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Hard interfaces with microstructure: the cases of strain gradient elasticity and micropolar elasticity

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As the size of a layered structure scales down, the adhesive layer thickness correspondingly decreases from macro to micro scale. The influence of the material microstructure of the adhesive becomes more pronounced, and possible size effect phenomena can appear. The paper describes the mechanical behavior of a composites made of two solids, bonded together by a thin layer, in the framework of strain gradient and micropolar elasticity. The adhesive layer is assumed to have the same stiffness properties of the adherents. By means of the asymptotic methods, the contact laws are derived at order 0 and order 1. These conditions represent a formal generalization of the hard elastic interface conditions. A simple benchmark equilibrium problem (a three-layer composite micro-bar subjected to an axial load) is developed to numerically assess the asymptotic model. Size effects and non-local phenomena, due to high strain concentrations at the edges, are highlighted. The example proves the efficiency of the proposed approach in designing micro-scale layered devices.

1. Introduction

Adhesive bonding offers numerous benefits, such as the ability to join various materials, enhance mechanical properties of assemblies, improve individual material characteristics, and meet specific strength and comfort needs.

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As a result, adhesive bonding technology is widely used in layered devices. When composites scale down in size, the influence of the material microstructure become more pronounced, and significantly affects the overall performance of the assemblies, as highlighted in various research papers (e.g., [1–3]). Thus, precise theoretical modeling of imperfect contact among the composite constituents is indispensable in engineering design. The adhesive layer is classically treated as an interphase with decreasing thickness. By letting the thickness tend to zero, the interphase reduces to a two-dimensional surface, called the imperfect interface, on which ad-hoc jump conditions of selected physical fields are considered, e.g. displacements and stresses. During the last decades, numerous interface models have emerged based on classical continuum mechanics. However, these theories are unable to accurately describe the mechanical behavior of materials with microstructure, such as molecules, grains, fibers, or pores, and also size-dependent phenomena. The effect of microstructure and size effects can be adequately taken into account with generalized continuum theories such as strain gradient elasticity and micropolar elasticity. The first one has been developed in the pioneering works by Mindlin [4,5] and, more recently, in [6–12]. These theories have been extended and generalized in several works by Polizzotto [13–17]. For a detailed review on gradient elasticity theories, the reader can refer to [18]. The second one establishes its roots in the milestone work of the Cosserat brothers [19], and various extensions have been explored in several books and papers, including those authored by Eringen [20,21] and Nowacki [22], as well as the overview analysis provided by Maugin [23]. These models are commonly used to characterize complex media, especially composite materials with internal microstructures. Examples include fibrous and granular materials, liquid crystals, cellular solids and foams, high-performance and glassy polymers, lattices, and masonry, among others [24–28]. In recent years, significant progress has been made in the theoretical understanding of thin adhesive modeling, with the development of imperfect interface laws using advanced mathematical techniques, such as the asymptotic analysis. These include linear elasticity [29–37], coupled thermoelasticity [38], piezoelectricity [39,40], magneto-electro-thermo-elasticity [41], poroelasticity [42] and general multiphysics theories [43,44]. Only a limited number of studies focus on investigating the impact of microstructure and size effects of micro-scale adhesives, particularly when considering higher order continua. For instance, [45,46] provide an initial contribution to the understanding of the mechanical behavior of a Cosserat bonded joint through the asymptotic analysis, by analysing a composite comprising two micropolar solids joined by a thin soft and rigid micropolar intermediate layer. In [47,48] the authors developed models for soft imperfect interfaces through the asymptotic methods for flexoelectricity and strain gradient elasticity, highlighting the size effect properties. The work [49] introduced a size-dependent model for adhesively bonded layered structures within the context of strain gradient elasticity, utilizing standard tools. Consequently, there remains a lack of theoretical modeling that adequately tackles the influence of microstructure and the size effects, particularly evident in micro-adhesive layers.

The goal of the present work is to identify a novel form of the contact laws, in the framework of strain gradient elasticity and micropolar elasticity, by means of an asymptotic analysis. The composite is constituted of a thin plate-like layer (adhesive), whose thickness depends on a small parameter ε, surrounded by two generic bodies (adherents). We analyze the case of a hard interface, where the material coefficients of both adhesive and adherents are of the same order of magnitude and are independent of ε. Following the asymptotic approach developed in [43], it is possible to define a novel form of hard interface at order 0 and order 1, within the context of strain gradient elasticity and micropolar elasticity.

The outline of the paper is the following. Section 2 presents the hard interface conditions in strain gradient elasticity. More precisely, the theoretical framework of the strain gradient elastic theory is presented, then the rescaling and the asymptotic expansions method are applied to the variational formulation of the problem; finally, the interface conditions at order 0 and 1 are identified in terms of the jumps of the displacements and microrotation, and their conjugated counterparts, namely the stress and couple stress vectors. Section 3 presents the hard interface conditions in micropolar elasticity. The same structure of Section 2 is maintained. In Section 4, we present a benchmark problem of a layered bar subjected to an axial load. Section 5 is devoted to the concluding remarks.

2. Hard interface conditions in strain gradient elasticity

(a) Theoretical settings

In the sequel, Greek indices range in the set $\{1,2\}$, Latin indices range in the set $\{1,2,3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted. Let us consider a three-dimensional Euclidean space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. We introduce the following notations for the scalar and vector products: $\mathbf{a}\cdot\mathbf{b} := a_i b_i$ and $\mathbf{a}\wedge\mathbf{b} = a_i\mathbf{e}_i \wedge b_j\mathbf{e}_j := a_i b_j \epsilon_{ijk}\mathbf{e}_k$, for all vectors $\mathbf{a}=(a_i)$ and $\mathbf{b}=(b_i)$ in $\mathbb{R}^3$, with ϵ_{ijk} the alternator Ricci's symbol.

Let us consider a material body Ω made of a strain gradient elastic material, whose constitutive law is defined as follows, [6–8]: $\sigma_{ij} = c_{ijhl}e_{hl}$, $\tau_{ijk} = \ell^2 c_{ijhl}\eta_{hlk} = \ell^2\sigma_{ij,k}$, where (σ_{ij}) and (τ_{ijk}) denote the Cauchy's stress tensor and the double-stress tensor, respectively, (c_{ijkl}) represents the elasticity tensor, ℓ is the internal length scale parameter, and $e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$, $\eta_{ijk} = e_{ij,k} = \frac{1}{2}(u_{i,jk}+u_{j,ik})$, with u_i the displacement field. We note with $\tilde{\sigma}_{ij} := \sigma_{ij} - \tau_{ijk,k} = c_{ijkl}e_{kl} - \ell^2 c_{ijhl}e_{hl,kk}$ the so-called total stress tensor. The body is subjected to body forces f_i, acting in Ω, and to surface forces g_i, surface double forces q_i, applied to the boundary $\Gamma_1 \subset \partial\Omega$. No edge forces are considered. The body is mechanically clamped on Γ_0. The work of external sources is given by

$$L(\mathbf{u}) := \int_\Omega f_i u_i dx + \int_{\Gamma_1} \{g_i u_i + q_i \partial_n u_i\} d\Gamma.$$

The variational formulation of the problem takes the following form:

$$\begin{cases} \text{Find } \mathbf{v} \in \mathcal{V}(\Omega) := \{\mathbf{v} \in H^2(\Omega,\mathbb{R}^3):\ \mathbf{v}=\mathbf{0},\ \partial_n\mathbf{v}=\mathbf{0} \text{ on } \Gamma_0\},\ \text{such that} \\ A(\mathbf{u},\mathbf{v}) = L(\mathbf{v}),\ \text{for all } \mathbf{v}\in\mathcal{V}(\Omega), \end{cases} \tag{2.1}$$

with

$$A(\mathbf{u},\mathbf{v}) := \int_\Omega \{\sigma_{ij}e_{ij}(\mathbf{v}) + \tau_{ijk}\eta_{ijk}(\mathbf{v})\}\, dx = \int_\Omega \left\{c_{ijhl}e_{hl}(\mathbf{u})e_{ij}(\mathbf{v}) + \ell^2 c_{ijhl}e_{hl,k}(\mathbf{u})e_{ij,k}(\mathbf{v})\right\} dx.$$

By virtue of Gauss-Green's theorem, the differential form of the governing equations can be easily derived:

$$\begin{aligned} &-\int_\Omega \tilde{\sigma}_{ij,j}v_i dx + \int_{\partial\Omega} \left\{\tilde{\sigma}_{ij}n_j + \mathcal{D}_l^t(n_l)n_k n_j\tau_{ijk} - \mathcal{D}_j^t(n_k\tau_{ijk})\right\} v_i d\Gamma \\ &+ \int_{\partial\Omega} \tau_{ijk}n_j n_k \partial_n v_i d\Gamma + \sum_m \oint_{C_m} \Delta(n_i k_j \tau_{ijk}) v_k ds = L(\mathbf{v}), \end{aligned}$$

where (n_i) denotes the outer unit normal vector to the boundary $\partial\Omega$, $\mathcal{D}_i^t(\cdot) := (\delta_{ij} - n_i n_j)(\cdot)_{,j}$ and $\partial_n(\cdot) := n_i(\cdot)_{,i}$ are the tangential and normal derivative operators on the boundary, respectively, (k_j) is the surface unit vector normal to the m-th sharp edge C_m of the boundary, and Δ represents the difference between the bracketed terms on the two sides of the edge. Thus, the equilibrium problem takes the following expression:

$$\begin{cases} \tilde{\sigma}_{ij,j} + f_i = 0, & \text{in } \Omega, \\ T_i := \tilde{\sigma}_{ij}n_j + \mathcal{D}_l^t(n_l)n_k n_j \tau_{ijk} - \mathcal{D}_j^t(n_k\tau_{ijk}) = g_i & \text{on } \Gamma_1, \\ R_i := \tau_{ijk}n_j n_k = q_i & \text{on } \Gamma_1, \\ P_i := \Delta(n_k k_j \tau_{kji}) = 0 & \text{on } C_m, \\ u_i = 0,\ \ \partial_n u_i = 0, & \text{on } \Gamma_0. \end{cases}$$

T_i, R_i and P_i are, respectively, the traction vector, the higher-order traction vector on Γ_1, and the line traction vector on the sharp edge C_m.

(b) Statement of the problem

Let us define a small parameter $0<\varepsilon<1$. We consider a composite structure made of two disjoint solids $\Omega_\pm^\varepsilon\subset\mathbb{R}^3$, bonded together by an intermediate thin layer $B^\varepsilon:=S\times(-\frac{h^\varepsilon}{2},\frac{h^\varepsilon}{2})$ of thickness $h^\varepsilon=\varepsilon h$, with cross-section $S\subset\mathbb{R}^2$. Let $S_\pm^\varepsilon$ be the planar interfaces between the interphase and the adherents and let $\Omega^\varepsilon:=\Omega_+^\varepsilon\cup B^\varepsilon\cup\Omega_-^\varepsilon$ denote the reference configuration of the composite, including the interphase and the adherents, (cf. Figure 1a). We note with $\gamma_\pm^\varepsilon=\partial S_\pm^\varepsilon$ the sharp edges, along the boundary of the surfaces $S_\pm^\varepsilon$. The equilibrium problem can be written in its

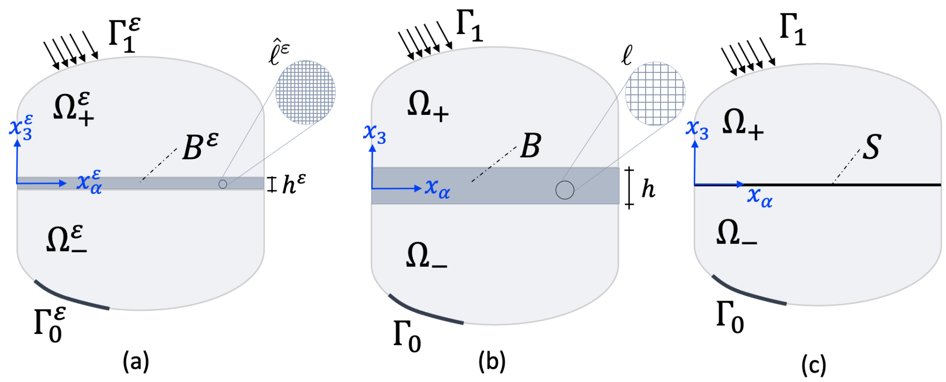

Figure 1. Initial (a), rescaled (b) and limit (c) configurations of the composite.

