Notation 2.1. Let $\mathbf{a} = (a_1, a_2, \ldots, a_k), \mathbf{b} = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$:

- For any $j \in \mathbb{N}$, we write $\mathbf{a}(j) = (a_1 - j, a_2, \ldots, a_k)$.
- We write $\mathbf{a} \succeq \mathbf{b}$ if $a_i \geq b_i$ for every $i \in \{1, 2, \ldots, k\}$.
- We write $|\mathbf{a}| = a_1 + a_2 + \cdots + a_k$.

If $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_k)$ are k-tuples of positive integers, then we set

$$N_{\mathbf{n},\mathbf{d}} = \prod_{i=1}^{k} \binom{n_i + d_i}{d_i},$$

and we define

$$r^*(\mathbf{n}, \mathbf{d}) = \left\lceil \frac{N_{\mathbf{n},\mathbf{d}}}{|\mathbf{n}| + 1} \right\rceil \quad \text{and} \quad r_*(\mathbf{n}, \mathbf{d}) = \left\lfloor \frac{N_{\mathbf{n},\mathbf{d}}}{|\mathbf{n}| + 1} \right\rfloor. \tag{2.1}$$

Remark 2.2. The two values defined in (2.1) are *critical* in the following sense:

- $r^*(\mathbf{n}, \mathbf{d})$ is the smallest positive integer m such that the mth secant variety is expected to fill the ambient space; namely, it is expected to have dimension $N_{\mathbf{n},\mathbf{d}} - 1$. Since $\dim \sigma_m(X)$ is increasing with respect to m, if $\mathrm{SV}_{\mathbf{n}}^{\mathbf{d}}$ is not $r^*(\mathbf{n}, \mathbf{d})$-defective then it is not m-defective for any $m \geq r^*(\mathbf{n}, \mathbf{d})$. For these values of m we say that $\sigma_m(\mathrm{SV}_{\mathbf{n}}^{\mathbf{d}})$ is *superabundant*.
- $r_*(\mathbf{n}, \mathbf{d})$ is the largest integer m such that the mth secant variety is expected to have dimension equal to the parameter count $m(|\mathbf{n}| + 1) - 1$. Since the difference of the dimensions of two consecutive secant varieties of $\mathrm{SV}_{\mathbf{n}}^{\mathbf{d}}$ is at most $|\mathbf{n}|+1$, if $\mathrm{SV}_{\mathbf{n}}^{\mathbf{d}}$ is not $r_*(\mathbf{n}, \mathbf{d})$-defective then it is not m-defective for any $m \leq r_*(\mathbf{n}, \mathbf{d})$. For these values of m we say that $\sigma_m(\mathrm{SV}_{\mathbf{n}}^{\mathbf{d}})$ is *subabundant*.

Therefore, in order to prove that a Segre–Veronese variety $\mathrm{SV}_{\mathbf{n}}^{\mathbf{d}}$ is never defective, it is enough to prove non-defectiveness at the critical values.

The *Horace method* is an inductive approach that goes back to Castelnuovo and was improved by Alexander and Hirschowitz, leading to the classification of defective Veronese varieties. This is a degeneration technique to study the dimension of complete linear systems of divisors with base points in general position with some multiplicities. We refer to [BCC+18, Section 2.2] for a detailed presentation of the method and its extensions. For the purpose of the present paper, we will employ the Horace method in the following formulation.

Theorem 2.3 ([AB13, Theorem 1.1])**.** *Let* $\mathbf{n}, \mathbf{d} \in \mathbb{N}^k$ *be such that* $d_1 \geq 3$ *and let* $r \in \mathbb{N}$. *Let*

$$s_r = s_r(\mathbf{n}, \mathbf{d}) = \left\lfloor \frac{(|\mathbf{n}| + 1)r - N_{\mathbf{n},\mathbf{d}(1)}}{|\mathbf{n}|} \right\rfloor \quad \textit{and}$$
$$\epsilon_r = \epsilon_r(\mathbf{n}, \mathbf{d}) = (|\mathbf{n}| + 1)r - N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}| s_r(\mathbf{n}, \mathbf{d}).$$

If all the following conditions are satisfied:

(1) $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}(1)}$ *is not* s_r*-defective;*
(2) $\mathrm{SV}^{\mathbf{d}(1)}_{\mathbf{n}}$ *is not* $(r-s_r)$*-defective and* $s_r \geq \epsilon_r$*;*
(3) $\mathrm{SV}^{\mathbf{d}(2)}_{\mathbf{n}}$ *is not* $(r-s_r-\epsilon_r)$*-defective and* $(r-s_r-\epsilon_r)(|\mathbf{n}|+1) \geq N_{\mathbf{n},\mathbf{d}(2)}$*;*

then $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ *is not* r*-defective.*

Remark 2.4. The numerical assumption in condition (2) guarantees that $\sigma_{r-s_r}(\mathrm{SV}^{\mathbf{d}(1)}_{\mathbf{n}})$ is a subabundant case. On the contrary, the numerical condition (3) implies that $\sigma_{r-s_r-\epsilon_r}(\mathrm{SV}^{\mathbf{d}(2)}_{\mathbf{n}})$ is a superabundant case.

