Received: 22 December 2022 | Revised: 1 June 2023 | Accepted: 9 June 2023

DOI: 10.1112/blms.12890

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Linear extensions and shelling orders

Davide Bolognini[1] | **Paolo Sentinelli**[2]

[1]Dipartimento di Ingegneria Industriale e Scienze Matematiche, Universitá Politecnica delle Marche, Ancona, Italy

[2]Dipartimento di Matematica, Politecnico di Milano, Milan, Italy

Correspondence
Paolo Sentinelli, Dipartimento di Matematica, Politecnico di Milano, Milan, Italy.
Email: paolosentinelli@gmail.com

Abstract

We prove that linear extensions of the Bruhat order of a matroid are shelling orders and that the barycentric subdivision of a matroid is a Coxeter matroid, viewing barycentric subdivisions as subsets of a parabolic quotient of a symmetric group. A similar result holds for order ideals in minuscule quotients of symmetric groups and in their barycentric subdivisions. Moreover, we apply promotion and evacuation for labeled graphs of Malvenuto and Reutenauer to dual graphs of simplicial complexes, introducing promotion and evacuation of shelling orders.

MSC 2020
05B35, 05E18, 05E45, 06A07, 20B30 (primary)

1 | INTRODUCTION

A pure simplicial complex is shellable if its facets admit a total order, called *shelling order*, such that each facet can be added gluing it along a subcomplex of codimension 1. Shellability is one of the most studied combinatorial properties of simplicial complexes. Its pivotal role in combinatorics and commutative algebra is due to the fact that a shellable simplicial complex is also Cohen–Macaulay over every field. It is combinatorial because there exist both shellable and nonshellable triangulations of the same topological space (for nonshellable triangulations of spheres and balls, see, e.g., [2]).

Examples of shellable simplicial complexes are vertex-decomposable ones (see, e.g., [16, Theorem 3.33]), boundaries of simplicial polytopes [24, Theorem 8.11], order complexes of Bruhat intervals in parabolic quotients of Coxeter groups [6] and of Bruhat intervals in their complements [20], order complexes of face posets of electrical networks [15], among others.

A subclass of vertex-decomposable simplicial complexes are independence complexes of matroids (see, for instance, [16, Theorem 13.1]), for which a shelling order is given by the lexicographic order of the facets. The set of facets of a pure k-dimensional simplicial complex on n vertices can be identified with a subset of the set $S_n^{(k)}$ of Grassmannian permutations, which can be endowed with the Bruhat order. Therefore, if $X \subseteq S_n^{(k)}$ is the set of bases of a matroid, we view X as a poset with the induced order, so we can speak about *the Bruhat order of the matroid* X (also called *Gale order*). Inspired by the fact that the lexicographic order is a linear extension of X, we state in Theorem 3.4 that all the linear extensions of X are shelling orders. Actually we prove this result for the larger class of simplicial complexes with the quasi-exchange property, introduced in [19]; this class includes also order ideals of $S_n^{(k)}$, see Corollary 3.5. As there are shellable simplicial complexes for which no linear extension is a shelling order (we checked it for the so-called Hachimori's complex, see, e.g., [7, Example 4.5] for a list of facets), this result provides a structural connection between shellings orders of matroids and linear extensions of their Bruhat orders. Nevertheless, as expected, there are shelling orders of matroids that are not linear extensions, also up to relabeling (see Example 3.10).

Coxeter matroids generalize, via the maximality property, standard matroids. By extending maximality property to different contexts, in [8] we generalized flag matroids to P-flag matroids and in [9] matroids to χ-matroids, where P is any finite poset and χ a one-dimensional character of a finite group. In this paper, we provide another connection between matroids and Coxeter matroids involving barycentric subdivisions of simplicial complexes (Theorem 4.1).

The interpretation of the facets of a pure simplicial complex X as elements of $S_n^{(k)}$ allows us to view the facets of the barycentric subdivision $\mathcal{B}(X)$ of X as permutations in S_n obtained by acting with S_k on the elements of X. In Definition 4.3, we introduce a notion of *flag shellability* for subsets of the barycentric subdivision $\mathcal{B}(S_n^{(k)})$. Flag shellability of $\mathcal{B}(X)$ coincides with shellability of the order complex of the face poset of X. In Theorem 4.5, we prove that the linear extensions of order ideals of $\mathcal{B}(S_n^{(k)})$ are flag shelling orders.

Although shellable simplicial complexes are extremely nice from a combinatorial point of view, also in this realm weird things may happen: for instance there exist shellable simplicial complexes such that every possible shelling order is forced to end with a specific facet (see [21, Appendix F]). For this reason, it is crucial to know if and how a shelling order can be rearranged to have a new shelling order. The promotion function was defined on linear extensions of posets (see [23] for a survey and [13, 14] for recent results and new developments): given a linear extension of a poset, its promotion is a new linear extension, obtained rearranging the first. By taking advantage of the generalization given in [18] and by considering the so-called dual graph of a pure simplicial complex (or, equivalently, its undirected Bruhat graph, see Remark 5.6), in Section 5 we introduce promotion and evacaution of shelling orders; see Theorems 5.4 and 5.11. The core of the proof is given by a structural property of shelling orders, which is interesting by itself, see Proposition 5.3. For simplicial complexes for which linear extensions are shelling orders, it is natural to ask if the promotion of shelling orders agrees with promotion of linear extensions: under a suitable assumption, in Proposition 5.7 we prove that this is the case; this assumption is fulfilled by interesting classes of simplicial complexes, see Corollary 5.8.

2 | NOTATION AND PRELIMINARIES

In this section, we fix notation and recall some definitions useful for the rest of the paper. We refer to [22] for posets, to [5] for Coxeter groups, and to [10] for matroids and Coxeter matroids.

