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Decisions in Economics and Finance manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract We analyze a financial market model with heterogeneous interacting agents where fundamentalists and chartists are considered. We assume that fundamentalists are homogeneous in their trading strategy but heterogeneous in their belief about the asset's fundamental value. On the other hand, we consider that chartists, when they are optimistic become overconfident and they trade more than when they are pessimistic. Consequently, our model dynamics are driven by a one-dimensional piecewise-linear continuous map with three linear branches. We investigate the bifurcation structures in the map's parameter space and describe the endogenous fear and greed market dynamics arising from our asset-pricing model.

1 Introduction

In this paper we contribute to the financial literature in understanding of what is driving the dynamics of financial markets. In particular, we focus on the role of qualitative Dynamical Systems Theory in modelling financial markets populated by heterogeneous agents as described in the pioneering works

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of [18], [9]. Several papers have been able to replicate the stylized facts of financial markets such as volatility clustering, asymmetry and excess of kurtosis documented in [38] and [16] among others. Most of these works rely on the introduction of a non-linearity component in the model, some examples are [4], [14], [32]. As explained in [50], non-linearity is used to build the structure of several heterogeneous interacting agents in financial markets. In particular, there are three main basic frameworks. First, introducing non-linearity in the trading rules as in [18] and [12]. Second, the Adaptive Belief System outlined by [9] where investors adapt their beliefs over time by choosing from different predictors of future values of endogenous variables, see e.g. [8], [25], [20]. Finally, a third mechanism of endogenous dynamics is based on market interactions where connections are due to the trading activity of heterogeneous speculators. Follow this line of research the works of [13], [46] and [21].

A further framework on building agent-based financial market models proposes the approach of the piecewise-linear maps. Piecewise-linear (PWL) maps can be considered as an approximation of more complicated non-linear maps, and this embodies an advantage given that they allow us an extensively analytical study of the model. Moreover, PWL maps exhibit a further bifurcation structures with respect to those occurring in standard maps: border collision bifurcations (see [43, 42]). Border collision bifurcations (BCB) are a peculiar phenomenon occurring only in non-smooth systems (continuous or discontinuous). These bifurcations are caused by the merging of some invariant set (i.e. fixed points, periodic cycles or boundary of any invariant set) with the kink point at which the function changes its definition (see [2] and [27]). Unlike the bifurcations occurring in standard maps, a feature of BCB consists of exhibiting a sharp transition to chaos. This is important in explaining some stylized facts, such as the boom-bust dynamics ([53], [51]), volatility clustering ([1]) and financial crises ([35], [26], [36]) . Only few models are represented by this kind of maps, some related contributions are [17], [50], [52].

In line with the studies on PWL maps set-up, we build a financial market populated with two groups of fundamentalists and two groups of chartists. Following the contribution of [39], [19] and [10] in the contest of non-linear smooth dynamical systems, we assume that fundamentalists are homogeneous in their trading strategy but heterogeneous in their beliefs about the fundamental value of the asset. This assumption introduces a degree of uncertainty in the market as outlined in [33]. This assumption is not new in the literature on heterogeneous agents in financial markets. [24] argues that different estimations concerning the fundamental value of an asset may originate from different techniques adopted by what they call financial "gurus", who are traders with a huge number of followers such to create several clusters of traders using different fundamental values in their decisions.
[40, 41] introduce two financial gurus followed by two groups of fundamentalists to show that the heterogeneity between these two groups can be such to create complex dynamics even in absence of technical traders.
[19] talks about "[...] two sets of beliefs emerged about the fundamental value of the U.S. dollar" and moving from this observation, they build a model of an

exchange rate market with biased and unbiased forecasts generating complex dynamics of the exchange rate with their interaction.
In our model we also consider two groups of fundamentalists that can be considered as followers of two different financial "gurus" who estimate the fundamental value of the asset price in a different way, because of the technique they use and/or the information they own. Also, we introduce two types of chartists, who are pessimistic when the price is lower than their estimated fundamental value (and sell the asset). In comparison, they are optimistic when the price is higher than their estimated fundamental value. We assume that the two fundamental values estimated by chartists are the same as the two fundamental values estimated by the two groups of fundamentalists. We let chartists share an optimistic market sentiment (or greed sentiment); i.e., when chartists are optimistic, they become overconfident and trade more than when they are pessimistic. The resulting map is piecewise-linear and continuous with three linear branches showing interesting Border Collision Bifurcations phenomena.

The contribution of the paper is as follows. First, we develop a financial market modeled as a one-dimensional continuous piecewise-linear map with three linear branches where two types of fundamentalists and chartists interact. We model the excess demand of fundamentalists as in [10]. Also, we include chartists in the model concerned with a prevailing sentiment of greed. Consequently, the more the price is above the fundamental value, the more they place orders because they believe in profitable opportunities. Second, we concentrate on the mathematical properties of the model providing a comprehensive analysis of the model dynamics. Depending on the slopes of the linear branches, we are able to describe three different cases: Case I where we consider that all three branches are increasing and the map can admit convergence to an equilibrium or divergence from it. Cases II and III where the map has at least one decreasing branch, then beyond the possible paths seen in the previous case, now also periodic and chaotic dynamics may emerge. The three linear branches of our map correspond to three market scenarios: fear scenario (period of declining prices), greed scenario (periods of rising prices), fear and greed scenario. We have chosen this terminology instead of the usual bear and bull market to stress the role of uncertainty in the market we introduce about the fundamental, and to be closely related to the terminology of market risk (see for example [22] and [54]). Finally, as remarked in [5], [45] and [2], a feature of piecewise-linear maps with two kinks is that they may exhibit coexisting attractors for opportune values of the parameters. For this purpose, we conduct numerical analysis to better investigate the multistability property of our model.

The rest of the paper is organized as follows. Section 2 reviews the related literature and highlights the contribution of the present paper. In Section 3 we introduce our financial market model. In Section 4 we conduct an in-depth analytical study of the model. In particular, we find the fixed points and examine their local and global stability. For this purpose, we also carry out a numerical analysis in order to support the analytical results. Section 5 concludes.

2 Literature review

In this section we give an overview of the most significant works analyzing the dynamics of a financial market by means of PWL maps, moreover we also highlight the main contributions of our study.

The contributions of PWL maps to the understanding of financial markets are very recent, with a few exceptions (see [17] and [34]). The relevant literature on PWL financial models considers mainly one-dimensional models due to the complexity and limitations to face models in higher dimensions. In this direction go the works of [44] [49],[53],[52], [51], [48], [50], [47]. All these contributions may be considered extensions of the models studied in the seminal papers of [18] and [34]. For example, [52] study an asset price model with chartists and fundamentalists resulting in a one-dimensional map with three linear branches and two discontinuity points. A feature of their model relies on the fact that only two or three branches of the map are involved in the dynamics, moreover, in the latter case they show that intricate bull and bear market dynamics may emerge. [53] extend the model of [52] finding a particular scenario characterized by the existence of periodic and quasi-periodic dynamic behaviors.

Other studies focus on asset price dynamics using two-dimensional PWL maps, this is the case of [7]. [31], [30]. In particular, the recent work of [1] analyzes an asset pricing model with three types of traders. Their model is characterized by an asymmetric propensity to trade between bull and bear markets. The multi-stability features underlined by the authors are able to describe a rich set of financial scenarios. Moreover, in their stochastic version of the model several empirical stylized fact are observed. Recently, [37] generalize the model of [18]. In particular they focus only on the asymmetry in no-trade intervals resulting in a discrete-time piecewise-linear model with 5-pieces linear branches.

