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FLIPCLASSES AND COMBINATORIAL INVARIANCE FOR KAZHDAN–LUSZTIG POLYNOMIALS

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Abstract. In this work, we investigate a novel approach to the Combinatorial Invariance Conjecture of Kazhdan–Lusztig polynomials for the symmetric group. Using the new concept of flipclasses, we introduce some combinatorial invariants of intervals in the symmetric group whose analysis leads us to a recipe to compute the coefficients of q^h of the Kazhdan–Lusztig $\widetilde{R}$-polynomials, for $h \leq 6$. This recipe depends only on the isomorphism class (as a poset) of the interval indexing the polynomial and thus provides new evidence for the Combinatorial Invariance Conjecture.

1. Introduction

The introduction of Kazhdan–Lusztig polynomials by Kazhdan and Lusztig in [21] has proved to be a major paradigm shift in Lie theory and representation theory, giving a fundamental contribution to the development of geometric representation theory. These polynomials, indexed by two elements u and v in a Coxeter group W, have soon found applications in several contexts.

Among the combinatorial problems regarding Kazhdan–Lusztig polynomials, arguably the most prominent one is what is usually referred to as the Combinatorial Invariance Conjecture. It was independently posed in the 80's by Lusztig, in private, and Dyer (see [15, Remark 7.31]).

Conjecture 1.1. *The Kazhdan–Lusztig polynomial $P_{u,v}(q)$ depends only on the isomorphism type of the interval $[u, v]$ as a partially ordered set under Bruhat order.*

Although widely open, the Combinatorial Invariance Conjecture has been settled in some special cases, among which: intervals of rank ≤ 4 (see, e.g., [15, 7.31], and [3, Chapter 5, Exercises 7 and 8]), intervals of rank ≤ 8 in type A and ≤ 6 in types B and D ([20]), intervals in type $\tilde{A}_2$ ([13]), intervals which are lattices ([15, 7.23], [7]), and intervals starting from the identity ([9]). We refer the reader to [8] for further details on Conjecture 1.1. Also a parabolic version of Conjecture 1.1 holds for intervals starting from the identity ([23], [24]). On the other hand, certain entirely poset-theoretical generalizations of Conjecture 1.1 fail ([10], [22]).

The Combinatorial Invariance Conjecture for Kazhdan–Lusztig polynomials is equivalent to the Combinatorial Invariance Conjecture for R-polynomials (also introduced in [21]), as well as for $\widetilde{R}$-polynomials, which are useful rescalings of R-polynomials. As a matter of fact, some of the above results have been achieved by means of these other families of polynomials.

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Recently, an even broader interest in the Combinatorial Invariance Conjecture has been sparked by the use of certain machine learning models to attack the problem (see [6], [14]). Blundell, Buesing, Davies, Veličković, and Williamson discovered a formula for the Kazhdan–Lusztig polynomials of the symmetric group, and propose a conjecture that implies the Combinatorial Invariance Conjecture for the symmetric group. This conjecture is based on the concept of a hypercube decomposition, which has already been the object of several studies (see [1], [12], [18]).

In this paper, we approach the Combinatorial Invariance Conjecture for the symmetric group $\mathfrak{S}_n$ from a new perspective. We believe that the intervals might not be the ultimate atoms for the combinatorial invariance and should be decomposed into smaller building blocks, viz. the *flipclasses*. The guiding principle is that, regarding the combinatorial invariance, all concepts associated with intervals should in fact be viewed as shadows of finer concepts associated with flipclasses. This work is intended to give first evidence in support of this new approach.

Let $u, v \in \mathfrak{S}_n$ with $u \leq v$ in Bruhat order. Let $B(\mathfrak{S}_n)$ denote the Bruhat graph of $\mathfrak{S}_n$. For each $h \in \mathbb{N}$, we partition the set of paths from u to v of length h in $B(\mathfrak{S}_n)$ into orbits under the action of the group generated by the flips. We call the orbits of this group *h-flipclasses*. The multiset of the unlabelled h-flipclasses of an interval is a combinatorial invariant. By a combinatorial invariant of an interval we mean an object that depends only on the isomorphism class of the interval as a poset. With each flipclass F, we associate two graphs, the support graph S_F and the time-support graph TS_F, and a polynomial, the ι-polynomial ι_F (see Section 3 for the definitions). These new concepts yield combinatorial invariants.

We study these new combinatorial objects and prove that, if two flipclasses have the same ι-polynomial, then they also have the same number of increasing paths (w.r.t. any reflection ordering) of length h, for $h \leq 6$. Thus, the multiset of the ι-polynomials of the h-flipclasses of paths from u to v determines the coefficient of q^h of the $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$, for $h \leq 6$. This implies that these coefficients are combinatorial invariants. We stress the facts that we find an explicit algorithm for computing such coefficients of the $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$ starting from the isomorphism class of Bruhat interval $[u, v]$ as a poset and that this algorithm works for any interval in any symmetric group.

Let us succintly describe the ingredients of the proof. Fix h in $\mathbb{N}$. The set whose elements are the h-flipclasses of any $\mathfrak{S}_n$ (letting n vary) is clearly infinite. On the other hand, we prove that the number of isomorphism classes of h-flipclasses is finite and that all such isomorphism classes can be constructed from h-flipclasses of $\mathfrak{S}_{h+1}$. Thus, we reduce the infinity of the general case to a finite analysis. By definition, isomorphic flipclasses have the same number of increasing paths w.r.t. the lexicographic order. But this alone is not applicable to the Combinatorial Invariance Conjecture because the multiset of the isomorphism classes of the h-flipclasses of an interval is not a combinatorial invariant of the interval since the flipclasses come with the labels of the edges. However, we show that, for $h \leq 6$, the ι-polynomial detects the number of increasing paths, and we construct the map sending a ι-polynomial to the number of increasing paths of any flipclass having that ι-polynomial. These computations can be done with the aid of a computer thanks to the above reduction to finite calculations, but unfortunately they become extremely unwieldy and we carry out this program up to $h = 6$. The multiset of the ι-polynomials of the h-flipclasses of a fixed interval $[u, v]$ is a

combinatorial invariant. Hence, the number of increasing paths from u to v of length h, for $h \leq 6$, is a combinatorial invariant: by a result of Dyer (see [17]), this number is equal to the coefficient of q^h of the $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$.

The algorithm derived from our study to compute the coefficient of q^h, for $h \leq 6$, of the $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$ from the isomorphism class of $[u, v]$ proceeds along the following steps.

(1) Construct the h-flipclasses of paths from u to v.
(2) For each of these flipclasses, compute its ι-polynomial.
(3) For each of these flipclasses, compute its number of increasing paths starting from its ι-polynomial.
(4) Sum the numbers obtained in the previous step to obtain the coefficient of q^h of $\widetilde{R}_{u,v}(q)$

To give an idea of the rate of growth, we note that the sequence of the number of h-flipclasses of $\mathfrak{S}_{h+1}$ is $(4, 4, 50, 1096, 36634, 1701056)$, for $h = 1, \ldots, 6$.

It is well known that an interval has the same number of increasing paths w.r.t. any reflection ordering. We prove that the same is true for flipclasses. This is first evidence in support of the principle that the appropriate setting in formulating the Combinatorial Invariance Conjecture could be that of flipclasses. Differently from the proofs in the literature of the analogous result for intervals, our proof is local in nature. It thus sheds new light also on the case of intervals.

A natural conjecture stems from our study, in line with the underlying philosophy. It is a refinement of the Combinatorial Invariance Conjecture and predicts that if two unlabelled h-flipclasses are isomorphic, then they have the same number of increasing paths.

The organization of the paper is as follows. In Section 2, we recall some definitions, notation, and results that are needed in the paper. In Section 3, we introduce the main new concepts of this work, namely support graphs, time-support graphs, ι-polynomials, and flipclasses. In Section 4, we prove that a flipclass has the same number of increasing paths w.r.t. any reflection ordering. In Section 5, we give some results on flipclasses, their support and time-support graphs, and their ι-polynomials. In Section 6, we apply results of Section 5 to reduce the infinite computations to a finite number, for all $h \in \mathbb{N}$. In Section 7, we provide a general strategy to prove the combinatorial invariance of the coefficient of q^h of the $\widetilde{R}$-polynomials., for all $h \in \mathbb{N}$, and apply it for $h \leq 6$. In Section 8, we propose a conjecture that implies the Combinatorial Invariance Conjecture.

2. Preliminaries

This section reviews the background material.

Given $n \in \mathbb{N}^+$, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and the symmetric group on $[n]$ by $\mathfrak{S}_n$. We write the elements of $\mathfrak{S}_n$ in one-line notation (so $u = [a_1, \ldots, a_n]$, or simply $u = a_1 \cdots a_n$, means $u(i) = a_i$ for all $i \in [n]$) as well as in disjoint cycle-form, omitting to write the 1-cycles. We view $\mathfrak{S}_n$ as a Coxeter group with the simple transpositions $(i, i + 1)$, for $i \in [n - 1]$, as Coxeter generators. It is a Coxeter group of type A. We fix our notation on the symmetric group in the following list:

e	identity of $\mathfrak{S}_n$,
S	$= \{(i, i+1) : i \in [n-1]\}$, the set of *simple transpositions* of $\mathfrak{S}_n$,
ℓ	the length function of $\mathfrak{S}_n$ with respect to S,
w_0	the longest element of $\mathfrak{S}_n$,
T	$= \{wsw^{-1} : w \in \mathfrak{S}_n,\ s \in S\}$, the set of *transpositions* of $\mathfrak{S}_n$,
$D_R(w)$	$= \{s \in S :\ \ell(ws) < \ell(w)\}$, the right descent set of $w \in \mathfrak{S}_n$,
$D_L(w)$	$= \{s \in S :\ \ell(sw) < \ell(w)\}$, the left descent set of $w \in \mathfrak{S}_n$,
$\leq$	Bruhat order on $\mathfrak{S}_n$ (as well as usual order on $\mathbb{R}$),
$[u, v]$	$= \{w \in \mathfrak{S}_n :\ u \leq w \leq v\}$, the (Bruhat) interval generated by $u, v \in \mathfrak{S}_n$,
$B(\mathfrak{S}_n)$	the *Bruhat graph of* $\mathfrak{S}_n$,
$B(X)$	the (directed) graph induced on the subset X by the Bruhat graph $B(\mathfrak{S}_n)$,
$P_h(u, v)$	the set of paths of length h from u to v in $B(\mathfrak{S}_n)$.

Recall that the *Bruhat graph* $B(\mathfrak{S}_n)$ is the edge-labelled directed graph with vertex set $\mathfrak{S}_n$ where $u \xrightarrow{t} v$ if and only if $vu^{-1} = t \in T$ and $\ell(u) < \ell(v)$. Recall also that the *Bruhat order* is the partial order on $\mathfrak{S}_n$ where $u \leq v$ provided that there is a (directed) path from u to v in $B(\mathfrak{S}_n)$. (All paths that we consider in a directed graph are tacitly assumed to be directed.)

The following result (see [16, Proposition 3.3]) implies that the Combinatorial Invariance Conjecture is equivalent to the conjecture that one obtains by considering the isomorphism type of the graph $B([u, v])$ induced by the Bruhat graph on the elements of the Bruhat interval $[u, v]$.

Theorem 2.1. *Let $u, v \in \mathfrak{S}_n$. The isomorphism class (as a poset) of the Bruhat interval $[u, v]$ determines the isomorphism class (as a directed graph) of the graph $B([u, v])$.*

Call a directed graph, respectively, a segment, a diamond, or a k-crown, if it is isomorphic, respectively, to the first, the second or the third directed graph depicted in Figure 1 (the third graph has $2k+2$ vertices). We call by the same name a poset whose Hasse diagram is a segment, a diamond, or a k-crown.

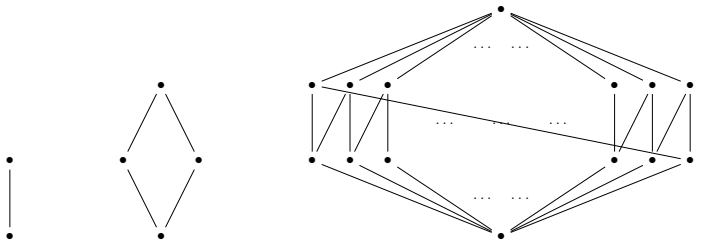


FIGURE 1. A segment, a diamond, and a k-crown

Lemma 2.2. *Let $u, v \in \mathfrak{S}_n$, with $u \leq v$ and $\ell(v) - \ell(u) = 3$ ($\ell(v) - \ell(u) = 2$, respectively). Then the Hasse diagram of the Bruhat interval $[u, v]$ is a k-crown for some $k \in \{2, 3, 4\}$ (a diamond, respectively).*

The following result-definition is specific to the symmetric group.