variational form as follows:

$$\left\{\begin{array}{l}\text{Find } \mathbf{u}^\varepsilon\in V(\Omega^\varepsilon), \ \text{ such that}\\ \bar{A}_-^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon)+\bar{A}_+^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon)+\hat{A}^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon)=L^\varepsilon(\mathbf{v}^\varepsilon), \ \text{for all } \mathbf{v}^\varepsilon\in V(\Omega^\varepsilon),\end{array}\right. \tag{2.2}$$

where $V(\Omega^\varepsilon):=\{\bar{\mathbf{v}}^\varepsilon\in H^2(\Omega_\pm^\varepsilon,\mathbb{R}^3),\ \hat{\mathbf{v}}^\varepsilon\in H^2(B^\varepsilon,\mathbb{R}^3):\ \bar{\mathbf{v}}^\varepsilon|_{S_\pm^\varepsilon}=\hat{\mathbf{v}}^\varepsilon|_{S_\pm^\varepsilon},\ \bar{\mathbf{v}}^\varepsilon_{,3}|_{S_\pm^\varepsilon}=\hat{\mathbf{v}}^\varepsilon_{,3}|_{S_\pm^\varepsilon},\ \bar{\mathbf{v}}^\varepsilon=\partial_n^\varepsilon\bar{\mathbf{v}}^\varepsilon=\mathbf{0}$ on $\Gamma_0^\varepsilon\}$, and $\bar{A}_\pm^\varepsilon(\cdot,\cdot)$ and $\hat{A}^\varepsilon(\cdot,\cdot)$ are the strain gradient elasticity bilinear forms on $\Omega_\pm^\varepsilon$ and B^ε, respectively:

$$\bar{A}_\pm^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon):=\int_{\Omega_\pm^\varepsilon}\left\{\bar{c}_{ijhl}^\varepsilon e_{hl}^\varepsilon(\mathbf{u}^\varepsilon)e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)+\bar{\ell}^{\varepsilon 2}\bar{c}_{ijhl}^\varepsilon e_{hl,k}^\varepsilon(\mathbf{u}^\varepsilon)e_{ij,k}^\varepsilon(\mathbf{v}^\varepsilon)\right\}dx^\varepsilon,$$

$$\hat{A}^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon):=\int_{B^\varepsilon}\left\{\hat{c}_{ijhl}^\varepsilon e_{hl}^\varepsilon(\mathbf{u}^\varepsilon)e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)+\hat{\ell}^{\varepsilon 2}\hat{c}_{ijhl}^\varepsilon e_{hl,k}^\varepsilon(\mathbf{u}^\varepsilon)e_{ij,k}^\varepsilon(\mathbf{v}^\varepsilon)\right\}dx^\varepsilon.$$

Only if necessary, $\bar{\mathbf{u}}^\varepsilon$ and $\hat{\mathbf{u}}^\varepsilon$ will note the restrictions of functions $\mathbf{u}^\varepsilon$ to $\Omega_\pm^\varepsilon$ and B^ε, respectively.

(c) Rescaling

In order to apply the asymptotic expansions method [50], problem (2.2), defined on an ε-dependent domain, must be reformulated on a fixed domain, independent of ε. We consider the coordinate transformation $\pi^\varepsilon: x\in\overline{\Omega}\mapsto x^\varepsilon\in\overline{\Omega}^\varepsilon$ given by

$$\pi^\varepsilon:\left\{\begin{array}{ll}\bar{\pi}^\varepsilon(x_1,x_2,x_3)=(x_1,x_2,x_3\mp\frac{h}{2}(1-\varepsilon)), & \text{for all } x\in\overline{\Omega}_\pm,\\ \hat{\pi}^\varepsilon(x_1,x_2,x_3)=(x_1,x_2,\varepsilon x_3), & \text{for all } x\in\overline{B},\end{array}\right. \tag{2.3}$$

where, after the change of variables, the adherents occupy $\Omega_\pm:=\Omega_\pm^\varepsilon\pm\frac{h}{2}(1-\varepsilon)\mathbf{e}_3$ and the interphase $B=\{x\in\mathbb{R}^3:\ (x_1,x_2)\in S,\ |x_3|<\frac{h}{2}\}$. The sets $S_\pm=\{x\in\mathbb{R}^3:\ (x_1,x_2)\in S,\ x_3=\pm\frac{h}{2}\}$ denote the interfaces between B and $\Omega_\pm$, with $\gamma_\pm=\partial S_\pm$ their sharp edges, and $\Omega=\Omega_+\cup\Omega_-\cup B$ is the rescaled configuration of the composite, see Fig. 1b. Lastly, Γ_0 and Γ_1 indicate the images through π^{ε} of Γ_0^{ε} and Γ_1^{ε} (cf. Figure 1.b). Consequently, $\frac{\partial}{\partial x_\alpha^\varepsilon} = \frac{\partial}{\partial x_\alpha}$ and $\frac{\partial}{\partial x_3^\varepsilon} = \frac{\partial}{\partial x_3}$ in $\Omega_\pm$, and $\frac{\partial}{\partial x_\alpha^\varepsilon} = \frac{\partial}{\partial x_\alpha}$ and $\frac{\partial}{\partial x_3^\varepsilon} = \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}$ in B.

In order to derive a hard interface law, the adherents $\Omega_\pm^\varepsilon$ and the adhesive B^ε present the same stiffness properties. Thus, the elastic coefficients are assumed to be independent of ε, namely $\bar{c}_{ijkl}^\varepsilon = \bar{c}_{ijkl}$ and $\hat{c}_{ijkl}^\varepsilon = \hat{c}_{ijkl}$. The internal length scale parameter measures the size of the microstructure. In this case, the internal length of the adherents is considered to be independent of ε, $\bar{\ell}^\varepsilon = \bar{\ell}$. The characteristic length of the adhesive is supposed to be of the same order of magnitude of its thickness ($h^\varepsilon = \varepsilon h$) and, thus it linearly depends on ε, so that $\hat{\ell}^\varepsilon = \varepsilon \ell$. The external loads present the following dependence with respect to ε: $f_i^\varepsilon(x^\varepsilon) = f_i(x)$, $x \in \Omega_\pm$, and $g_i^\varepsilon(x^\varepsilon) = g_i(x)$, $q_i^\varepsilon(x^\varepsilon) = q_i(x)$, $x \in \Gamma_1$. Thus, $L^\varepsilon(\mathbf{v}^\varepsilon) = L(\mathbf{v})$. The displacements and test functions depend on ε as follows, see [48,51]:

$$\begin{aligned}
&\bar{\mathbf{u}}^\varepsilon(x^\varepsilon) = \bar{\mathbf{u}}^\varepsilon(x), && \bar{\mathbf{v}}^\varepsilon(x^\varepsilon) = \bar{\mathbf{v}}(x) && x \in \Omega_\pm,\\
&\hat{\mathbf{u}}^\varepsilon(x^\varepsilon) = \mathbf{w}^\varepsilon(x) + \varepsilon \boldsymbol{\phi}^\varepsilon(x), && \hat{\mathbf{v}}^\varepsilon(x^\varepsilon) = \mathbf{s}(x) + \varepsilon \boldsymbol{\psi}(x) && x \in B,
\end{aligned}$$

with $\mathbf{w}^\varepsilon|_{S_\pm} = \bar{\mathbf{u}}^\varepsilon|_{S_\pm}$, $\mathbf{w}^\varepsilon_{,3}|_{S_\pm} = \mathbf{0}$, $\mathbf{s}|_{S_\pm} = \bar{\mathbf{v}}|_{S_\pm}$, $\mathbf{s}_{,3}|_{S_\pm} = \mathbf{0}$, $\boldsymbol{\phi}^\varepsilon_{,3}|_{S_\pm} = \bar{\mathbf{u}}^\varepsilon_{,3}|_{S_\pm}$, $\boldsymbol{\phi}^\varepsilon|_{S_\pm} = \mathbf{0}$, $\boldsymbol{\psi}_{,3}|_{S_\pm} = \bar{\mathbf{v}}_{,3}|_{S_\pm}$, and $\boldsymbol{\psi}|_{S_\pm} = \mathbf{0}$.

According to the previous rescalings, problem (2.2) can be rewritten on a fixed domain Ω, independent of ε:

$$\left\{\begin{array}{l}
\text{Find } \mathbf{u}^\varepsilon = (\bar{\mathbf{u}}^\varepsilon, \hat{\mathbf{u}}^\varepsilon = \mathbf{w}^\varepsilon + \varepsilon\boldsymbol{\phi}^\varepsilon) \in V(\Omega), \text{ such that}\\
\bar{A}_-(\mathbf{u}^\varepsilon, \mathbf{v}) + \bar{A}_+(\mathbf{u}^\varepsilon, \mathbf{v}) + \hat{A}(\mathbf{u}^\varepsilon, \mathbf{v}) = L(\mathbf{v}),\\
\text{for all } \mathbf{v} = (\bar{\mathbf{v}}, \hat{\mathbf{v}}^\varepsilon = \mathbf{s} + \varepsilon\boldsymbol{\psi}) \in V(\Omega),
\end{array}\right. \tag{2.4}$$

where $V(\Omega) := \{\bar{\mathbf{v}} \in H^2(\Omega_\pm, \mathbb{R}^3),\ \hat{\mathbf{v}} \in H^2(B, \mathbb{R}^3) :\ \bar{\mathbf{v}}|_{S_\pm} = \mathbf{s}|_{S_\pm},\ \mathbf{s}_{,3}|_{S_\pm} = \mathbf{0},\ \bar{\mathbf{v}}_{,3}|_{S_\pm} = \boldsymbol{\psi}_{,3}|_{S_\pm},\ \boldsymbol{\psi}|_{S_\pm} = \mathbf{0},\ \bar{\mathbf{v}} = \partial_n \bar{\mathbf{v}} = \mathbf{0} \text{ on } \Gamma_0\}$, and

$$\begin{aligned}
\bar{A}_\pm(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \int_{\Omega_\pm} \left\{\bar{c}_{ijhl} e_{hl}(\mathbf{u}^\varepsilon) e_{ij}(\mathbf{v}) + \bar{\ell}^2 \bar{c}_{ijhl} e_{hl,k}(\mathbf{u}^\varepsilon) e_{ij,k}(\mathbf{v})\right\} dx,\\
\hat{A}(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \tfrac{1}{\varepsilon} a_{-1}(\mathbf{u}^\varepsilon, \mathbf{v}) + a_0(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon a_1(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^2 a_2(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 a_3(\mathbf{u}^\varepsilon, \mathbf{v}),
\end{aligned}$$

with

$$\begin{aligned}
a_{-1}(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \int_B \left\{\hat{\mathbf{K}}_{33}\mathbf{u}^\varepsilon_{,3} \cdot \mathbf{v}_{,3} + \ell^2 \hat{\mathbf{K}}_{33}\mathbf{u}^\varepsilon_{,33} \cdot \mathbf{v}_{,33}\right\} dx,\\
a_0(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \int_B \left\{\hat{\mathbf{K}}_{\alpha 3}\mathbf{u}^\varepsilon_{,\alpha} \cdot \mathbf{v}_{,3} + \hat{\mathbf{K}}_{3\alpha}\mathbf{u}^\varepsilon_{,3} \cdot \mathbf{v}_{,\alpha} + \ell^2(\hat{\mathbf{K}}_{\alpha 3}\mathbf{u}^\varepsilon_{,\alpha 3} \cdot \mathbf{v}_{,33} + \hat{\mathbf{K}}_{3\alpha}\mathbf{u}^\varepsilon_{,33} \cdot \mathbf{v}_{,\alpha 3})\right\} dx,\\
a_1(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \int_B \left\{\hat{\mathbf{K}}_{\alpha\beta}\mathbf{u}^\varepsilon_{,\beta} \cdot \mathbf{v}_{,\alpha} + \ell^2(\hat{\mathbf{K}}_{33}\mathbf{u}^\varepsilon_{,3\beta} \cdot \mathbf{v}_{,3\beta} + \hat{\mathbf{K}}_{\alpha\beta}\mathbf{u}^\varepsilon_{,\beta 3} \cdot \mathbf{v}_{,\alpha 3}\right\} dx,\\
a_2(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \int_B \ell^2 \left\{\hat{\mathbf{K}}_{\alpha 3}\mathbf{u}^\varepsilon_{,\alpha\beta} \cdot \mathbf{v}_{,3\beta} + \hat{\mathbf{K}}_{3\alpha}\mathbf{u}^\varepsilon_{,3\beta} \cdot \mathbf{v}_{,\alpha\beta}\right\} dx,\\
a_3(\mathbf{u}^\varepsilon, \mathbf{v}) &:= \int_B \ell^2 \hat{\mathbf{K}}_{\alpha\beta}\mathbf{u}^\varepsilon_{,\beta\sigma} \cdot \mathbf{v}_{,\alpha\sigma} dx,
\end{aligned}$$

