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DYNAMICAL ANALYSIS OF A BANKING DUOPOLY MODEL WITH CAPITAL REGULATION AND ASYMMETRIC COSTS

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Abstract. It is well known that regulation and efficiency are two important issues on banking literature. The goal of the paper is to analyse them through a banking duopoly model with heterogeneous expectations. To this purpose, we consider two scenarios. In the first one, we focus on regulation effects. In particular, empirical literature on Italian banks finds evidence on the asymmetry of the costs of regulation that penalize small banks with respect to the large ones. In this direction, we analyse a duopoly model where small banks and large banks have different forecasting rules and we capture the differences of the regulations' effects assuming asymmetry in the cost functions. We introduce linear cost function for small banks and quadratic cost function for large banks. In the second scenario, we study the relation between regulation and bank efficiency highlighting empirical results showing that large banks register higher level of inefficiency than small banks. Moreover, in order to stress new evidences and to confirm empirical results on banking regulation and efficiency, we conduct an analytical and numerical analysis.

1. **Introduction.** The aim of this work is to analyse the efficiency of Italian banks in a duopoly model with capital regulation. Most researchers of the banking sector, face normally two main issues: efficiency and regulation. In this work, we focus mainly on the role of efficiency in the Italian banking sector. It is important to emphasize that banking systems vary from country to country and differ in relation with their specific economic structure too. Indeed, Italian economy, differently from other European economies, is characterized by dimensional fragmentation and productivity specialization (32) and this implies the presence of a high number of small firms. Empirical evidence shows the role of small banks in supporting small firms. We refer to the work of (33) that suggests how the amount of loans granted by small banks is increased with respect to that of large banks in the period $2007-2014$ playing a primary role in supporting families and small firms. The role of efficiency

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of Italian small banks is stressed, for example, by (25). The authors found evidence on higher efficiency of small banks to sustain small firms and local economies with a predominant role played by cooperative banks due to a deep knowledge of local customers and a better control of the credit risk. (2) analyse the cost and the profit efficiency of Italian banking sector over the period $2006-2011$ finding that efficiency is highly related to size, legal type and geographical area of banks. In particular, they show that efficiency decreases with size.
It is worth mentioning the work of (5) in order to emphasize the connection between efficiency and regulation after the recent financial crisis. The authors take into consideration the classification of size related to the amount of deposits, capital and managed external funds, provided by Bank of Italy (major, large, medium, small and minor). One of the results found by the authors in their work is the relation between size and efficiency explained through the impact of expansionary monetary policy and the asymmetric impact of regulation.
Empirical evidence suggests that higher costs of regulation may penalize a higher performance of efficiency of small banks than the large ones, as investigated by (6) and (4). For this reason, in our work we take into account both the elements. As observed in (2), the efficiency property can be studied in several ways; we are interested in the role of the size but the role of the geographical localization is important, too. To this end, the work of (21) analyses a sample of Italian banks located in different parts of the country, in the North, in the Centre and in the South, confirming that the banks located in the latter area are the less efficient. The work of (31) goes in the same direction: the author finds evidence of a higher level of efficiency of banks located in the North-Western of Italy than those situated in the South.
Among other works on efficiency of Italian banking system, we mention those of (24), (26), (23). In all the above cited works, the efficiency of banking industry has been studied by two techniques: Stochastic Frontier Analysis and Data Envelopment Analysis. A different approach, focused on Dynamical System Theory, is used by (18) although it is not related to the efficiency issue. In his paper, the author investigates a banking duopoly with heterogeneous and homogeneous banks in order to capture the effects of capital regulation with reference to a simplified version of the model of (30) and (27). The main result of this work is the stability of the banking system ensured by the capital regulation even if the latter reduces the equilibrium level of profits and loans.
In our work we analyse a duopoly banking model following (18) taking into account the main findings of recent empirical literature on bank efficiency and regulation. Most of all, we deal with the size of banks and their costs efficiency. Therefore, we study a duopoly banking model with heterogeneous banks and asymmetric costs. We consider large banks with bounded rational expectations and quadratic costs and small banks with naive expectations and linear costs. Unlike (18), we consider banks of different size and costs, and also we focus more on efficiency rather than the only regulation: these changes arise in a different bifurcation structure resulting from our model. As explained before, we consider different size of banks regarding evidence of a greater efficiency of small banks than the large ones ((2), (5), (25)), emphasizing its impact on cost function. Moreover, we stressed the role of expectations in relation to the size of the banks. In detail, from the results of (33) we endow small banks with naive expectations because these banks have maintained their level of loans almost constant over time, and they even have increased their volume especially in

the period of crisis, mainly in support of small firms and families. On the contrary, we consider large banks with bounded rational expectations because they follow the trend emerged in the period of crisis, that is, rationing the credit supply.

An interesting scenario described from our model concerns the boundary equilibrium. Indeed, differently from (18) where the boundary equilibrium is always a saddle, in our model we find parameters conditions which ensure its stability. From an economic point of view, the boundary equilibrium describes a situation where only one type of bank serves the market. In line with the literature, this could be the case where only small banks survive thanks to their deep knowledge of the environment. According to the previous considerations, we study a two dimensional map describing the evolution of loans of two banks. Moreover, we incorporate the efficiency effect in the cost function associating decreasing return to scale to the less efficient bank (large bank), and constant return to scale to the more efficient ones (small banks). To this regard, we introduce non linearity in the large banks cost function as (39), while we maintain a linear cost function for small banks as in (18). Our work highlights the role of efficiency between banks of different sizes, on the theoretical foundation it relies on the research field which studies the role of non-linearity in Economics and Finance (see (1), (16), (36), (11; 14) for example). As stressed by (15), convex cost function allows banks to increase their activity in two ways in the short period: establishing new branches or acquiring other banks. In both cases, the strategies generate anyway higher costs due to in-depth search of information and switching costs. In line with our assumptions, an increase in the size of banks causes a lower propensity to lend mainly to small businesses as empirically observed by (7) in the U.S. and (3) regarding the Italian banking sector.

The contribution of our work to the existing literature is twofold. First, it confirms empirical finding on higher inefficiency of large banks to manage loans demand in local territory with respect to small banks. Second, we deeply extend the analytical part of the model to both local and global dynamics of the map. Indeed, we focus on the role of the two cycle analysing its stability properties and emphasizing the economic consequences when it loses stability. Moreover, following (14; 13), we find conditions such that a global attractor exists, moreover we show numerically how its structure increases in complexity when marginal costs of small banks become larger.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we conduct the local stability analysis of the map proving the existence of two fixed points; the boundary equilibrium and the Nash equilibrium. Section 4 focuses on the role of the two cycle and its stability properties. In Section 5 we use both analytical and numerical tools to study the global properties of our map confirming the results obtained by the two groups of banks in terms of efficiency and regulations. In Section 6 we extend the deterministic model introducing stochastic shocks to the demand of loans of both banks and performing several Monte Carlo simulations. In this way, we check the robustness of our model and we test the sensitivity of the dynamical system to the key parameters, in order to give some policy suggestions to both the policy maker and regulator. Section 7 concludes our paper giving also some insight on further developments.