Our study is related to the strand of literature examining financial markets with heterogeneous agents using one-dimensional PWL maps. Unlike previous works outlined, we assume that fundamentalists are homogeneous in their trading strategy but heterogeneous in their beliefs about the fundamental value of the asset, In particular the two groups rely on two different fundamental values, because they refer to financial "gurus" who adopt different techniques to estimate the fundamental value and/or are endowed with different pieces of information. Moreover, we consider two groups of chartists endowed with an optimistic market sentiment; i.e., they trade more when the price is above the fundamental value than the price is below the fundamental value. They also are subdivided into the groups, according to the different fundamental value they rely on. We are able to show that depending on the slope of the branches of the map, the resulting dynamics consists of convergence to an equilibrium or divergence from it (when all the branches are increasing) or in the most interesting case, periodic and chaotic price dynamics (when at least one branch is decreasing). Although the model is deterministic, it replicates several aspects of stock market fluctuations quite well, such as bubbles and crashes. Indeed, one of the characteristics of PWL maps is that they may exhibit a sharp transition to chaotic dynamics, as we will see also in our analysis. This important feature of PWL maps resembles the sudden crisis analyzed in [36].

3 The model

Our model incorporates two types of fundamentalists: in particular, we assume that type-2 fundamentalists (f_2) underestimate the fundamental value with respect to type-1 fundamentalists (f_1), i.e. $F_2 < F_1$. This implies that when the price (P_t) is lower than F_2, type-2 fundamentalists think the price is underestimated less than type-1 fundamentalists. On the other hand, when price P_t is higher than F_1, type-2 fundamentalists think the price P_t is overestimated more than type-1 fundamentalists. If the price P_t is higher than F_2 but lower than F_1 then type-1 fundamentalists think that the price is underestimated while type-2 fundamentalists think at an overestimation and they behave in an opposite way. We assume that both types of fundamentalists have the same excess demand function:

$$D_t^{f_1} = f_1(F_1 - P_t) - \mu_1 \tag{1}$$

$$D_t^{f_2} = f_2(F_2 - P_t) - \mu_2 \tag{2}$$

where $D_t^{f_i}$ is the excess demand function for type-i fundamentalists; $i = 1, 2$, f_i is a positive parameter and indicates how aggressively the fundamentalist reacts to the distance of the price to the corresponding fundamental value (F_1, F_2); $i = 1, 2$. The positive parameters μ_1, μ_2 capture some general kind of pessimism of investors.

Similarly, we incorporate two types of chartists, who are pessimistic when the price is lower than the fundamental value they estimate (and sell the asset), while they are optimistic when the price is higher than their estimated fundamental value. We assume that the two fundamental values estimated by chartists are the same as the two fundamental values estimated by the two groups of fundamentalists. Moreover, we introduce the assumption that when chartists are optimistic they trade more than when they are pessimistic ([29]). It is in fact well known that when traders are optimistic they suddenly become overconfident, trading more and causing the phenomenon called as excess trading. In particular, financial literature on asset pricing converges on the importance of risk indices in investor trading decisions since they can be recognized as indicators of market sentiment. For example, the literature on volatility indices has found a negative relationship between volatility and index returns: an increase in volatility is associated with a decrease in returns. This is the reason why volatility indices have been labelled as *fear indices* ([55], [28],

[11]). Opposite to the fear indices, there are the greed indices. These reflect the fear of missing profitable opportunities ([23]). In line with these empirical findings, we assume that in the market, chartists share a prevailing sentiment of greed; that is, they are more concerned with the fear of missing profitable opportunities than the fear of losing money. So, we get:

$$D_t^{c_1} = \begin{cases} c_1(P_t - F_1) & \text{if } P_t < F_1 \\ (c_1 + h_1)(P_t - F_1) & \text{if } P_t > F_1 \end{cases}$$

$$D_t^{c_2} = \begin{cases} c_2(P_t - F_2) & \text{if } P_t < F_2 \\ (c_2 + h_2)(P_t - F_2) & \text{if } P_t > F_2 \end{cases}$$

where $D_t^{c_i}$ is the excess demand function for type-i chartists; $i = 1, 2$, c_i is a positive parameter and indicates how aggressively chartists of type-i react to the distance of the price to the corresponding fundamental value (F_1, F_2); $i = 1, 2$. The positive parameters h_1, h_2 capture the additional trading activity of chartists when they are optimistic. Depending on the price, we can distinguish the following regions:

a) $P_t < F_2 < F_1$, both types of fundamentalists become less pessimist and sell a less amount or buy, but type-2 fundamentalists are more pessimist than type-1 fundamentalists (greed predominance region). Both types of chartists are pessimist;
b) $F_2 < P_t < F_1$, type-2 fundamentalists sell, whereas type-1 fundamentalists sell a less amount or buy. This is similar to the bull and bear regime described in [51] (fear and greed mixed predominance region). Type-2 chartists become optimistic and trade more;
c) $F_2 < F_1 < P_t$, in this case both fundamentalists sell, type-2 fundamentalists sell more than type-1 (fear predominance region). Both chartists are optimistic and trade more.

The market maker adjusts the price at time $(t + 1)$ following this rule:

$$P_{t+1} = P_t + a(D_t^{f_1} + D_t^{f_2} + D_t^{c_1} + D_t^{c_2}) \tag{3}$$

where a is a positive parameter reflecting price adjustment (measuring market power of traders) and, being a scaling parameter, in the following is set equal to one without loss of generality.

The final map is:

$$P_{t+1} = T(P_t) = \begin{cases} f_L(P_t) = P_t + f_1(F_1 - P_t) + f_2(F_2 - P_t) & \text{if} \quad P_t < F_2 < F_1 \\ +c_1(P_t - F_1) + c_2(P_t - F_2) - m & \\ f_M(P_t) = P_t + f_1(F_1 - P_t) + f_2(F_2 - P_t) & \text{if} \quad F_2 < P_t < F_1 \\ +c_1(P_t - F_1) + (c_2 + h_2)(P_t - F_2) - m & \\ f_R(P_t) = P_t + f_1(F_1 - P_t) + f_2(F_2 - P_t) & \text{if} \quad F_2 < F_1 < P_t \\ +(c_1 + h_1)(P_t - F_1) + (c_2 + h_2)(P_t - F_2) - m & \end{cases} \tag{4}$$

where $m = \mu_1 + \mu_2$. This is a continuous piecewise-linear map with two kink points at $P_t = F_1$ and $P_t = F_2$. In order to better study it we rewrite the map with the equations of the three linear branches in explicit form:

$$P_{t+1} = T(P_t) = \begin{cases} f_L(P_t) = (1 - f_1 - f_2 + c_1 + c_2)P_t & \text{if} \quad P_t < F_2 < F_1 \\ +(f_1 - c_1)F_1 + (f_2 - c_2)F_2 - m & \\ f_M(P_t) = (1 - f_1 - f_2 + c_1 + c_2 + h_2)P_t & \text{if} \quad F_2 < P_t < F_1 \\ +(f_1 - c_1)F_1 + (f_2 - c_2 - h_2)F_2 - m & \\ f_R(P_t) = (1 - f_1 - f_2 + c_1 + h_1 + c_2 + h_2)P_t & \text{if} \quad F_2 < F_1 < P_t \\ +(f_1 - c_1 - h_1)F_1 + (f_2 - c_2 - h_2)F_2 - m & \\ & \end{cases} \tag{5}$$

The piecewise-linear form of the final Map (5) directly comes from the assumption of different volumes of trading when chartists are optimistic or pessimistic. Indeed, unlike [10], where the sentiment index was responsible for the prevailing scenario, in our work the main role is played by the two kink points that separate different regimes of the market (greed, fear-and-greed, fear). In addition, the piecewise-linear form of our map is in line with the research field focusing on different states or regimes of the market in a deterministic setting ([7], [15]).

