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Remarks on the nonlinear fractional Choquard equation

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Abstract

We analyze the following doubly nonlocal nonlinear elliptic problem:

$$\begin{cases} (-\Delta)^s u + \omega u = (I_\alpha * F(u))F'(u) \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$, $s \in (0,1)$, $\omega > 0$, $(-\Delta)^s$ denotes the fractional Laplacian, I_α is the Riesz potential of order $\alpha \in (0, N)$, and $F : \mathbb{R} \to \mathbb{R}$ is a C^1-nonlinearity of Berestycki-Lions type, exhibiting subcritical or critical growth in the sense of the Hardy-Littlewood-Sobolev inequality. By employing suitable variational techniques, we investigate the existence and qualitative properties of least energy solutions.

Keywords Nonlocal operators (primary) · Choquard equations · Variational methods · Qualitative properties of solutions

Mathematics Subject Classification 35R11 (primary) · 35B06 · 35B09 · 35B33 · 35B38 · 35B40 · 35B65

1 Introduction

In this paper, we consider the following doubly nonlocal nonlinear elliptic problem:

$$\begin{cases} (-\Delta)^s u + \omega u = (I_\alpha * F(u))f(u) \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 2$, $s \in (0,1)$, $\omega > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying suitable growth conditions, and $F(t) := \int_0^t f(\tau)d\tau$. The Riesz potential $I_\alpha : \mathbb{R}^N \setminus$

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$\{0\} \to \mathbb{R}$ of order $\alpha \in (0, N)$ is given by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{where} \quad A_\alpha := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{N/2}2^\alpha}.$$

The fractional Laplacian $(-\Delta)^s$ is defined for functions $u : \mathbb{R}^N \to \mathbb{R}$ in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ by the singular integral

$$(-\Delta)^s u(x) := C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{(u(x) - u(y))}{|x-y|^{N+2s}}\, dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $P.V.$ denotes the Cauchy principal value and

$$C_{N,s} := \frac{s 2^{2s} \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

Equivalently, $(-\Delta)^s$ admits the Fourier representation

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}u(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.$$

Fractional and nonlocal elliptic operators have been the subject of extensive study in recent years, owing to both their rich theoretical structure and their broad range of applications in fields such as mathematical finance, fluid mechanics, fractional quantum mechanics, plasma physics, image processing, and biological invasion models; see [3] and the references therein.

As $s \to 1^-$, the equation in (1.1) reduces to the following nonlinear Choquard equation:

$$-\Delta u + \omega u = (I_\alpha * F(u)) f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

which arises in various contexts of mathematical physics. For $N = 3$, $\omega = 1$, $F(u) = \frac{u^2}{2}$, and $\alpha = 2$, equation (1.2) simplifies to the classical Choquard-Pekar equation:

$$-\Delta u + u = \left(I_2 * \frac{u^2}{2}\right) u \text{ in } \mathbb{R}^3, \tag{1.3}$$

which was originally introduced in the works of Frölich [22] and Pekar [39] to model the quantum mechanics of a polaron at rest. In an approximation of the Hartree-Fock theory for a one-component plasma, Choquard used (1.3) to describe an electron trapped in its own hole [27]. Equation (1.3) is also known as the Schrödinger-Newton equation in the context of self-gravitating matter [33].

From a mathematical point of view, several contributions to Choquard equations have been established via variational methods. In [27], Lieb proved the existence and uniqueness (modulo translations) of the least energy solution to (1.3). Lions [30] obtained the existence of infinitely many radially symmetric solutions to (1.3). Later, Moroz and Van Schaftingen [34] investigated the existence, regularity, sign, symmetry, and decay of least energy solutions to (1.2) when $N \geq 3$, $\alpha \in (0, N)$, $\omega = 1$, and $F(u) = \frac{|u|^p}{p}$ with

$$p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right).$$

The endpoints $\frac{N+\alpha}{N}$ and $\frac{N+\alpha}{N-2}$ correspond to the extremal exponents of the Hardy-Littlewood-Sobolev inequality [28], often called the lower and upper critical H-L-S exponents, respectively. In [35], the same authors studied the existence of least energy solutions for (1.2) when f satisfies the following Berestycki-Lions type conditions [9]:

- $(F1)$ there exists $C_0 > 0$ such that $|tf(t)| \leq C_0(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha}{N-2}})$ for all $t \in \mathbb{R}$,
- $(F2)$ $\lim_{t\to 0} \frac{F(t)}{|t|^{\frac{N+\alpha}{N}}} = 0$ and $\lim_{|t|\to+\infty} \frac{F(t)}{|t|^{\frac{N+\alpha}{N-2}}} = 0$,
- $(F3)$ there exists $\tau_0 \in \mathbb{R} \setminus \{0\}$ such that $F(\tau_0) \neq 0$.

Additionally, if f is odd and has constant sign in $(0, +\infty)$, they showed that all least energy solutions are sign-definite and radially symmetric. These works deal with subcritical nonlinearities. In the critical case, the authors in [36] examined the existence and nonexistence of solutions to

$$-\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \text{ in } \mathbb{R}^N,$$

where $N \geq 3$ and $V \in L^\infty(\mathbb{R}^N)$ is a non-constant external potential satisfying appropriate assumptions. Gao and Yang [23] established a Brezis-Nirenberg type result [12] for a critical Choquard problem in a bounded domain. Cassani and Zhang [13] used the monotonicity trick and a decomposition technique to prove the existence of a least energy solution to (1.2) under the following assumptions on f:

- $(h1)$ $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$,
- $(h2)$ $\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{\alpha+2s}{N-2s}}} = 1$,
- $(h3)$ there exist $\mu > 0$ and $q \in (2, \frac{N+\alpha}{N-2})$ such that $f(t) \geq t^{\frac{\alpha+2}{N-2}} + \mu t^{q-1}$ for all $t > 0$.

Condition $(h3)$ ensures that f exhibits upper critical H-L-S growth. Using an approximation argument, Li and Tang [26] considered more general upper critical H-L-S nonlinearities f fulfilling the following conditions:

- $(k1)$ $\lim_{t\to 0^+} \frac{f(t)}{t^{\frac{\alpha}{N}}} = 0$,
- $(k2)$ $\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{\alpha+2}{N-2}}} = \mu_0 \in (0, +\infty)$,
- $(k3)$ F satisfies

  – $\lim_{t\to+\infty} \frac{F(t) - \mu_0 \frac{N-2}{N+\alpha} t^{\frac{N+\alpha}{N-2}}}{t^{\frac{N+\alpha}{N}}} = +\infty$ when $N > 4$,

  – $\lim_{t\to+\infty} \frac{F(t) - \mu_0 \frac{2}{4+\alpha} t^{\frac{4+\alpha}{2}}}{(t^2 \log t)^{\frac{\alpha+4}{8}}} = +\infty$ when $N = 4$,

– $\lim_{t\to+\infty} \frac{F(t)-\mu_0 \frac{1}{3+\alpha} t^{3+\alpha}}{t^{\frac{6+2\alpha}{3}}} = +\infty$ when $N = 3$,

$(k4)$ f is odd in $\mathbb{R}$ and has constant sign in $(0, +\infty)$.

For further developments, generalizations, and applications of Choquard equations, we refer the reader to the survey [40].

On the other hand, fractional Choquard equations have attracted significant attention over the past decade due to their applicability in modeling various physical phenomena, notably the gravitational collapse of boson stars; see [19] and [29]. Note that the presence of two nonlocal terms-the fractional Laplacian and the Riesz potential-entails substantial analytical challenges that distinguish these problems from their classical counterparts. In this context, d'Avenia et al. [17] investigated the existence and qualitative properties of least energy solutions to (1.1) when $N \geq 3, s \in (0, 1), \alpha \in (0, N)$, and $f(u) = |u|^{p-2}u$ with $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2s})$. Shen et al. [41] studied (1.1) with a more general subcritical nonlinearity f; see also [14, 15, 31] for further advances. In [1], the author examined the multiplicity and concentration of positive solutions for a nonlinear fractional Choquard equation. Giacomoni et al. [25] focused on the regularity of weak solutions for a class of doubly nonlocal problems in bounded domains. The existence and qualitative properties of least energy solutions for a fractional relativistic Choquard equation were analyzed in [7]. In the critical regime, Mukherjee and Sreenadh [38] established existence, nonexistence, and regularity results for a Brezis-Nirenberg type fractional Choquard problem in a bounded domain. Ma and Zhang [32] studied the existence and multiplicity of weak solutions to

$$(-\Delta)^s u + (\lambda V(x) - \beta)u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2s}})|u|^{\frac{N+\alpha}{N-2s}-2}u \text{ in } \mathbb{R}^N,$$

where $V \in C^0(\mathbb{R}^N)$ satisfies suitable conditions, $\beta > 0$ is sufficiently small, and $\lambda > 0$ is large. More recently, the existence of least energy solutions to (1.1) with upper critical H-L-S nonlinearities was obtained in [8].

In this paper, motivated by [7, 8, 26, 35], we aim to improve and extend existing results in the literature concerning the nonlocal problem (1.1). In particular, we investigate the existence and qualitative properties of least energy solutions to (1.1) under the assumption that the nonlinearity f exhibits either subcritical or critical growth. We begin with the subcritical case. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

$(f1)$ $\limsup_{t\to 0} \frac{|tf(t)|}{|t|^{\frac{N+\alpha}{N}}} < +\infty$ and $\limsup_{|t|\to+\infty} \frac{|tf(t)|}{|t|^{\frac{N+\alpha}{N-2s}}} < +\infty$,

$(f2)$ $\lim_{t\to 0} \frac{F(t)}{|t|^{\frac{N+\alpha}{N}}} = 0$ and $\lim_{|t|\to+\infty} \frac{F(t)}{|t|^{\frac{N+\alpha}{N-2s}}} = 0$,

$(f3)$ there exists $t_0 \in \mathbb{R} \setminus \{0\}$ such that $F(t_0) \neq 0$.

Since we seek weak solutions to (1.1), we recall that $u \in H^s(\mathbb{R}^N)$ is a weak solution to (1.1) if

$$\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}}\, dxdy + \omega \int_{\mathbb{R}^N} u\phi\, dx$$

$$= \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u)\phi \, dx \quad \text{for all } \phi \in H^s(\mathbb{R}^N).$$

Our first main result address the regularity of weak solutions to (1.1). Hereafter, we use the notation $C^\gamma(\mathbb{R}^N) := C^{[\gamma],\gamma-[\gamma]}(\mathbb{R}^N)$ for all $\gamma \in (0, +\infty) \setminus \mathbb{N}$.

Theorem 1 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ holds. Assume that one of the following conditions holds:*

(f4) $s \in \left[\frac{1}{2}, 1\right)$ *and* $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ *for some* $\sigma \in (0, 1]$,
(f5) $s \in (\frac{1}{4}, \frac{1}{2})$, $\alpha \in (1 - 2s, N)$, *and* $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ *for some* $\sigma \in (\frac{1-2s}{2s}, 1]$,
(f6) $s \in (0, \frac{1}{2})$, $\alpha \in (0, 1)$, *and* $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ *for some* $\sigma \in (1 - 2s, 1]$.

Let $u \in H^s(\mathbb{R}^N)$ be a weak solution to (1.1). Then:

- $u \in L^p(\mathbb{R}^N)$ *for all* $p \in [2, +\infty]$.
- $u \in C^{1,\theta}(\mathbb{R}^N)$ *for some* $\theta \in (0, 1)$.
- u *is a classical solution to (1.1), that is,* $u \in H^{2s}(\mathbb{R}^N) \cap C^{2s+\varepsilon}(\mathbb{R}^N)$ *for some* $\varepsilon \in (0, 1 + [2s] - 2s)$, *and u satisfies (1.1) pointwise.*

Moreover, if the following condition near the origin in $(f1)$ holds:

$$\limsup_{t\to 0} \frac{|tf(t)|}{|t|^2} < +\infty, \tag{1.4}$$

then there exists $C_1 > 0$ such that

$$|u(x)| \leq \frac{C_1}{1 + |x|^{N+2s}} \quad \textit{for all } x \in \mathbb{R}^N. \tag{1.5}$$

To prove Theorem 1, we first employ a nonlocal Brezis-Kato-type argument [10, 35] to show that every weak solution u to (1.1) belongs to $L^p(\mathbb{R}^N)$ for all $p \in [2, \frac{N}{\alpha} 2^*_s)$; see Proposition 3. This fact, together with Young's inequality, implies that $I_\alpha * F(u) \in L^\infty(\mathbb{R}^N)$. Hence, u is a weak solution to $(-\Delta)^s u + \omega u = g(x, u)$ in $\mathbb{R}^N$, where $|g(x, u)| \leq C(1 + |u|^{r-1})$ for some $r \in [1, 2^*_s - 1]$. We then utilize properties of the kernel $\mathcal{K}_{2s} := ((-\Delta)^s + 1)^{-1}$ to deduce that $u \in L^\infty(\mathbb{R}^N)$; see Proposition 4. We emphasize that our proof of the boundedness of weak solutions differs from classical approaches in the literature, which typically rely on Moser or De Giorgi iteration techniques; see [3]. Finally, by applying the Schauder estimates for the fractional Laplacian from [42, 43] and using the local Hölder regularity of f, we obtain the desired regularity result.

In light of Theorem 1 and an integration by parts formula established in [5], we show that every weak solution to (1.1) satisfies a Pohožaev-type identity.

Theorem 2 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ holds. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Let $u \in H^s(\mathbb{R}^N)$ be a weak solution to (1.1). Then u satisfies the following Pohožaev* *identity:*

$$P(u) := \frac{C_{N,s}}{4}(N-2s)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy + \frac{N\omega}{2}\int_{\mathbb{R}^N} u^2\,dx \\ -\left(\frac{N+\alpha}{2}\right)\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,dx = 0, \tag{1.6}$$

where $P : H^s(\mathbb{R}^N) \to \mathbb{R}$ is called Pohožaev functional.

We next focus on the existence of weak solutions to (1.1). Consider the energy functional $\mathcal{J}: H^s(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1):

$$\mathcal{J}(u) := \frac{C_{N,s}}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy + \frac{\omega}{2}\int_{\mathbb{R}^N} u^2\,dx \\ -\frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,dx \quad \text{for all } u \in H^s(\mathbb{R}^N).$$

By virtue of $(f1)$, the fractional Sobolev embeddings, and the Hardy-Littlewood-Sobolev inequality, it is standard to verify that $\mathcal{J} \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and that its critical points correspond to weak solutions of (1.1). Moreover, assumptions $(f1)$ and $(f3)$ ensure that $\mathcal{J}$ exhibits a mountain pass geometry (see Lemma 9), allowing us to define the mountain pass level

$$c_{\mathrm{MP}} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(\gamma(t)), \tag{1.7}$$

where

$$\Gamma := \{\gamma \in C^0([0,1], H^s(\mathbb{R}^N)) : \ \gamma(0) = 0 \text{ and } \mathcal{J}(\gamma(1)) < 0\}. \tag{1.8}$$

By constructing a bounded Palais-Smale sequence at level c_{MP} that asymptotically satisfies the Pohožaev identity (1.6) (see Proposition 6 and Lemma 10), and applying a concentration-compactness-type argument in $H^s(\mathbb{R}^N)$, we establish the existence of a weak solution to (1.1). To confirm that it is a least energy solution, we use the Pohožaev identity to build an optimal path (see Lemma 11), thereby showing that the least energy level coincides with c_{MP}. We say that $u \in H^s(\mathbb{R}^N)\setminus\{0\}$ is a least energy solution to (1.1) if $\mathcal{J}'(u) = 0$ and $\mathcal{J}(u) = c_{\mathrm{LE}}$, where the least energy level c_{LE} is defined by

$$c_{\mathrm{LE}} := \inf_{u\in\mathcal{S}} \mathcal{J}(u),$$

and

$$\mathcal{S} := \left\{u \in H^s(\mathbb{R}^N)\setminus\{0\} : \mathcal{J}'(u) = 0\right\}.$$

Our third result can now be stated as follows.

Theorem 3 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Then there exists a least energy solution $u \in H^s(\mathbb{R}^N)$ to (1.1). Moreover,*

$$c_{\mathrm{LE}} = c_{\mathrm{MP}} = c_{\mathrm{PO}}, \tag{1.9}$$

where

$$c_{\mathrm{PO}} := \inf\{\mathcal{J}(u) |\ u \in \mathcal{P}\}, \tag{1.10}$$

and $\mathcal{P}$ is the Pohožaev manifold defined by

$$\mathcal{P} := \{u \in H^s(\mathbb{R}^N) \setminus \{0\} : P(u) = 0\}.$$

Throughout the paper, we refer to $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ as a least energy Pohožaev minimizer for $\mathcal{J}$ if $u \in \mathcal{P}$ and $\mathcal{J}(u) = c_{\mathrm{PO}}$. By analyzing the sign and symmetry of least energy Pohožaev minimizers for $\mathcal{J}$ and leveraging the identity (1.9) (see Propositions 9 and 13), we deduce the following qualitative properties of least energy solutions to (1.1).

Theorem 4 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. In addition, we assume that f is odd in $\mathbb{R}$ and has constant sign in $(0, +\infty)$. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Then every least energy solution $u \in H^s(\mathbb{R}^N)$ of (1.1) has constant sign, is radially symmetric with respect to some point in $\mathbb{R}^N$, and is radially decreasing.*

Finally, we investigate (1.1) in the case where f is a general upper critical H-L-S nonlinearity. In this setting, we prove the following result.

Theorem 5 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in ((N-4s)_+, N)$. Let $f \in C^0(\mathbb{R})$ satisfy the following conditions:*

(f1)' $\lim_{t\to 0} \frac{tf(t)}{|t|^{\frac{N+\alpha}{N}}} = 0$,

(f2)' $\lim_{|t|\to +\infty} \frac{tf(t)}{|t|^{\frac{N+\alpha}{N-2s}}} = \mu \in (0, +\infty)$,

(f3)' *F satisfies*

- $\lim_{t\to+\infty} \frac{F(t)-\mu(\frac{N-2s}{N+\alpha})t^{\frac{N+\alpha}{N-2s}}}{t^{\frac{N+\alpha}{N}}} = +\infty$ *when* $N > 4s$,
- $\lim_{t\to+\infty} \frac{F(t)-\mu(\frac{2s}{4s+\alpha})t^{\frac{4s+\alpha}{2s}}}{(t^2 \log t^{\frac{1}{s}})^{\frac{\alpha+4s}{8s}}} = +\infty$ *when* $N = 4s$,
- $\lim_{t\to+\infty} \frac{F(t)-\mu(\frac{N-2s}{N+\alpha})t^{\frac{N+\alpha}{N-2s}}}{t^{\frac{2s(N+\alpha)}{N(N-2s)}}} = +\infty$ *when* $2s < N < 4s$.

Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Then, there exists a least energy solution $u \in H^s(\mathbb{R}^N)$ to (1.1). If, in addition, f is odd in $\mathbb{R}$ and has constant sign in $(0, +\infty)$, then every least energy solution to (1.1) has constant sign, is radially symmetric with respect to some point in $\mathbb{R}^N$, and is radially decreasing.

The proof of Theorem 5 proceeds via the variational strategy used in the subcritical case. However, the presence of a general upper critical H-L-S nonlinearity requires a more refined analysis to implement a concentration-compactness-type argument. A key step in our approach is deriving an upper bound for the mountain pass level c_{MP}, which we obtain by employing truncated extremal functions and establishing sharp estimates on the nonlinear Choquard term; see Lemmas 16, 17, and 18. We note that assumption $(f3)'$ plays a crucial role in achieving these estimates. In contrast to the method used in [26], our approach does not rely on subcritical approximation functionals. Instead, we work directly with the energy functional $\mathcal{J}$ in the fractional Sobolev space $H^s(\mathbb{R}^N)$, and our strategy also applies to the case $s = 1$. Consequently, we extend and improve upon the results in [8, 13] by covering a broader class of upper critical H-L-S nonlinearities.

The paper is organized as follows. In Section 2, we collect some preliminary results. In Section 3, we study the regularity of weak solutions and derive a Pohožaev-type identity. In Section 4, we prove Theorem 3. Section 5 focuses on the qualitative properties of least energy solutions to (1.1). Finally, Section 6 deals with the critical case and contains the proof of Theorem 5.

2 Preliminaries

We begin by fixing the notation. Let $s \in (0, 1)$ and $N > 2s$. We consider the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi < +\infty \right\}$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (|\xi|^{2s} + 1) |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

Setting

$$[u]_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

and noting that (see [3, formula (1.2.2), p. 7])

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \quad \text{for all } u \in H^s(\mathbb{R}^N),$$

we deduce

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_{\mathbb{R}^N} u^2 \, dx \quad \text{for all } u \in H^s(\mathbb{R}^N).$$

Evidently, $H^s(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^N)} := \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy + \int_{\mathbb{R}^N} uv\,dx \quad \text{for all } u, v \in H^s(\mathbb{R}^N).$$

Let us define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to $[\cdot]_{H^s(\mathbb{R}^N)}$. We recall that the following fractional Sobolev inequality holds (see [16, Theorem 1.1]):

$$S_*\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} \le [u]^2_{H^s(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \tag{2.1}$$

where

$$S_* := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)} \frac{[u]^2_{H^s(\mathbb{R}^N)}}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}},$$

and $2_s^* := \frac{2N}{N-2s}$ is the critical fractional Sobolev exponent. The best constant S_* is given explicitly by

$$S_* = 2^{2s}\pi^s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(\frac{N-2s}{2})}\left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right]^{\frac{2s}{N}}.$$

Then, the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ can also be characterized as

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \left\{u \in L^{2_s^*}(\mathbb{R}^N) : [u]_{H^s(\mathbb{R}^N)} < +\infty\right\}.$$

Put

$$\langle u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy \quad \text{for all } u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Thus, we can write

$$\langle u, v\rangle_{H^s(\mathbb{R}^N)} = \langle u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \langle u, v\rangle_{L^2(\mathbb{R}^N)} \quad \text{for all } u, v \in H^s(\mathbb{R}^N).$$

The next results are well known.

Proposition 1 *[3, Theorems 1.1.7 and 1.1.8] Let $N \ge 2$ and $s \in (0, 1)$. Then:*

- *$H^s(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2_s^*]$ and compactly embedded in $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2_s^*)$.*
- *$C_c^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$.*

Lemma 1 *[6, Lemma 2.1] Let $N \geq 2$, $s \in (0, 1)$, and $t \in [2, 2^*_s)$. Then there exists $C = C(N, s, t) > 0$ such that*

$$\|u\|^t_{L^t(\mathbb{R}^N)} \leq C \left(\sup_{x_0 \in \mathbb{R}^N} \|u\|^t_{L^t(B_1(x_0))} \right)^{1-\frac{2}{t}} \|u\|^2_{H^s(\mathbb{R}^N)} \quad \text{for all } u \in H^s(\mathbb{R}^N).$$

Lemma 2 *[3, Lemma 1.4.4] Let $N \geq 2$ and $s \in (0, 1)$. Let $(u_n) \subset H^s(\mathbb{R}^N)$ be a bounded sequence such that*

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^p \, dx = 0,$$

for some $R > 0$ and $p \in [2, 2^*_s)$. Then, $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*_s)$.

In view of the presence of the Choquard term, we recall the celebrated Hardy-Littlewood-Sobolev inequality.

Proposition 2 *[28, Theorem 4.3] Let $r, t \in (1, +\infty)$ and $\alpha \in (0, N)$ with $1/r + 1/t = 1 + \alpha/N$. Let $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then, there exists a sharp constant $C(N, \alpha, r) > 0$, independent of g and h, such that*

$$\left| \iint_{\mathbb{R}^{2N}} \frac{g(x)h(y)}{|x-y|^{N-\alpha}} \, dx dy \right| \leq C(N, \alpha, r) \|g\|_{L^r(\mathbb{R}^N)} \|h\|_{L^t(\mathbb{R}^N)}. \tag{2.2}$$

In particular, if $r = t = \frac{2N}{N+\alpha}$, then the sharp constant is given by

$$C(N, \alpha) := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right]^{-\frac{\alpha}{N}}.$$

In this case, equality holds in (2.2) if and only if $h \equiv (constant) g$ and

$$g(x) = \frac{A}{(\gamma^2 + |x-a|^2)^{\frac{N+\alpha}{2}}},$$

for some $A \in \mathbb{C}$, $\gamma \in \mathbb{R} \setminus \{0\}$, and $a \in \mathbb{R}^N$.