2. **The model.**

We consider a banking duopoly model, following (18), (27), (30). In order to maintain the tractability of the model, we assume the absence of open positions between banks in the interbank market, as in (18).

As a consequence, the balance sheet of bank i is composed by loans (L_i) on the asset side, capital (K_i) and deposits (D_i) on the liability side (with $i = 1, 2$).

The total demand function for loans is assumed linear, leading to the inverse demand function f:

$$f(L) = a - bL, \qquad L = L_1 + L_2$$

where $a, b > 0$.

Differently from (18), in this paper we consider heterogeneity in the bank size. More precisely, we assume quadratic costs for large banks ($i = 1$) and linear costs for small banks ($i = 2$). This is motivated by the recent empirical works on bank literature (see for example (5; 6), (2), (24)) which find evidence of the greater efficiency of small banks (in this work we mainly refer to cooperative banks) with respect to the larger ones, moreover this efficiency reflects the role played by small banks to economically sustain local firms and families. To this regard, we want to stress the social role of small banks, beyond the economic one, highlighting the inefficiency of large banks with quadratic costs.

As a consequence, the cost functions for loans are respectively given by $c_1 L_1^2$ and $c_2 L_2$, so that marginal costs for loans are increasing (decreasing returns to scale) for large banks ($i = 1$) and constant for small banks ($i = 2$).

Differently, costs for deposits are linear for both banks. In fact, following (18), we assume that capital regulation is based on the supply of loans and, then, deposit remuneration is not relevant in this case.[1] Anyway, notice that small banks have the same constant marginal cost for deposits and loans. Differently, large banks face greater marginal costs for loans than deposits.

As a consequence, profit functions are defined as:

$$\pi_1 = [a - b(L_1 + L_2)]L_1 - r_k K_1 - c_1 D_1 - c_1 L_1^2$$

$$\pi_2 = [a - b(L_1 + L_2)]L_2 - r_k K_2 - c_2 D_2 - c_2 L_2$$

for large and small banks, respectively. Notice that $r_k > 0$ is the exogenous capital remuneration which we assume high enough in order that capital remuneration is higher than marginal costs, i.e. $r_k > max[c_1, c_2]$.[2]

Following the benchmark model of (18), the capital requirement is binding: $K_i = \gamma L_i$ $(i = 1, 2)$, where γ is the percentage determined by the regulator. Moreover $D_i = L_i - K_i = (1 - \gamma)L_i$, $(i = 1, 2)$.

By substituting this last condition into profit functions, we obtain:

$$\begin{aligned} \pi_1(L_1, L_2) &= [a - b(L_1 + L_2)]L_1 - L_1[c_1(1 + L_1) + \gamma(r_k - c_1)] \\ \pi_2(L_1, L_2) &= [a - b(L_1 + L_2)]L_2 - L_2[2c_2 + \gamma(r_k - c_2)] \end{aligned} \tag{1}$$

From Equations (1), one can easily obtain marginal profits:

$$\begin{aligned} \frac{\partial \pi_1}{\partial L_1}(L_1, L_2) &= a - b(2L_1 + L_2) - (2L_1 + 1)c_1 - \gamma(r_k - c_1) \\ \frac{\partial \pi_2}{\partial L_2}(L_1, L_2) &= a - b(L_1 + 2L_2) - 2c_2 - \gamma(r_k - c_2) \end{aligned} \tag{2}$$

[1] We stress the fact that we are considering separable cost functions and in this case the optimal volume of loans is independent of the properties of the deposit market.

[2] Notice that it is possible to formalize this point by introducing an endogenous capital remuneration such that it is higher than marginal costs for any loan level. We leave this development for further research.

Dynamical set-up. Consider now a dynamic setting with time indexed by $t \in \mathbb{Z}_+$. Following (10), (20; 19), (34), (12), we assume that firm 1 has limited information. More specifically, large banks set the level of loans between two periods according to the following adjustment process:

$$L_{1,t+1} = L_{1,t} + \alpha L_{1,t} \frac{\partial \pi_1}{\partial L_{1,t}}(L_{1,t}, L_{2,t}), \quad t \in \mathbb{Z}_+$$

where $\alpha > 0$ is the speed of adjustment. Large banks increase or decrease their loans according to the marginal profit of the last period. This fact is motivated by the finding of empirical literature (see (33) and (5)), indeed large banks have cut supply of loans following the market trend in financial crisis and looking at the profitability of their investments.

Differently, small banks have maintained their level of loans almost constant over time, increasing its volume especially in the period of crisis (33). To this purpose, we assume that small banks expect that the level of loans of large banks will be equal to the last period's one and, under this assumption, they maximize their expected profit $\pi_2(L_1, L_2)$ of System (1), obtaining the following first order condition for maximization:

$$a - b(L_{1,t} + 2L_{2,t+1}) - 2c_2 - \gamma(r_k - c_2) = 0$$

The dynamics of loans is described by the following discrete time dynamical system:

$$\begin{cases} L_{1,t+1} = f(L_{1,t}, L_{2,t}) = L_{1,t} + \alpha L_{1,t}[a - b(2L_{1,t} + L_{2,t}) - (2L_{1,t} + 1)c_1 - \gamma(r_k - c_1)] \\ \\ L_{2,t+1} = g(L_{1,t}) = \dfrac{1}{2b}[a - bL_{1,t} - 2c_2 - \gamma(r_k - c_2)] = -\frac{1}{2}L_{1,t} + \frac{1}{2b}[a - \gamma r_k - (2-\gamma)c_2] \end{cases} \tag{3}$$

where $a, b, \alpha > 0$, $c_1, c_2 \geq 0$, $r_k > 0$ and $\gamma \in [0, 1]$.

We would like to underline that our model is able to show different dynamic outcomes with respect to (18). Moreover, many dynamic scenarios are related to the bank size, as we will see in the next section where the stability analysis is performed.