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{N}$ the set of positive integers. For $n \in \mathbb{N}$, we use the notation $[n] := \{1, 2, \dots, n\}$. For a finite set X, we denote by $|X|$ its cardinality and by $\mathcal{P}(X)$ its power set, which is an abelian group with the operation given by symmetric difference $A + B := (A \setminus B) \cup (B \setminus A)$, for all $A, B \subseteq X$. We denote by X^n the nth power under Cartesian product, by x_i the projection of $x \in X^n$ on the ith factor, and we set $N(x) := n$. For $k \in \mathbb{N}$, $k \leqslant |X|$, we define the kth *configuration space* of X by

$$\mathrm{Conf}_k(X) := \left\{ x \in X^k \, : \, x_i = x_j \Rightarrow i = j, \; \forall\, i, j \in [k] \right\},$$

and, if $<$ is a total order on X, the kth *unordered configuration space* of X by

$$X^k_< := \left\{ x \in X^k \, : \, i < j \Rightarrow x_i < x_j, \, \forall\, i, j \in [k] \right\}.$$

We also set

$$\mathrm{Conf}(X) := \bigcup_{k=1}^{|X|} \mathrm{Conf}_k(X).$$

Sometimes we write $a_1 \dots a_k \in \mathrm{Conf}_k(X)$ instead of $(a_1, \dots, a_k) \in \mathrm{Conf}_k(X)$.

We consider the symmetric group S_n of order $n!$ as a Coxeter group, with generators given by simple transpositions $S := \{s_1, \dots, s_{n-1}\}$, where, in one-line notation, $s_i := 12 \dots (i+1)i \dots n$, for all $i \in [n-1]$. The *right descent set* of a permutation $w \in S_n$ is defined by

$$D_R(w) := \{ i \in [n-1] \, : \, w(i) > w(i+1) \}.$$

For $J \subseteq [n-1]$, define

$$S_n^J := \{ w \in S_n \, : \, i \in J \; \Rightarrow \; w(i) < w(i+1) \}.$$

There is a function $P^J : S_n \to S_n^J$ defined by mapping a permutation w to an increasing rearrangement according to J, as described in [5, section 2.4]. The following example should make clear how to obtain the permutation $P^J(w)$.

Example 2.1. Let $n = 7$, $J = \{1, 2, 4, 6\}$ and $w = 4317625$. Therefore, we have to rearrange increasingly the blocks 431, 76 and 25. It follows that $P^J(w) = 1346725$.

If $k \in [n-1]$, the Bruhat order† $\leqslant$ on the *minuscule quotient* $S_n^{(k)} := S_n^{[n-1] \setminus \{k\}}$ is defined by setting $u \leqslant v$ if and only if $u(i) \leqslant v(i)$, for all $1 \leqslant i \leqslant k$ (see [5, Proposition 2.4.8]). We let $S_n^{(n)} := S_n^{[n-1]} = \{e\}$. The elements of $S_n^{(k)}$ are called *Grassmannian permutations*. The Bruhat order on S_n can be defined by setting

$$u \leqslant v \;\Leftrightarrow\; P^{[n-1]\setminus\{k\}}(u) \leqslant P^{[n-1]\setminus\{k\}}(v), \; \text{for all } k \in [n-1], \tag{1}$$

for all $u, v \in S_n$ (see [5, Theorem 2.6.1]). On the subset S_n^J of S_n we consider the induced order, and this leads to the definition of Coxeter matroid via the *maximality property*.

† The Bruhat order on a minuscule quotient of S_n is also known as *Gale order*.

Definition 2.2. A subset $X \subseteq S_n^J$ is a *Coxeter matroid* if the induced subposet $\{P^J(wx) : x \in X\} \subseteq S_n^J$ has a unique maximum (equivalently, has a unique minimum) for all $w \in S_n$.

For example, if $J = [n-1] \setminus \{k\}$, then a Coxeter matroid is a matroid of rank k on the set $[n]$ (see [10, section 1.3]). For $J = \varnothing$ a Coxeter matroid is a flag matroid (see [10, section 1.7]). In Section 4, we prove that some Coxeter matroids for $J = [n-1] \setminus [k]$ can be realized as barycentric subdivisions of independence complexes of matroids.

The kth configuration space of $[n]$ can be identified with the quotient $S_n^{[n-1]\setminus[k]}$, that is, as sets,

$$\mathrm{Conf}_k([n]) \simeq S_n^{[n-1]\setminus[k]}.$$

Then it makes sense to consider on $\mathrm{Conf}_k([n])$ the Bruhat order.

On $[n]^k_< \subseteq \mathrm{Conf}_k([n])$ we consider the induced order; this poset is isomorphic to $S_n^{(k)}$ with the Bruhat order. Then, as posets,

$$[n]^k_< \simeq S_n^{(k)}.$$

For example, in $[8]^4_<$ we have $3456 \leqslant 4568$ and $2568 \nleqslant 3478$. We also repeatedly use the identification

$$[n]^k_< \simeq \{X \subseteq [n] : |X| = k\},$$

where $[n]^0_< := \{\varnothing\}$. Then, identifying $U := \bigcup_{k=0}^{n} [n]^k_<$ with $\mathcal{P}(X)$, it makes sense to write $x \cap y$, $x \cup y$ and the symmetric difference $x + y$, for all $x, y \in U$.

As $\mathrm{Conf}([n]) \simeq \bigcup_{i=1}^n S_n^{[n-1]\setminus[i]}$, for $k \in [n]$ we have a function $P^{(k)} : \mathrm{Conf}([n]) \to [n]^k_<$ obtained by gluing the functions $P^{[n-1]\setminus\{k\}} : S_n^{[n-1]\setminus[i]} \to S_n^{(k)}$ for all $i \in [n]$. Notice that $x \leqslant y$ in the Bruhat order of $\mathrm{Conf}_k([n])$ if and only if $P^{(i)}(x) \leqslant P^{(i)}(y)$ in $[n]^i_<$ for all $i \in [k]$. For example, $3125 \leqslant 4251$ in $\mathrm{Conf}_4([5])$. On the other hand, $3152 \nleqslant 4215$ in $\mathrm{Conf}_4([5])$, as $P^{(3)}(3152) = 135 \nleqslant 124 = P^{(3)}(4215)$.

By our identifications, a matroid of rank k on the set $[n]$ is a subset of $[n]^k_<$, and a Coxeter matroid in the quotient $S_n^{[n-1]\setminus[k]}$ is a subset of $\mathrm{Conf}_k([n])$. We have defined a matroid by the maximality property, which is equivalent to the exchange property (see [10, Theorem 1.3.1]):

Definition 2.3 (Exchange property). A set $X \subseteq [n]^k_<$ is a matroid if and only if for all $A, B \in X$ and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $A + \{a, b\} \in X$.

