DYNAMICS OF AN EVOLUTION EQUATION WITH SINGULAR POTENTIAL

Dynamics of an evolution equation with singular potential

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Abstract. We consider a fourth order non linear evolution equation with a logarithmic potential. We study the asymptotic behaviour of the solutions and their regularity. Moreover, we provide an analysis of the set of stationary and travelling wave solutions.

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1. Introduction

We consider the following fourth order nonlinear evolution equation:

$$\begin{cases} u_t + u_{xxxx} = \left[F'(u_x)\right]_x, & x \in I, \quad t > 0, \\ u = u_{xx} = 0, & x \in \partial I, \\ u(x,0) = u^0(x), & x \in I, \end{cases} \tag{1}$$

where $I = \left(-\frac{L}{2}, \frac{L}{2}\right)$ is a bounded interval with $|I| \leq 1$ and $F : (-1,1) \to \mathbb{R}$ is given by

$$F(z) = -\theta_0 z^2 + \theta_1 \left[(1+z)\log(1+z) + (1-z)\log(1-z)\right],$$
$$F'(z) = -2\theta_0 z + \theta_1 \log\left(\frac{1+z}{1-z}\right),$$
$$F''(z) = -2\theta_0 + 2\theta_1 \frac{1}{1-z^2},$$

where $0 < \theta_1 < \theta_0$ are given real numbers (see figure 1 below).

The choice of a logarithmic potential is physically relevant from thermodynamical motivations (see [6]). Moreover, models involving non convex potentials arise in a large variety of scientific fields like image processing, population

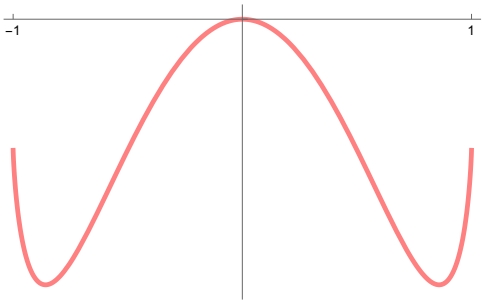

FIGURE 1. The potential $F(u)$, with $\theta_0 = \frac{3}{4}$ and $\theta_1 = \frac{1}{2}$.

dynamics, thin films and many others (see [4] and references cited therein). Equation (1) describes the L^2-gradient flows of the energy functional

$$E(u) = \frac{1}{2}\int_I u_{xx}^2 dx + \int_I F(u_x)dx,$$

which is a Lyapunov function, in fact multiplying the equation by u_t and integrating in I produces:

$$\frac{d}{dt}\left[\frac{1}{2}\int_I u_{xx}^2 dx + \int_I F(u_x)dx\right] = -\int_I u_t^2 dx \leq 0.$$

In order to prove the existence of solutions of (1) we can follow several approaches available in the literature for the Cahn-Hilliard equation (see [10], [12]) :

$$\begin{cases} v_t = -\left[v_{xx} - F'(v)\right]_{xx}, & x \in I, \quad t > 0, \\ v_x = v_{xxx} = 0, & x \in \partial I, \\ v(x,0) = v^0(x), & x \in I, \end{cases} \tag{2}$$

We note in fact that if $v(x,t)$ is the solution of (2) then $u(x,t) = \int_0^x v(y,t)dy$ is the solution of (1).
For example in [7] the solution is obtained as the limit of the solution of a regularized equation in which the singular potential is replaced by an approximating polynomial while in [11] the authors introduce a viscous term and prove convergence. Moreover, Miranville and Zelik prove that the solution is separated from the singular points of the potential. It is worth noting that it is possible to prove existence of solutions without considering the convergence of a regularized equation as is done in [8].
We refer to the above papers and to the references cited therein for the detailed proofs of the following results:

Theorem 1.1. *Let $u_0 \in D$, then equation* (1) *admits a unique solutions in the class $u \in L^\infty([0,T], D) \cap C([0,T], L^2(I))$ for all $T > 0$ where*

$$D = \{u \in H^2(I) : \quad u = u_{xx} = 0 \quad in \quad \partial I, \quad \|u_x\|_{L^\infty(I)} \leq 1, \quad F'(u_x) \in L^2(I)\}.$$

The following result is the so called separation property which will play a crucial role for the estimates provided in the next section (see also [9] for recent developments):

Theorem 1.2. *Let $u(t)$ be a solution of equation* (1) *with initial datum satisfying*

$$|u_x^0| < 1 - \delta_0, \quad \text{with} \quad \delta_0 \in (0,1).$$

Then, for every $T > 0$, there exists a positive constant $\delta = \delta(T, \delta_0)$ such that

$$\|u_x(t)\|_{L^\infty(I)} \leq 1 - \delta, \qquad \forall t \geq T.$$

The rest of the paper is organized as follows: in section 2 we investigate the asymptotic behaviour of solutions while in Section 3 we analyse the set of stationary solutions. Section 4 is concerned with the study of travelling waves while section 5 contains some conclusive remarks.

2. Energy estimates

In this section we study the asymptotic behaviour of the solutions together with their regularity. For simplicity we denote the norms of $L^2(I)$ and $L^\infty(I)$ by $\|\cdot\|$ and $\|\cdot\|_\infty$ respectively. The estimates of this section are based on the strict separation property that is given by the next theorem.

Theorem 2.1. *Let $u^0 \in C^1(I)$ such that $|u_x^0| < 1 - \delta_0$, with $\delta_0 > 0$ then the derivative u_x of the solution of* (1) *satisfies the strict separation property, that is, there exists $\delta = \delta(u^0, \theta_0, \theta_1) > 0$ such that*

$$|u_x(t)| < 1 - \delta, \qquad \forall t > 0.$$

Proof. Since $u \in H^2(I)$ then $u \in C^1$ from the continuous embedding of $H^1(I)$ in $C(I)$. Then the thesis is a consequence of theorem 1.2 (for more details see also [10], Remark 4.25). □

Theorem 2.2. *Let u be a solution of* (1) *which satisfies the hypotheses of Theorem* 2.1 *and with $\theta_0 - \theta_1 < \frac{1}{2}$. Then*

$$\lim_{t \to +\infty} \|u(t)\| = 0.$$

Proof. Multiplying the equation (1) by u and integrating in I we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u_{xx}\|^2 = \int_I [F'(u_x)]_x \, u dx,$$

from which

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u_{xx}\|^2 + \int_I F'(u_x) u_x dx = 0. \tag{3}$$

For simplicity we rewrite (3) in the following way

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u_{xx}\|^2 = \int_I h(u_x) dx, \tag{4}$$

where $h(p) = -pF'(p)$. Thanks to Lemma 2.3 below we have:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u_{xx}\|^2 \leq 2(\theta_0 - \theta_1)\|u_x\|^2 \\ &\leq (\theta_0 - \theta_1)\left\{\|u\|^2 + \|u_{xx}\|^2\right\}, \end{aligned} \tag{5}$$

that is

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + [1 - (\theta_0 - \theta_1)]\|u_{xx}\|^2 \leq (\theta_0 - \theta_1)\|u\|^2, \tag{6}$$

where we have used the interpolation inequality

$$\|u_x\|^2 \leq \frac{1}{2}\left\{\|u\|^2 + \|u_{xx}\|^2\right\}.$$

From the boundary condition we have

$$\|u\| \leq \|u_x\| \leq \|u_{xx}\|, \tag{7}$$

from which we can write

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + [1 - 2(\theta_0 - \theta_1)]\|u\|^2 \leq 0. \tag{8}$$

Then from Gronwall's lemma we have

$$\|u\|^2 \leq \|u_0\|^2 e^{-2\gamma t},$$

where we have set

$$\gamma := 1 - 2(\theta_0 - \theta_1) > 0.$$

From the previous inequality we obtain the thesis. □

The following Lemma has been used in the proof of the previous theorem.