Definition 2.3. Let $u, v \in \mathfrak{S}_n$. Given a path $\Gamma = \big(u \longrightarrow x \longrightarrow v\big)$ of length 2, there exists a unique path $f(\Gamma)$ of the form $\big(u \longrightarrow y \longrightarrow v\big)$, with $y \neq x$. We call f the *flip operator*.

Next result immediately follows by [11, Corollary 2.3].

Lemma 2.4. *Let $\preceq$ be a reflection order. Let Γ be a path of length 2 of the form $\big(\bullet \xrightarrow{t} \bullet \xrightarrow{r} \bullet\big)$, with $t \prec r$. Let $\big(\bullet \xrightarrow{p} \bullet \xrightarrow{q} \bullet\big)$ be the form of the flip of Γ. Then $q \prec p$, $t \prec p$, and $q \prec r$.*

A total order $\preceq$ on the set of transpositions T is a *reflection ordering* provided that, given $a, b, c \in [n]$ with $a < b < c$, either

$$(a, b) \preceq (a, c) \preceq (b, c) \qquad \text{or} \qquad (b, c) \preceq (a, c) \preceq (a, b). \tag{1}$$

The lexicographic order $(1,2) \preceq (1,3) \preceq \cdots \preceq (1,n) \preceq (2,3) \preceq \cdots \preceq (n-1,n)$ and the reverse lexicographic order $(n,n-1) \preceq (n,n-2) \preceq \cdots \preceq (n,1) \preceq (n-1,n-2) \preceq \cdots \preceq (2,1)$ are instances of reflection orderings.

Let $u, v \in \mathfrak{S}_n$, $u \leq v$, and $\Gamma = (u = x_0 \to x_1 \to \cdots \to x_h = v)$ be a path from u to v in $B(\mathfrak{S}_n)$. We define h to be the *length* of Γ, denoted $\ell(\Gamma)$. Fix a reflection ordering $\preceq$. A path $\Gamma = \big(x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{h-1}} x_{h-1} \xrightarrow{t_h} x_h\big)$ in $B(\mathfrak{S}_n)$ is *increasing (with respect to $\preceq$)* if $t_1 \preceq t_2 \preceq \cdots \preceq t_h$.

In [21], Kazhdan and Lusztig introduce two families of polynomials $\{P_{u,v}\}_{u,v \in \mathfrak{S}_n} \subseteq \mathbb{Z}[q]$ and $\{R_{u,v}\}_{u,v \in \mathfrak{S}_n} \subseteq \mathbb{Z}[q]$, which are now known as the Kazhdan–Lusztig and R-polynomials, respectively, of $\mathfrak{S}_n$ (actually, the group could be any Coxeter group, see [3, §5]). Kazhdan–Lusztig polynomials and R-polynomials are equivalent: more precisely, given $u, v \in \mathfrak{S}_n$, it is possible to compute the set $\{P_{x,y}\}_{x,y \in [u,v]}$ once one knows the set $\{R_{x,y}\}_{x,y \in [u,v]}$, and vice versa.

We do not define Kazhdan–Lusztig and R-polynomials here because we work with another family of polynomials, the $\widetilde{R}$-polynomials, which are a rescaling of the R-polynomials: the polynomial $\widetilde{R}_{u,v}(q)$ associated with u and v is the unique polynomial with natural coefficients satisfying

$$R_{u,v}(q) = q^{\frac{\ell(u,v)}{2}} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}),$$

(see, e.g., [3, Proposition 5.3.1]). The following combinatorial interpretation due to Dyer (see [17] and also [3, Theorem 5.3.4]) gives another definition of the $\widetilde{R}$-polynomials.

Theorem 2.5. *Let $\preceq$ be a reflection ordering, and $u, v \in \mathfrak{S}_n$. Then*

$$\widetilde{R}_{u,v}(q) = \sum q^{\ell(\Gamma)},$$

where the sum is over all increasing paths Γ from u to v.

3. Time-support graphs and flipclasses

In this section, we give the definitions of certain new combinatorial objects.

Given a directed graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Given $h \in \mathbb{N}$ and $u, v \in V(G)$, we let $P_h^G(u, v)$ denote the set of paths of length h in G from u to v.

Definition 3.1. Let G be an edge-labelled directed graph, $u, v \in V(G)$, and $h \in \mathbb{N}$. Let F be a subset of $P_h^G(u, v)$.

The *support graph of* F, denoted S_F, is the edge-labelled directed graph such that

(1) $V(S_F) = \{a \in V(G) : \text{ there exists a path in } F \text{ passing through } a\}$,

(2) $E(S_F) = \{a \xrightarrow{t} b \in E(G) : \text{ there exists a path in } F \text{ containing } a \xrightarrow{t} b\}$.

The *time-support graph of* F, denoted TS_F, is the edge-labelled directed graph such that

(1) $V(TS_F) = \{(a, i) \in G \times \{0, \dots, h\} : \exists\big(u = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_h = v\big) \text{ in } F \text{ with } x_i = a\}$

(2) $E(TS_F) = \{(a, i) \xrightarrow{t} (b, i+1) :$
$\exists\big(u = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_h = v\big) \text{ in } F \text{ with } x_i = a,\ x_{i+1} = b \text{ and containing } x_i \xrightarrow{t} x_{i+1}\}$.

For $i \in \{0, 1, \dots, h\}$, we say that the path $\big(u = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_h = v\big)$ passes through vertex x_i at time i.

As an example, consider the graph G depicted in Figure 2, on the left. Edges point upwards and can have arbitrary labels. Let F be the set of all paths from u to v of length 4. Then the support graph S_F is isomorphic to G, while the time-support graph TS_F is depicted on the right.

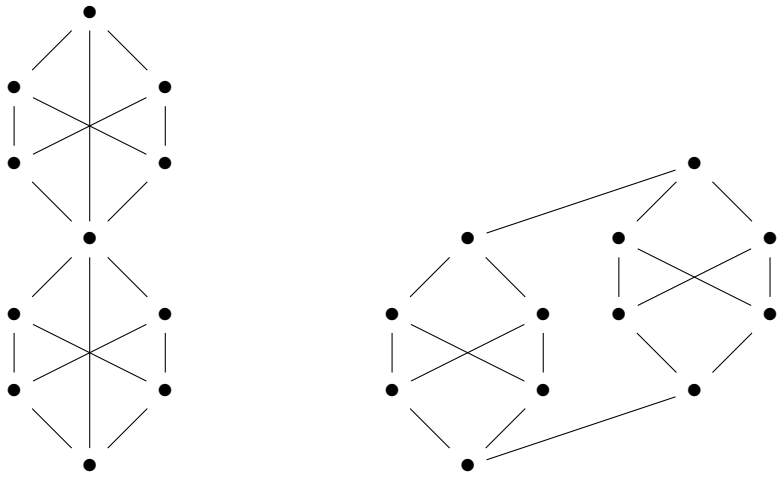


FIGURE 2. Support and time-support graphs

Remark 3.2.

(1) S_F and TS_F are both acyclic directed graphs, and so both induce a poset structure on the set of vertices. Moreover, TS_F is the Hasse diagram of the graded poset that it induces.

(2) The forgetful map $(x, i) \mapsto x$ from TS_F to S_F is a label preserving surjective map of directed graphs. It is an isomorphism of graphs if and only it is injective. Equivalently, it is an isomorphism if and only two paths of F sharing the same vertex x pass through x at the same i.

(3) S_F is a subgraph of G, while TS_F, in general, is not isomorphic to a subgraph of G.

(4) There are two natural label-preserving injective maps $i_{S_F} : F \to P_h^{S_F}(u, v)$ and $i_{TS_F} : F \to P_h^{TS_F}(u, v)$.

Definition 3.3. Let G be an edge-labelled directed graph, $u, v \in V(G)$, and $h \in \mathbb{N}$. Let F be a subset of $P_h^G(u,v)$.

- The *t-vector* of F, denoted t_F, is $t_F = (t_0, t_1, \ldots, t_h)$, where t_i is the cardinality of the set

  $$\{x : \text{ there exists a path } \Gamma = \left(x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_h\right) \text{ in } F \text{ such that } x_i = x\}.$$

- The *ι-polynomial* of F, denoted $\iota_F(x,y,t)$, is

$$\iota_F(x,y,t) = \sum_{(a,i) \in V(TS_F)} x^{\text{indegree}(a,i)}\ y^{\text{outdegree}(a,i)}\ t^i.$$

Remark 3.4. Notice that t_F is the rank vector of the poset induced by TS_F (see (1 of Remark 3.2), i.e. t_i is the cardinality of the set $\{(x,i) : (x,i) \in V(TS_F)\}$. Notice also that the t-vector of F is encoded in the ι-polynomial of F: indeed, the t-vector is the sequence of the coefficients of $\iota_F(1,1,t)$. Other specializations of the ι-polynomial of F give other pieces of information on TS_F: for example, $\frac{1}{i!}\frac{\partial^i \iota}{\partial t^i}(x,1,0)$ (respectively, $\frac{1}{i!}\frac{\partial^i \iota}{\partial t^i}(1,y,0)$) gives the sequences of the in-degrees (respectively, out-degrees) of the vertices of TS_F of time i.

Definition 3.5. We say that a path Δ of S_F (of TS_F, respectively) is *effective* if there exists a path Γ in F such that Δ is a subpath of $i_{S_F}(\Gamma)$ (of $i_{TS_F}(\Gamma), \textit{respectively}$).

Recall that $B(\mathfrak{S}_n)$ denotes the Bruhat graph of $\mathfrak{S}_n$. For short, we denote $P_h^{B(\mathfrak{S}_n)}(u,v)$ simply by $P_h(u,v)$. Recall that, given $u, v \in \mathfrak{S}_n$, the set $P_2(u,v)$ has cardinality either 0 or 2. If $P_2(u,v) = \{\Gamma_1, \Gamma_2\}$, we say that Γ_2 is the flip of Γ_1, and vice versa.

Proposition 3.6. *Let* $\Gamma_1 = \left(u \xrightarrow{p} x \xrightarrow{q} v\right)$ *be a path of length 2 in* $B(\mathfrak{S}_n)$. *Let* $\Gamma_2 = \left(u \xrightarrow{s} y \xrightarrow{t} v\right)$ *be the flip of* Γ_1. *Then:*

(1) if p and q commute, then $s = q$ and $t = p$;
(2) if $p = (a,b)$ and $q = (a,c)$:
- *if $a < b < c$ or $a > b > c$, then $s = (b,c)$ and $t = (a,b)$;*
- *if $a < c < b$ or $a > c > b$, then $s = (a,c)$ and $t = (b,c)$;*
- *if $b < a < c$ or $b > a > c$, then:*
  - *if $u^{-1}(b) < u^{-1}(a) < u^{-1}(c)$ or $u^{-1}(c) < u^{-1}(a) < u^{-1}(b)$, then $s = (a,c)$ and $t = (b,c)$;*
  - *otherwise, $s = (b,c)$ and $t = (a,b)$.*

Proof. In all these cases, one checks readily that $ts = qp$, and $u < su < tsu$ in the Bruhat order. □

Definition 3.7. Let $h \in \mathbb{N}$ and $u, v \in \mathfrak{S}_n$.

- For $i \in [h-1]$, the *i-th flip operator* $f_i : P_h(u,v) \to P_h(u,v)$ is the map sending a path $\Gamma = \left(u = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_h = v\right)$ to the path obtained from Γ by replacing the subpath $\left(x_{i-1} \longrightarrow x_i \longrightarrow x_{i+1}\right)$ with its flip.
- We call the *h-flip group of u and v*, denoted $\mathfrak{F}_{h,u,v}$, the subgroup of the symmetric group on $P_h(u,v)$ generated by the set $\{f_i : i \in [h-1]\}$ of all flip operators.
- Given $\Gamma \in P_h(u,v)$, the *flipclass of* Γ is the orbit of Γ under the action of $\mathfrak{F}_{h,u,v}$.
- If F is an orbit of $\mathfrak{F}_{h,u,v}$, we say that F is an h-flipclass of $\mathfrak{S}_n$.

We are interested in support graphs and time-support graphs associated with flipclasses.

