Superposing plane strain on anti-plane shear deformations in a special class of fiber-reinforced incompressible hyperelastic materials

Superposing Plane Strain on Anti-Plane Shear Deformations in a Special Class of Fiber-reinforced Incompressible Hyperelastic Materials

Marco Coco[a,1,*], Giuseppe Saccomandi[b,c]

[a] *Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, 60131, Ancona, Italy*

[b] *Dipartimento di Ingegneria, Università degli studi di Perugia, 06125 Perugia, Italy*

[c] *School of Mathematics, Statistics and Applied Mathematics, NUI Galway, University Road Galway, Ireland*

Abstract

The purpose of the present research is to investigate how the nonlinearity and the boundary conditions have the power to influence the coupling of the various modes of deformation when a plane strain deformation is superimposed on anti-plane shear deformation for a class of fiber-reinforced incompressible hyperelastic materials described by a strain energy density $\mathcal{W}$. Attention is confined when $\mathcal{W}$ depends only on the first invariant of the strain tensor and on the square of the stretch in the direction of the fibers.

We are able to write down the governing equations of equilibrium as a coupled system of three nonlinear partial differential equations for three displacement fields. Two displacements are the in-plane components and

* Corresponding author

Email address: m.coco@univpm.it (Marco Coco)

[1] tel: +39 071 220 4785, fax: +39 071 220 4771

one displacement is the anti-plane state. The system that we are able to deduce in a compact form is always compatible at variance with the case in which the anti-plane shear problem is analyzed. As explicit example of our findings we study the problem of the helical shear and we investigate into details the coupling of its axial and azimuthal components.
Keywords: fiber-reinforced materials, anti-plane shear deformations, axial shear, azimuthal shear.

1. Introduction

Anti-plane shear involves deformations of a cylindrical region in such a way that the displacement of each particle is parallel to the axial direction and independent of its axial coordinate. In this case, for linear isotropic elasticity, the full three-dimensional equations can be reduced to a single two-dimensional second order equation (the *axial equilibrium equation*) for a single scalar unknown, usually denoted as the *out-of-plane* displacement [1].

For anisotropic linear materials and in nonlinear materials the situation is different because in general the full balance equations reduce to two *in-plane* equilibrium equations and to the axial equation which are an overdetermined system in a single unknown. This latter is compatible only for certain restrictions on the constitutive laws or for certain classes of out-of-plane displacements. An overview on anti-plane deformations is provided by [7] and the solution of the compatibility problem for nonlinear isotropic and incompressible elastic material is summarized in [10]. The same problem for anisotropic materials in nonlinear elasticity has first been considered in [12].

On the other hand, when anti-plane shear deformation is coupled with in-plane deformation field, the equations of equilibrium are always reduced to a system of determined equations and therefore it is not necessary to face a compatibility problem.

In [8] it has been investigated the possibility of superimposing a plain strain deformation on an anti-plane shear deformation for a generalized neo-Hookean material. For this class of incompressible and isotropic elastic materials the strain-energy density is a function $\mathcal{W} = \mathcal{W}(I_1)$ where I_1 is the first principal invariant of the left Cauchy-Green strain tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, $\boldsymbol{F}$ being the gradient of deformation. In [8] the authors derive a compact format for the equations and they use this format to investigate the nature of the coupling between the two aforementioned modes of deformation.

The results of [8] have been fundamental to understand in a clear and rigorous way how the nonlinearity and the boundary conditions have the power to influence the coupling of the various modes of deformation. On the basis of the ideas contained in [8] a non trivial example of remarkable coupling in elasticity has been given in [2] where in the attempt to describe cork-pulling the authors investigate the appearance of secondary or latent deformations for special classes of elastic materials. Moreover, in [3] and [11] we find an application of the results of [8] in elastodynamics. The rigorous derivation of the Zabolotskaya model equation widely used in nonlinear acoustics is based on the extension of the results in [8] to elastodynamics.

If we consider anisotropic elastic materials all the remarks and considerations developed in [8] for the isotropic case are even more relevant. The paper devoted to linear elasticity [5] is a clear example of this situation.

In the light of this context the aim of the present paper is to investigate the same problem considered in [8] in the framework of a special class of fiber-reinforced materials, with transversely isotropic fibers oriented along the unit vector $\boldsymbol{M}$ representing the preferred direction of the material in the reference configuration. By considering the invariant $I_4 = \boldsymbol{FM} \cdot \boldsymbol{FM}$ we investigate a material whose strain-energy density is given by $\mathcal{W} = \mathcal{W}(I_1, I_4)$. We are able to write the balance equations also in this case in a compact form, which allows us to show that the decoupling of the two modes of deformation is rare and possible only in very specific situations, when anisotropic materials are considered.

The plan of the paper is the following. In the next Section we write down the basic equations we need for our investigation. In Section 3 we derive the determining equations for the in-plane and out-of-plane deformation field in a compact format similar to the one introduced in [8]. In Section 4 we give a notable application of our results to the problem of helical shear. By considering the solution of this problem we show that for a generic arrangement of the fibers the two mode of deformations (in-plane and anti-plane) are always coupled and it is never possible to reduce the solution of the given boundary value problem to a simple axial shear deformation. Section 5 is devoted to conclusions.

