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Abstract

This paper is concerned with regularity of minimizers of integral functionals with polyconvex potentials. In particular we obtain bounds on the difference $|u - u_*|_\infty$ for minimizers $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of problem

$$\min \left\{ \int_\Omega f(x, Dv(x))dx, \quad v \in u_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\}.$$

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1. Introduction

Let us consider mappings $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and the variational integral

$$\mathcal{F}(u) = \int_\Omega f(x, Du)dx, \tag{1.1}$$

where $f : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ with $n \geq 2$ and $N \geq 1$. After fixing a suitable boundary value u_*, we deal with the problem of minimizing $\mathcal{F}$ among mappings that agree with u_* on the boundary of Ω:

$$\min\{\mathcal{F}(u) : u = u_* \text{ on } \partial\Omega\}. \tag{1.2}$$

When we have p-coercivity

$$\nu|z|^p - a(x) \leq f(x, z), \tag{1.3}$$

for some exponent $p > 1$, then minimizing sequences for (1.2) are bounded in the Sobolev space $W^{1,p}$ and compact with respect to the weak convergence in $W^{1,p}$. In order to use direct methods, we need lower semicontinuity with respect to such a weak convergence. In the scalar case $N = 1$, $\mathcal{F}$ is sequentially weakly lower semicontinuous in $W^{1,p}$ if and only if $z \to f(x, z)$ is convex, see Theorem 1.3 in [15]. In the vectorial case $N \geq 2$, when we also have p-growth from above

$$f(x, z) \leq M|z|^p + b(x), \tag{1.4}$$

then $\mathcal{F}$ is sequentially weakly lower semicontinuous in $W^{1,p}$ if and only if $z \to f(x, z)$ is quasiconvex in the sense of Morrey, see Theorem 1.13 in [15]. Quasiconvexity is weaker then convexity in the vectorial case. Quasiconvexity is not a local condition, see [34]. A more friendly assumption is polyconvexity; this means that $f(x, z)$ can be written as a convex combination of minors taken from the $N \times n$ matrix z. Polyconvexity is weaker than convexity but stronger than quasiconvexity. When $N = n = 3$, polyconvexity can be stated as follows

$$f(x, z) = g(x, z, \mathrm{adj}_2 z, \det z), \tag{1.5}$$

with

$$(z, \xi, t) \to g(x, z, \xi, t) \quad \text{convex.} \tag{1.6}$$

In nonlinear elasticity, $f(x, z) = g(x, z, \mathrm{adj}_2 z, \det z)$ is the stored-energy function and $z, \mathrm{adj}_2 z, \det z$ govern the deformations of line, surface, volume elements respectively, see [1]. Existence results for minimization problem (1.2), when f is polyconvex, can be found in section 8.4.2 of [15].

This paper is concerned with regularity of minimizers of integral functional (1.1) when f is polyconvex. In this framework we recall partial regularity contained in [27], [25], [24], [45], [21], [32], [23], [9]. As far as everywhere regularity is concerned, only few contributions are recorded and they all consider the two dimensional case $n = 2$: [28], [17], [26], [5]. When L^∞ estimates are concerned, we mention global bounds in [35], [3], [4], [19], [38], [39], [37], [36], [6]. On the other hand, local L^∞ regularity is obtained in [12], [7], [13]. When the polyconvex term is of lower order, see [18], [14] and [30].

We focus our attention to the case where $N = n = 3$ and we are interested in studying the properties of solutions of the following minimization problem:

$$\min\left\{\int_\Omega f(x, Dv(x))dx, \quad v \in u_* + W_0^{1,p}(\Omega, \mathbb{R}^3)\right\}. \tag{1.7}$$

Our aim is to obtain bounds of the quantity $|u - u_*|$ for minimizers u of problem (1.7). Here Ω is a bounded open set in $\mathbb{R}^3$ and $u, u_* : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$. Our main result is stated in Theorem 2.2 and gives an explicit estimate of the L^∞ norm of $u-u_*$ under structural assumptions on f that are formalized in the next section.

We underline the fact that in the present paper a "good" boundary datum allows minimizers to be "regular" under quite general assumptions. At the end of section 2 we will discuss in detail the fact that our assumptions allow non standard growth of the type

$$\tilde{\nu}|z|^p - 3a(x) \le f(x, z) \le \tilde{M}[b(x) + 1 + |z|^{\hat{p}}], \tag{1.8}$$

where $\tilde{\nu}, \tilde{M}$ are positive constants, $a, b : \Omega \to \mathbb{R}$ are measurable functions with suitable summability and $p < \hat{p}$. In the case of the example discussed in Remark 2.3 we get $p = 5/2 < 4 = \hat{p}$. In the framework of non standard growth, regularity for minimizers is usually obtained when p and $\hat{p}$ are not too far apart, see [31], [40], [41], [29], [44], [33], [20], [22], [10], [11], [2], [16], sections 5 and 6 in [42] and sections 3,4 in [43]. In our case too a similar control on p and $\hat{p}$ appears. See Remark 2.3 for a deeper discussion. In section 5 we apply our main Theorem to the special case where the boundary datum is Lipschitz.

Let us end this introduction by mentioning that, in the scalar case $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, estimates of the difference between minimizers u and boundary values u_* are given in [8].

2. Notation and statement of the main result

2.1. Notation

In this paper we are concerned with $N = n = 3$, so $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$. We will use the following notation for u and for its Jacobian Matrix:

$$u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}, \qquad Du = \begin{pmatrix} D_1u^1 & D_2u^1 & D_3u^1 \\ D_1u^2 & D_2u^2 & D_3u^2 \\ D_1u^3 & D_2u^3 & D_3u^3 \end{pmatrix}.$$

For simplicity, we will also denote the Jacobian matrix using the gradients of functions u^α:

$$Du = \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix}.$$

Moreover, we consider the adjugate matrix of the Jacobian:

$$adj_2(Du) = \begin{pmatrix} adj_2(Du)^1_1 & adj_2(Du)^1_2 & adj_2(Du)^1_3 \\ adj_2(Du)^2_1 & adj_2(Du)^2_2 & adj_2(Du)^2_3 \\ adj_2(Du)^3_1 & adj_2(Du)^3_2 & adj_2(Du)^3_3 \end{pmatrix},$$

where

$$adj_2(Du)^\alpha_i = (-1)^{\alpha+i} det \begin{pmatrix} \frac{\partial u^\beta}{\partial x^r} & \frac{\partial u^\beta}{\partial x^s} \\ \\ \frac{\partial u^\gamma}{\partial x^r} & \frac{\partial u^\gamma}{\partial x^s} \end{pmatrix},$$

with

$$\beta < \gamma, \quad \beta, \gamma \in \{1, 2, 3\} \setminus \{\alpha\}, \quad r < s, \quad r, s \in \{1, 2, 3\} \setminus \{i\}.$$

We write $adj_2(Du)^\alpha$ for the $\alpha-$th row of the adjugate matrix.

Moreover we use also the following common notations: $|\Omega|$ is the Lebesgue measure of Ω; $\|g\|_s$ is the $L^s(\Omega)$ norm of the function g and p^* is the critical Sobolev exponent associated to p. We remark that in our special case $p^* = \frac{3p}{3-p}$.