Remark 1 From Proposition 2, we derive that if $q \in (1, \frac{N}{\alpha})$, then for every $v \in L^q(\mathbb{R}^N)$ we have $I_\alpha * v \in L^{\frac{Nq}{N-\alpha q}}(\mathbb{R}^N)$. Furthermore, the map $I_\alpha : L^q(\mathbb{R}^N) \to L^{\frac{Nq}{N-\alpha q}}(\mathbb{R}^N)$ is a bounded linear operator and

$$\|I_\alpha * v\|_{L^{\frac{Nq}{N-\alpha q}}(\mathbb{R}^N)} \leq C(N, \alpha, q) \|v\|_{L^q(\mathbb{R}^N)} \quad \text{for all } v \in L^q(\mathbb{R}^N).$$

Let us now define

$$S_{H,L} := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]^2_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{N+\alpha}{N-2s}})|u|^{\frac{N+\alpha}{N-2s}}\,dx\right)^{\frac{N-2s}{N+\alpha}}}.$$

The following useful result holds.

Lemma 3 *[8, Lemma 2.1] The constant $S_{H,L}$ is achieved if and only if*

$$u(x) = C\left(\frac{b}{b^2+|x-a|^2}\right)^{\frac{N-2s}{2}},$$

for some $C>0$, $a\in\mathbb{R}^N$, and $b>0$. Furthermore,

$$S_{H,L} = \frac{S_*}{(C(N,\alpha)A_\alpha)^{\frac{N-2s}{N+\alpha}}}.$$

By $(f1)$, we know that there exist $C_1, C_2 > 0$ such that

$$|f(t)| \le C_1(|t|^{\frac{\alpha}{N}} + |t|^{\frac{\alpha+2s}{N-2s}}) \quad \text{for all } t\in\mathbb{R}, \tag{2.3}$$

$$|F(t)| \le C_2(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha}{N-2s}}) \quad \text{for all } t\in\mathbb{R}. \tag{2.4}$$

Define $\mathscr{F} : H^s(\mathbb{R}^N)\to\mathbb{R}$ by setting

$$\mathscr{F}(u) := \int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,dx \quad \text{for all } u\in H^s(\mathbb{R}^N).$$

Clearly, (2.3), (2.4), and Proposition 2 imply that $\mathscr{F}\in C^1(H^s(\mathbb{R}^N),\mathbb{R})$ and

$$\langle\mathscr{F}'(u), v\rangle = \int_{\mathbb{R}^N}(I_\alpha * F(u))f(u)v\,dx \quad \text{for all } u, v\in H^s(\mathbb{R}^N).$$

Moreover, for all $u, v\in H^s(\mathbb{R}^N)$, we have the following estimates:

$$|\mathscr{F}(u)| \le C(N,\alpha)\|F(u)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C'\left(\|u\|^2_{L^2(\mathbb{R}^N)} + \|u\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)}\right)^{\frac{N+\alpha}{N}}, \tag{2.5}$$

and

$$\begin{aligned}|\langle\mathscr{F}'(u), v\rangle| &\le C(N,\alpha)\|F(u)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\|f(u)v\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\\ &\le C''\left(\|u\|^2_{L^2(\mathbb{R}^N)} + \|u\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)}\right)^{\frac{N+\alpha}{2N}}\left(\|u\|^{\frac{2\alpha}{N+\alpha}}_{L^2(\mathbb{R}^N)} + \|u\|^{2^*_s\frac{\alpha+2s}{N+\alpha}}_{L^{2^*_s}(\mathbb{R}^N)}\right)^{\frac{N+\alpha}{2N}}\|v\|_{H^s(\mathbb{R}^N)}.\end{aligned} \tag{2.6}$$

Finally, we prove a key convergence result for the nonlinear Choquard term.

Lemma 4 *Let $N \geq 2$, $s \in (0,1)$, and $\alpha \in (0,N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ holds. Let (u_n) be a bounded sequence in $H^s(\mathbb{R}^N)$. Assume that, up to a subsequence,*

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N),\\ &u_n \to u \quad \text{in } L^q_{loc}(\mathbb{R}^N) \text{ for all } q \in [1, 2^*_s),\\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{2.7}$$

Then, for all $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)\varphi \, dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u)\varphi \, dx.$$

Proof Utilizing (2.4) and the boundedness of (u_n) in $L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$, we know that $(F(u_n))$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Since F is continuous and $u_n \to u$ a.e. in $\mathbb{R}^N$, we have that $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^N$. Thus, applying [45, Proposition 5.4.7], we deduce that

$$F(u_n) \rightharpoonup F(u) \ \text{ in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \tag{2.8}$$

Combining (2.8) and Proposition 2, we find

$$I_\alpha * F(u_n) \rightharpoonup I_\alpha * F(u) \ \text{ in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \tag{2.9}$$

On the other hand, (2.3), (2.7), and the dominated convergence theorem yield

$$f(u_n) \to f(u) \ \text{ in } L^p_{loc}(\mathbb{R}^N) \ \text{ for all } p \in \left[1, \frac{2N}{\alpha + 2s}\right). \tag{2.10}$$

In light of (2.9) and (2.10), we obtain

$$\langle \mathscr{F}'(u_n), \varphi\rangle \to \langle \mathscr{F}'(u), \varphi\rangle \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^N). \tag{2.11}$$

Take $\varphi \in H^s(\mathbb{R}^N)$. Because $C^\infty_c(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$ (see Proposition 1), there exists $(\varphi_j) \subset C^\infty_c(\mathbb{R}^N)$ such that $\|\varphi_j - \varphi\|_{H^s(\mathbb{R}^N)} \to 0$ as $j \to +\infty$. Note that (2.6) and the boundedness of (u_n) in $H^s(\mathbb{R}^N)$ imply that

$$\left|\langle \mathscr{F}'(u_n) - \mathscr{F}'(u), \varphi - \varphi_j\rangle\right| \leq C_3 \|\varphi - \varphi_j\|_{H^s(\mathbb{R}^N)} \quad \text{for all } j, n \in \mathbb{N}. \tag{2.12}$$

Fix $\varepsilon > 0$. Then there exists $j_0 \in \mathbb{N}$ such that

$$\|\varphi_{j_0} - \varphi\|_{H^s(\mathbb{R}^N)} < \frac{\varepsilon}{2C_3}. \tag{2.13}$$

In view of (2.11), we may select $n_0 \in \mathbb{N}$ such that

$$\left|\langle \mathscr{F}'(u_n) - \mathscr{F}'(u), \varphi_{j0}\rangle\right| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0. \tag{2.14}$$

Combining (2.12), (2.13), and (2.14), we conclude that

$$\begin{aligned}&\left|\langle \mathscr{F}'(u_n), \varphi\rangle - \langle \mathscr{F}'(u), \varphi\rangle\right| \\ &\leq \left|\langle \mathscr{F}'(u_n) - \mathscr{F}'(u), \varphi_{j0}\rangle\right| + \left|\langle \mathscr{F}'(u_n) - \mathscr{F}'(u), \varphi - \varphi_{j0}\rangle\right| < \varepsilon \ \text{ for all } n \geq n_0.\end{aligned}$$

The proof is now complete. □

3 Regularity results and Pohožaev identity

We start by proving a Brezis-Kato-type estimate [10].

Lemma 5 *Let $N \geq 2$ and $s \in (0, 1)$. Let $V \in L^\infty(\mathbb{R}^N) + L^{\frac{N}{2s}}(\mathbb{R}^N)$. Then, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for every $v \in H^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \mathsf{V}|v|^2\, dx \leq \varepsilon^2 [v]^2_{H^s(\mathbb{R}^N)} + C_\varepsilon \|v\|^2_{L^2(\mathbb{R}^N)}.$$

Proof We argue as in [10, Lemma 2.1]. Write $\mathsf{V} = \mathsf{V}_1 + \mathsf{V}_2$ with $\mathsf{V}_1 \in L^\infty(\mathbb{R}^N)$ and $\mathsf{V}_2 \in L^{\frac{N}{2s}}(\mathbb{R}^N)$. Applying Hölder's inequality and (2.1), we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^N} \mathsf{V}|v|^2\, dx &\leq \|\mathsf{V}_1\|_{L^\infty(\mathbb{R}^N)} \|v\|^2_{L^2(\mathbb{R}^N)} + n \int_{\{|\mathsf{V}_2| \leq n\}} |v|^2\, dx + \int_{\{|\mathsf{V}_2| > n\}} \mathsf{V}_2 |v|^2\, dx \\ &\leq (\|\mathsf{V}_1\|_{L^\infty(\mathbb{R}^N)} + n)\|v\|^2_{L^2(\mathbb{R}^N)} + \|\mathsf{V}_2\|_{L^{\frac{N}{2s}}(\{|\mathsf{V}_2|>n\})} \|v\|^2_{L^{2^*_s}(\mathbb{R}^N)} \\ &\leq (\|\mathsf{V}_1\|_{L^\infty(\mathbb{R}^N)} + n)\|v\|^2_{L^2(\mathbb{R}^N)} + S_*^{-1} \|\mathsf{V}_2\|_{L^{\frac{N}{2s}}(\{|\mathsf{V}_2|>n\})} [v]^2_{H^s(\mathbb{R}^N)}.\end{aligned}$$

Fix $\varepsilon > 0$. Since $\mathsf{V}_2 \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, we can choose n large enough so that

$$S_*^{-1} \|\mathsf{V}_2\|_{L^{\frac{N}{2s}}(\{|\mathsf{V}_2|>n\})} < \varepsilon^2.$$

The proof is complete. □

Remark 2 More generally, in the proof of Lemma 5, we use the following fact: if $\mathsf{V} \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $1 < q < p \leq +\infty$ and $\mathsf{V} = \mathsf{V}_1 + \mathsf{V}_2$, where $\mathsf{V}_1 \in L^p(\mathbb{R}^N)$ and $\mathsf{V}_2 \in L^q(\mathbb{R}^N)$, then for each fixed $\varepsilon > 0$, we can write $\mathsf{V} = \bar{\mathsf{V}}_1 + \bar{\mathsf{V}}_2$, with $\bar{\mathsf{V}}_1 := \mathsf{V}_1 + \mathsf{V}_2 \chi_{|\mathsf{V}_2| \leq n_\varepsilon} \in L^p(\mathbb{R}^N)$, $\bar{\mathsf{V}}_2 := \mathsf{V}_2 \chi_{|\mathsf{V}_2| > n_\varepsilon} \in L^q(\mathbb{R}^N)$, and $n_\varepsilon \in \mathbb{N}$ such that $\|\bar{\mathsf{V}}_2\|_{L^q(\mathbb{R}^N)} \leq \varepsilon$.

The lemma below is proved in [35].

Lemma 6 *[35, Lemma 3.3] Let* $q, r, \tau, t \in [1, +\infty)$ *and* $\lambda \in [0, 2]$ *such that*

$$1 + \frac{\alpha}{N} - \frac{1}{\tau} - \frac{1}{t} = \frac{\lambda}{q} + \frac{2-\lambda}{r}.$$

If $\theta \in (0, 2)$ *satisfies*

$$\min(q, r)\left(\frac{\alpha}{N} - \frac{1}{\tau}\right) < \theta < \max(q, r)\left(1 - \frac{1}{\tau}\right),$$
$$\min(q, r)\left(\frac{\alpha}{N} - \frac{1}{t}\right) < 2 - \theta < \max(q, r)\left(1 - \frac{1}{t}\right),$$

then for every $H \in L^{\tau}(\mathbb{R}^N)$*,* $K \in L^{t}(\mathbb{R}^N)$*, and* $v \in L^{q}(\mathbb{R}^N) \cap L^{r}(\mathbb{R}^N)$*,*

$$\int_{\mathbb{R}^N} (I_\alpha * (H|v|^{\theta}))K|v|^{2-\theta}\, dx \leq C\|H\|_{L^{\tau}(\mathbb{R}^N)}\|K\|_{L^{\tau}(\mathbb{R}^N)}\|u\|^{\lambda}_{L^{q}(\mathbb{R}^N)}\|u\|^{2-\lambda}_{L^{r}(\mathbb{R}^N)}.$$

We now establish a nonlocal counterpart of Lemma 5.

Lemma 7 *Let* $N \geq 2$*,* $s \in (0, 1)$*, and* $\alpha \in (0, N)$*. Let* $\theta \in (\frac{\alpha}{N}, 2 - \frac{\alpha}{N})$ *and* $H, K \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$*. Then, for every* $\varepsilon > 0$ *there exists* $C_{\varepsilon,\theta} > 0$ *such that, for every* $v \in H^s(\mathbb{R}^N)$*,*

$$\int_{\mathbb{R}^N} (I_\alpha * (H|v|^{\theta}))K|v|^{2-\theta}\, dx \leq \varepsilon^2[v]^2_{H^s(\mathbb{R}^N)} + C_{\varepsilon,\theta}\|v\|^2_{L^2(\mathbb{R}^N)}.$$

Proof We adapt the proof of [35, Lemma 3.2]. Fix $v \in H^s(\mathbb{R}^N)$. Let $\mathsf{H} := \mathsf{H}_1 + \mathsf{H}_2$ and $\mathsf{K} := \mathsf{K}_1 + \mathsf{K}_2$ with $\mathsf{H}_1, \mathsf{K}_1 \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ and $\mathsf{H}_2, \mathsf{K}_2 \in L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$. Next, we apply Lemma 6 with suitable choices of q, r, τ, t, and λ. More precisely, taking $q = r = 2^*_s$, $\tau = t = \frac{2N}{\alpha+2s}$, and $\lambda = 0$, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_2|v|^{\theta}))\mathsf{K}_2|v|^{2-\theta}\, dx \leq C_1\|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|\mathsf{K}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|v\|^2_{L^{2^*_s}(\mathbb{R}^N)}. \tag{3.1}$$

When $t = \tau = \frac{2N}{\alpha}$, $q = r = 2$, and $\lambda = 2$, we find

$$\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_1|v|^{\theta}))\mathsf{K}_1|v|^{2-\theta}\, dx \leq C_2\|\mathsf{H}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\mathsf{K}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|v\|^2_{L^2(\mathbb{R}^N)}. \tag{3.2}$$

Choosing $t = \frac{2N}{\alpha}$, $\tau = \frac{2N}{\alpha+2s}$, $q = 2$, $r = 2^*_s$, and $\lambda = 1$, we deduce

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_2|v|^{\theta}))\mathsf{K}_1|v|^{2-\theta}\, dx \\ &\leq C_3\|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|\mathsf{K}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|v\|_{L^2(\mathbb{R}^N)}\|v\|_{L^{2^*_s}(\mathbb{R}^N)},\end{aligned} \tag{3.3}$$

and with $\tau = \frac{2N}{\alpha}$, $t = \frac{2N}{\alpha+2s}$, $q = 2$, $r = 2_s^*$, and $\lambda = 1$, we see

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_1|v|^\theta))\mathsf{K}_2|v|^{2-\theta}\,dx \\ &\le C_4\|\mathsf{H}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\mathsf{K}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|v\|_{L^2(\mathbb{R}^N)}\|v\|_{L^{2_s^*}(\mathbb{R}^N)}.\end{aligned} \tag{3.4}$$

Combining (3.1), (3.2), (3.3), and (3.4), and utilizing (2.1), we infer that

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}|v|^\theta))\mathsf{K}|v|^{2-\theta}\,dx \\ &\le C_5\|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|\mathsf{K}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}[v]^2_{H^s(\mathbb{R}^N)} \\ &\quad + C_2\|\mathsf{H}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\mathsf{K}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|v\|^2_{L^2(\mathbb{R}^N)} \\ &\quad + C_6\left(\|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|\mathsf{K}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} + \|\mathsf{H}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\mathsf{K}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\right) \\ &\qquad \times \|v\|_{L^2(\mathbb{R}^N)}[v]_{H^s(\mathbb{R}^N)}.\end{aligned}$$

Fix $\varepsilon > 0$. In view of Remark 2, we may choose H_2 and K_2 such that

$$C_5\|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|\mathsf{K}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)} \le \frac{\varepsilon^2}{2}.$$

Applying Young's inequality, we have

$$\begin{aligned}&C_6\left(\|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\|\mathsf{K}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} + \|\mathsf{H}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\mathsf{K}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)}\right) \\ &\quad \times \|v\|_{L^2(\mathbb{R}^N)}[v]_{H^s(\mathbb{R}^N)} \le \frac{\varepsilon^2}{2}[v]^2_{H^s(\mathbb{R}^N)} + C_\varepsilon\|v\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Taking the above facts into account, the assertion follows. □

Next, we provide a useful regularity result for a class of nonlocal linear equations.

Proposition 3 *Let $N \ge 2$, $s \in (0,1)$, and $\alpha \in (0,N)$. Let $H, K \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$ and $V \in L^\infty(\mathbb{R}^N) + L^{\frac{N}{2s}}(\mathbb{R}^N)$. Let $u \in H^s(\mathbb{R}^N)$ be a weak solution to*

$$(-\Delta)^s u + \omega u = (I_\alpha * (Hu))K + Vu \text{ in } \mathbb{R}^N.$$

Then, $u \in L^p(\mathbb{R}^N)$ for all $p \in [2, \frac{N}{\alpha}2_s^*)$. Furthermore, there exists a constant $C_p > 0$ independent of u such that $\|u\|_{L^p(\mathbb{R}^N)} \le C_p\|u\|_{L^2(\mathbb{R}^N)}$.

Proof Write $\mathsf{H} := \mathsf{H}_1 + \mathsf{H}_2$ and $\mathsf{K} := \mathsf{K}_1 + \mathsf{K}_2$ with $\mathsf{H}_1, \mathsf{K}_1 \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ and $\mathsf{H}_2, \mathsf{K}_2 \in L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$, and $\mathsf{V} := \mathsf{V}_1 + \mathsf{V}_2$ with $\mathsf{V}_1 \in L^\infty(\mathbb{R}^N)$ and $\mathsf{V}_2 \in L^{\frac{N}{2s}}(\mathbb{R}^N)$. Applying Lemma 5 to $\tilde{\mathsf{V}} := |\mathsf{V}_1| + |\mathsf{V}_2|$ with $\varepsilon = \frac{1}{2}$, and Lemma 7 to $\tilde{\mathsf{H}} := |\mathsf{H}_1| + |\mathsf{H}_2|$

and $\tilde{\mathsf{K}} := |\mathsf{K}_1| + |\mathsf{K}_2|$ with $\varepsilon = \frac{1}{2}$ and $\theta = 1$, we can find $\lambda_1 := 2C_{\frac{1}{2}} > 0$ and $\lambda_2 := 2C_{\frac{1}{2},1} > 0$ such that, for every $v \in H^s(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \tilde{\mathsf{V}}|v|^2\,dx \leq \frac{1}{4}[v]^2_{H^s(\mathbb{R}^N)} + \frac{\lambda_1}{2}\|v\|^2_{L^2(\mathbb{R}^N)}, \\ &\int_{\mathbb{R}^N} (I_\alpha * \tilde{\mathsf{H}}|v|)\tilde{\mathsf{K}}|v|\,dx \leq \frac{1}{4}[v]^2_{H^s(\mathbb{R}^N)} + \frac{\lambda_2}{2}\|v\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{3.5}$$

For every function $q(x)$ and for all $n \in \mathbb{N}$, we set

$$q^n(x) := \begin{cases} n & \text{if } q(x) > n, \\ q(x) & \text{if } |q(x)| \leq n, \\ -n & \text{if } q(x) < -n. \end{cases}$$

For all $n \in \mathbb{N}$, let $\mathsf{H}_n := \mathsf{H}_1 + \mathsf{H}_2^n$, $\mathsf{K}_n := \mathsf{K}_1 + \mathsf{K}_2^n$, and $\mathsf{V}_n := \mathsf{V}_1 + \mathsf{V}_2^n$. Note that $(\mathsf{H}_n), (\mathsf{K}_n) \subset L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ and $(\mathsf{V}_n) \subset L^\infty(\mathbb{R}^N)$ satisfy

$$\begin{aligned} &|\mathsf{H}_n| \leq \tilde{\mathsf{H}},\ \ |\mathsf{K}_n| \leq \tilde{\mathsf{K}},\ \ |\mathsf{V}_n| \leq \tilde{\mathsf{V}} \ \text{ for all } n \in \mathbb{N}, \\ &\mathsf{H}_n \to \mathsf{H},\ \ \mathsf{K}_n \to \mathsf{K},\ \ \mathsf{V}_n \to \mathsf{V} \ \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.6}$$

Put $\lambda := \lambda_1 + \lambda_2$. For each $n \in \mathbb{N}$, we consider the bilinear form $a_n : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned} a_n(\varphi, \psi) := \langle \varphi, \psi\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \lambda\langle \varphi, \psi\rangle_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_n\varphi))\mathsf{K}_n\psi\,dx \\ - \int_{\mathbb{R}^N} \mathsf{V}_n\varphi\psi\,dx \quad \text{for all } \varphi, \psi \in H^s(\mathbb{R}^N). \end{aligned}$$

By Proposition 2, we know that a_n is a continuous form on $H^s(\mathbb{R}^N)$. Furthermore, a_n is coercive because (3.5) and (3.6) guarantee that, for all $\varphi \in H^s(\mathbb{R}^N)$,

$$\begin{aligned} a_n(\varphi, \varphi) &\geq [\varphi]^2_{H^s(\mathbb{R}^N)} + \lambda\|\varphi\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}[\varphi]^2_{H^s(\mathbb{R}^N)} - \frac{\lambda}{2}\|\varphi\|^2_{L^2(\mathbb{R}^N)} \\ &= \frac{1}{2}[\varphi]^2_{H^s(\mathbb{R}^N)} + \frac{\lambda}{2}\|\varphi\|^2_{L^2(\mathbb{R}^N)} \\ &\geq \frac{1}{2}\min\{1, \lambda\}\|\varphi\|^2_{H^s(\mathbb{R}^N)} =: C_0\|\varphi\|^2_{H^s(\mathbb{R}^N)}. \end{aligned} \tag{3.7}$$

Then we can apply the Lax-Milgram theorem to a_n and $\mathscr{L} \in H^{-s}(\mathbb{R}^N)$ defined by $\mathscr{L}(\varphi) := (-\omega + \lambda)\langle u, \varphi\rangle_{L^2(\mathbb{R}^N)}$, to infer that there is a unique $u_n \in H^s(\mathbb{R}^N)$ such that

$$a_n(u_n, \varphi) = \mathscr{L}(\varphi) \quad \text{for all } \varphi \in H^s(\mathbb{R}^N). \tag{3.8}$$

Hence, u_n is the unique weak solution to

$$(-\Delta)^s u_n + \lambda u_n = (I_\alpha * (\mathsf{H}_n u_n))\mathsf{K}_n + \mathsf{V}_n u_n + (-\omega + \lambda)u \text{ in } \mathbb{R}^N. \tag{3.9}$$

Choosing $\varphi = u_n$ in (3.8), and employing (3.7), Hölder's inequality, and Young's inequality, we have

$$\begin{aligned} C_0 \|u_n\|^2_{H^s(\mathbb{R}^N)} &\le a_n(u_n, u_n) = \mathscr{L}(u_n) \\ &\le (\omega + \lambda)\|u_n\|_{H^s(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)} \\ &\le \frac{C_0}{2}\|u_n\|^2_{H^s(\mathbb{R}^N)} + \frac{(\omega+\lambda)^2}{2C_0}\|u\|^2_{L^2(\mathbb{R}^N)}, \end{aligned}$$

which implies that, for $C_1 := \frac{\omega+\lambda}{C_0} > 0$, it holds

$$\sup_{n\in\mathbb{N}} \|u_n\|_{H^s(\mathbb{R}^N)} \le C_1 \|u\|_{L^2(\mathbb{R}^N)}. \tag{3.10}$$

As a result, (u_n) is bounded in $H^s(\mathbb{R}^N)$. Up to a subsequence, we may assume that there exists $\tilde{u} \in H^s(\mathbb{R}^N)$ such that

$$\begin{aligned} &u_n \rightharpoonup \tilde{u} \text{ in } H^s(\mathbb{R}^N), \\ &u_n \to \tilde{u} \text{ in } L^q_{loc}(\mathbb{R}^N) \text{ for all } q \in [1, 2^*_s), \\ &u_n \to \tilde{u} \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.11}$$

Fix $\varphi \in H^s(\mathbb{R}^N)$. Thanks to the weak convergence in (3.11), we obtain

$$\langle u_n, \varphi\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \lambda\langle u_n, \varphi\rangle_{L^2(\mathbb{R}^N)} \to \langle \tilde{u}, \varphi\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \lambda\langle \tilde{u}, \varphi\rangle_{L^2(\mathbb{R}^N)}. \tag{3.12}$$