4 Study of the map

In this section we study analytically and numerically the main properties of the Map (5). First, we make some general considerations about the particular kind of piecewise-linear map we dealt with in this paper, and then we focus on the Border Collision Bifurcations and degenerate flip bifurcations that may originate by varying the model's parameters.

4.1 General Features

Map (5) is made up by three linear pieces, connected at the two kink points. From now on we refer to the three branches as L(eft), M(iddle) and R(ight) branch, respectively.

The slope of the right branch ($s_R = 1 + c_1 + h_1 + c_2 + h_2 - f_1 - f_2$) is larger than the slope of the middle one ($s_M = 1 + c_1 + c_2 + h_2 - f_1 - f_2$), which is larger than the slope of the left branch ($s_L = 1 + c_1 + c_2 - f_1 - f_2$). This is a consequence of the positive values that the reactivity parameters may assume. To limit the number of possible sub-cases we will assume that the reactivity of chartists of kind i (c_i, $i = 1, 2$) is lower than the corresponding reactivity of fundamentalists of the same kind but, at the opposite, it becomes larger than it when they are optimistic:

$$c_i < f_i < c_i + h_i, \; i = 1, 2$$

In particular, the slope of the left branch can be both positive and negative, but it cannot be higher than one. The middle branch can also be both increasing and decreasing and the slope may take any value. The right slope can only be increasing, with slope higher than one.

So, according to the (positive) values of the reactivity parameters we may have three possible scenarios:

Case I Increasing/Increasing/Increasing (if $f_1 < 1 - f_2 + c_1 + c_2$)
Case II Decreasing/Increasing/Increasing (if $1 - f_2 + c_1 + c_2 < f_1 < 1 - f_2 + c_1 + c_2 + h_2$)
Case III Decreasing/Decreasing/Increasing (if $f_1 > 1 - f_2 + c_1 + c_2 + h_2$)

This is sufficient to state that the map may admit up the two equilibria. Three coexisting equilibria are not possible. In fact, in Cases I and II, if the increasing middle branch intersects the bisector creating an equilibrium, the right branch, increasing with higher slope, cannot intersect the bisector. In Case III the left and middle branches, both decreasing, cannot both intersect the bisector. This intuitive result can be confirmed analytically as we do in the following Proposition.

Proposition 1 *Map (5) may admit either two equilibria or no equilibria at all. It admits no equilibria when:*

$$\frac{m}{F_1 - F_2} < \min(c_2 + h_2 - f_2,\ f_1 - c_1) \tag{6}$$

Otherwise, it admits two equilibria, one on the M and the other on the R branch when:

$$f_1 - c_1 < \frac{m}{F_1 - F_2} < c_2 + h_2 - f_2, \tag{7}$$

two equilibria, one on the L and the other on the M branch when:

$$c_2 + h_2 - f_2 < \frac{m}{F_1 - F_2} < f_1 - c_1, \tag{8}$$

two equilibria, one on the L and the other on the R branch, when:

$$\frac{m}{F_1 - F_2} > \max(c_2 + h_2 - f_2,\ f_1 - c_1) \tag{9}$$

Proof The left branch intersects the bisector at:

$$\overline{P}_L = \frac{F_1(f_1 - c_1) + F_2(f_2 - c_2) - m}{f_1 + f_2 - c_1 - c_2}, \tag{10}$$

which is an equilibrium of the map if it is lower than F_2, that is iff:

$$\frac{m}{F_1 - F_2} > f_1 - c_1. \tag{11}$$

The middle branch intersects the bisector at:

$$\overline{P}_M = \frac{F_1(f_1 - c_1) + F_2(f_2 - c_2 - h_2) - m}{f_1 + f_2 - c_1 - c_2 - h_2}, \tag{12}$$

which is an equilibrium of the map if it is higher than F_2 and lower than F_1, that is iff:

$$f_1 - c_1 \frac{m}{F_1 - F_2} < c_2 + h_2 - f_2 \tag{13}$$

or

$$c_2 + h_2 - f_2 < \frac{m}{F_1 - F_2} < f_1 - c_1 \tag{14}$$

The right branch intersects the bisector at:

$$\overline{P}_R = \frac{F_1(f_1 - c_1 - h_1) + F_2(f_2 - c_2 - h_2) - m}{f_1 + f_2 - c_1 - h_1 - c_2 - h_2}, \tag{15}$$

which is an equilibrium of the map if it is larger than F_1, that is iff:

$$\frac{m}{F_1 - F_2} > c_2 + h_2 - f_2. \tag{16}$$

By putting together the existence conditions of the equilibria we obtain what is stated in the Proposition. ■

Proposition 1 implies that it is not possible to have only one equilibrium or three coexisting equilibria. More in detail, two equilibria born via fold-BCB and one of them is located in the middle branch. Then the equilibrium of the middle branch, by moving a parameter, may move to the third branch.

As an example, let us consider the panels of Figure 1. In all the panels we keep fixed the parameters' values: $f_1 = 1.3$, $f_2 = 2.1$, $c_1 = 0.2$, $c_2 = 0.4$, $h_1 = 1.8$, $h_2 = 2.3$, $F_1 = 1.8$ and $F_2 = 1.4$. This implies that $c_2 + h_2 - f_2$ $(0.6) < f_1 - c_1$ (1.1). In panel (a) we have $m = 0$, so $\frac{m}{F_1 - F_2} = 0 < c_2 + h_2 - f_2$ and there are no equilibria. In panel (b) we have $m = 0.24$ and $\frac{m}{F_1 - F_2} = c_2 + h_2 - f_2 = 0.6$ and two coincident equilibria are created via fold-BCB at $P = F_1$ (that is in the kink between the middle and the right branch. In panel (c) we have $m = 0.33$ and $c_2 + h_2 - f_2 < \frac{m}{F_1 - F_2} = 0.825 < f_1 - c_1$ and we have two distinct equilibria in the middle and in the right branch. In panel (d) we have $m = 0.44$ and $\frac{m}{F_1 - F_2} = 1.1 = f_1 - c_1$ and the equilibrium of the middle branch merges with the kink at $P = F_2$. Finally, in panel (e) we have $m = 0.89$ and $f_1 - c_1 < \frac{m}{F_1 - F_2} = 2.225$ and the equilibrium from the middle branch is now on the left one, while the second equilibrium is still on the right branch.

4.2 Stability of the equilibria

In order to understand the outcome of the price dynamics, the study of the existence of the equilibria is not sufficient without the study of their (local) stability.