2.2. Structure assumptions on f.

Let p, q, r three strictly positive real numbers satisfying

$$0 < r < q < p < 3, \qquad p \geq 1, \quad p > \max\left\{\sqrt{3q}, \frac{3r}{q}\right\}. \tag{2.1}$$

Remark 2.1. *Note that* $\sqrt{3q} < 3$ *since* $q < 3$ *and* $3r/q < 3$ *since* $r < q$*; then, assumption* (2.1) *makes sense.*

We consider a polyconvex potential of the following form

$$\begin{aligned} f(x, Du) =& F_1(x, Du^1) + F_2(x, Du^2) + F_3(x, Du^3) \\ &+ G_1(x, adj_2(Du)^1) + G_2(x, adj_2(Du)^2) + G_3(x, adj_2(Du)^3) \\ &+ H(x, det(Du)), \end{aligned}$$

where we suppose that $F_\alpha, G_\alpha : \Omega \times \mathbb{R}^3 \to \mathbb{R}$, $\alpha = 1, 2, 3$, and $H : \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory functions. Moreover we will assume that there exist two constants

$$0 < \nu, M < +\infty. \tag{2.2}$$

and two functions

$$0 \leq a(x), b(x) < +\infty, \qquad a, b \in L^\sigma(\Omega), \qquad \sigma > \frac{3}{p}, \tag{2.3}$$

such that

$$\nu|\xi|^p - a(x) \leq \quad F_\alpha(x, \xi) \leq M|\xi|^p + b(x), \qquad F_\alpha(x, \xi) \geq 0, \tag{2.4}$$

$$\nu|\xi|^q - a(x) \leq \quad G_\alpha(x, \xi) \leq M|\xi|^q + b(x), \qquad G_\alpha(x, \xi) \geq 0, \tag{2.5}$$

$$0 \leq \quad H(x, t) \leq M|t|^r + b(x). \tag{2.6}$$

2.3. Statement of the main Theorem

We start defining some constants that will be used in Theorem 2.2:

- we denote by $\tilde{p}$ a fixed real number such that

$$\tilde{p} > \max\left\{3, \frac{3pq}{p^2 - 3q}, \frac{3qr}{qp - 3r}\right\};$$

- the real number γ is defined by

$$\gamma := \min\left\{ \underbrace{1-\frac{1}{\sigma}}_{\delta_1}, \underbrace{1-\frac{p}{\tilde{p}}}_{\delta_2}, \underbrace{1-\frac{q}{p}-\frac{q}{\tilde{p}}}_{\delta_3}, \underbrace{1-\frac{r}{q}-\frac{r}{\tilde{p}}}_{\delta_4} \right\},$$

and we underline that, thanks to previous choices, $\gamma > \frac{p}{p^*}$.

We consider a minimization problem with fixed boundary datum u_* and we assume that $u_* \in W^{1,\tilde{p}}(\Omega, \mathbb{R}^3)$ such that

$$x \to f(x, Du_*(x)) \in L^1(\Omega). \tag{2.7}$$

The last constant that we will need is

$$\begin{aligned} K := & \frac{4}{\nu}\left(\frac{2p}{3-p}\right)^{\frac{3p}{3-p}} 2^{\frac{3(p-1)}{3-p}} \\ & \cdot \Bigg(\max\Big\{ 4\|b\|_\sigma + \|a\|_\sigma,\ M\|Du_*\|^p_{\tilde{p}}, \\ & \quad 2M3^q\|Du_*\|^q_{\tilde{p}}\left(\frac{1}{\nu}\int_\Omega f(x,Du_*)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{q}{p}}, \\ & \quad M3^{\frac{r}{2}}\|Du_*\|^r_{\tilde{p}}\left(\frac{1}{\nu}\int_\Omega f(x,Du_*)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{r}{q}}\Big\} \\ & \cdot \max\left\{|\Omega|^{\delta_1-\gamma}, |\Omega|^{\delta_2-\gamma}, |\Omega|^{\delta_3-\gamma}, |\Omega|^{\delta_4-\gamma}\right\} + \|Du_*\|^p_{\tilde{p}}|\Omega|^{\delta_2-\gamma}\Bigg)^{\frac{3}{3-p}}. \end{aligned} \tag{2.8}$$

We are ready now to state our main Theorem that will be proved in the next section.

Theorem 2.2. *Let $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a minimizer of problem* (1.7) *such that hypotheses* (2.2)-(2.7) *are fulfilled. In addition, we suppose that boundary datum u_* has the following high degree of integrability*

$$u_* \in W^{1,\tilde{p}}(\Omega), \tag{2.9}$$

Then,

$$|u^\alpha - u^\alpha_*| \le L^*, \qquad in \quad \Omega, \quad \forall \alpha \in \{1,2,3\},$$

with

$$L^* = 2^{\frac{p^*\gamma}{p^*\gamma - p}} K^{\frac{1}{p^*}} |\Omega|^{\frac{\gamma}{p} - \frac{1}{p^*}}. \tag{2.10}$$

Remark 2.3. *We now give an application of the previous Theorem. We consider*

$$f(x, Du) = \sum_{\alpha=1}^{3} (|Du^\alpha|^p + |adj_2(Du)^\alpha|^q) + |\det Du|^r. \tag{2.11}$$

Assuming $p, q, r > 1$ we have that the functional is both polyconvex and coercive and, so, the corresponding minimum problem (1.7) *has a solution see Remark 8.32 (iii) in [15] and Theorem 3.1 in [12]. We choose $r = 4/3$, $q = 2$ and $p = 5/2$ and we observe that* (2.1) *is satisfied. For this choice we obtain $3pq/(p^2 - 3q) = 60$ and $3qr/(qp - 3r) = 8$; then our Theorem requires that the degree of integrability of the boundary datum u_* has to verify $\tilde{p} > 60$.*

Let note that (2.11) *has been considered in [12]: there, the conditions on p are more restrictive than the ones in the present paper, see section 4. We can guess it by recalling that [12] deals with local regularity and no boundary value u_* is fixed; on the contrary, in the present paper a "good" boundary value u_* allows minimizers to be "regular" under less restrictive assumptions.*

Let us come back to our general structure described in section 2.2; then, our growth assumptions (2.4)-(2.6) imply

$$\tilde{\nu}|z|^p - 3a(x) \le f(x, z) \le \tilde{M}[b(x) + 1 + |z|^{\hat{p}}], \tag{2.12}$$

where $\tilde{\nu}, \tilde{M}$ are positive constants and

$$\hat{p} := \max\{p; 2q; 3r\}. \tag{2.13}$$

So, we are in the framework of non standard growth, that is, when the exponent p of the growth from below is less than the exponent $\hat{p}$ of the growth from above: this happens in our previous example when $r = 4/3$, $q = 2$ and $p = 5/2$; in such a case $p = 5/2 < 4 = \hat{p}$. In the framework of non standard growth, regularity for minimizers is usually obtained when p and $\hat{p}$ are not too far apart, see [31], [40], [41], [29], [44], [33], [20], [22], [10], [11], [2], [16], sections 5 and 6 in [42] and sections 3,4 in [43]. So, it is not surprising that also in our case a similar control on p and $\hat{p}$ appears. Indeed, let us read the last inequality of (2.1) in a different way. Condition $p > \sqrt{3q}$ is equivalent to $2q < 2p^2/3$ and restriction $p > 3r/q$ can be seen as $3r < pq$; taking into account the previous condition on q, we get $3r < p^3/3$. This means that

$$\hat{p} = \max\{p; 2q; 3r\} \le \max\left\{p; \frac{2}{3}p^2; \frac{p^3}{3}\right\}. \tag{2.14}$$

3. Proof of Theorem 2.2

We will present some preliminary estimates that are resumed in the following technical lemma and that will be useful later.