Note that $u_n \rightharpoonup \tilde{u}$ in $L^2(\mathbb{R}^N)$ and $\mathsf{V}_1\varphi \in L^2(\mathbb{R})$ yield

$$\int_{\mathbb{R}^N} \mathsf{V}_1 u_n \varphi \, dx \to \int_{\mathbb{R}^N} \mathsf{V}_1 \tilde{u} \varphi \, dx,$$

while (3.11), $\mathsf{V}_2^n \to \mathsf{V}_2$ a.e. in $\mathbb{R}^N$, and $|\mathsf{V}_2^n| \le |\mathsf{V}_2|$ for all $n \in \mathbb{N}$ ensure that $\mathsf{V}_2^n u_n \to \mathsf{V}_2\tilde{u}$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned} \|\mathsf{V}_2^n u_n\|_{L^{\frac{2N}{N+2s}}(\mathbb{R}^N)} &\le \|\mathsf{V}_2 u_n\|_{L^{\frac{2N}{N+2s}}(\mathbb{R}^N)} \\ &\le \|\mathsf{V}_2\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \|u_n\|_{L^{2^*_s}(\mathbb{R}^N)} \le C_2 \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

and thus $\mathsf{V}_2^n u_n \rightharpoonup \mathsf{V}_2\tilde{u}$ in $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ owing to [45, Proposition 5.4.7]. Since $\varphi \in L^{2^*_s}(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} \mathsf{V}_2^n u_n \varphi \, dx \to \int_{\mathbb{R}^N} \mathsf{V}_2 \tilde{u} \varphi \, dx.$$

Consequently,

$$\int_{\mathbb{R}^N} \mathsf{V}_n u_n \varphi \, dx \to \int_{\mathbb{R}^N} \mathsf{V} \tilde{u} \varphi \, dx. \tag{3.13}$$

On the other hand, (3.6) and (3.11) show that $\mathsf{H}_n u_n \to \mathsf{H}\tilde{u}$ a.e. in $\mathbb{R}^N$, $\mathsf{K}_n \varphi \to \mathsf{K}\varphi$ a.e. in $\mathbb{R}^N$, and

$$\begin{aligned} \|\mathsf{H}_n u_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} &\le \|\tilde{\mathsf{H}} u_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &\le C_3(\|\mathsf{H}_1\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \|u_n\|_{L^2(\mathbb{R}^N)} + \|\mathsf{H}_2\|_{L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)} \|u_n\|_{L^{2^*_s}(\mathbb{R}^N)}) \\ &\le C_4 \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

and

$$|\mathsf{K}_n \varphi|^{\frac{2N}{N+\alpha}} \le |\tilde{\mathsf{K}} \varphi|^{\frac{2N}{N+\alpha}} \le C_5(|\mathsf{K}_1|^{\frac{2N}{\alpha}} + |\varphi|^2 + |\mathsf{K}_2|^{\frac{2N}{\alpha+2s}} + |\varphi|^{2^*_s}) \in L^1(\mathbb{R}^N).$$

Hence, [45, Proposition 5.4.7] implies $\mathsf{H}_n u_n \rightharpoonup \mathsf{H}\tilde{u}$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, while the dominated convergence theorem gives $\mathsf{K}_n \varphi \to \mathsf{K}\varphi$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Applying Proposition 2, we have that $I_\alpha * (\mathsf{H}_n u_n) \rightharpoonup I_\alpha * (\mathsf{H}\tilde{u})$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ and thus

$$\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_n u_n)) \mathsf{K}_n \varphi \, dx \to \int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}\tilde{u})) \mathsf{K}\varphi \, dx. \tag{3.14}$$

In light of (3.12), (3.13), and (3.14), and the arbitrariness of $\varphi \in H^s(\mathbb{R}^N)$, we conclude that $\tilde{u}$ is a weak solution to

$$(-\Delta)^s \tilde{u} + \lambda \tilde{u} = (I_\alpha * (\mathsf{H}\tilde{u}))\mathsf{K} + \mathsf{V}\tilde{u} + (-\omega + \lambda)u \text{ in } \mathbb{R}^N. \tag{3.15}$$

Since (3.15) possesses a unique weak solution by Lax-Milgram theorem, and u is a weak solution to (3.15) by assumption, we deduce that $\tilde{u} = u$.
For all $\mu > 0$, put

$$u_{n,\mu}(x) := \begin{cases} -\mu & \text{if } u_n(x) < -\mu, \\ u_n(x) & \text{if } |u_n(x)| \le \mu, \\ \mu & \text{if } u_n(x) > \mu. \end{cases}$$

Let $p \ge 2$ and use $|u_{n,\mu}|^{p-2} u_{n,\mu} \in H^s(\mathbb{R}^N)$ as test function in (3.9). Setting

$$t_\mu := \max\{-\mu, \min\{t, \mu\}\} = \text{sign}(t) \min\{|t|, \mu\} \quad \text{for all } t \in \mathbb{R},$$

we know that (see [24, Lemma 3.5])

$$\begin{aligned} &(a-b)(a_\mu |a_\mu|^{r-2} - b_\mu |b_\mu|^{r-2}) \ge \frac{4(r-1)}{r^2} (|a_\mu|^{\frac{r}{2}} - |b_\mu|^{\frac{r}{2}})^2 \\ &\quad \text{for all } a, b \in \mathbb{R} \text{ and } r \ge 2. \end{aligned} \tag{3.16}$$

Choosing $r = p$, $a = u_n(x)$, and $b = u_n(y)$ in (3.16), we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n \, (-\Delta)^{\frac{s}{2}} (|u_{n,\mu}|^{p-2} u_{n,\mu}) \, dx \\
&= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))((|u_{n,\mu}|^{p-2} u_{n,\mu})(x) - (|u_{n,\mu}|^{p-2} u_{n,\mu})(y))}{|x-y|^{N+2s}} dxdy \\
&\geq \frac{C_{N,s}}{2} \frac{4(p-1)}{p^2} \iint_{\mathbb{R}^{2N}} \frac{(|u_{n,\mu}(x)|^{\frac{p}{2}} - |u_{n,\mu}(y)|^{\frac{p}{2}})^2}{|x-y|^{N+2s}} \, dxdy \\
&= \frac{4(p-1)}{p^2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} (|u_{n,\mu}|^{\frac{p}{2}})|^2 \, dx = \frac{4(p-1)}{p^2} [|u_{n,\mu}|^{\frac{p}{2}}]^2_{H^s(\mathbb{R}^N)}.
\end{aligned}$$

Thus, since $u_n |u_{n,\mu}|^{p-2} u_{n,\mu} \geq |u_{n,\mu}|^p$, we have

$$\begin{aligned}
&\frac{4(p-1)}{p^2} [|u_{n,\mu}|^{\frac{p}{2}}]^2_{H^s(\mathbb{R}^N)} \\
&\leq \frac{4(p-1)}{p^2} [|u_{n,\mu}|^{\frac{p}{2}}]^2_{H^s(\mathbb{R}^N)} + \lambda \| |u_{n,\mu}|^{\frac{p}{2}} \|^2_{L^2(\mathbb{R}^N)} \\
&\leq \int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_n u_n)) \mathsf{K}_n |u_{n,\mu}|^{p-2} u_{n,\mu} \, dx + \int_{\mathbb{R}^N} \mathsf{V}_n u_n |u_{n,\mu}|^{p-2} u_{n,\mu} \, dx \\
&\quad + \int_{\mathbb{R}^N} (-\omega + \lambda) u |u_{n,\mu}|^{p-2} u_{n,\mu} \, dx.
\end{aligned} \tag{3.17}$$

Using (3.6) and that $|u_{n,\mu}| \leq |u_n|$, we see

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * (\mathsf{H}_n u_n)) \mathsf{K}_n |u_{n,\mu}|^{p-2} u_{n,\mu} \, dx \\
&\leq \int_{\mathbb{R}^N} (I_\alpha * (|\mathsf{H}_n| |u_n| \chi_{A_{n,\mu}})) |\mathsf{K}_n| |u_{n,\mu}|^{p-1} \, dx \\
&\quad + \int_{\mathbb{R}^N} (I_\alpha * (|\mathsf{H}_n| |u_n| \chi_{B_{n,\mu}})) |\mathsf{K}_n| |u_{n,\mu}|^{p-1} \, dx \\
&\leq \int_{\mathbb{R}^N} (I_\alpha * (|\mathsf{H}_n| |u_n| \chi_{A_{n,\mu}})) |\mathsf{K}_n| |u_n|^{p-1} \, dx \\
&\quad + \int_{\mathbb{R}^N} (I_\alpha * (\tilde{\mathsf{H}} |u_{n,\mu}|)) \tilde{\mathsf{K}} |u_{n,\mu}|^{p-1} \, dx,
\end{aligned} \tag{3.18}$$

where

$$A_{n,\mu} := \{x \in \mathbb{R}^N : |u_n(x)| > \mu\} \quad \text{and} \quad B_{n,\mu} := \{x \in \mathbb{R}^N : |u_n(x)| \leq \mu\}.$$

Let $p\in[2,\frac{2N}{\alpha})$. Applying Lemma 7 to $\tilde{\mathsf{H}}$ and $\tilde{\mathsf{K}}$ with $\theta=\frac{2}{p}$, $v=|u_{n,\mu}|^{\frac{p}{2}}$, and $\varepsilon^2=\frac{(p-1)}{p^2}$, we can find $C'_p>0$ such that

$$\int_{\mathbb{R}^N}(I_\alpha*(\tilde{\mathsf{H}}|u_{n,\mu}|))\tilde{\mathsf{K}}|u_{n,\mu}|^{p-1}\,dx \\ \leq \frac{(p-1)}{p^2}[|u_{n,\mu}|^{\frac{p}{2}}]^2_{H^s(\mathbb{R}^N)}+C'_p\||u_{n,\mu}|^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^N)}. \tag{3.19}$$

On the other hand, we have

$$\int_{\mathbb{R}^N}\mathsf{V}_n u_n|u_{n,\mu}|^{p-2}u_{n,\mu}\,dx \leq \int_{A_{n,\mu}}|\mathsf{V}_n||u_n|^p\,dx+\int_{B_{n,\mu}}|\mathsf{V}_n||u_{n,\mu}|^p\,dx \\ \leq \int_{A_{n,\mu}}|\mathsf{V}_n||u_n|^p\,dx+\int_{\mathbb{R}^N}\tilde{\mathsf{V}}|u_{n,\mu}|^p\,dx. \tag{3.20}$$

Applying Lemma 5 to $\tilde{\mathsf{V}}$ with $v=|u_{n,\mu}|^{\frac{p}{2}}$ and $\varepsilon^2=\frac{(p-1)}{p^2}$, there exists $C''_p>0$ such that

$$\int_{\mathbb{R}^N}\tilde{\mathsf{V}}|u_{n,\mu}|^p\,dx\leq\frac{(p-1)}{p^2}[|u_{n,\mu}|^{\frac{p}{2}}]^2_{H^s(\mathbb{R}^N)}+C''_p\||u_{n,\mu}|^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^N)}. \tag{3.21}$$

Using (3.17), (3.18), (3.19), (3.20), (3.21), and that $|u_{n,\mu}|\leq|u_n|$, we deduce

$$\frac{2(p-1)}{p^2}[|u_{n,\mu}|^{\frac{p}{2}}]^2_{H^s(\mathbb{R}^N)} \\ \leq (C'_p+C''_p)\int_{\mathbb{R}^N}|u_{n,\mu}|^p\,dx+|(-\omega+\lambda)|\int_{\mathbb{R}^N}|u||u_{n,\mu}|^{p-1}\,dx \\ +\int_{\mathbb{R}^N}(I_\alpha*|\mathsf{H}_n||u_n|\chi_{A_{n,\mu}})|\mathsf{K}_n||u_n|^{p-1}\,dx+\int_{A_{n,\mu}}|\mathsf{V}_n||u_n|^p\,dx. \tag{3.22}$$

Now, assume that $p\in[2,\frac{2N}{\alpha})$ and $u_n\in L^p(\mathbb{R}^N)$. Since $\mathsf{H}_n,\mathsf{K}_n\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$, it follows from the Hölder's inequality that $|\mathsf{H}_n u_n|\in L^{r_1}(\mathbb{R}^N)$ and $|\mathsf{K}_n||u_n|^{p-1}\in L^{r_2}(\mathbb{R}^N)$, where $\frac{1}{r_1}=\frac{\alpha}{2N}+\frac{1}{p}$ and $\frac{1}{r_2}=\frac{\alpha}{2N}+1-\frac{1}{p}$. Because $\frac{1}{r_1}+\frac{1}{r_2}=1+\frac{\alpha}{N}$, we derive from Proposition 2 that

$$\int_{\mathbb{R}^N}(I_\alpha*|\mathsf{H}_n||u_n|\chi_{A_{n,\mu}})|\mathsf{K}_n||u_n|^{p-1}\,dx \\ \leq C(N,\alpha,p)\left(\int_{A_{n,\mu}}|\mathsf{H}_n u_n|^{r_1}\,dx\right)^{\frac{1}{r_1}}\left(\int_{\mathbb{R}^N}||\mathsf{K}_n||u_n|^{p-1}|^{r_2}\,dx\right)^{\frac{1}{r_2}}.$$

Invoking the the dominated convergence theorem, we arrive at

$$\lim_{\mu\to+\infty}\int_{\mathbb{R}^N}(I_\alpha*|\mathsf{H}_n||u_n|\chi_{A_{n,\mu}})|\mathsf{K}_n||u_n|^{p-1}\,dx=0. \tag{3.23}$$

Utilizing the dominated convergence theorem again, and recalling that $|\mathsf{V}_n| \le n$, we also have

$$\lim_{\mu\to+\infty} \int_{A_{n,\mu}} |\mathsf{V}_n||u_n|^p \, dx = 0. \tag{3.24}$$

Combining (2.1), (3.22), (3.23), (3.24), and applying the monotone convergence theorem as $\mu \to +\infty$, we obtain

$$\begin{aligned}\|u_n\|^p_{L^{\frac{p}{2}2_s^*}(\mathbb{R}^N)} &= \||u_n|^{\frac{p}{2}}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \\ &\le C_p'''(\|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}\|u_n\|^{p-1}_{L^p(\mathbb{R}^N)}) \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Hence, $u_n \in L^{\frac{p}{2}2_s^*}(\mathbb{R}^N)$ and, for some $\tilde{C}_p > 0$, it holds

$$\|u_n\|^p_{L^{\frac{p}{2}2_s^*}(\mathbb{R}^N)} \le \tilde{C}_p \left(\|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)}\right) \quad \text{for all } n \in \mathbb{N}.$$

Note that (3.10) ensures that

$$\|u_n\|_{L^2(\mathbb{R}^N)} \le C_1\|u\|_{L^2(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N}.$$

Then, by iterating this process with respect to p a finite number of times and using Fatou's lemma, we infer that $u \in L^p(\mathbb{R}^N)$ for all $p \in [2, \frac{N}{\alpha}2_s^*)$, and that, for some $C_p > 0$ independent of u, $\|u\|_{L^p(\mathbb{R}^N)} \le C_p\|u\|_{L^2(\mathbb{R}^N)}$. The proof is now complete. □

Proposition 4 *Let $N \ge 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ holds. If $u \in H^s(\mathbb{R}^N)$ is a weak solution to (1.1), then $u \in L^\infty(\mathbb{R}^N)$.*

Proof Put $\mathsf{H}(x) := \frac{F(u(x))}{u(x)}$, $\mathsf{K}(x) := f(u(x))$, and $\mathsf{V}(x) := 0$. From $(f1)$ and $u \in L^2(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$, we know that $\mathsf{H}, \mathsf{K} \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$. Thanks to Proposition 3, we deduce that $u \in L^p(\mathbb{R}^N)$ for all $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2s})$. This fact and $(f1)$ yield $F(u) \in L^q(\mathbb{R}^N)$ for all $q \in [\frac{2N}{N+\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha})$. Now, we can write

$$I_\alpha * F(u) = (I_\alpha\chi_{B_1(0)}) * F(u) + (I_\alpha\chi_{B_1^c(0)}) * F(u).$$

It is clear that $I_\alpha\chi_{B_1(0)} \in L^r(\mathbb{R}^N)$ for all $r \in [1, \frac{N}{N-\alpha})$ and $I_\alpha\chi_{B_1^c(0)} \in L^t(\mathbb{R}^N)$ for all $t \in (\frac{N}{N-\alpha}, +\infty]$. Take $p_1 \in (\frac{2N^2}{2N^2-\alpha(N+\alpha)}, \frac{N}{N-\alpha}) \subset [1, \frac{N}{N-\alpha})$ and note that $\frac{p_1}{p_1-1} \in (\frac{N}{\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha}) \subset (\frac{2N}{N+\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha})$. From $I_\alpha\chi_{B_1(0)} \in L^{p_1}(\mathbb{R}^N)$, $F(u) \in L^{\frac{p_1}{p_1-1}}(\mathbb{R}^N)$, and Young's inequality we derive that $(I_\alpha\chi_{B_1(0)}) * F(u) \in L^\infty(\mathbb{R}^N)$. On the other hand, pick $t_1 \in (\frac{N}{N-\alpha}, \frac{2N}{N-\alpha}) \subset (\frac{N}{N-\alpha}, +\infty]$ and observe that $\frac{t_1}{t_1-1} \in (\frac{2N}{N+\alpha}, \frac{N}{\alpha}) \subset (\frac{2N}{N+\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha})$. It follows from $I_\alpha\chi_{B_1^c(0)} \in L^{t_1}(\mathbb{R}^N)$, $F(u) \in L^{\frac{t_1}{t_1-1}}(\mathbb{R}^N)$, and Young's inequality that $(I_\alpha\chi_{B_1^c(0)}) * F(u) \in L^\infty(\mathbb{R}^N)$. Consequently, $I_\alpha * F(u) \in L^\infty(\mathbb{R}^N)$.

Moreover, Young's inequality implies $I_\alpha * F(u) \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $(I_\alpha * F(u))(x) \to 0$ as $|x| \to +\infty$. By $I_\alpha * F(u) \in L^\infty(\mathbb{R}^N)$ and $(f1)$, we deduce that

$$|(I_\alpha * F(u)) f(u)| \leq C'(|u|^{\frac{\alpha}{N}} + |u|^{\frac{\alpha+2s}{N-2s}}).$$

Therefore, u is a weak solution to $(-\Delta)^s u + u = g(x,u)$ in $\mathbb{R}^N$, where $g(x,u) := (-\omega+1)u + (I_\alpha * F(u)) f(u)$ satisfies

$$|g(x,t)| \leq C''(1+|t|^r) \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } t \in \mathbb{R}, \tag{3.25}$$

for some $C'' > 0$ and $r \in [1, 2_s^*-1]$ (specifically, $r := \max\left\{1, \frac{\alpha+2s}{N-2s}\right\} \in [1, 2_s^*-1)$). Next, we prove that $u \in L^\infty(\mathbb{R}^N)$.

Claim 1 It holds

$$u \in L^q(B) \text{ for all measurable set } B \subset \mathbb{R}^N \text{ with } |B| < +\infty \text{ and for all } q \in [1,+\infty). \tag{3.26}$$

We first show that

$$|u| \leq \mathcal{K}_{2s} * |g(\cdot, u)| \quad \text{a.e. in } \mathbb{R}^N, \tag{3.27}$$

where the kernel $\mathcal{K}_{2s}$ is given by

$$\mathcal{K}_{2s}(x) := ((-\Delta)^s + 1)^{-1} = (2\pi)^{-\frac{N}{2}} \mathcal{F}^{-1}((|\xi|^{2s}+1)^{-1})(x). \tag{3.28}$$

In view of the fractional Kato inequality for $(-\Delta)^s$ (see [3, Theorem 17.3.5]), we obtain

$$\langle |u|, \phi \rangle_{H^s(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} |g(x,u)| \phi \, dx, \tag{3.29}$$

for all $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi \geq 0$ in $\mathbb{R}^N$. Pick $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\psi \geq 0$ in $\mathbb{R}^N$. Set $\varphi := \mathcal{K}_{2s} * \psi$. Observe that $\|\mathcal{K}_{2s}\|_{L^1(\mathbb{R}^N)} = (2\pi)^{\frac{N}{2}} \mathcal{F}\mathcal{K}_{2s}(0) = 1$. Furthermore, we recall the following useful properties (see [20, Theorem 3.3]):

- $(\mathcal{K}1)$ $\mathcal{K}_{2s}$ is a function of $|x|$ alone, $\mathcal{K}_{2s}$ is a decreasing function of $|x|$, $\mathcal{K}_{2s}$ is positive and smooth in $\mathbb{R}^N \setminus \{0\}$,
- $(\mathcal{K}2)$ there exists $C > 0$ such that

$$\mathcal{K}_{2s}(x) \leq C(|x|^{2s-N} \chi_{B_1(0)}(x) + |x|^{-(N+2s)} \chi_{B_1^c(0)}(x)) \text{ for all } x \in \mathbb{R}^N \setminus \{0\},$$

- $(\mathcal{K}3)$ if $v \in L^r(\mathbb{R}^N)$ with $r \in [1,+\infty]$, then $\mathcal{K}_{2s} * v \in L^r(\mathbb{R}^N)$,
- $(\mathcal{K}4)$ $\mathcal{K}_{2s} \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-2s})$,
- $(\mathcal{K}5)$ $\mathcal{K}_{2s} \in L_w^{\frac{N}{N-2s}}(\mathbb{R}^N)$, namely, $\sup_{t>0} t|\{\mathcal{K}_{2s} \geq t\}|^{\frac{N-2s}{N}} < +\infty$.

Note that $\varphi \geq 0$ in $\mathbb{R}^N$ (since $\mathcal{K}_{2s} > 0$ a.e. in $\mathbb{R}^N$ by ($\mathcal{K}1$), and $\psi \geq 0$ in $\mathbb{R}^N$), and that $\varphi \in C^\infty(\mathbb{R}^N)$, but φ does not have compact support. Select $h \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq h \leq 1$ in $\mathbb{R}^N$, $h = 1$ in $\overline{B}_1(0)$, and $h = 0$ in $B_2^c(0)$. For each $j \in \mathbb{N}$, define $h_j(x) := h(x/j)$ for all $x \in \mathbb{R}^N$, and set $\varphi_j := h_j\varphi = h_j(\mathcal{K}_{2s} * \psi)$. Clearly, $\varphi_j \in C_c^\infty(\mathbb{R}^N)$ and $\varphi_j \geq 0$ in $\mathbb{R}^N$. Taking $\phi = \varphi_j$ in (3.29), we find

$$\langle |u|, \varphi_j \rangle_{H^s(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} |g(x,u)|\varphi_j \, dx \quad \text{for all } j \in \mathbb{N}. \tag{3.30}$$

In light of Young's inequality and ($\mathcal{K}3$), we have that $\varphi = \mathcal{K}_{2s} * \psi$ and $\partial_{x_i}\varphi = \mathcal{K}_{2s} * \partial_{x_i}\psi$ belong to $L^r(\mathbb{R}^N)$ for all $r \in [1, +\infty]$ and $i = 1, \dots, N$. Thus, $\varphi_j \to \varphi$ in $H^1(\mathbb{R}^N)$, and hence in $H^s(\mathbb{R}^N)$, and $\varphi_j \to \varphi$ in $L^r(\mathbb{R}^N)$ for all $r \in [1, +\infty]$. Consequently,

$$\langle |u|, \varphi_j \rangle_{H^s(\mathbb{R}^N)} \to \langle |u|, \varphi \rangle_{H^s(\mathbb{R}^N)} = \langle |u|, \psi \rangle_{L^2(\mathbb{R}^N)}.$$

On the other hand, utilizing (3.25), and since $\varphi \in L^r(\mathbb{R}^N)$ for all $r \in [1, +\infty]$, we know that $|g(\cdot,u)|\varphi \in L^1(\mathbb{R}^N)$. Taking into account that $|g(\cdot,u)|\varphi_j \leq |g(\cdot,u)|\varphi \in L^1(\mathbb{R}^N)$ and that $\varphi_j \to \varphi$ a.e. in $\mathbb{R}^N$, we can apply the dominated convergence theorem to infer

$$\int_{\mathbb{R}^N} |g(x,u)|\varphi_j \, dx \to \int_{\mathbb{R}^N} |g(x,u)|\varphi \, dx.$$

The last two limit relations, together with (3.30), yield

$$\langle |u|, \psi \rangle_{L^2(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} |g(x,u)|\varphi \, dx.$$

Now, if we prove that

$$\mathcal{K}_{2s} * |g(\cdot,u)| \in L^1_{loc}(\mathbb{R}^N), \tag{3.31}$$

then we can use Fubini's theorem to confirm

$$\int_{\mathbb{R}^N} |g(x,u)|\varphi \, dx = \int_{\mathbb{R}^N} (\mathcal{K}_{2s} * |g(x,u)|)\psi \, dx,$$

and thus

$$\langle |u|, \psi \rangle_{L^2(\mathbb{R}^N)} \leq \langle \mathcal{K}_{2s} * |g(\cdot,u)|, \psi \rangle_{L^2(\mathbb{R}^N)}.$$

Due to the arbitrariness of $\psi \in C_c^\infty(\mathbb{R}^N)$, with $\psi \geq 0$, we deduce that (3.27) is valid. In what follows, we focus on (3.31). Since $\mathcal{K}_{2s} > 0$ a.e. in $\mathbb{R}^N$ (by ($\mathcal{K}1$)), and utilizing

$\|\mathcal{K}_{2s}\|_{L^1(\mathbb{R}^N)} = 1$, (3.25), and that $|u|^{2_s^*-1} = |u|^{2_s^*-1}\chi_{\{|u|\leq 1\}} + |u|^{2_s^*-1}\chi_{\{|u|>1\}}$ a.e. in $\mathbb{R}^N$, we see

$$\begin{aligned} 0 \leq \mathcal{K}_{2s} * |g(\cdot, u)| &\leq \mathcal{K}_{2s} * [C_0(1 + |u|^{2_s^*-1})] \\ &\leq \mathcal{K}_{2s} * (2C_0) + \mathcal{K}_{2s} * (C_0|u|^{2_s^*-1}\chi_{\{|u|>1\}}) \\ &= 2C_0 + \mathcal{K}_{2s} * (A(u)|u|) \quad \text{a.e. in } \mathbb{R}^N, \end{aligned} \tag{3.32}$$

where $C_0 := 2C'' > 0$ and $A(u) := C_0|u|^{2_s^*-2}\chi_{\{|u|>1\}}$. Because

$$\int_{\mathbb{R}^N} A(u)|u|\, dx = C_0 \int_{\mathbb{R}^N} |u|^{2_s^*-1}\chi_{\{|u|>1\}}\, dx \leq C_0 \int_{\mathbb{R}^N} |u|^{2_s^*}\, dx < +\infty,$$

we can exploit ($\mathcal{K}$3) with $r = 1$ to obtain $\mathcal{K}_{2s} * (A(u)|u|) \in L^1(\mathbb{R}^N)$. This and (3.32) imply that (3.31) holds. In view of (3.27) and (3.32), we arrive at

$$|u| \leq 2C_0 + \mathcal{K}_{2s} * (A(u)|u|) \quad \text{a.e. in } \mathbb{R}^N.$$

Given that $\mathcal{K}_{2s} \in L_w^{\frac{N}{N-2s}}(\mathbb{R}^N)$ (by ($\mathcal{K}$5)), $A(u) \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, and $|\text{supp}(A(u))| < +\infty$, we can apply [11, Lemma A.1] with $\alpha = \frac{N}{N-2s}$ and $\beta = \frac{2N}{N-2s}$ to conclude that (3.26) is true.