2. Basic Equations

Let us consider the following class of deformations

$$x_1 = X_1 + u(X_1, X_2),$$

$$x_2 = X_2 + v(X_1, X_2), \tag{2.1}$$

$$x_3 = X_3 + w(X_1, X_2),$$

where $u(X_1, X_2)$ and $v(X_1, X_2)$ are smooth functions denoted as the *in-plane* displacements, and $w(X_1, X_2)$ is the smooth function of the anti-plane deformation, in the following also named as *out-of-plane* displacement. These functions are defined on the plane cross-section $\mathcal{D}$ of a cylinder and they all have to be determined by the balance equations. (X_1, X_2, X_3) and (x_1, x_2, x_3) are the coordinates in the reference and actual configuration, respectively.

Denoted with $\boldsymbol{F}$ the gradient of deformation, the kinematical quantities of interest associated with (2.1) are

$$[\boldsymbol{F}] = \begin{bmatrix} 1 + \partial_{X_1} u & \partial_{X_2} u & 0 \\ \partial_{X_1} v & 1 + \partial_{X_2} v & 0 \\ \partial_{X_1} w & \partial_{X_2} w & 1 \end{bmatrix};$$

since we are interested in incompressible materials, we require that $\det \boldsymbol{F} = 1$ i.e.

$$\partial_{X_1} u + \partial_{X_2} v + \partial_{X_1} u \partial_{X_2} v - \partial_{X_1} v \partial_{X_2} u = 0, \tag{2.2}$$

and therefore

$$[\boldsymbol{F}^{-T}] = \begin{bmatrix} 1+\partial_{X_2}v & -\partial_{X_1}v & (\partial_{X_1}v\partial_{X_2}w - \partial_{X_2}v\partial_{X_1}w - \partial_{X_1}w) \\ -\partial_{X_2}u & 1+\partial_{X_1}u & (\partial_{X_2}u\partial_{X_1}w - \partial_{X_1}u\partial_{X_2}w - \partial_{X_2}w) \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, we compute the first principal invariant of $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, i.e. $I_1 = \operatorname{tr}\boldsymbol{B}$,

$$I_1 = (1+\partial_{X_1}u)^2 + \partial_{X_2}u^2 + \partial_{X_1}v^2 + (1+\partial_{X_2}v)^2 + \partial_{X_1}w^2 + \partial_{X_2}w^2 + 1.$$

Given the generic preferred direction in the reference configuration as

$$\boldsymbol{M} = M_1\boldsymbol{E}_1 + M_2\boldsymbol{E}_2 + M_3\boldsymbol{E}_3,$$

other quantities of interest are

$$\begin{aligned}[\boldsymbol{F}\boldsymbol{M}] &= ((1+\partial_{X_1}u)M_1 + \partial_{X_2}uM_2)\,\boldsymbol{E}_1 + (\partial_{X_1}vM_1 + (1+\partial_{X_2}v)M_2)\,\boldsymbol{E}_2 + \\ &\quad (\partial_{X_1}wM_1 + \partial_{X_2}wM_2 + M_3)\,\boldsymbol{E}_3, \end{aligned} \tag{2.3}$$

and $I_4 = \boldsymbol{F}\boldsymbol{M}\cdot\boldsymbol{F}\boldsymbol{M}$.

When the strain-energy density is defined as $\mathcal{W} = \mathcal{W}(I_1, I_4)$ the representation formula for the Cauchy stress is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mathcal{W}_1\boldsymbol{B} + 2\mathcal{W}_4(\boldsymbol{F}\boldsymbol{M}\otimes\boldsymbol{F}\boldsymbol{M}), \tag{2.4}$$

where the subscripts attached to $\mathcal{W}$ denote partial differentiation with respect the corresponding invariant and $p = p(X_1, X_2, X_3)$ is the unknown Lagrangian multiplier associated with the constraint of isochoricity (2.2).

It is convenient to use the Piola-Kirchhoff stress tensor $\boldsymbol{S} \equiv \boldsymbol{T}\boldsymbol{F}^{-T}$. We split $\boldsymbol{S}$ in two parts: the reaction stress

$$\boldsymbol{S}^r = -p\boldsymbol{F}^{-T},$$

and the extra stress

$$\boldsymbol{S}^{ex} = 2\mathcal{W}_1\boldsymbol{F} + 2\mathcal{W}_4(\boldsymbol{FM} \otimes \boldsymbol{FM})\boldsymbol{F}^{-T}.$$

The balance equations in absence of body forces are

$$\mathrm{Div}\boldsymbol{S} \equiv \mathrm{Div}\boldsymbol{S}^r + \mathrm{Div}\boldsymbol{S}^{ex} = \boldsymbol{0}.$$