Remark 3.8. (1) Each flip operator is a non trivial involution, i.e. it has order 2.
(2) If F is a flipclass, then the poset induced on $V(S_F)$ by S_F injects in $\mathfrak{S}_n$ under Bruhat order (however, given two incomparable vertices in $V(S_F)$, they could be comparable in Bruhat order).
(3) Let F be a flipclass. If $(a,i) \xrightarrow{t} (b,i+1) \in E(TS_F)$, then $a \xrightarrow{t} b \in E(S_F)$ (as well as $a \xrightarrow{t} b \in E(B(\mathfrak{S}_n))$.
(4) Recall that, if $x \longrightarrow y \in E(B(\mathfrak{S}_n))$, then $\ell(y) - \ell(x) \equiv 1 \pmod 2$. Hence, given a flipclass F, if $(a,i),(a,j) \in V(TS_F)$, then $i \equiv j \pmod 2$.

Definition 3.9. Let $n, m \in \mathbb{N}^+$ and $h \in \mathbb{N}$. Let F be an h-flipclass of $\mathfrak{S}_n$ and F' be an h-flipclass of $\mathfrak{S}_m$. We say that (f,g) is an isomorphism from F to F' provided that:

- f is a bijection from the set of elements of the paths in F to the set of elements of the paths in F' that induces a flips-preserving bijection from F to F';
- g is a bijection from the set of transpositions labeling the paths in F to the set of transpositions labeling the paths in F';
- if $(t_1, \dots t_h)$ is the ordered sequence of the labels of a path Γ in F, then $(g(t_1), \dots g(t_h))$ is the ordered sequence of the labels of the path $f(\Gamma)$ of F';
- g is order preserving w.r.t. the lexicographical orders on $\mathfrak{S}_n$ and $\mathfrak{S}_m$.

Analogously, we may introduce the notion of isomorphism between support graphs and time-support graphs. Clearly, isomorphic flipclasses have isomorphic support graphs, isomorphic time-support graphs, and same ι-polynomial.

Let F be a flipclass. We also consider the unlabelled version F^u, S_F^u, and TS_F^u obtained by forgetting the labels of the edges of F, S_F, and TS_F, respectively, together with the natural notions of isomorphisms. In particular, given two h-flipclasses F and G, an isomorphism from F^u to G^u is a bijection from the set of elements of the paths in F to the set of elements of the paths in G that induces a flips-preserving bijection from F to G.

Remark 3.10. Let f be a function with the set of Bruhat intervals in $\mathfrak{S}_n$ as domain. Recall from the introduction that we say that f is a *combinatorial invariant* provided that f is constant on the isomorphism classes (in the category of posets). Given $u, v \in \mathfrak{S}_n$, with $u \leq v$, and $h \in \mathbb{N}$, the set of the isomorphism classes of the unlabelled h-flipclasses of paths from u to v, the set of the isomorphism classes of their unlabelled support graphs, the set of the isomorphism classes of the their unlabelled time-support graphs, and the set of their ι-polynomials are combinatorial invariants.

4. Flipclasses and number of increasing paths

In this section, we prove a first basic result in support of our guiding principle that what happens for intervals should in fact be viewed as shadows of finer facts for flipclasses. Theorem 2.5 implies that an interval has the same number of increasing paths w.r.t. any reflection ordering. As far as we know, there are no direct proofs of this fact. Here we give an elementary and conceptual proof of the finer fact that a flipclass has the same number of increasing paths w.r.t. any reflection ordering. This sheds new light also on the case of intervals.

Theorem 4.1. *A flipclass F has the same number of increasing paths w.r.t. any reflection ordering.*

Proof. It is well known (see [15], Remark 6.16,(iv)) that the reflection orderings of $\mathfrak{S}_n$ are in bijection with the reduced expressions of the longest permutation: $t_1 \preceq t_2 \preceq \cdots \preceq t_{\binom{n}{2}}$ is a reflection ordering if and only if there is a reduced expression $s_1 \cdots s_{\binom{n}{2}}$ of the longest permutation such that $t_i = s_1 \cdots s_i \cdots s_1$. It is also well known (see [26]) that two reduced expressions of the same permutation are linked by a finite sequence of braid moves. A braid move is the replacement of a consecutive subsequence $ss's\cdots$ with $s'ss'\cdots$, where the two sequences have as many factors as the order of the product ss'. We call a braid move short if s and s' commute.

Let $\preceq$ and $\preceq'$ be two reflection orderings, and let $\lhd$ and $\lhd'$ denote the covering relations of $\preceq$ and $\preceq'$, respectively. Denote by t_i (respectively, t'_i) the i-th reflection in the order $\preceq$ (respectively, $\preceq'$). We find a bijection ϕ from the set $I_{\preceq}$ of increasing paths of F w.r.t. $\preceq$ to the set $I_{\preceq'}$ of increasing paths of F w.r.t. $\preceq'$. By transitivity, we may suppose that the reduced expressions associated with $\preceq$ and $\preceq'$ differ by a braid move. Let $s_1 \cdots s_n$ be the reduced expression associated with $\preceq$.

First suppose that the reduced expression associated with $\preceq'$ is obtained from this expression by a short braid move, say swapping s_k and s_{k+1} with $s_k s_{k+1} = s_{k+1} s_k$ Let σ denote $s_1 \cdots s_{k-1}$. Then $t_i = t'_i$ for $i \notin \{k, k+1\}$, $t_k = \sigma s_k \sigma^{-1}$, $t'_k = \sigma s_{k+1} \sigma^{-1}$, $t_{k+1} = \sigma s_k s_{k+1} s_k \sigma^{-1} = t'_k$, $t'_{k+1} = \sigma s_{k+1} s_k s_{k+1} \sigma^{-1} = t_k$, i.e., $\preceq'$ is obtained from $\preceq$ by changing the order of t_k and t_{k+1} ($t_k \lhd t_{k+1}$ while $t_{k+1} \lhd t_k$). Note

$$t_k t_{k+1} = t_{k+1} t_k \tag{2}$$

Let $\Gamma \in I_{\preceq}$. If t_k and t_{k+1} are not both labels of Γ, then $\Gamma \in I_{\preceq'}$ and we set $\phi(\Gamma) = \Gamma$. Suppose t_k and t_{k+1} are both labels of Γ. Then, since $t_k \lhd t_{k+1}$ and Γ is increasing w.r.t. $\preceq$, the target of the edge labeled by t_k coincides with the source of the edge labelled by t_{k+1} (and clearly there are no other edges with labels in $\{t_k, t_{k+1}\}$). The flip of Γ in correspondence with these two edges gives a path not only in F, but also in $I_{\preceq'}$ by Lemma 2.4, Proposition 3.6, and (2). We set this path be $\phi(\Gamma)$.

Now suppose that the reduced expression associated with $\preceq'$ is obtained from the expression associated with $\preceq$ by a braid move that is not short, say by replacing $s_k s_{k+1} s_{k+2}$ with $s_{k+2} = s_k$ by $s_{k+1} s_k s_{k+1}$. For short, let $s = s_k$ (so $s = s_{k+2}$) and $r = s_{k+1}$. Let σ denote $s_1 \cdots s_{k-1}$. Then $t_i = t'_i$ for $i \notin \{k, k+2\}$, $t_k = \sigma s \sigma^{-1}$, $t'_k = \sigma r \sigma^{-1}$, $t_{k+2} = \sigma s r s r s \sigma^{-1} = \sigma r \sigma^{-1}$, $t'_{k+2} = \sigma r s r s r \sigma^{-1} = \sigma s \sigma^{-1}$, i.e. $\preceq'$ is obtained from $\preceq$ by swapping t_k and t_{k+2}. Indeed, $t_{k+1} = \sigma s r s \sigma^{-1} = \sigma r s r \sigma^{-1} = t'_{k+1}$. Note

$$t_k t_{k+1} t_k = t_{k+1} t_k t_{k+1} \tag{3}$$

Let $\Gamma \in I_{\preceq}$. If none of t_k, t_{k+1}, t_{k+2} are labels of Γ, then $\Gamma \in I_{\preceq'}$ and we set $\phi(\Gamma) = \Gamma$. If exactly one of t_k, t_{k+1}, t_{k+2} is a label of Γ, then again $\Gamma \in I_{\preceq'}$ and we set $\phi(\Gamma) = \Gamma$. If exactly two of t_k, t_{k+1}, t_{k+2} are labels of Γ, then, as above, the two edges must be consecutive in Γ. The flip of Γ in correspondence with these two edges gives a path in $I_{\preceq'}$ (by Lemma 2.4 and Proposition 3.6) that we set to be $\phi(\Gamma)$.

If exactly three of t_k, t_{k+1}, t_{k+2} are labels of Γ, then Γ has a subpath of the form $\left(\bullet \xrightarrow{t_k} \bullet \xrightarrow{t_{k+1}} \bullet \xrightarrow{t_{k+2}} \bullet\right)$. By flipping twice (in any order) in correspondence with the two

permutations in the middle (similarly to what happens in $\mathfrak{S}_3$), we obtain a path in $I_{\preceq'}$ by Proposition 3.6 and (3). We set this path to be $\phi(\Gamma)$.

The map ϕ so defined is an involution by Proposition 3.6. □

After Theorem 4.1, we can give the following definition.

Definition 4.2. Given a flipclass F, we let $c(F)$ be the number of increasing paths in F (w.r.t. any reflection ordering).

5. Some results on flipclasses

In this section, we provide some results on flipclassess, their support graphs, and their time-support graphs that are needed in what follows.

The following result is basically an adjustment of [5, Proposition 3.9] and [2, Proposition 6.3] to flipclasses. Its proof is therefore omitted.

Proposition 5.1. *Let $u, v \in \mathfrak{S}_n$ and $h \in \mathbb{N}$. Let F be an h-flipclass of paths from u to v. Then the number $c(F)$ satisfies*

$$1 \leq c(F) \leq \frac{|F|}{2^{h-1}}.$$

An increasing path of F w.r.t. a reflection ordering $\preceq$ is the lexicographically-first one.

Remark 5.2. We notice that also the colexicographically-last path of F is an increasing path (possibly coinciding with the lexicographically-first one).

Lemma 5.3. *Let $h \in \mathbb{N}$ and $u, v \in \mathfrak{S}_n$. Let F be an h-flipclass of paths from u to v. Each vertex (x, i) of TS_F with $i > 1$ has in-degree at least 2. Each vertex (x, i) of TS_F with $i < h - 1$ has out-degree at least 2.*

Proof. By upside down symmetry, we need to prove only the first assertion.

By the definition of the vertex set of TS_F, there exists a path in F passing through x at time i. Since $i > 1$, we can apply the $(i-1)$-th flip operator to this path and obtain another path passing through x at time i. The edges with target (x, i) in these two paths have different sources. □

We shall use several times the following property: we give it the name $P_{\bar{x},\bar{y}}$ to easily refer to it.

Lemma 5.4. *Let $h \in \mathbb{N}$ and $i \in [h-1]$. Let $u, v \in \mathfrak{S}_n$, with $u < v$. Let F be an h-flipclass of paths from u to v. The following property holds.*

($P_{\bar{x},\bar{y}}$) Given two vertices $\bar{x} = (x, i-1)$ and $\bar{y} = (y, i+1)$ of TS_F, the number of paths of length 2 from $\bar{x}$ to $\bar{y}$ in TS_F is 0, 1 or 2.

Proof. By definition, every edge of TS_F is effective. Hence a path $\bar{\Delta}$ of length 2 from $\bar{x}$ to $\bar{y}$ in TS_F determines a path Δ of length 2 from x to y in $B(\mathfrak{S}_n)$. Moreover, fix i; if (a, i) and (b, i) are distinct vertices of TS_F, then a and b are distinct element of $\mathfrak{S}_n$, so distinct paths $\bar{\Delta}$ and $\bar{\Delta}'$ determine distinct paths Δ and Δ'. Since the number of paths of length 2 from x to y in $B(\mathfrak{S}_n)$ is either 0 or 2, we have the assertion. □

Remark 5.5. We notice explicitely that we cannot exclude that there is only one path of length 2 from $\bar{x} = (x, i-1)$ to $\bar{y} = (y, i+1)$ in TS_F. Indeed, given such a path $\bar{\Delta}$, the two edges of it could (a priori) come from two different paths in F, so $\bar{\Delta}$ could be non-effective, and there could be no path in F we can apply the i-th flip to in order to obtain a second path of length 2 from $\bar{x}$ to $\bar{y}$ in TS_F.