Lemma 2.3. *Let $h(p) = -pF'(p)$ then there exists $k \in (0,1)$ such that*

$$h(p) \leq kp^2, \qquad \forall p \in (-1,1).$$

Proof. In details we need to prove that there exists $k \in (0,1)$ such that

$$(k - 2\theta_0)p^2 + \theta_1 p \log\left(\frac{1+p}{1-p}\right) \geq 0, \qquad \forall p \in (-1,1). \tag{9}$$

It is easy to see that

$$p \log\left(\frac{1+p}{1-p}\right) \geq 0, \qquad \forall p \in (-1,1),$$

and we should conclude that the choice $k \geq 2\theta_0$ is sufficient to obtain the thesis. However, this choice implies a restriction on the values of the parameters, that is $\theta_1 < \theta_0 < \frac{1}{2}$. In order to obtain a more general result we prove the following equality

$$p \log\left(\frac{1+p}{1-p}\right) = qp^2 + R(p), \qquad \forall p \in (-1,1), \tag{10}$$

where $R(p) \geq 0$ in $(-1,1)$ and $q \in (0,2]$. We prove (10) for $p \in (0,1)$ since the other case is similar. From $p > 0$ we have

$$p \log\left(\frac{1+p}{1-p}\right) - qp^2 \geq 0 \iff \log\left(\frac{1+p}{1-p}\right) - qp \geq 0.$$

We observe that $\alpha(p) = \log\left(\frac{1+p}{1-p}\right) - qp$ is zero for $p = 0$ and $\alpha'(p) = \frac{2}{1-p^2} - q > 0$ if and only if $p^2 > 1 - \frac{2}{q}$ which is true for any $p \in (0,1)$ if $q \leq 2$. This proves (10). Thanks to the above discussion we can rewrite (9) in the following way

$$(k + \theta_1 q - 2\theta_0)p^2 + \theta_1 R(p) \geq 0, \qquad \forall p \in (-1,1). \tag{11}$$

Then, in order to obtain the thesis it is sufficient that k satisfies

$$k > 2\theta_0 - q\theta_1.$$

This choice of k produces the following restriction on the parameters :

$$\theta_1 < \theta_0 < \frac{q}{2}\theta_1 + \frac{1}{2}.$$

We observe that the optimal choice is $q = 2$ and as a consequence $k = 2(\theta_0 - \theta_1) \in (0,1)$. □

As a trivial consequence of theorem 2.2 we have the existence of an absorbing set in $L^2(I)$.

Proposition 2.4. *Under the hypotheses of Theorems* 2.1 *and* 2.2 *there exists* $T_1 := \frac{1}{\gamma}\log\|u_0\|$ *and* $R_1 > 0$ *such that*

$$\|u(t)\| \leq R_1 := 1, \qquad \forall t \geq T_1.$$

In the following lines we provide some estimates that will be useful for the study of regularity of the solutions. We start by recalling an important classical result:

Lemma 2.5. *(The Uniform Gronwall's Lemma). Let* f, g, h *be three positive locally integrable functions on* $(t_0, +\infty)$ *which satisfy*

$$f' \leq fg + h, \qquad \forall t \geq t_0,$$

and for $s > 0$

$$\int_t^{t+s} f(\tau)d\tau \leq a_1, \qquad \int_t^{t+s} g(\tau)d\tau \leq a_2, \qquad \int_t^{t+s} h(\tau)d\tau \leq a_3, \qquad \forall t \geq t_0,$$

where a_1, a_2, a_3 *are positive constants. Then*

$$f(t+s) \leq \left(\frac{a_1}{s} + a_3\right)e^{a_2}, \qquad \forall t \geq t_0.$$

Proposition 2.6. *Under the hypotheses of Theorems* 2.1 *and* 2.2 *there exists* $T_2 := T_1 + \frac{1}{2\gamma} > 0$ *and* $R_2 > 0$ *such that*

$$\|u_x(t)\| \leq R_2 := R_1 e^{\frac{\theta_0^2}{2\gamma}}, \qquad \forall t \geq T_2.$$

Proof. We start the proof showing an estimate that will be useful for proving the thesis. By using (6) and (7) we can write

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|u_{xx}\|^2 \leq 0. \tag{12}$$

We integrate (12) in $(t, t+\rho)$, with $\rho > 0$, obtaining

$$\int_t^{t+\rho} \|u_x(s)\|^2 ds \leq \int_t^{t+\rho} \|u_{xx}(s)\|^2 ds \leq \frac{1}{2\gamma}\|u(t)\|^2 \leq \frac{1}{2\gamma}R_1^2, \qquad \forall t > T_1. \tag{13}$$

We multiply the equation (1) by u_{xx} and integrate over I:

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \|u_{xxx}\|^2 + \int_I F''(u_x)u_{xx}^2 dx = 0, \tag{14}$$

from which

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \|u_{xxx}\|^2 + 2\theta_1 \int_I \frac{u_{xx}^2}{1-u_x^2} dx = 2\theta_0 \|u_{xx}\|^2. \tag{15}$$

Using and interpolating inequality we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \|u_{xxx}\|^2 + 2\theta_1 \int_I \frac{u_{xx}^2}{1-u_x^2} dx \leq \theta_0 \left\{ \frac{1}{\beta}\|u_x\|^2 + \beta\|u_{xxx}\|^2 \right\}, \tag{16}$$

where $\beta \in \left(0, \frac{1}{\theta_0}\right]$. Thanks to Lemma 2.3 we can write

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + (1-\beta\theta_0)\|u_{xxx}\|^2 \leq \frac{\theta_0}{\beta}\|u_x\|^2. \tag{17}$$