where $\hat{\mathbf{K}}_{33} := (\hat{c}_{i3j3})$, $\hat{\mathbf{K}}_{\alpha 3} := (\hat{c}_{i3j\alpha})$, $\hat{\mathbf{K}}_{3\alpha} := (\hat{c}_{i\alpha j3}) = \hat{\mathbf{K}}_{\alpha 3}^T$ and $\hat{\mathbf{K}}_{\alpha\beta} := (\hat{c}_{i\alpha j\beta})$. We can now perform an asymptotic analysis of the rescaled problem (2.4). Since the rescaled problem (2.4) has a polynomial structure with respect to ε, its solution can be expressed as a series of powers of ε:

$$\begin{aligned}
&\bar{\mathbf{u}}^\varepsilon = \bar{\mathbf{u}}^0 + \varepsilon\bar{\mathbf{u}}^1 + \varepsilon^2\bar{\mathbf{u}}^2 + \ldots, && \hat{\mathbf{u}}^\varepsilon = \hat{\mathbf{u}}^0 + \varepsilon\hat{\mathbf{u}}^1 + \varepsilon^2\hat{\mathbf{u}}^2 + \ldots,\\
&\hat{\mathbf{w}}^\varepsilon = \mathbf{w}^0 + \varepsilon\mathbf{w}^1 + \varepsilon^2\mathbf{w}^2 + \ldots, && \hat{\boldsymbol{\phi}}^\varepsilon = \boldsymbol{\phi}^0 + \varepsilon\boldsymbol{\phi}^1 + \varepsilon^2\boldsymbol{\phi}^2 + \ldots.
\end{aligned} \tag{2.5}$$

where $\bar{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon \circ \bar{\pi}^\varepsilon$ and $\hat{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon \circ \hat{\pi}^\varepsilon$. Recalling that $\hat{\mathbf{u}}^\varepsilon = \mathbf{w}^\varepsilon + \varepsilon\boldsymbol{\phi}^\varepsilon$ and using (2.5), one has: $\hat{\mathbf{u}}^n = \sum_{n>0} = \mathbf{w}^n + \boldsymbol{\phi}^{n-1}$, with $\hat{\mathbf{u}}^0 = \mathbf{w}^0$. By substituting expressions (2.5) into equation (2.4) and by identifying the terms with identical powers of ε, we can finally characterize the limit problems at order 0 and the first order corrector term.

(d) The asymptotic expansions method

The present section is devoted to the computations associated with the asymptotic analysis, which will allow us to identify the limit model for a hard strain gradient interface. Let us inject (2.5) into (2.4) and obtain the following set of variational problems $\mathcal{P}_n$:

$$\begin{aligned}
&\mathcal{P}_{-1}: \ a_{-1}(\mathbf{w}^0,\mathbf{s}) = 0,\\
&\mathcal{P}_0: \ \bar{A}_-(\mathbf{u}^0,\mathbf{v}) + \bar{A}_+(\mathbf{u}^0,\mathbf{v}) + a_{-1}(\hat{\mathbf{u}}^1,\mathbf{s}) + a_0(\mathbf{w}^0,\mathbf{s}) = L(\mathbf{v}),\\
&\mathcal{P}_1: \ \bar{A}_-(\mathbf{u}^1,\mathbf{v}) + \bar{A}_+(\mathbf{u}^1,\mathbf{v}) + a_{-1}(\hat{\mathbf{u}}^2,\mathbf{s}) + a_{-1}(\hat{\mathbf{u}}^1,\boldsymbol{\psi}) + a_0(\hat{\mathbf{u}}^1,\mathbf{s}) + a_0(\mathbf{w}^0,\boldsymbol{\psi}) + a_1(\mathbf{w}^0,\mathbf{s}) = 0.
\end{aligned}$$

Let us consider variational problem $\mathcal{P}_{-1}$. After an integration by parts along the x_3-coordinate, one has:

$$\begin{aligned}
&\hat{\mathbf{K}}_{33}\left(\mathbf{w}^0_{,33} - \ell^2\mathbf{w}^0_{,3333}\right) = \mathbf{0} && \text{in } B,\\
&\hat{\mathbf{K}}_{33}\left(\mathbf{w}^0_{,3} - \ell^2\mathbf{w}^0_{,333}\right)|_{x_3=\pm\frac{h}{2}} = \mathbf{0} && \text{on } S_\pm.
\end{aligned} \tag{2.6}$$

Recalling that $\mathbf{w}^0_{,3}|_{x_3=\pm\frac{h}{2}} = \mathbf{0}$, one obtains that $\mathbf{w}^0 = \mathbf{w}^0(\tilde{x})$, $\tilde{x} = x_\alpha$, is independent of x_3 and thus, by the continuity on the upper and lower interfaces $S_\pm$, $[\mathbf{w}^0] = \mathbf{0}$ and $\langle \mathbf{w}^0\rangle = \langle \bar{\mathbf{u}}^0\rangle$, where $[\varphi] := \varphi^+ - \varphi^-$ and $\langle\varphi\rangle := \frac{1}{2}(\varphi^+ + \varphi^-)$, $\varphi^\pm = \varphi|_{S_\pm}$, denote, respectively, the jump and mean values of the restrictions of φ at the interface $S_\pm$.

Let us consider the variational problem $\mathcal{P}_0$ and perform the integration by parts by means of the Gauss-Green's theorem, as follows:

$$\begin{aligned}
&-\int_{\Omega_\pm} \bar{\bar{\sigma}}^0_{ij,j} v_i dx + \int_{\Gamma_1}\left\{\bar{T}^0_i v_i + \bar{R}^0_i \partial_n v_i\right\} d\Gamma + \sum_m \oint_{C_m\cup\gamma_\pm} \bar{P}^0_i v_i ds\\
&-\int_B \hat{\mathbf{K}}_{33}\left(\hat{\mathbf{u}}^1_{,33} - \ell^2\hat{\mathbf{u}}^1_{,3333}\right)\cdot \mathbf{s} dx \mp \int_{S_\pm}\left(\bar{\mathbf{t}}^0 - \hat{\mathbf{K}}_{33}\left(\hat{\mathbf{u}}^1_{,3} - \ell^2\hat{\mathbf{u}}^1_{,333}\right) - \hat{\mathbf{K}}_{\alpha 3}\mathbf{w}^0_\alpha\right)\cdot \mathbf{s} d\Gamma = L(\mathbf{v}),
\end{aligned} \tag{2.7}$$

where $\mathbf{n}(\tilde{x}, \pm\frac{h}{2}) = \mp\mathbf{e}_3$ on $S_\pm$. From equation (2.7), using standard variational arguments, the following set of equilibrium equations is derived:

$$\begin{cases}
\bar{\bar{\sigma}}^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm,\\
\bar{T}^0_i = g_i, \ \ \bar{R}^0_i = q_i & \text{on } \Gamma_1,\\
\bar{P}^0_i = 0 & \text{on } C_m \cup \gamma_\pm,\\
\bar{u}^0_i = 0, \ \ \partial_n \bar{u}^0_i = 0, & \text{on } \Gamma_0,\\
\hat{\mathbf{K}}_{33}\left(\hat{\mathbf{u}}^1_{,33} - \ell^2\hat{\mathbf{u}}^1_{,3333}\right) = \mathbf{0} & \text{in } B,\\
\mp\left(\bar{\mathbf{t}}^0 - \hat{\mathbf{K}}_{33}\left(\hat{\mathbf{u}}^1_{,3} - \ell^2\hat{\mathbf{u}}^1_{,333}\right) - \hat{\mathbf{K}}_{\alpha 3}\mathbf{w}^0_{,\alpha}\right)\Big|_{x_3=\pm\frac{h}{2}} = \mathbf{0} & \text{on } S_\pm,\\
\bar{\mathbf{r}}^0\Big|_{x_3=\pm\frac{h}{2}} = \mathbf{0} & \text{on } S_\pm,
\end{cases} \tag{2.8}$$

where $\bar{\mathbf{t}}^0 := (\bar{\bar{\sigma}}^0_{i3} + \bar{\tau}^0_{i\alpha 3,\alpha})$ and $\bar{\mathbf{r}}^0 := (\bar{\tau}^0_{i33})$ represent, respectively, the traction and the double-traction vectors evaluated at the interfaces $S_\pm$.

Equations $(2.8)_{1,2,3}$ represent the equilibrium equations at order 0 on the adherents with the suitable boundary conditions. By integrating equation $(2.8)_5$, the characterization of $\hat{\mathbf{u}}^1$ within the adhesive B as an exponential function of the thickness coordinate is obtained:

$$\hat{\mathbf{u}}^1(\tilde{x}, x_3) = \mathbf{c}_0 + x_3\mathbf{c}_1 + \mathbf{c}_2 e^{\frac{x_3}{\ell}} + \mathbf{c}_3 e^{-\frac{x_3}{\ell}}, \tag{2.9}$$

where the integration constants $\mathbf{c}_K$ can be found explicitly imposing the continuity conditions $\hat{\mathbf{u}}^1\Big|_{x_3=\pm\frac{h}{2}} = \bar{\mathbf{u}}^1|_{x_3=\pm\frac{h}{2}}$ and $\hat{\mathbf{u}}^1_{,3}\Big|_{x_3=\pm\frac{h}{2}} = \boldsymbol{\phi}^0_{,3}\Big|_{x_3=\pm\frac{h}{2}} = \bar{\mathbf{u}}^0_{,3}\Big|_{x_3=\pm\frac{h}{2}}$. By summing and subtracting equations $(2.8)_{6,7}$, combined with the expressions (2.9), one can evaluate the jumps and mean values of the traction and double-traction vectors at the interface:

$$[\bar{\mathbf{t}}^0] = \mathbf{0}, \ \ \langle\bar{\mathbf{t}}^0\rangle = \frac{1}{h} f(l)\hat{\mathbf{K}}_{33}[\bar{\mathbf{u}}^1] + (1 - f(l))\hat{\mathbf{K}}_{33}\langle\bar{\mathbf{u}}^0_{,3}\rangle + \hat{\mathbf{K}}_{\alpha 3}\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle, \ \ [\bar{\mathbf{r}}^0] = \mathbf{0}, \ \ \langle\bar{\mathbf{r}}^0\rangle = \mathbf{0}, \tag{2.10}$$

with $f(l) := \frac{1}{1 - \frac{2}{l}\tanh\frac{l}{2}}$, $l := \frac{h}{\ell}$, the internal length scale function.