While it may be difficult to prove that a variety is not defective, in the literature, several varieties have been proven to not be m-defective when m is sufficiently far from the critical ones. One example is [BC23], where Taveira Blomenhofer and Casarotti generalize a result by Ådlandsvik [Åd88] and prove non-defectivity for varieties that are invariant under the action of a group G and contained in irreducible G-module. The precise statement that we need in the case of Segre–Veronese varieties is as follows.

Theorem 2.5 ([BC23, Theorem 4.8])**.** *Let* $\mathbf{n}, \mathbf{d} \in \mathbb{N}^k$*. If* $m \leq r_*(\mathbf{n},\mathbf{d}) - |\mathbf{n}| - 1$ *or* $m \geq r^*(\mathbf{n},\mathbf{d}) + |\mathbf{n}| + 1$*, then* $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ *is not* m*-defective.*

3. Proof of Theorem 1.1

We prove Theorem 1.1 by induction on the number k of factors, on the dimension n_1, and on the degree d_1. First we give the necessary results to deal with the base case for $n_1 = 1$ and two base cases for $d_1 \in \{3, 4\}$.

We recall the following result by Ballico.

Theorem 3.1 ([Bal23, Theorem 2])**.** *Let* $X \subset \mathbb{P}^N$ *be an irreducible non-degenerate variety with* $\dim(X) \geq 3$*. Let*

$$r = \left\lfloor \frac{N+1}{\dim(X)} \right\rfloor,$$

and assume that X *is not* r*-defective. Let* $d \geq 2$ *and consider* $Y = \mathbb{P}^1 \times X$ *embedded in* $\mathbb{P}^{(d+1)(N+1)-1}$ *by the line bundle* $\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathcal{O}_X(1)$*. If* $N+1 > \dim(X)^2$*, then* Y *is not defective.*

By applying this theorem, we prove a technical result that will be useful in the proof of Theorem 1.1.

Corollary 3.2. *Let* $k \geq 2$*. Let* $\mathbf{n}' = (n_2, n_3, \ldots, n_k)$ *and* $\mathbf{d}' = (d_2, d_3, \ldots, d_k) \succeq 3^{k-1}$ *be* $(k-1)$*-tuples of positive integers. If* $\mathrm{SV}^{\mathbf{d}'}_{\mathbf{n}'}$ *is not defective and* $d_1 \geq 2$*, then* $\mathrm{SV}^{(d_1,\mathbf{d}')}_{(1,\mathbf{n}')}$ *is never defective.*

Proof. We start by proving that

$$\prod_{i=2}^{k} \binom{n_i + d_i}{n_i} > (n_2 + n_3 + \cdots + n_k)^2. \tag{3.1}$$

The left-hand side is increasing with respect to $d_2, d_3, \ldots, d_k$, so it is enough to prove (3.1) for $\mathbf{d} = 3^{k-1}$. On the left-hand side n_i^2 appears with coefficient 1, while $n_i n_j$ $(i \neq j)$ appears with coefficient $11^2/6^2 > 2$.

If $k = 2$ and $n_2 = n_3 = 1$, then $\mathrm{SV}^{(d_1,d_2,d_3)}_{(1,1,1)}$ is not defective by [LP13, Theorem 3.1]. In any other cases, we have $n_2+n_3+\cdots+n_k \geq 3$, so we can apply Proposition 3.1 to the variety $X = \mathrm{SV}^{\mathbf{d}'}_{\mathbf{n}'}$, which is not defective by hypothesis. □

In Theorem 2.3, there are no assumptions about the order of all the d_i and n_i. Up to permuting the factors, it is not restrictive to suppose that $n_1 \leq n_2 \leq \cdots \leq n_k$. This is crucial for some of our numerical computations.

Now we deal with the case $d_1 = 3$. In order to make our arguments easier to read, we postpone some of the arithmetic computations to Appendix A.

Proposition 3.3. *Let $k \geq 2$ and let $\mathbf{n}' = (n_2 \leq n_3 \leq \cdots \leq n_k)$ and $\mathbf{d}' = (d_2, d_3, \ldots, d_k) \succeq 3^{k-1}$ be $(k-1)$-tuples of positive integers. If $\mathrm{SV}^{\mathbf{d}'}_{\mathbf{n}'}$ is not defective and n_1 is a positive integer, then $\mathrm{SV}^{(3,\mathbf{d}')}_{(n_1,\mathbf{n}')}$ is not defective.*

Proof. We argue by induction on n_1. The initial case $n_1 = 1$ is Lemma 3.2. For $n_1 \geq 2$, we prove that $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ is not r-defective for the critical values $r \in \{r_*(\mathbf{n},\mathbf{d}), r^*(\mathbf{n},\mathbf{d})\}$ by applying Theorem 2.3. We check all conditions:

(1) $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}(1)}$ is not defective by our inductive hypothesis on n_1.
(2) By Lemma A.1, $r - s_r \leq r_*(\mathbf{n},\mathbf{d}(1)) - |\mathbf{n}| - 1$, so $\mathrm{SV}^{\mathbf{d}(1)}_{\mathbf{n}}$ is not $(r - s_r)$-defective by Theorem 2.5. The numerical condition of Theorem 2.3(2) is Lemma A.2.
(3) By Lemma A.3, $r-s_r-\epsilon_r \geq r^*(\mathbf{n},\mathbf{d}(2))+|\mathbf{n}|+1$, so $\mathrm{SV}^{\mathbf{d}(2)}_{\mathbf{n}}$ is not $(r-s_r-\epsilon_r)$-defective by Theorem 2.5. The numerical condition of Theorem 2.3(3) also follows from Theorem A.3, because $r^*(\mathbf{n},\mathbf{d}(2)) + |\mathbf{n}| + 1 \geq \frac{N_{\mathbf{n},\mathbf{d}(2)}}{|\mathbf{n}|+1}$. □

Next, we consider the case $d_1 = 4$. The proof is very similar to the previous one. The only difference is that we apply Proposition 3.3 to check the second condition in Theorem 2.3.