For simplicity we set $\beta = \frac{1}{\theta_0}$, then we can apply Uniform Gronwall's lemma to the following inequality

$$\frac{d}{dt}\|u_x\|^2 \leq 2\theta_0^2 \|u_x\|^2, \tag{18}$$

with $f = \|u_x\|^2$, $g = 2\theta_0^2$, $h = 0$ and

$$\int_t^{t+\rho} 2\theta_0^2 d\tau = 2\theta_0^2 \rho := a_1, \quad a_2 := 0, \int_t^{t+\rho} \|u_x\|^2 d\tau \leq \frac{1}{2\gamma}R_1^2 := a_3.$$

Then we obtain

$$\|u_x(t+\rho)\|^2 \leq \frac{a_3}{\rho}e^{a_1} = \frac{1}{\rho}\frac{1}{2\gamma}R_1^2 e^{2\theta_0^2\rho}, \qquad \forall t \geq T_1,$$

and setting $\rho = \frac{1}{2\gamma}$ we conclude

$$\|u_x(t)\| \leq R_1 e^{\frac{\theta_0^2}{2\gamma}} := R_2, \qquad \forall t \geq T_1 + \frac{1}{2\gamma} := T_2. \tag{19}$$

□

Proposition 2.7. *Under the hypotheses of Theorems* 2.1 *and* 2.2 *there exists* $T_2 := T_1 + \frac{1}{2\gamma}$ *and* $R_3 > 0$ *such that*

$$\|u_{xx}(t)\| \leq R_3 := (\sqrt{1+4\theta_1})R_1 e^{\frac{\bar{\theta}}{4\gamma}}, \qquad \forall t \geq T_2.$$

Proof. We integrate inequality (17) in $(t, t+\rho)$ with $\beta = \frac{1}{2\theta_0}$ and obtain

$$\begin{aligned}\int_t^{t+\rho} \|u_{xxx}(s)\|^2 ds &\leq 4\theta_0^2 \int_t^{t+\rho} \|u_x(s)\|^2 ds + \|u_x(t)\|^2 \\ &\leq R_1^2 \left[e^{\frac{\theta_0^2}{\gamma}} + 2\frac{\theta_0^2}{\gamma} \right], \qquad \forall t \geq T_2,\end{aligned} \tag{20}$$

where we have used (13) and Proposition (2.6).
Multiplying the equation (1) by u_t and integrating over I produces

$$\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \theta_1 \int_I \log\left(\frac{1+u_x}{1-u_x}\right) u_{xt} dx = \theta_0 \frac{d}{dt}\|u_x\|^2. \tag{21}$$

For simplicity we introduce the function

$$\Gamma(z) = F(z) + 2\theta_0 z^2, \qquad z \in (-1, 1).$$

We observe that $\Gamma(0) = 0$ while $\Gamma'(z)$ is positive in $(0, 1)$ and negative in $(-1, 0)$. This means that $\Gamma(z) \geq 0$ in $(-1, 1)$, then we can write:

$$\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \frac{d}{dt}\int_I \Gamma(u_x) dx \leq \theta_0 \frac{d}{dt}\|u_x\|^2 + \int_I \Gamma(u_x) dx,$$

using (7) and (17) we have

$$\frac{d}{dt}\|u_{xx}\|^2 + 2\frac{d}{dt}\int_I \Gamma(u_x) dx \leq 2\int_I \Gamma(u_x) dx + 4\frac{\theta_0^2}{\beta}\|u_{xx}\|^2.$$

If we set $\bar{\theta} := \max\left\{1, \frac{4\theta_0^2}{\beta}\right\}$, we obtain the inequality

$$\frac{d}{dt}\left\{\|u_{xx}\|^2 + 2\int_I \Gamma(u_x) dx\right\} \leq \bar{\theta}\left\{\|u_{xx}\|^2 + 2\int_I \Gamma(u_x) dx\right\}.$$

We introduce for simplicity the following function

$$S(z) = \Gamma(z) - 2\theta_1 z^2.$$

It is easy to see that $S(z) \leq 0$ in $(-1, 1)$ in fact we have $S''(z) \leq 0$ in $(-1, 1)$ with $S''(0) = 0$ which means that $S'(z)$ is decreasing. Since $S'(0) = 0$ we have that $S'(0) < 0$ in $(0, 1)$ and as a consequence $S(z)$ is decreasing in $(0, 1)$ with $S(0) = 0$. In the same manner we deal with the case $z \in (0, 1)$.
From the previous reasoning we obtain

$$\int_I \Gamma(u_x) dx \leq 2\theta_1 \|u_x\|^2. \tag{22}$$

Then, using also (7), we have

$$\|u_{xx}\|^2 + 2\int_I \Gamma(u_x) dx \leq \|u_{xx}\|^2 + 4\theta_1 \|u_x\|^2 \leq (1 + 4\theta_1)\|u_{xx}\|^2.$$

Integrating the previous inequality in $(t, t+\rho)$ and using (13) we obtain

$$\int_t^{t+\rho} \|u_{xx}(s)\|^2 ds + 2\int_t^{t+\rho}\int_I \Gamma(u_x(x,s))dx \le (1+4\theta_1)\int_t^{t+\rho} \|u_{xx}(s)\|^2 ds$$
$$\le (1+4\theta_1)\frac{1}{2\gamma}R_1^2, \qquad \forall t \ge T_1. \tag{23}$$

Now we are able to apply uniform Gronwall's lemma with $f(t) = \|u_{xx}\|^2 + 2\int_I \Gamma(u_x)dx$, $g = \bar\theta$ and $h = 0$, with

$$a_1 = (1+4\theta_1)\frac{1}{2\gamma}R_1^2, \qquad a_2 = \bar\theta\rho, \qquad a_3 = 0.$$

Then:

$$\|u_{xx}(t+\rho)\|^2 + 2\int_I \Gamma(u_x(t+\rho))dx \le \left[(1+4\theta_1)\frac{1}{2\gamma\rho}R_1^2\right] e^{\bar\theta\rho}, \qquad \forall t \ge T_1,$$

from which, setting $\rho = \frac{1}{2\gamma}$, we conclude

$$\|u_{xx}(t)\| \le (\sqrt{1+4\theta_1})R_1 e^{\frac{\bar\theta}{4\gamma}} := R_3, \qquad \forall t \ge T_1 + \frac{1}{2\gamma} := T_2. \tag{24}$$

□

Theorem 2.8. *Under Hypothesis of theorem* (2.2) *the solution of* (1) *is in* $L^\infty([T_2, +\infty), H^4(I))$, *where* $T_2 > 0$ *is as in* (24).