3. **Stability Analysis of equilibrium points.** Once the equilibrium conditions $L_{i,t} = L_i \ \forall i = 1, 2$ are set, we obtain two fixed points. In particular, the interior Nash Equilibrium $L^\star = (L_1^\star, L_2^\star)$ given by:

$$L_1^\star = \frac{(2-\gamma)c_2 - 2(1-\gamma)c_1 + a - \gamma r_k}{3b + 4c_1}$$

$$L_2^\star = \frac{1}{2b}[a - 2c_2 - \gamma(r_k - c_2)] - \frac{a + 2(c_2 - c_1) - \gamma r_k + \gamma(2c_1 - c_2)}{2(3b + 4c_1)} = -\frac{1}{2}L_1^\star + \frac{1}{2b}[a - 2c_2 - \gamma(r_k - c_2)],$$

and a boundary fixed point L^0, located on the invariant vertical axis, where only small banks are presented in the market:

$$L_1^0 = 0$$

$$L_2^0 = \frac{1}{2b}[a - 2c_2 - \gamma(r_k - c_2)].$$

As a consequence for $(2-\gamma)c_2 - 2(1-\gamma)c_1 + a - \gamma r_k = 0$ there exists a unique fixed point (the boundary one), otherwise there exist two equilibria. We underline that from a mathematical point of view, these equilibria exist for any admissible range of the parameter values. Nevertheless, because their economic meaning, we have to introduce conditions under which System (3) admits non-negative fixed points.[3] To this end, observe that the steady states are linked by the relationship $L_2^\star = -\frac{1}{2}L_1^\star + L_2^0$.[4] A preliminary consideration is that $L_i^\star > 0\ \forall i = 1,2$ implies $L_2^0 > 0$ as well, in other words, a positive interior equilibrium implies the presence of the boundary fixed point. Moreover:

1. If $L_2^0 \leq 0$ then $L_i^\star < 0$ for some i (no equilibria has economic sense)
2. If $L_2^0 > 0$ then three cases are possible:
   (a) $L_1^\star < 0$ (only the boundary steady state does exist from an economic point of view)
   (b) $L_1^\star > 0$
   (c) $L_1^\star = 0$ (repeated steady states).

The following proposition makes the economically interesting scenarios clear in terms of the parameter values, in order to guarantee non-negative output of firms for the interior steady state.

Proposition 1. *Assume $a - \gamma r_k > 0$. In order to have positive equilibrium values for loans, the following range of the parameters has to be assumed:*

$$\begin{cases} c_2 > 2\left(\frac{1-\gamma}{2-\gamma}\right)c_1 + \frac{\gamma r_k - a}{2-\gamma} \quad (i.e.\ \ L_1^\star > 0) \\ c_2 < \frac{1-\gamma}{2-\gamma} \cdot \frac{b}{2b+2c_1}c_1 + \frac{a-\gamma r_k}{2-\gamma} \quad (i.e.\ \ L_2^\star > 0) \end{cases} \tag{4}$$

Proof. The following system directly comes from previous considerations:

$$\begin{cases} c_2 > 2\left(\frac{1-\gamma}{2-\gamma}\right)c_1 + \frac{\gamma r_k - a}{2-\gamma} \quad (i.e.\ \ L_1^\star > 0) \\ L_1^\star < \frac{1}{b}[a - 2c_2 - \gamma(r_k - c_2)] \quad (i.e.\ \ L_2^\star > 0) \end{cases}$$

where the assumption $a - \gamma r_k > 0$ (i.e. $L_2^0 > 0$) guarantees that it is not empty. After some algebra, it is possible to obtain System (4). □

Parameters' conditions established in Proposition (1) are in line with our assumptions on the difference regarding the bank dimensions. In detail, if we consider the first inequality of System (4) we see that for the positivity of the demand of loans of large banks, it occurs that small banks face higher costs (c_2) than large banks. This is an important proposition because it resumes the essence of our work, that is large banks introduce a greater inefficiency to the system. In particular, due to economy of scale arguments, large banks are able to lower their costs more than small banks do, however this is not sufficient to guarantee the appropriate level of loans in the market. Indeed, from our assumptions, supported by (33), large banks have rationed the credit supply mostly in the period of crisis unlikely from small banks that have maintained their level of loans almost constant over time.

As it is well known, in order to perform the local stability analysis of fixed points, we have to compute the partial derivatives of System (3): $\frac{\partial f}{\partial L_1}(L_1, L_2) = 1 + \alpha[-4(b+c_1)L_1 - bL_2 + a - c_1 - \gamma(r_k - c_1)]$, $\frac{\partial f}{\partial L_2}(L_1, L_2) = -\alpha b L_1$, $\frac{\partial g}{\partial L_1}(L_1, L_2) = -\frac{1}{2}$,

[3] Anyway, in our model negative fixed points are always unstable.

[4] Notice that this relationship can be rewritten only in terms of $L^\star$ as follows: $L_2^\star = -\frac{2(b+c_1)}{b}L_1^\star + \frac{1}{b}[a - \gamma r_k - (1-\gamma)c_1]$.

$\frac{\partial g}{\partial L_2}(L_1, L_2) = 0$. By considering the Jacobian matrix J evaluated at an equilibrium point, the stability conditions are given by:[5]

1. $1 + tr(J) + det(J) > 0$
2. $1 - tr(J) + det(J) > 0$
3. $1 - det(J) > 0$

In particular, by studying the interior fixed point $L^\star$, some algebra proves that Condition 1 yields the following result:

$$c_2 < \frac{4}{\alpha(2-\gamma)} \frac{3b + 4c_1}{5b + 4c_1} + 2\left(\frac{1-\gamma}{2-\gamma}\right) c_1 + \frac{\gamma r_k - a}{2-\gamma}, \tag{5}$$

while Conditions 2 and 3 are always fulfilled given the positivity of $L^\star$ (defined by Proposition 1). Hence the following remark holds:

Remark 1. Under the assumptions of Proposition 1, the interior steady state loses stability via period-doubling bifurcation. The flip bifurcation curve is defined in the parameter space by: $c_2 = \frac{4}{\alpha(2-\gamma)} \frac{3b+4c_1}{5b+4c_1} + 2\left(\frac{1-\gamma}{2-\gamma}\right) c_1 + \frac{\gamma r_k - a}{2-\gamma}$.

Notice that this result is different from what happens in the case of symmetric costs, where a Neimark-Sacker bifurcation of the Nash equilibrium can occur (see (18)).

About the boundary equilibrium L_0, we would like to point out that it is always locally unstable when both the steady states coexist in the model. More precisely, in the following proposition we investigate the bifurcation responsible for the loss of the stability of L_0 and the positiveness of $L^\star$, where an equilibrium point is positive iff so is any component.

Proposition 2. *Let $L_2^0 > 0$ and $(2-\gamma)c_2 - 2(1-\gamma)c_1 + a - \gamma r_k \leq 0$ then the non-negative boundary equilibrium L_0 loses stability via transcritical bifurcation which occurs at $c_2 = 2\left(\frac{1-\gamma}{2-\gamma}\right) c_1 + \frac{\gamma r_k - a}{2-\gamma}$. Immediately after this bifurcation (i.e. $(2-\gamma)c_2 - 2(1-\gamma)c_1 + a - \gamma r_k > 0$) the interior fixed point becomes positive and stable.*

Observe that L_0 cannot undergo a Neimark-Sacker bifurcation, as well (being $det(J(L_0)) = 0$). Moreover, it can be characterized by a period-doubling bifurcation for $(2-\gamma)c_2 - 2(1-\gamma)c_1 + a - \gamma r_k < 0$, but we do not consider this case because the interior equilibrium is not positive. Hence, we restrict the analysis to parameter values satisfying conditions defined by Proposition 1. Notice that in the case under study, the following system is verified:

$$\begin{cases} 1 - tr(J(L_0)) + det(J(L_0)) < 0 \\ 1 + tr(J(L_0)) + det(J(L_0)) > 0 \\ 1 - det(J(L_0)) > 0 \end{cases}$$

where $J(L_0)$ is the Jacobian matrix of the map evaluated at the boundary equilibrium.