Let us consider the variational problem $\mathcal{P}_1$. After an integration by parts, we get:

$$
\begin{aligned}
&-\int_{\Omega_\pm} \bar{\tilde{\sigma}}^1_{ij,j} v_i dx + \int_{\Gamma_1} \left\{ \bar{T}^1_i v_i + \bar{R}^1_i \partial_n v_i \right\} d\Gamma + \sum_m \oint_{C_m} \bar{P}^1_i v_i ds \\
&-\int_B \left\{ \hat{\mathbf{K}}_{33} \left(\hat{\mathbf{u}}^2_{,33} - \ell^2 \hat{\mathbf{u}}^2_{,3333} \right) + (\hat{\mathbf{K}}_{\alpha 3} + \hat{\mathbf{K}}_{3\alpha}) \left(\hat{\mathbf{u}}^1_{,\alpha 3} - \ell^2 \hat{\mathbf{u}}^1_{,\alpha 333} \right) + \hat{\mathbf{K}}_{\alpha\beta} \mathbf{w}^0_{,\alpha\beta}) \right\} \cdot \mathbf{s} dx \\
&\mp \int_{S_\pm} \left(\bar{\mathbf{t}}^1 - \hat{\mathbf{K}}_{33} \left(\hat{\mathbf{u}}^2_{,3} - \ell^2 \hat{\mathbf{u}}^2_{,333} \right) - \hat{\mathbf{K}}_{\alpha 3} \hat{\mathbf{u}}^1_{,\alpha} + \ell^2 (\hat{\mathbf{K}}_{\alpha 3} + \hat{\mathbf{K}}_{3\alpha}) \hat{\mathbf{u}}^1_{,\alpha 33} \right) \cdot \mathbf{s} d\Gamma \\
&\mp \int_{S_\pm} \left(\bar{\mathbf{r}}^1 - \ell^2 \hat{\mathbf{K}}_{33} \hat{\mathbf{u}}^1_{,33} \right) \cdot \boldsymbol{\psi}_{,3} d\Gamma \pm \oint_{\gamma_\pm} \left(\bar{\mathbf{p}}^1 - \ell^2 \hat{\mathbf{K}}_{3\alpha} \nu_\alpha \hat{\mathbf{u}}^1_{,33} \right) \cdot \mathbf{s} d\gamma \\
&+ \int_{\Gamma_{lat}} \nu_\alpha \left(\hat{\mathbf{K}}_{3\alpha} (\mathbf{u}^1_{,3} - \ell^2 \mathbf{u}^1_{,333}) + \hat{\mathbf{K}}_{\alpha\beta} \mathbf{w}^0_{,\beta} \right) \cdot \mathbf{s} d\Gamma = 0,
\end{aligned}
\tag{2.11}
$$

where $\bar{\mathbf{p}}^1 := (\Delta(\nu_\beta \tau_{3\beta i}))$ denotes the edge traction vector on $\gamma_\pm$, and Γ_{lat} represents the lateral surface of B, with unit normal vector (ν_α). This yields to the following differential system:

$$
\begin{cases}
\bar{\tilde{\sigma}}^1_{ij,j} = 0 & \text{in } \Omega_\pm, \\
\bar{T}^1_i = 0, \quad \bar{R}^1_i = 0 & \text{on } \Gamma_1, \\
\bar{P}^1_i = 0 & \text{on } C_m, \\
\bar{u}^1_i = 0, \quad \partial_n \bar{u}^1_i = 0 & \text{on } \Gamma_0, \\
\hat{\mathbf{K}}_{33} \left(\hat{\mathbf{u}}^2_{,33} - \ell^2 \hat{\mathbf{u}}^2_{,3333} \right) + (\hat{\mathbf{K}}_{\alpha 3} + \hat{\mathbf{K}}_{3\alpha}) \left(\hat{\mathbf{u}}^1_{,\alpha 3} - \ell^2 \hat{\mathbf{u}}^1_{,\alpha 333} \right) + \hat{\mathbf{K}}_{\alpha\beta} \mathbf{w}^0_{,\alpha\beta} = \mathbf{0} & \text{in } B, \\
\mp \left(\bar{\mathbf{t}}^1 - \hat{\mathbf{K}}_{33} \left(\hat{\mathbf{u}}^2_{,3} - \ell^2 \hat{\mathbf{u}}^2_{,333} \right) - \hat{\mathbf{K}}_{\alpha 3} \hat{\mathbf{u}}^1_{,\alpha} + \ell^2 (\hat{\mathbf{K}}_{\alpha 3} + \hat{\mathbf{K}}_{3\alpha}) \hat{\mathbf{u}}^1_{,\alpha 33} \right) \Big|_{x_3 = \pm \frac{h}{2}} = \mathbf{0} & \text{on } S_\pm, \\
\mp \left(\bar{\mathbf{r}}^1 - \ell^2 \hat{\mathbf{K}}_{33} \hat{\mathbf{u}}^1_{,33} \right) \Big|_{x_3 = \pm \frac{h}{2}} = \mathbf{0} & \text{on } S_\pm, \\
\pm \left(\bar{\mathbf{p}}^1 - \ell^2 \hat{\mathbf{K}}_{3\alpha} \hat{\mathbf{u}}^1_{,33} \nu_\alpha \right) |_{x_3 = \pm \frac{h}{2}} = \mathbf{0} & \text{on } \gamma_\pm.
\end{cases}
\tag{2.12}
$$

The governing equations at order 1 are homogeneous since no external forces are applied. The order 0 terms act as known source charges, obtained by solving the order 0 problem. By integrating (2.12)$_5$ along x_3, we can characterize $\hat{\mathbf{u}}^2$ as follows:

$$
\hat{\mathbf{u}}^2(\tilde{x}, x_3) = \mathbf{d}_0 + x_3 \mathbf{d}_1 + \mathbf{d}_2 e^{\frac{x_3}{\ell}} + \mathbf{d}_3 e^{-\frac{x_3}{\ell}} - \hat{\mathbf{H}}_\alpha \left(x_3 \mathbf{c}_{0,\alpha} + \frac{x_3^2}{2} \mathbf{c}_{1,\alpha} \right) - \frac{x_3^2}{2} \hat{\mathbf{H}}_{\alpha\beta} \mathbf{w}^0_{,\alpha\beta}, \tag{2.13}
$$

with $\hat{\mathbf{H}}_\alpha := (\hat{\mathbf{K}}_{33})^{-1} (\hat{\mathbf{K}}_{\alpha 3} + \hat{\mathbf{K}}_{3\alpha})$ and $\hat{\mathbf{H}}_{\alpha\beta} := (\hat{\mathbf{K}}_{33})^{-1} \hat{\mathbf{K}}_{\alpha\beta}$. Constants $\mathbf{d}_K$ are independent of x_3 and can be found by virtue of the continuity conditions $\hat{\mathbf{u}}^2_{,3}\Big|_{x_3 = \pm \frac{h}{2}} = \boldsymbol{\phi}^1_{,3}\Big|_{x_3 = \pm \frac{h}{2}} = \bar{\mathbf{u}}^2_{,3}\Big|_{x_3 = \pm \frac{h}{2}}$ and $\hat{\mathbf{u}}^2\Big|_{x_3 = \pm \frac{h}{2}} = \bar{\mathbf{u}}^2|_{x_3 = \pm \frac{h}{2}}$ on $S_\pm$.

From (2.12)$_{6,7,8}$, it is possible to obtain the expressions of the jump and mean values of the stress vector, the double-forces and edge forces at the interface, respectively:

$$
\begin{aligned}
&[\bar{\mathbf{t}}^1] = -\hat{\mathbf{K}}_{3\alpha} [\bar{\mathbf{u}}^1_{,\alpha}] - h \hat{\mathbf{K}}_{\alpha\beta} \langle \bar{\mathbf{u}}^0_{,\alpha\beta} \rangle, \\
&\langle \bar{\mathbf{t}}^1 \rangle = \tfrac{1}{h} f(l) \hat{\mathbf{K}}_{33} [\bar{\mathbf{u}}^2] + (1 - f(l)) \hat{\mathbf{K}}_{33} \langle \bar{\mathbf{u}}^1_{,3} \rangle + \hat{\mathbf{K}}_{\alpha 3} \langle \bar{\mathbf{u}}^1_{,\alpha} \rangle - \tfrac{\ell}{2} g(l) (\hat{\mathbf{K}}_{\alpha 3} + \hat{\mathbf{K}}_{3\alpha}) [\bar{\mathbf{u}}^0_{,3\alpha}], \\
&[\bar{\mathbf{r}}^1] = (1 - f(l)) \hat{\mathbf{K}}_{33} ([\bar{\mathbf{u}}^1] - h \langle \bar{\mathbf{u}}^0_{,3} \rangle), \quad \langle \bar{\mathbf{r}}^1 \rangle = \tfrac{\ell}{2} g(l) \hat{\mathbf{K}}_{33} [\bar{\mathbf{u}}^0_{,3}], \\
&[\bar{\mathbf{p}}^1] = (1 - f(l)) \hat{\mathbf{K}}_{3\alpha} \nu_\alpha ([\bar{\mathbf{u}}^1] - h \langle \bar{\mathbf{u}}^0_{,3} \rangle), \quad \langle \bar{\mathbf{p}}^1 \rangle = \ell g(l) \hat{\mathbf{K}}_{3\alpha} \nu_\alpha [\bar{\mathbf{u}}^0_{,3}].
\end{aligned}
\tag{2.14}
$$

where $g(l) := \frac{1}{2} \coth \frac{l}{2}$.

By using the approach developed in [43], it is possible to obtain a condensed form of the transmission conditions summarizing both orders 0 and 1. To this end, we denote $\tilde{\mathbf{t}}^\varepsilon = \mathbf{t}^0 + \varepsilon \mathbf{t}^1$, $\tilde{\mathbf{r}}^\varepsilon = \mathbf{r}^0 + \varepsilon \mathbf{r}^1$, $\tilde{\mathbf{p}}^\varepsilon = \mathbf{p}^0 + \varepsilon \mathbf{p}^1$ and $\tilde{\mathbf{u}}^\varepsilon = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2$, four suitable approximations of $\mathbf{t}^\varepsilon$, $\mathbf{r}^\varepsilon$, $\mathbf{p}^\varepsilon$ and $\mathbf{u}^\varepsilon$. After rescaling back to the initial domain Ω^ε, thus $\frac{\partial}{\partial x_3} = \varepsilon \frac{\partial}{\partial x_3^\varepsilon}$ in B^ε and $h = \frac{1}{\varepsilon} h^\varepsilon$, and by considering that the constitutive coefficients $\hat{\mathbf{K}}_{ij} = \hat{\mathbf{K}}^\varepsilon_{ij}$, $\ell = \frac{1}{\varepsilon} \hat{\ell}^\varepsilon$, $l^\varepsilon = \frac{h^\varepsilon}{\hat{\ell}^\varepsilon}$ in B^ε, one can obtain an explicit form of the transmission conditions, comprising order 0 and 1:

$$
\begin{aligned}
&[\tilde{\mathbf{t}}^\varepsilon] = -\hat{\mathbf{K}}^\varepsilon_{3\alpha}[\tilde{\mathbf{u}}^\varepsilon_{,\alpha}] - h^\varepsilon \hat{\mathbf{K}}^\varepsilon_{\alpha\beta}\langle\tilde{\mathbf{u}}^\varepsilon_{,\alpha\beta}\rangle + o(\varepsilon),\\
&\langle\tilde{\mathbf{t}}^\varepsilon\rangle = \tfrac{1}{h^\varepsilon} f(l^\varepsilon)\hat{\mathbf{K}}^\varepsilon_{33}[\tilde{\mathbf{u}}^\varepsilon] + (1-f(l^\varepsilon))\hat{\mathbf{K}}^\varepsilon_{33}\langle\tilde{\mathbf{u}}^\varepsilon_{,3}\rangle + \hat{\mathbf{K}}^\varepsilon_{\alpha 3}\langle\tilde{\mathbf{u}}^\varepsilon_{,\alpha}\rangle - \tfrac{\ell^\varepsilon}{2} g(l^\varepsilon)(\hat{\mathbf{K}}^\varepsilon_{\alpha 3} + \hat{\mathbf{K}}^\varepsilon_{3\alpha})[\tilde{\mathbf{u}}^\varepsilon_{,3\alpha}] + o(\varepsilon),\\
&[\tilde{\mathbf{r}}^\varepsilon] = (1-f(l^\varepsilon))\hat{\mathbf{K}}^\varepsilon_{33}([\tilde{\mathbf{u}}^\varepsilon] - h^\varepsilon\langle\tilde{\mathbf{u}}^\varepsilon_{,3}\rangle) + o(\varepsilon),\;\; \langle\tilde{\mathbf{r}}^\varepsilon\rangle = \tfrac{\ell^\varepsilon}{2} g(l^\varepsilon)\hat{\mathbf{K}}^\varepsilon_{33}[\tilde{\mathbf{u}}^\varepsilon_{,3}] + o(\varepsilon),\\
&[\tilde{\mathbf{p}}^\varepsilon] = (1-f(l^\varepsilon))\hat{\mathbf{K}}^\varepsilon_{3\alpha}\nu_\alpha([\tilde{\mathbf{u}}^\varepsilon] - h^\varepsilon\langle\tilde{\mathbf{u}}^\varepsilon_{,3}\rangle) + o(\varepsilon),\;\; \langle\tilde{\mathbf{p}}^\varepsilon\rangle = \ell^\varepsilon g(l^\varepsilon)\hat{\mathbf{K}}^\varepsilon_{3\alpha}\nu_\alpha[\tilde{\mathbf{u}}^\varepsilon_{,3}] + o(\varepsilon).
\end{aligned}
\tag{2.15}
$$

(e) The interface conditions

Thanks to the results of the asymptotic methods, derived in Section 2(d), we can now recover the expression of the hard contact laws for a strain gradient elastic interface. Using relations (2.10), the transmission problem at order 0 can be summarized as follows:

Governing equations:

$$
\begin{cases}
\bar{\sigma}^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm,\\
\bar{T}^0_i = g_i, \;\; \bar{R}^0_i = q_i & \text{on } \Gamma_1,\\
\bar{P}^0_i = 0 & \text{on } C_m \cup \gamma_\pm,\\
\bar{u}^0_i = 0, \;\; \partial_n \bar{u}^0_i = 0, & \text{on } \Gamma_0.
\end{cases}
$$

Interface conditions on $S_\pm$:

$$
\begin{cases}
[\bar{\mathbf{u}}^0] = \mathbf{0}, \quad [\bar{\mathbf{t}}^0] = \mathbf{0},\\
\bar{\mathbf{r}}^0 = \mathbf{0}.
\end{cases}
$$

Concerning the order 0 problem, the transmission conditions provide the continuity of the displacements and the traction vector at the interface, which is typical for an adhesive having the same stiffness as the upper and lower bodies. In this case, from a macroscopic point of view, the adherents are perfectly bonded together and no interactions with the intermediate layer can be highlighted. The double traction vector $\bar{\mathbf{r}}^0$ vanishes at the interface and no conditions can be imposed on the normal derivative of the displacement $\bar{\mathbf{u}}^0_{,3}$. Hence, it is necessary to look at the order 1 transmission problem, in order to get a better approximation of the interface mechanical behavior. Combining (2.10) and (2.14), the order 1 equations read as follows:

Governing equations:

$$
\begin{cases}
\bar{\sigma}^1_{ij,j} = 0 & \text{in } \Omega_\pm,\\
\bar{T}^1_i = 0, \;\; \bar{R}^1_i = 0 & \text{on } \Gamma_1,\\
\bar{P}^1_i = 0 & \text{on } C_m,\\
\bar{u}^1_i = 0, \;\; \partial_n \bar{u}^1_i = 0, & \text{on } \Gamma_0.
\end{cases}
$$

Interface conditions on $S_\pm$:

$$
\begin{cases}
[\bar{\mathbf{u}}^1] = \frac{h}{f(l)}\left(\hat{\mathbf{K}}^{-1}_{33}\langle\bar{\mathbf{t}}^0\rangle - (1-f(l))\langle\bar{\mathbf{u}}^0_{,3}\rangle - \hat{\mathbf{K}}^{-1}_{33}\hat{\mathbf{K}}_{\alpha 3}\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle\right),\\
[\bar{\mathbf{t}}^1] = -\frac{h}{f(l)}\left(\hat{\mathbf{K}}_{3\alpha}\hat{\mathbf{K}}^{-1}_{33}\langle\bar{\mathbf{t}}^0_{,\alpha}\rangle - (1-f(l))\hat{\mathbf{K}}_{3\alpha}\langle\bar{\mathbf{u}}^0_{,\alpha 3}\rangle\right)\\
\qquad\quad -h\hat{\mathbf{L}}_{\alpha\beta}\langle\bar{\mathbf{u}}^0_{,\alpha\beta}\rangle,\\
[\bar{\mathbf{u}}^1_{,3}] = [\bar{\mathbf{u}}^0_{,3}],\\
[\bar{\mathbf{r}}^1] = (1-f(l))\hat{\mathbf{K}}_{33}([\bar{\mathbf{u}}^1] - h\langle\bar{\mathbf{u}}^0_{,3}\rangle),
\end{cases}
$$

with $\hat{\mathbf{L}}_{\alpha\beta} := \hat{\mathbf{K}}_{\alpha\beta} - \frac{1}{f(l)}\hat{\mathbf{K}}_{3\alpha}\hat{\mathbf{K}}^{-1}_{33}\hat{\mathbf{K}}_{\beta 3}$. Alternatively, one can also impose the conditions related to the edge tractions defined on the boundary of the interface $\gamma_\pm$: $[\bar{\mathbf{p}}^1] = (1-f(l))\hat{\mathbf{K}}_{3\alpha}\nu_\alpha([\bar{\mathbf{u}}^1] - h\langle\bar{\mathbf{u}}^0_{,3}\rangle)$. At order 1, we obtain a mixed interface model in which both the kinematical quantities, namely displacements and their normal derivatives, and the conjugated counterparts are discontinuous through the interface and their jumps depend on the order 0 solution. The hard strain gradient elastic transmission problems above present the same structures of the analogous linear elastic hard interface models [34,35,43]. Indeed, fixing h and letting $l \to \infty$, i.e. $\ell \to 0$, $\lim_{l\to\infty} f(l) = 1$ and $\mathbf{t}^1$ coincides with the Cauchy's traction vector, and the double forces $\mathbf{r}^1$ and all the strain gradient contributions can be neglected. Thus, the classical hard interface conditions at order 1 are recovered (see [43]):

$$
[\bar{\mathbf{u}}^1] = h\hat{\mathbf{K}}^{-1}_{33}\left(\langle\bar{\mathbf{t}}^0\rangle - \hat{\mathbf{K}}_{\alpha 3}\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle\right), \quad [\bar{\mathbf{t}}^1] = -h\hat{\mathbf{K}}_{3\alpha}\hat{\mathbf{K}}^{-1}_{33}\langle\bar{\mathbf{t}}^0_{,\alpha}\rangle - h\hat{\mathbf{L}}_{\alpha\beta}\langle\bar{\mathbf{u}}^0_{,\alpha\beta}\rangle.
$$

It is worth-mentioning that the above contact laws establish a generalization to the case of strain gradient elasticity of the so-called coherent or Gurtin-Murdoch's interface model, according to which the traction vector suffers a jump discontinuity, satisfying a two-dimensional Laplace-Young equation. This particular model has been firstly developed for continuum theories with surface effects by the early work of M.E. Gurtin and A.I. Murdoch [52], and applied to surface and interfacial waves propagation by [53–55]. Indeed, the term $h\hat{\mathbf{L}}_{\alpha\beta}\langle\bar{\mathbf{u}}^0_{,\alpha\beta}\rangle$ represents the divergence of a surface stress tensor defined on the plane of the interface, equivalent to the one introduced in [52]. Finally, the hard strain gradient elastic conditions show new features with respect to the mechanical behavior of the interface. The formal asymptotic derivation allowed us to consider the intrinsic second gradient nature of the adhesive layer and size-dependent phenomena, typical of micro- and nano-structures, as also shown in [48].

3. Hard interface conditions in micropolar elasticity

(a) Theoretical settings

Let us consider a material body Ω constituted by a centrosymmetric micropolar linear elastic material, whose constitutive law is defined as follows: $\sigma_{ji} = c_{ijhl}e_{lh}$ and $\mu_{ji} = a_{ijhl}\chi_{lh}$, where (σ_{ij}) denotes the nonsymmetric stress tensor, associated with the nonsymmetric strain $e_{ji} = u_{i,j} + \epsilon_{ijk}\omega_k$, (μ_{ij}) is the couple stress tensor, conjugated with the curvature tensor $\chi_{ji} = \omega_{i,j}$, with u_i and ω_i the displacement and microrotations fields, respectively. (c_{ijkl}) and (a_{ijkl}) represent the fourth-order micropolar elasticity tensors,satisfying the following major symmetry property: $c_{ijkl} = c_{klij}$ and $a_{ijkl} = a_{klij}$. The micropolar elastic state at the equilibrium is fully determined by the pair $s = (\mathbf{u}, \boldsymbol{\omega})$. The material body is clamped on a portion of the boundary Γ_0, so that $u_i = \omega_i = 0$, and it is subjected to body forces f_i and body couples m_i, acting in Ω, surface forces g_i and surface couples l_i, applied to the boundary Γ_1. The work of external sources is given by

$$\mathcal{L}(s) := \int_\Omega \{f_i u_i + m_i\omega_i\}dx + \int_{\Gamma_1}\{g_i u_i + l_i\omega_i\}d\Gamma.$$

The variational formulation of the problem takes the following form:

$$\begin{cases} \text{Find } s = (\mathbf{u}, \boldsymbol{\omega}) \in W(\Omega) \times W(\Omega), \text{ such that} \\ \mathcal{A}(s, r) = \mathcal{L}(s), \text{ for all } r = (\mathbf{v}, \mathbf{w}) \in W(\Omega) \times W(\Omega), \end{cases} \tag{3.1}$$

with $W(\Omega) := \{\mathbf{v} \in H^1(\Omega, \mathbb{R}^3):\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}$ and

$$\mathcal{A}(s, r) := \int_\Omega \{\sigma_{ji}e_{ji}(r) + \mu_{ji}\chi_{ji}(\mathbf{w})\}\, dx = \int_\Omega \{c_{ijhl}e_{lh}(s)e_{ji}(r) + a_{ijhl}\chi_{lh}(\boldsymbol{\omega})\chi_{ji}(\mathbf{w})\}\, dx.$$

By virtue of Gauss-Green's theorem, the differential form of the governing equations can be easily derived:

$$-\int_\Omega \sigma_{ji,j}v_i dx - \int_\Omega (\mu_{ji,j} - \epsilon_{kji}\sigma_{kj})\omega_i dx + \int_{\partial\Omega}(\sigma_{ji}n_j v_i + \mu_{ji}n_j\omega_i)d\Gamma = \mathcal{L}(r).$$

Thus, the equilibrium problem can be written, as customary:

$$\begin{cases} \sigma_{ji,j} + f_i = 0, & \text{in } \Omega, \\ \mu_{ji,j} - \epsilon_{kji}\sigma_{kj} + m_i = 0, & \text{in } \Omega, \\ \sigma_{ji}n_j = g_i, & \text{on } \Gamma_1, \\ \mu_{ji}n_j = l_i, & \text{on } \Gamma_1, \\ u_i = 0, \ \ \omega_i = 0, & \text{on } \Gamma_0. \end{cases}$$

(b) Statement of the problem and rescaling

In the sequel, the composite assembly presents the same geometry and features shown in Section 2(b) and illustrated in Figure 1. Considering the micropolar case, the variation formulation of the problem can be written as follows:

$$\left\{\begin{array}{l}\text{Find } s^\varepsilon \in W(\Omega^\varepsilon)\times W(\Omega^\varepsilon),\ \text{ such that}\\ \bar{\mathcal{A}}^\varepsilon_-(s^\varepsilon,r^\varepsilon)+\bar{\mathcal{A}}^\varepsilon_+(s^\varepsilon,r^\varepsilon)+\hat{\mathcal{A}}^\varepsilon(s^\varepsilon,r^\varepsilon)=\mathcal{L}^\varepsilon(r^\varepsilon),\ \text{for all } r^\varepsilon\in W(\Omega^\varepsilon)\times W(\Omega^\varepsilon),\end{array}\right. \tag{3.2}$$

where

$$\begin{aligned}\bar{\mathcal{A}}^\varepsilon_\pm(s^\varepsilon,r^\varepsilon)&:=\int_{\Omega^\varepsilon_\pm}\left\{\bar{c}^\varepsilon_{ijhl}e^\varepsilon_{lh}(s^\varepsilon)e^\varepsilon_{ji}(r^\varepsilon)+\bar{a}^\varepsilon_{ijhl}\chi^\varepsilon_{lh}(\boldsymbol{\omega}^\varepsilon)\chi^\varepsilon_{ji}(\mathbf{w}^\varepsilon)\right\}dx^\varepsilon,\\ \hat{\mathcal{A}}^\varepsilon(s^\varepsilon,r^\varepsilon)&:=\int_{B^\varepsilon}\left\{\hat{c}^\varepsilon_{ijhl}e^\varepsilon_{lh}(s^\varepsilon)e^\varepsilon_{ji}(r^\varepsilon)+\hat{a}^\varepsilon_{ijhl}\chi^\varepsilon_{lh}(\boldsymbol{\omega}^\varepsilon)\chi^\varepsilon_{ji}(\mathbf{w}^\varepsilon)\right\}dx^\varepsilon.\end{aligned}$$

In order to apply the asymptotic expansions method [50], problem (3.2), defined on an ε-dependent domain, must be reformulated on a fixed domain, independent of ε, using the change of variables defined in (2.3). The rescaled configuration of the composite domain has the same geometry characteristics shown in Section 2(c).

We assume that the elastic and micropolar coefficients of the adherents and adhesive are independent of ε, namely $\bar{c}^\varepsilon_{ijkl}=\bar{c}_{ijkl}$, $\hat{c}^\varepsilon_{ijkl}=\hat{c}_{ijkl}$, $\bar{a}^\varepsilon_{ijkl}=\bar{a}_{ijkl}$ and $\hat{a}^\varepsilon_{ijkl}=\hat{a}_{ijkl}$.