Lemma 3.1. *Let u be a minimizer of* (1.7) *satisfying* (2.2)-(2.7)*; then:*

$$x \to f(x, Du(x)) \in L^1(\Omega), \tag{3.1}$$

$$\|Du\|_p^q \le 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1 \right)^{\frac{q}{p}}, \tag{3.2}$$

and

$$\|adj_2(Du)\|_q^r \le 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1 \right)^{\frac{r}{q}}. \tag{3.3}$$

Proof. For the first claim we observe that:

$$0 \le \int_\Omega f(x, Du(x))dx \le \int_\Omega f(x, Du_*(x))dx < +\infty.$$

Moreover,

$$|Du|^p = \left(\sum_{\alpha=1}^{3} \sum_{i=1}^{3} |D_i u^\alpha|^2 \right)^{\frac{p}{2}} \le \left(\sum_{\alpha=1}^{3} \sum_{i=1}^{3} |D_i u^{\tilde{\alpha}}|^2 \right)^{\frac{p}{2}} = 3^{\frac{p}{2}} \left(\sum_{i=1}^{3} |D_i u^{\tilde{\alpha}}|^2 \right)^{\frac{p}{2}},$$

where the index $\tilde{\alpha}$ corresponds to the maximum of the quantity $\sum_{i=1}^{3} |D_i u^\alpha|^2$ with $\alpha = 1, 2, 3$. From the previous inequality we have

$$|Du|^p \le 3^{\frac{p}{2}} \sum_{\alpha=1}^{3} \left(\sum_{i=1}^{3} |D_i u^\alpha|^2 \right)^{\frac{p}{2}} = 3^{\frac{p}{2}} \sum_{\alpha=1}^{3} \left(|Du^\alpha|^2 \right)^{\frac{p}{2}} = 3^{\frac{p}{2}} \sum_{\alpha=1}^{3} |Du^\alpha|^p. \tag{3.4}$$

Then, using (3.4), (2.2)-(2.7) we can write:

$$\begin{aligned}
\|Du\|_p^q &= \left(\int_\Omega |Du|^p\right)^{\frac{q}{p}} \\
&\le \left(\int_\Omega 3^{\frac{p}{2}} \sum_{\alpha=1}^{3} |Du^\alpha|^p\right)^{\frac{q}{p}} = 3^{\frac{q}{2}} \left(\sum_{\alpha=1}^{3} \int_\Omega |Du^\alpha|^p\right)^{\frac{q}{p}} \\
&\le 3^{\frac{q}{2}} \left(\sum_{\alpha=1}^{3} \frac{1}{\nu} \int_\Omega [F_\alpha(x, Du^\alpha) + a(x)]\, dx\right)^{\frac{q}{p}} \\
&\le 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_\Omega f(x, Du)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{q}{p}} \\
&\le 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{q}{p}},
\end{aligned}$$

where in the last inequality we have used the minimality of u.
Now we pass to the second estimate. From a similar argument used for estimating $|Du|^p$ we obtain that

$$|adj_2(Du)|^q \le 3^{\frac{q}{2}} \sum_{\alpha=1}^{3} |adj_2(Du)^\alpha|^q, \tag{3.5}$$

and then, using (3.5), (2.2)-(2.7) and the minimality of u:

$$\begin{aligned}
\|adj_2(Du)\|_q^r &= \left(\int_\Omega |adj_2(Du)|^q dx\right)^{\frac{r}{q}} \\
&\le \left(\int_\Omega 3^{\frac{q}{2}} \sum_{\alpha=1}^{3} |adj_2(Du)^\alpha|^q\, dx\right)^{\frac{r}{q}} \\
&= 3^{\frac{r}{2}} \left(\sum_{\alpha=1}^{3} \int_\Omega |adj_2(Du)^\alpha|^q dx\right)^{\frac{r}{q}} \\
&\le 3^{\frac{r}{2}} \left(\sum_{\alpha=1}^{3} \int_\Omega \frac{1}{\nu} [G_\alpha(x, adj_2(Du)^\alpha) + a(x)]\, dx\right)^{\frac{r}{q}} \\
&\le 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_\Omega f(x, Du)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{r}{q}} \\
&\le 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{r}{q}}.
\end{aligned}$$

$\square$

Remark 3.2. *We observe that*

$$\frac{3pq}{p^2 - 3q} > \frac{pq}{p - q}, \tag{3.6}$$

and

$$\frac{3qr}{qp - 3r} > \frac{qr}{q - r}. \tag{3.7}$$

The previous inequalities will be useful for the proof of Theorem 2.2. Moreover, (2.9) *implies* $\gamma > \frac{p}{p^*}$*, see* (3.14)*.*

Now we are ready to prove Theorem 2.2

Proof. (Theorem 2.2) For $L > 0$ we introduce the following test function

$$v = \begin{pmatrix} v^1 \\ u^2 \\ u^3 \end{pmatrix},$$

where

$$v^1 = \begin{cases} u^1, & \text{on} \quad \Omega \setminus A^1_L, \\ u^1_* + L, & \text{on} \quad A^1_L, \end{cases}$$

where

$$A^1_L = \{x \in \Omega : \qquad u^1(x) > u^1_*(x) + L\}.$$

We first show that v is an admissible function for the minimum problem. Note that on $\partial\Omega$ we have

$$u^1(x) = u^1_*(x) < u^1_*(x) + L,$$

then

$$v^1(x) = u^1(x) \qquad \text{on} \quad \partial\Omega.$$

We observe that

$$Dv^1 = \begin{cases} Du^1, & \text{on} \quad \Omega \setminus A^1_L, \\ Du^1_*, & \text{on} \quad A^1_L, \end{cases}$$

and

$$Dv^2 = Du^2, \qquad Dv^3 = Du^3.$$

Regarding the functions F_α we have:

$$F_1(x,Dv^1)=\begin{cases} F_1(x,Du^1), & \text{on} \quad \Omega\setminus A^1_L, \\ F_1(x,Du^1_*), & \text{on} \quad A^1_L, \end{cases},$$

$$F_2(x,Dv^2)=F_2(x,Du^2), \quad F_3(x,Dv^3)=F_3(x,Du^3).$$

On $\Omega\setminus A^1_L$ we have $Dv=Du$ and as a consequence $f(x,Dv)=f(x,Du)$. On the set A^1_L we have

$$Dv=\begin{pmatrix} Du^1_* \\ Du^2 \\ Du^3 \end{pmatrix},$$

and for the first row of the adjugate matrix we have

$$adj_2(Dv)^1=adj_2(Du)^1,$$

while for the other rows we have

$$adj_2(Dv)^2_i=(-1)^{2+i}det\begin{pmatrix} \frac{\partial u^1_*}{\partial x^r} & \frac{\partial u^1_*}{\partial x^s} \\ \\ \frac{\partial u^3}{\partial x^r} & \frac{\partial u^3}{\partial x^s} \end{pmatrix},$$

with

$$r<s, \qquad r,s\in\{1,2,3\}\setminus\{i\}.$$

The previous quantity can be estimated in the following way:

$$\begin{aligned} |adj_2(Dv)^2_i| &= \left|\frac{\partial u^1_*}{\partial x^r}\frac{\partial u^3}{\partial x^s}-\frac{\partial u^1_*}{\partial x^s}\frac{\partial u^3}{\partial x^r}\right| \le \left|\frac{\partial u^1_*}{\partial x^r}\right|\left|\frac{\partial u^3}{\partial x^s}\right|+\left|\frac{\partial u^1_*}{\partial x^s}\right|\left|\frac{\partial u^3}{\partial x^r}\right| \\ &\le \left(\left|\frac{\partial u^1_*}{\partial x^r}\right|^2+\left|\frac{\partial u^1_*}{\partial x^s}\right|^2\right)^{\frac{1}{2}}\left(\left|\frac{\partial u^3}{\partial x^r}\right|^2+\left|\frac{\partial u^3}{\partial x^s}\right|^2\right)^{\frac{1}{2}} \le |Du^1_*||Du^3|, \end{aligned} \tag{3.8}$$

where we have used Cauchy-Schwartz inequality. For the third row we have:

$$adj_2(Dv)^3_i=(-1)^{3+i}det\begin{pmatrix} \frac{\partial u^1_*}{\partial x^r} & \frac{\partial u^1_*}{\partial x^s} \\ \\ \frac{\partial u^2}{\partial x^r} & \frac{\partial u^2}{\partial x^s} \end{pmatrix},$$

with

$$r<s, \qquad r,s\in\{1,2,3\}\setminus\{i\},$$

for which we obtain in a similar way the following bound:

$$|adj_2(Dv)^3_i| \le |Du^1_*||Du^2|. \tag{3.9}$$

Then we have

$$G_1(adj_2(Dv)^1) = G_1(adj_2(Du)^1),$$
$$G_2(adj_2(Dv)^2) \le M|adj_2(Dv)^2|^q + b(x) \le M|Du^1_*|^q|Du^3|^q + b(x).$$