Since we are interested in the role of asymmetric costs, in Figure 1 we represent the curves previously obtained in the $(c_1, c_2)-$plane. The positivity of $L^\star$ is guaranteed between the red and blue curves, while the flip curve defined in Remark 1 is depicted in yellow. Being the fixed point locally stable (unstable) below (above) this curve, we can observe that for the chosen set of parameters and for a given value of c_2, large values of c_1 implies stability, while instability arises when c_1 decreases.

[5] See e.g. (29).

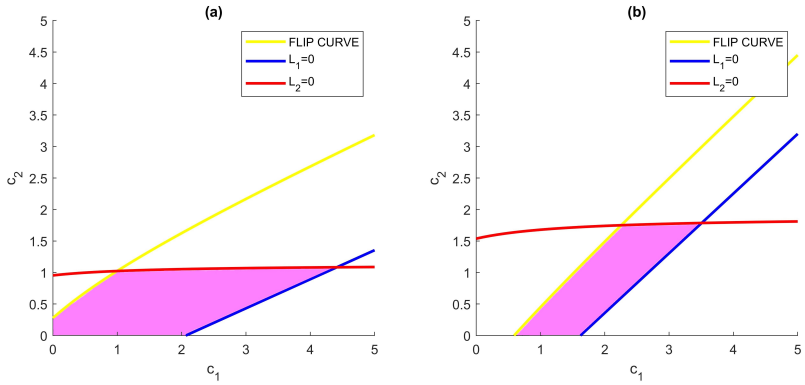

FIGURE 1. In the $(c_1, c_2)-$plane the set of points between the red and blue curve ensures positive equilibrium loans, while the yellow curve represents the parameter values (c_1, c_2) for which a flip bifurcation occurs. In Panel (a) the parameters are: $a = 3.2$, $\alpha = 1.5$, $b = 1.5$, $\gamma = 0.7$, $r_k = 2.8$. In panel (b) $\gamma = 0.1$ while the other parameters as in Panel (a).

The scenario described in Figure 1 helps us to understand the role of the costs of the two banks. Large banks inefficiency[6] clearly results from the stability induced by an increasing of parameter c_1 and γ as well. In Panel (a) of Figure 1 we observe a greater stability zone, highlighted in lilac (between the red and blue lines and below the yellow line) when $\gamma = 0.7$ and costs of large banks are sufficiently high. In Panel (b), instead, when γ decreases the stability region restricts. In both scenarios, apart the role of γ that confirms the stability role of regulatory system, we can note that if small banks control the market alone, the stability zone should become very large. Moreover the costs of stability for small banks are constants over time. On the contrary, the solitary control of the market by large banks is possible only increasing consistently their costs. This last scenario could represent the situation of local banks that offer their services to specific area. As stressed by (25) small banks have a fundamental role to sustain local economies thanks to a deep knowledge of local customers and a better control of the credit risk.

We want to deeply investigate the role of marginal costs for both banks, in detail we describe the consequences of an increase in the costs. In Figure 2 we show two $1-$dimensional bifurcation diagrams with respect to the parameter c_2 (the marginal cost of small banks) for two different values of marginal cost of large banks. Most fully, in Panel (a) we have assumed that marginal costs of small banks belong to the interval $(0.01, 0.6)$ and large banks face costs $c_1 = 0.15$. As we can see, complex dynamics emerge when c_2 is higher than 0.5 (approximately), while a stability region is possible when c_2 is small enough. Differently, in Figure 2 (b) it results that stability region decreases when c_1 decreases. In this case, both banks face a heavy period of competition and as we can see in Figure 2 (b), small banks are

[6]In this paper we consider cost efficiency, that is we consider that banks offer the maximum level of loans at the minimum cost. In particular, we consider efficient a bank that is able to guarantee a certain level of loans over time with a limited increase in its costs. On the contrary, we consider inefficient a bank if it provides loans only consistently increasing its costs.

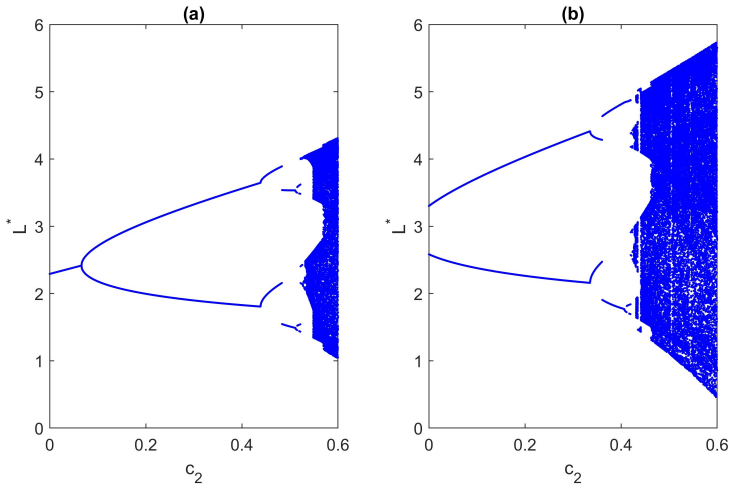

FIGURE 2. Two interesting economic scenarios through bifurcation diagrams. In Panel (a) the parameters are: $a = 3$, $\alpha = 1.38$, $b = 0.12$, $\gamma = 0.2$, $r_k = 2.8$, $c_1 = 0.15$ and $c_2 \in (0.01, 0.6)$. In Panel (b) $c_1 = 0.1$ and $c_2 \in (0.01, 0.6)$ while the other parameters as in panel (a).

able to compete for very small values of their costs (i.e. for small values of c_2 the equilibrium is stable), and it is evident relying on economy of scale arguments.

4. **Periodic points and route to complexity.** When the positive equilibrium $L^\star$ exists, it loses stability via period doubling bifurcation after which a stable 2-cycle is created.

In order to study how the 2−cycle loses stability (and hence the route to more complex dynamical behaviour), assume that $\left\{\bar{L} = (\bar{L}_1, \bar{L}_2), \bar{\bar{L}} = (\bar{\bar{L}}_1, \bar{\bar{L}}_2)\right\}$ is the period 2 orbit observed immediately after the flip bifurcation of the interior fixed point $L^\star$. Hence, thanks to the definition of periodic point, we obtain the following system of equations:

$$\begin{cases} \bar{L}_1 = f(\bar{\bar{L}}_1, g(\bar{L}_1)) \\ \bar{\bar{L}}_1 = f(\bar{L}_1, g(\bar{\bar{L}}_1)) \end{cases}$$

by subtracting member to member, we arrive to:

$$\bar{\bar{L}}_1 = -\bar{L}_1 + K$$

with $K = \frac{1}{4\alpha(b+c_1)} \{4 + \alpha[a - \gamma r_k + (2-\gamma)c_2 - 2(1-\gamma)c_1]\}$.[7]

From the last equation, it is possible to arrive to a second order equation which admits real solutions if and only if $c_2 \geq \frac{4}{\alpha(2-\gamma)} \frac{3b+4c_1}{5b+4c_1} + 2\left(\frac{1-\gamma}{2-\gamma}\right) c_1 + \frac{\gamma r_k - a}{2-\gamma}$ (according to Remark 1). This confirms analytically the flip bifurcation of the interior positive fixed point. Moreover, some algebra proves that distinct and real solutions of the previous equation produce the same periodic point, as a consequence our system admits only one 2−cycle. This result is very interesting, because it excludes the

[7]Notice that K is positive under the assumptions of Proposition 1.

occurrence of some types of bifurcations (such as fold or transcritical bifurcation of the 2−cycle). Final results of our analysis are summarized in the following remark.