Proposition 3.4. *Let $k \geq 2$ and let $\mathbf{n}' = (n_2 \leq n_3 \leq \cdots \leq n_k)$ and $\mathbf{d}' = (d_2, d_3, \ldots, d_k) \succeq 3^{k-1}$ be $(k-1)$-tuples of positive integers. If $\mathrm{SV}^{\mathbf{d}'}_{\mathbf{n}'}$ is not defective and n_1 is a positive integer, then $\mathrm{SV}^{(4,\mathbf{d}')}_{(n_1,\mathbf{n}')}$ is not defective.*

Proof. We argue by induction on n_1. The initial case $n_1 = 1$ is Lemma 3.2. For $n_1 \geq 2$, we prove that $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ is not r-defective for $r \in \{r_*(\mathbf{n},\mathbf{d}), r^*(\mathbf{n},\mathbf{d})\}$ by applying Theorem 2.3. We check all conditions:

(1) $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}(1)}$ is not defective by the inductive hypothesis.
(2) $\mathrm{SV}^{\mathbf{d}(1)}_{\mathbf{n}}$ is not defective by Proposition 3.3. The numerical condition of Theorem 2.3(2) is Lemma A.2.
(3) By Lemma A.3, we have $r - s_r - \epsilon_r \geq r^*(\mathbf{n},\mathbf{d}(2)) + |\mathbf{n}| + 1$, so $\mathrm{SV}^{\mathbf{d}(2)}_{\mathbf{n}}$ is not $(r - s_r - \epsilon_r)$-defective by Theorem 2.5. The numerical condition of Theorem 2.3(3) also follows from Lemma A.3, because $r^*(\mathbf{n},\mathbf{d}(2))+|\mathbf{n}|+1 \geq \frac{N_{\mathbf{n},\mathbf{d}(2)}}{|\mathbf{n}|+1}$. □

Proof of Theorem 1.1. As we pointed out, without loss of generality, we may assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. We argue by induction on $k \geq 2$. The base case $k = 2$ is [GO22, Theorem 1.2]. We assume that $k \geq 3$ and that $\mathrm{SV}^{d_2,\ldots,d_k}_{n_2,\ldots,n_k}$ is not defective, and we prove that $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}}$ is not defective. We proceed by one-step induction on n_1 and by two-step induction on d_1. The base case $n_1 = 1$ is a consequence of Lemma 3.2, while the base cases $d_1 \in \{3, 4\}$ follow from Propositions 3.3 and 3.4. Now we suppose that $n_1 \geq 2$, $d_1 \geq 5$ and that $\mathrm{SV}^{\mathbf{d}}_{\mathbf{n}(1)}$, $\mathrm{SV}^{\mathbf{d}(1)}_{\mathbf{n}}$ and $\mathrm{SV}^{\mathbf{d}(2)}_{\mathbf{n}}$ are not defective. Thanks to Theorem 2.3, in order to conclude Theorem 1.1, it is enough to show that the two numerical conditions hold. The numerical condition of Theorem 2.3(2) is Lemma A.2. The numerical condition of Theorem 2.3(3) follows from Lemma A.3, because $r^*(\mathbf{n}, \mathbf{d}(2)) + |\mathbf{n}| + 1 \geq \frac{N_{\mathbf{n},\mathbf{d}(2)}}{|\mathbf{n}|+1}$. □

4. The splitting lemma and proof of Theorem 1.2

Let V and W be $\Bbbk$-vector spaces with $\dim V = n_0 + 1$ and $\dim W = \alpha + 1$. Let $X \subseteq \mathbb{P}W$ be a non-degenerate algebraic variety and let $Y = \mathbb{P}V \times X \subset \mathbb{P}(V \otimes W)$ be the Segre product. In this section we describe an inductive method useful to derive non-defectivity of Y from the non-defectivity of X. Using this method, we prove Theorem 1.2.

Let $\widehat{T}_p Y$ denote the affine cone over the tangent space to Y at p. It is immediate to observe that if $p = [v \otimes w] \in Y$, then $\widehat{T}_p Y = V \otimes w + v \otimes \widehat{T}_{[w]} X$.

Proposition 4.1 (Segre induction)**.** *Let*

$$a_* = \left\lfloor \frac{\alpha + 1}{n_0 + \dim X + 1} \right\rfloor \text{ and } a^* = \left\lceil \frac{\alpha + 1}{n_0 + \dim X + 1} \right\rceil .$$

Suppose that X is neither a_*-defective nor a^*-defective. If $m \leq (n_0 + 1)a_*$ or $m \geq (n_0 + 1)a^*$, then Y is not m-defective.