Lemma 5.6. *Let $h \in \mathbb{N}$. Let $u, v \in \mathfrak{S}_n$, with $u < v$. Let F be an h-flipclass of paths from u to v. Suppose that, for all $i \in [h-1]$ and all $(a, i-1), (b, i+1) \in V(TS_F)$, the interval $[(a, i-1), (b, i+1)]$ in the poset induced by TS_F, if nonempty, is a diamond. Then, for $i \in [h-1]$, there is a natural concept of i-th flip operator also in $P_h^{TS_F}((u,0),(v,h)))$, and the map i_{TS_F} is equivariant with respect to the action of the flip operators.*

Proof. Let $p = \big((u = a_0, 0), (a_1, 1), \ldots, (a_{h-1}, h-1), (v = a_h, h)\big) \in P_h^{TS_F}((u,0),(v,h))$. Let $i \in [h-1]$. Then, the interval $[(a_{i-1}, i-1), (a_{i+1}, i+1)]$ in the poset induced by TS_F is nonempty. Hence, it is a diamond. The two middle elements are (a_i, i) and another element (a_i', i). The i-th flip of p is defined to be the path $p' = \big((u,0), \ldots, (a_{i-1}, i-1), (a_i', i), (a_{i+1}, i+1), \ldots, (v, h)\big)$. Suppose now that p comes from a path $\tilde{p} \in F$, i.e., $p = i_{TS_F}(\tilde{p})$. Then, clearly one has $\tilde{p} = (u, a_1, \ldots, a_{h-1}, v)$ and its i-th flip is $\tilde{p}' = (u, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, v)$. Thus one has $p' = i_{TS_F}(\tilde{p}')$. □

Lemma 5.7. *Let $h \in \mathbb{N}$. Let $u, v \in \mathfrak{S}_n$, with $u < v$. Let F be an h-flipclass of paths from u to v. TFAE.*

(1) The map i_{TS_F} is bijective, i.e. every path in TS_F is effective.
(2) Given $(a, i), (b, i+2) \in V(TS_F)$ with $(a, i) \leq (b, i+2)$, the interval $[(a, i), (b, i+2)]$ is a diamond. Moreover, the flip operators act transitively on $P_h^{TS_F}((u,0),(v,h))$.

Proof. Suppose (1). Let $(a, i), (b, i+2) \in V(TS_F)$ with $(a, i) \leq (b, i+2)$, and consider a path $\bar{\Delta}$ from $(u, 0)$ to (v, h) passing through both (a, i) and $(b, i+2)$. By hypothesis, there exists Δ in F such that $i_{TS_F}(\Delta) = \bar{\Delta}$. The $(i+1)$-th flip operator sends Δ to a path Δ' that also passes through a and b at time i and $i+2$, respectively. This implies that the path $i_{TS_F}(\Delta')$ passes through both (a, i) and $(b, i+2)$. Hence, by Lemma 5.4, the interval $[(a, i), (b, i+2)]$ is a diamond. Thus, by Lemma 5.6, the map i_{TS_F} is equivariant with respect to the action of the flip operators. Hence, transitivity of the flip group on F implies transitivity of the flip group on $P_h^{TS_F}((u,0),(v,h))$. So, (2) holds.

Vice versa, suppose (2). Recall that i_{TS_F} is always injective. By Lemma 5.6, the map i_{TS_F} is equivariant with respect to the action of the flip operators. Thus, transitivity of the flip group on $P_h^{TS_F}((u,0),(v,h))$ implies surjectivity of the map i_{TS_F}. Hence, (1) holds. □

The following result treats the case of h-flipclasses of intervals of length h.

Lemma 5.8. *Let $h \in \mathbb{N}$. Let $u, v \in \mathfrak{S}_n$, with $u < v$ and $\ell(v) - \ell(u) = h$. Then*

(1) there is only one h-flipclass F of paths from u to v;
(2) S_F and TS_F are both isomorphic to the Hasse diagram of the Bruhat interval $[u, v]$;
(3) all paths of S_F and TS_F are effective;
(4) c(F)=1;
(5) the t-vector $t_F = (t_0, \ldots, t_h)$ satisfies $\sum(-1)^i t_i = 0$.

Proof. A path from u to v of length h corresponds to a maximal chain in the Bruhat interval $[u, v]$. The fact that the order complex of a Bruhat interval is shellable (see [4]) implies

that every two paths from u to v of length h are flip-equivalent, hence (1) holds. Since $\ell(v) - \ell(u) = h$, every edge of S_F (and of TS_F) corresponds to a cover relation of Bruhat order since an edge of the form $x \longrightarrow y$, with $\ell(y) - \ell(x) > 1$, cannot be part of a maximal chain. Vice versa, every cover relation belongs to a maximal chain. Hence (2) holds. Property (3) follows by (1) and(2). Property (4) is implied by [17, Proposition 4.3]. Property (5) follows by the fact that Bruhat intervals are Eulerian (see [3, Corollary 2.7.11]). □

6. From infinite to finite

Fix h in $\mathbb{N}$. Consider the h-flipclasses of any $\mathfrak{S}_n$ (letting n vary). These are clearly infinite in number. The aim of this section is to show that, on the other hand, the number of isomorphism classes of (labelled) h-flipclasses is finite and that all such isomorphism classes can be constructed from flipclasses of $\mathfrak{S}_{h+1}$.

Let $n \in \mathbb{N}^+$ and $u, v \in \mathfrak{S}_n$, with $u < v$. Let $\Gamma = \big(u = x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{h-1}} x_{h-1} \xrightarrow{t_h} x_h = v\big)$ be a path from u to v in $B(\mathfrak{S}_n)$ of length h. We denote by $W(\Gamma)$ the reflection subgroup of $\mathfrak{S}_n$ generated by the transpositions $t_1, \ldots, t_h$. Moreover, we denote by $G(\Gamma)$ the edge-labeled undirected graph such that:

(1) $\{a :$ there exists i in $[h]$ such that $t_i(a) \neq a\}$ is its vertex set;
(2) there is an edge between a and b with label i if $t_i = (a, b)$.

Notice that $G(\Gamma)$ may have multiple edges.

Lemma 6.1. *Let Γ and Γ' be two paths in $B(\mathfrak{S}_n)$ in the same flipclass. Then:*

- *the groups $W(\Gamma)$ and $W(\Gamma')$ coincide;*
- *the vertex sets of $G(\Gamma)$ and $G(\Gamma')$ coincide;*
- *the number of connected components of $G(\Gamma)$ and $G(\Gamma')$ coincide.*

Proof. By transitivity, it is enough to prove the statements when Γ' is a flip of Γ, i.e., $\Gamma' = f_i(\Gamma)$ for some i in $[h-1]$. In this case, the assertions readily follow by Proposition 3.6. □

Definition 6.2. Let F be a flipclass.

- We denote by $W(F)$ the reflection subgroup generated by the transpositions labeling the edges of Γ, for any $\Gamma \in F$.
- We denote by $E(F)$ the vertex set of $G(\Gamma)$, for any Γ in F.
- We say that a flipclass F is reducible if $W(F)$ is a nontrivial direct product, i.e., $G(\Gamma)$ is not connected, for any $\Gamma \in F$. Otherwise, we say that F is irreducible.

Remark 6.3. (1) The group $W(F)$ is a direct product of symmetric groups.
(2) The partition of $E(F)$ given by the connected components of $G(\Gamma)$ does not depend on the choice of Γ in F.

Recall that, given a subset X of $\mathfrak{S}_n$, we denote by $B(X)$ the (directed) graph induced on X by the Bruhat graph $B(\mathfrak{S}_n)$.

Lemma 6.4. *Let W be a reflection subgroup of $\mathfrak{S}_n$. Let F be a flipclass of $\mathfrak{S}_n$ for which $W(F) = W$. Then*

- *each path in F is contained in a left coset of W,*
- *there is a flips and edge labeling preserving bijection between F and a flipclass F' of paths in $B(W)$.*

Proof. The first assertion is straightforward. Let C denote this left coset.

By [16, Theorem 1.4], there is an edge labeling preserving isomorphism between the graph $B(C)$ and the graph $B(W)$. This isomorphism clearly preserves the flips. Hence, the second assertion follows. □

The cartesian product of two edge-labelled directed graphs G_1 and G_2 is the edge-labelled directed graph $G_1 \times G_2$ satisfying $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(x_1, x_2) \xrightarrow{t} (y_1, y_2) : \ x_1 = y_1 \text{ and } x_2 \xrightarrow{t} y_2 \in E(G_2), \text{ or } x_1 \xrightarrow{t} y_1 \in E(G_1) \text{ and } x_2 = y_2\}$. Given a set of paths F_i in G_i, for $i \in \{1, 2\}$, we denote the set of paths in $G_1 \times G_2$ obtained by all possible shuffles of paths of F_1 and F_2 by $F_1 * F_2$.

Lemma 6.5. *Let W be a reflection subgroup of $\mathfrak{S}_n$. Suppose $W = W_1 \times W_2$ (direct product). Then the following hold.*

(1) $B(W) = B(W_1) \times B(W_2)$.
(2) The set of paths of $B(W)$ is the set of shuffles of the sets of paths of $B(W_1)$ and $B(W_2)$.
(3) Let Γ be a shuffle of Γ_1 and Γ_2:
  (a) $G(\Gamma)$ (as unlabelled graph) is the disjoint union of $G(\Gamma_1)$ and $G(\Gamma_2)$ (as unlabelled graphs) and the labels are determined by the shuffle;
  (b) any flip of Γ corresponds to either a flip of Γ_1, or a flip of Γ_2, or a change of a position in the shuffle.

Proof. The proof is easy and left to the reader. □

Given a path Γ in $B(W_1) \times B(W_2)$, denote by Γ_i the projection of Γ on $B(W_i)$, for $i \in \{1, 2\}$.

Lemma 6.6. *Let W be a reflection subgroup of $\mathfrak{S}_n$. Suppose $W = W_1 \times W_2$ (direct product). Let $u = (u_1, u_2), v = (v_1, v_2) \in W$, and F be a flipclass of paths from u to v. Let $F_i = \{\Gamma_i : \Gamma \in F\}$, for $i \in \{1, 2\}$. Then the following hold.*

(1) F_i is a flipclass in $B(W_i)$, for $i \in \{1, 2\}$;
(2) $F = F_1 * F_2$;
(3) $S_F = S_{F_1} \times S_{F_2}$;
(4) $TS_F = TS_{F_1} \times TS_{F_2}$;
(5) $\iota_F = \iota_{F_1} \cdot \iota_{F_2}$;
(6) $c(F) = c(F_1) \cdot c(F_2)$ (and analogously for the increasing paths of of S_F and TS_F w.r.t. any reflection ordering).

Proof. (1) follows by Lemma 6.5, (3b).
(2) follows by Lemma 6.5, (2) and Lemma 6.5,(3b).
(3) and (4) follows by (2) and the definitions of support graph and time-support graph.
(5) follows by (4).
(6) follows by (2), (3), and (4). □

Definition 6.7. Let E be a subset of $[n]$ of cardinality m. Denote by r_E the unique order preserving bijection from E to $[m]$, as well as the following maps that it induces.

(1) Given $w \in \mathfrak{S}_n$, we denote by $r_E(w)$ the permutation in $\mathfrak{S}_m$ whose one-line notation is obtained from the one-line notation of w by deleting the numbers not in E and applying r_E to the numbers in E.

(2) Let Γ be a path in $B(\mathfrak{S}_n)$ with labels of the type (a, b) with $a, b \in E$. We denote by $r_E(\Gamma)$ the path in $B(\mathfrak{S}_m)$ obtained from Γ by applying r_E to its nodes and labels.
(3) Given a total order on the set of transpositions in $\mathfrak{S}_n$, we denote by $\preceq_{r_E}$ the total order on the set of transpositions in $\mathfrak{S}_m$ defined by $(a, b) \preceq_{r_E} (c, d)$ if and only if $\big(r_E^{-1}(a), r_E^{-1}(b)\big) \preceq \big(r_E^{-1}(c), r_E^{-1}(d)\big)$.