Proof. We first estimate the norm $L^2((0,t), L^2(I))$ of u_t. By integrating (21) on $(0,t)$ and using that $\Gamma(z) \ge 0$ for $z \in (-1,2)$ we obtain

$$\begin{aligned}\int_0^t \|u_t(s)\|^2 ds &\le \frac{1}{2}\|u_{xx}^0\|^2 + \int_I \Gamma(u_x^0)dx + \theta_0\|u_x(t)\|^2 \\ &\le \frac{1}{2}\|u_{xx}^0\|^2 + \int_I \Gamma(u_x^0)dx + \theta_0 R_2^2 := R_4, \qquad \forall t \ge T_2,\end{aligned} \tag{25}$$

where we have also used Proposition 2.6.
We differentiate equation (1) with respect to t, multiply by $t^2 u_t$ and integrate in I:

$$\int_I t^2 u_{tt}u_t dx + \int_I t^2 u_{xxxxt}u_t dx = \int_I [F'(u_x)]_{xt}\, t^2 u_t dx,$$

that is

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|tu_t\|^2 - t\|u_t\|^2 + t^2\|u_{xxt}\|^2 = -t^2\int_I F''(u_x)u_{tx}^2 dx \\ &= 2\theta_0 t^2\|u_{tx}\|^2 - 2\theta_1 t^2\int_I \frac{u_{tx}^2}{1-u_x^2}dx \le 2(\theta_0-\theta_1)t^2\|u_{tx}\|^2 \\ &\le t^2\|u_{txx}\|^2 + t^2(\theta_0-\theta_1)^2\|u_t\|^2,\end{aligned}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\|tu_t\|^2 \le t^2(\theta_0-\theta_1)^2\|u_t\|^2 + t\|u_t\|^2.$$

Integrating the previous inequality with respect to time in $(0,t)$ we have

$$\frac{1}{2}\|tu_t\|^2 \leq \int_0^t \left\{s^2(\theta_0-\theta_1)^2+s\right\}\|u_t(s)\|^2 ds,$$

then setting $t=1$ produces

$$\|u_t(1)\|^2 \leq 2\left\{(\theta_0-\theta_1)^2+1\right\}\int_0^1 \|u_t(s)\|^2 ds \leq 2\left\{(\theta_0-\theta_1)^2+1\right\}R_4,$$

where we have used (25). We observe that if we set $\tilde{u}(t,x) := u(s+t-1,x)$, with $s>0$, then $\tilde{u}_t(t,x) := u_t(s+t-1,x)$ and $\tilde{u}_t(1,x) := u_t(s,x)$. By rescaling time we conclude that

$$\|u_t(s)\|^2 \leq 2\left\{(\theta_0-\theta_1)^2+1\right\}R_4, \qquad s>0,$$

that is $u_t \in L^\infty(\mathbb{R}, L^2(I))$.
Now we pass to estimate the L^2 norm of $F''(u_x)u_{xx}$:

$$\int_I \left[F''(u_x)\right]^2 u_{xx}^2 dx = \int_I \left[\frac{2\theta_1}{1-u_x^2}-2\theta_0\right]^2 u_{xx}^2 dx \leq 8\int_I \left[\frac{\theta_1^2}{(1-u_x^2)^2}+\theta_0^2\right]u_{xx}^2 dx$$
$$\leq 8\left[\frac{\theta_1^2}{\delta^4}+\theta_0^2\right]\|u_{xx}\|^2 \leq 8\left[\frac{\theta_1^2}{\delta^4}+\theta_0^2\right]R_3^2 := R_5, \qquad \forall t>T_2,$$

where we have used proposition (2.7) and the hypothesis of theorem (2.1) from which

$$1-u_x^2 > \delta^2, \qquad \text{and} \qquad u_x^2 < (1-\delta)^2.$$

We recall that the real number δ depends on u_x^0.
From the expression of the equation (1) we have that

$$u_{xxxx} = -u_t + F''(u_x)u_{xx},$$

from which we obtain that $u_{xxxx} \in L^\infty(\mathbb{R}, L^2(I))$, in particular

$$\|u_{xxxx}\| \leq 2\left\{(\theta_0-\theta_1)^2+1\right\}^{\frac{1}{2}} R_4 + R_5^{\frac{1}{2}} := R_6, \qquad \forall t \geq T_2.$$

Moreover, using an interpolating inequality we obtain

$$\|u_{xxx}\| \leq \|u_{xx}\|^{\frac{1}{2}}\|u_{xxxx}\|^{\frac{1}{2}} \leq (R_3R_6)^{\frac{1}{2}}, \qquad \forall t \geq T_2,$$

and this concludes the proof. □

We end this section providing the uniform bound for the $H^4(I)$ norm of u with a different method.

Proposition 2.9. *Under the hypotheses of Theorems* 2.1 *and* 2.2 *there exists* $T_3>0$ *and* $R_7>0$ *such that*

$$\|u_{xxxx}(t)\| \leq R_7, \qquad \forall t \geq T_3.$$

Proof. We differentiate the equation four times with respect to x, multiply by u_{xxxx} and integrate over I:

$$\frac{1}{2}\frac{d}{dt}\|u_{xxxx}\|^2 + \|u_{xxxxxx}\|^2 = \int_I [F'(u_x)]_{xxx} u_{xxxxxx} dx$$
$$:= \int_I G(u) u_{xxxxxx} dx \leq \|G(u)\| \|u_{xxxxxx}\| \leq \frac{1}{2}\left\{\|G(u)\|^2 + \|u_{xxxxxx}\|^2\right\},$$

that is

$$\frac{d}{dt}\|u_{xxxx}\|^2 \leq \|G(u)\|^2.$$

Then we need to estimate $\|G(u)\|^2$, in details we have:

$$\begin{aligned}\int_I G(u)^2 dx &= \int_I \left\{F^{iv}(u_x)u_{xx}^3 + 2F''(u_x)u_{xx}u_{xxx} + F'''(u_x)u_{xx}u_{xxx}\right.\\ &\quad \left. + F''(u_x)u_{xxxx}\right\}^2 dx\\ &\leq \int_I \left\{4[F^{iv}(u_x)]^2 u_{xx}^6 + 4[2F''(u_x) + F'''(u_x)]^2 u_{xx}^2 u_{xxx}^2\right.\\ &\quad \left. + 4[F''(u_x)]^2 u_{xxxx}^2\right\} dx := A_1 + A_2 + A_3.\end{aligned}$$

We estimate the three terms separately. We start with the first:

$$\begin{aligned}A_1 &= \int_I 4[F^{iv}(u_x)]^2 u_{xx}^6 dx = 16\theta_1 \int_I \frac{1+3u_x^2}{(1-u_x^2)^3} u_{xx}^6 dx \leq 16\theta_1 \frac{1+3(1-\delta)^2}{\delta^6}\|u_{xx}\|_6^6\\ &\leq 16\theta_1 \frac{1+3(1-\delta)^2}{\delta^6}\|u_{xx}\|^2\|u_{xx}\|_\infty^4 \leq 16\theta_1 \frac{1+3(1-\delta)^2}{\delta^6}\|u_{xx}\|^4\|u_{xxx}\|^2\\ &\leq 16\theta_1 \frac{1+3(1-\delta)^2}{\delta^6} R_3^5 R_6, \qquad \forall t \geq T_2.\end{aligned}$$

where we have used an interpolation inequality (see [3]), the inequality $\|v\|_\infty^2 \leq \|v\|\|v_x\|$ and the previous results.
For the second term we have:

$$\begin{aligned}A_2 &:= \int_I 4[2F''(u_x) + F'''(u_x)]^2 u_{xx}^2 u_{xxx}^2 dx\\ &= 64\int_I \left[-\theta_0 + \frac{\theta_1}{1-u_x^2} + \frac{\theta_1 u_x}{(1-u_x^2)^2}\right]^2 u_{xx}^2 u_{xxx}^2 dx\\ &\leq 256\int_I \left[\theta_0^2 + \frac{\theta_1^2}{(1-u_x^2)^2} + \frac{\theta_1^2 u_x^2}{(1-u_x^2)^4}\right] u_{xx}^2 u_{xxx}^2 dx\\ &\leq 256\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4} + \frac{\theta_1^2(1-\delta)^2}{\delta^8}\right]\int_I u_{xx}^2 u_{xxx}^2 dx\\ &\leq 256\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4} + \frac{\theta_1^2(1-\delta)^2}{\delta^8}\right]\|u_{xx}\|_\infty^2\|u_{xxx}\|^2\\ &\leq 256\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4} + \frac{\theta_1^2(1-\delta)^2}{\delta^8}\right]\|u_{xx}\|\|u_{xxx}\|^3\\ &\leq 256\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4} + \frac{\theta_1^2(1-\delta)^2}{\delta^8}\right] R_3(R_3R_6)^{\frac{3}{2}}, \qquad \forall t \geq T_2.\end{aligned}$$

For the third term we have:

$$A_3 = \int_I 4[F''(u_x)]^2 u_{xxxx}^2 dx = 4\int_I \left[-2\theta_0 + \frac{2\theta_1}{1-u_x^2}\right]^2 u_{xxxx}^2$$
$$\leq 8\int_I \left[4\theta_0^2 + \frac{4\theta_1^2}{(1-u_x^2)^2}\right] u_{xxxx}^2 dx \leq 32\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4}\right]\|u_{xxxx}\|^2.$$

Then, putting all together, we obtain the inequality

$$\frac{d}{dt}\|u_{xxxx}\|^2 \leq C_1\|u_{xxxx}\|^2 + C_2, \qquad \forall t \geq T_2,$$

where

$$C_1 = C_1(\delta,\theta_0,\theta_1) = 32\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4}\right],$$
$$C_2 = C_2(\delta,\theta_0,\theta_1) = 16\theta_1\frac{1+3(1-\delta)^2}{\delta^6}R_3^5R_6 + 256\left[\theta_0^2 + \frac{\theta_1^2}{\delta^4} + \frac{\theta_1^2(1-\delta)^2}{\delta^8}\right]R_3(R_3R_6)^{\frac{3}{2}}.$$

In order to apply Uniform Gronwall's lemma we need that $u_{xxxx} \in L^1_{Loc}(\mathbb{R}^+, L^2(I))$, for this purpose we multiply the equation by u_{xxxx} and integrate over I:

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \|u_{xxxx}\|^2 = \int_I [F'(u_x)]_x u_{xxxx} dx \leq \frac{1}{2}\|u_{xxxx}\|^2 + \frac{1}{2}\int_I [F''(u_x)]^2 u_{xx}^2 dx$$
$$\leq \frac{1}{2}\|u_{xxxx}\|^2 + \frac{1}{2}\int_I \left[\frac{2\theta_1}{1-u_x^2} - 2\theta_0\right]^2 u_{xx}^2 dx$$
$$\leq \frac{1}{2}\|u_{xxxx}\|^2 + 4\int_I \left[\frac{\theta_1^2}{(1-u_x^2)^2} + \theta_0^2\right] u_{xx}^2 dx \leq \frac{1}{2}\|u_{xxxx}\|^2 + 4\left[\frac{\theta_1^2}{\delta^4} + \theta_0^2\right]\|u_{xx}\|^2$$

from which we have

$$\frac{d}{dt}\|u_{xx}\|^2 + \|u_{xxxx}\|^2 \leq 8\left[\frac{\theta_1^2}{\delta^4} + \theta_0^2\right]\|u_{xx}\|^2.$$

We integrate the previous inequality with respect to time in $(t, t+\rho)$, with $\rho > 0$, obtaining

$$\int_t^{t+\rho} \|u_{xxxx}(s)\|^2 ds \leq \|u_{xx}(t)\|^2 + 8\left[\frac{\theta_1^2}{\delta^4} + \theta_0^2\right]\int_t^{t+\rho}\|u_{xx}(s)\|^2 ds$$
$$\leq 8\left[\frac{\theta_1^2}{\delta^4} + \theta_0^2\right](1+4\theta_1)\frac{1}{2\gamma}R_1^2 + R_3^2 = a_1, \qquad \forall t \geq T_2,$$

where we have used (23) and Proposition (2.7). We are now able to apply uniform Gronwall's lemma with a_1 as above and

$$a_2 = C_1\rho, \qquad\qquad a_3 = C_2\rho,$$

from which we obtain

$$\|u_{xxxx}(t+\rho)\|^2 \leq \left(\frac{a_1}{s} + C_2 s\right)e^{C_1\rho}, \qquad \forall t \geq T_2.$$

By setting $\rho = 1$ we conclude the proof:

$$\|u_{xxxx}(t)\| \leq \left\{ 8 \left[\frac{\theta_1^2}{\delta^4} + \theta_0^2 \right] (1 + 4\theta_1) \frac{1}{2\gamma} R_1^2 + R_3^2 + C_2 \right\}^{\frac{1}{2}} e^{\frac{C_1}{2}} := R_7,$$

$$\forall t \geq T_3 := T_2 + 1.$$

□

3. Stationary solutions

In this section we study the set of stationary solutions, that is, the solutions of the following ODE:

$$u_{xxxx} = [F'(u_x)]_x,$$

that is

$$u_{xxx} = F'(u_x) + K. \tag{26}$$

For simplicity we set $K = 0$ (so we have the constant solution $u = 0$), moreover, if we introduce the function $v = u_x$, we can write

$$v_{xx} = F'(v),$$

that is

$$v_{xx} = -2\theta_0 v + \theta_1 \log\left(\frac{1+v}{1-v}\right), \qquad v \in (-1,1).$$

We can rewrite the previous equation in hamiltonian form:

$$\begin{cases} v_x = h = \frac{\partial H}{\partial h}, \\ h_x = F'(v) = -\frac{\partial H}{\partial v}, \end{cases} \tag{27}$$

where $v \in (-1,1)$ and with hamiltonian

$$H(v,h) = \frac{1}{2}h^2 + \theta_0 v^2 - \theta_1 \left[(1+v)\log(1+v) + (1-v)\log(1-v)\right].$$