Claim 2 $u \in L^\infty(\mathbb{R}^N)$.

Employing (3.27), the fact that $\mathcal{K}_{2s} > 0$ a.e. in $\mathbb{R}^N$, (3.25), and noting that $(1 + |u|^{2_s^*-1}) \leq 2(1 + |u|^{2_s^*}\chi_{\{|u|>1\}})$ a.e. in $\mathbb{R}^N$, we have

$$|u| \leq \mathcal{K}_{2s} * [C_0(1 + |u|^{2_s^*-1})] \leq \mathcal{K}_{2s} * (2C_0) + (\mathcal{K}_{2s} * \tilde{h}(u)) \quad \text{a.e. in } \mathbb{R}^N, \tag{3.33}$$

where $\tilde{h}(u)(x) := 2C_0|u(x)|^{2_s^*}\chi_{\{|u|>1\}}(x)$. Considering that $|\{|u| > 1\}| \leq \|u\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} < +\infty$, we derive from (3.26) that $|u|\chi_{\{|u|>1\}} \in L^q(\mathbb{R}^N)$ for all $q \in [1, +\infty)$, and hence $\tilde{h}(u) \in L^{q_0}(\mathbb{R}^N)$ for some $q_0 \in (\frac{N}{2s}, +\infty)$. As $\mathcal{K}_{2s} \in L^{q_0'}(\mathbb{R}^N)$ (by ($\mathcal{K}$4)), where $q_0' \in (1, \frac{N}{N-2s})$ is the conjugate exponent of q_0, it follows from Young's inequality that $\mathcal{K}_{2s} * \tilde{h}(u) \in L^\infty(\mathbb{R}^N)$. This, together with (3.33) and the fact that $\|\mathcal{K}_{2s}\|_{L^1(\mathbb{R}^N)} = 1$, shows that

$$|u| \leq 2C_0 + \mathcal{K}_{2s} * \tilde{h}(u) \quad \text{a.e. in } \mathbb{R}^N,$$

and thus $u \in L^\infty(\mathbb{R}^N)$. This finishes the proof of Proposition 4. □

Next, we recall the Schauder-Zygmund estimates in Hölder-Zygmund spaces Λ_α, with $\alpha > 0$, for the operator $(-\Delta)^s$ (see [42, 43]).

Proposition 5 *[42, Propositions 2.8 and 2.9] [43, Theorem 15] Let $N \geq 2$ and $s \in (0, 1)$. Let $u \in L^\infty(\mathbb{R}^N)$.*

(a) *Assume that* $(-\Delta)^s u = f \in C^{0,\alpha}(\mathbb{R}^N)$ *for some* $\alpha \in (0, 1]$. *Then,* $u \in \Lambda_{\alpha+2s}$ *and the following Schauder-Zygmund estimate holds:*

$$\|u\|_{\Lambda_{\alpha+2s}} \leq C_1(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^N)}),$$

where the constant $C_1 > 0$ *depends only on* N, s, *and* α. *In particular:*

- *If* $\alpha + 2s < 1$, *then* $u \in C^{0,\alpha+2s}(\mathbb{R}^N)$.
- *If* $1 < \alpha + 2s < 2$, *then* $u \in C^{1,\alpha+2s-1}(\mathbb{R}^N)$.
- *If* $2 < \alpha + 2s < 3$, *then* $u \in C^{2,\alpha+2s-2}(\mathbb{R}^N)$.
- *If* $\alpha + 2s = k \in \{1, 2\}$, *then* $u \in \Lambda_k$.

(b) *Assume that* $(-\Delta)^s u = f \in L^\infty(\mathbb{R}^N)$. *Then,* $u \in \Lambda_{2s}$ *and the following Schauder-Hölder-Zygmund estimate holds:*

$$\|u\|_{\Lambda_{2s}} \leq C_2(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(\mathbb{R}^N)}),$$

where the constant $C_2 > 0$ *depends only on* N *and* s. *In particular:*

- *If* $2s < 1$, *then* $u \in C^{0,2s}(\mathbb{R}^N)$.
- *If* $2s = 1$, *then* $u \in \Lambda_1$.
- *If* $2s > 1$, *then* $u \in C^{1,2s-1}(\mathbb{R}^N)$.

We are now ready to establish the first result of this paper.

Proof of Theorem 1 We divide the proof into several steps.

Step 1: $I_\alpha * F(u) \in C^{0,\alpha-\frac{N}{q}}(\mathbb{R}^N)$ for all $q \in (\frac{N}{\alpha}, +\infty)$ such that $\alpha - \frac{N}{q} \in (0, 1)$.

From Proposition 4, we derive that $u \in L^\infty(\mathbb{R}^N)$. Consequently, $F(u) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and by interpolation, $F(u) \in L^r(\mathbb{R}^N)$ for all $r \in \left[\frac{2N}{N+\alpha}, +\infty\right]$. Now, fix $q \in (\frac{N}{\alpha}, +\infty)$ such that $\alpha - \frac{N}{q} \in (0, 1)$. Such a q exists: if $\alpha \in (0, 1]$, we may take any $q \in \left(\frac{N}{\alpha}, +\infty\right)$; whereas if $\alpha \in (1, N)$, any $q \in \left(\frac{N}{\alpha}, \frac{N}{\alpha-1}\right)$ is admissible. Since $\frac{N}{\alpha} > \frac{2N}{N+\alpha}$, it follows that $F(u) \in L^q(\mathbb{R}^N)$. Then, recalling that $I_\alpha * F(u)$ is finite almost everywhere on $\mathbb{R}^N$ (as $I_\alpha * F(u) \in C_0^0(\mathbb{R}^N)$, by Proposition 4), we conclude from [18, Theorem 2] that $I_\alpha * F(u) \in C^{0,\alpha-\frac{N}{q}}(\mathbb{R}^N)$.

Step 2: $u \in C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon \in (0, 1 + [2s] - 2s)$.

We know that $u \in L^\infty(\mathbb{R}^N)$ and $I_\alpha * F(u) \in L^\infty(\mathbb{R}^N)$. Put

$$\mathcal{G}(x) := -\omega u(x) + (I_\alpha * F(u))(x) f(u(x)).$$

Thus, $\mathcal{G} \in L^\infty(\mathbb{R}^N)$, and u is a bounded weak solution to $(-\Delta)^s u = \mathcal{G}$ in $\mathbb{R}^N$. In view of Proposition 5-(*b*), we have that if $s \in (0, \frac{1}{2})$, then $u \in C^{0,2s}(\mathbb{R}^N)$; if $s = \frac{1}{2}$, then $u \in \Lambda_1$; and if $s \in (\frac{1}{2}, 1)$, then $u \in C^{1,2s-1}(\mathbb{R}^N)$. Hence, if $s \in \left(0, \frac{1}{2}\right]$, then $u \in C^{0,\beta}(\mathbb{R}^N)$ for all $\beta \in (0, 2s)$, while if $s \in \left(\frac{1}{2}, 1\right)$, then $u \in C^{1,\beta}(\mathbb{R}^N)$ for all $\beta \in (0, 2s - 1)$. Take $\gamma \in (0, 1)$ such that $u \in C^{0,\gamma}(\mathbb{R}^N)$. Accordingly, $f(u) \in C^{0,\sigma\gamma}(\mathbb{R}^N)$, which, together with Step 1, ensures that $\mathcal{G} \in C^{0,\gamma^*}(\mathbb{R}^N)$, where

$\gamma^* := \min\left\{\alpha - \frac{N}{q}, \sigma\gamma\right\}$. By Proposition 5-$(a)$, we deduce that if $\gamma^* + 2s \in (0, 1)$, then $u \in C^{0,\gamma^*+2s}(\mathbb{R}^N)$; if $\gamma^* + 2s = 1$, then $u \in \Lambda_1$; and if $\gamma^* + 2s \in (1, 2)$, then $u \in C^{1,\gamma^*+2s-1}(\mathbb{R}^N)$. In particular, if $\gamma^* + 2s \in (1, 2)$, then $u \in C^{1,\gamma^*+2s-1}(\mathbb{R}^N)$, while if $\gamma^* + 2s \in (0, 1]$, then $u \in C^{0,\delta+2s}(\mathbb{R}^N)$ for all $\delta \in (0, \gamma^*)$. Therefore, $u \in C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon \in (0, 1 + [2s] - 2s)$.

Step 3: $u \in C^{1,\vartheta}(\mathbb{R}^N)$ for some $\vartheta \in (0, 1)$.

If $s \in \left(\frac{1}{2}, 1\right)$, then we have already observed that $u \in C^{1,\eta}(\mathbb{R}^N)$ for some $\eta \in (0, 1)$. Let us focus on the case $s = \frac{1}{2}$. Take $\nu \in (0, 1)$. Given that $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ and $u \in C^{0,\nu}(\mathbb{R}^N)$, we see $f(u) \in C^{0,\sigma\nu}(\mathbb{R}^N)$. This fact, together with Step 1, shows that $(I_\alpha * F(u))f(u) \in C^{0,\lambda^*}(\mathbb{R}^N)$, where $\lambda^* := \min\left\{\alpha - \frac{N}{q}, \sigma\nu\right\}$. As a result, $\mathcal{G} \in C^{0,\lambda^*}(\mathbb{R}^N)$. Considering that $\lambda^* + 2s = \lambda^* + 1 \in (1, 2)$, we can apply Proposition 5-(a) to infer that $u \in C^{1,\lambda^*+2s-1}(\mathbb{R}^N)$.

Now, we suppose that $s \in \left(\frac{1}{4}, \frac{1}{2}\right)$, $\alpha \in (1 - 2s, N)$, and $\sigma \in \left(\frac{1-2s}{2s}, 1\right]$. Choose $q \in \left(\frac{N}{\alpha+2s-1}, +\infty\right)$ when $\alpha \in (0, 1]$, and $q \in \left(\frac{N}{\alpha+2s-1}, \frac{N}{\alpha-1}\right)$ when $\alpha \in (1, N)$. By Step 1, we know that $I_\alpha * F(u) \in C^{0,\alpha-\frac{N}{q}}(\mathbb{R}^N)$. Because $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ and $u \in C^{0,2s}(\mathbb{R}^N)$, we find $f(u) \in C^{0,2s\sigma}(\mathbb{R}^N)$. Therefore, $\mathcal{G} \in C^{0,\delta^*}(\mathbb{R}^N)$, where $\delta^* := \min\left\{\alpha - \frac{N}{q}, 2s\sigma\right\}$. Since $q > \frac{N}{\alpha+2s-1}$, $s \in \left(\frac{1}{4}, \frac{1}{2}\right)$, and $\sigma \in \left(\frac{1-2s}{2s}, 1\right]$, it follows that $\alpha - \frac{N}{q} + 2s \in (1, 2)$ and $2s\sigma + 2s \in (1, 2)$, and so $\delta^* + 2s \in (1, 2)$. In light of Proposition 5-(a), we conclude that $u \in C^{1,\delta^*+2s-1}(\mathbb{R}^N)$.

Finally, we assume that $s \in \left(0, \frac{1}{2}\right)$, $\alpha \in (0, 1)$, and $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ for some $\sigma \in (1 - 2s, 1]$. Take $q \in \left(\frac{N}{\alpha}, +\infty\right)$. According to Step 1, $I_\alpha * F(u) \in C^{0,\alpha-\frac{N}{q}}(\mathbb{R}^N)$. As $u \in C^{0,2s}(\mathbb{R}^N)$ and $f(u) \in C^{0,2s\sigma}(\mathbb{R}^N)$, we obtain $\mathcal{G} \in C^{0,\varpi}(\mathbb{R}^N)$, where $\varpi := \min\left\{\alpha - \frac{N}{q}, 2s\sigma\right\} \in (0, 1)$. Put $\theta_0 := \varpi + 2s \in (0, 2)$. From Proposition 5-$(a)$, we deduce that if $\theta_0 \in (1, 2)$, then $u \in C^{1,\theta_0-1}(\mathbb{R}^N)$, while if $\theta_0 \in (0, 1]$, then $u \in C^{0,\gamma}(\mathbb{R}^N)$ for all $\gamma \in (0, \theta_0)$. Hence, if $\theta_0 \in (1, 2)$, then we are done. Assume now $\theta_0 \in (0, 1]$. In this case, $u \in C^{0,\gamma}(\mathbb{R}^N)$, $F(u) \in C^{0,\gamma}(\mathbb{R}^N)$, and $f(u) \in C^{0,\sigma\gamma}(\mathbb{R}^N)$ for all $\gamma \in (0, \theta_0)$. Since $\alpha \in (0, 1)$, it follows from [18, Theorem 1] that $I_\alpha * F(u) \in C^{0,\delta}(\mathbb{R}^N)$ for all $\delta \in (0, \min\{\alpha + \theta_0, 1\})$. Pick $\varepsilon \in \left(\frac{1-2s}{\sigma}, 1\right)$. Set $\mu_0 := \varepsilon\sigma\theta_0$ and note that $\mu_0 < \min\{\alpha + \theta_0, \sigma\theta_0\}$ as $\varepsilon \in (0, 1)$ and $\sigma \in (0, 1]$. Consequently, $\mathcal{G} \in C^{0,\mu_0}(\mathbb{R}^N)$. Put $\theta_1 := \mu_0 + 2s$. Applying Proposition 5-(a) again, we have that if $\theta_1 \in (1, 2)$, then $u \in C^{1,\theta_1-1}(\mathbb{R}^N)$, while if $\theta_1 \in (0, 1]$, then $u \in C^{0,\eta}(\mathbb{R}^N)$ for all $\eta \in (0, \theta_1)$. If $\theta_1 \in (1, 2)$, then the desired regularity is achieved. Otherwise, when $\theta_1 \in (0, 1]$, we argue as before to arrive at $\mathcal{G} \in C^{0,\mu_1}(\mathbb{R}^N)$, where $\mu_1 := \varepsilon\sigma\theta_1$. Setting $\theta_2 := \mu_1 + 2s$, we deduce that if $\theta_2 \in (1, 2)$, then $u \in C^{1,\theta_2-1}(\mathbb{R}^N)$, while if $\theta_2 \in (0, 1]$, then $u \in C^{0,\lambda}(\mathbb{R}^N)$ for all $\lambda \in (0, \theta_2)$. Iterating this procedure, we can define $\mu_n := \varepsilon\sigma\theta_n$ and $\theta_n := \mu_{n-1} + 2s$ for all $n \in \mathbb{N}$. Thus, $\theta_n = \varepsilon\sigma\theta_{n-1} + 2s$ for all $n \in \mathbb{N}$, and so $\theta_n = (\varepsilon\sigma)^n\mu_0 + 2s\sum_{k=0}^{n-1}(\varepsilon\sigma)^k$ for all $n \in \mathbb{N}$. Because $\varepsilon\sigma \in (0, 1)$ and $\frac{2s}{1-\varepsilon\sigma} > 1$, we conclude that $\theta_n \to \frac{2s}{1-\varepsilon\sigma} > 1$. Then we can find $n_0 \in \mathbb{N}$ such that $\theta_{n_0} > 1$. Therefore, $u \in C^{1,\gamma}(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$.

Step 4: u is a classical solution to (1.1).

Note that (f_1) and Proposition 2 ensure that $I_\alpha * F(u) \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, while (f_1) and $u \in L^2(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$ for all $t \in [2_s^*, +\infty]$ imply that $f(u) \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$. Thus, by Hölder's inequality, $(I_\alpha * F(u))f(u) \in L^2(\mathbb{R}^N)$, and so $\mathcal{G} \in L^2(\mathbb{R}^N)$. Given that $u \in H^s(\mathbb{R}^N)$ is a weak solution to $(-\Delta)^s u = \mathcal{G}$ in $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} [|\xi|^{2s}\mathcal{F}u(\xi) - \mathcal{F}\mathcal{G}(\xi)]\overline{\mathcal{F}\phi}(\xi)\,d\xi = 0 \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^N),$$

which guarantees that $|\xi|^{2s}\mathcal{F}u(\xi) = \mathcal{F}\mathcal{G}(\xi)$ for a.e. $\xi \in \mathbb{R}^N$. As a result, $u \in H^{2s}(\mathbb{R}^N)$. This fact allows us to integrate by parts, yielding

$$\int_{\mathbb{R}^N} (-\Delta)^s u\,\phi\,dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u\,(-\Delta)^{\frac{s}{2}}\phi\,dx = \int_{\mathbb{R}^N} \mathcal{G}\phi\,dx \ \text{ for all } \phi \in H^s(\mathbb{R}^N),$$

and hence $(-\Delta)^s u(x) = \mathcal{G}(x)$ for a.e. $x \in \mathbb{R}^N$. In view of Step 2 and [42, Proposition 2.4], we know that $(-\Delta)^s u \in C^0(\mathbb{R}^N)$, which, combined with $\mathcal{G} \in C^0(\mathbb{R}^N)$, confirms that $(-\Delta)^s u(x) = \mathcal{G}(x)$ for all $x \in \mathbb{R}^N$.

Step 5: The estimate in (1.5) is valid when (1.4) holds in $(f1)$.

By Step 4, u is a classical solution to

$$(-\Delta)^s u + \omega u = (I_\alpha * F(u))f(u) \text{ in } \mathbb{R}^N.$$

Employing the pointwise fractional Kato inequality for $(-\Delta)^s$, we infer that

$$(-\Delta)^s |u| + \omega|u| \le |I_\alpha * F(u)||f(u)| \text{ in } \mathbb{R}^N.$$

Fix $\varepsilon \in (0, \omega)$. Recalling that $I_\alpha * F(u) \in C_0^0(\mathbb{R}^N)$ and $|u(x)| \to 0$ as $|x| \to +\infty$, and using (1.4), we can find $R > 2$ such that

$$|(I_\alpha * F(u))(x)||f(u(x))| \le \varepsilon\,|u(x)| \quad \text{for all } |x| \ge R.$$

Consequently,

$$(-\Delta)^s |u| + (\omega - \varepsilon)|u| \le 0 \text{ in } B_R^c(0).$$

Pick $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \phi \le 1$, $\phi = 1$ in $\overline{B}_{\frac{1}{2}}(0)$ and $\phi = 0$ in $B_1^c(0)$. Let $w \in H^s(\mathbb{R}^N)$ the unique weak solution to

$$(-\Delta)^s w + (\omega - \varepsilon)w = \phi \text{ in } \mathbb{R}^N. \tag{3.34}$$

Since $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi \ge 0$, and $\phi \not\equiv 0$, it is easy to verify that w is a positive classical solution of (3.34). In particular,

$$(-\Delta)^s w + (\omega - \varepsilon)w = 0 \text{ in } B_{R-1}^c(0).$$

Moreover, as $\omega-\varepsilon>0$, we can apply [20, Lemma 4.3] to see that

$$0<w(x)\leq \frac{C}{1+|x|^{N+2s}} \quad \text{for all } x\in\mathbb{R}^N,$$

for some $C>0$. Consider

$$z:=\left(\frac{\max_{\overline{B}_R(0)}|u|}{\min_{\overline{B}_R(0)} w}\right) w-|u|\in H^s(\mathbb{R}^N).$$

Note that $z\geq 0$ in $\overline{B}_R(0)$ and that

$$(-\Delta)^s z+(\omega-\varepsilon)z\geq 0 \text{ in } B_R^c(0).$$

Multiplying the above equation by $z_-:=\max\{0,-z\}\in H^s(\mathbb{R}^N)$ and integrating over $B_R^c(0)$, we deduce

$$\int_{\mathbb{R}^N}(-\Delta)^s z\, z_-\,dx\geq(\omega-\varepsilon)\int_{\mathbb{R}^N}(z_-)^2\,dx,$$

where we have used the fact that $z_-=0$ in $B_R(0)$. Since $(x-y)(x_--y_-)\leq 0$ for all $x,y\in\mathbb{R}$, we obtain

$$\int_{\mathbb{R}^N}(-\Delta)^s z\, z_-\,dx=\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(z(x)-z(y))(z_-(x)-z_-(y))}{|x-y|^{N+2s}}\,dxdy\leq 0,$$

which leads to

$$(\omega-\varepsilon)\int_{\mathbb{R}^N}(z_-)^2\,dx\leq 0.$$

Because $\varepsilon\in(0,\omega)$, we find $z_-=0$ in $\mathbb{R}^N$, that is, $z\geq 0$ in $\mathbb{R}^N$. As a result,

$$\begin{aligned}|u(x)|&\leq\left(\frac{\max_{\overline{B}_R(0)}|u|}{\min_{\overline{B}_R(0)} w}\right) w(x)\\&\leq\left(\frac{\max_{\overline{B}_R(0)}|u|}{\min_{\overline{B}_R(0)} w}\right)\frac{C}{1+|x|^{N+2s}} \quad \text{for all } x\in\mathbb{R}^N.\end{aligned}$$

The proof is now complete. □

Next, we prove that every weak solution to (1.1) satisfies a Pohožaev-type identity. To this end, we recall a useful integration by parts formula established in [5].