Lemma 6.8. *Let $\preceq$ be a reflection ordering on $\mathfrak{S}_n$ and E be a subset of $[n]$ of cardinality m. The total order $\preceq_{r_E}$ on the set of the reflections of $\mathfrak{S}_m$ is a reflection ordering. Moreover, if $\preceq$ is the lexicographical order for $\mathfrak{S}_n$, then $\preceq_{r_E}$ is the lexicographical order for $\mathfrak{S}_m$.*

Proof. Since r_E is order preserving, condition (1) of the definition of a reflection order is satisfied by $\preceq_{r_E}$ if it is satisfied by $\preceq$. The second statement is straightforward. □

Lemma 6.9. *Let F be a flipclass. Denote $E(F)$ simply by E. Let $|E| = m$. The following hold.*

(1) The set $F' = \{r_E(\Gamma) : \Gamma \in F\}$ is a flipclass of $\mathfrak{S}_m$. The map r_E is a flips preserving bijection from F to F' satisfying $r_E(l_i(\Gamma)) = l_i(r_E(\Gamma))$ for all $\Gamma \in F$ (where $l_i(\Delta)$ denote the i-th label of the path Δ).
(2) The map r_E induces an isomorphism of edge-labelled directed graphs from S_F to $S_{F'}$ and an isomorphism of edge-labelled directed graphs from TS_F to $TS_{F'}$. The labels also are matched by r_E.
(3) Fix any reflection ordering $\preceq$ for $\mathfrak{S}_n$. The increasing paths of F, S_F, and TS_F (w.r.t. $\preceq$) correspond to the increasing paths of F', $S_{F'}$, and $TS_{F'}$ (w.r.t. $\preceq_{r_E}$), respectively.
(4) The map r_E induces an isomorphism from F to F'.

Proof. (1). The one-line notations of the permutations in a path in F have each number of $[n]\backslash E$ in the same position. By Proposition 3.6, the map r_E is flips preserving and hence F' is a flipclass. It is now straightforward that r_E is a bijection and satisfies $r_E(l_i(\Gamma)) = l_i(r_E(\Gamma))$ for all $\Gamma \in F$.

(2), (3). Straightforward.

(4). Follows by (1), (3), and Lemma 6.8. □

Proposition 6.10. *Let $h \in \mathbb{N}$. Let F be an h-flipclass of $\mathfrak{S}_n$. Let $E_1, \dots, E_s$ be the vertices of the connected components of $G(\Gamma)$, for any Γ in F, and let e_i be the cardinality of E_i, for $i \in [s]$. Then the following hold.*

(1) $W(F) = \mathfrak{S}_{E_1} \times \cdots \times \mathfrak{S}_{E_s}$;
(2) there exist (unique up to isomorphism and order) irreducible flipclasses F_i of $\mathfrak{S}_{e_i}$, for $i \in [s]$, such that

$$F \cong F_1 * \cdots * F_s.$$

(3) Denoting h_i the length of the paths in F_i, for $i \in [s]$, we have $\sum h_i = h$.

Proof. (1). If a subgroup of $\mathfrak{S}_n$ contains the transpositions (i, j) and (j, k), then it contains also (i, k) since $(i, k) = (i, j)(j, k)(i, j)$. Hence, for all $r \in [s]$, the subgroup $W(F)$ contains all transpositions (i, j) with $i, j \in E_r$, so $W(F)$ contains $\mathfrak{S}_{E_r}$. The assertion follows.

(2). By Lemma 6.9, the flipclass F is isomorphic to a flipclass in $\mathfrak{S}_{\sum e_i}$. The assertion follows by Lemma 6.6,

(3). This is clear. □

Remark 6.11. The isomorphism of Proposition 6.10 can be described explicitely.

Theorem 6.12. *Let $h \in \mathbb{N}$. Let F be an h-flipclass of $\mathfrak{S}_n$, and $e = |E(F)|$. Then there exists an h-flipclass F' of $\mathfrak{S}_e$ that is isomorphic to F. In particular, if F is irreducible, such F' may be found in $\mathfrak{S}_{h+1}$.*

Proof. Denote $E(F)$ simply by E. The group $W(F)$ is a subgroup of the subgroup $\mathfrak{S}_E$ of $\mathfrak{S}_n$. By Lemma 6.4, the flipclass F is isomorphic to some flipclass F' in the graph $B(W(F))$, which is a subgraph of $B(\mathfrak{S}_E)$. By applying the map r_E, we get the first assertion.

Suppose that F is irreducible. Then $e \leq h+1$ since a connected graph with h edges has at most $h+1$ vertices, and the second assertion follows. □

The following now readily follows.

Corollary 6.13. *Fix h in $\mathbb{N}$. There are finitely many isomorphism classes of h-flipclasses.*

Proof. For any h-flipclass F, we have the obvious bound $|E(F)| \leq 2h$. Hence, the result follows by Theorem 6.12. □

7. Combinatorial invariance of the coefficients of $\widetilde{R}$-polynomials

In this section, we propose a novel approach to tackle the problem of the Combinatorial Invariance of the coefficients of the $\widetilde{R}$-polynomials. This strategy allows us to prove the combinatorial invariance of the coefficient of q^h of the $\widetilde{R}$-polynomial $\widetilde{R}_{u,v}(q)$, for $h \leq 6$. For short, we denote this coefficient by $[q^h]\widetilde{R}_{u,v}(q)$.

This is achieved by showing that, given $h \leq 6$ and $u, v \in \mathfrak{S}_n$, with $u \leq v$, the multiset of time-support graphs of the h-flipclasses of paths from u to v determines $[q^h]\widetilde{R}_{u,v}(q)$. More precisely, the multiset of ι-polynomials determines the coefficient $[q^h]\widetilde{R}_{u,v}(q)$, and in fact, for $h \leq 4$, already the multiset of t-vectors suffices. Indeed, being the multiset of ι-polynomials of the h-flipclasses of paths from u to v a combinatorial invariant (see Remark 3.10), we get the combinatorial invariance of $[q^h]\widetilde{R}_{u,v}(q)$, for $h \leq 6$.

We note that, in theory, we provide explicit formulas for the coefficients $[q^h]\widetilde{R}_{u,v}(q)$, for $h \leq 6$, for any interval $[u, v]$ of any symmetric group $\mathfrak{S}_n$.

We now describe our strategy.

Let $u, v \in \mathfrak{S}_n$. By Theorem 2.5, the coefficient $[q^h]\widetilde{R}_{u,v}(q)$ is the number of paths from u to v of length h that are increasing w.r.t. to any reflection ordering. Hence, we have

$$[q^h]\widetilde{R}_{u,v}(q) = \sum_F c(F),$$

where F ranges over all h-flipclasses of paths from u to v. Proving that $c(F_1) = c(F_2)$ holds for all h-flipclasses F_1 and F_2 such that F_1^u and F_2^u are isomorphic shows the Combinatorial Invariance of the coefficient of q^h of the $\widetilde{R}$-polynomial.

The following result provides a general possible way to show the desired equality above. For $r \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{F}_r &= \{F : F \text{ is an } r\text{-flipclass in some } \mathfrak{S}_n\} \\ \mathcal{F}_{r,r+1} &= \{F : F \text{ is an } r\text{-flipclass in } \mathfrak{S}_{r+1}\} \end{aligned}$$

Definition 7.1. Let $h \in \mathbb{N}$. A function INV with domain $\mathcal{F}_h$ is a *h-combinatorial invariant* provided that $INV(F_1) = INV(F_2)$ for all h-flipclasses F_1 and F_2 such that F_1^u and F_2^u are isomorphic. Moreover, we say that a combinatorial invariant INV is *h-good* provided

that it detects the function c, i.e., $c(F_1) = c(F_2)$ for all h-flipclasses F_1 and F_2 satifying $INV(F_1) = INV(F_2)$.

Theorem 7.2. *Fix h in $\mathbb{N}$. Let INV be an r-combinatorial invariant for each r in $[h]$, and suppose that*

- *$INV(F * G)$ can be obtained as a function of $INV(F)$ and $INV(G)$, say $INV(F * G) = \mu(INV(F), INV(G))$.*

Let

$$\begin{aligned} \mathcal{A}c_r &= \{(INV(F), c(F)) : F \in \mathcal{F}_r\} \\ \mathcal{I}c_r &= \{(INV(F), c(F)) : F \in \mathcal{F}_{r,r+1}\}. \\ \mathcal{R}c_r &= \{\Big(\mu(i_1, i_2), c_1c_2\Big) : (i_1, c_1) \in \mathcal{A}c_j, (i_2, c_2) \in \mathcal{A}c_k, \text{ and } j + k = r\}. \end{aligned}$$

Then, $\mathcal{A}c_r$ is finite, and

$$\mathcal{A}c_r = \mathcal{I}c_r \cup \mathcal{R}c_r,$$

for each r in $[h]$.

Hence, letting

$$\begin{aligned} \mathcal{I}_h &= \{INV(F) : F \in \mathcal{F}_{h,h+1}\}. \\ \mathcal{R}_h &= \{\mu(i_1, i_2) : (i_1, c_1) \in \mathcal{A}c_j, (i_2, c_2) \in \mathcal{A}c_k, \text{ and } j + k = h\}, \end{aligned}$$

if the cardinality of $\mathcal{I}c_h \cup \mathcal{R}c_h$ is equal to the cardinality of $\mathcal{I}_h \cup \mathcal{R}_h$, then INV is an r-good invariant, for each $r \in [h]$.

Proof. Let $r \in [h]$. Let $F \in \mathcal{F}_r$. Suppose that F is irreducible. By Theorem 6.12, there exists $G \in \mathcal{F}_{r,r+1}$ that is isomorphic to F: in particular $INV(F) = INV(G)$ and $c(F) = c(G)$. So $(INV(F), c(F)) \in \mathcal{I}c_r$.

Suppose that F is reducible. Then, by Proposition 6.10, there exist $j, k \in \mathbb{N}^+$ with $j + k = h$, $G \in \mathcal{F}_j$, and $G' \in \mathcal{F}_k$ such that F is isomorphic to $G * G'$. In particular, $INV(F) = INV(G * G') = \mu(INV(G), INV(G'))$. Clearly, $(INV(G), c(G)) \in \mathcal{A}c_j$ and $(INV(G'), c(G')) \in \mathcal{A}c_k$. Moreover, by Lemma 6.6(6), we have $c(F) = c(G)c(G')$. So $(INV(F), c(F)) \in \mathcal{R}c_r$. By induction, we can show that $\mathcal{A}c_r$ is finite.

The second statement follows from the first since being r-good is equivalent to requiring that the projection of $\mathcal{A}c_r$ to the first factor is injective. □

The strategy of Theorem 7.2 is viable for all h in $\mathbb{N}$, but the complexity of the computations increases very rapidly: we have carried out the prescribed calculation by computer up to $h = 6$.

In fact, for $h \leq 5$, we present here also a different approach that allows us to prove more than what follows by the strategy. Given an h-flipclass F, we determine an explicit formula for $c(F)$ that depends only on the ι-polynomial. Actually, for $h \leq 4$, the formula for $c(F)$ depends only on the t-vector of F.

7.1. $h = 3$. The first unknown case is for $h = 3$ since $[q^0]\widetilde{R}_{u,v}(q)$ is 1 if $u = v$ and 0 if $u \neq v$, $[q^1]\widetilde{R}_{u,v}(q)$ is 1 if $u \to v \in E(B(\mathfrak{S}_n))$ and 0 otherwise, $[q^2]\widetilde{R}_{u,v}(q)$ is 1 if there is a path from u to v in $B(\mathfrak{S}_n)$ of length 2 (and, in fact, there are two such paths) and 0 otherwise.

Theorem 7.3. *Let $u, v \in \mathfrak{S}_n$, with $u < v$. The group $\mathfrak{F}_{3,u,v}$ is a dihedral group. Moreover, given a 3-flipclass F, the following hold.*

(1) *The unlabelled support graph S_F^u and the unlabelled time-support graph TS_F^u are both isomorphic to a k-crown, for some $k \in \{2,3,4,5\}$.*
(2) *k coincides with the order of the product f_2f_1 on F.*
(3) *The maps i_{S_F} and i_{TS_F} are bijections (i.e, all paths in S_F and TS_F are effective) and send the increasing paths to the increasing paths (w.r.t. any reflection ordering).*
(4) *The number $c(F)$ is equal to 1 when $k \in \{2,3,4\}$, to 2 when $k = 5$.*

Proof. Since $\mathfrak{F}_{3,u,v}$ is generated by two involutions, it is a dihedral group.

Let $\Gamma = \big(u \longrightarrow x_1 \longrightarrow x_2 \longrightarrow v\big)$ be a path in F. Let us construct F from Γ by alternately applying the flips f_1 and f_2 (i.e., we consider Γ, $f_1(\Gamma)$, $f_2f_1(\Gamma)$, $f_1f_2f_1(\Gamma)$, and so on). When in this process, the action of f_1 produces a new element, then also the subsequent action of f_2 produces a new element, since there are at most two paths of length 2 with the same starting element and ending element. When, in this process, the action of f_1 does not produce a new element, and so we necessarily come back to x_1 (again since there are at most two paths of length 2 with the same starting element and ending element), then with the subsequent action of f_2 we get back to Γ. This implies that S_F is a k-crown, where k is the order of the product f_2f_1 on F. Since two paths sharing the same vertex x pass through x at the same time, S_F^u and TS_F^u are isomorphic.