Remark 2. Let $K = \frac{1}{4\alpha(b+c_1)}\{4+\alpha[a-\gamma r_k+(2-\gamma)c_2-2(1-\gamma)c_1]\}$, then:

1. for $c_2 > \frac{4}{\alpha(2-\gamma)}\frac{3b+4c_1}{5b+4c_1}+2\left(\frac{1-\gamma}{2-\gamma}\right)c_1+\frac{\gamma r_k-a}{2-\gamma}$ there exists only one $2-cycle$, $\left\{(\bar{L}_1, g(\bar{\bar{L}}_1)), (\bar{\bar{L}}_1, g(\bar{L}_1))\right\}$, with $\bar{L}_1 = \frac{K}{2}+\frac{1}{2}\sqrt{K^2-\frac{8K}{\alpha(5b+4c_1)}}$ and $\bar{\bar{L}}_1 = -\bar{L}+K$,
2. for $c_2 \leq \frac{4}{\alpha(2-\gamma)}\frac{3b+4c_1}{5b+4c_1}+2\left(\frac{1-\gamma}{2-\gamma}\right)c_1+\frac{\gamma r_k-a}{2-\gamma}$ the system cannot admit any 2−cycle.

Remembering that the Jacobian matrix of the second iterate of the map is given by the product of the Jacobian matrices J evaluated at the two periodic points of the cycle, differently from the fixed point, the 2−cycle may be characterized by complex conjugate eigenvalues.

More precisely, the Jacobian matrix $J^2 = J(\bar{L})J(\bar{\bar{L}})$ is given by:

$$J^2 = \begin{pmatrix} f_{L_1}(\bar{L})f_{L_1}(\bar{\bar{L}})+\frac{1}{2}\alpha\beta\bar{L}_1 & -\alpha\beta\bar{\bar{L}}_1 f_{L_1}(\bar{L}) \\ -\frac{1}{2}f_{L_1}(\bar{\bar{L}}) & \frac{1}{2}\alpha\beta\bar{\bar{L}}_1 \end{pmatrix}$$

The characteristic equation for the eigenvalues of J^2 is $\lambda^2 - tr(J^2)+det(J^2)=0$, where:

$$tr(J^2) = f_{L_1}(\bar{L})\cdot f_{L_1}(\bar{\bar{L}})+\frac{1}{2}\alpha b(\bar{L}_1+\bar{\bar{L}}_1)$$

$$det(J^2) = \frac{1}{4}(\alpha b)^2\bar{L}_1\bar{\bar{L}}_1.$$

Consider the 2−cycle defined by Remark 2, hence: (i) $1-det(J^2)>0$ if and only if $K < \frac{2(5b+4c_1)}{\alpha b^2}$; (ii) $det(J^2)>0$ for any parameter values. Moreover $1+tr(J^2)+det(J^2)<0$ implies $1-tr(J^2)+det(J^2)>0$ since $det(J^2)>0$ (anyway we remember that the 2−cycle is unique). This study proves the following proposition, which defines the set of parameters such that the secondary bifurcation is a period doubling bifurcation (as the primary bifurcation).

Proposition 3. *Assume Case* 1 *of Remark 2. If* $(0<)K < \frac{2(5b+4c_1)}{\alpha b^2}$ *then a loss of stability of the* 2−*cycle is due to a period doubling bifurcation.*

In this case numerical simulations confirm the appearing of a 4−cycle so that, even though our result makes use of necessary conditions, it constitutes strong evidence for the existence of the period doubling bifurcation of the 2−cycle.

Differently for $K > \frac{2(5b+4c_1)}{\alpha b^2}$ (while Case 1 of Remark 2 continues to hold), we have $1-det(J^2)<0$. Notice that if the determinant of the Jacobian matrix of the second iterate is equal to one then the modulus of complex eigenvalues is equal to one. Given the high number of the parameters involved in our model, we investigate this case numerically. On the other hand, now we are able to select a restricted set of the parameters, in order to explore the open case. As we can see in Figure 3, we show under Proposition 3 that the 2−cycle loses stability via flip bifurcation. In Figure 4 we analyse numerically what happens to the attractors of the system when Proposition 3 is violated. In detail, the structure of the basins of attraction becomes more complex, indeed we assist to the loss of stability of the 2−cycle and a stable 4−cycle appears.

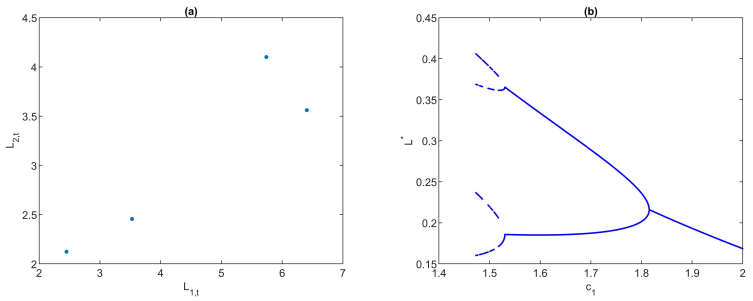

FIGURE 3. On the left is depicted the attractor of the 4-cycle for $\alpha = 1.25$, $\gamma = 0.35$, $r_k = 2.8$, $a = 3.2$, $b = 0.1$, $c_1 = 0.08$, $c_2 = 0.7$. On the right panel a bifurcation diagram for $c_1 \in (1.4, 2.1)$ showing the period-doubling of the 2-cycle for $\alpha = 2$, $\gamma = 0.25$, $r_k = 2.3$, $a = 3.2$, $b = 0.4$, $c_2 = 1.1$.