Lemma 8 *[5, Lemma 2.2] Let $u\in H^s(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. If $s\in(0,\frac{1}{2})$ then we assume that $u\in C^{0,2s+\varepsilon}(\mathbb{R}^N)\cap C^{0,1}_{loc}(\mathbb{R}^N)$ for some $\varepsilon\in(0,1-2s)$, while if $s\in[\frac{1}{2},1)$*

then we assume that $u \in C^{1,2s+\varepsilon-1}(\mathbb{R}^N)$ *for some* $\varepsilon \in (0, 2-2s)$*. Then, for all* $\mathcal{X} \in C_c^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$*, it holds that*

$$\frac{C_{N,s}}{4} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{K}_{\mathcal{X}}(x,y)dxdy = -\int_{\mathbb{R}^N} \mathcal{X}(x)\cdot \nabla u(x)\,(-\Delta)^s u(x)dx, \tag{3.35}$$

where

$$\mathrm{K}_{\mathcal{X}}(x,y) := \operatorname{div}(\mathcal{X}(x)) + \operatorname{div}(\mathcal{X}(y)) - (N+2s)\frac{(\mathcal{X}(x)-\mathcal{X}(y))\cdot(x-y)}{|x-y|^2}$$

for all $x, y \in \mathbb{R}^N$ *with* $x \neq y$*.*

Proof of Theorem 2 Take $\varphi \in C_c^1(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ in $\mathbb{R}^N$, $\varphi(x) = 1$ for $|x| \le 1$ and $\varphi(x) = 0$ for $|x| \ge 2$. Set $\varphi_\lambda(x) := \varphi(\lambda x)$ for all $x \in \mathbb{R}^N$ and $\lambda > 0$. Observe that, for all $x \in \mathbb{R}^N$ and $\lambda > 0$,

$$0 \le \varphi_\lambda(x) \le 1 \quad \text{and} \quad |x||\nabla \varphi_\lambda(x)| \le C_1, \tag{3.36}$$

for some constant $C_1 > 0$ independent of λ. By Theorem 1, we have that $u \in C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$ and that $\nabla u \in L^\infty_{loc}(\mathbb{R}^N)$. This means that all the assumptions of Lemma 8 are valid. Set $\mathcal{X}_\lambda(x) := \varphi_\lambda(x)x \in C_c^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$. Applying (3.35) with $\mathcal{X} = \mathcal{X}_\lambda$, and using the fact that u is a classical solution to (1.1) (by Lemma 8), we obtain

$$\begin{aligned} &\frac{C_{N,s}}{4} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{K}_{\mathcal{X}_\lambda}(x,y)dxdy \\ &= -\int_{\mathbb{R}^N} \mathcal{X}_\lambda(x)\cdot \nabla u(x)\,(-\Delta)^s u(x)\,dx \\ &= -\int_{\mathbb{R}^N} \mathcal{X}_\lambda(x)\cdot \nabla u(x)\,[-\omega u(x) + (I_\alpha * F(u))(x) f(u(x))]\,dx. \end{aligned} \tag{3.37}$$

In light of (3.36), it follows that

$$\left|\mathrm{K}_{\mathcal{X}_\lambda}(x,y)\right| \le C_2 \quad \text{for all } x, y \in \mathbb{R}^N \text{ with } x \neq y \text{ and } \lambda > 0,$$

for some constant $C_2 > 0$ independent of λ. Employing this estimate, the fact that $u \in H^s(\mathbb{R}^N)$, (3.36), and the pointwise convergences $\mathcal{X}_\lambda(x) \to x$ and $\operatorname{div} \mathcal{X}_\lambda(x) \to N$ for all $x \in \mathbb{R}^N$ as $\lambda \to 0^+$, we can apply the dominated convergence theorem to deduce that, as $\lambda \to 0^+$,

$$\begin{aligned} &\frac{C_{N,s}}{4} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{K}_{\mathcal{X}_\lambda}(x,y)dxdy \\ &\to \frac{C_{N,s}}{4}(N-2s) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy. \end{aligned} \tag{3.38}$$

On the other hand, utilizing integration by parts, the fact that div $\mathcal{X}_\lambda(x) \to N$ for all $x \in \mathbb{R}^N$, as $\lambda \to 0^+$, and the dominated convergence theorem, we find that, as $\lambda \to 0^+$,

$$\begin{aligned}\omega \int_{\mathbb{R}^N} \mathcal{X}_\lambda(x) \cdot \nabla u(x) u(x)\, dx &= \omega \int_{\mathbb{R}^N} \mathcal{X}_\lambda(x) \cdot \nabla \left(\frac{u^2(x)}{2}\right) dx \\ &= -\omega \int_{\mathbb{R}^N} \operatorname{div} \mathcal{X}_\lambda(x) \left(\frac{u^2(x)}{2}\right) dx \\ &\to -\frac{N}{2}\omega \int_{\mathbb{R}^N} u^2(x)\, dx.\end{aligned} \tag{3.39}$$

Finally, proceeding as in [35, p. 6571], we see that integration by parts and the dominated convergence theorem yield

$$\begin{aligned}&\int_{\mathbb{R}^N} \mathcal{X}_\lambda(x) \cdot \nabla u(x) (I_\alpha * F(u))(x) f(u(x))\, dx \\ &\to -\left(\frac{N+\alpha}{2}\right) \int_{\mathbb{R}^N} (I_\alpha * F(u))(x) F(u(x))\, dx \quad \text{as } \lambda \to 0^+.\end{aligned} \tag{3.40}$$

Combining (3.37), (3.38), (3.39), and (3.40), we arrive at

$$\begin{aligned}&\frac{C_{N,s}}{4}(N-2s) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \\ &= -\frac{N}{2}\omega \int_{\mathbb{R}^N} u^2(x)\, dx + \left(\frac{N+\alpha}{2}\right) \int_{\mathbb{R}^N} (I_\alpha * F(u))(x) F(u(x))\, dx.\end{aligned}$$

The proof of Theorem 2 is now complete. □

4 Existence of least energy solutions

In this section, we deal with the existence of least energy solutions to (1.1). Let us recall that the energy functional $\mathcal{J} : H^s(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1) is given by

$$\mathcal{J}(u) := \frac{1}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\mathscr{F}(u),$$

where

$$\mathscr{F}(u) = \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u)\, dx.$$

We know that $\mathcal{J} \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and

$$\langle \mathcal{J}'(u), \varphi \rangle = \langle u, \varphi \rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \omega \langle u, \varphi \rangle_{L^2(\mathbb{R}^N)} - \langle \mathscr{F}'(u), \varphi \rangle \text{ for all } u, \varphi \in H^s(\mathbb{R}^N).$$

We now show that $\mathcal{J}$ possesses a mountain-pass geometry.

Lemma 9 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ and $(f3)$ hold. Then, $\mathcal{J}$ satisfies the following properties:*

(MP1) $\mathcal{J}(0) = 0$,
(MP2) *there exist $\rho, \delta > 0$ such that $\mathcal{J}(u) \geq \delta$ for all $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_{H^s(\mathbb{R}^N)} = \rho$,*
(MP3) *there exists $w \in H^s(\mathbb{R}^N)$ such that $\|w\|_{H^s(\mathbb{R}^N)} > \rho$ and $\mathcal{J}(w) < 0$.*

Proof Clearly, $(MP1)$ is true. Let us verify $(MP2)$. Utilizing (2.5) and the fractional Sobolev embeddings (see Proposition 1), we have that, for all $u \in H^s(\mathbb{R}^N)$,

$$\mathcal{F}(u) \leq C(N,\alpha)\|F(u)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C_1\left(\|u\|^{\frac{2(N+\alpha)}{N}}_{H^s(\mathbb{R}^N)} + \|u\|^{\frac{2(N+\alpha)}{N-2s}}_{H^s(\mathbb{R}^N)}\right). \tag{4.1}$$

From (4.1), we deduce that, for every $u \in H^s(\mathbb{R}^N)$,

$$\begin{aligned}
\mathcal{J}(u) &\geq \frac{1}{2}\min\{1,\omega\}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{2}\mathcal{F}(u) \\
&\geq C_2\|u\|^2_{H^s(\mathbb{R}^N)} - C_3\|u\|^{\frac{2(N+\alpha)}{N}}_{H^s(\mathbb{R}^N)} - C_3\|u\|^{\frac{2(N+\alpha)}{N-2s}}_{H^s(\mathbb{R}^N)} \\
&= \|u\|^2_{H^s(\mathbb{R}^N)}\left(\frac{C_2}{2} - C_3\|u\|^{\frac{2\alpha}{N}}_{H^s(\mathbb{R}^N)} + \frac{C_2}{2} - C_3\|u\|^{\frac{2(\alpha+2s)}{N-2s}}_{H^s(\mathbb{R}^N)}\right).
\end{aligned}$$

Let $\rho > 0$ be such that

$$0 < \rho < \min\left\{\left(\frac{C_2}{4C_3}\right)^{\frac{N}{2\alpha}}, \left(\frac{C_2}{4C_3}\right)^{\frac{N-2s}{2(\alpha+2s)}}\right\}.$$

Therefore, $\mathcal{J}(u) \geq \frac{C_2}{2}\|u\|^2_{H^s(\mathbb{R}^N)}$ for all $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_{H^s(\mathbb{R}^N)} \leq \rho$. It follows that there exist $\rho, \delta > 0$ such that $\mathcal{J}(u) \geq \delta$ for all $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_{H^s(\mathbb{R}^N)} = \rho$.

Finally, we confirm $(MP3)$. Set $v_0(x) := t_0\chi_{B_1(0)}(x)$, where t_0 is given in $(f3)$. Observe that

$$\mathcal{F}(v_0) = (F(t_0))^2\int_{B_1(0)}\int_{B_1(0)} I_\alpha(x-y)\,dxdy > 0.$$

Since the map $v \in L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N) \mapsto \mathcal{F}(v)$ is continuous, and $H^s(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$, we can find $w \in H^s(\mathbb{R}^N)$ such that $\mathcal{F}(w) > 0$. Let $w_\tau(x) := w(\frac{x}{\tau})$ for a.e. $x \in \mathbb{R}^N$ and for all $\tau > 0$. Hence,

$$\mathcal{J}(w_\tau) = \frac{\tau^{N-2s}}{2}[w]^2_{H^s(\mathbb{R}^N)} + \frac{\tau^N\omega}{2}\|w\|^2_{L^2(\mathbb{R}^N)} - \frac{\tau^{N+\alpha}}{2}\mathcal{F}(w) \to -\infty \text{ as } \tau \to +\infty.$$

Taking $w_0(x) := w_T(x)$ with $T > 0$ large enough, we obtain the assertion. □

Remark 3 Note that $c_{\mathrm{MP}} > 0$. To verify this, fix $\gamma \in \Gamma$. From the proof of Lemma 9-($MP2$), we know that $\|\gamma(0)\|_{H^s(\mathbb{R}^N)} = 0 < \rho < \|\gamma(1)\|_{H^s(\mathbb{R}^N)}$. By the intermediate value theorem, there exists $\bar{\tau} \in (0, 1)$ such that $\|\gamma(\bar{\tau})\|_{H^s(\mathbb{R}^N)} = \rho$. Consequently,

$$\frac{C_2}{2}\rho^2 \leq \mathcal{J}(\gamma(\bar{\tau})) \leq \sup_{t\in[0,1]} \mathcal{J}(\gamma(t)).$$

Because $\gamma \in \Gamma$ was arbitrary, we conclude that $c_{\mathrm{MP}} \geq \frac{C_2}{2}\rho^2 > 0$.

Next, we construct a Palais-Smale sequence for $\mathcal{J}$ at the mountain pass level c_{MP} that asymptotically satisfies the Pohožaev identity.

Proposition 6 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ and $(f3)$ hold. Then, there exists a Pohožaev-Palais-Smale sequence $(u_n) \subset H^s(\mathbb{R}^N)$ for $\mathcal{J}$ at level c_{MP}, namely,*

$$\mathcal{J}(u_n) \to c_{\mathrm{MP}}, \ \mathcal{J}'(u_n) \to 0 \text{ in } H^{-s}(\mathbb{R}^N), \ \text{ and } P(u_n) \to 0. \tag{4.2}$$

Proof Following [7, 8, 35], we define $\Phi : \mathbb{R} \times H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ by $\Phi(\theta, u)(x) := u\left(\frac{x}{e^\theta}\right)$ for $\theta \in \mathbb{R}$, $u \in H^s(\mathbb{R}^N)$, and $x \in \mathbb{R}^N$, where $\mathbb{R} \times H^s(\mathbb{R}^N)$ is endowed with the norm

$$\|(\theta, u)\|_{\mathbb{R}\times H^s(\mathbb{R}^N)} := |\theta| + \|u\|_{H^s(\mathbb{R}^N)}.$$

We use the notation

$$\mathrm{dist}_{\mathbb{R}\times H^s(\mathbb{R}^N)}((\theta, u), A) := \inf_{(\tau,v)\in A}(|\theta-\tau|+\|u-v\|_{H^s(\mathbb{R}^N)}) \text{ for all } A \subset \mathbb{R}\times H^s(\mathbb{R}^N).$$

Consider $\tilde{\mathcal{J}} : \mathbb{R} \times H^s(\mathbb{R}^N) \to \mathbb{R}$ given by $\tilde{\mathcal{J}} := \mathcal{J} \circ \Phi$. Hence, for every $\theta \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^N)$, we have

$$\tilde{\mathcal{J}}(\theta, u) := \frac{1}{2}e^{(N-2s)\theta}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}e^{N\theta}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{e^{(N+\alpha)\theta}}{2}\mathscr{F}(u).$$

Denote by $\tilde{\mathcal{J}}_\theta$ and $\tilde{\mathcal{J}}_u$ the partial derivatives of $\tilde{\mathcal{J}}$. Observe that, for all $\theta \in \mathbb{R}$ and $u, w \in H^s(\mathbb{R}^N)$,

$$\begin{aligned}\tilde{\mathcal{J}}_\theta(\theta, u) &= \frac{(N-2s)}{2}e^{(N-2s)\theta}[u]^2_{H^s(\mathbb{R}^N)} + \frac{N\omega}{2}e^{N\theta}\|u\|^2_{L^2(\mathbb{R}^N)} \\ &\quad - \frac{(N+\alpha)}{2}e^{(N+\alpha)\theta}\mathscr{F}(u) \\ &= P(\Phi(\theta, u)),\end{aligned}$$

and

$$\langle \tilde{\mathcal{J}}_u(\theta, u), w\rangle = e^{(N-2s)\theta}\langle u, w\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \omega\langle u, w\rangle_{L^2(\mathbb{R}^N)}$$

$$- e^{(N+\alpha)\theta} \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) w \, dx$$
$$= \langle \mathcal{J}'(\Phi(\theta, u)), \Phi(\theta, w) \rangle.$$

Therefore, for all $(h, w) \in \mathbb{R} \times H^s(\mathbb{R}^N)$,

$$\langle \tilde{\mathcal{J}}'(\theta, u), (h, w) \rangle = \langle \mathcal{J}'(\Phi(\theta, u)), \Phi(\theta, w) \rangle + P(\Phi(\theta, u))h. \tag{4.3}$$

Arguing as in the proof of Lemma 9, we can verify that $\tilde{\mathcal{J}}$ possesses a mountain pass geometry. Then, we can consider the mountain pass level of $\tilde{\mathcal{J}}$ given by

$$\widetilde{c}_{\mathrm{MP}} := \inf_{\widetilde{\gamma} \in \widetilde{\Gamma}} \max_{t \in [0,1]} \tilde{\mathcal{J}}(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma} := \{\widetilde{\gamma} \in C^0([0, 1], \mathbb{R} \times H^s(\mathbb{R}^N)) : \widetilde{\gamma}(0) = (0, 0) \text{ and } \tilde{\mathcal{J}}(\widetilde{\gamma}(1)) < 0\}.$$

It is easy to check that $\widetilde{c}_{\mathrm{MP}} = c_{\mathrm{MP}}$. Let us prove that there exists $((\theta_n, v_n)) \subset \mathbb{R} \times H^s(\mathbb{R}^N)$ such that, as $n \to +\infty$,

(i) $\tilde{\mathcal{J}}(\theta_n, v_n) \to c_{\mathrm{MP}}$,
(ii) $\tilde{\mathcal{J}}'(\theta_n, v_n) \to 0$ in $\mathbb{R} \times H^{-s}(\mathbb{R}^N)$,
(iii) $\theta_n \to 0$.

In fact, (1.7) and (1.8) ensure that there exists $(\gamma_n) \subset \Gamma$ such that

$$\max_{t \in [0,1]} \mathcal{J}(\gamma_n(t)) \leq c_{\mathrm{MP}} + \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Set $\tilde{\gamma}_n(t) := (0, \gamma_n(t)) \in \widetilde{\Gamma}$ for all $n \in \mathbb{N}$. Hence,

$$\max_{t \in [0,1]} \tilde{\mathcal{J}}(\tilde{\gamma}_n(t)) = \max_{t \in [0,1]} \mathcal{J}(\gamma_n(t)) \leq c_{\mathrm{MP}} + \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

In view of [44, Theorem 2.8], we can find $((\theta_n, v_n)) \subset \mathbb{R} \times H^s(\mathbb{R}^N)$ such that, as $n \to +\infty$, (i) and (ii) hold, and

$$\mathrm{dist}_{\mathbb{R} \times H^s(\mathbb{R}^N)}((\theta_n, v_n), \{0\} \times \gamma_n([0, 1])) \to 0,$$

which yields (iii). Next, we put $u_n := \Phi(\theta_n, v_n)$ for all $n \in \mathbb{N}$. By (i), we see $\mathcal{J}(u_n) \to c_{\mathrm{MP}}$. Taking $h = 1$ and $w = 0$ in (4.3), and using (ii), we obtain $P(u_n) \to 0$. Now, for all $\varphi \in H^s(\mathbb{R}^N)$ and $n \in \mathbb{N}$, we have

$$\|\varphi(e^{\theta_n} \cdot)\|^2_{H^s(\mathbb{R}^N)} = e^{-(N-2s)\theta_n} [\varphi]^2_{H^s(\mathbb{R}^N)} + e^{-N\theta_n} \|\varphi\|^2_{L^2(\mathbb{R}^N)}$$
$$\leq e^{(N-2s)|\theta_n|} [\varphi]^2_{H^s(\mathbb{R}^N)} + e^{N|\theta_n|} \|\varphi\|^2_{L^2(\mathbb{R}^N)}$$

$$\leq e^{N|\theta_n|}\|\varphi\|^2_{H^s(\mathbb{R}^N)}.$$

Thus, choosing $h = 0$ in (4.3), we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|\mathcal{J}'(u_n)\|_{H^{-s}(\mathbb{R}^N)} &= \sup_{\|\varphi\|_{H^s(\mathbb{R}^N)}\leq 1} |\langle \mathcal{J}'(u_n), \varphi\rangle| \\
&= \sup_{\|\varphi\|_{H^s(\mathbb{R}^N)}\leq 1} \left|\langle \tilde{\mathcal{J}}_u(\theta_n, u_n), \varphi(e^{\theta_n}\cdot)\rangle\right| \\
&\leq \sup_{\|\varphi\|_{H^s(\mathbb{R}^N)}\leq 1} \|\tilde{\mathcal{J}}_u(\theta_n, u_n)\|_{H^{-s}(\mathbb{R}^N)}\|\varphi(e^{\theta_n}\cdot)\|_{H^s(\mathbb{R}^N)} \\
&\leq \sup_{\|\varphi\|_{H^s(\mathbb{R}^N)}\leq 1} \|\tilde{\mathcal{J}}_u(\theta_n, u_n)\|_{H^{-s}(\mathbb{R}^N)} e^{\frac{N}{2}|\theta_n|}\|\varphi\|_{H^s(\mathbb{R}^N)} \\
&\leq \|\tilde{\mathcal{J}}_u(\theta_n, u_n)\|_{H^{-s}(\mathbb{R}^N)} e^{\frac{N}{2}|\theta_n|}.
\end{aligned}$$

This estimate, combined with (ii) and (iii), implies that $\mathcal{J}'(u_n) \to 0$ in $H^{-s}(\mathbb{R}^N)$. Therefore, (u_n) has the properties stated in (4.2). □

We now establish the boundedness of Pohožaev-Palais-Smale sequences for $\mathcal{J}$.

Lemma 10 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ and $(f3)$ hold. Let $(u_n) \subset H^s(\mathbb{R}^N)$ be such that*

$$\sup_{n\in\mathbb{N}} \mathcal{J}(u_n) \leq C_1 \quad \text{and} \quad \inf_{n\in\mathbb{N}} P(u_n) \geq -C_2, \tag{4.4}$$

for some $C_1, C_2 > 0$. Then, (u_n) is bounded in $H^s(\mathbb{R}^N)$. In particular, every sequence $(u_n) \subset H^s(\mathbb{R}^N)$ satisfying (4.2) is bounded in $H^s(\mathbb{R}^N)$.

Proof Using (4.4), we see that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
C_1 + \frac{C_2}{N+\alpha} &\geq \mathcal{J}(u_n) - \frac{1}{N+\alpha}P(u_n) \\
&= \frac{\alpha+2s}{2(N+\alpha)}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\omega\alpha}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)} \\
&\geq \min\left\{\frac{\alpha+2s}{2(N+\alpha)}, \frac{\omega\alpha}{2(N+\alpha)}\right\}\|u_n\|^2_{H^s(\mathbb{R}^N)}.
\end{aligned}$$

which implies that (u_n) is bounded in $H^s(\mathbb{R}^N)$. □

Hereafter, we examine the convergence of Pohožaev-Palais-Smale sequences for $\mathcal{J}$.

Proposition 7 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. Let $(u_n) \subset H^s(\mathbb{R}^N)$ be a sequence such that*

(i) $(\mathcal{J}(u_n))$ *is bounded in* $\mathbb{R}$,

(ii) $\mathcal{J}'(u_n) \to 0$ *in* $H^{-s}(\mathbb{R}^N)$ *and* $P(u_n) \to 0$ *as* $n \to +\infty$.

Then, up to a subsequence, one of the following alternatives holds:

(1) $u_n \to 0$ *in* $H^s(\mathbb{R}^N)$ *as* $n \to +\infty$;
(2) there exist $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ *and a sequence* $(x_n) \subset \mathbb{R}^N$ *such that* $\mathcal{J}'(u) = 0$ *and* $u_n(\cdot - x_n) \rightharpoonup u$ *in* $H^s(\mathbb{R}^N)$ *as* $n \to +\infty$.

Proof We note that Lemma 10 ensures that (u_n) is bounded in $H^s(\mathbb{R}^N)$. Suppose that (1) is not true, that is,

$$\liminf_{n\to+\infty} \|u_n\|_{H^s(\mathbb{R}^N)} > 0. \tag{4.5}$$

We first prove that, for every $r \in (2, 2_s^*)$,

$$\liminf_{n\to+\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^r(B_1(x_0))} > 0. \tag{4.6}$$

Assume by contradiction that, for some $r \in (2, 2_s^*)$, it holds

$$\liminf_{n\to+\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^r(B_1(x_0))} = 0. \tag{4.7}$$

Using the definition of the Pohožaev functional P, (4.5) and that $P(u_n) \to 0$ as $n \to \infty$, we deduce

$$\begin{aligned} &\liminf_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)\,dx \\ &= \liminf_{n\to+\infty} \left[\frac{N-2s}{N+\alpha}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{N}{N+\alpha}\|u_n\|^2_{L^2(\mathbb{R}^N)} - \frac{2}{N+\alpha}P(u_n)\right] \\ &= \liminf_{n\to+\infty} \left[\frac{N-2s}{N+\alpha}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{N}{N+\alpha}\|u_n\|^2_{L^2(\mathbb{R}^N)}\right] > 0, \end{aligned} \tag{4.8}$$

where we have used the fact that, for all $(a_n), (b_n) \subset \mathbb{R}$ such that $\lim_{n\to+\infty} b_n$ exists,

$$\liminf_{n\to+\infty}(a_n + b_n) = \liminf_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n.$$

By $(f1)$ and $(f2)$, for each fixed $\varepsilon > 0$ there exists $C_{\varepsilon,r} > 0$ such that

$$|F(t)|^{\frac{2N}{N+\alpha}} \le \varepsilon(|t|^2 + |t|^{2_s^*}) + C_{\varepsilon,r}|t|^r \quad \text{for all } t \in \mathbb{R}. \tag{4.9}$$

Employing (4.9), the boundedness of (u_n) in $H^s(\mathbb{R}^N)$, the fractional Sobolev embeddings (see Proposition 1), and Lemma 1 with $t = r$, we obtain

$$\|F(u_n)\|^{\frac{2N}{N+\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C'\varepsilon + C''_{\varepsilon,r}\left(\sup_{x_0\in\mathbb{R}^N} \|u_n\|^r_{L^r(B_1(x_0))}\right)^{1-\frac{2}{r}} \quad \text{for all } n \in \mathbb{N},$$

which, combined with (4.7) and the arbitrariness of $\varepsilon > 0$, leads to

$$\liminf_{n\to+\infty} \|F(u_n)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = 0. \tag{4.10}$$

Since Proposition 2 guarantees that

$$\left|\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)\,dx\right| \le C(N,\alpha)\|F(u_n)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N},$$

it follows from (4.10) that

$$\liminf_{n\to+\infty} \left|\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)\,dx\right| = 0. \tag{4.11}$$

From (4.8) and (4.11), we derive

$$0 < \liminf_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)\,dx \le 0,$$

that is a contradiction. Consequently, (4.6) is valid. Thus, up to a translation, we may assume that, for some $r \in (2, 2^*_s)$,

$$\liminf_{n\to+\infty} \|u_n\|_{L^r(B_1(0))} > 0. \tag{4.12}$$

Since (u_n) is bounded in $H^s(\mathbb{R}^N)$, we may suppose that there exists $u \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N),\\ &u_n \to u \quad \text{in } L^q_{loc}(\mathbb{R}^N) \text{ for all } q \in [1, 2^*_s),\\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{4.13}$$

In view of (4.12) and (4.13), we see $u \not\equiv 0$. Employing (4.13), Lemma 4, and that $\mathcal{J}'(u_n) \to 0$ in $H^{-s}(\mathbb{R}^N)$, we conclude that $\langle \mathcal{J}'(u), \varphi\rangle = 0$ for all $\varphi \in H^s(\mathbb{R}^N)$. □

Next, we prove the existence of an optimal path in the spirit of [25, Lemma 2.1].