Moreover, the paths corresponding to the paths of F exhaust all paths in the k-crown, and increasing paths correspond to increasing paths.

It remains to prove $k \in \{2,3,4,5\}$ and (4).

If F is reducible, then $F = F_1 * F_2$ for some 1-flipclass F_1 and some 2-flipclass F_2, and S_F^u and TS_F^u are both isomorphic to a 3-crown. By Lemma 6.6(6), we have $c(F) = 1$ since $c(F_1) = 1$ and $c(F_2) = 1$.

Suppose F is irreducible. By Theorem 6.12, there exists a 3-flipclass G of $\mathfrak{S}_4$ that is isomorphic to F.

Recall that, for every element x in a Coxeter group, the number of reflections t such that $xt \leq x$ is equal to the length of x. Hence, as a general fact, we note that, if S_F^u is a k-crown, then $\ell(v) \geq k$ and $\ell(u) \leq |T| - k$. Since 6 is the length of the longest element w_0 of $\mathfrak{S}_4$, which is 4321, the graph S_F^u is a k-crown with $k \leq 6$. Actually, k cannot be 6 because the only pair of elements (u, v) with $\ell(v) \geq 6$ and $\ell(u) \leq |T| - 6 = 0$ is (e, w_0) and the interval between this two elements has length 6, thus even, and hence has no paths of length 3, since 3 is odd.

By Lemmas 2.2 and 5.8, we need to consider only those intervals $[u, v]$ with $\ell(v) - \ell(u) = 5$. So $[u, v] \in \{[e, 4312], [e, 4231], [e, 3421], [1243, w_0], [1324, w_0], [2134, w_0]\}$. The intervals $[e, 4231]$ and $[1324, w_0]$ are anti-isomorphic: each of them has two 3-flipclasses and each of these two flipclasses has unlabelled support and time-support graphs isomorphic to a 2-crown. By Proposition 5.1, each of them has only one increasing path.

Consider the remaining four intervals. Each of them has one 3-flipclass F, which contains two increasing paths. Moreover, in all these four cases, S_F^u and TS_F^u are both isomorphic to a 5-crown. □

Remark 7.4. Let F be a 3-flipclass of paths from u to v. Then

(1) if $|E(F)| \in \{5, 6\}$, then S_F^u and TS_F^u are both isomorphic to a 3-crown for all u and v,
(2) if $|E(F)| = 3$, then S_F^u and TS_F^u are both isomorphic to a 2-crown for all u and v,
(3) if $|E(F)| = 4$ and $\ell(v) - \ell(u) = 5$, then S_F^u and TS_F^u are both isomorphic to a 5-crown.

A further analysis completing all cases shows that, if $|E(F)| = 4$ and $\ell(v) - \ell(u) = 3$, then S_F is a 3-crown except in one case, namely when F corresponds to the flipclass of the path $\big(2143 \stackrel{(4,2)}{\longrightarrow} 4123 \stackrel{(3,2)}{\longrightarrow} 4132 \stackrel{(2,1)}{\longrightarrow} 4231\big)$ of $\mathfrak{S}_4$, which is a 4-crown.

Remark 7.5. Note that, for a general flipclass F of paths of length 3, the support and time-support graphs might be a 5-crown, while this is forbidden for intervals $[u, v]$ with $\ell(v) - \ell(u) = 3$ (see Lemma 2.2).

Corollary 7.6. *Let $u, v \in \mathfrak{S}_n$. The coefficient $[q^3]\widetilde{R}_{u,v}$ of q^3 in $\widetilde{R}_{u,v}$ is*

$$[q^3]\widetilde{R}_{u,v} = \sum_F c_{t_F},$$

where F ranges over all 3-flipclasses of paths from u to v, and c_{t_F} depends only on the t-vector $t_F = (t_0, t_1, t_2, t_3)$ of F:

$$c_{t_F} = \begin{cases} 2, & \text{if } t_1 = 5 \text{ (or, equivalently, } t_2 = 5) \\ 1, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.5, the coefficient of q^3 of $\widetilde{R}_{u,v}$ is the number of increasing paths of length 3 (with respect to any reflection ordering), which, by Theorem 7.3, is given by the displayed formula. □

7.2. $h = 4$**.** We need the followig result.

Proposition 7.7. *Let F be a 4-flipclass in $\mathfrak{S}_n$. Then S_F and TS_F are isomorphic.*

Proof. If F is reducible, then the assertion follows by Lemma 6.6 and Proposition 6.10, since S_G and TS_G are isomorphic for all h-flipclasses G with $h \leq 3$.

Suppose that F is irreducible. By Theorem 6.12, the flipclass F is isomorphic to a 4-flipclass G of $\mathfrak{S}_5$.

Let u (respectively, v) be the first (respectively, last) element of the paths in G. If $\ell(v)-\ell(u) = 4$, then the assertion follows by Lemma 5.8. Hence, we may suppose $\ell(v)-\ell(u) \geq 6$, since $\ell(v) - \ell(u)$ must be even. The group $\mathfrak{S}_5$ has 276 intervals $[u, v]$ with $\ell(v) - \ell(u)$ even and ≥ 6. We checked by computer calculations that the statement holds in all of these cases. □

Remark 7.8. For $h = 5$, there are flipclasses having support graph and time-support graph not isomorphic. For example, the paths $\Gamma = \big(e \longrightarrow 2 \longrightarrow 32 \longrightarrow 232 \longrightarrow 2342 \longrightarrow 234123121\big)$ and $\Delta = \big(e \longrightarrow 232 \longrightarrow 3423 \longrightarrow 34123 \longrightarrow 34123121 \longrightarrow 234123121\big)$ are in the same flipclass F. Since Γ and Δ pass through 232 at time 1 and at time 3, respectively, the support graph S_F and the time-support graph TS_F are not isomorphic.

The following result is needed in the proof of Theorem 7.10. It treats the flipclasses having t-vector equal to $(1, 3, 4, 3, 1)$. As we see in the proof of Theorem 7.10, when an interval of length 4 has more than one flipclass and one of them has more than one increasing path, all the other flipclasses have t-vector equal to $(1, 3, 4, 3, 1)$.

Proposition 7.9. *Let F be a 4-flipclass having t-vector equal to $t_F = (t_0, t_1, t_2, t_3, t_4) = (1, 3, 4, 3, 1)$. Then S_F^u and TS_F^u are both isomorphic to the directed graph in Figure 3. Every path of TS_F (or, equivalently, S_F) is effective. Furthermore, $c(F) = 1$.*

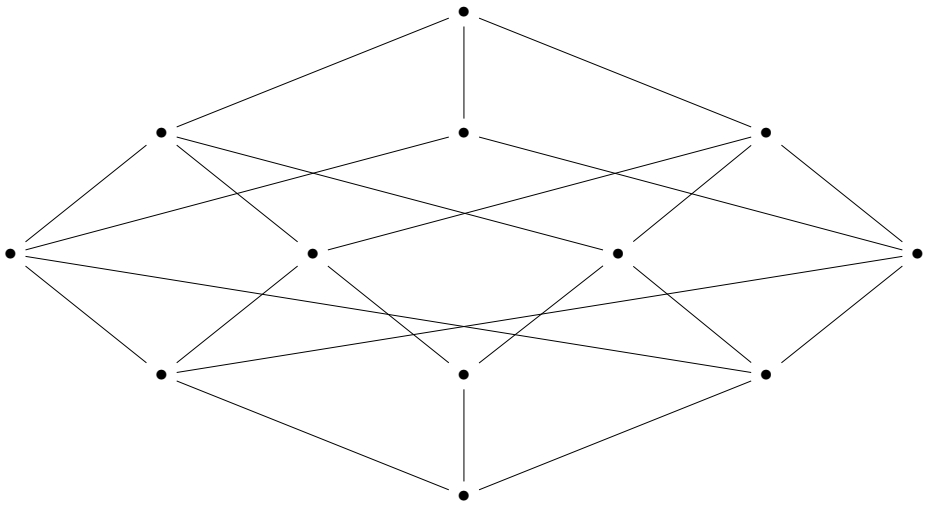


FIGURE 3. The time-support graph of a flipclass with t-vector $(1, 3, 4, 3, 1)$

Proof. Let u (respectively, v) be the first (respectively, last) element of the paths in F. Let $TS^u_F = (V, E)$ and $V = \{(u, 0), (a_i, 1)$ for $i \in \{1, 2, 3\}, (b_j, 2)$ for $j \in \{1, 2, 3, 4\}, (c_k, 3)$ for $k \in \{1, 2, 3\}, (v, 4)\}$. For short, we let $\bar{x} = (x, i)$.

We shall use several times the property $P_{\bar{x},\bar{y}}$ of Lemma 5.4.

Given $\bar{x}, \bar{y} \in V$, we denote by $A_{\bar{x},\bar{y}}$ the subgraph of TS_F given by the vertices and the edges of the paths of TS^u_F from $\bar{x}$ to $\bar{y}$. Equivalently, we can define $A_{\bar{x},\bar{y}}$ as the subgraph of TS^u_F given by the interval $[\bar{x}, \bar{y}]$ in the poset induced by TS_F.

We claim that $A_{\bar{u},\bar{c}_i}$, for $i \in \{1, 2, 3\}$, is either a 2-crown or a 3-crown. Let us prove the claim.

Let us study, say, $A_{\bar{u},\bar{c}_1}$. Fix a path from u to v in F that passes through the vertex c_1, say $\big(u \longrightarrow a_1 \longrightarrow b_1 \longrightarrow c_1 \longrightarrow v\big)$. Since every path in the 3-flipclass F' of the subpath $\big(u \longrightarrow a_1 \longrightarrow b_1 \longrightarrow c_1\big)$ can be extended to a path in F, we have that $A_{\bar{u},\bar{c}_1}$ contains $TS_{F'}$ as a subgraph. By Theorem 7.3, since $t_1 = 3$, the time-support graph $TS^u_{F'}$ is either a 2-crown or a 3-crown. A priori, $TS^u_{F'}$ does not coincide with $A_{\bar{u},\bar{c}_1}$: we show that actually it does.

Suppose first that $TS^u_{F'}$ is a 2-crown. We may suppose that its vertex set is $\{\bar{u}, \bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2, \bar{c}_1\}$, and so $\bar{u} \longrightarrow \bar{a}_i \in E$ for $i \in \{1, 2\}$, $\bar{a}_i \longrightarrow \bar{b}_j \in E$ for $i \in \{1, 2\}$ and $j \in \{1, 2\}$, $\bar{b}_i \longrightarrow \bar{c}_1 \in E$ for $i \in \{1, 2\}$. We have $\bar{b}_3 \longrightarrow \bar{c}_1 \notin E$. Indeed, if $\bar{b}_3 \longrightarrow \bar{c}_1 \in E$, then we could find a path in F of the form $\big(u \longrightarrow a_k \longrightarrow b_3 \longrightarrow c_1 \longrightarrow v\big)$ for some k in $\{1, 2, 3\}$. If $k \in \{1, 2\}$, then $(P_{\bar{a}_k,\bar{c}_1})$ would not be true. So $k = 3$ but, if we apply the 1-st flip operator to this path, we obtain a path $\big(u \longrightarrow a_k \longrightarrow b_3 \longrightarrow c_1 \longrightarrow v\big)$ for some k in $\{1, 2\}$, again contradicting $(P_{\bar{a}_k,\bar{c}_1})$. Similarly, $\bar{b}_4 \longrightarrow \bar{c}_1 \notin E$. Moreover, $a_3 \longrightarrow b_i \notin E$ by $(P_{\bar{u},\bar{b}_i})$, for $i \in \{1, 2\}$. Hence, $A_{\bar{u},\bar{c}_1}$ is a 2-crown.