To write down the components of these equations when (2.4) is in force and for the deformation field in (2.1) we need to take into account that the functions u, v and w depends only on X_1 and X_2 and therefore the only components of the extra stress tensor we need are

$$\begin{aligned}
S_{11}^{ex} &= 2\mathcal{W}_1(1+\partial_{X_1}u) + 2\mathcal{W}_4(\partial_{X_1}uM_1 + \partial_{X_2}uM_2 + M_1)M_1,\\
S_{12}^{ex} &= 2\mathcal{W}_1\partial_{X_2}u + 2\mathcal{W}_4(\partial_{X_1}uM_1 + \partial_{X_2}uM_2 + M_1)M_2,\\
S_{21}^{ex} &= 2\mathcal{W}_1\partial_{X_1}v + 2\mathcal{W}_4(\partial_{X_1}vM_1 + \partial_{X_2}vM_2 + M_2)M_1,\\
S_{22}^{ex} &= 2\mathcal{W}_1(1+\partial_{X_2}v) + 2\mathcal{W}_4(\partial_{X_1}vM_1 + \partial_{X_2}vM_2 + M_2)M_2,\\
S_{31}^{ex} &= 2\mathcal{W}_1\partial_{X_1}w + 2\mathcal{W}_4(\partial_{X_1}wM_1 + \partial_{X_2}wM_2 + M_3)M_1,\\
S_{32}^{ex} &= 2\mathcal{W}_1\partial_{X_2}w + 2\mathcal{W}_4(\partial_{X_1}wM_1 + \partial_{X_2}wM_2 + M_3)M_2.
\end{aligned}$$

The scalar version of the balance equations is therefore

$$\begin{aligned}
&\frac{\partial}{\partial X_1}[p(1+\partial_{X_2}v)] - \frac{\partial}{\partial X_2}[p\partial_{X_1}v] + \\
&+\frac{\partial p}{\partial X_3}(\partial_{X_1}v\partial_{X_2}w - \partial_{X_1}w - \partial_{X_2}v\partial_{X_1}w) = \frac{\partial S_{11}^{ex}}{\partial X_1} + \frac{\partial S_{12}^{ex}}{\partial X_2},
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&\frac{\partial}{\partial X_2}[p(1+\partial_{X_1}u)] - \frac{\partial}{\partial X_1}[p\partial_{X_2}u] + \\
&+\frac{\partial p}{\partial X_3}(\partial_{X_2}u\partial_{X_1}w - \partial_{X_2}w - \partial_{X_1}u\partial_{X_2}w) = \frac{\partial S_{21}^{ex}}{\partial X_1} + \frac{\partial S_{22}^{ex}}{\partial X_2},
\end{aligned} \tag{2.6}$$

and

$$\frac{\partial p}{\partial X_3} = \frac{\partial S_{31}^{ex}}{\partial X_1} + \frac{\partial S_{32}^{ex}}{\partial X_2}. \tag{2.7}$$

3. Determining Equations

To determine a set of equations in the unknowns u, v and w we consider the following steps. First from (2.5)-(2.6) we obtain that it must be

$$p(X_1, X_2, X_3) = kX_3 + p_0(X_1, X_2), \tag{3.1}$$

where k is a constant (the gradient of the pressure in the axial coordinate) and $p_0(X_1, X_2)$ is a smooth function on $\mathcal{D}$.

We denote the RHSs of the two equations (2.5)-(2.6) as

$$\mathcal{A} = \frac{\partial S_{11}^{ex}}{\partial X_1} + \frac{\partial S_{12}^{ex}}{\partial X_2}, \quad \mathcal{B} = \frac{\partial S_{21}^{ex}}{\partial X_1} + \frac{\partial S_{22}^{ex}}{\partial X_2},$$

and we introduce the notation

$$\Gamma^{(1)} = \partial_{X_1} v \partial_{X_2} w - \partial_{X_1} w - \partial_{X_2} v \partial_{X_1} w_1,$$
$$\Gamma^{(2)} = \partial_{X_2} u \partial_{X_1} w - \partial_{X_2} w - \partial_{X_1} u \partial_{X_2} w.$$

We substitute (3.1) in the two equations (2.5)-(2.6) and as second step we use the Cramer rule to obtain

$$\begin{aligned} \frac{\partial p_0}{\partial X_1} &= (1 + \partial_{X_1} u)\mathcal{A} + \partial_{X_1} v \mathcal{B} - k[\Gamma^{(1)}(1 + \partial_{X_1} u) + \Gamma^{(2)} \partial_{X_1} v], \\ \frac{\partial p_0}{\partial X_2} &= (\partial_{X_2} u)\mathcal{A} + (1 + \partial_{X_2} v)\mathcal{B} - k[\Gamma^{(1)} \partial_{X_2} u + \Gamma^{(2)}(1 + \partial_{X_2} v)]. \end{aligned} \tag{3.2}$$