Lemma 11 *Let $N \ge 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ holds. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Let $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ be a weak solution to (1.1). Then, there exists $\gamma \in \Gamma$ such that $u \in \gamma([0,1])$ and*

$$\max_{t\in[0,1]} \mathcal{J}(\gamma(t)) = \mathcal{J}(u).$$

Proof Put $u_t(x) := u(\frac{x}{t})$ for a.e. $x \in \mathbb{R}^N$ and for all $t > 0$. Consider $\tilde{\gamma} : [0, +\infty) \to H^s(\mathbb{R}^N)$ given by

$$\tilde{\gamma}(t)(x) := \begin{cases} u_t(x) & \text{for } t > 0,\\ 0 & \text{for } t = 0. \end{cases}$$

It is evident that $\tilde{\gamma} \in C^0([0, +\infty), H^s(\mathbb{R}^N))$. By Theorem 2, we know that $P(u) = 0$. Then, for all $t > 0$, we have

$$\begin{aligned}
\mathcal{J}(\tilde{\gamma}(t)) &= \frac{t^{N-2s}}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{t^N\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{t^{N+\alpha}}{2}\mathscr{F}(u) \\
&= \frac{t^{N-2s}}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{t^N\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} \\
&\quad - \frac{t^{N+\alpha}}{N+\alpha}\left[\frac{(N-2s)}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{N\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)}\right] \\
&= \left[\frac{t^{N-2s}}{2} - \frac{t^{N+\alpha}(N-2s)}{2(N+\alpha)}\right][u]^2_{H^s(\mathbb{R}^N)} + \omega\left[\frac{t^N}{2} - \frac{Nt^{N+\alpha}}{2(N+\alpha)}\right]\|u\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Differentiating with respect to t, we deduce that, for all $t > 0$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}(\tilde{\gamma}(t)) &= \frac{N-2s}{2}\left(t^{N-2s-1} - t^{N+\alpha-1}\right)[u]^2_{H^s(\mathbb{R}^N)} \\
&\quad + \frac{N}{2}\omega\left(t^{N-1} - t^{N+\alpha-1}\right)\|u\|^2_{L^2(\mathbb{R}^N)} \\
&= t^{N-1}\beta(t),
\end{aligned}$$

where

$$\beta(t) := \frac{(N-2s)}{2}(t^{-2s} - t^{\alpha})[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega N}{2}(1 - t^{\alpha})\|u\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all } t > 0.$$

Due to $\beta(1) = 0$ and

$$\begin{aligned}
\beta'(t) &= \frac{(N-2s)}{2}(-2st^{-2s-1} - \alpha t^{\alpha-1})[u]^2_{H^s(\mathbb{R}^N)} \\
&\quad + \frac{\omega N}{2}(-\alpha t^{\alpha-1})\|u\|^2_{L^2(\mathbb{R}^N)} < 0 \quad \text{for all } t > 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathcal{J}(\tilde{\gamma}(t)) = 0 \ \text{ for } t = 1, \\
&\frac{d}{dt}\mathcal{J}(\tilde{\gamma}(t)) > 0 \ \text{ for all } t \in (0, 1), \\
&\frac{d}{dt}\mathcal{J}(\tilde{\gamma}(t)) < 0 \ \text{ for all } t \in (1, +\infty).
\end{aligned}$$

Hence,

$$\max_{t \geq 0}\mathcal{J}(\tilde{\gamma}(t)) = \mathcal{J}(\tilde{\gamma}(1)) = \mathcal{J}(u).$$

Considering that $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ and $P(u) = 0$ imply $\mathscr{F}(u) > 0$, we see

$$\mathcal{J}(\tilde{\gamma}(t)) = \frac{t^{N-2s}}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{t^N \omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{t^{N+\alpha}}{2}\mathscr{F}(u) \to -\infty \text{ as } t \to +\infty.$$

Consequently, we can find $L > 1$ such that $\mathcal{J}(\tilde{\gamma}(L)) < 0$. Putting $\gamma(t)(x) := \tilde{\gamma}(tL)(x)$ for a.e. $x \in \mathbb{R}^N$ and for all $t \in [0, 1]$, we conclude that γ satisfies the required properties. □

Next, we show that (1.1) has a least energy solution.

Proof of Theorem 3 In light of Propositions 6 and 7, there exists a Pohožaev-Palais-Smale sequence $(u_n) \subset H^s(\mathbb{R}^N)$ for $\mathcal{J}$ at level $c_{\rm MP} > 0$ such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, for some $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ that weakly solves (1.1). Utilizing that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, $\mathcal{J}(u_n) \to c_{\rm MP}$, $P(u_n) \to 0$, $P(u) = 0$, and that

$$\liminf_{n\to+\infty} a_n + \liminf_{n\to+\infty} b_n \le \liminf_{n\to+\infty}(a_n + b_n) \quad \text{for all } (a_n), (b_n) \subset \mathbb{R},$$

we deduce

$$\begin{aligned}
\mathcal{J}(u) &= \mathcal{J}(u) - \frac{1}{N+\alpha}P(u) \\
&= \frac{\alpha+2s}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u\|^2_{L^2(\mathbb{R}^N)} \\
&\le \frac{\alpha+2s}{2(N+\alpha)}\liminf_{n\to+\infty}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\liminf_{n\to+\infty}\|u_n\|^2_{L^2(\mathbb{R}^N)} \\
&\le \liminf_{n\to+\infty}\left[\frac{\alpha+2s}{2(N+\alpha)}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)}\right] \\
&= \liminf_{n\to+\infty}\left[\mathcal{J}(u_n) - \frac{1}{N+\alpha}P(u_n)\right] = c_{\rm MP}.
\end{aligned} \tag{4.14}$$

Considering that u is a weak solution to (1.1) and that $u \not\equiv 0$, it follows from the definition of $c_{\rm LE}$ and (4.14) that

$$c_{\rm LE} \le \mathcal{J}(u) \le c_{\rm MP}. \tag{4.15}$$

Now, let $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ be an arbitrary weak solution to (1.1). Thanks to Theorem 2, $P(v) = 0$. Applying Lemma 11 to v and recalling the definition of $c_{\rm MP}$, we infer that $\mathcal{J}(v) \ge c_{\rm MP}$. Due to the arbitrariness of v, we find $c_{\rm LE} \ge c_{\rm MP}$. This fact and (4.15) imply $\mathcal{J}(u) = c_{\rm MP} = c_{\rm LE}$. Lastly, we observe that $u \in \mathcal{S} \subset \mathcal{P}$, and thus $c_{\rm MP} = \mathcal{J}(u) \ge c_{\rm LE} \ge c_{\rm PO}$. On the other hand, $c_{\rm PO} \ge c_{\rm MP}$ by Lemma 11. As a result, $c_{\rm MP} = \mathcal{J}(u) = c_{\rm LE} = c_{\rm PO}$. □

In addition, we establish the strong convergence of the translated Pohožaev-Palais-Smale sequence in Proposition 7.

Corollary 1 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Under the assumptions (i) and (ii) of Proposition 7, if*

$$\liminf_{n\to+\infty} \|u_n\|_{H^s(\mathbb{R}^N)} > 0 \quad \textit{and} \quad \limsup_{n\to+\infty} \mathcal{J}(u_n) \leq c_{\mathrm{LE}},$$

then there exists $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{J}'(u) = 0$, and a sequence $(x_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $u_n(\cdot - x_n) \to u$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$.

Proof From Proposition 7, up to a subsequence and translations, we may assume that there exists $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, 2_s^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$. Moreover, u is a weak solution to (1.1). Hereafter, we confirm that $u_n \to u$ in $H^s(\mathbb{R}^N)$. As in the proof of Theorem 3, we have

$$\begin{aligned}
c_{\mathrm{LE}} &\leq \mathcal{J}(u) = \mathcal{J}(u) - \frac{1}{N+\alpha} P(u) \\
&= \frac{\alpha+2s}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u\|^2_{L^2(\mathbb{R}^N)} \\
&\leq \frac{\alpha+2s}{2(N+\alpha)} \liminf_{n\to+\infty}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)} \liminf_{n\to+\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} \\
&\leq \liminf_{n\to+\infty} \left[\frac{\alpha+2s}{2(N+\alpha)}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)} \right] \\
&\leq \limsup_{n\to+\infty} \left[\frac{\alpha+2s}{2(N+\alpha)}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)} \right] \\
&= \limsup_{n\to+\infty} \left(\mathcal{J}(u_n) - \frac{1}{N+\alpha} P(u_n) \right) = \limsup_{n\to+\infty} \mathcal{J}(u_n) \leq c_{\mathrm{LE}},
\end{aligned}$$

from which $\mathcal{J}(u) = c_{\mathrm{LE}}$ and

$$\begin{aligned}
&\frac{\alpha+2s}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u\|^2_{L^2(\mathbb{R}^N)} \\
&= \lim_{n\to+\infty} \left[\frac{\alpha+2s}{2(N+\alpha)}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\alpha\omega}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)} \right].
\end{aligned}$$

Consequently, $u_n \to u$ in $H^s(\mathbb{R}^N)$, as desired. □

Finally, we prove that, up to translations, the set of least energy solutions to (1.1) is compact in $H^s(\mathbb{R}^N)$.

Proposition 8 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Up to translations in $\mathbb{R}^N$, the set*

$$\mathcal{L}_{\mathrm{LE}} := \left\{ u \in H^s(\mathbb{R}^N) : \mathcal{J}(u) = c_{\mathrm{LE}},\ \mathcal{J}'(u) = 0 \right\}$$

is compact in $H^s(\mathbb{R}^N)$ *endowed with the strong topology. Furthermore, if we assume that* (1.4) *holds, then there exists* $K_1 > 0$ *such that, for all* $u \in \mathcal{L}_{\rm LE}$*,*

$$|u(x)| \leq \frac{K_1}{1+|x|^{N+2s}} \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof Let $(u_n) \subset \mathcal{L}_{\rm LE}$. Thus, $\mathcal{J}(u_n) = c_{\rm LE}$ and $\mathcal{J}'(u_n) = 0$ for all $n \in \mathbb{N}$. By Theorem 2, we see $P(u_n) = 0$ for all $n \in \mathbb{N}$. Arguing as in the proof of Corollary 1, we have that, up to a subsequence and translations, $u_n \to u$ in $H^s(\mathbb{R}^N)$ for some $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{J}(u) = c_{\rm LE}$ and $\mathcal{J}'(u) = 0$. Thus, up to translations in $\mathbb{R}^N$, $\mathcal{L}_{\rm LE}$ is compact in $H^s(\mathbb{R}^N)$. We now assume the additional condition (1.4) and demonstrate the uniform polynomial decay estimate. Proceeding as in the proof of Proposition 3, and because $\mathcal{L}_{\rm LE}$ is bounded in $H^s(\mathbb{R}^N)$, we obtain that, for all $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2s})$, $\|u_n\|_{L^p(\mathbb{R}^N)} \leq C_p$ for all $n \in \mathbb{N}$. This and $(f1)$ show that, for all $q \in [\frac{2N}{N+\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha})$, $\|F(u_n)\|_{L^q(\mathbb{R}^N)} \leq C_q$ for all $n \in \mathbb{N}$. By Young's inequality, we deduce that $\|I_\alpha * F(u_n)\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. This estimate, together with the fractional Kato inequality for $(-\Delta)^s$ (see [3, Theorem 17.3.5]), ensures that $|u_n|$ satisfies

$$(-\Delta)^s|u_n| \leq -\omega|u_n| + C|f(u_n)| =: g(u_n) \text{ in } \mathbb{R}^N.$$

In view of $(f1)$ and (1.4), we know that $|g(t)| \leq C(|t| + |t|^{2^*_s-1})$ for all $t \in \mathbb{R}$. Given that (u_n) is bounded in $H^s(\mathbb{R}^N)$ and $u_n \to u$ in $L^{2^*_s}(\mathbb{R}^N)$, we can use a classical Moser iteration argument (see [1, Lemma 5.1] and [4, Theorem 1.1]) to infer that, for some $K > 0$, $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq K$ for all $n \in \mathbb{N}$. Hence, $\mathcal{L}_{\rm LE}$ is bounded in $L^\infty(\mathbb{R}^N)$. Since $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq K$ and $\|f(u_n)\|_{L^\infty(\mathbb{R}^N)} \leq C_K$ for all $n \in \mathbb{N}$, we derive from Corollary 5-(b) that $(u_n) \subset C^{0,\sigma}(\mathbb{R}^N)$, for some $\sigma \in (0,1)$ independent of n, and that there exists $C = C(N, s, \sigma, m, K) > 0$ such that $\|u_n\|_{C^{0,\sigma}(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Consequently, (u_n) is uniformly equicontinuous in $\mathbb{R}^N$, namely, for every $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that, if $x, y \in \mathbb{R}^N$ are such that $|x-y| < \delta$, then $|u_n(x) - u_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$. This, combined with $u_n \to u$ in $L^2(\mathbb{R}^N)$, yields $\lim_{|x|\to+\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0$ (see [4, Lemma 2.1]). Then we proceed as in Theorem 1-Step 5 to achieve the uniform decay estimate. In this case, we utilize the comparison function

$$z_n := \left(\frac{K}{\min_{\overline{B}_R(0)} w}\right) w - |u_n| \in H^s(\mathbb{R}^N).$$

The proof of Proposition 8 is now complete.

□

5 Qualitative properties of least energy solutions

In this section, we examine the sign and symmetry of least energy solutions to (1.1). We begin by proving a useful identity involving $\mathcal{J}(u)$, $\mathcal{J}(u_t)$, and $P(u)$, where $u_t = u(\frac{\cdot}{t})$ for $t > 0$.

Lemma 12 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0,N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ holds. Then, for all $u \in H^s(\mathbb{R}^N)$ and $t > 0$,*

$$\mathcal{J}(u) = \mathcal{J}(u_t) + \frac{(1-t^{N+\alpha})}{N+\alpha}P(u) + \frac{\mathfrak{g}(t)[u]^2_{H^s(\mathbb{R}^N)} + \omega\mathfrak{h}(t)\|u\|^2_{L^2(\mathbb{R}^N)}}{2(N+\alpha)}, \tag{5.1}$$

where $\mathfrak{g}, \mathfrak{h} : [0,+\infty) \to \mathbb{R}$ are given by

$$\mathfrak{g}(t) := 2s + \alpha - (N+\alpha)t^{N-2s} + (N-2s)t^{N+\alpha},$$
$$\mathfrak{h}(t) := \alpha - (N+\alpha)t^N + Nt^{N+\alpha}.$$

In particular, for all $u \in H^s(\mathbb{R}^N)$ and $t > 0$,

$$\mathcal{J}(u) \geq \mathcal{J}(u_t) + \frac{(1-t^{N+\alpha})}{N+\alpha}P(u) + \frac{\mathfrak{g}(t)[u]^2_{H^s(\mathbb{R}^N)}}{2(N+\alpha)}. \tag{5.2}$$

Proof From the definitions of $\mathcal{J}$ and P, we obtain

$$\begin{aligned}
&\mathcal{J}(u) - \mathcal{J}(u_t)\\
&= \frac{1}{2}(1-t^{N-2s})[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}(1-t^N)\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}(1-t^{N+\alpha})\mathscr{F}(u)\\
&= \frac{(1-t^{N+\alpha})}{N+\alpha}\left\{\left(\frac{N-2s}{2}\right)[u]^2_{H^s(\mathbb{R}^N)} + \frac{N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \left(\frac{N+\alpha}{2}\right)\mathscr{F}(u)\right\}\\
&\quad + \frac{1}{2}(1-t^{N-2s})[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}(1-t^N)\|u\|^2_{L^2(\mathbb{R}^N)}\\
&\quad - \frac{1}{2}\left(\frac{N-2s}{N+\alpha}\right)(1-t^{N+\alpha})[u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2(N+\alpha)}(1-t^{N+\alpha})\|u\|^2_{L^2(\mathbb{R}^N)}\\
&= \frac{(1-t^{N+\alpha})}{N+\alpha}P(u) + \frac{1}{2}\left[(1-t^{N-2s}) - \left(\frac{N-2s}{N+\alpha}\right)(1-t^{N+\alpha})\right][u]^2_{H^s(\mathbb{R}^N)}\\
&\quad + \frac{\omega}{2}\left[1-t^N - \frac{N}{N+\alpha}(1-t^{N+\alpha})\right]\|u\|^2_{L^2(\mathbb{R}^N)}\\
&= \frac{(1-t^{N+\alpha})}{N+\alpha}P(u) + \frac{\mathfrak{g}(t)[u]^2_{H^s(\mathbb{R}^N)} + \omega\mathfrak{h}(t)\|u\|^2_{L^2(\mathbb{R}^N)}}{2(N+\alpha)},
\end{aligned}$$

that is, (5.1) is valid. Since, for all $t > 0$,

$$\mathfrak{g}'(t) = (N+\alpha)(N-2s)\left[-t^{N-2s-1} + t^{N+\alpha-1}\right],$$
$$\mathfrak{h}'(t) = N(N+\alpha)\left[-t^{N-1} + t^{N+\alpha-1}\right],$$

we easily deduce that

$$\mathfrak{g}'(1) = 0, \ \ \mathfrak{g}'(t) < 0 \ \text{ for all } t \in (0,1), \ \ \mathfrak{g}'(t) > 0 \ \text{ for all } t \in (1,+\infty), \tag{5.3}$$

and

$$\mathfrak{h}'(1) = 0, \ \mathfrak{h}'(t) < 0 \ \text{ for all } t \in (0, 1), \ \mathfrak{h}'(t) > 0 \ \text{ for all } t \in (1, +\infty).$$

Consequently,

$$\mathfrak{g}(t) > \mathfrak{g}(1) = 0 \quad \text{and} \quad \mathfrak{h}(t) > \mathfrak{h}(1) = 0 \quad \text{for all } t \in [0, 1) \cup (1, \infty). \tag{5.4}$$

From (5.1) and (5.4), we derive (5.2). □

Remark 4 Using (5.2) and (5.4), we obtain

$$\mathcal{J}(u) \geq \mathcal{J}(u_t) \quad \text{for all } u \in \mathcal{P} \text{ and } t > 0, \tag{5.5}$$

which leads to

$$\mathcal{J}(u) = \max_{t>0} \mathcal{J}(u_t) \quad \text{for all } u \in \mathcal{P}.$$

Now, we prove a helpful lemma.

Lemma 13 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ and $(f3)$ hold. Define*

$$\Lambda := \left\{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : \mathscr{F}(u) > 0 \right\}.$$

Then, $\Lambda \neq \emptyset$ and

$$\left\{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : P(u) \leq 0 \right\} \subset \Lambda.$$

Proof By Lemma 9-$(MP3)$, we know that $\Lambda \neq \emptyset$. Let $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ be such that $P(u) \leq 0$. Then,

$$\begin{aligned} \frac{(N+\alpha)}{2} \mathscr{F}(u) &\geq \frac{(N-2s)}{2} [u]^2_{H^s(\mathbb{R}^N)} + \frac{N\omega}{2} \|u\|^2_{L^2(\mathbb{R}^N)} \\ &\geq \min\left\{ \frac{N-2s}{2}, \frac{N\omega}{2} \right\} \|u\|^2_{H^s(\mathbb{R}^N)} > 0, \end{aligned}$$

which implies that $\mathscr{F}(u) > 0$. Hence, $u \in \Lambda$. □

Lemma 14 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ and $(f3)$ hold. Then, for all $u \in \Lambda$ there exists a unique $t_u > 0$ such that $u_{t_u} \in \mathcal{P}$. Moreover, t_u is the unique global maximum point of the function $\zeta_u : (0, +\infty) \to \mathbb{R}$ defined by $\zeta_u(t) := \mathcal{J}(u_t)$.*

Proof Take $u \in \Lambda$. Since

$$\zeta_u(t) = \frac{t^{N-2s}}{2}[u]^2_{H^s(\mathbb{R}^N)} + \omega\frac{t^N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{t^{N+\alpha}}{2}\mathscr{F}(u),$$

we deduce that, for all $t > 0$,

$$\begin{aligned}
\zeta_u'(t) &= \frac{d}{dt}\mathcal{J}(u_t) \\
&= \left(\frac{N-2s}{2}\right)t^{N-2s-1}[u]^2_{H^s(\mathbb{R}^N)} + \omega\frac{N}{2}t^{N-1}\|u\|^2_{L^2(\mathbb{R}^N)} \\
&\quad - t^{N+\alpha-1}\left(\frac{N+\alpha}{2}\right)\mathscr{F}(u) \\
&= \frac{1}{t}P(u_t).
\end{aligned}$$

Consequently,

$$\zeta_u'(t) = 0 \Longleftrightarrow P(u_t) = 0 \Longleftrightarrow u_t \in \mathcal{P}.$$

Now, we show

$$\zeta_u(t) > 0 \quad \text{for all } t > 0 \text{ small.} \tag{5.6}$$

For all $t \in (0, 1)$, we see

$$\begin{aligned}
\zeta_u(t) &= \frac{t^{N-2s}}{2}[u]^2_{H^s(\mathbb{R}^N)} + \omega\frac{t^N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{t^{N+\alpha}}{2}\mathscr{F}(u) \\
&\geq \frac{t^N}{2}\left[[u]^2_{H^s(\mathbb{R}^N)} + \omega\|u\|^2_{L^2(\mathbb{R}^N)} - t^\alpha\mathscr{F}(u)\right].
\end{aligned}$$

Because $u \in \Lambda$, we know that $\mathscr{F}(u) > 0$. Accordingly,

$$\zeta_u(t) > 0 \quad \text{if } 0 < t < \min\left\{1, \left(\frac{[u]^2_{H^s(\mathbb{R}^N)} + \omega\|u\|^2_{L^2(\mathbb{R}^N)}}{\mathscr{F}(u)}\right)^{\frac{1}{\alpha}}\right\}.$$

Thus, (5.6) is valid. Next we prove

$$\zeta_u(t) < 0 \quad \text{for all } t > 0 \text{ large.} \tag{5.7}$$

Given that $u \in \Lambda$, we obtain

$$\zeta_u(t) = \frac{t^{N-2s}}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{t^N\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{t^{N+\alpha}}{2}\mathscr{F}(u) \to -\infty \quad \text{as } t \to +\infty.$$

and this ensures that (5.7) is true. By the continuity of ζ_u, (5.6), and (5.7), it follows that $\max_{t>0}\zeta_u(t)$ is attained at some $t_u > 0$, so that $\zeta_u'(t_u) = 0$ and $u_{t_u} \in \mathcal{P}$. Let us demonstrate that such a t_u is unique. Let $t_1, t_2 > 0$ be such that $u_{t_1}, u_{t_2} \in \mathcal{P}$. Using $P(u_{t_1}) = P(u_{t_2}) = 0$, and applying (5.2) first with $u = u_{t_1}$ and $t = t_2/t_1$, and then with $u = u_{t_2}$ and $t = t_1/t_2$, we find

$$\begin{aligned}\mathcal{J}(u_{t_1}) &\geq \mathcal{J}(u_{t_2}) + \frac{(t_1^{N+\alpha} - t_2^{N+\alpha})}{(N+\alpha)t_1^{N+\alpha}} P(u_{t_1}) + \frac{\mathfrak{g}(t_2/t_1)}{2(N+\alpha)}[u_{t_1}]^2_{H^s(\mathbb{R}^N)} \\ &= \mathcal{J}(u_{t_2}) + t_1^{N-2s}\frac{\mathfrak{g}(t_2/t_1)}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}(u_{t_2}) &\geq \mathcal{J}(u_{t_1}) + \frac{(t_2^{N+\alpha} - t_1^{N+\alpha})}{(N+\alpha)t_2^{N+\alpha}} P(u_{t_2}) + \frac{\mathfrak{g}(t_1/t_2)}{2(N+\alpha)}[u_{t_2}]^2_{H^s(\mathbb{R}^N)} \\ &= \mathcal{J}(u_{t_1}) + t_2^{N-2s}\frac{\mathfrak{g}(t_1/t_2)}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)}.\end{aligned}$$

Consequently,

$$0 \geq \left(t_1^{N-2s}\mathfrak{g}(t_2/t_1) + t_2^{N-2s}\mathfrak{g}(t_1/t_2)\right)\frac{1}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)},$$

which, together with $u \in H^s(\mathbb{R}^N)\setminus\{0\}$ and (5.4), implies that $t_1 = t_2$. This concludes the proof of the lemma. □

Remark 5 Assuming only conditions $(f1)$ and $(f3)$, an inspection of the proof of Lemma 11 reveals that for each $u \in \mathcal{P}$ there exists $\gamma \in \Gamma$ such that $u \in \gamma([0,1])$ and $\max_{t\in[0,1]}\mathcal{J}(\gamma(t)) = \mathcal{J}(u)$.