The data verify the following scaling assumptions: $f^\varepsilon_i(x^\varepsilon)=f_i(x)$ and $m^\varepsilon_i(x^\varepsilon)=m_i(x)$, $x\in\Omega_\pm$, and $g^\varepsilon_i(x^\varepsilon)=g_i(x)$ and $l^\varepsilon_i(x^\varepsilon)=l_i(x)$, $x\in\Gamma_1$. Thus, $\mathcal{L}^\varepsilon(r^\varepsilon)=\mathcal{L}(r)$. The unknowns and test functions depend on ε as follows:

$$\begin{array}{lllll}\bar{\mathbf{u}}^\varepsilon(x^\varepsilon)=\bar{\mathbf{u}}^\varepsilon(x), & \bar{\boldsymbol{\omega}}^\varepsilon(x^\varepsilon)=\bar{\boldsymbol{\omega}}^\varepsilon(x) & \bar{\mathbf{v}}^\varepsilon(x^\varepsilon)=\bar{\mathbf{v}}(x) & \bar{\mathbf{w}}^\varepsilon(x^\varepsilon)=\bar{\mathbf{w}}(x) & x\in\Omega_\pm,\\ \hat{\mathbf{u}}^\varepsilon(x^\varepsilon)=\hat{\mathbf{u}}^\varepsilon(x), & \hat{\boldsymbol{\omega}}^\varepsilon(x^\varepsilon)=\hat{\boldsymbol{\omega}}^\varepsilon(x) & \bar{\mathbf{v}}^\varepsilon(x^\varepsilon)=\bar{\mathbf{v}}(x) & \hat{\mathbf{w}}^\varepsilon(x^\varepsilon)=\hat{\mathbf{w}}(x) & x\in B.\end{array}$$

According to the previous rescaling hypothesis, problem (3.2) can be reformulated on a fixed domain Ω:

$$\left\{\begin{array}{l}\text{Find } s^\varepsilon\in W(\Omega)\times W(\Omega),\ \text{ such that}\\ \bar{\mathcal{A}}_-(s^\varepsilon,r)+\bar{\mathcal{A}}_+(s^\varepsilon,r)+\hat{\mathcal{A}}(s^\varepsilon,r)=\mathcal{L}(r),\ \text{for all } r\in W(\Omega)\times W(\Omega),\end{array}\right. \tag{3.3}$$

where

$$\begin{aligned}\bar{\mathcal{A}}_\pm(s^\varepsilon,r)&:=\int_{\Omega_\pm}\left\{\bar{c}_{ijhl}e_{lh}(s^\varepsilon)e_{ji}(r)+\bar{a}_{ijhl}\chi_{lh}(\boldsymbol{\omega}^\varepsilon)\chi_{ji}(\mathbf{w})\right\}dx,\\ \hat{\mathcal{A}}(s^\varepsilon,r)&:=\tfrac{1}{\varepsilon}b_{-1}(s^\varepsilon,r)+b_0(s^\varepsilon,r)+\varepsilon b_1(s^\varepsilon,r),\end{aligned}$$

with

$$\begin{aligned}b_{-1}(s^\varepsilon,r)&:=\int_B\Big\{\hat{\mathbf{C}}_{33}\mathbf{u}^\varepsilon_{,3}\cdot\mathbf{v}_{,3}+\hat{\mathbf{A}}_{33}\boldsymbol{\omega}^\varepsilon_{,3}\cdot\mathbf{w}_{,3}\Big\}\,dx,\\ b_0(s^\varepsilon,r)&:=\int_B\Big\{\hat{\mathbf{C}}_{33}\mathbf{u}^\varepsilon_{,3}\cdot(\mathbf{e}_3\wedge\mathbf{w})+\hat{\mathbf{C}}_{33}(\mathbf{e}_3\wedge\boldsymbol{\omega}^\varepsilon)\cdot\mathbf{v}_{,3}++\hat{\mathbf{C}}_{\alpha3}(\mathbf{u}^\varepsilon_{,\alpha}+\mathbf{e}_\alpha\wedge\boldsymbol{\omega}^\varepsilon)\cdot\mathbf{v}_{,3}\\ &\qquad+\hat{\mathbf{C}}_{3\alpha}\mathbf{u}^\varepsilon_{,3}\cdot(\mathbf{v}_{,\alpha}+\mathbf{e}_\alpha\wedge\mathbf{w})+\hat{\mathbf{A}}_{\alpha3}\boldsymbol{\omega}^\varepsilon_{,\alpha}\cdot\mathbf{w}_{,3}+\hat{\mathbf{A}}_{3\alpha}\boldsymbol{\omega}^\varepsilon_{,3}\cdot\mathbf{w}_{,\alpha}\Big\}\,dx,\\ b_1(s^\varepsilon,r)&:=\int_B\Big\{\hat{\mathbf{C}}_{33}(\mathbf{e}_3\wedge\boldsymbol{\omega}^\varepsilon)\cdot(\mathbf{e}_3\wedge\mathbf{w})+\hat{\mathbf{C}}_{\alpha3}(\mathbf{u}^\varepsilon_{,\alpha}+\mathbf{e}_\alpha\wedge\boldsymbol{\omega}^\varepsilon)\cdot(\mathbf{e}_3\wedge\mathbf{w})+\hat{\mathbf{A}}_{\alpha\beta}\boldsymbol{\omega}^\varepsilon_{,\beta}\cdot\mathbf{w}_{,\alpha}\\ &\qquad+\hat{\mathbf{C}}_{3\alpha}(\mathbf{e}_3\wedge\boldsymbol{\omega}^\varepsilon)\cdot(\mathbf{v}_{,\alpha}+\mathbf{e}_\alpha\wedge\mathbf{w})+\hat{\mathbf{C}}_{\alpha\beta}(\mathbf{u}^\varepsilon_{,\beta}+\mathbf{e}_\beta\wedge\boldsymbol{\omega}^\varepsilon)\cdot(\mathbf{v}_{,\alpha}+\mathbf{e}_\alpha\wedge\mathbf{w})\Big\}\,dx,\end{aligned}$$

where $\hat{\mathbf{C}}_{33}:=(\hat{c}_{i3j3})$, $\hat{\mathbf{C}}_{\alpha3}:=(\hat{c}_{i3j\alpha})$, $\hat{\mathbf{C}}_{3\alpha}:=(\hat{c}_{i\alpha j3})=\hat{\mathbf{C}}^T_{\alpha3}$, $\hat{\mathbf{C}}_{\alpha\beta}:=(\hat{c}_{i\alpha j\beta})$, $\hat{\mathbf{A}}_{33}:=(\hat{a}_{i3j3})$, $\hat{\mathbf{A}}_{\alpha3}:=(\hat{a}_{i3j\alpha})$, $\hat{\mathbf{A}}_{3\alpha}:=(\hat{a}_{i\alpha j3})=\hat{\mathbf{A}}^T_{\alpha3}$ and $\hat{\mathbf{A}}_{\alpha\beta}:=(\hat{a}_{i\alpha j\beta})$. In order to apply the asymptotic expansions method to (3.3), the solution is expanded as a series of powers of ε:

$$s^\varepsilon=s^0+\varepsilon s^1+\varepsilon^2s^2+\ldots\ \Rightarrow\ \begin{array}{l}\bar{\mathbf{u}}^\varepsilon=\bar{\mathbf{u}}^0+\varepsilon\bar{\mathbf{u}}^1+\varepsilon^2\bar{\mathbf{u}}^2+\ldots,\\ \hat{\mathbf{u}}^\varepsilon=\hat{\mathbf{u}}^0+\varepsilon\hat{\mathbf{u}}^1+\varepsilon^2\hat{\mathbf{u}}^2+\ldots,\\ \bar{\boldsymbol{\omega}}^\varepsilon=\bar{\boldsymbol{\omega}}^0+\varepsilon\bar{\boldsymbol{\omega}}^1+\varepsilon^2\bar{\boldsymbol{\omega}}^2+\ldots,\\ \hat{\boldsymbol{\omega}}^\varepsilon=\hat{\boldsymbol{\omega}}^0+\varepsilon\hat{\boldsymbol{\omega}}^1+\varepsilon^2\hat{\boldsymbol{\omega}}^2+\ldots.\end{array} \tag{3.4}$$

Hence, by substituting expressions (3.4) into equation (3.3) and by identifying the terms with identical powers, we can finally characterize the limit problems at order 0 and order 1.

(c) The asymptotic expansions method

Following the steps of Section 3(b), the following set of variational problems $\mathcal{P}_n$ can be defined:

$$\begin{aligned}
&\mathcal{P}_{-1}:\ b_{-1}(s^0,r)=0,\\
&\mathcal{P}_0:\ \bar{\mathcal{A}}_-(s^0,r)+\bar{\mathcal{A}}_+(s^0,r)+b_{-1}(s^1,r)+b_0(s^0,r)=\mathcal{L}(r),\\
&\mathcal{P}_1:\ \bar{\mathcal{A}}_-(s^1,r)+\bar{\mathcal{A}}_+(s^1,r)+b_{-1}(s^2,r)+b_0(s^1,r)+b_1(s^0,r)=0.
\end{aligned}$$

Let us consider variational problem $\mathcal{P}_{-1}$. After an integration by parts along x_3, we get that both $\hat{\mathbf{u}}^0$ and $\hat{\boldsymbol{\omega}}^0$ are constant with respect to the thickness coordinate. By applying the continuity conditions at the interfaces, one has $[\bar{\mathbf{u}}^0]=\mathbf{0}$ and $[\bar{\boldsymbol{\omega}}^0]=\mathbf{0}$. Following the same procedure of Section 2(d), let us apply the Gauss-Green's formulae to problem $\mathcal{P}_0$. We obtain a set of equilibrium equations at order 0 on $\Omega_\pm$, coupled with additional conditions on the traction vectors:

$$\begin{cases}
\bar{\sigma}^0_{ji,j}+f_i=0, & \text{in } \Omega_\pm,\\
\bar{\mu}^0_{ji,j}-\epsilon_{kji}\bar{\sigma}^0_{kj}+m_i=0, & \text{in } \Omega_\pm,\\
\bar{\sigma}^0_{ji}n_j=g_i,\ \bar{\mu}^0_{ji}n_j=l_i & \text{on } \Gamma_1,\\
u^0_i=0,\ \ \omega^0_i=0, & \text{on } \Gamma_0,\\
\hat{\mathbf{u}}^1_{,33}=\mathbf{0},\ \hat{\boldsymbol{\omega}}^1_{,33}=\mathbf{0} & \text{in } B,\\
\mp\left(\bar{\boldsymbol{\sigma}}^0-\hat{\mathbf{C}}_{33}\left(\hat{\mathbf{u}}^1_{,3}+\mathbf{e}_3\wedge\hat{\boldsymbol{\omega}}^0\right)-\hat{\mathbf{C}}_{\alpha 3}(\hat{\mathbf{u}}^0_{,\alpha}+\mathbf{e}_\alpha\wedge\hat{\boldsymbol{\omega}}^0)\right)\Big|_{x_3=\pm\frac{h}{2}}=\mathbf{0} & \text{on } S_\pm,\\
\mp\left(\bar{\boldsymbol{\mu}}^0-\hat{\mathbf{A}}_{33}\hat{\boldsymbol{\omega}}^1_{,3}-\hat{\mathbf{A}}_{\alpha 3}\hat{\boldsymbol{\omega}}^0_{,\alpha}\right)\Big|_{x_3=\pm\frac{h}{2}}=\mathbf{0} & \text{on } S_\pm,
\end{cases}\tag{3.5}$$

where $\bar{\boldsymbol{\sigma}}^0=(\bar{\sigma}^0_{i3})$ and $\bar{\boldsymbol{\mu}}^0=(\bar{\mu}^0_{i3})$ are the stress and couple-stress vectors evaluated at the interface $S_\pm$. From equations (3.5)$_5$ and thanks to the continuity properties at the interface, the displacements and microrotations are affine functions of x_3, i.e. $\hat{\mathbf{u}}^1(\tilde{x},x_3)=\langle\bar{\mathbf{u}}^1\rangle(\tilde{x})+\frac{x_3}{h}[\bar{\mathbf{u}}^1](\tilde{x})$ and $\hat{\boldsymbol{\omega}}^1(\tilde{x},x_3)=\langle\bar{\boldsymbol{\omega}}^1\rangle(\tilde{x})+\frac{x_3}{h}[\bar{\boldsymbol{\omega}}^1](\tilde{x})$. Moreover, from (3.5)$_{6,7}$, we obtain

$$\begin{aligned}
&[\bar{\boldsymbol{\sigma}}^0]=\mathbf{0},\quad \langle\bar{\boldsymbol{\sigma}}^0\rangle=\tfrac{1}{h}\hat{\mathbf{C}}_{33}\left([\bar{\mathbf{u}}^1]+h\mathbf{e}_3\wedge\langle\bar{\boldsymbol{\omega}}^0\rangle\right)+\hat{\mathbf{C}}_{\alpha 3}(\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle+\mathbf{e}_\alpha\wedge\langle\bar{\boldsymbol{\omega}}^0\rangle),\\
&[\bar{\boldsymbol{\mu}}^0]=\mathbf{0},\quad \langle\bar{\boldsymbol{\mu}}^0\rangle=\tfrac{1}{h}\hat{\mathbf{A}}_{33}[\bar{\boldsymbol{\omega}}^1]+\hat{\mathbf{A}}_{\alpha 3}\langle\bar{\boldsymbol{\omega}}^0_{,\alpha}\rangle.
\end{aligned}$$