Since a simple direct computation allows to check that

$$\Gamma^{(1)}(1+\partial_{X_1}u)+\Gamma^{(2)}\partial_{X_1}v=-\partial_{X_1}w,$$
$$\Gamma^{(1)}\partial_{X_2}u+\Gamma^{(2)}(1+\partial_{X_2}v)=-\partial_{X_2}w,$$

from (3.2) it is possible to eliminate the function $p_0(X_1,X_2)$ obtaining the higher order partial differential equation

$$\frac{\partial}{\partial_{X_2}}[\mathcal{A}(1+\partial_{X_1}u)+\mathcal{B}\partial_{X_1}v]-\frac{\partial}{\partial_{X_1}}[\mathcal{A}\partial_{X_2}u+\mathcal{B}(1+\partial_{X_2}v)]=0. \qquad (3.3)$$

This latter plus the equation

$$\frac{\partial S_{31}^{ex}}{\partial X_1}+\frac{\partial S_{32}^{ex}}{\partial X_2}=k, \qquad (3.4)$$

and the isochoricity (2.2) are a *well determined coupled* system of three nonlinear partial differential equations for u, v and w. These are the determining equations for the deformation field.

3.1. The compatibility problem

If we set $u\equiv v\equiv 0$ the equation (2.2) is satisfied automatically, the principal invariants I_1 and I_4 are defined as

$$I_1=3+\partial_{X_1}w^2+\partial_{X_2}w^2, \quad I_4=M_1^2+M_2^2+(\partial_{X_1}wM_1+\partial_{X_2}wM_2+M_3)^2,$$

and the relevant components of the extra-stress tensor simplify as

$$S_{11}^{ex}=2\mathcal{W}_1+2\mathcal{W}_4M_1^2,$$
$$S_{12}^{ex}=2\mathcal{W}_4M_1M_2,$$
$$S_{21}^{ex}=2\mathcal{W}_4M_1M_2,$$
$$S_{22}^{ex}=2\mathcal{W}_1+2\mathcal{W}_4M_2^2,$$
$$S_{31}^{ex}=2\mathcal{W}_1\partial_{X_1}w+2\mathcal{W}_4(\partial_{X_1}wM_1+\partial_{X_2}wM_2+M_3)M_1,$$
$$S_{32}^{ex}=2\mathcal{W}_1\partial_{X_2}w+2\mathcal{W}_4(\partial_{X_1}wM_1+\partial_{X_2}wM_2+M_3)M_2.$$

Besides, equation (3.3) in this case reduces to

$$\partial_{X_2}\mathcal{A} - \partial_{X_1}\mathcal{B} = 0,$$

which reads

$$2(M_1^2 - M_2^2)\frac{\partial^2 \mathcal{W}_4}{\partial X_1 \partial X_2} + 2\left[\frac{\partial^2 \mathcal{W}_4}{\partial X_2^2} - \frac{\partial^2 \mathcal{W}_4}{\partial X_1^2}\right] M_1 M_2 = 0. \tag{3.5}$$

The system composed by equations (3.4) and (3.5) is now an *overdetermined* system of partial differential equations in the single unknown $w = w(X_1, X_2)$.

The problem of the compatibility of such system of equations is quite complex. A summary about this problem when we consider an overdetermined system of two equations in an unknown function of two variables may be found in [10], in the framework of nonlinear isotropic and incompressible elasticity. The full solution of the actual compatibility problem may be recovered with some cumbersome computation along the same lines.

If we denote equations (3.4), (3.5) and their system as $\mathcal{E}^{(1)} = 0$, $\mathcal{E}^{(2)} = 0$ and $\mathcal{S}$, respectively, following [10] we can say that:

- The system $\mathcal{S}$ is *strain-energy fully compatible* if there is a solution $w = w(X_1, X_2)$ of $\mathcal{E}^{(1)} = 0$ and $\mathcal{E}^{(2)} = 0$ and this for any choice of the strain-energy density function $\mathcal{W} = \mathcal{W}(I_1, I_4)$.

- The system $\mathcal{S}$ is *deformation fully compatible* if there exist a strain-energy density $\mathcal{W} = \mathcal{W}(I_1, I_4)$ such that all the solutions $w = w(X_1, X_2)$ of $\mathcal{E}^{(1)} = 0$ are solutions of $\mathcal{E}^{(2)} = 0$. Because, $\mathcal{E}^{(2)} = 0$ is an equation of higher order with respect $\mathcal{E}^{(1)} = 0$ this means that $\mathcal{E}^{(2)} = 0$ must be a differential consequence of $\mathcal{E}^{(1)} = 0$.

- The system $\mathcal{S}$ is *simply compatible* if there is a specific deformation field solution of this system only for a special choice of a strain-energy density $\mathcal{W}(I_1, I_4)$.