Let $M \subseteq [n]^k_<$ be a matroid and $i \in [n-1]$. Then $\{P^{(i)}(x) : x \in M\}$ is a matroid, called the *shift* of M to $[n]^i_<$ (see [10, section 6.12.1]). The *underlying flag matroid* of M is the union of cosets $\biguplus_{x \in M} x(S_n)_{S\setminus\{s_k\}}$, where $(S_n)_{S\setminus\{s_k\}}$ is the parabolic subgroup of S_n generated by $S \setminus \{s_k\}$ (see [10, section 6.6]).

Example 2.4. Let $M := \{13, 34\} \subseteq [4]^2_<$. Then the shift of the matroid M to $[4]^3_<$ is the matroid $\{123, 134\}$. The underlying flag matroid of M is $\{1324, 3124, 1342, 3142, 3412, 4312, 3421, 4321\} \subseteq \mathrm{Conf}_4([4]) \simeq S_4$.

In general, for $I, J \subseteq [n-1]$, the shift of a Coxeter matroid $M \subseteq S_n^J$ to S_n^I is the Coxeter matroid $\{P^I(x) : x \in M\}$.

3 | LINEAR EXTENSIONS OF PURE SIMPLICIAL COMPLEXES

Let $k, n \in \mathbb{N}$ be such that $k \leqslant n$. We identify a pure simplicial complex X of dimension $k-1$ on n vertices with the set of its facets. As any facet of X corresponds to a subset of $[n]$ of cardinality k, we can view the X as a subset of $[n]^k_<$. On the other hand, any subset of $[n]^k_<$ provides a pure simplicial complex of dimension $k-1$ on n vertices. Therefore, matroids of rank k on the set $[n]$ are pure simplicial complexes of dimension $k-1$.

Definition 3.1. An element $L \in \mathrm{Conf}([n]^k_<)$ is a *linear extension* if $L_i < L_j$ in the Bruhat order implies $i < j$, for all $i, j \in [N(L)]$.

For example, $(357, 268, 468) \in \mathrm{Conf}([8]^3_<)$ is a linear extension. We provide now the definition of shelling order.

Definition 3.2. An element $C \in \mathrm{Conf}([n]^k_<)$ is a *shelling order* if $i < j$ implies that there exists $z < j$ such that $|C_z \cap C_j| = |C_j| - 1$ and $C_i \cap C_j \subseteq C_z \cap C_j$, for all $i, j \in [N(C)]$.

A pure simplicial complex $X \subseteq [n]^k_<$ is said to be *shellable* if there exists a shelling order $C \in \mathrm{Conf}([n]^k_<)$ such that $X = \{C_1, \dots, C_{N(C)}\}$. It is well-known that, if $X \subseteq [n]^k_<$ is a matroid, then the lexicographic order on X is a shelling order (see [4, Theorems 7.3.3 and 7.3.4]) and a linear extension of the Bruhat order of X.

In the following theorem, we prove that for a wide class of simplicial complexes, including matroids and order ideals in $[n]^k_<$, actually any linear extension of the Bruhat order provides a shelling order. This class is defined by the following property (see [19, Definition 4.1]).

Definition 3.3. A subset $X \subseteq [n]^k_<$ has the *quasi-exchange* property if, given $x, y \in X$, then $i \in x \setminus y$ and $i > \max(y \setminus x)$ imply that there exists $j \in y \setminus x$ such that $x + \{i, j\} \in X$.

Notice that if $i \in x$, $i > \max(y \setminus x)$ and $j \in y \setminus x$, then $x + \{i, j\} < x$ in the Bruhat order, for all $x, y \in [n]^k_<$.

Theorem 3.4. *If $X \subseteq [n]^k_<$ has the quasi-exchange property, then any linear extension of X is a shelling order.*

Proof. If $k = n$ the statement is trivial. So we may assume $k < n$. Let $h := |X|$ and $L = (L_1, \dots, L_h)$ be a linear extension of X. If $h = 1$ we have nothing to show. So let $h > 1$. Assume that $(L_1, \dots, L_r)$ is a shelling order for $r < h$ and consider the linear extension $(L_1, \dots, L_r, L_{r+1})$. Let $i \in [r]$. As L is a linear extension we have that $L_i \ngeq L_{r+1}$. We are going to show that there exists L_z with $z \in [r]$ such that $|L_z \cap L_{r+1}| = |L_{r+1}| - 1$ and $L_i \cap L_{r+1} \subseteq L_z \cap L_{r+1}$. Let $v := \max\{j \in [k] : L_{r+1}(j) \neq L_i(j)\}$. We have two cases.

(1) $L_{r+1}(v) > L_i(v)$: in this case $L_{r+1}(v) > L_i(v) = \max(L_i \setminus L_{r+1})$ and $L_{r+1}(v) \notin L_i$. By the quasi-exchange property, there exists $y \in L_i \setminus L_{r+1}$ such that $Y := L_{r+1} + \{L_{r+1}(v), y\} \in X$. Hence,

$Y < L_{r+1}$ in the Bruhat order, that is, there exists $z \in [r]$ such that $Y = L_z$, as L is a linear extension of the Bruhat order of X. Therefore, L_z has the required properties.

(2) $L_{r+1}(v) < L_i(v)$: in this case $L_i(v) > L_{r+1}(v) = \max(L_{r+1} \setminus L_i)$ and $L_i(v) \notin L_{r+1}$. By the quasi-exchange property, there exists $y \in L_{r+1} \setminus L_i$ such that $Y := L_i + \{y, L_i(v)\} \in X$, and $Y < L_i$ in the Bruhat order. Then $i > 1$ and there exists $j \in [i-1]$ such that $Y = L_j$, as L is a linear extension of X. Moreover, if $u := \max\{j \in [k] : L_{r+1}(j) \neq L_1(j)\}$, then $L_1(u) < L_{r+1}(u)$. In fact, if $L_1(u) > L_{r+1}(u)$, then $L_{r+1}(u) = \max(L_{r+1} \setminus L_1)$ and there exists $m \in L_{r+1} \setminus L_1$ such that $M := L_1 + \{m, L_1(u)\} \in X$ with $M < L_1$ in the Bruhat order, a contradiction. So, $L_{r+1}(u) > L_1(u) = \max(L_1 \setminus L_{r+1})$. By the previous case, there exists $w \in L_{r+1} \setminus L_1$ and $w' \in L_1 \setminus L_{r+1}$ such that $w' < w$ and $L_{r+1} + \{w, w'\} \in X$. Assume that, for all $j < i$, there exist $w \in L_{r+1} \setminus L_j$ and $w' \in L_j \setminus L_{r+1}$ such that $w' < w$ and $L_{r+1} + \{w, w'\} \in X$. By our inductive assumption, there exists $w \in L_{r+1} \setminus Y$ and $w' \in Y \setminus L_{r+1}$ such that $w' < w$ and $W := L_{r+1} + \{w, w'\} \in X$; therefore $W < L_{r+1}$ in the Bruhat order. This implies that there exists $z \in [r]$ such that $W = L_z$. Notice that $w \notin L_i$; in fact, as $Y = L_i + \{L_i(v), y\}$, if $w \in L_i$ we have that $w = L_i(v) \notin L_{r+1}$, a contradiction. Then $|L_z \cap L_{r+1}| = |L_{r+1}| - 1$ and $L_i \cap L_{r+1} \subseteq L_z \cap L_{r+1}$. □