Then in order to have $G_2(adj_2(Dv)^2) \in L^1(\Omega)$ it is sufficient to note that

$$\begin{aligned}\int_\Omega G_2(adj_2(Dv)^2)dx &\le M\int_\Omega |Du^1_*|^q|Du^3|^q dx + \|b\|_1 \\ &\le M\|Du^3\|^q_{L^p(\Omega)}\left(\int_\Omega |Du^1_*|^{\frac{pq}{p-q}}dx\right)^{\frac{p-q}{p}} + \|b\|_1 \\ &\le c\|Du^3\|^q_p + \|b\|_1,\end{aligned}$$

where we have used (2.9) and (3.6) which ensure that

$$u^1_* \in W^{1,\tilde{p}}, \qquad \tilde{p} \ge \frac{pq}{p-q}.$$

The same reasoning can be used for G_3.
We now consider the last term of the functional; we observe that the determinant of the Jacobian matrix can be computed using the first row:

$$det(Dv) = D_1u^1_* \cdot adj_2(Du)^1_1 + D_2u^1_* \cdot adj_2(Du)^1_2 + D_3u^1_* \cdot adj_2(Du)^1_3,$$

then by Cauchy-Schwartz inequality we have

$$|det(Dv)| \le |Du^1_*||adj_2(Du)^1|, \tag{3.10}$$

and as a consequence

$$0 \le H(det(Dv)) \le M|det(Dv)|^r + b(x) \le M|Du^1_*|^r|adj_2(Du)^1|^r + b(x).$$

We observe that $adj_2(Du)^1 \in L^q(\Omega)$: indeed, from (2.5) we have

$$\nu|adj_2(Du)^1|^q \le a(x) + G_1(x, adj_2(Du)^1) \le a(x) + f(x, Du).$$

Then, using Hölder's inequality we obtain:

$$\begin{aligned}\int_\Omega H(det(Dv))dx &\le M\int_\Omega |Du^1_*|^r|adj_2(Du)^1|^r dx + \|b\|_1 \\ &\le M\left(\int_\Omega |Du^1_*|^{\frac{qr}{q-r}}dx\right)^{\frac{q-r}{q}}\left(\int_\Omega |adj_2(Du)^1|^q dx\right)^{\frac{r}{q}} + \|b\|_1,\end{aligned}$$

then we obtain $H(det(Dv)) \in L^1(\Omega)$ by using hypothesis (2.9) and (3.7) for which we have

$$u^1_* \in W^{1,\tilde{p}}, \qquad \tilde{p} \geq \frac{rq}{q-r}.$$

Then we conclude that v is an admissible function. Now we pass to the bound of u. By minimality of u we can write

$$\int_\Omega f(x, Du(x))dx \leq \int_\Omega f(x, Dv(x))dx,$$

so that

$$\int_{\Omega\setminus A^1_L} f(x, Du)dx + \int_{A^1_L} f(x, Du)dx \leq \int_{\Omega\setminus A^1_L} f(x, Dv)dx + \int_{A^1_L} f(x, Dv)dx.$$

Since $u = v$ in $\Omega \setminus A^1_L$, we drop $\int_{\Omega\setminus A^1_L} f(x, Du)dx = \int_{\Omega\setminus A^1_L} f(x, Dv)dx$ from both sides and we obtain

$$\int_{A^1_L} f(x, Du(x))dx \leq \int_{A^1_L} f(x, Dv(x))dx.$$

We rewrite the previous inequality in details emphasising the terms that are equal (keep in mind that $v^2 = u^2, v^3 = u^3$ and $adj_2(Dv)^1 = adj_2(Du)^1$):

$$\begin{aligned}
&\int_{A^1_L} F_1(x, Du^1)dx + \underbrace{\int_{A^1_L} F_2(x, Du^2)dx}_{(I)} + \underbrace{\int_{A^1_L} F_3(x, Du^3)dx}_{(II)} \\
&+ \underbrace{\int_{A^1_L} G_1(x, adj_2(Du)^1)dx}_{(III)} + \int_{A^1_L} G_2(x, adj_2(Du)^2)dx \\
&+ \int_{A^1_L} G_3(x, adj_2(Du)^3)dx + \int_{A^1_L} H(x, det(Du))dx \\
\leq &\int_{A^1_L} F_1(x, Dv^1)dx + \underbrace{\int_{A^1_L} F_2(x, Dv^2)dx}_{(I)} + \underbrace{\int_{A^1_L} F_3(x, Dv^3)dx}_{(II)} \\
&+ \underbrace{\int_{A^1_L} G_1(x, adj_2(Dv)^1)dx}_{(III)} + \int_{A^1_L} G_2(x, adj_2(Dv)^2)dx \\
&+ \int_{A^1_L} G_3(x, adj_2(Dv)^3)dx + \int_{A^1_L} H(x, det(Dv))dx;
\end{aligned}$$

then, dropping the equal terms and using $Dv^1 = Du^1_*$ in A^1_L, we have

$$\int_{A^1_L} F_1(x, Du^1)dx + \int_{A^1_L} G_2(x, adj_2(Du)^2)dx + \int_{A^1_L} G_3(x, adj_2(Du)^3)dx$$
$$+ \int_{A^1_L} H(x, det(Du))dx$$
$$\leq \int_{A^1_L} F_1(x, Du^1_*)dx + \int_{A^1_L} G_2(x, adj_2(Dv)^2)dx + \int_{A^1_L} G_3(x, adj_2(Dv)^3)dx$$
$$+ \int_{A^1_L} H(x, det(Dv))dx,$$

and, using growth assumption (2.4), (2.5), (2.6) we obtain that the right hand side of the estimate above is less or equal to

$$M \int_{A^1_L} |Du^1_*|^p dx + \int_{A^1_L} b(x)dx + M \int_{A^1_L} |adj_2(Dv)^2|^q dx + \int_{A^1_L} b(x)dx$$
$$+ M \int_{A^1_L} |adj_2(Dv)^3|^q dx + \int_{A^1_L} b(x)dx + M \int_{A^1_L} |det(Dv)|^r dx + \int_{A^1_L} b(x)dx.$$

Finally, by Hölder's inequality, (3.8), (3.9) and (3.10), the previous quantity can be be majorized by

$$4 \left(\int_\Omega b(x)^\sigma dx \right)^{\frac{1}{\sigma}} |A^1_L|^{1-\frac{1}{\sigma}} + M \left(\int_\Omega |Du^1_*|^{\tilde{p}} dx \right)^{\frac{p}{\tilde{p}}} |A^1_L|^{1-\frac{p}{\tilde{p}}}$$
$$+ 2M3^{\frac{q}{2}} \int_{A^1_L} |Du_*|^q |Du|^q dx + M \int_{A^1_L} |Du_*|^r |adj_2(Du)|^r dx.$$