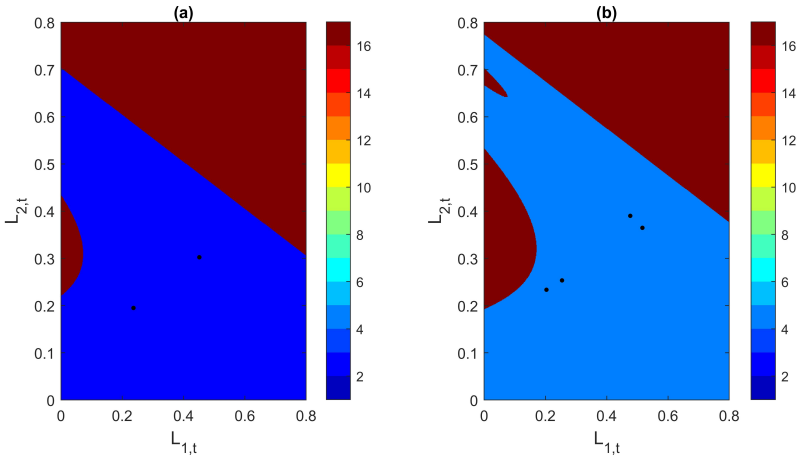

FIGURE 4. On the left, basins of attraction of the two-cycle, given the following values of the parameters, $\alpha = 1.35$, $\gamma = 0.7$, $r_k = 1.5$, $a = 3$, $b = 1.9$, $c_1 = 0.01$, $c_2 = 0.27$. On the right, basins of attraction showing the birth and the stability of a 4-cycle for $a = 3.27$ and the other parameters as in the left panel.

5. **Global dynamics and strange attractors.** The main purpose of this section is the study of the global dynamics of our system in order to better understand the final behaviour of the model. In particular, we will show that it admits attractors characterized by complex structures. As stressed by (8) and (9), the bifurcation which transforms the basins from simply connected to disconnected sets causes a loss of predictability concerning the long-run outcome. In this line of reasoning, this causes sensitivity with respect to the initial levels of loans' demand, that is a small perturbation may lead to trajectories that converge to different Nash equilibria.

Definition 5.1. A nonempty compact set $A \subset \mathbb{R}^2_+$ is the global attractor for the system $(T, \mathbb{R}^2_+)$ if the following conditions are fulfilled:

1. A is invariant,
2. A attracts all the bounded subsets from $\mathbb{R}^2_+$.[8]

Remembering that: (i) a set $A \subseteq \mathbb{R}^2_+$ is positively (negatively) invariant if $T^i(A) \subseteq A$ $(T^i(A) \supseteq A)$ $\forall i \in \mathbb{Z}_+$, (ii) A is invariant when it is both positively and negatively invariant, (iii) a closed and positively invariant region is called trapping; we are able to enunciate the following theorem.

Theorem 5.2. *For* $\frac{1}{2-\gamma}[2(1-\gamma)c_1 - (a-\gamma r_k)] < c_2 < \frac{1}{2-\gamma}\{\, a-\gamma r_k - \frac{b}{8\alpha(b+c_1)}[1+\alpha(a-\gamma r_k-(1-\gamma)c_1)]^2 \,\}$ *our system admits the compact global attractor which is contained in the set* $[0, 2B] \times [0, B]$ *with* $B = \frac{1}{2b}[a - \gamma r_k - (2-\gamma)c_2]$.

Proof. The proof is geometrical rather than analytical. Rewrite our system as:

$$L_{1,t+1} = f(L_{1,t}, g(L_{1,t-1})) = [1+\alpha(a-\gamma r_k-(1-\gamma)c_1)]L_{1,t}-2\alpha(b+c_1)L_{1,t}^2-\alpha b L_{1,t}g(L_{1,t-1})$$

where $g(L_{1,t-1}) = -\frac{1}{2}L_{1,t-1} + B$. If $L_{1,t-1} < 2B$ then $f(L_{1,t}, g(L_{1,t-1})) < [1 + \alpha(a - \gamma r_k - (1-\gamma)c_1)]L_{1,t} - 2\alpha(b + c_1)L_{1,t}^2 - \alpha b L_{1,t} g(2B) = [1 + \alpha(a - \gamma r_k - (1-\gamma)c_1)]L_{1,t} - 2\alpha(b + c_1)L_{1,t}^2$ which is a concave parabola. The condition $c_2 < \frac{1}{2-\gamma}\left\{a - \gamma r_k - \frac{b}{8\alpha(b+c_1)}[1 + \alpha(a - \gamma r_k - (1-\gamma)c_1)]^2\right\}$ guarantees that the vertex has ordinate less than $2B$ so that $L_{1,t+1} = f(L_{1,t}, g(L_{1,t-1})) < 2B$. This proves that $[0, 2B]$ is positively invariant for L_1 and hence $[0, 2B] \times [0, B]$ is positively invariant for the system. Suppose now $L_{1,t-1} > 2B$ hence $g(L_{1,t-1}) < 0$, but there exists $T > t - 1$ such that $g(L_{1,T}) > 0$ (and, consequently, $L_{1,T} < 2B$). In fact, if we suppose that the last statement is false, then $g(L_{1,T}) < 0$ at any time T. Hence at any time L_1 is defined by a parabola with the same geometrical properties, this means that there exists some iteration of the map which becomes smaller than $2B$ and we obtain a contradiction. This proves that $[0, 2B] \times [0, B]$ is trapping.

Finally, being $[0, 2B]\times[0, B]$ compact, positively invariant and trapping, then $A = \bigcap_{t\geq 0} T^t([0, 2B] \times [0, B])$ is a compact invariant set which attracts $[0, 2B] \times [0, B]$, that is $\forall \epsilon > 0$ $\exists t_\epsilon$ s.t. $T^t([0, 2B] \times [0, B]) \in B(A, \epsilon)$ $\forall t > t_\epsilon$, where $B(A, \epsilon)$ is a $\epsilon-$neighbourhood of A.

□

Now we want to explore numerically the asymptotic behaviour of our map. Thanks to Theorem 5.2 we have now conditions that allow to well define the long run path of the map. Following (13) we perform a graphical analysis showing the complexity of our model. In detail, in Figures 5 and 6 we report different kinds of attractors depending on the costs of the two banks. Keeping fixed $\alpha = 1.38$, $\gamma = 0.35$, $r_k = 2.8$, $a = 3$, $b = 0.12$ and $c_1 = 0.13$ we show how the structure of the attractor changes varying c_2 (the cost of small banks). In particular, in Figure 5 (a) we find a four-pieces chaotic attractor where the long-run behaviour of the map is confined. Increasing $c_2 = 0.8$ the attractor reduces to two pieces chaotic attractor (Figure 5 (b)) until it becomes of only one piece (see Figure 6 (a)). In this last figure, we have the occurrence of a global bifurcation (also called homoclinic bifurcation) and the evolution of the map is confined in the compact, positively invariant and trapping set found in Theorem 5.2. Finally, Figure 6 (b) shows the basin of attraction in the plane (L_1, L_2) of the attractor of Figure 6 (a). We can

[8] See (13).