Corollary 3.5. *Let $X \subseteq [n]^k_<$ be an order ideal or a matroid. Then any linear extension of X is a shelling order.*

Proof. Clearly any matroid has the quasi-exchange property. Moreover by [19, Theorem 4.11], any order ideal of $[n]^k_<$ has the quasi-exchange property. So, the result follows by Theorem 3.4. □

Remark 3.6. Recall that there exist matroids that are not order ideals, for example, the nonrepresentable ones. Analogously, by the maximality property of matroids, nonprincipal order ideals are not matroids.

Remark 3.7. In a private communication, J. A. Samper pointed out to us that the statement of Corollary 3.5 for matroids can be deduced by combining [1, Theorem 1.3] and [19, Theorem 4.14].

We formalize now a notion of isomorphism between shelling orders. A permutation $\sigma \in S_n$ induces a function

$$\sigma : \mathrm{Conf}\big([n]^k_<\big) \to \mathrm{Conf}\big([n]^k_<\big),$$

defined by $\sigma(X) = ((P^{(k)} \circ \sigma)(X_1), \dots, (P^{(k)} \circ \sigma)(X_k))$, for all $X \in \mathrm{Conf}([n]^k_<)$, where $\sigma : [n]^k_< \to \mathrm{Conf}_k([n])$ is the function defined by $\sigma(x) = (\sigma(x_1), \dots, \sigma(x_k))$, for all $x \in [n]^k_<$.

Definition 3.8. Two elements $A, B \in \mathrm{Conf}([n]^k_<)$ are *isomorphic* if there exists $\sigma \in S_n$ such that $\sigma(A) = B$.

Essentially, two shelling orders are isomorphic if they are the same up to relabeling. For example, all shelling orders in $\mathrm{Conf}_2([n]^k_<)$ are isomorphic; on the other hand, the shelling orders $A_1 := (123, 124, 125)$, $A_2 := (123, 124, 135)$ and $A_3 := (123, 124, 145)$ are pairwise not isomorphic in $\mathrm{Conf}_3([5]^3_<)$.

In the following example, we observe that there exist linear extensions of a matroid that are not isomorphic to a lexicographic order.

Example 3.9. The Bruhat interval $[12, 24] = \{12, 13, 14, 23, 24\} \subseteq [4]^2_<$ is a matroid and it has two linear extensions: the lexicographic order and $L := (12, 13, 23, 14, 24)$. As the linear extension L is a shelling order, $\sigma(L)$ is a shelling order; it is different from the lexicographic order, for all $\sigma \in S_4$.

In the following example, we show that there exist shelling orders of a matroid not isomorphic to any linear extension.

Example 3.10. The tuple $C := (12, 23, 13, 14, 24)$ is a shelling order for the matroid $[12, 24] \subseteq [4]^2_<$ and $\sigma(C)$ is not a linear extension, for all $\sigma \in S_4$.

4 | BARYCENTRIC SUBDIVISIONS AND FLAG SHELLABILITY

The barycentric subdivision of a simplicial complex is the order complex of its face poset; see for instance [11]. Let $X \subseteq [n]^k_<$ and F_X be the face poset of X; we denote by $\mathcal{MC}(F_X)$ the set of maximal chains of F_X. There exists an injective function $B : \mathcal{MC}(F_X) \to \mathrm{Conf}_k([n])$ defined as follows. Let $c \in \mathcal{MC}(F_X)$; then c corresponds to a flag $\{x_1\} \subset \{x_1, x_2\} \subset \ldots \subset \{x_1, \ldots, x_k\}$ of subsets of the facet $\{x_1, \ldots, x_k\}_< \in X$, where $\{x_1, \ldots, x_k\}_< \in [n]^k_<$ is the tuple obtained by ordering $x_1, \ldots, x_k$. Hence, we set

$$B(c) := (x_1, \ldots, x_k) \in \mathrm{Conf}_k([n]).$$

Therefore, maximal chains in F_X with maximum $x = (x_1, \ldots, x_k) \in X \subseteq [n]^k_<$ are in bijection with permutations of the set $\{x_1, \ldots, x_k\}$. We introduce a new definition of *barycentric subdivision* $\mathcal{B}(X)$ of X as a union of cosets of the symmetric group S_k, viewing elements of $[n]^k_<$ as permutations:

$$\mathcal{B}(X) := \biguplus_{x \in X} \{x\sigma : \sigma \in S_k\} \subseteq \mathrm{Conf}_k([n]).$$

In particular, the barycentric subdivision of $[n]^k_<$ is $\mathrm{Conf}_k([n])$.

The standard way to subdivide barycentrically a matroid in $[n]^k_<$ provides a simplicial complex in a suitable $[m]^k_<$, which is almost never a matroid. The following theorem shows that barycentric subdivisions of matroids, in our interpretation, are Coxeter matroids.

Theorem 4.1. *A simplicial complex $X \subseteq [n]^k_<$ is a matroid if and only if the barycentric subdivision $\mathcal{B}(X) \subseteq \mathrm{Conf}_k([n])$ is a Coxeter matroid.*

Proof. Let $\mathcal{B}(X)$ be a Coxeter matroid; then $X = \{P^{(k)}(y) : y \in \mathcal{B}(X)\}$ is the shift of $\mathcal{B}(X)$ to $[n]^k_<$ and so it is a matroid (see [10, Lemma 6.12.1]).