By combining Proposition 4.1 and Theorem 1.1, we immediately deduce Theorem 1.2. The rest of this section is devoted to proving Proposition 4.1. We will employ the so-called splitting lemma, which is a variation of the inductive approach successfully employed in studying secant varieties of various classically known varieties such as Segre varieties [BCS13, AOP09] and Segre–Veronese varieties with two factors embedded in bi-degree $(1, 2)$ [AB09]. The splitting lemma is based on the classical *Terracini's lemma.*

Lemma 4.2 (Terracini's lemma, [Ter11])**.** *Let $Z \subset \mathbb{P}^N$ be an algebraic variety. Let $p_1, \ldots, p_m \in Z$ be generic and let $q \in \langle p_1, \ldots, p_m \rangle$ be generic. Then*

$$\widehat{T}_q \sigma_m(Z) = \sum_{i=1}^{m} \widehat{T}_{p_i} Z.$$

Notation 4.3. Fixed X and Y as above, we set $x = \dim X$ and we denote by $T(n_0, s, t)$ the following property:

> *For generic $p_1, \ldots, p_m, q_1, \ldots, q_t \in Y$, with $p_i = [v_i \otimes w_i]$ and $q_i = [v'_i \otimes w'_i]$,*

$$\dim \left(\sum_{i=1}^{m} \widehat{T}_{p_i} Y + \sum_{i=1}^{t} V \otimes w'_i \right) = \min\{(n_0 + 1)(\alpha + 1), m\,(n_0 + x + 1) + t\,(n_0 + 1)\}.$$

Moreover, analogously to the terminology introduced in Section 2, we say that the triple (n_0, m, t) is *subabundant* if $m(n_0 + x + 1) + t(n_0 + 1) \leq (n_0 + 1)(\alpha + 1)$; while we say that it is *superabundant* if $m(n_0 + x + 1) + t(n_0 + 1) \geq (n_0 + 1)(\alpha + 1)$.

Remark 4.4. By Terracini's lemma, the property $T(n_0, m, 0)$ is equivalent to saying that Y is not m-defective. For example, Remark 2.2 can be rephrased as follows:

- $T(n_0, m, 0)$ implies $T(n_0, m', 0)$ for every $m' \leq m$ whenever $(n_0, m, 0)$ is subabundant;
- $T(n_0, m, 0)$ implies $T(n_0, m', 0)$ for every $m' \geq m$ whenever $(n_0, m, 0)$ is superabundant.

Lemma 4.5 (Splitting lemma)**.** *Let $m' \in \{0, 1, \ldots, m\}$ and let $n' \in \{0, 1, \ldots, n_0 - 1\}$.*

(1) *If $(n', m', t+m-m')$ and $(n_0 - n' - 1, m - m', t + m')$ are both subabundant (resp., superabundant), then (n_0, m, t) is subabundant (resp., superabandant).*
(2) *If $T(n', m', t + m - m')$ and $T(n_0 - n' - 1, m - m', t + m')$ are both true, then $T(n_0, m, t)$ is true.*

Proof. For (1), if $(n', m', t+m-m')$ and $(n_0 - n' - 1, m - m', t + m')$ are subabundant, then

$$\begin{aligned} m(n_0 + x + 1) + t(n_0 + 1) &= m'(n' + x + 1) + (t + m - m')(n' + 1) \\ &\qquad + (m - m')(n_0 - n' + x) + (t + m')(n_0 - n') \\ &\leq (n' + 1)(\alpha + 1) + (n_0 - n')(\alpha + 1) \\ &= (n_0 + 1)(\alpha + 1), \end{aligned}$$

and hence (n_0, m, t) is subabundant. An analogous proof holds for the superabundant case.

For (2), by semicontinuity, in order to prove $T(n_0, s, t)$ it is enough to prove that the property holds for a *special* choice of the points. Let V_1 be of dimension $(n' + 1)$ and let V_2 be a subspace of V such that $V = V_1 \oplus V_2$. Let $Y_i = \mathbb{P}V_i \times X$ be the Segre product in $\mathbb{P}(V_i \otimes W)$ for $i = 1, 2$. If we specialize all the p_i such that $v_1, \ldots, v_{m'}$ are generic in V_1 and $v_{m'+1}, \ldots, v_m$ are generic in V_2, then

$$\begin{aligned} \widehat{T}_{p_i} Y &= (V_1 \oplus V_2) \otimes w_i + v_i \otimes \widehat{T}_{[w_i]} X \\ &= \begin{cases} \widehat{T}_{p_i} Y_1 + V_2 \otimes w_i & \text{for each } i \in \{1, 2, \ldots, m'\}, \\ \widehat{T}_{p_i} Y_2 + V_1 \otimes w_i & \text{for each } i \in \{s' + 1, \ldots, m\}. \end{cases} \end{aligned}$$

Thus, $\sum_{i=1}^{m} \widehat{T}_{p_i} Y + \sum_{i=1}^{t} V \otimes w'_i$ is the direct sum

$$\begin{aligned} &\left(\sum_{i=1}^{m'} \widehat{T}_{p_i} Y_1 + \sum_{i=1}^{t} V_1 \otimes w'_i + \sum_{i=m'+1}^{m} V_1 \otimes w_i \right) \\ &\qquad \oplus \left(\sum_{i=m'+1}^{m} \widehat{T}_{p_i} Y_2 + \sum_{i=1}^{t} V_2 \otimes w'_i + \sum_{i=1}^{m'} V_2 \otimes w_i \right). \end{aligned}$$

By the assumption that both $T(n', m', t+m-m')$ and $T(n_0 - n' - 1, m - m', t + m')$ hold, we have that both summands have the expected dimensions and then also $T(n_0, m, t)$ holds. □

***Proof of Proposition* 4.1**. By Remark 4.4, it is enough to show that

(1) $(n_0, (n_0 + 1)\, a_*, 0)$ is subabundant and $T(n_0, (n_0 + 1)\, a_*, 0)$ is true;
(2) $(n_0, (n_0 + 1)\, a^*, 0)$ is superabundant and $T(n_0, (n_0 + 1)\, a^*, 0)$ is true.