The trajectories are given by the level curves of the hamiltonian, while the fixed points are solutions of the following system

$$\begin{cases} h = 0, \\ F'(v) = 0, \end{cases}$$

that is the points of the set:

$$\left\{ v \in (-1,1) : \qquad \rho v = \log\left(\frac{1+v}{1-v}\right) \right\},$$

where for simplicity we have introduced $\rho = \frac{2\theta_0}{\theta_1} > 2$. In order to study the previous set we can rewrite the above equality in the following way

$$\frac{1+v}{1-v} = e^{\rho v}. \tag{28}$$

We observe that for any $\rho > 0$, $v = 0$ is a solution of (28). We set

$$h(v) := \frac{1+v}{1-v} - e^{\rho v},$$

and look for $v \in (-1,1)$ such that $h(v) = 0$. Its derivative is

$$h'(v) = \frac{2}{(1-v)^2} - \rho e^{\rho v},$$

then we pass to study the equation $h'(v) = 0$. If $\rho = 2$, the condition $h'(v) = 0$ becomes

$$\frac{1}{(1-v)^2} = e^{2v},$$

that is

$$\frac{1}{(1-v)} = e^v.$$

We check that

$$\frac{1}{(1-v)} > e^v, \qquad \forall v \in (0,1),$$

that is

$$(1-v) < e^{-v}, \qquad \forall v \in (0,1).$$

For $v = 0$ we have the equality while

$$\left[(1-v) - e^{-v}\right]' = e^{-v} - 1 < 0, \qquad v \in (0,1).$$

Moreover, by a similar reasoning it is easy to show that:

$$\frac{1}{(1-v)} < e^v, \qquad v \in (-1,0).$$

From the above discussion we have obtained that $h'(v) \geq 0$ in $(-1,1)$ and it is zero only for $v = 0$. This means that $h(v)$ is strictly increasing in $(-1,1)$ and it is negative in (-1,0), positive in (0,1) with

$$h(-1) = -e^{-2}, \qquad \text{and} \qquad \lim_{v \to 1^-} h(v) = +\infty.$$

We conclude that for $\rho = 2$ we have a unique fixed point, that is, $(v,h) = (0,0)$.

Case $\rho > 2$.
For any $\rho > 0$ we have

$$h(-1) = -e^{-\rho}, \qquad \text{and} \qquad \lim_{v \to 1^-} h(v) = +\infty.$$

The study of $h'(v)$ will clarify the number of fixed points of system (27). At first we show that the equation $h'(v) = 0$ has exactly two solutions. We observe that

$$h'(-1) = \frac{1}{2} - \rho e^{-\rho} > 0 \qquad \Longleftrightarrow \qquad e^\rho > 2\rho, \tag{29}$$

and since

$$(e^\rho - 2\rho)' = e^\rho - 2 > 0, \qquad \forall \rho > 2,$$

we conclude that (29) is true for $\rho > 2$. We study the zeroes of $h'(v)$ that is solutions of the equation

$$\frac{2}{(1-v)^2} = \rho e^{\rho v}. \tag{30}$$

The function on the r.h.s. of (30) and that on l.h.s are strictly increasing since

$$\left[\frac{2}{(1-v)^2}\right]' = \frac{4}{(1-v)^3} > 0.$$

In the interval $(-1,0)$ the function on the r.h.s. takes values in $[\rho e^{-\rho}, \rho]$ while the other in $[\frac{1}{2}, 2]$. We observe that

$$\frac{\rho}{e^\rho} < \frac{2}{e^2} < \frac{1}{2}, \qquad \text{and} \quad \rho > 2,$$

there is a unique intersection that we called x_M.
For $v \in (0,1)$ since

$$\lim_{v\to 1^-} \frac{2}{(1-v)^2} = +\infty, \tag{31}$$

thanks to the intermediate value theorem there is only one intersection in $(0,1)$, that we call x_m. Then for $\rho > 2$ the function $h'(v)$ has two zeros, that is $x_{M,m}$ which are a local maximum and minimum for $h(v)$ respectively. Since $h(0) = 0$ and $x_M < 0 < x_m$, from the above analysis we have $h(x_M) > 0$ and $h_(x_m) < 0$ and as a consequence the function $h(v)$ has three zeroes:

$$x_- \in (-1, x_M), \qquad x = 0, \qquad x_+ \in (x_m, 1),$$

which correspond to three fixed points for system (27)

$$O = (0,0), \qquad P_\pm = (x_\pm, 0).$$

The functional Jacobian is:

$$J(v,h) = \begin{pmatrix} 0 & 1 \\ -2\theta_0 + \frac{2\theta_1}{1-v^2} & 0 \end{pmatrix},$$

whose trace and determinant are respectively:

$$Tr(J) = 0, \qquad det(J) = 2\left[\theta_0 - \frac{\theta_1}{1-v^2}\right].$$

The fixed point $O = (0,0)$ has two complex conjugate pure imaginary eigenvalues

$$\lambda_{1,2} = \pm i\sqrt{2}\sqrt{\theta_0 - \theta_1},$$

and it is always a centre surrounded by periodic orbits.
The fixed points $P_\pm$ have two real eigenvalues with opposite sign:

$$\lambda_{1,2} = \pm\sqrt{2}\sqrt{\left[\frac{\theta_1}{1-x_\pm^2} - \theta_0\right]},$$

and they are saddles. This follows from the following computation:

$$x_+ > x_m > 0 \Rightarrow 1 - x_+^2 < 1 - x_m^2,$$

then using that $h'(x_m) = 0$ and that $x_m > 0$ we have

$$\frac{2}{1-x_+^2} > \frac{2}{1-x_m^2} = \rho e^{\rho x_m} > \rho = \frac{2\theta_0}{\theta_1},$$

that is

$$\frac{\theta_1}{1-x_+^2} > \theta_0.$$

The same computations can be performed for x_-.

Remark 3.1. *We observe that , for any value of the Hamiltonian, the orbits of* (27) *are symmetric with respect both horizontal and vertical axis:*

$$h_\pm(v) = \pm\sqrt{2}\sqrt{H - \theta_0 v^2 + \theta_1\left[(1+v)\log(1+v) + (1-v)\log(1-v)\right]}.$$

In fact, using that

$$(1+v)\log(1+v) + (1-v)\log(1-v) = \left[(1-v)\log(1-v) + (1+v)\log(1+v)\right],$$

we obtain $h_\pm(v) = h_\pm(-v)$.

Theorem 3.2. *Let* $\rho > 4\log(2)$*, then system* (27) *admits a family of periodic orbits for* $H \in (0, \tilde{H})$ *where* $\tilde{H} := \theta_1 \left[\frac{\rho}{2} - 2\log(2)\right]$.