Considering that our map is piecewise-linear this means that in order to have a stable equilibrium, existence conditions must be combined with the condition that the slope of the branch where the equilibrium is located should be lower than one in absolute value. This permits to obtain the following result:

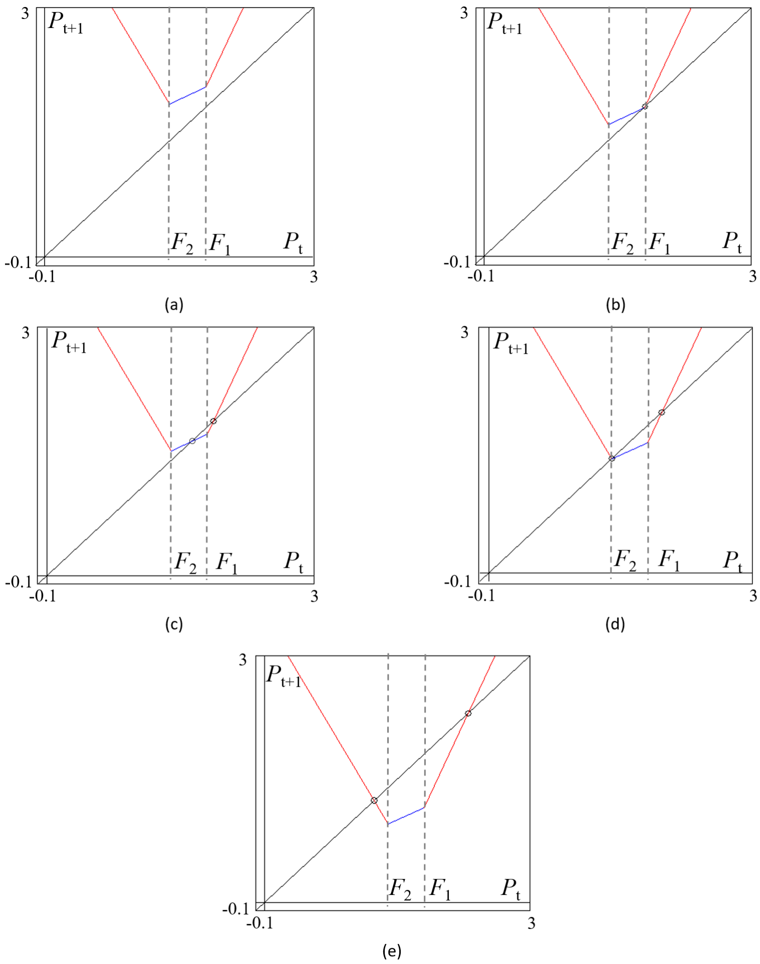

Fig. 1 Parameter values: $f_1 = 1.3$, $f_2 = 2.1$, $c_1 = 0.2$, $c_2 = 0.4$, $h_1 = 1.8$, $h_2 = 2.3$ $F_1 = 1.8$ and $F_2 = 1$. In panel (a) we have $m = 0$. In panel (b) $m = 0.24$. In panel (c) $m = 0.33$. In panel (d) $m = 0.44$, while in panel (e) $m = 0.89$.

Proposition 2 *Map (5) may admit only one locally stable equilibrium, the one located on the left branch or the one located on the middle branch.*

Proof The equilibrium located on the left branch ($\overline{P}_{\mathrm{L}}$) and on the middle branch ($\overline{P}_{\mathrm{M}}$) coexist provided that:

$$f_1 - c_1 < \frac{m}{F_1 - F_2} < c_2 + h_2 - f_2 \tag{17}$$

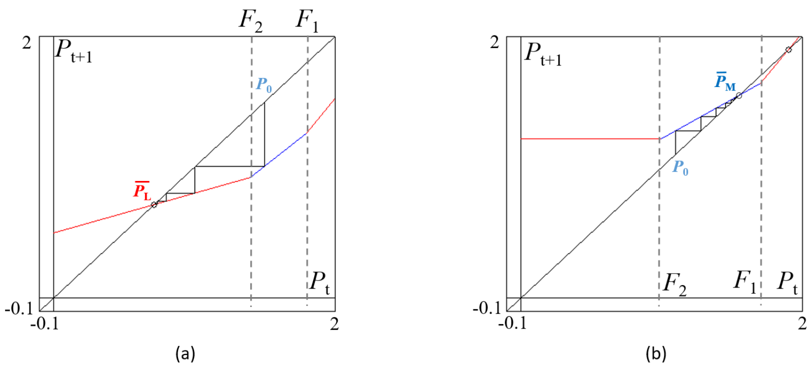

Fig. 2 In panel (a) parameter values $m = 0.6$, $f_1 = 0.5$, $f_2 = 0.8$, $c_1 = 0.2$, $c_2 = 0.4$, $h_1 = 0.5$, $h_2 = 0.55$, $F_1 = 1.8$, $F_2 = 1.4$. In panel (b) $m = 0.2$, $f_1 = 1.4$, $f_2 = 0.8$, $c_1 = 0.8$, $c_2 = 0.4$, $h_1 = 0.7$, $h_2 = 0.6$, $F_1 = 1.7$, $F_2 = 1$.

$\overline{P}_{\mathrm{L}}$ is locally stable, if the slope of the left branch is larger than -1 (remember that it is always lower than +1), that is iff:

$$f_1 - c_1 < 2 - f_2 + c_2, \tag{18}$$

while $\overline{P}_M$ is locally stable if the absolute value of the slope of the middle branch is lower than one, that is iff:

$$c_2 + h_2 - f_2 < f_1 - c_1 < 2 + c_2 + h_2 - f_2 \tag{19}$$

which clearly violates the coexistence condition (17), so they cannot coexist and be both locally stable.

Moreover, considering that the slope of the right branch is always larger than one, the equilibrium located on the right branch ($\overline{P}_{\mathrm{R}}$), when it exists, is always unstable. ■

In Figure 2 we have an example of converging trajectory towards $\overline{P}_{\mathrm{L}}$ (panel (a)) and converging trajectory towards $\overline{P}_{\mathrm{M}}$ (panel (b)).

In order to better investigate the kind of dynamics that may occur and the attractors involved, we start by considering Scenario I and then we will consider the other two scenarios.

4.3 Dynamics in Scenario I (Incr/Incr/Incr)

As we know this scenario occurs when:

$$f_1 < 1 - f_2 + c_1 + c_2 \tag{20}$$

In this case when there are no equilibria, the three branches are all over the bisector and only diverging trajectories may exist (Figure 3, panel (a)).

At the opposite, when there are two coexisting equilibria, if one of them is located on the right branch, then the other one (which can be either on the left or in the middle branch) is locally stable. Initial conditions $P_0 > \overline{P}_{\mathrm{R}}$ lead to divergence, while with $P_0 < \overline{P}_{\mathrm{R}}$ trajectories converge to the stable equilibrium (Figure 3, panel (b) and (c)).

If the two coexisting equilibria are located one the left and on the middle branch, then $\overline{P}_{\mathrm{M}}$ is necessarily unstable, while $\overline{P}_{\mathrm{L}}$ is locally stable. Initial conditions $P_0 < \overline{P}_{\mathrm{M}}$ lead to convergence to $\overline{P}_{\mathrm{L}}$, otherwise we have divergence (Figure 3, panel (d)).

So in this scenario dynamics either converge to an equilibrium or diverge and in case of coexistence of equilibria the initial condition is fundamental to have one kind of dynamics or the other.