Let us consider problem $\mathcal{P}_1$. Using the Gauss-Green's theorem, we get an homogenous equilibrium system at order 1 and we can also characterize the jumps and mean values of the stress vector and couple stress vector at order 1, respectively:

$$\begin{aligned}
&[\bar{\boldsymbol{\sigma}}^1]=-\hat{\mathbf{C}}_{3\alpha}\left([\bar{\mathbf{u}}^1_{,\alpha}]+h\mathbf{e}_3\wedge\langle\bar{\boldsymbol{\omega}}^0_{,\alpha}\rangle\right)-h\hat{\mathbf{C}}_{\alpha\beta}\left(\langle\bar{\mathbf{u}}^0_{,\alpha\beta}\rangle+\mathbf{e}_\beta\wedge\langle\bar{\boldsymbol{\omega}}^0_{,\alpha}\rangle\right)=-\mathbf{T}^1_{\alpha,\alpha},\\
&\langle\bar{\boldsymbol{\sigma}}^1\rangle=\tfrac{1}{h}\hat{\mathbf{C}}_{33}\left([\bar{\mathbf{u}}^2]+h\mathbf{e}_3\wedge\langle\bar{\boldsymbol{\omega}}^1\rangle\right)+\hat{\mathbf{C}}_{\alpha 3}(\langle\bar{\mathbf{u}}^1_{,\alpha}\rangle+\mathbf{e}_\alpha\wedge\langle\bar{\boldsymbol{\omega}}^1\rangle),\\
&[\bar{\boldsymbol{\mu}}^1]=-\hat{\mathbf{A}}_{3\alpha}[\bar{\boldsymbol{\omega}}^1_{,\alpha}]-h\hat{\mathbf{A}}_{\alpha\beta}\langle\bar{\boldsymbol{\omega}}^0_{,\alpha\beta}\rangle+h\langle\bar{\boldsymbol{\sigma}}^0\rangle\wedge\mathbf{e}_3-\mathbf{T}^1_\alpha\wedge\mathbf{e}_\alpha,\\
&\langle\bar{\boldsymbol{\mu}}^1\rangle=\tfrac{1}{h}\hat{\mathbf{A}}_{33}[\bar{\boldsymbol{\omega}}^2]+\hat{\mathbf{A}}_{\alpha 3}\langle\bar{\boldsymbol{\omega}}^1_{,\alpha}\rangle,
\end{aligned}$$

where $\mathbf{T}^1_\alpha:=\hat{\mathbf{C}}_{3\alpha}\left([\bar{\mathbf{u}}^1]+h\mathbf{e}_3\wedge\langle\bar{\boldsymbol{\omega}}^0\rangle\right)+\hat{\mathbf{C}}_{\alpha\beta}\left(\langle\bar{\mathbf{u}}^0_{,\beta}\rangle+\mathbf{e}_\beta\wedge\langle\bar{\boldsymbol{\omega}}^0\rangle\right)$ can be interpreted as a generalized membrane stress vector, (see [43]). By using the same methodology of Section 2(d), it is possibile to combine the order 0 and order 1 conditions in order to have a better approximation of the interface transmission problem. Noting with $\tilde{\boldsymbol{\sigma}}^\varepsilon=\boldsymbol{\sigma}^0+\varepsilon\boldsymbol{\sigma}^1$, $\tilde{\boldsymbol{\mu}}^\varepsilon=\boldsymbol{\mu}^0+\varepsilon\boldsymbol{\mu}^1$, $\tilde{\boldsymbol{\omega}}^\varepsilon=\boldsymbol{\omega}^0+\varepsilon\boldsymbol{\omega}^1+\varepsilon^2\boldsymbol{\omega}^2$ and $\tilde{\mathbf{u}}^\varepsilon=\mathbf{u}^0+\varepsilon\mathbf{u}^1+\varepsilon^2\mathbf{u}^2$, one has:

$$\begin{aligned}
&[\tilde{\boldsymbol{\sigma}}^\varepsilon]=-\hat{\mathbf{C}}^\varepsilon_{3\alpha}\left([\tilde{\mathbf{u}}^\varepsilon_{,\alpha}]+h\mathbf{e}_3\wedge\langle\tilde{\boldsymbol{\omega}}^\varepsilon_{,\alpha}\rangle\right)-h^\varepsilon\hat{\mathbf{C}}^\varepsilon_{\alpha\beta}\left(\langle\tilde{\mathbf{u}}^\varepsilon_{,\alpha\beta}\rangle+\mathbf{e}_\beta\wedge\langle\tilde{\boldsymbol{\omega}}^\varepsilon_{,\alpha}\rangle\right)+o(\varepsilon)=-\mathbf{T}^\varepsilon_{\alpha,\alpha}+o(\varepsilon),\\
&\langle\tilde{\boldsymbol{\sigma}}^\varepsilon\rangle=\tfrac{1}{h^\varepsilon}\hat{\mathbf{C}}^\varepsilon_{33}\left([\tilde{\mathbf{u}}^\varepsilon]+h^\varepsilon\mathbf{e}_3\wedge\langle\tilde{\boldsymbol{\omega}}^\varepsilon\rangle\right)+\hat{\mathbf{C}}^\varepsilon_{\alpha 3}(\langle\tilde{\mathbf{u}}^\varepsilon_{,\alpha}\rangle+\mathbf{e}_\alpha\wedge\langle\tilde{\boldsymbol{\omega}}^\varepsilon\rangle)+o(\varepsilon),\\
&[\tilde{\boldsymbol{\mu}}^\varepsilon]=-\hat{\mathbf{A}}^\varepsilon_{3\alpha}[\tilde{\boldsymbol{\omega}}^\varepsilon_{,\alpha}]-h^\varepsilon\hat{\mathbf{A}}^\varepsilon_{\alpha\beta}\langle\tilde{\boldsymbol{\omega}}^\varepsilon_{,\alpha\beta}\rangle+h^\varepsilon\langle\tilde{\boldsymbol{\sigma}}^\varepsilon\rangle\wedge\mathbf{e}_3-\mathbf{T}^\varepsilon_\alpha\wedge\mathbf{e}_\alpha+o(\varepsilon),\\
&\langle\tilde{\boldsymbol{\mu}}^\varepsilon\rangle=\tfrac{1}{h^\varepsilon}\hat{\mathbf{A}}^\varepsilon_{33}[\tilde{\boldsymbol{\omega}}^\varepsilon]+\hat{\mathbf{A}}^\varepsilon_{\alpha 3}\langle\tilde{\boldsymbol{\omega}}^\varepsilon_{,\alpha}\rangle+o(\varepsilon).
\end{aligned}\tag{3.6}$$

(d) The interface conditions

Using the results of Section 3(c), the contact conditions for hard micropolar interface can be recovered. The transmission problem at order 0 can be summarized as follows:

Governing equations:

$$\begin{cases} \bar{\sigma}^0_{ji,j} + f_i = 0, & \text{in } \Omega_\pm, \\ \bar{\mu}^0_{ji,j} - \epsilon_{kji}\bar{\sigma}^0_{kj} + m_i = 0, & \text{in } \Omega_\pm, \\ \bar{\sigma}^0_{ji} n_j = g_i, \ \bar{\mu}^0_{ji} n_j = l_i, & \text{on } \Gamma_1, \\ u^0_i = 0, \ \omega^0_i = 0, & \text{on } \Gamma_0. \end{cases}$$

Interface conditions on $S_\pm$:

$$\begin{cases} [\bar{\mathbf{u}}^0] = \mathbf{0}, & [\bar{\mathbf{t}}^0] = \mathbf{0}, \\ [\bar{\boldsymbol{\omega}}^0] = \mathbf{0}, & [\bar{\boldsymbol{\mu}}^0] = \mathbf{0}. \end{cases}$$

The zero-th order transmission conditions provide the continuity of the displacements and microrotation, and of their conjugated counterparts, namely the stress vector and couple stress vector. As shown in [43], this result is typical of adhesives having similar rigidities of the adherents. In this case, the interface is perfect.

The order 1 transmission problem states:

Governing equations:

$$\begin{cases} \bar{\sigma}^1_{ji,j} = 0, & \text{in } \Omega_\pm, \\ \bar{\mu}^1_{ji,j} - \epsilon_{kji}\bar{\sigma}^1_{kj} = 0, & \text{in } \Omega_\pm, \\ \bar{\sigma}^1_{ji} n_j = 0, \ \bar{\mu}^1_{ji} n_j = 0, & \text{on } \Gamma_1, \\ u^1_i = 0, \ \omega^1_i = 0, & \text{on } \Gamma_0. \end{cases}$$

Interface conditions on $S_\pm$:

$$\begin{cases} [\bar{\mathbf{u}}^1] = h\left(\hat{\mathbf{C}}^{-1}_{33}\langle\bar{\boldsymbol{\sigma}}^0\rangle - \mathbf{e}_3 \wedge \langle\bar{\boldsymbol{\omega}}^0\rangle \right. \\ \qquad \left. -\hat{\mathbf{C}}^{-1}_{33}\hat{\mathbf{C}}_{\alpha 3}(\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle + \mathbf{e}_\alpha \wedge \langle\bar{\boldsymbol{\omega}}^0\rangle)\right), \\ [\bar{\boldsymbol{\sigma}}^1] = -h\hat{\mathbf{C}}_{3\alpha}\hat{\mathbf{C}}^{-1}_{33}\langle\bar{\boldsymbol{\sigma}}^0_{,\alpha}\rangle - h\hat{\mathbf{D}}_{\alpha\beta}(\langle\bar{\mathbf{u}}^0_{,\alpha\beta}\rangle + \mathbf{e}_\beta \wedge \langle\bar{\boldsymbol{\omega}}^0_{,\alpha}\rangle), \\ [\bar{\boldsymbol{\omega}}^1] = h\hat{\mathbf{A}}^{-1}_{33}\langle\bar{\boldsymbol{\mu}}^0\rangle - h\hat{\mathbf{A}}^{-1}_{33}\hat{\mathbf{A}}_{\alpha 3}\langle\bar{\boldsymbol{\omega}}^0_{,\alpha}\rangle, \\ [\bar{\boldsymbol{\mu}}^1] = -h\hat{\mathbf{A}}_{3\alpha}\hat{\mathbf{A}}^{-1}_{33}\langle\bar{\boldsymbol{\mu}}^0_{,\alpha}\rangle - h\hat{\mathbf{B}}_{\alpha\beta}\langle\bar{\boldsymbol{\omega}}^0_{,\alpha\beta}\rangle \\ \qquad + h\langle\bar{\boldsymbol{\sigma}}^0\rangle \wedge \mathbf{e}_3 - \mathbf{T}^1_\alpha \wedge \mathbf{e}_\alpha, \end{cases}$$

with $\hat{\mathbf{D}}_{\alpha\beta} := \hat{\mathbf{C}}_{\alpha\beta} - \hat{\mathbf{C}}_{3\alpha}\hat{\mathbf{C}}^{-1}_{33}\hat{\mathbf{C}}_{\beta 3}$ and $\hat{\mathbf{B}}_{\alpha\beta} := \hat{\mathbf{A}}_{\alpha\beta} - \hat{\mathbf{A}}_{3\alpha}\hat{\mathbf{A}}^{-1}_{33}\hat{\mathbf{A}}_{\beta 3}$. At order 1, we obtain a mixed interface model in which the displacements, microrotations, traction vector, and couple stress vector are discontinuous across the interface. The right-hand sides of the above relations depend all on the micropolar state at order 0 and its in-plane derivatives. The contact model provides a generalization of the conditions obtained for elastic hard interfaces to the case of micropolar elasticity, see [43]. Following the same approach adopted in [43], it can be shown that the above relations comprise also the soft micropolar interface model, derived in [48]. The inspection of order 1 conditions highlights that the jumps of $[\bar{\boldsymbol{\sigma}}^1]$ and $[\bar{\boldsymbol{\mu}}^1]$ present a discontinuity at the interface, depending, respectively, on the divergence of a surface stress tensor, namely $\hat{\mathbf{D}}_{\alpha\beta}\langle(\bar{\mathbf{u}}^0_{,\alpha\beta}\rangle + \mathbf{e}_\beta \wedge \langle\bar{\boldsymbol{\omega}}^0_{,\alpha}\rangle)$, and of a surface couple stress tensor, namely $\hat{\mathbf{B}}_{\alpha\beta}\langle\bar{\boldsymbol{\omega}}^0_{,\alpha\beta}\rangle$. This yields to a micropolar membrane equilibrium problem defined on the the interface plane. We can state that the above conditions represent a micropolar version of the Gurtin-Murdoch's interface model [52–55], which take into account not only the presence of surface stresses, but also the effects of surface couple stresses. Moreover, $[\bar{\boldsymbol{\mu}}^1]$ contains also bending and torsional moments contributions coming from the stress vectors $\langle\bar{\boldsymbol{\sigma}}^0\rangle$ and $\mathbf{T}^1_\alpha$.