Now suppose that $TS^u_{F'}$ is a 3-crown. We may suppose that its vertex set is the set $\{\bar{u}, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{c}_1\}$ and $\bar{a}_1 \longrightarrow \bar{b}_i \in E$ for $i \in \{1, 2\}$, $\bar{a}_2 \longrightarrow \bar{b}_i \in E$ for $i \in \{2, 3\}$, and $\bar{a}_3 \longrightarrow \bar{b}_i \in E$ for $i \in \{1, 3\}$. We also have $\bar{u} \longrightarrow \bar{a}_i \in E$ for $i \in \{1, 2, 3\}$ and $\bar{b}_i \longrightarrow \bar{c}_1 \in E$ for $i \in \{1, 2, 3\}$. We have $\bar{b}_4 \longrightarrow \bar{c}_1 \notin E$. Indeed, if $\bar{b}_4 \longrightarrow \bar{c}_1 \in E$, then $\bar{a}_i \longrightarrow \bar{b}_4 \notin E$ by $(P_{\bar{a}_i,\bar{c}_1})$ for $i \in \{1, 2, 3\}$, and this contradicts the fact that each vertex except $\bar{u}$ has target-valence at least 1. Moreover, $a_1 \longrightarrow b_3 \notin E$ by $(P_{\bar{a}_1,\bar{c}_1})$, $a_2 \longrightarrow b_1 \notin E$ by $(P_{\bar{a}_2,\bar{c}_1})$, and $a_3 \longrightarrow b_2 \notin E$ by $(P_{\bar{a}_3,\bar{c}_1})$. Hence, $A_{\bar{u},\bar{c}_1}$ is a 3-crown.

The claim is proved.

Let $i \in \{1,2,3,4\}$. The element $\bar{b}$ is the source of exactly two edges by Lemma 5.3 and $(P_{\bar{b}_i,\bar{v}})$. Since the unique partition of 8 as a sum of 2's and 3's is $8 = 3+3+2$, among the $A_{\bar{u},\bar{c}_i}$, for $i \in \{1,2,3\}$, there are two 3-crowns and one 2-crown.

Since the t-vector is symmetric, we can use an upside down argument to show that, among the $A_{\bar{a}_i,\bar{v}}$, for $i \in \{1,2,3\}$, there are two 3-crowns and one 2-crown.

We may suppose that $A_{\bar{u},\bar{c}_1}$ and $A_{\bar{u},\bar{c}_3}$ are 3-crowns and $A_{\bar{u},\bar{c}_2}$ is a 2-crown.

Let us consider the 2-crown $A_{\bar{u},\bar{c}_2}$. We may suppose that the vertex set of $A_{\bar{u},\bar{c}_2}$ is $\{\bar{u},\bar{a}_1,\bar{a}_3,\bar{b}_1,\bar{b}_4,\bar{c}_2\}$, and so $\bar{u} \longrightarrow \bar{a}_i \in E$ for $i \in \{1,3\}$, $\bar{a}_i \longrightarrow \bar{b}_j \in E$ for $i \in \{1,3\}$, $j \in \{1,4\}$, $\bar{b}_i \longrightarrow \bar{c}_2 \in E$ for $i \in \{1,4\}$, $\bar{a}_2 \longrightarrow \bar{b}_i \notin E$, for $i \in \{1,4\}$, and $\bar{b}_i \longrightarrow \bar{c}_2 \notin E$, for $i \in \{2,3\}$. Hence, $\bar{a}_2 \longrightarrow \bar{b}_i \in E$, for $i \in \{2,3\}$, by Lemma 5.3. By $(P_{\bar{u},\bar{b}_i})$, for $i \in \{2,3\}$ and Lemma 5.3, we may suppose $\bar{a}_1 \longrightarrow \bar{b}_2 \in E$, $\bar{a}_3 \longrightarrow \bar{b}_2 \notin E$, $\bar{a}_3 \longrightarrow \bar{b}_3 \in E$, and $\bar{a}_1 \longrightarrow \bar{b}_3 \notin E$.

By Lemma 5.3, $\bar{b}_i \longrightarrow \bar{c}_j \in E$, for $i \in \{2,3\}$ and $j \in \{1,3\}$.

By $(P_{\bar{b}_1,\bar{v}})$ and Lemma 5.3, we may suppose $\bar{b}_1 \longrightarrow \bar{c}_1 \in E$ and $\bar{b}_1 \longrightarrow \bar{c}_3 \notin E$. Since $A_{\bar{u},\bar{c}_1}$ is a 3-crown, we have $\bar{b}_4 \longrightarrow \bar{c}_1 \notin E$. Hence, By Lemma 5.3, $\bar{b}_4 \longrightarrow \bar{c}_3 \in E$.

We have described all edges of TS_F^u and obtained the graph in Figure 3. By Proposition 7.7, it coincides also with S_F^u. Hence the first statement holds.

Every path is effective by Lemma 5.7.

Fix a reflection ordering. It remains to show that there exists a unique increasing path from u to v.

Let Γ be the lexicographically-first path from u to v in the flipclass F, and let $\bar{\Gamma}$ be the corresponding path $\bar{u}$ to $\bar{v}$ in TS_F. By Proposition 5.1, the path Γ is increasing, and so is $\bar{\Gamma}$, since Γ and $\bar{\Gamma}$ have the same labels in the same order.

Let Δ be an increasing path from $\bar{u}$ to $\bar{v}$ in TS_F. We want to show $\Delta = \bar{\Gamma}$. Keep the notations as above. Suppose $\bar{c}_i \in \Delta$, for an index i in $\{1,3\}$. Then, by Proposition 5.1 and Theorem 7.3, the restriction of Δ from $\bar{u}$ to $\bar{c}_i$ is the lexicographically-first of the 3-crown $A_{\bar{u},\bar{c}_i}$. Then, the first edge of Δ coincides with the first edge of $\bar{\Gamma}$. Let $i \in \{1,2,3\}$ be such that this common edge is $\bar{u} \longrightarrow a_i$. The restrictions of Δ and $\bar{\Gamma}$ from $\bar{a}_i$ to $\bar{v}$ give two increasing paths of the 3-flipclass given by $A_{\bar{a}_i,\bar{v}}$, which is either a 2- or a 3-crown.These two restricted paths hence coincide, by Theorem 7.3. So $\Delta = \bar{\Gamma}$.

Suppose $\bar{c}_2 \in \Delta$. Then, $\bar{a}_i \in \Delta$ for an index i in $\{1,3\}$, the subgraph $A_{\bar{a}_i,\bar{v}}$ is a 3-crown and an upside down argument considering the colexicographically-last path shows $\Delta = \bar{\Gamma}$. □

Theorem 7.10. *Let F be a 4-flipclass. The number $c(F)$ depends only on the t-vector $t_F = (t_0,t_1,t_2,t_3,t_4)$ of F and is given by the following formula:*

$$c(F) = \begin{cases} 3, & \text{if } t_F \in \{(1,5,10,5,1),(1,8,14,8,1)\}, \\ 2, & \text{if } t_1 + t_3 \geq 12 \text{ and } t_F \neq (1,8,14,8,1) \\ 1, & \text{otherwise.} \end{cases}$$

Proof. If F is reducible, by Proposition 6.10 and Theorem 7.3, the graphs S_F^u and TS_F^u are both isomorphic to the cartesian product of either

(1) a segment and a k-crown for some k in $\{2,3,4,5\}$, or
(2) two diamonds.

Then, by Lemma 6.6(5)(6), the flipclass F satisfies $c(F) = 1$ and $t_1 + t_3 \leq 10$ except when TS_F is isomorphic to the cartesian product of a segment and a 5-crown, in which case $c(F) = 2$ and and the t-vector is equal to $(1, 6, 10, 6, 1)$.

Suppose that F is irreducible. By Theorem 6.12, the flipclass F is isomorphic to a 4-flipclass G of $\mathfrak{S}_5$.

Let u (respectively, v) be the first (respectively, last) element of the paths in G.

Suppose $\ell(v) - \ell(u) = 4$. By Lemma 5.8, one has $c(F) = 1$ and that the graph TS_F^u is given by the Hasse diagram of the Bruhat interval $[u, v]$. By [19, Theorem 3.6] for the symmetric group, the t-vector $(t_0, t_1, t_2, t_3, t_4)$ of F satisfies

$$(t_1, t_3) \in \{(3, 3), (3, 4), (4, 3), (4, 4), (5, 5), (6, 5), (5, 6)\}.$$

Hence $t_1 + t_3 < 12$; furthermore, also when $(t_1, t_3) = (5, 5)$, the t-vector t_F cannot be $(1, 5, 10, 5, 1)$ by Lemma 5.8(5).

Hence we may suppose $\ell(v) - \ell(u) \geq 6$, since $\ell(v) - \ell(u)$ must be even.

The group $\mathfrak{S}_5$ has 276 intervals $[u, v]$ with $\ell(v) - \ell(u)$ even and ≥ 6. For each of these intervals $[u, v]$, we asked the computer to determine:

- the coefficient $[q^4]\widetilde{R}_{u,v}$ (which coincides with the number of increasing paths $i_{u,v}$ from u to v of length 4);
- the number $n_{u,v}$ of 4-flipclasses of paths from u to v;
- for each of these flipclasses, its t-vector.

The computations show $i_{u,v} \leq 5$. If $i_{u,v} = 1$, then $n_{u,v} = 1$ and the t-vector of the only flipclass satisfies $t_1 + t_3 \leq 8$. If $i_{u,v} = 2$, then either $n_{u,v} = 1$ and the t-vector t of the only flipclass satisfies both $t_1 + t_3 \geq 12$ and $t \neq (1, 8, 14, 8, 1)$, or $n_{u,v} = 2$ and both flipclasses have t-vector satisfying $t_1 + t_3 \leq 7$. If $i_{u,v} = 3$, then either $n_{u,v} = 3$ and always all three flipclasses have $(1, 3, 4, 3, 1)$ as t-vector, or $n_{u,v} = 1$ and the t-vector of the only flipclass is either $(1, 5, 10, 5, 1)$ or $(1, 8, 14, 8, 1)$. If $i_{u,v} = 4$, then $n_{u,v} = 2$ and the two t-vectors are $(1, 3, 4, 3, 1)$ and $(1, 5, 10, 5, 1)$. If $i_{u,v} = 5$, then $n_{u,v} = 3$ and the three t-vectors are $(1, 3, 4, 3, 1)$ with multiplicity 2 and $(1, 5, 10, 5, 1)$.

The assertion follows since, by Proposition 7.9, the flipclasses with $(1, 3, 4, 3, 1)$ as t-vector contain exactly one increasing path and, hence, the flipclasses with t-vector $(1, 5, 10, 5, 1)$ contain exactly three increasing paths. □

Corollary 7.11. *Given $u, v \in \mathfrak{S}_n$, the coefficient $[q^4]\widetilde{R}_{u,v}$ of q^4 in $\widetilde{R}_{u,v}$ is*

$$[q^4]\widetilde{R}_{u,v} = \sum_F c_{t_F},$$

where F ranges over all orbits of $\mathfrak{F}_{4,u,v}$ and c_{t_F} depends only on the t-vector $t_F = (t_0, t_1, t_2, t_3, t_4)$ of F:

$$c_{t_F} = \begin{cases} 3, & \text{if } t_F \in \{(1, 5, 10, 5, 1), (1, 8, 14, 8, 1)\}, \\ 2, & \text{if } t_1 + t_3 \geq 12 \text{ and } t_F \neq (1, 8, 14, 8, 1) \\ 1, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.5, the coefficient of q^4 of $\widetilde{R}_{u,v}$ is the number of increasing paths of length 4 from u to v, which, by Theorem 7.10, is given by the displayed formula. □

7.3. $h = 5$**.** For $h = 5$, the t-vector is not a h-good invariant (i.e., there are flipclasses with same t-vector but different number of increasing paths), but the ι-polynomial is. Actually, the triple given by the t-vector, the sequence of the out-degrees of the vertices of the time-support graph of time 1, and the sequence of the in-degrees of the vertices of the time-support graph of time 4 is a 5-good invariant. This triple is encoded in the ι-polynomial since it is equivalent to $\big(\iota(1,1,t), \frac{\partial \iota}{\partial t}(1,y,0), \frac{1}{(4)!}\frac{\partial^4 \iota}{\partial t^4}(x,1,0)\big)$. For short, given a flipclass F, we let succ_F and prec_F denote, respectively, $\frac{\partial \iota_F}{\partial t}(1,y,0)$ and $\frac{1}{(4)!}\frac{\partial^4 \iota_F}{\partial t^4}(x,1,0)$.

For each $h \in \mathbb{N}$, denote by $tVec_h$ the set of vectors occurring as a t-vector of an h-flipclass F in the symmetric group $\mathfrak{S}_n$, for some n in $\mathbb{N}$. By Corollary 6.13, the set $tVec_h$ is finite, for all $h \in \mathbb{N}$. The following proposition is the outcome of computer calculations.