In literature only the deformation fully compatible problem has been considered in [12]. Here we provide an example of simple compatibility to show how special are this kind of solutions. For this purpose we consider as specific strain-energy density the standard model

$$\mathcal{W} = \frac{\mu_0}{2}(I_1 - 3) + \frac{\mu_1}{4}(I_4 - 1)^2, \tag{3.6}$$

where μ_0, μ_1 are stress-like materials parameters, and the out-of-plane deformation in the special form

$$w(X_1, X_2) = \alpha\frac{X_1^2}{2} + \beta\frac{X_2^2}{2} + \gamma X_1 X_2. \tag{3.7}$$

In this case

$$\begin{aligned} I_4 - 1 = &(\alpha M_1 + \gamma M_2)^2 X_1^2 + (\gamma M_1 + \beta M_2)^2 X_2^2 \\ &+ 2(\alpha M_1 + \gamma M_2)(\gamma M_1 + \beta M_2) X_1 X_2 \\ &+ 2[(\alpha M_1 + \gamma M_2) X_1 + (\gamma M_1 + \beta M_2) X_2] M_3 \end{aligned}$$

and therefore (3.5) reduces to

$$\begin{aligned} &(\alpha M_1 + \gamma M_2)(\gamma M_1 + \beta M_2)(M_1^2 - M_2^2) \\ &+ [(\gamma M_1 + \beta M_2)^2 - (\alpha M_1 + \gamma M_2)^2] M_1 M_2 = 0. \end{aligned} \tag{3.8}$$

A direct computation shows that equation (3.4) is solved by choosing $k = \mu_0(\alpha + \beta)$ and

$$\alpha = -\gamma\frac{M_2}{M_1}, \quad \beta = -\gamma\frac{M_1}{M_2},$$

a choice satisfying even (3.8).

This example shows that the possible solutions of the compatibility problem are very specific and of minor mechanical interest.

We do not pursue anymore the problem of the compatibility stressing once again that the system composed by equations (4.1), (4.2) and (2.2) for *any* choice of the strain-energy is well determined and may be used to solve a large class of significant boundary value problems.

In the next Section we will give in the case of a cylindrical domain an example to stress out the interplay of nonlinearity, boundary conditions and fiber direction in the coupling of the in-plane and anti-plane deformations.

4. The Helical Shear

The importance of the results of our investigation is clearly noticed in the framework of the helical shear problem.

In this context it is convenient to consider two systems of cylindrical coordinates: (R, Θ, Z) in the reference configuration and (r, θ, z) in the actual configuration. The helical shear deformation is defined as

$$r = R, \quad \theta = \Theta + \psi(R), \quad z = Z + w(R). \tag{4.1}$$

This deformation is composed by an axial shear $w(R)$ (the out-of-plane displacement) and a rotational shear displacement $\psi(R)$ (the in-plane displacement) and is usually denoted as an *helical shear* deformation. This class of deformation has been considered by many authors, see for example [9] for the case of isotropic materials and [4] for the anisotropic case.

If $\boldsymbol{e}_i$ and $\boldsymbol{E}_\alpha$, with $i = (r, \theta, z)$ and $\alpha = (R, \Theta, Z)$, are the othonormal vector bases associated with the corresponding cylindrical coordinates in the reference and actual configurations, the deformation gradient, $\boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}$, associated with (4.1) is given by

$$\boldsymbol{F} = \boldsymbol{I} + \kappa_\theta \boldsymbol{e}_\theta \otimes \boldsymbol{E}_R + \kappa_a \boldsymbol{e}_z \otimes \boldsymbol{E}_R, \tag{4.2}$$

where $\boldsymbol{I}$ is the identity tensor and

$$\kappa_\theta(R) = R\psi', \quad \kappa_a(R) = w',$$

the prime denoting differentiation with respect to R, are the azimuthal and axial shear, respectively.

The deformation (4.1) is isochoric, i.e. $\det \boldsymbol{F} = 1$, and the components of the Cauchy-Green tensor read as

$$[\boldsymbol{B}] = \begin{bmatrix} 1 & R\psi' & w' \\ R\psi' & 1 + R^2\psi'^2 & R\psi' w' \\ w' & R\psi' w' & 1 + w'^2 \end{bmatrix},$$

The first principal invariant of such tensor is

$$I_1 = 3 + \kappa^2,$$

where $\kappa = \sqrt{\kappa_\theta^2 + \kappa_a^2}$ is the *amount of shear*.

By considering the components of the unit vector of the preferred direction as

$$\boldsymbol{M} = M_R \boldsymbol{E}_R + M_\Theta \boldsymbol{E}_\Theta + M_Z \boldsymbol{E}_Z,$$

we find

$$\boldsymbol{F}\boldsymbol{M} = M_R \boldsymbol{e}_r + (\kappa_\theta M_R + M_\Theta)\boldsymbol{e}_\theta + (\kappa_a M_R + M_Z)\boldsymbol{e}_z, \tag{4.3}$$

and

$$I_4 = 1 + \kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta + 2\kappa_a M_R M_Z.$$

The stress components are functions of R alone and the equilibrium equations in absence of body force thus simplify as

$$RT'_{rr} = T_{\theta\theta} - T_{rr}, \quad (R^2 T_{r\theta})' = 0, \quad (RT_{rz})' = 0. \tag{4.4}$$