4.4 Dynamics in Scenarios II and III

The other two scenarios, with at least one decreasing branch, may originate more interesting dynamics. In particular, not only convergence to an equilibrium or divergence are possible, but also periodic and chaotic dynamics. Let us consider the two-dimensional bifurcation diagram of Figure 4a, obtained by keeping fixed the parameters $c_1 = 0.2$, $c_2 = 0.4$, $h_1 = 1.8$, $h_2 = 2.3$, $F_1 = 1.8$, $F_2 = 1.4$, $m = 1$ and by letting vary f_1 and f_2. As we can see, besides the blue regions denoting convergence to an equilibrium and the dark grey one denoting divergence, there is a yellow region where dynamics converge to a cycle of period 2 (with a point on the left branch and one in the middle), a red region corresponding to a cycle of period 3 (with one point on the left branch, one on the middle and the third on the right branch) and a small light grey region corresponding to an attracting cycle of period 4 (with two points on the left branch, one on the middle and the forth on the right branch). Moreover, the white region denotes chaotic trajectories or cycles of high period.

If we focus on the directions of the arrows A1 and A2 we obtain the one-dimensional bifurcation diagrams shown in Figure 4b.

Let us start by considering the bifurcations occurring moving along arrow A1 of Figure 4b. In point A we have the sudden transition from convergence to the equilibrium $\overline{P}_{\mathrm{L}}$ on the left branch, to chaos. Figure 6a shows that in that case the equilibrium becomes unstable via degenerate flip (the slope of the left branch is equal to -1). In the (f_1, f_2) parameter plane, the equation of the degenerate flip bifurcation curve of the equilibrium $\overline{P}_{\mathrm{L}}$, that we denote with Ψ_{L}, is the following:

$$\Psi_{\mathrm{L}} : f_2 = 2 + c_1 + c_2 - f_1 \tag{21}$$

After that, dynamics become chaotic. In particular, the bifurcation diagram of Figure 5a permits to conclude that the occurring chaotic attractor is made up by four bands. Figure 6b is an example of it. By keep increasing the value of f_1 we reach point B, where from chaos we move to an attracting cycle of period 2, with one point on the left branch and the second one in the middle

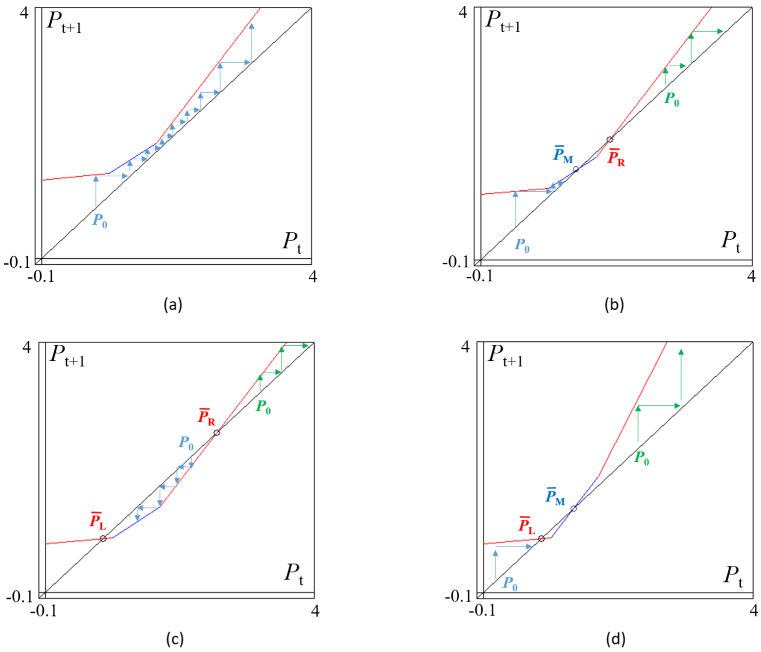

Fig. 3 Parameter values: $f_1 = 1.3$, $f_2 = 0.8$, $c_1 = 0.8$, $c_2 = 0.4$, $h_1 = 0.7$, $h_2 = 0.6$, $F_1 = 1.7$, $F_2 = 1$. In panel (a) we have that $m = 0$, in panel (b) $m = 0.22$, in panel (c) $m = 0.48$, while in panel (d) $h_2 = 1.3$ and $m = 0.48$.

branch. The cycle originated through degenerate flip, in fact, in B the product of the slopes of the left and middle branch is equal to -1 (see Figure 6c). The equation, in implicit form, of the degenerate flip bifurcation curve of the 2-cycle, that we denote with Ψ_{LM}, is the following:

$$\Psi_{\mathrm{LM}} : (1 + c_1 + c_2 - f_1 - f_2)(1 + c_1 + c_2 + h_2 - f_1 - f_2) = -1 \qquad (22)$$

The cycle of period 2 undergoes a BCB for a value of f_1 denoted by point C in Figure 4b. Here, a point of the 2-cycle collides with the border F_1 (see

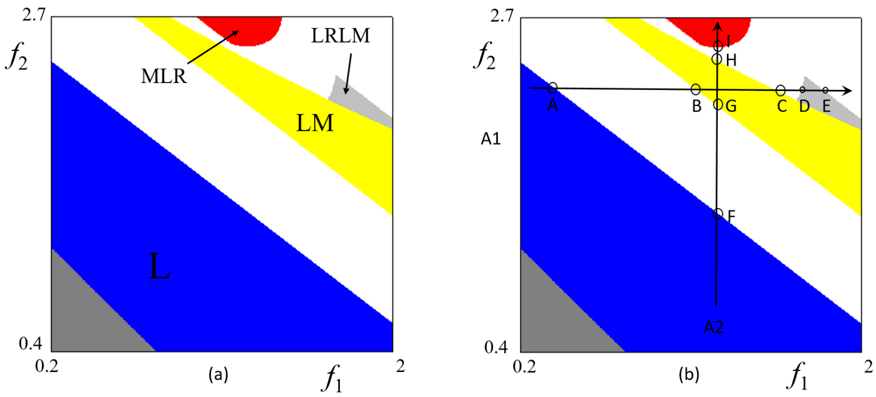

Fig. 4 In blue combinations of parameters leading to an equilibrium, in yellow combinations leading to a cycle of period 2, in red to a cycle of period 3, in light grey to a cycle of period 4. In white chaotic dynamics or periodic cycles of high period, while in dark grey diverging trajectories.

Figure 6d). The BCB curve of the 2-cycle (that we denote with Φ_{LM}) cannot be obtained analytically, but we can obtain it numerically. After the BCB of the 2-cycle we have again chaos (in two bands, see the bifurcation diagram in Figure 5a) until we reach point D, where a cycle of period 4 comes out, via BCB (curve denotes as Φ_{LRLM}) with the border F_2 (see Figure 6e). The 4-cycle loses stability via degenerate flip in point E, where the product of the three slopes is equal to -1:

$$\Psi_{\mathrm{LRLM}} : (1+c_1+c_2-f_1-f_2)^2(1+c_1+c_2+h_2-f_1-f_2)(1+c_1+h_1+c_2+h_2-f_1-f_2) = -1 \tag{23}$$

After that, in the final part of arrow A1, we have a one-band chaotic attractor.

Let us consider now the arrow A2 of Figure 4b. In that case f_1 is fixed at a value of 1.3 and we increase the value of f_2.