We only prove the first statement because the proof of (2) is similar.

Note that $(0, a_*, n_0 a_*)$ is subabundant by the definition of a_*. Moreover, since $T(0, a_*, 0)$ is true by the assumption of non-defectivity of X and adding generic points always imposes the expected number of conditions, $T(0, a_*, n_0 a_*)$ is true. Thus, by Lemma 4.5, it is enough to prove that $(n_0 - 1, n_0 a_*, a_*)$ is subabundant and $T(n_0 - 1, n_0 a_*, a_*)$ is true.

In order to prove this, we show that $(n_0 - i, (n_0 - i + 1)a_*, ia_*)$ is subabundant and $T(n_0 - i, (n_0 - i + 1)a_*, ia_*)$ is true for all $i \in \{1, 2, \dots, n_0\}$. We proceed by backward induction on i. The case $i = n_0$ is true, as commented above. If we assume that $(n_0 - i, (n_0 - i + 1)a_*, ia_*)$ is subabundant for any $i \in \{1, 2, \dots, n_0\}$ and $T(n_0 - i, (n_0 - i + 1)a_*, ia_*)$ is true, then it follows from Lemma 4.5 that $(n_0 - (i-1), (n_0 - (i-1) + 1)a_*, (i-1)a_*)$ is subabundant and $T(n_0 - (i-1), (n_0 - (i-1) + 1)a_*, (i-1)a_*)$ holds. In particular, it holds for $i = 1$. □

Remark 4.6. Recall that $(n_0, (n_0{+}1)\, a_*, 0)$ is subabundant and that $(n_0, (n_0{+}1)\, a^*, 0)$ is superabundant. Furthermore,

$$(n_0 + 1)\, a^* - (n_0 + 1)\, a_* = (n_0 + 1)(a^* - a_*) \leq n_0 + 1.$$

Thus, $(n_0 + 1)\, a_*$ is the greatest multiple of $n_0 + 1$ which is smaller than or equal to $\lfloor \frac{(n_0+1)(\alpha+1)}{n_0+\dim X+1} \rfloor$, while $(n_0 + 1)\, a^*$ is the least multiple of $n_0 + 1$ which is greater than or equal to $\lceil \frac{(n_0+1)(\alpha+1)}{n_0+\dim X+1} \rceil$. Observe that the gap between the thresholds is $2|\mathbf{n}| + 2$ in Theorem 2.5, while it is $n_0 + 1$ in Theorem 1.2.

Remark 4.7. If X is a dth Veronese embedding of $\mathbb{P}^{n_1}$ with $n_1, d \geq 5$, then the Alexander-Hirschowitz Theorem implies that $\sigma_{a_*}(X)$ and $\sigma_{a^*}(X)$ have the expected dimensions. By Proposition 4.1, if $m \leq (n_0 + 1)\, a_*$ or $m \geq (n_0 + 1)\, a^*$, then $\sigma_m(Y)$ has the expected dimension. This gives an alternative proof to almost all cases of [BCC11, Corollary 2.2], and it extends it to any number of factors.

Appendix A. Numerical computations

In this section, we prove the numerical conditions needed in the main proofs. Let $k \geq 3$, let $\mathbf{n} = (n_1 \leq n_2 \leq \cdots \leq n_k)$, and let $\mathbf{d} \succeq 3^k$ be k-tuples of positive integers such that $n_1 \geq 2$. As in Theorem 2.3, we write s_r and ϵ_r for $s_r(\mathbf{n}, \mathbf{d})$ and $\epsilon_r(\mathbf{n}, \mathbf{d})$, respectively.

Lemma A.1. *If $r \in \{r^*(\mathbf{n}, \mathbf{d}), r_*(\mathbf{n}, \mathbf{d})\}$, then $r - s_r \leq r_*(\mathbf{n}, \mathbf{d}(1)) - |\mathbf{n}| - 1$.*

Proof. We prove that $r - s_r - r_*(\mathbf{n}, \mathbf{d}(1)) + |\mathbf{n}| + 1 \leq 0$. By the definitions of s_r and $r_*(\mathbf{n}, \mathbf{d}(1))$, and by the fact that $-\lfloor \frac{a}{b} \rfloor \leq -\frac{a-b+1}{b}$, it suffices to show that

$$r - \frac{(|\mathbf{n}| + 1)r - N_{\mathbf{n},\mathbf{d}(1)} - (|\mathbf{n}| - 1)}{|\mathbf{n}|} - \frac{N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}|}{|\mathbf{n}| + 1} + |\mathbf{n}| + 1 \leq 0.$$

Clearing the denominators, one gets

$$-(|\mathbf{n}| + 1)r + N_{\mathbf{n},\mathbf{d}(1)} + |\mathbf{n}|^3 + 4|\mathbf{n}|^2 + |\mathbf{n}| - 1 \leq 0.$$