Next, we show that every least energy Pohožaev minimizer of $\mathcal{J}$ is a weak solution to (1.1).

Lemma 15 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0,N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$ and $(f3)$ hold. If $u \in \mathcal{P}$ is such that $\mathcal{J}(u) = c_{\mathrm{PO}}$, then $\mathcal{J}'(u) = 0$.*

Proof Assume by contradiction that $\mathcal{J}'(u) \neq 0$. By continuity, we can find $\delta > 0$ and $\rho > 0$ such that

$$\|\mathcal{J}'(v)\|_{H^{-s}(\mathbb{R}^N)} \geq \rho \quad \text{for all } v \in H^s(\mathbb{R}^N) \text{ such that } \|v-u\|_{H^s(\mathbb{R}^N)} \leq 3\delta. \tag{5.8}$$

Snce $\lim_{t\to 1}\|u_t - u\|_{H^s(\mathbb{R}^N)} = 0$, there exists $\delta_1 \in (0,1)$ sufficiently small such that

$$\|u_t - u\|_{H^s(\mathbb{R}^N)} < \delta \quad \text{for all } t \in \mathbb{R} \text{ such that } |t-1| < \delta_1. \tag{5.9}$$

From (5.2), $u \in \mathcal{P}$, and $\mathcal{J}(u) = c_{\text{PO}}$, we derive

$$\begin{aligned}\mathcal{J}(u_t) &\le \mathcal{J}(u) - \frac{\mathfrak{g}(t)}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} \\ &= c_{\text{PO}} - \frac{\mathfrak{g}(t)}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} \quad \text{for all } t > 0.\end{aligned} \tag{5.10}$$

Note that, for all $t > 0$,

$$P(u_t) = \frac{(N-2s)}{2} t^{N-2s}[u]^2_{H^s(\mathbb{R}^N)} + \frac{N\omega}{2} t^N \|u\|^2_{L^2(\mathbb{R}^N)} - \frac{(N+\alpha)}{2} t^{N+\alpha} \mathscr{F}(u).$$

Arguing as in the proof of Lemma 14, we see that $P(u_t) > 0$ for all $t > 0$ small and $P(u_t) < 0$ for all $t > 0$ large. In fact, for all $t \in (0, 1)$,

$$\begin{aligned}P(u_t) &\ge t^N \left[\frac{(N-2s)}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{N\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{(N+\alpha)}{2} t^\alpha \mathscr{F}(u) \right] \\ &= t^N(1-t^\alpha)\left[\frac{(N-2s)}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{N\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)} \right] > 0,\end{aligned}$$

while $u \in \mathcal{P} \subset \Lambda$ implies

$$P(u_t) \to -\infty \quad \text{as } t \to +\infty.$$

Therefore, there exist $t_1 \in (0, 1)$ and $t_2 \in (1, +\infty)$ such that

$$P(u_{t_1}) > 0 > P(u_{t_2}). \tag{5.11}$$

Set

$$\varepsilon := \min\left\{ \frac{\mathfrak{g}(t_1)}{5(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)}, \frac{\mathfrak{g}(t_2)}{5(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)}, 1, \frac{\rho\delta}{8} \right\} > 0.$$

Let $S := B_\delta(u)$ and $S_r := \{v \in H^s(\mathbb{R}^N) : \text{dist}(v, S) \le r\}$ for all $r > 0$. By (5.8) and the definition of ε, we have

$$\|\mathcal{J}'(v)\|_{H^{-s}(\mathbb{R}^N)} \ge \frac{8\,\varepsilon}{\delta} \quad \text{for all } v \in \mathcal{J}^{-1}([c_{\text{PO}} - 2\,\varepsilon, c_{\text{PO}} + 2\,\varepsilon]) \cap S_{2\delta}.$$

Applying [44, Lemma 2.3], we can find $\eta \in C^0([0, 1] \times H^s(\mathbb{R}^N), H^s(\mathbb{R}^N))$ such that:

(i) $\eta(t, v) = v$ if $t = 0$ or if $v \notin \mathcal{J}^{-1}([c_{\text{PO}} - 2\,\varepsilon, c_{\text{PO}} + 2\,\varepsilon]) \cap S_{2\delta}$,
(ii) $\eta(1, \mathcal{J}^{c_{\text{PO}}+\varepsilon} \cap S) \subset \mathcal{J}^{c_{\text{PO}}-\varepsilon}$,
(iii) $\eta(t, \cdot)$ is a homeomorphism of $H^s(\mathbb{R}^N)$ for all $t \in [0, 1]$,
(iv) $\|\eta(t, v) - v\|_{H^s(\mathbb{R}^N)} \le \delta$ for all $v \in H^s(\mathbb{R}^N)$ and $t \in [0, 1]$,
(v) $\mathcal{J}(\eta(\cdot, v))$ is non increasing for all $v \in H^s(\mathbb{R}^N)$,
(vi) $\mathcal{J}(\eta(t, v)) < c_{\text{PO}}$ for all $v \in \mathcal{J}^{c_{\text{PO}}} \cap S_\delta$ and $t \in (0, 1]$.

Let us observe that, by (5.5),

$$\mathcal{J}(u_t) \leq \mathcal{J}(u) = c_{\mathrm{PO}} < c_{\mathrm{PO}} + \varepsilon \quad \text{for all } t > 0.$$

This, combined with (5.9) and (ii), gives

$$\mathcal{J}(\eta(1, u_t)) \leq c_{\mathrm{PO}} - \varepsilon \quad \text{for all } t > 0 \text{ such that } |t-1| < \delta_1. \tag{5.12}$$

Employing (i), (v), (5.10), and the fact that, by (5.3),

$$\mathfrak{g}(t) \geq \delta_2 := \min\{\mathfrak{g}(1-\delta_1), \mathfrak{g}(1+\delta_1)\} > 0 \ \text{ for all } t > 0 \text{ such that } |t-1| \geq \delta_1,$$

we obtain

$$\begin{aligned}\mathcal{J}(\eta(1, u_t)) &\leq \mathcal{J}(\eta(0, u_t)) \\ &= \mathcal{J}(u_t) \\ &\leq c_{\mathrm{PO}} - \frac{\mathfrak{g}(t)}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} \\ &\leq c_{\mathrm{PO}} - \frac{\delta_2}{2(N+\alpha)}[u]^2_{H^s(\mathbb{R}^N)} \quad \text{for all } t > 0 \text{ such that } |t-1| \geq \delta_1.\end{aligned} \tag{5.13}$$

Taking (5.12) and (5.13) into account, we deduce

$$\max_{t \in [t_1, t_2]} \mathcal{J}(\eta(1, u_t)) < c_{\mathrm{PO}}. \tag{5.14}$$

Put $\Upsilon(t) := P(\eta(1, u_t))$ for all $t > 0$. From (5.10) and the definition of ε, we derive that $\mathcal{J}(u_t) < c_{\mathrm{PO}} - 2\varepsilon$ for all $t \in \{t_1, t_2\}$, which, together with (i), ensures that $\eta(1, u_t) = u_t$ for all $t \in \{t_1, t_2\}$. This fact, combined with (5.11), yields

$$\Upsilon(t_1) = P(u_{t_1}) > 0 > P(u_{t_2}) = \Upsilon(t_2).$$

By the continuity of Υ in $(0, +\infty)$, there exists $\tau \in [t_1, t_2]$ such that $\Upsilon(\tau) = P(\eta(1, u_\tau)) = 0$. Thus, $\mathcal{J}(\eta(1, u_\tau)) \geq c_{\mathrm{PO}}$, which contradicts (5.14). □

Hereafter, we show that every least energy Pohožaev minimizer of $\mathcal{J}$ has constant sign.

Proposition 9 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Moreover, we assume that f is odd in $\mathbb{R}$ and has constant sign in $(0, +\infty)$. Then, every least energy Pohožaev minimizer of $\mathcal{J}$ has constant sign.*

Proof Take $u \in \mathcal{P}$ with $\mathcal{J}(u) = c_{\mathrm{PO}}$. From (5.5), we see

$$\mathcal{J}(u_t) \leq \mathcal{J}(u) \quad \text{for all } t > 0. \tag{5.15}$$

Since $u \in \mathcal{P} \subset \Lambda$ and F is even, it follows that

$$\mathscr{F}(|u|) = \mathscr{F}(u) > 0, \tag{5.16}$$

which implies that $|u| \in \Lambda$. In light of Lemma 14, we can find a unique $t_{|u|} > 0$ such that $w := |u|_{t_{|u|}} \in \mathcal{P}$. By the definition of c_{PO} in (1.10), we have

$$c_{\text{PO}} = \mathcal{J}(u) \leq \mathcal{J}(w). \tag{5.17}$$

From (5.16) and $[|u|]_{H^s(\mathbb{R}^N)} \leq [u]_{H^s(\mathbb{R}^N)}$, we derive

$$\begin{aligned}\mathcal{J}(w) &= \frac{1}{2}\left(t_{|u|}\right)^{N-2s}[|u|]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\left(t_{|u|}\right)^N \|u\|^2_{L^2(\mathbb{R}^N)} - \frac{\left(t_{|u|}\right)^{N+\alpha}}{2}\mathscr{F}(|u|) \\ &\leq \frac{1}{2}\left(t_{|u|}\right)^{N-2s}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\left(t_{|u|}\right)^N \|u\|^2_{L^2(\mathbb{R}^N)} - \frac{\left(t_{|u|}\right)^{N+\alpha}}{2}\mathscr{F}(u) \\ &= \mathcal{J}(u_{t_{|u|}}).\end{aligned} \tag{5.18}$$

Combining (5.15), (5.17), and (5.18), we obtain

$$c_{\text{PO}} = \mathcal{J}(u) \leq \mathcal{J}(w) \leq \mathcal{J}(u_{t_{|u|}}) \leq \mathcal{J}(u),$$

which implies $\mathcal{J}(w) = c_{\text{PO}}$. Therefore, $w \in \mathcal{P}$ and $\mathcal{J}(w) = c_{\text{PO}}$, that is, w is a least energy Pohožaev minimizer for $\mathcal{J}$. By Lemma 15, we deduce that w is a nontrivial nonnegative weak solution to (1.1). Applying Theorem 1, we infer that $w \in H^{2s}(\mathbb{R}^N) \cap C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon \in (0, 1 + [2s] - 2s)$ and that w is a classical solution to

$$(-\Delta)^s w + \omega w = (I_\alpha * F(w)) f(w) \quad \text{in } \mathbb{R}^N.$$

Considering that $w \geq 0$ in $\mathbb{R}^N$, $w \not\equiv 0$, and $f(0) = 0$, we can apply the strong maximum principle [6, Theorem 3.2] to conclude that $w > 0$ in $\mathbb{R}^N$. Since u is continuous, we obtain that u has constant sign. □

In order to study the symmetry of least energy solutions to (1.1), we recall some useful results on the symmetric-decreasing rearrangement; see [28, Chapter 3] for more details. Let $E \subset \mathbb{R}^N$ be a Borel set such that $|E| < +\infty$. The symmetric-decreasing rearrangement of E is defined as $E^* := B_r(0)$, where $r := (|E|/|B_1(0)|)^{1/N}$. Set $\chi^*_E := \chi_{E^*}$. Let $u : \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable function such that $|\{|u| > t\}| < +\infty$ for all $t > 0$. The symmetric-decreasing rearrangement of u is given by

$$u^*(x) := \int_0^{+\infty} \chi^*_{\{|u|>t\}}(x)\, dt \quad \text{for } x \in \mathbb{R}^N.$$

Observe that u^* is nonnegative, radially symmetric, nonincreasing, and lower semicontinuous. Furthermore, $|\{u^* > t\}| = |\{|u| > t\}|$ for all $t > 0$.

Proposition 10 *[28, Chapter 3] Let $N \geq 2$, $p \in [1, +\infty)$, and $u \in L^p(\mathbb{R}^N)$ be a nonnegative function. Then, $u^* \in L^p(\mathbb{R}^N)$ and*

$$\|u^*\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)}.$$

Proposition 11 *(Pólya-Szegö type inequality) [21, Theorem A.1] Let $s \in (0, 1)$ and $N \geq 2$. Let $u \in H^s(\mathbb{R}^N)$ be a nonnegative function. Then,*

$$[u^*]_{H^s(\mathbb{R}^N)} \leq [u]_{H^s(\mathbb{R}^N)}. \tag{5.19}$$

The inequality in (5.19) is strict unless u is a translate of a radially symmetric decreasing function.

Proposition 12 *(Riesz's rearrangement inequality) [28, Theorems 3.7 and 3.9] Let f_1, f_2, and f_3 be three nonnegative measurable functions on $\mathbb{R}^N$. Set*

$$\mathcal{I}(f_1, f_2, f_3) := \iint_{\mathbb{R}^{2N}} f_1(x) f_2(x - y) f_3(y)\, dxdy.$$

Then,

$$\mathcal{I}(f_1, f_2, f_3) \leq \mathcal{I}(f_1^*, f_2^*, f_3^*). \tag{5.20}$$

Moreover, if f_2 is radially symmetric and strictly decreasing, then equality in (5.20) holds only if $f_1(x) = f_1^*(x - y)$ and $f_3(x) = f_3^*(x - y)$ for some $y \in \mathbb{R}^N$.

Now, we prove that every least energy Pohožaev minimizer of $\mathcal{J}$ is radially symmetric, up to a translation.

Proposition 13 *Let $N \geq 2$, $s \in (0, 1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $f \in C^0(\mathbb{R})$ be such that $(f1)$-$(f3)$ hold. Assume that one of the conditions $(f4)$, $(f5)$, or $(f6)$ holds. Moreover, we assume that f is odd in $\mathbb{R}$ and has constant sign in $(0, +\infty)$. Then, every least energy Pohožaev minimizer of $\mathcal{J}$ is radially symmetric with respect to some point in $\mathbb{R}^N$ and is radially decreasing.*

Proof Without loss of generality, we suppose that $f \geq 0$ in $(0, +\infty)$. Arguing by contradiction, assume that there exists a least energy Pohožaev minimizer $u \in \mathcal{P}$ for $\mathcal{J}$ which is not radially symmetric up to a translation. In light of Proposition 9, we may assume that $u > 0$. Let u^* be the symmetric rearrangement of u. From Propositions 10, 11, and 12, it follows that

$$\|u^*\|_{L^2(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)}, \ [u^*]_{H^s(\mathbb{R}^N)} < [u]_{H^s(\mathbb{R}^N)} \text{ and } \mathscr{F}(u^*) \geq \mathscr{F}(u), \tag{5.21}$$

which shows that $P(u^*) < 0$. Since $P(u^*_t) > 0$ for $t > 0$ sufficiently small, there exists $\tau_1 > 0$ such that $P(u^*_{\tau_1}) = 0$. Combining (5.5), (5.21), and $\mathcal{J}(u) = c_{\rm PO}$, we deduce

$$\mathcal{J}(u^*_{\tau_1}) = \frac{1}{2}\tau_1^{N-2s}[u^*]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\tau_1^N \|u^*\|^2_{L^2(\mathbb{R}^N)} - \frac{\tau_1^{N+\alpha}}{2}\mathscr{F}(u^*)$$

$$
\begin{aligned}
&< \frac{1}{2}\tau_1^{N-2s}[u]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\tau_1^N\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{\tau_1^{N+\alpha}}{2}\mathscr{F}(u) \\
&= \mathcal{J}(u_{\tau_1}) \\
&\leq \mathcal{J}(u) = c_{\mathrm{PO}},
\end{aligned}
$$

hence $\mathcal{J}(u^*_{\tau_1}) < c_{\mathrm{PO}}$, which contradicts $\mathcal{J}(u^*_{\tau_1}) \geq c_{\mathrm{PO}}$. The proof of Proposition 13 is complete. □

Proof of Theorem 4 The result is a consequence of (1.9), Proposition 9, and Proposition 13. □

6 Fractional Choquard equations with upper critical H-L-S exponent

In this section, we focus on (1.1) when f satisfies $(f1)'$-$(f3)'$. Define $g(t) := f(t) - \mu|t|^{\frac{N+\alpha}{N-2s}-2}t$ and $G(t) := \int_0^t g(\tau)\,d\tau$. Then, g fulfills the following conditions:

$(g1)$ $\lim_{t\to 0} \frac{tg(t)}{|t|^{\frac{N+\alpha}{N}}} = 0$,

$(g2)$ $\lim_{|t|\to +\infty} \frac{tg(t)}{|t|^{\frac{N+\alpha}{N-2s}}} = 0$,

$(g3)$ G satisfies

- $\lim_{t\to +\infty} \frac{G(t)}{t^{\frac{N+\alpha}{N}}} = +\infty$ when $N > 4s$,
- $\lim_{t\to +\infty} \frac{G(t)}{(t^2 \log t^{\frac{1}{s}})^{\frac{\alpha+4s}{8s}}} = +\infty$ when $N = 4s$,
- $\lim_{t\to +\infty} \frac{G(t)}{t^{\frac{2s(N+\alpha)}{N(N-2s)}}} = +\infty$ when $2s < N < 4s$.

For simplicity, we assume that $\mu = 1$. Since $(g1)$-$(g3)$ imply that $(f1)$ and $(f3)$ hold, it is easy to check that $\mathcal{J}$ has a mountain pass geometry (note that $(g3)$ ensures that there exists $t_0 > 0$ such that $F(t_0) \neq 0$). Moreover, the conclusions of Proposition 6, Lemma 10, and Lemma 11 remain valid. In order to prove the existence of a least energy solution to (1.1), we only need to give a suitable variant of Proposition 7. Due to the presence of the upper critical H-L-S exponent, we provide an upper bound for the mountain pass value c_{MP}. For this end, as in [40], we consider $\tilde{v}(x) := \kappa(\mu^2 + |x|^2)^{-\frac{N-2s}{2}}$ with $\kappa \in \mathbb{R}\setminus\{0\}$ and $\mu > 0$, $\bar{v} := \tilde{v}/\|\tilde{v}\|_{L^{2^*_s}(\mathbb{R}^N)}$, and $v^*(x) := \bar{v}(x/S_*^{\frac{1}{2s}})$. Let $v_\varepsilon(x) := \varepsilon^{-\frac{N-2s}{2}} v^*(x/\varepsilon)$. Observe that $[v_\varepsilon]^2_{H^s(\mathbb{R}^N)} = \|v_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} = S_*^{\frac{N}{2s}}$. Set $V_\varepsilon := \varphi v_\varepsilon$, where $\varphi \in C_c^\infty(\mathbb{R}^N)$ is a cut-off function such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi = 1$ in $\overline{B_1}(0)$ and $\varphi = 0$ in $B_2^c(0)$. Then, as $\varepsilon \to 0^+$, we have (see [40, Propositions 21 and 22]):

$$
[V_\varepsilon]^2_{H^s(\mathbb{R}^N)} \leq S_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \tag{6.1}
$$

$$
\|V_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} = S_*^{\frac{N}{2s}} + O(\varepsilon^N), \tag{6.2}
$$

$$\|V_\varepsilon\|^2_{L^2(\mathbb{R}^N)} = \begin{cases} C_s\,\varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if } N > 4s, \\ C_s\,\varepsilon^{2s}\,|\log\varepsilon| + O(\varepsilon^{2s}) & \text{if } N = 4s, \\ C_s\,\varepsilon^{N-2s} + O(\varepsilon^{2s}) & \text{if } 2s < N < 4s, \end{cases} \tag{6.3}$$

for some $C_s > 0$. We also recall that (see [8, Lemma 4.6])

$$\int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{N+\alpha}{N-2s}})|V_\varepsilon|^{\frac{N+\alpha}{N-2s}}\,dxdy \geq A_\alpha C(N,\alpha) S_*^{\frac{N+\alpha}{2s}} - O(\varepsilon^{\frac{N+\alpha}{2}}). \tag{6.4}$$

Let us now define $U_\varepsilon := V_\varepsilon/\|V_\varepsilon\|_{L^{2^*_s}(\mathbb{R}^N)}$. Thus, (6.1), (6.2), and (6.3) yield, as $\varepsilon \to 0^+$,

$$[U_\varepsilon]^2_{H^s(\mathbb{R}^N)} \leq S_* + O(\varepsilon^{N-2s}), \tag{6.5}$$

$$\gamma(\varepsilon) := \|U_\varepsilon\|^2_{L^2(\mathbb{R}^N)} = \begin{cases} O(\varepsilon^{2s}) & \text{if } N > 4s, \\ O(\varepsilon^{2s}(\log(\frac{1}{\varepsilon})) & \text{if } N = 4s, \\ O(\varepsilon^{N-2s}) & \text{if } 2s < N < 4s, \end{cases} \tag{6.6}$$

while (6.2), (6.4), and $\alpha \in (0, N)$ imply

$$\int_{\mathbb{R}^N} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})|U_\varepsilon|^{\frac{N+\alpha}{N-2s}}\,dxdy \geq A_\alpha C(N,\alpha) - O(\varepsilon^{\frac{N+\alpha}{2}}). \tag{6.7}$$

Next, we collect some auxiliary lemmas.