In general this is a set of three ordinary differential equations in the unknown functions $p = p(R)$, $\psi = \psi(R)$ and $w = w(R)$. Clearly the pressure p appears linearly only in the first equation in (4.4), the radial balance equation, and therefore the crucial point is to discuss the nature of the azimuthal and axial components. In this case

$$T_{\theta r} = S_{\theta R}, \quad T_{zr} = S_{zR}.$$

Using (4.2) in the definition of the Piola-Kirchhoff stress tensor we obtain

$$T_{\theta r} \equiv \tau_\theta(\kappa_\theta, \kappa_a) = \frac{\partial \mathcal{W}}{\partial \kappa_\theta}, \quad T_{zr} \equiv \tau_a(\kappa_\theta, \kappa_a) = \frac{\partial \mathcal{W}}{\partial \kappa_a}, \tag{4.5}$$

and a direct integration of the second and the third equation in (4.4) gives

$$R^2 \tau_\theta = A, \quad R\tau_a = B. \tag{4.6}$$

The integration constants A and B will be not zero because we consider as domain $\mathcal{D}$ a hollow cylinder.

From (2.4) we point out that the components τ_θ and τ_a we need are defined via

$$T_{\theta r} = 2\mathcal{W}_1 \kappa_\theta + 2\mathcal{W}_4 M_R(\kappa_\theta M_R + M_\Theta),$$

and

$$T_{zr} = 2\mathcal{W}_1\kappa_a + 2\mathcal{W}_4 M_R(\kappa_a M_R + M_Z).$$

We assume $M_R \neq 0$ otherwise the material response in the class of deformations we are considering is just isotropic.

We solve the (4.4) by considering a typical boundary value problem for a hollow circular cylindrical tube. The inner surface, $R = R_i$, is bonded to a rigid cylinder and an uniformly distributed axial shear traction is applied to the outer surface, $R = R_o$. Therefore we impose the following Neumann boundary conditions for $R = R_o$

$$T_{rr}(R_o) = 0, \quad T_{\theta r}(R_o) = 0, \quad T_{zr}(R_o) = T, \tag{4.7}$$

whereas on the other inner surface $R = R_i$ we require the Dirichlet boundary condition

$$w(R_i) = 0. \tag{4.8}$$

Clearly to maintain such deformation we also need to apply an axial force

$$N = \int_0^{2\pi}\int_{R_i}^{R_o} T_{zz} r dr d\theta.$$

In the general framework of anisotropic materials, the boundary values problem can be reformulated by considering the second and third equations in (4.4) and (4.14), along with definitions (4.5):

$$\tau_\theta(\kappa_\theta, \kappa_a) = 0, \quad \tau_a(\kappa_\theta, \kappa_a) = \frac{TR_o}{R}, \tag{4.9}$$

$$w(R_i) = \psi(R_i) = 0. \tag{4.10}$$

We point out that equations (4.9) may be rewritten as

$$2\mathcal{W}_1\kappa_\theta + 2\mathcal{W}_4 M_R(\kappa_\theta M_R + M_\Theta) = 0,$$
$$2\mathcal{W}_1\kappa_a + 2\mathcal{W}_4 M_R(\kappa_a M_R + M_Z) = \frac{TR_o}{R}. \tag{4.11}$$

From (4.11) we deduce an interesting relation

$$2\mathcal{W}_1\left(\kappa_a M_\Theta - \kappa_\theta M_Z\right) = \frac{TR_o}{R}(\kappa_\theta M_R + M_\Theta), \tag{4.12}$$

valid for any strain-energy density $\mathcal{W} = \mathcal{W}(I_1, I_4)$. Since the (4.12) does not contains $\mathcal{W}_4$ but still depends on $\mathcal{W}_1$, we denote it as a *pseudo-universal relation*. The system (4.9) is just a Cauchy problem but it is not in normal form. To this aim we need to ensure that the Hessian of the strain-energy density function $W = W(k_\theta, k_a)$ is not singular; in this case, it is possible to solve the boundary value problem (4.9)-(4.10) in a direct way and in the following we deduce some solutions for special class of strain-energy density functions.

In the isotropic case the boundary conditions may be satisfied when $\kappa_\theta \equiv 0$. Indeed, in this case we have that the first of the boundary conditions in (4.13) is satisfied by an appropriate choice of the pressure field and the second one by the fact that $T_{r\theta} \equiv 0$ for any $R \in [R_i, R_o]$. On the other hand, in the anisotropic case when $\kappa_\theta \equiv 0$ the stress component $T_{r\theta}$ is equal to $2\mathcal{W}_4 M_R M_\Theta$ and therefore it is necessary to require $M_\Theta \equiv 0$. This is the only arrangement of fibers that allows us to solve the boundary value problem we are considering using only an axial shear deformation.

If $M_\Theta \neq 0$ the boundary conditions (4.7) couples the axial, κ_a, and azimuthal, κ_θ, shears. This is a very interesting phenomenon which allows us to observe a remarkable fact: when an anisotropic material is pulled in a direction not only the material moves axially but we notice also a rotation around this same direction. This phenomenon can be observed also in the framework of the linear theory and is associated to the possibility of the design of special soft elastic machines.