We estimate the last two terms of the previous expression separately. We first observe that the coefficient $3^{\frac{q}{2}}$ appears since (3.8) implies

$$|adj_2(Dv)^2| \leq \sqrt{3}|Du^1_*||Du^3| \leq \sqrt{3}|Du_*||Du|,$$

moreover (3.9) implies

$$|adj_2(Dv)^3| \leq \sqrt{3}|Du^1_*||Du^2| \leq \sqrt{3}|Du_*||Du|.$$

Note that (2.9) implies $\tilde{p} > \frac{pq}{p-q}$, see (3.6); then, for the first term we have

$$\int_{A_L^1} |Du_*|^q |Du|^q dx \le \left(\int_{A_L^1} |Du_*|^{\frac{qp}{p-q}} dx\right)^{\frac{p-q}{p}} \left(\int_\Omega |Du|^p dx\right)^{\frac{q}{p}}$$

$$\le \left[\left(\int_{A_L^1} |Du_*|^{\tilde{p}} dx\right)^{\frac{qp}{(p-q)\tilde{p}}} |A_L^1|^{1-\frac{qp}{(p-q)\tilde{p}}}\right]^{\frac{p-q}{p}} \left(\int_\Omega |Du|^p dx\right)^{\frac{q}{p}}$$

$$\le \left(\int_\Omega |Du_*|^{\tilde{p}} dx\right)^{\frac{q}{\tilde{p}}} \left(\int_\Omega |Du|^p dx\right)^{\frac{q}{p}} \left[|A_L^1|^{1-\frac{qp}{(p-q)\tilde{p}}}\right]^{\frac{p-q}{p}}$$

$$= \|Du_*\|_{\tilde{p}}^q \|Du\|_p^q |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}}.$$

Lemma 3.1 implies that the last term is less than

$$\|Du_*\|_{\tilde{p}}^q \, 3^{\frac{q}{2}} \left(\frac{1}{\nu}\int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{q}{p}} |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}} := c_1 |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}}.$$

Note that (2.9) implies $\tilde{p} > \frac{qr}{q-r}$, see (3.7), then, thanks again to Lemma 3.1 we can estimate the second term as follows

$$\int_{A_L^1} |Du_*|^r |adj_2(Du)|^r dx \le \left(\int_\Omega |adj_2(Du)|^q dx\right)^{\frac{r}{q}} \left(\int_{A_L^1} |Du_*|^{\frac{rq}{q-r}} dx\right)^{\frac{q-r}{q}}$$

$$\le \left(\int_\Omega |adj_2(Du)|^q dx\right)^{\frac{r}{q}} \left[\left(\int_\Omega |Du_*|^{\tilde{p}}\right)^{\frac{rq}{(q-r)\tilde{p}}} |A_L^1|^{1-\frac{rq}{(q-r)\tilde{p}}}\right]^{\frac{q-r}{q}}$$

$$\le \left(\int_\Omega |adj_2(Du)|^q dx\right)^{\frac{r}{q}} \left(\int_\Omega |Du_*|^{\tilde{p}}\right)^{\frac{r}{\tilde{p}}} \left[|A_L^1|^{1-\frac{rq}{(q-r)\tilde{p}}}\right]^{\frac{q-r}{q}}$$

$$= \|adj_2(Du)\|_q^r \, \|Du_*\|_{\tilde{p}}^r \, |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}}$$

$$\le \|Du_*\|_{\tilde{p}}^r \, 3^{\frac{r}{2}} \left(\frac{1}{\nu}\int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1\right)^{\frac{r}{q}} |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}} := c_2 |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}}.$$

Then

$$\int_{A_L^1} F_1(x, Du^1)dx + \int_{A_L^1} G_2(x, adj_2(Du)^2)dx + \int_{A_L^1} G_3(x, adj_2(Du)^3)dx$$

$$+ \int_{A_L^1} H(x, det(Du))dx$$

$$\le 4\|b\|_\sigma |A_L^1|^{1-\frac{1}{\sigma}} + M\|Du_*^1\|_{\tilde{p}}^p |A_L^1|^{1-\frac{p}{\tilde{p}}} + 2M3^{\frac{q}{2}} c_1 |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}} + Mc_2 |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}}.$$

By using the positiveness of G_α, H and the left hand side of (2.4) we can write:

$$\nu \int_{A_L^1} |Du^1|^p dx - \int_{A_L^1} a(x)dx \le \int_{A_L^1} F_1(x, Du^1)dx$$
$$\le k_1|A_L^1|^{1-\frac{1}{\sigma}} + k_2|A_L^1|^{1-\frac{p}{\tilde{p}}} + k_3|A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}} + k_4|A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}},$$

where

$$k_1 = 4\|b\|_\sigma, \qquad k_2 = M\|Du_*\|_{\tilde{p}}^p, \qquad k_3 = 2M3^{\frac{q}{2}}c_1, \qquad k_4 = Mc_2.$$

We can estimate the term depending on the function $a(\cdot)$ in the following way:

$$\int_{A_L^1} a(x)dx \le \left(\int_\Omega a(x)^\sigma dx\right)^{\frac{1}{\sigma}} |A_L^1|^{1-\frac{1}{\sigma}}.$$

Finally, we obtain

$$\int_{A_L^1} |Du^1|^p \le k|A_L^1|^\gamma,$$

where

$$k = \frac{4}{\nu}\max\{k_1 + \|a\|_\sigma, k_2, k_3, k_4\}\max\{|\Omega|^{\delta_1-\gamma}, |\Omega|^{\delta_2-\gamma}, |\Omega|^{\delta_3-\gamma}, |\Omega|^{\delta_4-\gamma}\},$$

and

$$\gamma := \min\left\{\underbrace{1-\frac{1}{\sigma}}_{\delta_1}, \underbrace{1-\frac{p}{\tilde{p}}}_{\delta_2}, \underbrace{1-\frac{q}{p}-\frac{q}{\tilde{p}}}_{\delta_3}, \underbrace{1-\frac{r}{q}-\frac{r}{\tilde{p}}}_{\delta_4}\right\}.$$

The previous estimate follows from the following reasoning,

$$|A_L^1|^\delta = |A_L^1|^\gamma |A_L^1|^{\delta-\gamma} \le |A_L^1|^\gamma |\Omega|^{\delta-\gamma},$$

where $\delta \ge \gamma$. From Hölder's inequality we have

$$\int_{A_L^1} |Du_*^1|^p dx \le \left(\int_\Omega |Du_*^1|^{\tilde{p}}\right)^{\frac{p}{\tilde{p}}} |A_L^1|^{1-\frac{p}{\tilde{p}}},$$

and as a consequence

$$\int_{A_L^1} |Du_*^1|^p dx \le \|Du_*\|_{\tilde{p}}^p \; |A_L^1|^{1-\frac{p}{\tilde{p}}} \le \tilde{k}|A_L^1|^\gamma,$$

where γ is as above and

$$\tilde{k} := \|Du_*\|_{\tilde{p}}^{p} \, |\Omega|^{\delta_2-\gamma}.$$

Finally, we can write

$$\begin{aligned}\int_{A_L^1} |D[u^1(x)-u_*^1(x)]|^p dx &\leq 2^{p-1}\int_{A_L^1} |Du^1(x)|^p dx + 2^{p-1}\int_{A_L^1} |Du_*^1(x)|^p dx \\ &\leq 2^{p-1}(k+\tilde{k})|A_L^1|^{\gamma} := \hat{k}|A_L^1|^{\gamma}.\end{aligned}$$