Conversely, $\mathcal{B}(X)$ is the shift to $\mathrm{Conf}_k([n])$ of the underlying flag matroid of X, so it is a Coxeter matroid (see [10, Lemmas 6.6.1 and 6.6.2]). ☐

Example 4.2. An interval $[x, y] \subseteq [n]^k_<$ is a matroid and its barycentric subdivision is the interval $[x, y_k y_{k-1} \ldots y_1] \subseteq \mathrm{Conf}_k([n])$, which is a Coxeter matroid. In general, it is proved in [12] that any Bruhat interval of a parabolic quotient of a finite Coxeter group is a Coxeter matroid.

We now provide a notion of shellability for subsets of $\mathrm{Conf}_k([n])$, which agrees with the standard notion in case of barycentric subdivisions.

For $y \in Y \subseteq \mathrm{Conf}_k([n])$ let us define

$$P(y) := \{P^{(1)}(y), \dots, P^{(k)}(y)\}$$

and the simplicial complex $\Delta(Y)$ whose set of facets is $\{P(y) : y \in Y\}$.

Definition 4.3. We say that a set $Y \subseteq \mathrm{Conf}_k([n])$ is *flag shellable* if $\Delta(Y)$ is shellable.

Let $Y = \{a, b, \dots\} \subseteq \mathrm{Conf}_k([n])$. We say that $(a, b, \dots)$ is a *flag shelling order* for Y if $(P(a), P(b), \dots)$ is a shelling order for $\Delta(Y)$.

Example 4.4. Consider the set $Y = \{132, 435\} \subseteq \mathrm{Conf}_3([5])$. Then $\Delta(Y) = \{\{1, 13, 123\}, \{4, 34, 345\}\}$; hence it is not flag shellable. On the other hand, $Y = \{142, 143\} \subseteq \mathrm{Conf}_3([4])$ is flag shellable, because $(\{1, 14, 124\}, \{1, 14, 134\})$ is a shelling order.

We observe that, if $X \subseteq [n]^k_<$, then the simplicial complex $\Delta(\mathcal{B}(X))$ is the order complex of the face poset F_X. Therefore, according to Definition 4.3, the barycentric subdivision $\mathcal{B}(X)$ is flag shellable if and only if the order complex of F_X is shellable. The following theorem is the analogue of Corollary 3.5 for order ideals of $\mathrm{Conf}_k([n])$.

Theorem 4.5. *Let $Y \subseteq \mathrm{Conf}_k([n])$ be an order ideal; then any linear extension of Y is a flag shelling order.*

Proof. Let $h := |Y|$ and $L := (L_1, \dots, L_h)$ be a linear extension of Y. If $h = 1$ the result is trivial. Let $h \geqslant 2$ and assume $(L_1, \dots, L_{h-1})$ is a flag shelling order. Let $i \in [h-1]$. We have that $L_h \neq (1, 2, \dots, k)$ and $L_i \ngeq L_h$, as L is a linear extension. Notice that there exists $r \in D_R(L_h)$ such that $P^{(r)}(L_i) \neq P^{(r)}(L_h)$. In fact, if $P^{(r)}(L_i) = P^{(r)}(L_h)$ for all $r \in D_R(L_h)$, then $L_h = L_i$, by [5, Corollary 2.6.2], a contradiction. Hence, let $j := \min\{r \in D_R(L_h) : P^{(r)}(L_i) \neq P^{(r)}(L_h)\}$. If $j < k$ we have that $L_h s_j \in X$, because $L_h > L_h s_j \in \mathrm{Conf}_k([n])$ and X is an order ideal, and then there exists $z \in [h-1]$ such that $L_h s_j = L_z$. Moreover, $P^{(j)}(L_h) \notin P(L_i)$ and $|P(L_z) \cap P(L_h)| = |P(L_h)| - 1$. Therefore $(L_1, \dots, L_h)$ is a flag shelling order for Y. If $j = k$ then the result follows analogously, by considering $L_z = P^{[n-1]\setminus[k]}(L_h s_j) \in \mathrm{Conf}_k([n])$, as $P^{[n-1]\setminus[k]}$ is order preserving (see [5, Proposition 2.5.1]) and then $L_z \leqslant L_h s_j < L_h$. □

Although principal order ideals in $\mathrm{Conf}_k([n])$ are Coxeter matroids by [12, Theorem 6.3], the result of Theorem 4.5 is not true for all Coxeter matroids in $\mathrm{Conf}_k([n])$, as the following example shows.

Example 4.6. Let $Y := \{24, 42, 34, 43\} \subseteq \mathrm{Conf}_2([4])$. This is the barycentric subdivision of the matroid $\{24, 34\} \subseteq [4]^2_<$, hence it is a Coxeter matroids by Theorem 4.1. It is also a Bruhat interval. We have that $\Delta(Y) = \{\{2, 24\}, \{4, 24\}, \{3, 34\}, \{4, 34\}\}$. The linear extensions of Y are $L_1 := (24, 34, 42, 43)$ and $L_2 := (24, 42, 34, 43)$; but $(\{2, 24\}, \{3, 34\}, \{4, 24\}, \{3, 34\})$ and $(\{2, 24\}, \{4, 24\}, \{3, 34\}, \{4, 34\})$ are not shelling orders, and hence L_1 and L_2 are not flag shelling orders.

In the following example, we list the flag shelling orders provided by the linear extensions of an order ideal of $\mathrm{Conf}_2([4])$.

Example 4.7. Let $Y := \{12, 13, 21, 23, 14\} \subseteq \mathrm{Conf}_2([4])$. This is an order ideal and $\Delta(Y) = \{\{1, 12\}, \{1, 13\}, \{2, 12\}, \{2, 23\}, \{1, 14\}\}$. The linear extensions of Y are $L_1 := (12, 13, 21, 23, 14)$, $L_2 := (12, 21, 13, 23, 14)$, $L_3 := (12, 13, 21, 14, 23)$, $L_4 := (12, 21, 13, 14, 23)$ and $L_5 := (12, 13, 14, 21, 23)$. They correspond to the following shelling orders of $\Delta(Y)$:

(1) $(\{1, 12\}, \{1, 13\}, \{2, 12\}, \{2, 23\}, \{1, 14\})$,
(2) $(\{1, 12\}, \{2, 12\}, \{1, 13\}, \{2, 23\}, \{1, 14\})$,
(3) $(\{1, 12\}, \{1, 13\}, \{2, 12\}, \{1, 14\}, \{2, 23\})$,
(4) $(\{1, 12\}, \{2, 12\}, \{1, 13\}, \{1, 14\}, \{2, 23\})$,
(5) $(\{1, 12\}, \{1, 13\}, \{1, 14\}, \{2, 12\}, \{2, 23\})$.