In point F of Figure 4b we pass from convergence to $\overline{P}_L$ to chaos through a degenerate flip of the equilibrium, similarly to the previously analyzed point A (in fact points A and F belong to the same bifurcation curve Ψ_{L}) When the value of f_2 is such to be in point G, a cycle of period 2 originates through degenerate flip, as in point B (again these points belong to the same bifurcation curve Ψ_{LM}). Point H is instead similar to point C, both belonging to the BCB curve Φ_{LM}. The novelty is in point I, where a cycle of period 3 with a point on each branch is created via BCB. In fact in I a point is equal to F_2 (see Figure 6f). The BCB curve, obtainable numerically, is denoted by Φ_{MLR}.

As we can see from Figure 7ab the bifurcation curves that we have obtained permit to explain the two-dimensional bifurcation diagram of Figure 4.

After this outline of the possible dynamics occurring in these scenarios, we concentrate on their economic intuition. An inspection of Figure 4 casts some

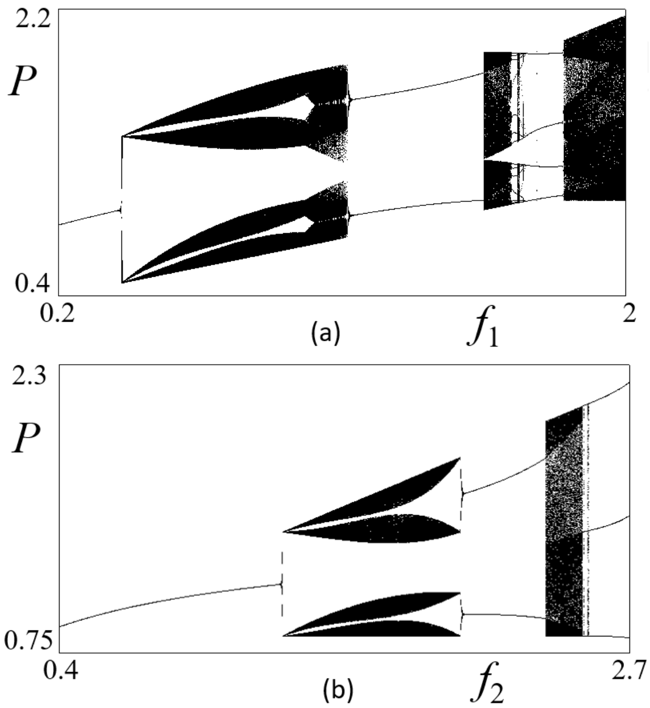

Fig. 5 In both panels we used the same fixed parameters used for Figure 4. In panel (a) we also fix $f_2 = 2.2$, while in panel (b) we fix $f_1 = 1.3$.

light on the role of PWL maps in explaining the stylized facts of financial markets. In particular, we observe a sharp transition to chaos when uncertainty is excessive, leading to the well-known fear and greed scenarios ([54], [51]). Indeed, looking at the blue region of the two-cycle, the more both types of fundamentalists react to price misalignment, the greater instability characterizes the market. Consequently, uncertainty about fundamental can spread when fundamentalists heavily believe in their expectations about the fundamental value. Chartists exacerbate the course of events attracted to higher gains, facilitating complex dynamics.

4.5 Multistability

A feature of a piecewise-linear maps with two kinks is that it is possible to have values of the parameters at which two attractors coexist ([2]). An example is shown in Figure 8, where we have coexistence of a chaotic attractor (panel a) and a cycle of period 3 (panel b). It is known that each attractor must attract one kink point. Here F_1 belongs to the chaotic interval while F_2 converges to the periodic cycle.

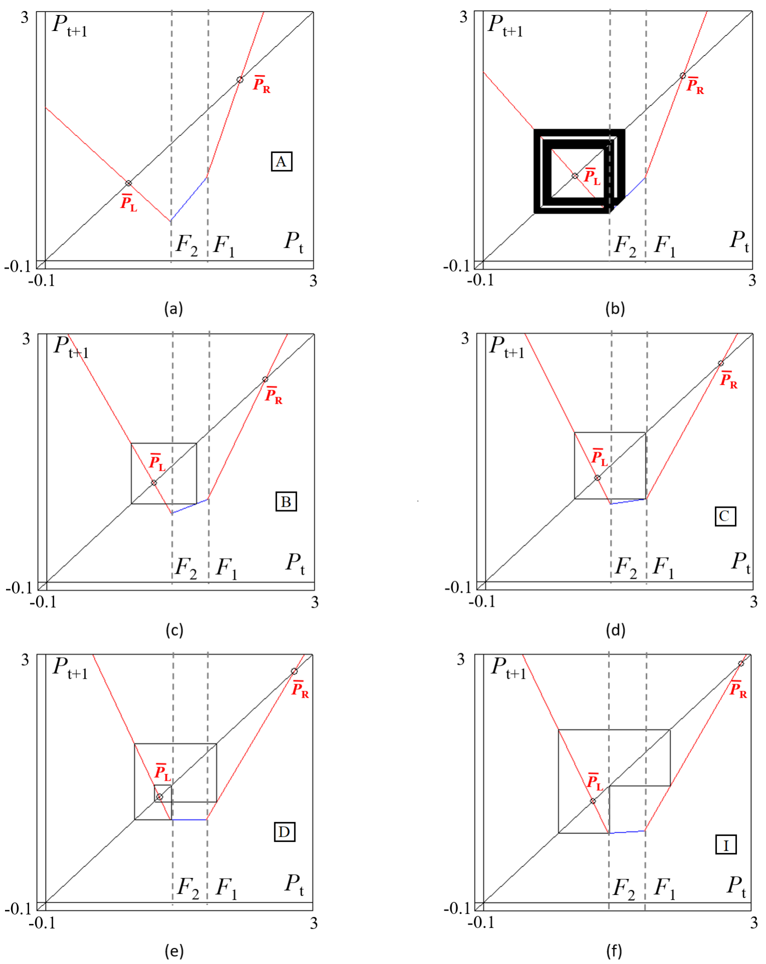

Fig. 6 leading to an equilibrium, in yellow combinations leading to a cycle of period 2, in red to a cycle of period 3, in light grey to a cycle of period 4. In white chaotic dynamics or periodic cycles of high period, while in dark grey diverging trajectories.

The basin of attraction of the periodic cycle is given by the interval [F_2, F_2^{-1}], where F_2^{-1} is the preimage of the kink point in the middle branch, which consists in the immediate basin, and all its preimages of any rank give the total basin. The complementary set gives the basin of the chaotic attractor. As a consequence, even if the two initial conditions are close each other (1.79 and 1.81 in panels a and b, respectively) it is possible that while one trajectory converges to chaos, the other one to the periodic cycle. This causes a sensitivity with respect to the initial levels of price, that is a small perturbation like an

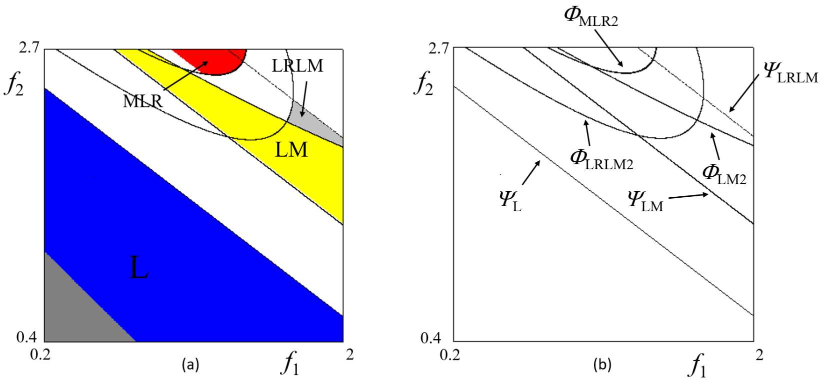

Fig. 7 Bifurcation curves, some obtained analytically and some numerically.