We use Sobolev inequality for $1 \leq p < 3$:

$$\begin{aligned}&\int_{A_L^1} |(u^1-u_*^1)-L|^{p^*} dx = \int_{\Omega} |[(u^1-u_*^1)-L]\vee 0|^{p^*} dx \\ &\leq C_{S,3}\left(\int_{\Omega} |D\{[(u^1-u_*^1)-L]\vee 0\}|^p dx\right)^{\frac{p^*}{p}} \\ &= C_{S,3}\left(\int_{A_L^1} |D(u^1-u_*^1)|^p dx\right)^{\frac{p^*}{p}}, \qquad (3.11)\end{aligned}$$

where $C_{S,3} = \left(\frac{2p}{3-p}\right)^{\frac{3p}{3-p}}$ is the Sobolev constant in dimension 3, while p^* is the Sobolev exponent in dimension 3: $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. From the previous computation we obtain

$$\int_{A_L^1} |[u^1(x)-u_*^1(x)]-L|^{p^*} dx \leq K|A_L^1|^{\frac{p^*}{p}\gamma},$$

where $K = C_{S,3}\,(\hat{k})^{\frac{p^*}{p}}$. Now we take $L_1 > L$; from $A_{L_1}^1 \subset A_L^1$ we have

$$|A_{L_1}^1|[L_1-L]^{p^*} \leq \int_{A_{L_1}^1} |[u^1(x)-u_*^1(x)]-L|^{p^*} dx \leq \int_{A_L^1} |[u^1(x)-u_*^1(x)]-L|^{p^*} dx.$$

Finally, we have obtained the following inequality:

$$|A_{L_1}^1| \leq \frac{K}{[L_1-L]^{p^*}}|A_L^1|^{\frac{p^*}{p}\gamma}.$$

The previous inequality falls within the hypotheses of Stampacchia's Lemma that we write for the convenience of the reader.

Lemma 3.3. *(see page 93 in[46]) Let $\varphi(t)$ a non negative and non increasing function defined for $t \geq k_0$ such that for $h > k \geq k_0$ we have*

$$\varphi(h) \leq \frac{c}{(h-k)^{\alpha}}[\varphi(k)]^{\beta},$$

with c, α, β positive constants. If $\beta > 1$ then we have

$$\varphi(k_0 + d) = 0,$$

where

$$d^{\alpha} = c[\varphi(k_0)]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}.$$

We use such a Lemma with

$$\alpha = p^* > 0, \qquad \beta = \frac{p^*}{p}\gamma > 1,$$

where the second inequality is proved below.

Then, we have $|A^1_{\tilde{L}}| = 0$ for

$$\tilde{L} = L + 2^{\frac{\beta}{\beta-1}} \left\{ K |A^1_L|^{\beta-1} \right\}^{\frac{1}{\alpha}} \leq L + 2^{\frac{\beta}{\beta-1}} \left\{ K |\Omega|^{\beta-1} \right\}^{\frac{1}{\alpha}}.$$

Since L was free in $(0, +\infty)$, taking $L = \frac{1}{N}$, with $N \in \mathbb{N}$, we have

$$|A^1_{L^*}| = 0, \quad \text{with} \quad L^* = 2^{\frac{\beta}{\beta-1}} \left\{ K |\Omega|^{\beta-1} \right\}^{\frac{1}{\alpha}}.$$

Then we conclude that

$$u^1 - u^1_* \leq L^* = 2^{\frac{\beta}{\beta-1}} \left\{ K |\Omega|^{\beta-1} \right\}^{\frac{1}{\alpha}} = 2^{\frac{p^*\gamma}{p^*\gamma - p}} K^{\frac{1}{p^*}} |\Omega|^{\frac{\gamma}{p} - \frac{1}{p^*}} \tag{3.12}$$

where

$$\begin{aligned} K := {} & \frac{4}{\nu}\left(\frac{2p}{3-p}\right)^{\frac{3p}{3-p}} 2^{\frac{3(p-1)}{3-p}} \\ & \cdot \Bigg(\max\Bigg\{ 4\|b\|_\sigma + \|a\|_\sigma,\ M\|Du_*\|^p_{\tilde{p}}, \\ & \quad 2M3^q \|Du_*\|^q_{\tilde{p}} \left(\frac{1}{\nu}\int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1 \right)^{\frac{q}{p}}, \\ & \quad M3^{\frac{r}{2}} \|Du_*\|^r_{\tilde{p}} \left(\frac{1}{\nu}\int_\Omega f(x, Du_*)dx + \frac{3}{\nu}\|a\|_1 \right)^{\frac{r}{q}} \Bigg\} \\ & \quad \cdot \max\left\{ |\Omega|^{\delta_1 - \gamma}, |\Omega|^{\delta_2-\gamma}, |\Omega|^{\delta_3-\gamma}, |\Omega|^{\delta_4-\gamma} \right\} + \|Du_*\|^p_{\tilde{p}} |\Omega|^{\delta_2 - \gamma} \Bigg)^{\frac{3}{3-p}}. \end{aligned} \tag{3.13}$$

It remains to check the inequality

$$\beta = \frac{p^*}{p}\gamma = \frac{3}{3-p}\gamma > 1; \tag{3.14}$$

we recall that γ may take four different values then we check the inequality for each value:

$$\begin{aligned}
&(1) \quad \frac{3}{3-p}\left(1-\frac{1}{\sigma}\right) > 1, \quad \text{for} \quad \sigma > \frac{3}{p},\\
&(2) \quad \frac{3}{3-p}\left(1-\frac{p}{\tilde{p}}\right) > 1, \quad \text{for} \quad \tilde{p} > 3,\\
&(3) \quad \frac{3}{3-p}\left(1-\frac{q}{p}-\frac{q}{\tilde{p}}\right) > 1, \quad \text{for} \quad \tilde{p} > \frac{3pq}{p^2-3q}, \quad \text{with} \quad p > \sqrt{3q},\\
&(4) \quad \frac{3}{3-p}\left(1-\frac{r}{q}-\frac{r}{\tilde{p}}\right) > 1, \quad \text{for} \quad \tilde{p} > \frac{3qr}{qp-3r}, \quad \text{with} \quad p > \frac{3r}{q}.
\end{aligned}$$

They are all verified thanks to the hypotheses (2.3),(2.9) and (2.1).
In order to provide a bound from below for the quantity $u^1(x) - u^1_*(x)$ we observe that $\tilde{u} = -u$ solves the following minimum problem:

$$\min\left\{\int_\Omega \tilde{f}(x, Dh(x))dx, \quad h \in \tilde{u}_* + W^{1,p}_0(\Omega, \mathbb{R}^3)\right\},$$

where $\tilde{f}(x,z) = f(x,-z)$ and $\tilde{u}_*(x) = -u_*(x)$. We note that $D\tilde{u} = D(-u) = -Du$, $adj_2(D\tilde{u})^\alpha_i = adj_2(-Du)^\alpha_i = adj_2(Du)^\alpha_i$ and $det(D\tilde{u}) = det(-Du) = -det(Du)$. Then

$$\tilde{f}(x,z) = \sum_\alpha F_\alpha(x, -z^\alpha) + \sum_\alpha G_\alpha(x, adj_2(z)^\alpha) + H(x, -det(z)).$$

Since $F_\alpha(x,-\xi)$, $H(x,-t)$ satisfy again (2.4) and (2.6) we can repeat the same computations made for u without any changes. Then we obtain the following upper bound:

$$-u^1(x) + u^1_*(x) = \tilde{u}^1(x) - \tilde{u}^1_*(x) \le L^*,$$

that is

$$u^1(x) - u^1_*(x) \ge -L^*. \tag{3.15}$$

Note that L^* is given by the right hand side of (3.12) and (3.13) with $\tilde{u}_*$ instead of u_*; since $\tilde{u}_* = -u_*$ then $\|D\tilde{u}_*\|_s = \|Du_*\|_s$; since $\tilde{f}(x,z) = f(x,-z)$, then

$$\int_\Omega f(x, D\tilde{u}_*)dx = \int_\Omega f(x, Du_*)dx.$$

This means that L^* in (3.15) and L^* in (3.12) are the same. From the previous inequality and from (3.12) we conclude:

$$|u^1(x) - u^1_*(x)| \leq L^*. \tag{3.16}$$

Now we pass to analyse the second component of u. We introduce the following function by changing the order of the components of u:

$$\hat{u} = \begin{pmatrix} u^2 \\ u^1 \\ u^3 \end{pmatrix}, \qquad \text{for which} \qquad D\hat{u} = \begin{pmatrix} Du^2 \\ Du^1 \\ Du^3 \end{pmatrix}.$$