4.1. The standard solid

We first start for considering the case of the standard model. In this case

$$\begin{aligned} &\kappa_\theta + \mu(\kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta + 2\kappa_a M_R M_Z) M_R(\kappa_\theta M_R + M_\Theta) = 0, \\ &\kappa_a + \mu(\kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta + 2\kappa_a M_R M_Z) M_R(\kappa_a M_R + M_Z) = \frac{T R_o}{\mu_0 R}, \end{aligned} \tag{4.13}$$

where $\mu = \mu_1/\mu_0$.

When $M_\Theta = 0$ the solution of the BVP is clearly given by $\kappa_\theta \equiv 0$. When $M_\Theta \neq 0$ it is fundamental in the case of the standard model to use the pseudo-universal relation (4.12)

$$\kappa_\theta = \frac{\mu R \kappa_a - T R_o}{T R_o M_R + \mu R M_Z} M_\Theta. \tag{4.14}$$

Using in $(4.13)_2$ the (4.14) we obtain a cubic equation in κ_a. Computing the real solution of this cubic equation, let us say $\kappa_a = \Phi(R, \boldsymbol{M}, T)$, whose existence is ensured by the Descartes signs rule, we obtain the solution of our problem by quadrature as

$$w(R) = \int_{R_i}^{R} \Phi(x, \boldsymbol{M}, T) dx.$$

In any case a direct numerical solution is easy to obtain since in (4.13) the BVP has been already been reduced to a simple first order initial value problem. To this end we simplify the system considering

$$M_R = \cos\Lambda, \quad M_\Theta = \sin\Lambda, \quad M_Z = 0,$$

where $\Lambda \in [0, \pi/2[$, and the dimensionless variables $\xi = R/R_o$ (i.e. $\xi \in [R_i/R_o, 1]$), $\hat{w} = w/R_o$, $\hat{\psi} = \psi/R_o$ and $\hat{T} = T/\mu_0$. Clearly in this case we have that

$$\kappa_\theta(\xi) = \xi\frac{d\hat{\psi}}{d\xi}, \quad \kappa_a = \frac{d\hat{w}}{d\xi}.$$

In so doing we have to solve the equations

$$\begin{aligned} &\kappa_\theta + \mu(\kappa^2\cos\Lambda + 2\kappa_\theta\sin\Lambda)(\kappa_\theta\cos\Lambda + \sin\Lambda)\cos^2\Lambda = 0, \\ &\kappa_a + \mu(\kappa^2\cos\Lambda + 2\kappa_\theta\sin\Lambda)\kappa_a\cos^3\Lambda = \frac{\hat{T}}{\xi}, \end{aligned} \tag{4.15}$$

with the "initial" conditions $\hat{\psi}(R_i/R_o) = \hat{w}(R_i/R_o) = 0$ for $\xi \in [R_i/R_o, 1]$.

In figure 1 we plot the two limiting case $\Lambda = 0$ (blue lines) and $\Lambda = \pi/2$ (black lines), which recover isotropic solutions and we have that the deformations corresponding to the boundary conditions we choose are just an axial shear and only $w(\xi)$ is different from zero. We see that the azimuthal deformation increases for $\Lambda \in]0, \pi/4]$ and then it decreases (see the cases $\Lambda = 3\pi/8,\ 7\pi/16$).

We highlight the fact that an axial tension "wakes up" not only the axial deformation but also the in-plane azimuthal deformation saying that the material act like an *elastic machine* [5]: you pull in a direction and the material is able not only to move in this direction but also to rotate around this direction.

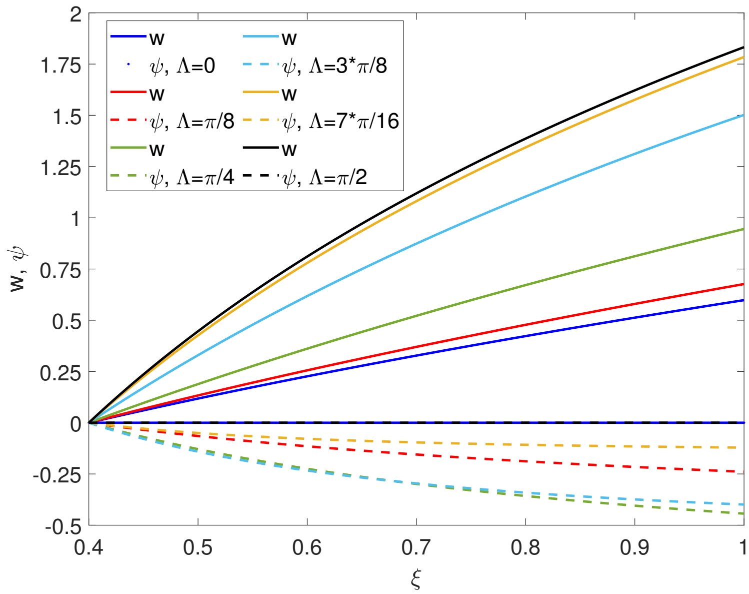

Figure 1: Solutions of (4.15) for $\mu = 2$, $\hat{T} = 2$ and $\xi \in [0.4, 1]$ for several angles Λ.