Lemma 16 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0, N)$. Let $g \in C^0(\mathbb{R})$ be such that $(g1)$-$(g3)$ hold. Then, there exists $C_1 > 0$ such that, for all $\varepsilon > 0$ small enough,*

$$-C_1[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}} \leq \int_{\mathbb{R}^N} (I_\alpha * G(U_\varepsilon))G(U_\varepsilon)\,dx.$$

Proof Let $G^+ := \max\{G, 0\}$ and $G^- := \min\{G, 0\}$. Using $(g1)$, $(g2)$, and (6.6), we see that, for all $\varepsilon > 0$ small,

$$\int_{\mathbb{R}^N} |G^\pm(U_\varepsilon)|^{\frac{2N}{N+\alpha}}\,dx \leq \int_{\mathbb{R}^N} |G(U_\varepsilon)|^{\frac{2N}{N+\alpha}}\,dx \leq C_1 \int_{\mathbb{R}^N} (|U_\varepsilon|^2 + |U_\varepsilon|^{2^*_s})\,dx \leq C_2. \tag{6.8}$$

On the other hand, due to $(g1)$ and $(g3)$, we can find $C_2 > 0$ such that

$$G(t) \geq -C_2 t^{\frac{N+\alpha}{N}} \quad \text{for all } t \geq 0. \tag{6.9}$$

In particular, $G^-(t) \geq -C_2 t^{\frac{N+\alpha}{N}}$ for all $t \geq 0$. Employing Proposition 2 and (6.8), we have, for all $\varepsilon > 0$ small,

$$\int_{\mathbb{R}^N} (I_\alpha * G^-(U_\varepsilon))G^+(U_\varepsilon)\,dx$$

$$\geq -C_2 \iint_{\mathbb{R}^{2N}} I_\alpha(x-y)|U_\varepsilon|^{\frac{N+\alpha}{N}} G^+(U_\varepsilon(x))\,dxdy$$
$$\geq -C_3 \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^{\frac{N+\alpha}{N}} \|G^+(U_\varepsilon)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}$$
$$\geq -C_4[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}.$$

Applying Fubini's theorem, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * G(U_\varepsilon))G(U_\varepsilon)\,dx \\
&= \int_{\mathbb{R}^N} (I_\alpha * G^-(U_\varepsilon))G^-(U_\varepsilon)\,dx + \int_{\mathbb{R}^N} (I_\alpha * G^+(U_\varepsilon))G^+(U_\varepsilon)\,dx \\
&\quad + 2\int_{\mathbb{R}^N} (I_\alpha * G^-(U_\varepsilon))G^+(U_\varepsilon)\,dx \\
&\geq 2\int_{\mathbb{R}^N} (I_\alpha * G^-(U_\varepsilon))G^+(U_\varepsilon)\,dx \\
&\geq -2C_4[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}.
\end{aligned}$$

The proof is now complete. □

Lemma 17 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in (0,N)$. Let $g \in C^0(\mathbb{R})$ be such that $(g1)$-$(g3)$ hold. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{\int_{B_\varepsilon(0)} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\,dx}{[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}} = +\infty. \tag{6.10}$$

Moreover, there exists $C_2 > 0$ such that, for all $\varepsilon > 0$ small enough,

$$\int_{B_\varepsilon^c(0)} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\,dx \geq -C_2[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}.$$

Proof From (6.2), there exists $\varepsilon_0 > 0$ such that $\|V_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)} \leq K$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence, if $\varepsilon \in (0, \min\{1, \varepsilon_0\})$, we obtain

$$U_\varepsilon(x) \geq \frac{1}{K} V_\varepsilon(x) = \frac{1}{K} v_\varepsilon(x) \geq C_0\,\varepsilon^{-\frac{N-2s}{2}} \quad \text{for all } |x| < \varepsilon, \tag{6.11}$$

for some $C_0 > 0$. In view of $(g3)$, for every fixed $R > 0$ there exists $C_R > 0$ such that, for all $t \geq C_R$,

$$G(t) \geq \begin{cases} Rt^{\frac{N+\alpha}{N}} & \text{if } N > 4s, \\ R(t^2 \log t^{\frac{1}{s}})^{\frac{\alpha+4s}{8s}} & \text{if } N = 4s, \\ Rt^{\frac{2s(N+\alpha)}{N(N-2s)}} & \text{if } 2s < N < 4s. \end{cases} \tag{6.12}$$

Take $\varepsilon_1 > 0$ such that $C_0\,\varepsilon_1^{-\frac{N-2s}{2}} > C_R$. Then, for all $\varepsilon \in (0, \min\{1, \varepsilon_0, \varepsilon_1\})$, we have $U_\varepsilon(x) \geq C_R$ for all $|x| < \varepsilon$. Let $\varepsilon_2 > 0$ be such that $C_0\,\varepsilon_2^{-\frac{N-2s}{2}} > 1$, and choose $\varepsilon_3 \in (0, \min\{C_0^{\frac{1}{s}}, 1\})$ such that $\log(C_0\,\varepsilon_3^{-s}) \geq \frac{s}{2}\log\varepsilon_3^{-1}$. Hence, for all $\varepsilon \in (0, \min\{1, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\})$, (6.11) and (6.12) ensure that the following estimates hold: When $N > 4s$, we derive

$$\begin{aligned}
&\int_{B_\varepsilon(0)} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\,dx \\
&\geq C_1 R\,\varepsilon^{-\frac{N+\alpha}{2}}\,\varepsilon^{-\frac{(N-2s)(N+\alpha)}{2N}} \int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)} \frac{1}{|x-y|^{N-\alpha}}\,dxdy \\
&= C_1 R\,\varepsilon^{-\frac{N+\alpha}{2}}\,\varepsilon^{-\frac{(N-2s)(N+\alpha)}{2N}}\,\varepsilon^{N+\alpha} \int_{B_1(0)}\int_{B_1(0)} \frac{1}{|x-y|^{N-\alpha}}\,dxdy \\
&\geq C_2 R\,\varepsilon^{\frac{(N+\alpha)s}{N}};
\end{aligned}$$

When $N = 4s$, we find

$$\begin{aligned}
&\int_{B_\varepsilon(0)} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\,dx \\
&\geq C_3 R\,\varepsilon^{-\frac{4s+\alpha}{2}}\,(\varepsilon^{-2s}\log\varepsilon^{-1})^{\frac{\alpha+4s}{8s}} \int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)} \frac{1}{|x-y|^{4s-\alpha}}\,dxdy \\
&= C_3 R\,\varepsilon^{-\frac{4s+\alpha}{2}}\,(\varepsilon^{-2s}\log\varepsilon^{-1})^{\frac{\alpha+4s}{8s}}\,\varepsilon^{4s+\alpha} \int_{B_1(0)}\int_{B_1(0)} \frac{1}{|x-y|^{4s-\alpha}}\,dxdy \\
&\geq C_4 R(\varepsilon^{2s}\log\varepsilon^{-1})^{\frac{\alpha+4s}{8s}};
\end{aligned}$$

When $N < 4s$, we obtain

$$\begin{aligned}
&\int_{B_\varepsilon(0)} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\,dx \\
&\geq C_5 R\,\varepsilon^{-\frac{N+\alpha}{2}}\,\varepsilon^{-\frac{s(N+\alpha)}{N}} \int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)} \frac{1}{|x-y|^{N-\alpha}}\,dxdy \\
&= C_5 R\,\varepsilon^{-\frac{N+\alpha}{2}}\,\varepsilon^{-\frac{s(N+\alpha)}{N}}\,\varepsilon^{N+\alpha} \int_{B_1(0)}\int_{B_1(0)} \frac{1}{|x-y|^{N-\alpha}}\,dxdy \\
&\geq C_6 R\,\varepsilon^{\frac{(N+\alpha)(N-2s)}{2N}}.
\end{aligned}$$

Combining the above estimates with (6.5), and since $R > 0$ was arbitrary, we arrive at (6.10). Finally, (6.9), Proposition 2, and $\|U_\varepsilon\|_{L^{2^*_s}(\mathbb{R}^N)} = 1$ yield

$$\begin{aligned}
&\int_{B^c_\varepsilon(0)} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\,dx \\
&\geq \int_{B^c_\varepsilon(0)}\int_{B^c_\varepsilon(0)} I_\alpha(x-y)|U_\varepsilon(y)|^{\frac{N+\alpha}{N-2s}}G(U_\varepsilon(x))\,dx
\end{aligned}$$

$$\geq -C_7 \int_{B_\varepsilon^c(0)} \int_{B_\varepsilon^c(0)} I_\alpha(x-y)|U_\varepsilon(y)|^{\frac{N+\alpha}{N-2s}} |U_\varepsilon(x)|^{\frac{N+\alpha}{N}}\, dx$$
$$\geq -C_8 \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^{\frac{N+\alpha}{N}} \|U_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)}^{\frac{N+\alpha}{N-2s}}$$
$$= -C_8[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}.$$

This finishes the proof. □

Lemma 18 *Let $N \geq 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in ((N-4s)_+, N)$. Let $g \in C^0(\mathbb{R})$ be such that $(g1)$-$(g3)$ hold. Define*

$$c_* := \frac{\alpha + 2s}{2(N+\alpha)} \left(\frac{N+\alpha}{N-2s}\right)^{\frac{N-2s}{\alpha+2s}} S_{H,L}^{\frac{N+\alpha}{\alpha+2s}}.$$

Then, $c_{\mathrm{MP}} \in (0, c_*)$.

Proof Observe that Proposition 2 and (6.6) imply that, for all $\varepsilon > 0$ small enough,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(U_\varepsilon))F(U_\varepsilon)\, dx \right| \leq A_\alpha C(N,\alpha) \|F(U_\varepsilon)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}$$
$$\leq C_1 \left[\|U_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \|U_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} \right]^{\frac{N+\alpha}{N}} \leq C_2.$$

On the other hand, Lemmas 16 and 17 ensure that, for all $\varepsilon > 0$ small enough,

$$\int_{\mathbb{R}^N} (I_\alpha * F(U_\varepsilon))F(U_\varepsilon)\, dx > 0. \tag{6.13}$$

Applying Lemma 14, we can find a unique $t_\varepsilon > 0$ such that $(U_\varepsilon)_{t_\varepsilon} \in \mathcal{P}$. In view of Remark 5, we have that $\mathcal{J}((U_\varepsilon)_{t_\varepsilon}) \geq c_{\mathrm{PO}} \geq c_{\mathrm{MP}}$. Hence,

$$0 < c_{\mathrm{MP}} \leq \mathcal{J}((U_\varepsilon)_{t_\varepsilon}) = \frac{t_\varepsilon^{N-2s}}{2} [U_\varepsilon]^2_{H^s(\mathbb{R}^N)} + \omega \frac{t_\varepsilon^N}{2} \|U_\varepsilon\|^2_{L^2(\mathbb{R}^N)}$$
$$- \frac{t_\varepsilon^{N+\alpha}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(U_\varepsilon))F(U_\varepsilon)\, dx,$$

and using (6.5), (6.6), and (6.13), we deduce that $t_\varepsilon \not\to 0$ and $t_\varepsilon \not\to +\infty$ as $\varepsilon \to 0$. Consequently, there exist $0 < t_1 < t_2 < +\infty$ such that $t_\varepsilon \in [t_1, t_2]$ for all $\varepsilon > 0$ small enough. Combining this fact with (6.5), (6.7), Lemma 16, Lemma 17, and noting that, by Lemma 3, it holds

$$\max_{t \geq 0} \left[\frac{t^{N-2s}}{2} S_* - \frac{t^{N+\alpha}}{2} \left(\frac{N-2s}{N+\alpha}\right)^2 A_\alpha C(N,\alpha) \right] = c_*,$$

we obtain

$$
\begin{aligned}
&\mathcal{J}((U_\varepsilon)_{t_\varepsilon})\\
&= \frac{t_\varepsilon^{N-2s}}{2}[U_\varepsilon]^2_{H^s(\mathbb{R}^N)} - \frac{t_\varepsilon^{N+\alpha}}{2}\left(\frac{N-2s}{N+\alpha}\right)^2 \int_{\mathbb{R}^N} (I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})|U_\varepsilon|^{\frac{N+\alpha}{N-2s}}\, dxdy\\
&+ \omega\frac{t_\varepsilon^N}{2}\|U_\varepsilon\|^2_{L^2(\mathbb{R}^N)} - \frac{t_\varepsilon^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * G(U_\varepsilon))G(U_\varepsilon)\, dx\\
&- t_\varepsilon^{N+\alpha}\left(\frac{N-2s}{N+\alpha}\right)\int_{\mathbb{R}^N}(I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\, dx\\
&\le \frac{t_\varepsilon^{N-2s}}{2}S_* - \frac{t_\varepsilon^{N+\alpha}}{2}\left(\frac{N-2s}{N+\alpha}\right)^2 A_\alpha C(N,\alpha) + O(\varepsilon^{N-2s}) + O(\varepsilon^{\frac{N+\alpha}{2}})\\
&+ \omega\frac{t_2^N}{2}\gamma(\varepsilon) - t_1^{N+\alpha}\left(\frac{N-2s}{N+\alpha}\right)\int_{B_\varepsilon(0)}(I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\, dx + C_8[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}\\
&\le c_* + \zeta(\varepsilon) - t_1^{N+\alpha}\left(\frac{N-2s}{N+\alpha}\right)\int_{B_\varepsilon(0)}(I_\alpha * |U_\varepsilon|^{\frac{N+\alpha}{N-2s}})G(U_\varepsilon)\, dx + C_8[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}},
\end{aligned}
$$

where

$$\zeta(\varepsilon) := O(\varepsilon^{N-2s}) + O(\varepsilon^{\frac{N+\alpha}{2}}) + \omega\frac{t_2^N}{2}\gamma(\varepsilon).$$

Now, since $\alpha \in ((N-4s)_+, N)$, it follows from (6.6) that, as $\varepsilon \to 0^+$,

$$\zeta(\varepsilon) = \begin{cases} O(\varepsilon^{2s}) & \text{if } N > 4s,\\ O(\varepsilon^{2s}(\log(\frac{1}{\varepsilon}) + 1)) & \text{if } N = 4s,\\ O(\varepsilon^{N-2s}) & \text{if } 2s < N < 4s. \end{cases}$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \frac{\zeta(\varepsilon)}{[\gamma(\varepsilon)]^{\frac{N+\alpha}{2N}}} = 0.$$

Using this fact and Lemma 17, we conclude that, for $\varepsilon > 0$ small enough, $\mathcal{J}((U_\varepsilon)_{t_\varepsilon}) < c_*$. As a result, $0 < c_{\mathrm{MP}} < c_*$. The proof is now complete. □

Next, we provide an analogue of Proposition 7 in the critical case.

Proposition 14 *Let $N \ge 2$, $s \in (0,1)$, $\omega > 0$, and $\alpha \in ((N-4s)_+, N)$. Let $g \in C^0(\mathbb{R})$ be such that (g1)-(g3) hold. Let $(u_n) \subset H^s(\mathbb{R}^N)$ be a sequence such that*

(i) $\mathcal{J}(u_n) \to c < c_*$,
(ii) $\mathcal{J}'(u_n) \to 0$ in $H^{-s}(\mathbb{R}^N)$ and $P(u_n) \to 0$ as $n \to +\infty$.

Then, up to a subsequence, one of the following alternatives holds:

(1) $u_n \to 0$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$;
(2) there exist $u \in H^s(\mathbb{R}^N)\setminus\{0\}$ and a sequence $(x_n) \subset \mathbb{R}^N$ such that $\mathcal{J}'(u) = 0$ and $u_n(\cdot - x_n) \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$.

Proof According to Lemma 10, (u_n) is bounded in $H^s(\mathbb{R}^N)$. Suppose that (1) is not valid, namely,

$$\liminf_{n\to+\infty} \|u_n\|_{H^s(\mathbb{R}^N)} > 0. \tag{6.14}$$

We start by showing that

$$\liminf_{n\to+\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^2(B_1(x_0))} > 0. \tag{6.15}$$

Arguing by contradiction, we assume that, up to a subsequence,

$$\lim_{n\to+\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^2(B_1(x_0))} = 0.$$

From Lemma 2, it follows that

$$u_n \to 0 \text{ in } L^q(\mathbb{R}^N) \quad \text{for all } q \in (2, 2_s^*). \tag{6.16}$$

Utilizing $\mathcal{J}(u_n) = c + o_n(1)$ as $n \to +\infty$, we know that

$$\begin{aligned}
&\frac{1}{2}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u_n))G(u_n)\,dx \\
&\quad - \frac{1}{2}\left(\frac{N-2s}{N+\alpha}\right)\int_{\mathbb{R}^N} (I_\alpha * G(u_n))|u_n|^{\frac{N+\alpha}{N-2s}}\,dx \\
&\quad - \frac{1}{2}\left(\frac{N-2s}{N+\alpha}\right)\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})G(u_n)\,dx \\
&\quad - \frac{1}{2}\left(\frac{N-2s}{N+\alpha}\right)^2\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})|u_n|^{\frac{N+\alpha}{N-2s}}\,dx = c + o_n(1),
\end{aligned} \tag{6.17}$$

while $\langle \mathcal{J}'(u_n), u_n\rangle = o_n(1)$ as $n \to +\infty$ yields

$$\begin{aligned}
&[u_n]^2_{H^s(\mathbb{R}^N)} + \omega\|u_n\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (I_\alpha * G(u_n))g(u_n)u_n\,dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * G(u_n))|u_n|^{\frac{N+\alpha}{N-2s}}\,dx - \left(\frac{N-2s}{N+\alpha}\right)\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})g(u_n)u_n\,dx \\
&\quad - \left(\frac{N-2s}{N+\alpha}\right)\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})|u_n|^{\frac{N+\alpha}{N-2s}}\,dx = o_n(1).
\end{aligned} \tag{6.18}$$

By virtue of $(g1)$ and $(g2)$, for each fixed $\varepsilon > 0$ and $r \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2s})$, there exists $C_{\varepsilon,r} > 0$ such that

$$|g(t)t|^{\frac{2N}{N+\alpha}}, |G(t)|^{\frac{2N}{N+\alpha}} \le \varepsilon(|t|^2 + |t|^{2_s^*}) + C_{\varepsilon,r}|t|^{\frac{2Nr}{N+\alpha}} \quad \text{for all } t \in \mathbb{R}. \tag{6.19}$$

Thus, using (6.19), the boundedness of (u_n) in $H^s(\mathbb{R}^N)$, and the fractional Sobolev embeddings from Proposition 1, we see

$$\|g(u_n)u_n\|^{\frac{2N}{N+\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}, \|G(u_n)\|^{\frac{2N}{N+\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C'\,\varepsilon + C_{\varepsilon,r}\|u_n\|^{\frac{2Nr}{N+\alpha}}_{L^{\frac{2Nr}{N+\alpha}}(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N}.$$

Noting that $\frac{2Nr}{N+\alpha} \in (2, 2^*_s)$, it follows from (6.16) that $u_n \to 0$ in $L^{\frac{2Nr}{N+\alpha}}(\mathbb{R}^N)$, which implies

$$\limsup_{n\to+\infty} \|g(u_n)u_n\|^{\frac{2N}{N+\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C'\,\varepsilon,$$

and

$$\limsup_{n\to+\infty} \|G(u_n)\|^{\frac{2N}{N+\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C'\,\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, we deduce

$$\lim_{n\to+\infty} \|g(u_n)u_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = \lim_{n\to+\infty} \|G(u_n)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = 0. \tag{6.20}$$

Now, applying Proposition 2, we have

$$\left|\int_{\mathbb{R}^N} (I_\alpha * G(u_n))G(u_n)\,dx\right| \leq A_\alpha C(N,\alpha)\|G(u_n)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N},$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} (I_\alpha * G(u_n))g(u_n)u_n\,dx\right| \\ &\leq A_\alpha C(N,\alpha)\|G(u_n)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\|g(u_n)u_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

which, together with (6.20), leads to

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * G(u_n))G(u_n)\,dx = 0 = \lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * G(u_n))g(u_n)u_n\,dx. \tag{6.21}$$

Employing (6.20), Proposition 2, and the boundedness of (u_n) in $L^{2^*_s}(\mathbb{R}^N)$, we also see

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * G(u_n))|u_n|^{\frac{N+\alpha}{N-2s}}\,dx = 0 = \lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})G(u_n)\,dx. \tag{6.22}$$

Combining (6.17), (6.18), (6.21), and (6.22), we obtain

$$\frac{1}{2}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{\omega}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\left(\frac{N-2s}{N+\alpha}\right)^2 \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})|u_n|^{\frac{N+\alpha}{N-2s}}\, dx$$
$$= c + o_n(1), \tag{6.23}$$

and

$$[u_n]^2_{H^s(\mathbb{R}^N)} + \omega\|u_n\|^2_{L^2(\mathbb{R}^N)} = \left(\frac{N-2s}{N+\alpha}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})|u_n|^{\frac{N+\alpha}{N-2s}}\, dx + o_n(1). \tag{6.24}$$

Let $\ell \in (0, +\infty)$ be such that

$$\begin{aligned}\ell &:= \lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} + \omega\|u_n\|^2_{L^2(\mathbb{R}^N)} \\ &= \lim_{n\to+\infty} \left(\frac{N-2s}{N+\alpha}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})|u_n|^{\frac{N+\alpha}{N-2s}}\, dx.\end{aligned}$$

By the definition of $S_{H,L}$, we have

$$\begin{aligned}& S_{H,L}\left(\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2s}})|u_n|^{\frac{N+\alpha}{N-2s}}\, dx\right)^{\frac{N-2s}{N+\alpha}} \\ &\le [u_n]^2_{H^s(\mathbb{R}^N)} \le [u_n]^2_{H^s(\mathbb{R}^N)} + \omega\|u_n\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

which, along with (6.23), implies

$$\ell \ge \left(\frac{N+\alpha}{N-2s}\right)^{\frac{N-2s}{\alpha+2s}} S_{H,L}^{\frac{N+\alpha}{\alpha+2s}}.$$

Since (6.24) gives

$$\frac{1}{2}\left(\frac{\alpha+2s}{N+\alpha}\right)\ell = c,$$

we deduce that

$$c \ge \frac{1}{2}\left(\frac{\alpha+2s}{N+\alpha}\right)\left(\frac{N+\alpha}{N-2s}\right)^{\frac{N-2s}{\alpha+2s}} S_{H,L}^{\frac{N+\alpha}{\alpha+2s}} = c_*,$$

which is a contradiction. As a result, (6.15) is true. Therefore, up to a translation, we may assume that

$$\liminf_{n\to+\infty} \|u_n\|_{L^2(B_1(0))} > 0. \tag{6.25}$$

Because (u_n) is bounded in $H^s(\mathbb{R}^N)$, we may assume that there exists $u \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N),\\ &u_n \to u \quad \text{in } L^q_{loc}(\mathbb{R}^N) \text{ for all } q \in [1, 2^*_s),\\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{6.26}$$

Thanks to (6.25) and (6.26), we see that $u \not\equiv 0$. Finally, we check that $\mathcal{J}'(u) = 0$. Utilizing (2.4) and the boundedness of (u_n) in $L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$, we know that $(F(u_n))$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Since F is continuous and $u_n \to u$ a.e. in $\mathbb{R}^N$, we have that $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^N$. Thus, applying [45, Proposition 5.4.7], we conclude that

$$F(u_n) \rightharpoonup F(u) \ \text{ in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \tag{6.27}$$

Combining (6.27) with Proposition 2, we find

$$I_\alpha * F(u_n) \rightharpoonup I_\alpha * F(u) \ \text{ in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N), \tag{6.28}$$

which yields

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u)\varphi \, dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u) f(u)\varphi \, dx. \tag{6.29}$$

On the other hand, by Proposition 2, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n))[f(u_n) - f(u)]\varphi \, dx \right| \\ &\le C_2 \|[f(u_n) - f(u)]\varphi\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{6.30}$$

Note that $2^*_s(\frac{\alpha+2s}{N+\alpha}) \in (1, 2^*_s)$ due to $\alpha \in ((N-4s)_+, N)$. Then, (6.26) ensures that $u_n \to u$ in $L^{2^*_s(\frac{\alpha+2s}{N+\alpha})}(\mathrm{supp}(\varphi))$. Thus, up to a subsequence, there exists $w \in L^{2^*_s(\frac{\alpha+2s}{N+\alpha})}(\mathrm{supp}(\varphi))$ such that $|u_n| \le w$ a.e. in $\mathrm{supp}(\varphi)$. Considering that

$$|f(t)|^{\frac{2N}{N+\alpha}} \le C_1(1 + |t|^{2^*_s(\frac{\alpha+2s}{N+\alpha})}) \quad \text{for all } t \in \mathbb{R},$$

we see

$$\begin{aligned} &|[f(u_n) - f(u)]\varphi|^{\frac{2N}{N+\alpha}} \\ &\le C_2(1 + |u_n|^{2^*_s(\frac{\alpha+2s}{N+\alpha})} + |u|^{2^*_s(\frac{\alpha+2s}{N+\alpha})})|\varphi|^{\frac{2N}{N+\alpha}} \\ &\le C_2|\varphi|^{\frac{2N}{N+\alpha}} + C_2(|w|^{2^*_s(\frac{\alpha+2s}{N+\alpha})} + |u|^{2^*_s(\frac{\alpha+2s}{N+\alpha})})|\varphi|^{\frac{2N}{N+\alpha}} \in L^1(\mathrm{supp}(\varphi)). \end{aligned}$$

Since $f(u_n) \to f(u)$ a.e. in $\mathbb{R}^N$, it follows from the dominated convergence theorem that

$$\int_{\mathbb{R}^N} |[f(u_n) - f(u)]\varphi|^{\frac{2N}{N+\alpha}}\,dx \to 0 \quad \text{as } n \to +\infty,$$

which, combined with (6.30), implies

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))[f(u_n) - f(u)]\varphi\,dx \to 0. \tag{6.31}$$

In light of (6.29) and (6.31), we arrive at

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)\varphi\,dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u) f(u)\varphi\,dx. \tag{6.32}$$

Utilizing $\mathcal{J}'(u_n) \to 0$ in $H^{-s}(\mathbb{R}^N)$, (6.26), and (6.32), we obtain that $\langle \mathcal{J}'(u), \varphi\rangle = 0$ for every $\varphi \in C_c^\infty(\mathbb{R}^N)$. Because $C_c^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$ (see Proposition 1), we conclude that u is a weak solution to (1.1). □

Proof of Theorem 5 It suffices to proceed as in the proof of Theorem 3 replacing Proposition 7 by Proposition 14. If, in addition, we assume that g is odd in $\mathbb{R}$ and that $g(t) + \mu t^{\frac{\alpha+2s}{N-2s}} \geq 0$ for all $t \in (0, +\infty)$, then we can argue as in the proof of Theorem 4 to deduce the sign and symmetry of least energy solutions to (1.1). □

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Declarations

Conflicts of Interest The author declares that he has no conflict of interest.

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