We denote by $C_{1,2}(z)$ the matrix obtained from z by inverting line 1 and line 2. Then

$$D\hat{u} = C_{1,2}(Du).$$

Since $adj_2(D\hat{u})^1 = -adj_2(Du)^2$, $adj_2(D\hat{u})^2 = -adj_2(Du)^1$ and $adj_2(D\hat{u})^3 = -adj_2(Du)^3$ we have

$$adj_2(D\hat{u}) = -\begin{pmatrix} adj_2(Du)^2 \\ adj_2(Du)^1 \\ adj_2(Du)^3 \end{pmatrix} = -C_{1,2}(adj_2(Du)),$$

moreover $det(D\hat{u}) = -det(Du)$. Then we have that $\hat{u}$ solves

$$\min\left\{ \int_\Omega \hat{f}(x, Dh(x))dx, \quad h \in \hat{u}_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\},$$

where $\hat{u}_*(x) = \begin{pmatrix} u^2_* \\ u^1_* \\ u^3_* \end{pmatrix}$ and

$$\begin{aligned} \hat{f}(x,z) =& f(x, C_{1,2}(z)) \\ =& F_1(x, z^2) + F_2(x, z^1) + F_3(x, z^3) + G_1(x, -adj_2(z)^2) \\ &+ G_2(x, -adj_2(z)^1) + G_3(x, -adj_2(z)^3) + H(x, -det(z)). \end{aligned}$$

Again, $\hat{f}(x,y)$ satisfies (2.4), (2.5), (2.6), and all the computations made for u can be repeated without any changes and we obtain:

$$|u^2(x) - u^2_*(x)| = |\hat{u}^1(x) - \hat{u}^1_*(x)| \leq L^*. \tag{3.17}$$

Note that L^* is given by the right hand side of (3.12) and (3.13) with $\hat{u}_*$ instead of u_*; since $\hat{u}_* = C_{1,2}(u_*)$, then $\|D\hat{u}_*\|_s = \|Du_*\|_s$; since $\hat{f}(x,z) = f(x, C_{1,2}(z))$, then

$$\int_\Omega \hat{f}(x, D\hat{u}_*)dx = \int_\Omega f(x, Du_*)dx.$$

This means that L^* in (3.17) and L^* in (3.12) are the same.
For the third component we introduce the function

$$\mathring{u} = \begin{pmatrix} u^3 \\ u^2 \\ u^1 \end{pmatrix}.$$

In this case we denote by $C_{1,3}(z)$ the matrix obtained from z by inverting line 1 and line 3. Then

$$D\mathring{u} = C_{1,3}(Du), \qquad adj_2(D\mathring{u}) = -C_{1,3}(adj_2(Du)), \qquad det(D\mathring{u}) = -det(Du).$$

Then we have that $\mathring{u}$ solves

$$\min \left\{ \int_\Omega \mathring{f}(x, Dh(x))dx, \quad h \in \mathring{u}_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\},$$

where $\mathring{u}_*(x) = \begin{pmatrix} u_*^3 \\ u_*^2 \\ u_*^1 \end{pmatrix}$ and

$$\begin{aligned} \mathring{f}(x,z) =& f(x, C_{1,3}(z)) \\ =& F_1(x, z^3) + F_2(x, z^2) + F_3(x, z^1) + G_1(x, -adj_2(z)^3) \\ &+ G_2(x, -adj_2(z)^2) + G_3(x, -adj_2(z)^1) + H(x, -det(z)). \end{aligned}$$

Then we obtain as above that:

$$|u^3(x) - u_*^3(x)| = |\mathring{u}^1(x) - \mathring{u}_*^1(x)| \leq L^*. \tag{3.18}$$

Note that L^* is given by the right hand side of (3.12) and (3.13) with $\mathring{u}_*$ instead of u_*; since $\mathring{u}_* = C_{1,3}(u_*)$, then $\|D\mathring{u}_*\|_s = \|Du_*\|_s$; since $\mathring{f}(x,z) = f(x, C_{1,3}(z))$, then

$$\int_\Omega \mathring{f}(x, D\mathring{u}_*)dx = \int_\Omega f(x, Du_*)dx.$$

This means that L^* in (3.18) and L^* in (3.12) are the same.
This concludes the proof. □

4. Remarks on growth assumptions

We now consider model density (2.11) and we compare the restrictions on the growth assumptions considered in the present paper with the ones contained in [12]. Both requires r to be less than q which, in turn, must be less than p. Moreover, we are in the polyconvex case, so $1 \leq r$. The present paper requires $p < 3$, then $p^* = 3p/(3-p)$; in [12] also the case $p = 3$ is dealt with. Then, we confine ourselves to

$$1 \leq r < q < p < 3.$$

We recall that [12] asks for two conditions:

$$\frac{p}{p^*} < 1 - \frac{qp^*}{p(p^*-q)};$$

and

$$\frac{p}{p^*} < 1 - \frac{rp^*}{q(p^*-r)};$$

they can be written as follows

$$\frac{p}{3} > \frac{qp^*}{p(p^*-q)}; \tag{4.1}$$

and

$$\frac{p}{3} > \frac{rp^*}{q(p^*-r)}. \tag{4.2}$$

Let us study condition (4.1): it can be written as

$$\frac{p}{3} > \frac{q3p}{p(3p-q(3-p))}; \tag{4.3}$$

this amounts to say

$$p(3p-q(3-p)) > 3q3; \tag{4.4}$$

exploiting calculations we get

$$(3+q)p^2 - 3qp - 9q > 0; \tag{4.5}$$

the corresponding second order equation has two roots: $p_1 := \frac{3q-3\sqrt{5q^2+12q}}{2(3+q)} < 0$ and $p_2 := \frac{3q+3\sqrt{5q^2+12q}}{2(3+q)}$; then, inequality (4.5) is equivalent, in our case, to $p > p_2$. We claim that

$$p_2 > \sqrt{3q}; \tag{4.6}$$

indeed, squaring both sides gives

$$\frac{9q^2 + 6q\sqrt{5q^2 + 12q} + 45q^2 + 108q}{4(3 + q)^2} > 3q; \tag{4.7}$$

q can be cut from both sides; eliminating the denominator and rearranging elements result in

$$3\sqrt{5q^2 + 12q} > 2q^2 + 3q; \tag{4.8}$$

squaring again both sides, rearranging elements and cutting q give

$$0 > q^3 + 3q^2 - 9q - 27 := g(q). \tag{4.9}$$

We study the function g: we have $g(3) = 0$ and $g'(q) = 3q^2+6q-9$; equation $3q^2 + 6q - 9 = 0$ has two roots $q_1 = -3$ and $q_2 = 1$; then $g'(q) > 0$ for every $q > 1$ and g strictly increases in $[1, +\infty)$; in particular, $g(q) < g(3) = 0$ for $1 \leq q < 3$ and (4.9) holds true: this proves our claim (4.6). Now we turn our attention to (4.2): it can be written as

$$\frac{p}{3} > \frac{r3p}{q(3p - r(3 - p))}; \tag{4.10}$$

this amounts to say

$$q(3p - r(3 - p)) > 3r3; \tag{4.11}$$

dividing by q and exploiting calculations give

$$p > 3\frac{r(q + 3)}{q(r + 3)}. \tag{4.12}$$

Note that

$$3\frac{r(q + 3)}{q(r + 3)} > 3\frac{r}{q}, \tag{4.13}$$

since $q > r$. Let us summarize: in [12] conditions on p are

$$\begin{cases} p > p_2, \\ p > 3\frac{r(q+3)}{q(r+3)}; \end{cases} \tag{4.14}$$

if we keep in mind (4.6) and (4.13) we see that (4.14) implies

$$\begin{cases} p > \sqrt{3q}, \\ p > 3\frac{r}{q}, \end{cases} \tag{4.15}$$

and these are our conditions on p. This shows that our conditions are weaker than the ones in [12] when dealing with (2.11) and $1 \leq r < q < p < 3$.