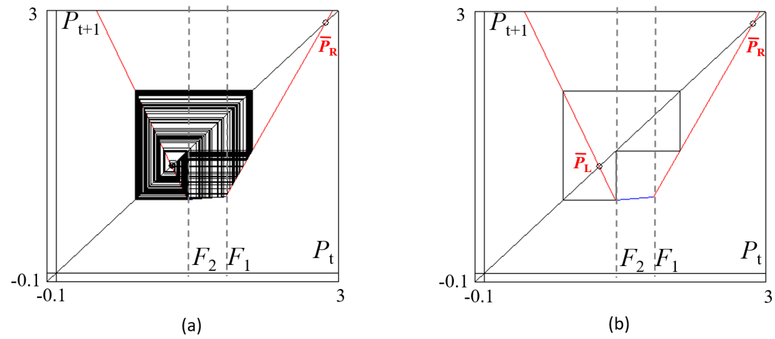

Fig. 8 For the two panels we use the same parameter values used for the bifurcation diagram of Figure 5b, with $f_2 = 2.525$. In (a) the initial condition is $P(0) = 1.79$, while in (b) we used $P(0) = 1.81$.

external shock may lead to trajectories converging to a different attractor (see [6] and [3] for example). The analysis of multistability can be important for regulators and policy makers, as in [51], in order to prevent a market from collapsing. Moreover, it gives a first idea of the possible consequences of the introduction of some stochastic elements in the picture, producing a switching from the basin of attractions of a stable cycle (a relatively calm scenario) to the basin of attraction of a chaotic attractor (turbulent scenario), and vice versa.

5 Conclusions

In this paper we have studied a financial market model where interaction comes from two types of fundamentalists and two types of chartists that display heterogeneous beliefs in the fundamental value of the asset. The model differs from the researches in the field of the PWL maps provided in the last years in the role that agents play in the market. Indeed, considering fundamentalists, we are assuming a marginal role of speculation. Instead, introducing uncertainty about the fundamental value in the market we are focused on the analysis of market risk inherent to the fundamentals of the market. Moreover, for chartists we also take into consideration the behavioral feature of the traders that when are optimistic trade more than when they are pessimistic.

Our model is able to generate interesting endogenous price dynamics. The final map is piecewise-linear and composed of three linear branches. We have conducted an in-depth analytical study of the model. From the general properties of the map, we conclude on the stability and location of the fixed points. As a result, depending on the slope of each branch, we are able to analyze three different cases. The first one is characterized by three linear increasing branches and we have seen the occurrence of three kinds of scenarios. No equilibria exist or either convergence to an equilibrium or divergence occur. In the last case the initial condition controls for the kind of dynamics.

The second and third case show more interesting dynamics. As we stressed in the paper, with PWL maps the structure of bifurcations is different than those occurring in regular maps. We analytically obtained the most of the bifurcation curves in the f_1, f_2 parameter plane.

The final part of the work concerns the role of multistability. This scenario is characterized by the coexistence of multiple attractors. We shown that depending on the initial condition, the model exhibits a convergence to a quite different attractor, being possible to shift from a stable periodic attractor to a chaotic attractor. In this case, we are able to replicate several fear and greed market dynamics where we observe more/less price drops.
We have also highlighted the role of the analysis of coexisting attractors for regulators and policy makers. Indeed, tuning the parameters of the model they may prevent dangerous market crashes and guarantee a stability period.

Our model may be extended in several ways. First, it would be interesting to calibrate or even estimate a stochastic version of the model in order to determine how it is able to replicate the stylized facts of real financial markets. Second, we may introduce a speculative component in the model considering other kinds of chartists and imitators following [7] and [52]. Finally, one could assume more complex demand functions or that some traders are not always active in the market in line with [52].

Acknowledgements The authors thank the Editor and the anonymous reviewers for their helpful and valuable comments.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Anufriev, M., Gardini, L., Radi, D.: Chaos, border collisions and stylized empirical facts in an asset pricing model with heterogeneous agents. Nonlinear Dynamics **102**, 993–1017 (2020)
2. Avrutin, V., Gardini, L., Sushko, I., Tramontana, F.: Continuous and Discontinuous Piecewise Smooth One-Dimensional Maps: Invariant Sets and Bifurcation structures, vol. 95. World Scientific (2019)
3. Bischi, G.I., Chiarella, C., Kopel, M., Szidarovszky, F., et al.: Nonlinear oligopolies. Springer (2010)
4. Bischi, G.I., Gallegati, M., Gardini, L., Leombruni, R., Palestrini, A.: Herd behavior and nonfundamental asset price fluctuations in financial markets. Macroeconomic Dynamics **10**(4), 502 (2006)
5. Bischi, G.I., Gardini, L., Tramontana, F.: Bifurcation curves in discontinuous maps. Discrete & Continuous Dynamical Systems-B **13**(2), 249 (2010)
6. Bischi, G.I., Lamantia, F.: Coexisting attractors and complex basins in discrete-time economic models. In: Nonlinear dynamical systems in economics, pp. 187–231. Springer (2005)
7. Brianzoni, S., Campisi, G.: Dynamical analysis of a financial market with fundamentalists, chartists, and imitators. Chaos, Solitons & Fractals **130**, 109434 (2020)
8. Brianzoni, S., Mammana, C., Michetti, E.: Updating wealth in an asset pricing model with heterogeneous agents. Discrete Dynamics in Nature and Society **2010** (2010)
9. Brock, W.A., Hommes, C.H.: Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control **22**(8-9), 1235–1274 (1998)
10. Campisi, G., Muzzioli, S.: Investor sentiment and trading behavior. Chaos: An Interdisciplinary Journal of Nonlinear Science **30**(9), 093103 (2020). Doi: 10.1063/5.0011636
11. Campisi, G., Muzzioli, S.: Designing volatility indices for austria, finland and spain. Financial Markets and Portfolio Management pp. 1–87 (2021)
12. Chiarella, C., Dieci, R., Gardini, L.: Speculative behaviour and complex asset price dynamics: a global analysis. Journal of Economic Behavior & Organization **49**(2), 173–197 (2002)
13. Chiarella, C., Dieci, R., Gardini, L.: The dynamic interaction of speculation and diversification. Applied Mathematical Finance **12**(1), 17–52 (2005)
14. Chiarella, C., Dieci, R., Gardini, L.: Asset price and wealth dynamics in a financial market with heterogeneous agents. Journal of Economic Dynamics and Control **30**(9-10), 1755–1786 (2006)
15. Chiarella, C., He, X.Z., Huang, W., Zheng, H.: Estimating behavioural heterogeneity under regime switching. Journal of Economic Behavior & Organization **83**(3), 446 – 460 (2012). DOI http://dx.doi.org/10.1016/j.jebo.2012.02.014. The Great Recession: motivation for re-thinking paradigms in macroeconomic modeling
16. Cont, R.: Empirical properties of asset returns: stylized facts and statistical issues. Quantitative Finance **1**, 223–236 (2001)
17. Day, R.H.: Complex dynamics, market mediation and stock price behavior. North American Actuarial Journal **1**(3), 1–16 (1997)
18. Day, R.H., Huang, W.: Bulls, bears and market sheep. Journal of Economic Behavior & Organization **14**(3), 299–329 (1990)
19. De Grauwe, P., Kaltwasser, P.R.: Animal spirits in the foreign exchange market. Journal of Economic Dynamics and Control **36**(8), 1176–1192 (2012)
20. Dercole, F., Radi, D.: Does the "uptick rule" stabilize the stock market? insights from adaptive rational equilibrium dynamics. Chaos, Solitons & Fractals **130**, 109426 (2020)
21. Dieci, R., Westerhoff, F.: Heterogeneous speculators, endogenous fluctuations and interacting markets: a model of stock prices and exchange rates. Journal of Economic Dynamics and Control **34**(4), 743–764 (2010)
22. Elyasiani, E., Gambarelli, L., Muzzioli, S.: Moment risk premia and the cross-section of stock returns in the european stock market. Journal of Banking & Finance **111**, 105732 (2020)
23. Elyasiani, E., Gambarelli, L., Muzzioli, S.: The skewness index: uncovering the relationship with volatility and market returns. Applied Economics pp. 1–17 (2021)