Recall that $r \geq r_*(\mathbf{n}, \mathbf{d})$. So, again by $-\lfloor \frac{a}{b} \rfloor \leq -\frac{a-b+1}{b}$, it is enough to show that

$$-N_{\mathbf{n},\mathbf{d}} + N_{\mathbf{n},\mathbf{d}(1)} + |\mathbf{n}|^3 + 4|\mathbf{n}|^2 + 2|\mathbf{n}| - 1 \leq 0. \tag{A.1}$$

Since

$$-N_{\mathbf{n},\mathbf{d}} + N_{\mathbf{n},\mathbf{d}(1)} = -\binom{n_1 + d_1 - 1}{d_1} \prod_{i=2}^{k} \binom{n_i + d_i}{d_i}$$

is decreasing with respect to $d_1, d_2, \dots, d_k$, it is enough to prove (A.1) for $\mathbf{d} = 3^k$. We do it by induction on k.

Base case: we prove (A.1) for $\mathbf{d} = (3, 3, 3)$. Since $n_1 \le n_2 \le n_3$, it is enough to prove that

$$-\binom{n_1+2}{3}\binom{n_1+3}{3}\binom{n_3+3}{3} + (n_1+2n_3)^3 + 4(n_1+2n_3)^2 + 2(n_1+2n_3) - 1 \le 0. \tag{A.2}$$

As a univariate polynomial in $\mathbb{Q}[n_1][n_3]$, the left-hand side is equal to

$$\begin{aligned}&\left(-\frac{1}{216}n_1^6 - \frac{1}{24}n_1^5 - \frac{31}{216}n_1^4 - \frac{17}{72}n_1^3 - \frac{5}{27}n_1^2 - \frac{1}{18}n_1 + 8\right) n_3^3\\ &\quad + \left(-\frac{1}{36}n_1^6 - \frac{1}{4}n_1^5 - \frac{31}{36}n_1^4 - \frac{17}{12}n_1^3 - \frac{10}{9}n_1^2 + \frac{35}{3}n_1 + 16\right) n_3^2\\ &\quad + \left(-\frac{11}{216}n_1^6 - \frac{11}{24}n_1^5 - \frac{341}{216}n_1^4 - \frac{187}{72}n_1^3 + \frac{107}{27}n_1^2 + \frac{277}{18}n_1 + 4\right) n_3\\ &\quad - \frac{1}{36}n_1^6 - \frac{1}{4}n_1^5 - \frac{31}{36}n_1^4 - \frac{5}{12}n_1^3 + \frac{26}{9}n_1^2 + \frac{5}{3}n_1 - 1.\end{aligned}$$

It is immediate that all coefficients are negative for $n_1 \ge 3$, allowing us to conclude that (A.2), and hence (A.1), holds for $\mathbf{d} = (3, 3, 3)$ and $n_1 \ge 3$.

We are left with the case $n_1 = 2$, for which (A.2) does not hold for $n_3 \gg 0$. Therefore, we prove directly (A.1) by substituting $n_1 = 2$, i.e., we consider

$$-4\binom{n_2+3}{3}\binom{n_3+3}{3} + (2+n_2+n_3)^3 + 4(2+n_2+n_3)^2 + 2(2+n_2+n_3) - 1 \le 0. \tag{A.3}$$

As a univariate polynomial in $\mathbb{Q}[n_2][n_3]$, the left-hand side is equal to

$$\begin{aligned}&\left(-\frac{1}{9}n_2^3 - \frac{2}{3}n_2^2 - \frac{11}{9}n_2 + \frac{1}{3}\right) n_3^3 + \left(-\frac{2}{3}n_2^3 - 4\,n_2^2 - \frac{13}{3}n_2 + 6\right) n_3^2\\ &\quad + \left(-\frac{11}{9}n_2^3 - \frac{13}{3}n_2^2 + \frac{59}{9}n_2 + \frac{68}{3}\right) n_3 + \frac{1}{3}n_2^3 + 6\,n_2^2 + \frac{68}{3}n_2 + 23\end{aligned}$$

For $n_2 \ge 2$ the first and the second coefficients are negative and the fourth is positive, so, independently on the sign of the third one, there is only one change of sign in the coefficients. Hence, by Descartes' rule of signs, it has only one positive real root. In order to show that (A.3) holds for every $n_3 \ge 2$, it is enough to show that such a polynomial is negative for $n_3 = 2$. For $n_3 = 2$ it becomes

$$-\frac{17}{3}{n_2}^3 - 24{n_2}^2 + \frac{26}{3}n_2 + 95 \le 0.$$

Hence (A.1) holds for $\mathbf{d} = (3, 3, 3)$ and $n_1 = 2$.