Hence, L_1, L_2, L_3, L_4 and L_5 are flag shelling orders of Y.

5 | PROMOTION AND EVACUATION OF SHELLING ORDERS

In this section, we introduce promotion and evacuation of shelling orders. Promotion and evacuation functions, ∂_P and ϵ_P respectively, can be defined on the set of linear extensions of a finite poset P (see [23]); we consider the generalizations ∂_G and ϵ_G for a labeled graph G, introduced in [18]. They coincide with ∂_P and ϵ_P if G is the Hasse diagram of P.

For the following construction see [18]. Let $h \in \mathbb{N}$. Given a graph $G = (V, E)$ such that $V = [h]$, define the *track* $T_G = \{v_1, \dots, v_r\} \subseteq [h]$ by:

(1) $v_1 = 1$,
(2) for $i \geqslant 2$, $v_i = \min\{j \in [h] : j > v_{i-1}, \{v_{i-1}, j\} \in E\}$ if this minimum exists, otherwise $r = i - 1$.

The *promotion* of the labeled graph G is the permutation $\partial_G \in S_h$ defined by:

(1) $\partial_G(i) = i - 1$, if $i \in [h] \setminus T_G$;
(2) $\partial_G(v_j) = v_{j+1} - 1$, if $j \in [r - 1]$;
(3) $\partial_G(v_r) = h$.

To introduce promotion and evacuation of shelling orders, we consider the so-called dual graph of $X \subseteq [n]^k_<$ (for an overview on dual graphs, see [3]).

Definition 5.1. Let $X \subseteq [n]^k_<$. The *dual graph* $D(X)$ of X is the graph whose vertex set is X and $\{x, y\}$ is an edge if and only if $|x \cap y| = k - 1$, for all $x, y \in X$.

An element $C \in \mathrm{Conf}([n]^k_<)$ uniquely determines a simplicial complex $\{C_1, \dots, C_h\} \subseteq [n]^k_<$, where $h := N(C)$. The *dual graph* of C, denoted by $D(C)$, is the graph $([h], E)$, where $\{i, j\} \in E$ if and only if $|C_i \cap C_j| = k - 1$, for all $i, j \in [h]$. Let us define a function

$$\partial_D : \mathrm{Conf}([n]^k_<) \to \mathrm{Conf}([n]^k_<)$$

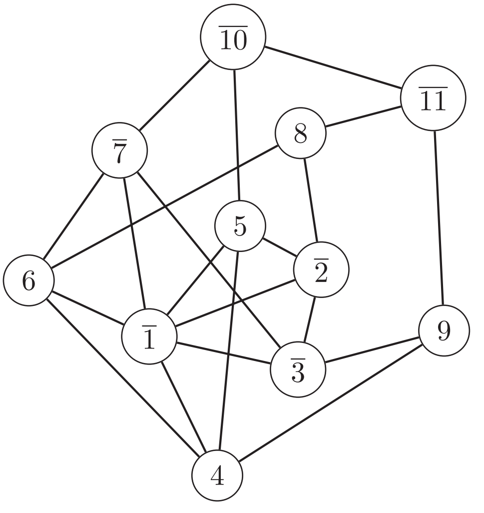

FIGURE 1 Dual graph of the Björner's example. The labeling is given by the shelling order C of Example 5.2.

by setting $\partial_D C := \partial_{D(C)} C$, where, for a permutation $\sigma \in S_h$, we let

$$\sigma C = \left(C_{\sigma^{-1}(1)}, \dots, C_{\sigma^{-1}(h)}\right).$$

Notice that $\partial_D C$ is simply obtained from C by changing the positions of the elements in the track. Moreover, $C \in \mathrm{Conf}_h([n]^k_<)$ implies $\partial_D C \in \mathrm{Conf}_h([n]^k_<)$, for all $h \geqslant 1$.

Similarly, we can define

$$\partial_H : \mathrm{Conf}([n]^k_<) \to \mathrm{Conf}([n]^k_<),$$

by setting $\partial_H C := \partial_{H(C)} C$, where $H(C) = ([N(C)], E)$ and $\{i, j\} \in E$ if and only if $\{C_i, C_j\}$ is an edge of the Hasse diagram of the Bruhat order, for all $i, j \in [N(C)]$.

Example 5.2. Let $k = 3$ and $n = 6$. Consider the so-called Björner's example (see [4, Exercise 7.7.1]), a 2-dimensional shellable simplicial complex obtained by adding a suitable facet to the minimal triangulation of the real projective plane. We consider the shelling order

$$C := (123, 125, 126, 234, 235, 134, 136, 145, 246, 356, 456);$$

the dual graph of C is depicted in Figure 1. The dual graph track is $T_{D(C)} = \{1, 2, 3, 7, 10, 11\}$ and, in Figure 1, it is denoted by overlined labels. Then $\partial_{D(C)} = (1, 2, 6, 3, 4, 5, 9, 7, 8, 10, 11) \in S_{11}$. We have that

$$\partial_D C = (123, 125, 234, 235, 134, 126, 145, 246, 136, 356, 456)$$

and it is not difficult to see that $\partial_D C$ is a shelling order. The Hasse track of C is $T_{H(C)} = \{1, 2, 3, 7, 9, 10, 11\}$ and then $\partial_{H(C)} = (1, 2, 6, 3, 4, 5, 8, 7, 9, 10, 11) \in S_{11}$. Hence,

$$\partial_H C = (123, 125, 234, 235, 134, 126, 145, 136, 246, 356, 456).$$

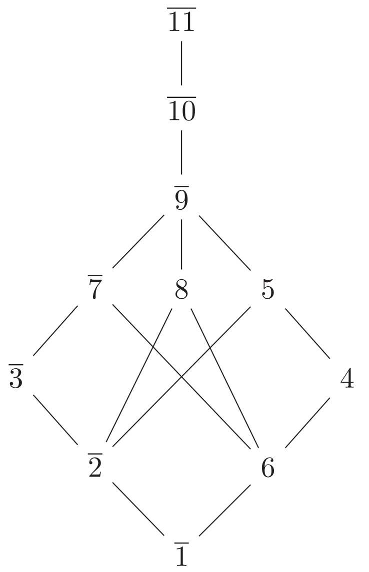

FIGURE 2 Hasse diagram of the Björner's example. The labeling is given by the shelling order C of Example 5.2.