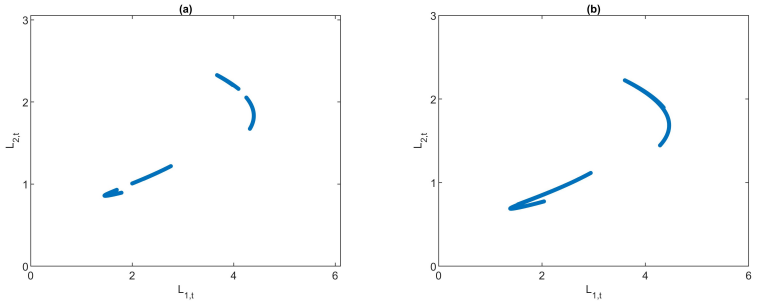

FIGURE 5. In (a), four pieces chaotic attractor for $\alpha = 1.38$, $\gamma = 0.35$, $r_k = 2.8$, $a = 3$, $b = 0.12$, $c_1 = 0.13$, $c_2 = 0.78$. In (b), two pieces chaotic attractor for $c_2 = 0.8$ and the other parameters as in Panel (a).

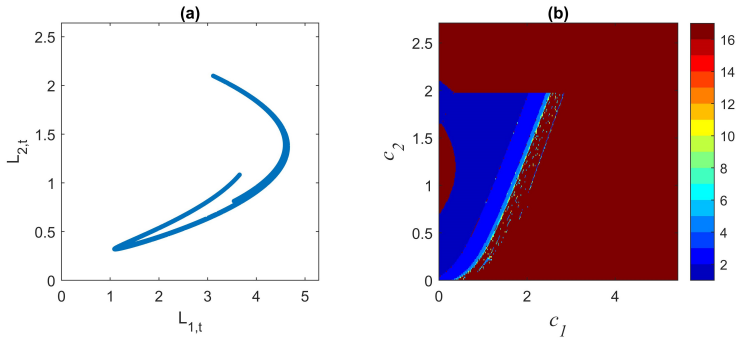

FIGURE 6. In (a), connected chaotic attractor for $c_2 = 0.84$ and the other parameters as in Panel (a) of Figure 5. On the right, cycle cartogram in the (c_1, c_2)-plane for parameter values as in Panel (a) of Figure 5.

observe that while small banks keep their costs constant, stability region increases with higher costs of large banks.

6. **Introducing stochastic shocks.** We now proceed with an analysis on the effect that the parameters of interest, i.e. the marginal costs c_1, c_2 and the regulation γ, produce on the dynamic of loans of the large and small Italian banks. To this regard, we add a noise term to each of the demand component of the Dynamical System (3). We remember that the two types of banks are heterogeneous in their expectations on future loans supplied and in the cost functions assumed in their profit maximization problem. As a further element of heterogeneity and robustness of our model, we perturb the two demand components with two independent and normally distributed random variables $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$, for the large and small banks, respectively. Combining the deterministic and stochastic elements, the large banks' demand of loans is:

$$L_{1,t+1} = f(L_{1,t}, L_{2,t}) + \varepsilon_{1,t} \qquad \varepsilon_{1,t} \sim N(0, (\sigma_{1,t})^2) \tag{6}$$

and similarly the small banks' demand of loans is:

$$L_{2,t+1} = g(L_{1,t}) + \varepsilon_{2,t} \qquad \varepsilon_{2,t} \sim N(0, (\sigma_{2,t})^2) \tag{7}$$

where $\sigma_{1,t}$ and $\sigma_{2,t}$ are two positive parameters representing the standard deviations of the normal random variables. Note that Equation (6) depends on both the level of loans, then the disturbance term includes the variances of both components, i.e. $\varepsilon_{1,t} \sim N(0, (\sigma_1)^2)$ where $\sigma_{1,t}^2 = (\sigma_t^{L_1})^2 + (\sigma_t^{L_2})^2$. On the other hand, the term $\varepsilon_{2,t} \sim N(0, (\sigma_2)^2)$ depends only on the variance of the large banks, that is $\sigma_{2,t}^2 = (\sigma_t^{L_1})^2$.

In our analysis, we benchmark on the quarterly data of Bank of Italy concerning loans granted to families and firms by large and small banks from March 31, 2015 to September 30, 2019. The baseline parametrization, reported in Table 1, has been chosen in order to replicate the benchmark set-up of the original data-set of Bank of Italy. In particular, following (38) and (37), the parameter values shown in Table 1 have been identified via a trial-and-error calibration exercise, i.e. we have systematically varied these parameters till the model dynamics appeared satisfactory to us. Moreover, regarding the choice of the parameter costs σ_1 and σ_2, we proceed in order to maintain the assumptions of our study corroborated by (33), i.e. small banks have maintained their level of loans almost constant over time with respect to large banks. For this purpose we assume that volatility associated with the demand of loans of small banks (σ_2) is lower than the volatility associated with the demand of large banks (σ_1). In what follows, we first study whether in the baseline parametrization the time series of loans generated simulating the model display statistical properties found in the real data-set. Then, we test the robustness of the model through a sensitivity analysis conducted to analyse the impact of the three key parameters c_1, c_2, γ. As we can see in Figure 7, we present the time series generated from the real data of Bank of Italy, L_1^{real} and L_2^{real}, for large and small banks (blue line in Panels (a) and (b), respectively) and the simulated time series of the loans, L_1^{SS} and L_2^{SS}, generated by the stochastic version of our model for large (red line in Panel (a)) and small (red line in Panel (b)) banks, respectively. Although the loans' trajectories look quite different, we do observe that, on average the model captures the main stylized facts. In detail, from Figure 7, we observe a constant decline of the loans of both banks but it is more evident for large banks than for small ones, in line with empirical evidence (see annual reports of Bank of Italy). Moreover, for the data-set considered, we highlight from Table 2 the value of the asymmetry of both banks. In particular, we can see that the asymmetry is negative for both banks and greater for large banks. From an economic point of view this data reflects a greater restriction of the credit of the loans of large banks. Indeed, when both banks restrict the output of loans, this restriction is greater for large banks than for small banks.

As a further step, we are also interested in how the stylized facts depend on the choice of some key parameters of the model. To this purpose, following (11) and (17) we conduct a Monte Carlo analysis on the robustness of the results of our model exploring its sensitivity to the costs of large and small banks (c_1, c_2) and to the regulation parameter (γ). In Figure 8 we have conducted several Monte Carlo simulations, we run 1000 simulations over 5000 time periods and discards the first 1000 time periods to wash out possible initial noise effects. In Panel (a) of Figure 8 we leave c_1, c_2 vary on their range, i.e. $c_1, c_2 \in [0.01, 0.4]$, keeping fixed all other parameters as in Table 1. We discover that the difference between the loans of the two banks increases with the costs and for all the range of the costs considered,

α	γ	r_k	a	b	c_1	c_2	σ_1	σ_1
1.04	0.18	3.6	1.68	0.043	0.08	0.38	0.12	0.06

TABLE 1. Parameter setting and initial values

	Mean	**sd**	**Min**	**Max**	**Skewness**	**Kurtosis**
L_1^{real}	291,440,000	64,858,000	192,020,000	356,490,000	-0.4471	1.6507
L_2^{real}	224,430,000	25,598,000	186,240,000	263,480,000	-0.0867	1.7266
L_1^{SS}	345,330,000	7,315,000	331,320,000	355,280,000	-0.4075	2.2085
L_2^{SS}	224,080,000	3,652,000	216,660,000	230,890,000	-0.1410	2.9930