Inductive step: we prove (A.1) for $\mathbf{d} = 3^k$ and $k \ge 4$. Let $\mathbf{n}' = (n_1, n_2, \ldots, n_{k-1})$. By inductive assumption

$$\begin{aligned}&-\binom{n_1+2}{3}\prod_{i=2}^{k-1}\binom{n_i+3}{3}\binom{n_k+3}{3} + (|\mathbf{n}|^3 + 4|\mathbf{n}|^2 + 2|\mathbf{n}| - 1)\\ &\qquad \le -\binom{n_k+3}{3}(|\mathbf{n}'|^3 + 4|\mathbf{n}'|^2 + 2|\mathbf{n}'| - 1) + (|\mathbf{n}|^3 + 4|\mathbf{n}|^2 + 2|\mathbf{n}| - 1).\end{aligned}$$

We express this as univariate polynomial in $\mathbb{Q}[|\mathbf{n}'|][n_k]$:

$$\left(-\frac{1}{6}|\mathbf{n}'|^3 - \frac{2}{3}|\mathbf{n}'|^2 - \frac{1}{3}|\mathbf{n}'| + \frac{7}{6}\right) n_k^3 \\ + \left(-|\mathbf{n}'|^3 - 4\,|\mathbf{n}'|^2 + |\mathbf{n}'| + 5\right) n_k^2 \\ + \left(-\frac{11}{6}|\mathbf{n}'|^3 - \frac{13}{3}|\mathbf{n}'|^2 + \frac{13}{3}|\mathbf{n}'| + \frac{23}{6}\right) n_k.$$

Since $2 \le n_1 \le n_2 \le \cdots \le n_{k-1}$ and $k \ge 4$, we have $|\mathbf{n}'| \ge 6$. Under this condition, all coefficients of this polynomial are negative, and hence (A.1) also holds for $\mathbf{d} = 3^k$ for any $k \ge 4$. □

Lemma A.2. *If $r \in \{r^*(\mathbf{n},\mathbf{d}), r_*(\mathbf{n},\mathbf{d})\}$, then $s_r \ge \epsilon_r$.*

Proof. Note that

$$\begin{aligned} s_r - \epsilon_r &= s_r - (|\mathbf{n}|+1)r + N_{\mathbf{n},\mathbf{d}(1)} + |\mathbf{n}|s_r \\ &= (|\mathbf{n}|+1)(s_r - r) + N_{\mathbf{n},\mathbf{d}(1)} \\ &\ge (|\mathbf{n}|+1)(-r_*(\mathbf{n},\mathbf{d}(1)) + |\mathbf{n}| + 1) + N_{\mathbf{n},\mathbf{d}(1)} \\ &\ge (|\mathbf{n}|+1)\left(-\frac{N_{\mathbf{n},\mathbf{d}(1)}}{|\mathbf{n}|+1} + |\mathbf{n}| + 1\right) + N_{\mathbf{n},\mathbf{d}(1)} \\ &= (|\mathbf{n}|+1)^2, \end{aligned}$$

where the first inequality follows from Lemma A.1 and the second one follows from the definition of r_*. □

Lemma A.3. *If $r \in \{r^*(\mathbf{n},\mathbf{d}), r_*(\mathbf{n},\mathbf{d})\}$, then $r^*(\mathbf{n},\mathbf{d}(2)) + |\mathbf{n}| + 1 \le r - s_r - \epsilon_r$.*

Proof. By definition, $\epsilon_r \le |\mathbf{n}| - 1$. So $r - s_r - \epsilon_r - r^*(\mathbf{n},\mathbf{d}(2)) - |\mathbf{n}| - 1 \ge r - s_r - r^*(\mathbf{n},\mathbf{d}(2)) - 2|\mathbf{n}|$. We prove that the latter is greater than equal to zero. By the definition of s_r

$$r - s_r - r^*(\mathbf{n},\mathbf{d}(2)) - 2|\mathbf{n}| \ge r - \frac{(|\mathbf{n}|+1)r - N_{\mathbf{n},\mathbf{d}(1)}}{|\mathbf{n}|} - r^*(\mathbf{n},\mathbf{d}(2)) - 2|\mathbf{n}|.$$

Clear the denominator. It suffices to show that

$$-r + N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}|r^*(\mathbf{n},\mathbf{d}(2)) - 2|\mathbf{n}|^2 \ge 0.$$

Since $r \le r^*(\mathbf{n},\mathbf{d})$, we obtain

$$\begin{aligned} &-r + N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}|r^*(\mathbf{n},\mathbf{d}(2)) - 2|\mathbf{n}|^2 \\ &\qquad\ge -\left\lceil \frac{N_{\mathbf{n},\mathbf{d}}}{|\mathbf{n}|+1} \right\rceil + N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}| \left\lceil \frac{N_{\mathbf{n},\mathbf{d}(2)}}{|\mathbf{n}|+1} \right\rceil - 2|\mathbf{n}|^2 \\ &\qquad\ge -\frac{N_{\mathbf{n},\mathbf{d}} + |\mathbf{n}|}{|\mathbf{n}|+1} + N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}|\frac{N_{\mathbf{n},\mathbf{d}(2)} + |\mathbf{n}|}{|\mathbf{n}|+1} - 2|\mathbf{n}|^2. \end{aligned}$$

Clearing the denominator, we are left to prove that

$$-N_{\mathbf{n},\mathbf{d}} + (|\mathbf{n}|+1)N_{\mathbf{n},\mathbf{d}(1)} - |\mathbf{n}|N_{\mathbf{n},\mathbf{d}(2)} - |\mathbf{n}|(|\mathbf{n}|+1)(2|\mathbf{n}|+1) \ge 0. \tag{A.4}$$

Observe that

$$-N_{\mathbf{n},\mathbf{d}}+(|\mathbf{n}|+1)N_{\mathbf{n},\mathbf{d}(1)}-|\mathbf{n}|N_{\mathbf{n},\mathbf{d}(2)} = \frac{(n_1+d_1-2)!}{(n_1-1)!d_1!}\prod_{i=2}^{k}\binom{n_i+d_i}{n_i}\left(|\mathbf{n}|d_1-n_1-d_1+1\right).$$

The left-hand side of (A.4) is increasing when all the d_i are positive and increasing. Therefore, it is enough to prove (A.4) for $\mathbf{d}=3^k$. We do it by induction on k.