The Hasse diagram of C is depicted in Figure 2, where the overlined vertices correspond to the Hasse track. Notice that C is a linear extension and then $\partial_H C$ is a linear extension; it is also a shelling order.

Given a graph $G = (V, E)$ such that $V = [h]$, for $i \in [h-1]$ we define a permutation $s_i^G \in S_h$ by setting

$$s_i^G = \begin{cases} s_i, & \text{if } \{i, i+1\} \notin E; \\ e, & \text{otherwise,} \end{cases}$$

where s_i is the simple transposition $12 \ldots (i+1)i \ldots h$. Then, for $C \in \mathrm{Conf}_h([n]^k_<)$ and $i \in [N(C) - 1]$, we define $s_i^D C := s_i^{D(C)} C$. By [18, Lemma 1], we have that

$$\partial_D C = s_{h-1}^D \cdots s_1^D C, \tag{2}$$

for all $C \in \mathrm{Conf}_h([n]^k_<)$.

The following result essentially states that, if C is a shelling order, then $s_i^D C$ is a shelling order, for all $1 \leqslant i \leqslant N(C) - 1$.

Proposition 5.3. *Let $C \in \mathrm{Conf}_h([n]^k_<)$ be a shelling order, with $h \geqslant 3$. If $|C_{h-1} \cap C_h| < k - 1$ then $(C_1, \ldots, C_h, C_{h-1})$ is a shelling order.*

Proof. Consider $i < h - 1$. For the pair (C_i, C_{h-1}) we have nothing to show. For the pair (C_i, C_h), there exists $x \in C_h \setminus C_i$ and $j < h$ such that $C_j = C_h + \{x, y\}$, for some $y \in [n]$. By our assumption, $j \neq h - 1$ and the shellability condition on this pair follows.

It remains to verify the shellability condition for (C_h, C_{h-1}). By the fact that C is a shelling order and by our assumption, there exists $z \in C_h \setminus C_{h-1}$ and $j < h-1$ such that $C_j = C_h + \{z, y\}$, for some $y \in [n]$. As C is a shelling order, there exists $c \in C_{h-1} \setminus C_j$ and $r < h-1$ such that $C_r = C_{h-1} + \{c, v\}$, for some v. As $c \notin C_j = C_h + \{z, y\}$ and $c \neq z$, hence $c \in C_{h-1} \setminus C_h$ and $C_r = C_{h-1} + \{c, v\}$, with $r < h-1$, and this concludes the proof. □

The statement of the following theorem is the main result of this section.

Theorem 5.4. *Let $C \in \mathrm{Conf}([n]^k_<)$ be a shelling order. Then the promotion $\partial_D C$ is a shelling order.*

Proof. The result is a direct consequence of (2) and Proposition 5.3. □

In the following example we show that Theorem 5.4 does not hold for ∂_H.

Example 5.5. Let $C := (235, 234, 246) \in \mathrm{Conf}([6]^3_<)$; then C is a shelling order and $\partial_D C = C$; on the other hand, $\partial_H C = (235, 246, 234)$ is not a shelling order.

Remark 5.6. Let $X \subseteq [n]^k_<$ be a pure simplicial complex. Notice that $\{x, y\}$ is an edge of $D(X)$ if and only if there exists a reflection $t \in S_n$ such that $x = P^{(k)}(ty)$, as elements of S_n, that is, $D(X)$ is the undirected Bruhat graph of X (for a definition of the Bruhat graph in the parabolic setting, see, e.g., [17, Definition 2.5]). Hence, if $\{x, y\}$ is an edge of $D(X)$, the elements x and y are comparable in the Bruhat order.

In the next result, we prove that if a linear extension L of $X \subseteq [n]^k_<$ is a shelling order, promotion of L viewed as a linear extension and promotion of L viewed as a shelling order coincide, under a suitable assumption.

Proposition 5.7. *Let $L \in \mathrm{Conf}([n]^k_<)$ be a linear extension. Assume that the Hasse diagram of L is a subgraph of the dual graph of L. Then $\partial_D L = \partial_H L$.*

Proof. Recall that the promotion of L as a linear extension is the linear extension $\partial_H L$. By our assumption, if $L_i \lhd L_j$ then $\{i, j\}$ is an edge of $D(L)$, for all $i, j \in [N(L)]$. We are going to prove that the dual graph track $T_{D(L)} = \{i_1, \dots, i_r\}$ is equal to the Hasse track $T_{H(L)} = \{j_1, \dots, j_s\}$.

If $r = 1$, then $T_{D(L)} = \{L_1\} = T_{H(L)}$, because $H(L)$ is a subgraph of $D(L)$. Hence we may assume $r > 1$. Suppose that $i_a = j_a$, for some $a \leqslant r-1$. Hence $i_{a+1} \leqslant j_{a+1}$, because $H(L)$ is a subgraph of $D(L)$. Assume $i_{a+1} < j_{a+1}$. As $\{i_a, i_{a+1}\}$ is an edge of $D(L)$ and L is a linear extension, $L_{i_a} < L_{i_{a+1}}$. From the fact that $\{i_a, i_{a+1}\}$ is not an edge of $H(L)$ (i.e., $L_{i_a} < L_{i_{a+1}}$ is not a covering relation), there exists $z \in [h]$ such that $L_{i_a} \lhd L_z < L_{i_{a+1}}$. As L is a linear extension, $z < i_{a+1}$. But this is a contradiction, because in this way $\{i_a, z\}$ is an edge of $D(L)$, against the fact that $i_{a+1} \in T_{D(L)}$. Therefore $i_{a+1} = j_{a+1}$. Starting with $a = 1$ and proceeding inductively, we proved that $i_a = j_a$ for every $a \in [r]$, that is, the first elements of the Hasse track $T_{H(L)}$ are the elements of the dual track $T_{D(L)}$. As $H(L)$ is a subgraph of $D(L)$, $r = s$ and $T_{D(L)} = T_{H(L)}$. □

For order ideals or intervals of $[n]^k_<$, the assumption of Proposition 5.7 is fulfilled.