TABLE 2. Summary statistics of banks' loans including mean, standard deviation (sd), skewness, minimum and maximum value for real time series of the loans of large banks (L_1^{real}), real time series of the loans of small banks (L_2^{real}) and the simulated stochastic time series of the loans for large (L_1^{SS}) and small (L_2^{SS}) banks.

moreover, the quantity of loans of small banks is always greater than that of large banks. Second, we explore the effects of the regulation on the loans' demand (see Panel (d) of Figure 8) when it varies between 0.05 and 0.4. Differently from the previous case, now we see that the effects on loans are reversed. Indeed, higher level of regulation diminishes the difference between the two banks (in terms of output) but an increase in regulation affects heavily small banks. This facts can be viewed as a violation of the "proportionality principle" according to which the regulatory and supervisory activities should be graded according to the size, business model and actual risk of supervised entities (see (5)).

Finally, we now analyse the joint changes of parameters because in this way we can observe the combined effect of the costs and regulations. In Figure 8 (b) and (c) we have plotted the loans of large and small banks with respect to the cost and the regulation parameters, respectively. In both panels, we have varied $c_1, c_2 \in [0.01, 0.3]$ and $\gamma \in [0.05, 0.4]$. This last simulation explain us other important facts. When we consider simultaneously changes in costs and regulation not only the demand of loans of all the banks in the economic system increases but, also, the differences among banks (in terms of loans output) decrease.

In this section the performed simulation analysis has allowed us to better understand the effect of parameters on loans' demand of large and small banks and efficiency (both the isolate effect of the costs change and the joint effect of costs and regulation parameters varying together). The value added of this computational exercise is then to suggest some effective interventions to both policy makers and regulators.

7. **Conclusions.** In this paper we investigated the role of the asymmetric costs between small and large banks. The aim of the paper is to show the higher efficiency of small banks (in particular cooperative banks) with respect to the larger ones (as found by (2)). This efficiency is not intended in absolute terms, but we refer to the predominant role played by small banks to sustain local economies ((25), (33)). To this purpose, we first analysed local properties of our model showing that different equilibria are possible. Then, we focused on the global dynamics of the demand of loans of the two banks and we were able to detect a global compact attractor

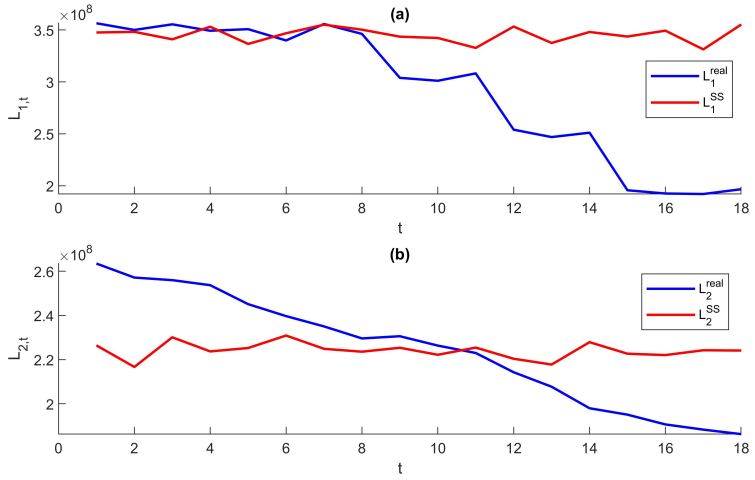

FIGURE 7. The time series of the loans of large (Panel (a)) and small (Panel (b)) Italian banks. The real time series are depicted in blue, while in red the time series of the loans resulting from the stochastic model.

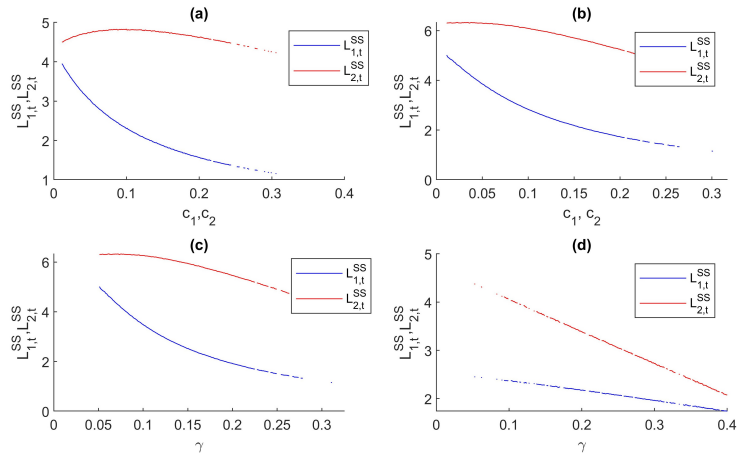

FIGURE 8. Monte Carlo simulations for analyse differences in loans' demand of the two banks for several combinations of costs and regulation parameters. In (a), $c_1, c_2 \in [0.01, 0.4]$ and the other parameters as in Table 1. In (b) and (c), joint effect of the costs and regulation keeping fixed all other parameters as in Table 1. In (b), the loans' demand are plotted with respect to the costs of the two banks when $c_1, c_2 \in [0.01, 0.3]$ and $\gamma \in [0.05, 0.4]$. In (c) the loans' demand are plotted with respect to the parameter of regulation γ when $c_1, c_2 \in [0.01, 0.3]$ and $\gamma \in [0.05, 0.4]$. In (d), the effects of regulation on the loans' demand when $\gamma \in [0.05, 0.4]$ and the other parameters as in Table 1.

where all the dynamics is confined (for opportune parameters values). This last case is possible when the costs of small banks belong to a specific interval, that is for not too high values of their costs. Finally, we extended our model by introducing stochastic shocks in the demand of loans of large and small banks. This analysis allowed us to further explore the joint role of costs and regulation on the demand of loans of both banks, but also we were able to give some policy suggestions. In particular, we found that tuning the effects of costs and regulation the demand of loans in the economic system increases and the difference among banks in terms of loans output reduces.

Our work could be extended in several ways. Introducing further non-linearity in the model could be beneficial. Indeed, we have highlighted in our work the role of non-linearity to produce complex dynamics. In particular, it has been possible thanks to the introduction of quadratic costs for large banks. It is worth to stress that also discontinuity plays a central role in the emergence of complex dynamics although the model continues to remain linear. To this purpose, following (22), (35), (28), we will attempt to introduce discontinuity in the model given by different levels of marginal costs, in order to analyse the interaction between banks with different size and expectations. A further development of the work comes from the consideration that the duopoly only considers small bank and large bank as representatives of the total sample. Following (8), it is possible to employ a semi-symmetrical set-up in order to consider an oligopoly with N firms (m small banks and $N - m$ large banks).

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