Base case: we prove (A.4) for $\mathbf{d}=(3,3,3)$. We employ the fact that $n_1\le n_2\le n_3$ to deduce that the left-hand side of (A.4) for $\mathbf{d}=(3,3,3)$ is greater than or equal to

$$\begin{aligned}
&\frac{1}{3}\binom{n_1+1}{2}\binom{n_1+3}{3}\binom{n_3+3}{3}(5n_1+3n_3-2)\\
&\qquad -(n_1+2n_3)(n_1+2n_3+1)(2n_1+4n_3+1)\\
&\quad=\left(\frac{1}{72}n_1^5+\frac{7}{72}n_1^4+\frac{17}{72}n_1^3+\frac{17}{72}n_1^2+\frac{1}{12}n_1\right)n_3^4\\
&\qquad+\left(\frac{5}{216}n_1^6+\frac{17}{72}n_1^5+\frac{197}{216}n_1^4+\frac{119}{72}n_1^3+\frac{151}{108}n_1^2+\frac{4}{9}n_1-16\right)n_3^3\\
&\qquad+\left(\frac{5}{36}n_1^6+\frac{77}{72}n_1^5+\frac{73}{24}n_1^4+\frac{289}{72}n_1^3+\frac{179}{72}n_1^2-\frac{281}{12}n_1-12\right)n_3^2\\
&\qquad+\left(\frac{55}{216}n_1^6+\frac{127}{72}n_1^5+\frac{907}{216}n_1^4+\frac{289}{72}n_1^3-\frac{1\,165}{108}n_1^2-\frac{109}{9}n_1-2\right)n_3\\
&\qquad+\frac{5}{36}n_1^6+\frac{11}{12}n_1^5+\frac{71}{36}n_1^4-\frac{7}{12}n_1^3-\frac{28}{9}n_1^2-\frac{4}{3}n_1.
\end{aligned}$$

Regarding it as a univariate polynomial in $\mathbb{Q}[n_1][n_3]$, we observe that each coefficient is positive under our assumption that $n_1\ge 2$. Hence, (A.4) holds for $\mathbf{d}=(3,3,3)$.

Inductive step: we prove (A.4) for $\mathbf{d}=3^k$ with $k\ge 4$. Let $\mathbf{n}'=(n_1,n_2,\ldots,n_{k-1})$. By inductive assumption and by replacing $|\mathbf{n}|=|\mathbf{n}'|+n_k$, we have

$$\begin{aligned}
&\frac{1}{3}\binom{n_1+1}{2}\prod_{i=2}^{k}\binom{n_i+3}{n_i}(3|\mathbf{n}|-n_1-2)-|\mathbf{n}|(|\mathbf{n}|+1)(2|\mathbf{n}|+1)\\
&\ge\binom{n_k+3}{3}\cdot\frac{1}{3}\binom{n_1+1}{2}\prod_{i=2}^{k-1}\binom{n_i+3}{n_i}(3|\mathbf{n}'|-n_1-2)-|\mathbf{n}|(|\mathbf{n}|+1)(2|\mathbf{n}|+1)\\
&\ge\binom{n_k+3}{n_k}|\mathbf{n}'|(|\mathbf{n}'|+1)(2|\mathbf{n}'|+1)-(|\mathbf{n}'|+n_k)(|\mathbf{n}'|+n_k+1)(2|\mathbf{n}'|+2n_k+1).
\end{aligned}$$

We express this latter polynomial as a univariate polynomial in $\mathbb{Q}[|\mathbf{n}'|][n_k]$:

$$\begin{aligned}
&\left(\frac{1}{3}|\mathbf{n}'|^3+\frac{1}{2}|\mathbf{n}'|^2+\frac{1}{6}|\mathbf{n}'|-2\right)n_k^3\\
&\qquad+\left(2\,|\mathbf{n}'|^3+3\,|\mathbf{n}'|^2-5\,|\mathbf{n}'|-3\right)n_k^2\\
&\qquad+\left(\frac{11}{3}|\mathbf{n}'|^3-\frac{1}{2}|\mathbf{n}'|^2-\frac{25}{6}|\mathbf{n}'|-1\right)n_k.
\end{aligned}$$

Since $2 \le n_1 \le n_2 \le \cdots \le n_{k-1}$ and $k \ge 4$, we have $|\mathbf{n}'| \ge 6$. Under this condition, all coefficients of the latter polynomial are positive, and hence (A.4) also holds for $\mathbf{d} = 3^k$ for any $k \ge 4$. □

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Department of Mathematics, University of Idaho, Moscow, Idaho 83844-1103 United States of America
Email address: abo@uidaho.edu

Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy
Email address: m.c.brambilla@univpm.it

Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
Email address: galuppi@mimuw.edu.pl (ORCID 0000-0001-5630-5389)

Department of Mathematics, Università di Trento, Via Sommarive 14, 38123 Povo (Trento), Italy
Email address: alessandro.oneto@unitn.it