Proof. The existence of closed orbits can be shown by proving that the functions $h_\pm(v)$ have two intersections with the horizontal axis. Then, we must solve the equation:

$$\theta_1 \left[(1+v)\log(1+v) + (1-v)\log(1-v)\right] = \theta_0 v^2 - H,$$

for simplicity we divide by θ_1, set $\tilde{H} := \frac{H}{\theta_1}$ and rewrite the previous equality in the following form:

$$(1+v)\log(1+v) + (1-v)\log(1-v) = \frac{\rho}{2}v^2 - \tilde{H}.$$

We call the left and right hand side of the previous equality $L(v)$ and $R(v)$ respectively. The existence of solutions of the equation

$$L(v) = R(v), \qquad v \in (-1, 1), \tag{32}$$

depends on the parameters $\rho > 2$ and $\tilde{H} > 0$. We observe that $L(0) = 0$, $R(0) = -\tilde{H} < 0$ while $L(\pm 1) = 2\log(2)$ and $R(\pm 1) = \frac{\rho}{2} - \tilde{H}$. Then if

$$\frac{\rho}{2} - \tilde{H} > 2\log(2),$$

from continuity, there should be at least one intersection between $R(v)$ and $L(v)$ for $v \in (0, 1)$. By the symmetry of the curves $h_\pm(v)$ we obtain the symmetric intersections in (-1,0) and this concludes the proof. □

Remark 3.3. *We observe that the hypotheses of the previous theorem are not sharp.*
In fact, we can obtain periodic orbits, that is, solutions of the equation $L(v) = R(v)$ *for higher values of* H*. The critical value of* H*, that we call* H_c*, corresponds to the case in which the solution of* (32) *correspond to the fixed points* $P_\pm$*. Then, the critical value can be computed by solving the following system for any fixed value of* $\rho > 2$*:*

$$\begin{cases} L(v) = R(v), \\ F'(v) = 0. \end{cases} \tag{33}$$

We note that the second equation can be rewritten as $L'(v) = R'(v)$ *that is the limit value of* H *correspond to the case in which the curve* $L(v)$ *and* $R(v)$ *are tangent.*
Since for $H = H_c$ *the intersections of* $h_\pm(v)$ *with the horizontal axis are the fixed points* $P_\pm$*, then the corresponding solutions are* $P_\pm$ *and their stable and unstable curves. In particular we have two heteroclinic connections between fixed points.*
The system (33) *can be solved numerically for any fixed value of the parameters. For example for* $\theta_1 = \frac{1}{2}, \theta_0 = \frac{3}{4}$*, that is* $\rho = 3 > 2\log(2)$*, we obtain the critical value* $H_c \approx 0.23038$ *while the fixed points are* $P_\pm = (x_\pm, 0)$ *with* $x_\pm \approx \pm 0.85856$ *(see figure* 2 *below).*

Remark 3.4. *We note that an upper bound for the critical value H_c is given by the quantity $\theta_1 \frac{\rho}{2} = \theta_0$. In fact for $H \geq \theta_0$ we have that $R(\pm 1) \leq 0$ and as a consequence $R(v) \leq 0$ for all $v \in [-1, 1]$. Since $L(v) \geq 0$ and it is zero only for $z = 0$ there are no intersections between $L(v)$ and $R(v)$ (and as a consequence no possibility to be tangent at some point).*

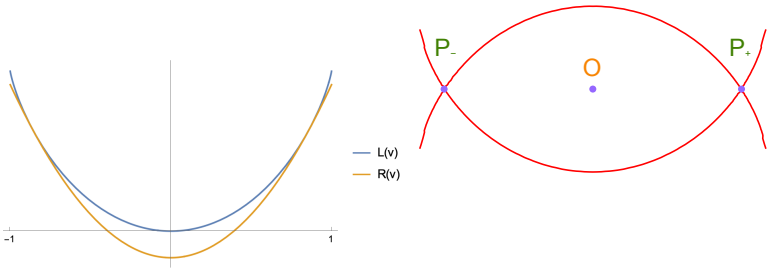

FIGURE 2. We have set $\theta_1 = \frac{1}{2}, \theta_0 = \frac{3}{4}$, that is $\rho = 3 > 2\log(2)$. For $H = H_c$ the curves $L(v)$ and $R(v)$ are tangent (left panel), solutions are the fixed points $P_\pm$ and their stable and unstable curves (right panel).

Theorem 3.5. *Let $\rho > 2$ then there exists a value $H = H_c(\rho)$ for the Hamiltonian which corresponds to heteroclinic solutions and fixed points $P_\pm$ of system* (27)*. If $H > H_c$ solutions are unbounded.*

Proof. The thesis follows form the previous arguments and observing that for $H > H_c$ solutions do not intersect the horizontal axis in the plane (v, h). In fact, since $h = v'$, if $h(t) > \delta > 0$ (or $h(t) < -\delta < 0$) for all $t > 0$ then $v(t)$ is unbounded. □

In the last result we come back to consider the original equation (26):

Theorem 3.6. *The set of stationary solutions of equation* (1) *consists in the following classes:*

1. *constant solutions $u(x) = c$, $c \in \mathbb{R}$;*
2. *linear solutions of the form $\ell_\pm(x) = d_\pm + x_\pm \cdot x$ where $d_\pm \in \mathbb{R}$.*
3. *periodic solutions;*
4. *solutions connecting two linear functions of the form $\ell_\pm(x)$;*
5. *unbounded solutions.*

Proof.

1. The constant solutions $u(x) = c$, $c \in \mathbb{R}$ correspond to the value $H = 0$ of the hamiltonian system.

2. Linear solutions $\ell_\pm(x) = d_\pm + x_\pm \cdot x$ correspond to to the fixed points $P_\pm = (x_\pm, 0)$ of the hamiltonian system.
3. Since periodic solutions $h(v)$ of the hamiltonian systems (corresponding to the values $H \in (0, H_c)$) are symmetric with respect to both horizontal and vertical axis also the function $u(x) = \int v(x)dx$ is periodic.
4. For $H = H_c$ the heteroclinic connection satisfy $v \to x_\pm$ for $x \to \pm\infty$ that is $u_x \to x_\pm$ and as a consequence $u(x)$ connects at $\pm\infty$ two linear solutions of the form $\ell_\pm(x) = d_\pm + x_\pm \cdot x$ where $d_\pm$ are proper constants.
5. For $H > H_c$, since $v(x) = u_x(x)$ is unbounded also $u(x)$ is too.

□

4. Travelling waves

In this section we study another important class of special solutions, that is, travelling wave solutions of equation:

$$u_t + u_{xxxx} - [F'(u_x)]_x = 0. \tag{34}$$

We observe that every constant function $u \equiv a$ with $a \in \mathbb{R}$ is a solution of (34). In order to obtain travelling waves solutions we make the usual ansatz

$$u(x,t) = v(x - ct) = v(\xi),$$

from which we find the equation

$$cv' + v^{iv} - [F'(v')]' = 0.$$

Integrating the equation on $(0, \xi)$ we obtain

$$v''' - [F'(v')] + cv = v'''(0) - [F'(v'(0))] + cv(0) := K.$$

For simplicity we set $K = 0$ without affecting the subsequent analysis. Then, we study the following equation

$$v''' - F'(v') + cv = 0, \tag{35}$$

which has a unique constant solution $v \equiv 0$. We observe that if v is solution of (35), also $-v$ solves the same equation while $v(-\xi)$ and -$v(-\xi)$ are not solutions of the same equations. For this reason we expect that solutions are neither odd nor even.