4. A benchmark problem

In the present section, the equilibrium problem of a composite bar subjected to an axial load is studied in the framework of classical (C), micropolar (M) and strain gradient (SG) elasticity. The composite beam comprises two adherents and an intermediate adhesive. The closed-form solutions are known (see [56,57]) and will be employed to analytically and numerically assess the imperfect contact problem in the framework of micropolar and strain gradient elasticity, respectively. In the sequel, the exact solution of the equilibrium problem for a three-phase bar is compared with the solution of the asymptotic model, i.e., a two-phase bar with an imperfect interface. For the sake of simplicity, we omitted the dependences on ε of the unknown functions and constitutive coefficients. The geometry of the composite beam is illustrated in Fig. 2. The bar axis is identified with the abscissa x_3, while x_α denotes the cross-section coordinates. The

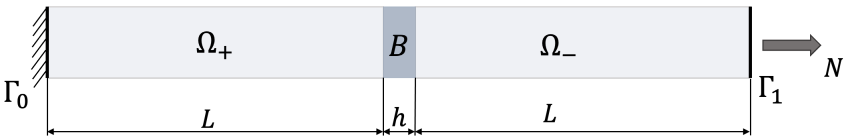

Figure 2. The composite bar geometry under an axial load

adherents are $\Omega_- := A \times (0, L)$, $\Omega_+ := A \times (L+h, 2L+h)$, and the adhesive $B := A \times (L, L+h)$, with $h << L$. We note with A the beam cross-sectional area. The boundary $\Gamma_0 = \{0\}$ is fully clamped, while $\Gamma_1 = A \times \{2L+h\}$ is a free end. The materials are assumed to be isotropic with Lamé's constants λ and μ, and Cosserat's couple modulus κ. The classical E and micropolar Young's moduli $\bar{E}$ are defined in terms of the Lamé's constants as follows: $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and $\bar{E} = \frac{(2\mu+\kappa)(3\lambda+2\mu+\kappa)}{(2\lambda+2\mu+\kappa)}$. The material properties of the adherents $\Omega_\pm$ and adhesive B are shown in Table 1. In the absence of distributed longitudinal loads, the solution of a bar under an axial load

Table 1. Constitutive material properties, [58].

Moduli	$\Omega_\pm$	B
λ (MPa)	78500	75900
μ (MPa)	13700	13500
κ (MPa)	139	149

takes the following form [56,57]:

$$\text{SG solution}: \ u(x_3) = a_0 + a_1 x_3 + a_2 e^{x_3/\ell} + a_3 e^{-x_3/\ell}, \quad \text{C, M solution}: \ u(x_3) = b_0 + b_1 x_3,$$

where u represents the axial displacement, with integration constants a_K and b_K. The analytical solution is valid for both the adherents $\Omega_\pm$ and the adhesive B. The chosen boundary conditions at the extremities of the composite beam are:

$$(BC)_{SG}: \begin{cases} u(0) = 0 & \tau(0) = 0 & \text{on } \Gamma_0, \\ \tilde{\sigma}(2L+h) = N & u'(2L+h) = \epsilon_0 & \text{on } \Gamma_1, \end{cases} \quad (BC)_{C,M}: \begin{cases} u(0) = 0 & \text{on } \Gamma_0, \\ \sigma(2L+h) = N & \text{on } \Gamma_1, \end{cases}$$

where $\tau = EA\ell^2 u''$ and $\tilde{\sigma} = EA(u' - \ell^2 u''')$ denote the boundary axial double force and the axial force for the SG model, respectively, $\sigma = EAu'$ is the axial force for the C model (for the M case, E must be switched with its micropolar version $\bar{E}$), and ϵ_0 a prescribed boundary strain. Note that the boundary conditions are valid for both the exact three-phase model and two-phase model with imperfect contact. For the three-layer composite bar, classical interface continuity conditions (CC) are considered on $x_3 = L$ and $x_3 = L+h$ so that

$$(CC)_{SG}: \begin{cases} [u] = 0, & [u'] = 0, \\ [\tilde{\sigma}] = 0, & [\tau] = 0, \end{cases} \quad (CC)_{C,M}: \begin{cases} [u] = 0, \\ [\sigma] = 0. \end{cases}$$

Concerning the two-layer bar with imperfect contact (IC), the hard strain gradient (2.15) and micropolar interface conditions (3.6) are adapted for the one-dimensional case and reduce to:

$$(IC)_{SG}: \begin{cases} [\tilde{\sigma}] = 0, \\ \langle \tilde{\sigma} \rangle = (\hat{\lambda} + 2\hat{\mu}) \left(\frac{1}{h} f(l)[u] + (1 - f(l)) \langle u' \rangle \right), \\ [\tau] = (1 - f(l))(\hat{\lambda} + 2\hat{\mu})([u] - h\langle u' \rangle), \\ \langle \tau \rangle = \frac{\ell}{2} g(l)(\hat{\lambda} + 2\hat{\mu})[u'], \end{cases} \quad (IC)_M: \begin{cases} [\sigma] = 0, \\ \langle \sigma \rangle = \frac{1}{h}(\hat{\lambda} + 2\hat{\mu} + \hat{\kappa})[u]. \end{cases}$$

The adherents length is $L = 1$ mm the thickness of the adhesive h and the internal length of the adherents $\bar{\ell}$ are kept constant and both equal to 0.1 mm, while the internal length of the adhesive ℓ can take different values. Following [44], the results are given in terms of the dimensionless units. For a prescribed normal force N, we set $\bar{x} = \frac{x_3}{2L+h}$, and $\bar{u} = \frac{EA}{NL} u$.

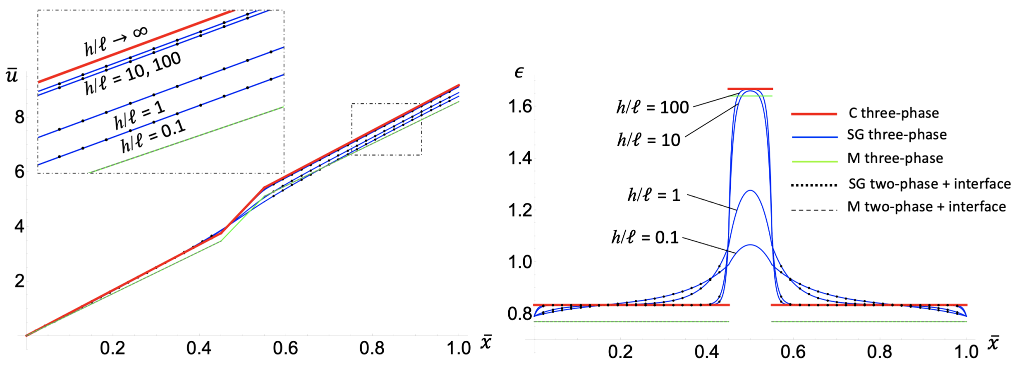

Figure 3. Dimensionless axial displacement $\bar{u}$ and strain ϵ vs $\bar{x}$, for $h/L = 0.1$

Figure 3 show the comparison between the exact three-phase SG and M solutions (continuous blue and green lines) and the asymptotic solutions of the two-phase problem with SG and M imperfect contact (dotted black and dashed grey lines), in terms of the dimensionless axial displacement $\bar{u}$ and longitudinal strain $\bar{\epsilon}$. The plots highlight the influence of the adhesive characteristic length of the microstructure compared with the classical elastic solution of a three-layer beam (red continuous line), for which $h/\ell \to \infty$, i.e., $\ell = 0$. The diagrams reveal the following results:

(i) The imperfect interface solution perfectly matches the solution of the three-phase problem, for both strain gradient and micropolar cases; as expected, by letting $\ell \to 0$ (or $\kappa \to 0$), the strain gradient (or micropolar) solution converges towards the classical elastic one.

(ii) The internal length parameter influences the trend of the general solution and its effect becomes more prominent when $h/\ell = 1$. Indeed, when the characteristic length is of the same order of magnitude as the adhesive thickness, a stiffening phenomenon is highlighted. In this case, the strain gradient beam appears to be more rigid with respect to the elastic one. A similar stiffening effect is also highlighted by the micropolar solution, which is typical of micro-scale devices.

(iii) The longitudinal strain is strongly affected by the size ratio h/ℓ, implying that the model is able to describe size effects and, also, can capture high strain gradients at the extremities.

5. Conclusions

An original form of the hard imperfect interface laws have been derived, through the asymptotic expansions method, based on the strain gradient elastic theory and the micropolar elasticity theory. The order 0 and order 1 transmission problem for hard interfaces have been presented in Section 2(e), for the case of gradient elasticity, and in Section 3(d), for the case of micropolar elasticity, respectively. The transmission conditions at order 0 and 1 presented a similar structure compared to other hard adhesives in other different multiphysic frameworks [43].

Concerning the strain gradient elastic model, the order 0 conditions provide the continuity of the displacements and of the traction vector at the interface, typical of hard adhesives. The interface model at order 1 represented a mixed model and highlighted a discontinuity of the whole set of kinematical and conjugated fields, depending on the scale length function $f(l)$. This function has been proved to produce size effect phenomena in [48].

Concerning the micropolar model, the order 0 conditions provide the continuity of the displacements, microrotations, stress vector and couple stress vector at the interface level, generalizing the typical behavior of hard elastic interfaces. At order 1, a mixed interface model is obtained. Morerover, both the jumps of the order 1 stress vector and couple stress vector depend on an in-plane micropolar membrane equilibrium problem, containing bending and torsional moments contributions coming from the stress vectors at order 0.

In order to have a better approximation of the interface transmission problem, a condensed form of the contact laws (2.15) and (3.6) has been given for both cases, comprising the order 0 and order 1 models.

The benchmark problem of a composite layered bar subjected to a tensile load, considering both cases of strain gradient and micropolar elasticity, allowed us to assess the validity of the previous asymptotic procedures. The diagrams show that the solution of the three-phase model perfectly coincides and is well-approximated by the solution of the two-phase model with imperfect contact. The internal length parameter has a significant influence on the mechanical response, especially when $h/\ell = 1$. A stiffening phenomenon, typical of structural components at the micro-scale, is captured by both the strain gradient and micropolar asymptotic models. Moreover, the asymptotic models, comprising the order 0 and order 1, allow to predict the behavior of the bonded joint even for moderately thick adhesives, having relative thickness $h/L = 0.1$.

Although simple in its design, this analytical/numerical example proves to be efficient in assessing the impact of the thin adhesive on the structural behavior of the overall system, taking also into account materials presenting an internal microstructure. As proposed in [43], the condensed form of the interface conditions can be employed to write the variational formulation of the transmission problem in the case of strain gradient and micropolar elasticity. The weak formulation can be easily implemented in a general FE code and represents a key step towards simulating numerically imperfect interface effects inside composite materials and constructing structured finite elements with non-classical contact conditions.

Finally, the suggested methodology has shown effectiveness and simplicity, suitable for various applications involving thin films and interfaces at micro- and nano-scales and considering the influence of their microstructure.

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