Corollary 5.8. *Let $X \subseteq [n]^k_<$ be an order ideal or an interval. If L is a linear extension of X then $\partial_D L = \partial_H L$.*

Proof. If $X \subseteq [n]^k_<$ is an order ideal or an interval then the Hasse diagram X is a subgraph of the dual graph of X. In fact, as elements of S_n, $x \lhd y$ in X if and only if $x = ty$, for some reflection $t \in S_n$ (see [5, Theorem 2.5.5]). Then the result follows by Proposition 5.7. □

Remark 5.9. Any Bruhat interval I in $[n]^k_<$ is a matroid. Then by Theorem 3.4, a linear extension of I is a shelling order. By Corollary 5.8, the promotion of a linear extension L of I is equal to the promotion of L as shelling order.

In the following example, we show that Proposition 5.7 does not hold if $H(L)$ is not a subgraph of $D(L)$. Moreover, it shows that this assumption does not hold in general for matroids.

Example 5.10. Consider the linear extension $L := (123, 124, 135, 145)$. This is a linear extension of a matroid that is not a Bruhat interval. We have that $\partial_H L = L$ but $\partial_D L = (123, 135, 124, 145)$. Hence, $\partial_D L \neq \partial_H L$.

We end the article by introducing the evacuation function with respect to the dual graph. Let $h \geqslant 1$ and $r \in [h]$; the r-promotion $\partial_{r,D} : \mathrm{Conf}_h([n]^k_<) \to \mathrm{Conf}_h([n]^k_<)$ is defined as follows:

$$\partial_{r,D} C = \partial_D (C_1 \dots C_r) C_{r+1} \dots C_h,$$

for all $C \in \mathrm{Conf}_h([n]^k_<)$. The *evacuation* $\epsilon_D : \mathrm{Conf}([n]^k_<) \to \mathrm{Conf}([n]^k_<)$ is the function defined by setting

$$\epsilon_D C = \left(\partial_{2,D} \circ \dots \circ \partial_{h-1,D} \circ \partial_{h,D}\right)(C),$$

for all $C \in \mathrm{Conf}_h([n]^k_<)$, $h \geqslant 1$. The function ϵ_D is an involution, as stated in [18, Theorem 1]. The last theorem follows directly from Theorem 5.4 and the definition of ϵ_D.

Theorem 5.11. *Let $C \in \mathrm{Conf}([n]^k_<)$ be a shelling order. Then the evacuation $\epsilon_D C$ is a shelling order.*

ACKNOWLEDGMENTS

The second author is grateful to Dipartimento di Ingegneria Industriale e Scienze Matematiche of Universitá Politecnica delle Marche, for its hospitality and financial support on October 2022. He is also grateful to the town of Castelfidardo for its accordions.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Paolo Sentinelli https://orcid.org/0000-0002-5040-9168

REFERENCES

1. F. Ardila, F. Castillo, and J. A. Samper, *The topology of the external activity complex of a matroid*, Electron. J. Combin. **23** (2016), no. 3.
2. B. Benedetti and F. H. Lutz, *Knots in collapsible and non-collapsible balls*, Electron. J. Combin. **20** (2013), no. 3.
3. B. Benedetti and M. Varbaro, *On the dual graphs of Cohen–Macaulay algebras*, Int. Math. Res. Not. **2015** (2015), no. 17, 8085–8115.
4. A. Björner, *The homology and shellability of matroids and geometric lattices*, Matroid Applications **40** (1992), 226–283.
5. A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.
6. A. Björner and M. Wachs, *Bruhat order of Coxeter groups and shellability*, Adv. Math. **43** (1982), no. 1, 87–100.
7. D. Bolognini, *Recursive Betti numbers for Cohen–Macaulay d-partite clutters arising from posets*, J. Pure Appl. Algebra **220** (2016), no. 9, 3102–3118.
8. D. Bolognini and P. Sentinelli, *P-flag spaces and incidence stratifications*, Selecta Math. (N.S.) **27** (2021), 72.
9. D. Bolognini and P. Sentinelli, *Immanant varieties*, arXiv:2211.11634, 2022.
10. A. Borovik, I. M. Gelfand, and N. White, *Coxeter matroids*, Progress in Mathematics, vol. 216, Birkhäuser, 2003.
11. F. Brenti and V. Welker, *f-Vectors of barycentric subdivisions*, Math. Z. **259** (2008), no. 4, 849–865.
12. F. Caselli, M. D'Adderio, and M. Marietti, *Weak generalized lifting property*, Bruhat intervals and Coxeter matroids, Int. Math. Res. Not. **2021** (2021), no. 3, 1678–1698.
13. C. Defant and N. Kravitz, *Promotion sorting*, Order **40** (2022), 199–216.
14. C. Defant, *Toric promotion*, Proc. Amer. Math. Soc. **151** (2023), no. 01, 45–57.
15. P. Hersh and R. Kenyon, *Shellability of face posets of electrical networks and the CW poset property*, Adv. Appl. Math. **127** (2021), 102178.
16. J. Jonsson, *Simplicial complexes of graphs*, Lecture Notes in Mathematics, vol. 1928, Springer, Berlin, 2008.
17. M. Lanini, *Kazhdan–Lusztig combinatorics in the moment graph setting*, J. Algebra **370** (2012), 152–170.
18. C. Malvenuto and C. Reutenauer, *Evacuation of labelled graphs*, Discrete Math. **132** (1994), no. 1–3, 137–143.
19. J. A. Samper, *Quasi-matroidal classes of ordered simplicial complexes*, J. Combin. Theory Ser. A **175** (2020), 105274.
20. P. Sentinelli, *Complements of Coxeter group quotients*, J. Algebraic Combin. **41** (2015), no. 3, 727–750.
21. R. S. Simon, *Combinatorial properties of cleanness*, J. Algebra **167** (1994), 361–388.
22. R. P. Stanley, *Enumerative combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.
23. R. P. Stanley, *Promotion and evacuation*, Electron. J. Combin. **16** (2009), no. 2.
24. G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer, New York, 1995.