Proposition 7.12. *The set $tVec_5$ has cardinality 104 and is the disjoint union of the following sets C_i, for $i \in [7]$, and D:*

$$\begin{aligned}
C_1 &= \{(1,3,5,5,3,1),(1,3,5,6,4,1),(1,4,6,5,3,1),(1,4,7,7,4,1),(1,4,8,9,5,1),\\
&\quad (1,4,9,10,5,1),(1,4,10,12,6,1),(1,5,9,8,4,1),(1,5,10,9,4,1),(1,5,10,10,5,1),\\
&\quad (1,5,10,11,6,1),(1,6,11,10,5,1),(1,6,12,10,4,1),(1,6,13,13,6,1),(1,6,14,15,7,1),\\
&\quad (1,6,14,16,8,1),(1,6,15,18,9,1),(1,7,15,14,6,1),(1,7,17,17,7,1),(1,8,16,14,6,1),\\
&\quad (1,9,18,15,6,1)\}\\
C_2 &= \{(1,6,12,12,6,1),(1,6,13,14,7,1),(1,7,14,13,6,1),(1,7,16,16,7,1),(1,7,17,19,9,1),\\
&\quad (1,7,18,20,9,1),(1,8,19,19,8,1),(1,8,20,20,8,1),(1,8,20,21,9,1),(1,8,21,22,9,1),\\
&\quad (1,8,21,23,10,1),(1,8,21,24,11,1),(1,8,22,24,10,1),(1,8,22,25,11,1),(1,9,19,17,7,1),\\
&\quad (1,9,20,18,7,1),(1,9,21,20,8,1),(1,9,22,21,8,1),(1,9,23,23,9,1),(1,10,23,21,8,1),\\
&\quad (1,10,24,22,8,1),(1,11,24,21,8,1),(1,11,25,22,8,1)\}\\
C_3 &= \{(1,6,15,15,6,1),(1,6,17,19,7,1),(1,7,17,18,7,1),(1,7,18,17,7,1),(1,7,18,19,7,1),\\
&\quad (1,7,18,20,7,1),(1,7,19,17,6,1),(1,7,19,18,7,1),(1,7,19,21,8,1),(1,7,20,18,7,1),\\
&\quad (1,8,21,19,7,1),(1,9,22,22,9,1),(1,9,23,24,10,1),(1,9,24,26,11,1),(1,9,25,27,11,1),\\
&\quad (1,10,24,23,9,1),(1,10,25,25,10,1),(1,10,26,27,11,1),(1,10,27,27,10,1),\\
&\quad (1,10,27,28,11,1),(1,10,27,29,12,1),(1,10,28,29,11,1),(1,10,28,30,12,1),\\
&\quad (1,11,26,24,9,1),(1,11,27,25,9,1),(1,11,27,26,10,1),(1,11,28,27,10,1),\\
&\quad (1,11,29,28,10,1),(1,11,30,30,11,1),(1,12,29,27,10,1),(1,12,30,28,10,1)\}\\
C_4 &= \{(1,9,26,27,10,1),(1,9,27,29,11,1),(1,10,27,26,9,1),(1,10,28,28,10,1),\\
&\quad (1,10,29,30,11,1),(1,10,31,34,13,1),(1,11,28,28,11,1),(1,11,29,27,9,1),\\
&\quad (1,11,29,30,12,1),(1,11,30,29,10,1),(1,11,31,32,12,1),(1,12,30,29,11,1),\\
&\quad (1,12,32,31,11,1),(1,13,34,31,10,1)\}\\
C_5 &= \{(1,9,24,25,9,1),(1,9,25,24,9,1),(1,12,34,34,12,1),(1,12,35,36,13,1),\\
&\quad (1,13,36,35,12,1)\}\\
C_6 &= \{(1,8,27,28,9,1),(1,9,28,27,8,1)\}\\
C_7 &= \{(1,13,40,40,13,1)\}\\
D &= \{(1,11,31,31,11,1),(1,7,17,18,8,1),(1,9,24,25,10,1),(1,8,18,17,7,1),\\
&\quad (1,11,32,33,12,1),(1,10,25,24,9,1),(1,12,33,32,11,1)\}.
\end{aligned}$$

Definition 7.13. We define a function $\bar{c}$ from $\mathcal{F}_5$ to $\mathbb{N}$. The function $\bar{c}(F)$ depends only on the triple (t-vector $t_F, \mathrm{succ_F}(y), \mathrm{prec_F}(x)$); thus it depends only on the time-support graph, or, more precisely, on the ι-polynomial of F. (Actually, $\bar{c}(F)$ depends only on the t-vector except in 7 cases out of 104.)

Let F be a 5-flipclass. If $t_F \in C_i$, for $i \in [7]$, we let

$$\bar{c}(F) = i.$$

Otherwise, by Proposition 7.12, we have $t_F \in D$.

If $t_F = (1, 11, 31, 31, 11, 1)$, then

$$\bar{c}(F) = \begin{cases} 4, & \text{if } \mathrm{succ}_F \in \{4y^4 + 4y^6 + 2y^7 + y^8, 4y^4 + 3y^6 + 4y^7\}; \\ 5, & \text{if } \mathrm{succ}_F = 3y^4 + 7y^6 + y^8. \end{cases}$$

If $t_F = (1, 7, 17, 18, 8, 1)$, then

$$\bar{c}(F) = \begin{cases} 1, & \text{if } \mathrm{succ}_F = y^4 + 6y^5; \\ 2, & \text{if } \mathrm{succ}_F = 4y^4 + 3y^6. \end{cases}$$

If $t_F = (1, 9, 24, 25, 10, 1)$, then

$$\bar{c}(F) = \begin{cases} 2, & \text{if } \mathrm{succ}_F \in \{3y^4 + 4y^5 + 2y^8, 3y^4 + 2y^5 + 2y^6 + 2y^7, 2y^4 + 4y^5 + y^6 + 2y^7\}; \\ 3, & \text{if } \mathrm{succ}_F = 5y^4 + 2y^6 + 2y^8. \end{cases}$$

If $t_F = (1, 8, 18, 17, 7, 1)$, then

$$\bar{c}(F) = \begin{cases} 1, & \text{if } \mathrm{succ}_F = 4y^4 + 4y^5; \\ 2, & \text{if } \mathrm{succ}_F = 6y^4 + y^5 + y^7. \end{cases}$$

If $t_F = (1, 11, 32, 33, 12, 1)$, then

$$\bar{c}(F) = \begin{cases} 4, & \text{if } \mathrm{succ}_F = 3y^4 + 4y^6 + 4y^7; \\ 5, & \text{if } \mathrm{succ}_F \in \{3y^4 + 6y^6 + 2y^8, 2y^4 + 8y^6 + y^8\}. \end{cases}$$

If $t_F = (1, 10, 25, 24, 9, 1)$, then

$$\bar{c}(F) = \begin{cases} 2, & \text{if } \mathrm{succ}_F \in \{5y^4 + y^5 + 3y^6 + y^7, 2y^4 + 7y^5 + y^7, 2y^4 + 6y^5 + 2y^6\}; \\ 3, & \text{if } \mathrm{succ}_F = 6y^4 + y^5 + 3y^7. \end{cases}$$

If $t_F = (1, 12, 33, 32, 11, 1)$, then $\bar{c}(F)$ is either 4 or 5. The function succ does not distinguish the two cases, but its dual function prec does.

$$\bar{c}(F) = \begin{cases} 4, & \text{if } \mathrm{prec}_F = 3y^4 + 4y^6 + 4y^7; \\ 5, & \text{if } \mathrm{prec}_F \in \{3y^4 + 6y^6 + 2y^8, 2y^4 + 8y^6 + y^8\}. \end{cases}$$

The given values above exhaust all possibilities.

Theorem 7.14. *Let F be a 5-flipclass of $\mathfrak{S}_n$. The number $c(F)$ is given by the function $\bar{c}(F)$ as defined in Definition 7.13.*

Proof. Suppose that F is irreducible. By Theorem 6.12, the flipclass F is isomorphic to a 5-flipclass G of $\mathfrak{S}_6$. By computer analysis, we construct all 5-flipclasses in $\mathfrak{S}_6$ and, for each of them, its t-vector (see Proposition 7.12) and its number of increasing paths. We see that the number of increasing paths is determined by the t-vector except in 7 cases out of 104, when it is determined by the t-vector together with the functions succ and prec. This analysis leads to $c(F) = \bar{c}(F)$.

Suppose that F is reducible. Then, its t-vector belongs to t-Vec$_1$ $*$ t-Vec$_4$ or t-Vec$_2$ $*$ t-Vec$_3$ (convolution product) by Lemma 6.6(5), and its number of increasing paths is obtained by the product. We check that all possible convolutions belong to C_i and have i increasing paths except when one of the following two occur:

(1) $F = F_1 * F_2$ for a 1-flipclass F_1 and a 4-flipclass F_2 with $t_{F_2} = (1, 6, 11, 7, 1)$
(2) $F = F_1 * F_2$ for a 1-flipclass F_1 and a 4-flipclass F_2 with $t_{F_2} = (1, 7, 11, 6, 1)$

In the first case, $c(F) = 2$, $t_F = (1, 7, 17, 18, 8, 1)$ and $\mathrm{succ}_F = 4y^4 + 3y^6$. In the second case, $c(F) = 2$, $t_F = (1, 8, 18, 17, 7, 1)$ and $\mathrm{succ}_F = 6y^4 + y^5 + y^7$. Hence, $c(F) = \bar{c}(F)$. □

Corollary 7.15. *Given $u, v \in \mathfrak{S}_n$, the coefficient $[q^5]\widetilde{R}_{u,v}$ of q^5 in $\widetilde{R}_{u,v}$ is*

$$[q^5]\widetilde{R}_{u,v} = \sum_F \bar{c}(F),$$

where F ranges over all 5-flipclasses of paths from u to v and $\bar{c}(F)$ is defined in Definition 7.13.

Proof. By Theorem 2.5, the coefficient of q^5 of $\widetilde{R}_{u,v}$ is the number of increasing paths of length 5 from u to v, which, by Theorem 7.14, is given by the displayed formula. □

7.4. $h = 6$. To tackle the case $h = 6$, we use Theorem 7.2.

Theorem 7.16. *The ι-polynomial is an h-good invariant for each $h \leq 6$.*

Proof. We apply the strategy of Theorem 7.2. The ι-polynomial is an r-combinatorial invariant, for all $r \in \mathbb{N}$, and satifies $\iota_{F*G} = \iota_F \iota_G$.

Recall that $\mathcal{I}c_h$, $\mathcal{R}c_h$, $\mathcal{I}_h$, and $\mathcal{R}_h$ are finite sets. By computer analysis, we checked that the condition of Theorem 7.2 is satisfied. □

To give an idea of the entity of the computation, we note that $|\mathcal{F}_{6,7}| = 1701056$ and $|\mathcal{I}c_6| = 4515$. We also note that the computer calculations show $\mathcal{I}c_6 \supset \mathcal{R}c_6$.

Corollary 7.17. *Given $u, v \in \mathfrak{S}_n$, the coefficient $[q^6]\widetilde{R}_{u,v}$ of q^6 in $\widetilde{R}_{u,v}$ depends only on the ι-polynomials of the 6-flipclasses of paths from u to v.*

7.5. **Consequences.** The first and main consequence of Corollaries 7.6, 7.11, 7.15, and 7.17 is the following.

Theorem 7.18. *For any $n \in \mathbb{N}^+$ and for $h \leq 6$, the coefficient of q^h of the $\widetilde{R}$-polynomials of the symmetric group $\mathfrak{S}_n$ is a combinatorial invariant.*

As a byproduct, we obtain a new proof of the main result of [20] for the symmetric group.

Theorem 7.19. *For any $n \in \mathbb{N}^+$, the Combinatorial Invariance Conjecture holds for all intervals in the symmetric group $\mathfrak{S}_n$ up to length 8.*

Proof. Given $u, v \in \mathfrak{S}_n$, the polynomial $\widetilde{R}_{u,v}$ is monic, has degree $\ell(v) - \ell(u)$, and satisfies $[q^s]\widetilde{R}_{u,v} = 0$ whenever $s \not\equiv \ell(v) - \ell(u) \pmod 2$. Hence the result follows by the results in this section. □

8. A conjecture

We believe that the following conjecture might hold for all $h \in \mathbb{N}$. The results in the previous section settle it for $h \leq 6$.

Conjecture 8.1. *Let $h \in \mathbb{N}$. Let F and G be two h-flipclasses such that F^u and G^u are isomorphic. Then $c(F) = c(G)$.*

Conjecture 8.1 implies the Combinatorial Invariance Conjecture. To give an idea of the rate of growth, we note that the sequence of the number $|\mathcal{F}_{h,h+1}|$ of h-flipclasses of $\mathfrak{S}_{h+1}$ is $(4, 4, 50, 1096, 36634, 1701056)$, for $h = 1, \ldots, 6$.

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