Theorem 4.1. *Equation* (34) *does not admit solitary waves solutions.*

Proof. We consider the following functional:

$$E(\xi) = H(\xi) + cvv', \tag{36}$$

where $H(\xi) = \frac{1}{2}[v''(\xi)]^2 - F(v')$ is the Hamiltonian of the system (27) (obtained from (35) with $c = 0$). We observe that

$$\frac{d}{d\xi}E(\xi) = v''v''' - F'(v')v'' + c(v')^2 + cvv'' = c(v')^2, \tag{37}$$

from which we obtain that the function $E(\xi)$ is increasing for $c > 0$ and decreasing for $c < 0$ along the solutions of equation (35). We recall that a solitary wave solution of equation (34) corresponds to a homoclinic solution of equation (35). If v is a homoclinic orbit, we should have

$$\lim_{\xi \to \pm\infty} E(\xi) = 0,$$

since $E \equiv 0$ on $v \equiv 0$. However, since the function E is monotone we conclude that on such orbits we should have that $E(\xi)$ is constant, that is, using (37) $v' \equiv 0$, that means $v \equiv 0$. Then we conclude that there are no non trivial homoclinic solutions. □

Since for $c = 0$ equation (35) admits a family of periodic solutions one could ask if in a neighbourhood of $c = 0$ periodic orbits persist. However, using again monotonicity of $E(\xi)$ we have the following result.

Theorem 4.2. *Equation* (35) *does not admit periodic solutions for* $c \neq 0$.

We observe that from (36) and (37) that if v is unbounded then

$$\lim_{\xi \to +\infty} E(\xi) = \pm\infty,$$

where the sign of the limit depends on the sign of c. In particular we have

Lemma 4.3. *$E(\xi)$ is unbounded if and only if $v(\xi)$ is unbounded.*

Proof. We prove the result for $c > 0$ since the other case is similar.
(i) If $v \to \infty$ as $\xi \to +\infty$ then $v' > 0$ for ξ large and from the expression (36) of $E(\xi)$ we obtain

$$\lim_{\xi \to +\infty} E(\xi) = +\infty.$$

We have the same situation for $\xi \to -\infty$. If $v \to -\infty$ as $\xi \to +\infty$ then $v' < 0$ for ξ large and using (36) we have again

$$\lim_{\xi \to +\infty} E(\xi) = +\infty. \tag{38}$$

(ii) If (38) is true we have that for ξ large $v' > 0$ and as a consequence v is unbounded. □

Then if we look for bounded solutions we need that $E(\xi)$ is bounded and this happens if and only if

$$\lim_{\xi \to +\infty} \frac{d}{d\xi} E(\xi) = 0,$$

that is if and only if

$$\lim_{\xi \to +\infty} v'(\xi) = 0. \tag{39}$$

In particular we have:

Proposition 4.4. *Let $c \neq 0$ and let $v(\xi)$ be a solution of* (35) *that satisfies* (39) *then $v(\xi) \to 0$ as $\xi \to \infty$.*

Proof. We suppose that (39) is satisfied, then

$$\lim_{\xi\to+\infty} v(\xi) = d,$$

while from the equation (35) we obtain

$$\lim_{\xi\to+\infty} v'''(\xi) = -dc,$$

and since $v' \to 0$ we must have $d = 0$. □

From the above discussion we obtain the following result that characterises the set of bounded solutions.

Theorem 4.5. *Let $c \neq 0$. All bounded solutions of equation* (35) *satisfy*

$$\lim_{\xi\to+\infty} v(\xi) = 0.$$

Proof. From the above discussion we have that if $v' \to 0$ then $v(\xi)$ is bounded for $\xi > 0$. On the contrary, if $v(\xi)$ is bounded for $\xi > 0$ then $E(\xi)$ is bounded for $\xi > 0$ and, as a consequence, $v' \to 0$. Then $v(\xi)$ is bounded for $\xi > 0$ if and only if $v' \to 0$ is satisfied. We obtain the thesis using Proposition 4.4. □

Before we conclude the analysis of bounded and unbounded solutions we need some considerations on the linearised equation around $v = 0$:

$$V''' - F''(0)V - cV = 0, \tag{40}$$

whose characteristic polynomial is

$$\lambda^3 + 2(\theta_0 - \theta_1)\lambda - c = 0.$$

We observe that we have

$$\Delta := \frac{c^2}{4} + \frac{8}{27}(\theta_0 - \theta_1)^2 > 0,$$

for any choice of the parameters c and $\theta_0 > \theta_1 > 0$. As a consequence the roots of the polynomial are given by

$$\begin{aligned}
\lambda_1 &= A_1 + A_2,\\
\lambda_2 &= -\tfrac{1}{2}(A_1 + A_2) + i\tfrac{\sqrt{3}}{2}(A_1 - A_2),\\
\lambda_2 &= -\tfrac{1}{2}(A_1 + A_2) - i\tfrac{\sqrt{3}}{2}(A_1 - A_2),
\end{aligned}$$

where

$$A_1 = \sqrt[3]{\tfrac{c}{2} + \sqrt{\Delta}}, \qquad A_2 = \sqrt[3]{\tfrac{c}{2} - \sqrt{\Delta}}.$$

We note that

$$A_1 + A_2 > 0 \quad \Longleftrightarrow \quad c > 0.$$

In this case, if we consider the system of three first order equations associated to (35) we obtain the following results on the dimension of the local stable and unstable manifolds of O:

$$dim(W^s_{Loc}(0)) = 2, \qquad dim(W^u_{Loc}(0)) = 1.$$

From the above results we obtain that, for $c > 0$ the only bounded solutions are the solutions converging to 0 on the stable manifold. For $c < 0$, we have the following characterisation of bounded and unbounded solutions.

Theorem 4.6. *For $c < 0$, equation* (35) *admits only one bounded solution which converges to 0 as $\xi \to +\infty$.*

Proof. Since all the bounded solutions approach 0 as $\xi \to +\infty$ and since we have a $1-$dimensional stable manifold we conclude that there is only one solution which converges to 0, that is the $1-$dimensional stable manifold of the origin O. □

5. Conclusion

In this work we have studied a fourth order nonlinear evolution with logarithmic nonlinearity. Existence and uniqueness of solutions are obtained with similar arguments used in [11] while we concentrate on the regularity and asymptotic behaviour of solutions. The analysis of stationary solutions reveals the existence of periodic solutions with different amplitude and period. As for other pattern-formation (see also [2], [5]) equation they represent metastable patterns and are an indication of the presence of coarsening phenomena (see for example [14], [13]). Finally, travelling waves analysis produces the non existence of solitons.

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