24. Föllmer, H., Horst, U., Kirman, A.: Equilibria in financial markets with heterogeneous agents: a probabilistic perspective. Journal of Mathematical Economics **41**(1-2), 123–155 (2005)
25. Franke, R., Westerhoff, F.: Why a simple herding model may generate the stylized facts of daily returns: explanation and estimation. Journal of Economic Interaction and Coordination **11**(1), 1–34 (2016)
26. Gallegati, M., Palestrini, A., Rosser, J.B.: The period of financial distress in speculative markets: interacting heterogeneous agents and financial constraints. Macroeconomic Dynamics **15**(01), 60–79 (2011)
27. Gardini, L., Tramontana, F.: Border collision bifurcation curves and their classification in a family of 1d discontinuous maps. Chaos, Solitons & Fractals **44**(4-5), 248–259 (2011)
28. Giot, P.: Relationships between implied volatility indexes and stock index returns. The Journal of Portfolio Management **31**(3), 92–100 (2005)
29. Glaser, M., Weber, M.: Overconfidence and trading volume. The Geneva Risk and Insurance Review **32**(1), 1–36 (2007)
30. Gu, E.G.: On the existence of chaos in a discontinuous area-preserving map arising in financial markets. International Journal of Bifurcation and Chaos **28**(14), 1850177 (2018)
31. Gu, E.G., Guo, J.: Bcb curves and contact bifurcations in piecewise linear discontinuous map arising in a financial market. International Journal of Bifurcation and Chaos **29**(02), 1950022 (2019)
32. He, X.Z., Li, Y.: Power-law behaviour, heterogeneity, and trend chasing. Journal of Economic Dynamics and Control **31**(10), 3396–3426 (2007)
33. He, X.Z., Zheng, H.: Trading heterogeneity under information uncertainty. Journal of Economic Behavior & Organization **130**, 64–80 (2016)
34. Huang, W., Day, R.: Chaotically switching bear and bull markets: the derivation of stock price distributions from behavioral rules. Nonlinear dynamics and evolutionary economics pp. 169–182 (1993)
35. Huang, W., Zheng, H.: Financial crises and regime-dependent dynamics. Journal of Economic Behavior & Organization **82**(2), 445–461 (2012)
36. Huang, W., Zheng, H., Chia, W.M.: Financial crises and interacting heterogeneous agents. Journal of Economic Dynamics and Control **34**(6), 1105–1122 (2010)
37. Jungeilges, J., Maklakova, E., Perevalova, T.: Asset price dynamics in a "bull and bear market". Structural Change and Economic Dynamics **56**, 117–128 (2021)
38. Lux, T., Marchesi, M.: Volatility clustering in financial markets: a microsimulation of interacting agents. International journal of Theoretical and Applied Finance **3**(04), 675–702 (2000)
39. Naimzada, A.K., Ricchiuti, G.: Heterogeneous fundamentalists and imitative processes. Applied Mathematics and Computation **199**(1), 171 – 180 (2008)
40. Naimzada, A.K., Ricchiuti, G.: Dynamic effects of increasing heterogeneity in financial markets. Chaos, Solitons & Fractals **41**(4), 1764–1772 (2009)
41. Naimzada, A.K., Ricchiuti, G.: Complexity with heterogeneous fundamentalists and a multiplicative price mechanism. Economic Notes: Review of Banking, Finance and Monetary Economics **43**(3), 233–247 (2014)
42. Nusse, H.E., Ott, E., Yorke, J.A.: Border-collision bifurcations: An explanation for observed bifurcation phenomena. Physical Review E **49**(2), 1073 (1994)
43. Nusse, H.E., Yorke, J.A.: Border-collision bifurcations including "period two to period three" for piecewise smooth systems. Physica D: Nonlinear Phenomena **57**(1-2), 39–57 (1992)
44. Panchuk, A., Sushko, I., Westerhoff, F.: A financial market model with two discontinuities: Bifurcation structures in the chaotic domain. Chaos: An Interdisciplinary Journal of Nonlinear Science **28**(5), 055908 (2018)
45. Sushko, I., Tramontana, F., Westerhoff, F., Avrutin, V.: Symmetry breaking in a bull and bear financial market model. Chaos, Solitons & Fractals **79**, 57–72 (2015)
46. Tramontana, F., Gardini, L., Dieci, R., Westerhoff, F.: The emergence of bull and bear dynamics in a nonlinear model of interacting markets. Discrete Dynamics in Nature and Society **2009** (2009)

47. Tramontana, F., Gardini, L., Westerhoff, F.: Intricate asset price dynamics and one-dimensional discontinuous maps. Nonlinear Economic Dynamics pp. 43–57 (2010)
48. Tramontana, F., Westerhoff, F.: One-dimensional discontinuous piecewise-linear maps and the dynamics of financial markets. In: Global analysis of dynamic models in economics and finance, pp. 205–227. Springer (2013)
49. Tramontana, F., Westerhoff, F.: Piecewise-linear maps and their application to financial markets. Frontiers in Applied Mathematics and Statistics **2**, 10 (2016)
50. Tramontana, F., Westerhoff, F., Gardini, L.: On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. Journal of Economic Behavior & Organization **74**(3), 187–205 (2010)
51. Tramontana, F., Westerhoff, F., Gardini, L.: The bull and bear market model of Huang and Day: Some extensions and new results. Journal of Economic Dynamics and Control **37**(11), 2351–2370 (2013)
52. Tramontana, F., Westerhoff, F., Gardini, L.: One-dimensional maps with two discontinuity points and three linear branches: mathematical lessons for understanding the dynamics of financial markets. Decisions in Economics and Finance **37**(1), 27–51 (2014)
53. Tramontana, F., Westerhoff, F., Gardini, L.: A simple financial market model with chartists and fundamentalists: market entry levels and discontinuities. Mathematics and Computers in Simulation **108**, 16–40 (2015)
54. Westerhoff, F.H.: Greed, fear and stock market dynamics. Physica A: Statistical Mechanics and its Applications **343**, 635–642 (2004)
55. Whaley, R.E.: The investor fear gauge. The Journal of Portfolio Management **26**(3), 12–17 (2000)