5. Lipschitz boundary datum

In the case where the boundary datum is a Lipschitz function it is possible to obtain a sharper bound for the minimizers. In this section we discuss in details this situation. We start fixing some notation.

Definition 5.1. *Let $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ and let $u_* : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be such that u_*^i is Lipschitz with Lipschitz constant less than or equal to $\mathcal{K}_i$, $i = 1, 2, 3$. We will say that such a u_* is a $\mathcal{K}$-Lipschitz mapping. Let u_* be a $\mathcal{K}$-Lipschitz mapping, we will denote with $\mathrm{Lip}_{\mathcal{K}_i, u_*^i}$ the set of Lipschitz functions $g : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ with Lipschitz constant less or equal to $\mathcal{K}_i$ that coincide with u_*^i on $\partial\Omega$. We will denote by $\mathrm{Lip}_{\mathcal{K}_{u_*}}$ the set of functions $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ such that $u(x) = (u^1(x), u^2(x), u^3(x))$ with $u^i \in \mathrm{Lip}_{\mathcal{K}_i, u_*^i}$ for every $i = 1, 2, 3$.*

It is straightforward that, since we assume u_* is $\mathcal{K}$-Lipschitz, the set $\mathrm{Lip}_{\mathcal{K}_i, u_*^i}$ is non empty. We also remark that it is compact with respect to the uniform convergence.
We fix $i \in \{1, 2, 3\}$ and we consider the functional

$$\mathrm{Lip}_{\mathcal{K}_i, u_*^i} \ni g \to \int_\Omega g(x)\, dx. \tag{5.1}$$

The functional (5.1) is continuous with respect to the uniform convergence and then it admits both a minimum and a maximum on $\mathrm{Lip}_{\mathcal{K}_i, u_*^i}$.
We denote the minimizer by u_-^i and the maximizer by u_+^i and we consider the functions $u_-, u_+ : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ defined by $u_- = (u_-^1, u_-^2, u_-^3)$ and $u_+ = (u_+^1, u_+^2, u_+^3)$.

Proposition 5.2. *Let $u \in \mathrm{Lip}_{\mathcal{K}_{u_*}}$. Then*

$$u_-^i(x) \le u^i(x) \le u_+^i(x) \qquad \text{for all } x \in \Omega, \qquad \text{for all } i \in \{1, 2, 3\}.$$

Proof. It is sufficient to prove that $u_-^1(x) \le u^1(x)$ for all $x \in \Omega$ since the other inequalities follow similarly. We argue by contradiction assuming that there exist $u \in \mathrm{Lip}_{\mathcal{K}_1, u_*^1}$ and $x \in \Omega$ such that $u^1(x) < u_-^1(x)$. The continuity of both u^1 and u_-^1 implies that the set $A = \{x \in \Omega : u^1(x) < u_-^1(x)\}$ has positive measure. Let us consider the function

$$w = \min\{u^1; u_-^1\} = \begin{cases} u^1, & \text{if } x \in A, \\ u_-^1, & \text{otherwise.} \end{cases}$$

We obtain that $w \in \mathrm{Lip}_{\mathcal{K}_1, u^1_*}$ and

$$\begin{aligned}
&\int_\Omega w(x)\,dx \\
&= \int_{\Omega\setminus A} u^1_-(x)\,dx + \int_A u^1(x)\,dx \\
&< \int_{\Omega\setminus A} u^1_-(x)\,dx + \int_A u^1_-(x)\,dx \\
&= \int_\Omega u^1_-(x)\,dx.
\end{aligned} \tag{5.2}$$

This is in contrast with the minimality of u^1_- for the functional (5.1). □

Remark 5.3. *The previous proposition shows also that the minimizer u^i_- is unique and, of course, the same is true for the maximizer u^i_+, for $i = 1, 2, 3$. There is only one case in which u^i_- and u^i_+ coincide in Ω and it happens when there exists just one function with Lipschitz constant $\mathcal{K}_i$ assuming the boundary datum u^i_*.*

Corollary 5.4. *Let u_* be a Lipschitz mapping with Lipschitz constant $\mathcal{K}$. Let u be a solution of the minimization problem 1.7. Then for almost every $x \in \Omega$ we have*

$$u^i_+(x) - L^+ \le u^i(x) \le u^i_-(x) + L^-, \tag{5.3}$$

for every $i \in \{1, 2, 3\}$, where L^- is the constant L^* that appears in (2.10) and (2.8) with u_- instead of u_* and, analogously, L^+ is the constant L^* with u_+ instead of u_* .

Proof. Since $u_- = u_+ = u_*$ on $\partial\Omega$, we have $u_* + W^{1,p}_0(\Omega) = u_- + W^{1,p}_0(\Omega) = u_+ + W^{1,p}_0(\Omega)$ and we can apply Theorem 2.2 both with u_- and u_+ instead of u_*. Then using just one side of the inequality (2.2) we get the estimate. □

Remark 5.5. *Let us give an estimate for the constants L^- and L^+. Looking at (2.10) and (2.8), we have to control the following items*

$$\|Du_-\|_{\tilde{p}} \qquad \text{and} \qquad \int_\Omega f(x, Du_-)dx.$$

Let us set

$$\mathcal{K}_* := \max\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}.$$

We have

$$|Du_-|^{\tilde{p}} = \left(\sum_{\alpha=1}^{3}\sum_{i=1}^{3}|D_i u_-^{\alpha}|^2\right)^{\tilde{p}/2} \leq \left(\sum_{\alpha=1}^{3}\sum_{i=1}^{3}|\mathcal{K}_{\alpha}|^2\right)^{\tilde{p}/2}$$
$$\leq \left(\sum_{\alpha=1}^{3}\sum_{i=1}^{3}|\mathcal{K}_{*}|^2\right)^{\tilde{p}/2} = (3\mathcal{K}_*)^{\tilde{p}},$$

then

$$\|Du_-\|_{\tilde{p}} \leq |\Omega|^{\frac{1}{\tilde{p}}} 3\mathcal{K}_*.$$

Moreover, using (2.4)-(2.2)*, we can write*

$$\int_{\Omega} f(x, Du_-)dx \leq 7\|b\|_1 + M\left\{\sum_{\alpha=1}^{3}\int_{\Omega}|Du_-^{\alpha}|^p dx + \sum_{\alpha=1}^{3}\int_{\Omega}|adj_2(Du_-)^{\alpha}|^q dx + \right.$$
$$\left.\int_{\Omega}|det(Du_-)|^r dx\right\} \leq 7\|b\|_1 + M|\Omega|\left\{3^{1+\frac{p}{2}}\mathcal{K}_*^p + 3\cdot 12^{\frac{q}{2}}\mathcal{K}_*^{2q} + 6^r\mathcal{K}_*^{3r}\right\},$$

where we have used the estimates

$$|adj_2(Du_-)^{\alpha}|^q = \left[\sum_{k=1}^{3}|adj_2(Du_*)_k^{\alpha}|^2\right]^{\frac{q}{2}} \leq \left[\sum_{k=1}^{3}(2\mathcal{K}_*^2)^2\right]^{\frac{q}{2}} = 12^{\frac{q}{2}}\mathcal{K}_*^{2q},$$

and

$$|det(Du_-)|^r \leq (3!\mathcal{K}_*^3)^r = 6^r\mathcal{K}_*^{3r}.$$

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