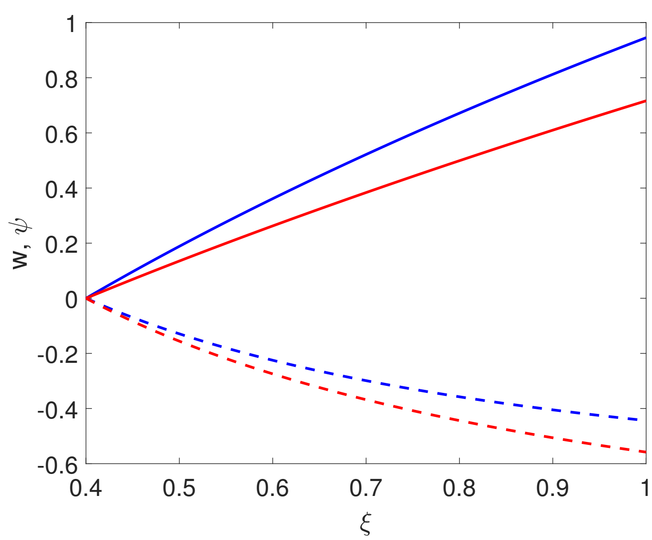

Figure 2: Solutions of (4.15) for $\Lambda = \pi/4$, $\hat{T} = 2$ in $\xi \in [0.4, 1]$ for $\mu = 2$ (blue lines) and $\mu = 10$ (red lines).

If we increase the relative stiffness of the fibers with respect the stiffness of the matrix an interesting phenomena appears. In figure 2 you can see that because the overall stiffness of the materials is increased the amplitude of the axial deformation decreases (the material is more stiff) but the amplitude of the azimuthal deformation increases. Therefore stiff fibers produce a more pronounced elastic machine effect. This effect is interesting in application or design of experimental tests.

On the other hand, it is clear that the amplitude of the deformation is proportional to the amount of the pulling tension $\hat{T}$.

The use of more complex strain-energy densities is effective on the numerical results and, for strain hardening materials, the elastic machine effect is definitely more pronounced. As an example, we consider the strain energy density model with exponentials proposed in [6]:

$$\mathcal{W} = \frac{\mu_0}{2}(I_1 - 3) + \frac{1}{2}\left(\frac{\mu_1}{\mu_2}\exp\mu_2(I_4-1)^2 - 1\right). \tag{4.16}$$

In this case, the two equations (4.13) read as

$$\kappa_\theta + 2\frac{\mu}{\mu_2}(\kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta - M_Z^2)M_R(\kappa_\theta M_R + M_\theta)\times$$
$$e^{\mu_2(\kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta - M_Z^2)^2} = 0, \tag{4.17}$$

$$\kappa_a + 2\frac{\mu}{\mu_2}(\kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta - M_Z^2)M_R(\kappa_a M_R + M_Z)\times$$
$$e^{\mu_2(\kappa^2 M_R^2 + 2\kappa_\theta M_R M_\Theta - M_Z^2)^2} = \frac{TR_o}{\mu_0 R}, \tag{4.18}$$

that, in the case

$$M_R = \cos\Lambda, \quad M_\Theta = \sin\Lambda, \quad M_Z = 0,$$

become

$$\kappa_\theta + 2\frac{\mu}{\mu_2}(\kappa^2\cos\Lambda + 2\kappa_\theta\sin\Lambda)(\kappa_\theta\cos\Lambda + \sin\Lambda)\cos^2\Lambda\times \\ e^{\mu_2\cos^2(\kappa^2\cos\Lambda + 2\kappa_\theta\sin\Lambda)^2} = 0, \tag{4.19}$$

$$\kappa_a + 2\frac{\mu}{\mu_2}(\kappa^2\cos\Lambda + 2\kappa_\theta\sin\Lambda)\kappa_a\cos^3\Lambda\times \\ e^{\mu_2\cos^2(\kappa^2\cos\Lambda + 2\kappa_\theta\sin\Lambda)^2} = \frac{\hat{T}}{\xi}, \tag{4.20}$$

with the "initial" conditions $\hat{\psi}(R_i/R_o) = \hat{w}(R_i/R_o) = 0$ for $\xi \in [R_i/R_o, 1]$. As suggested in [6], the case $\mu_2 = 0.8393$ is used. In figure 4.1 we show the values of the axial and azimuthal displacement, w and ψ, respectively, for several values of Λ. When Λ is about $\pi/2$ and 0, i.e. we are near the isotropic behavior, the higher hardening is not so effective with respect to the standard model, but the difference becomes evident in the other cases; in particular, the axial displacement w is lower than in the standard model whereas the azimuthal one is higher in absolute value, and the elastic machines-like behavior increases.

5. Concluding Remarks

We have determined, in a compact format, the determining equations describing the superimposition of a plane strain on anti-plane shear deformation for a special class of transverse isotropic materials. In principle these

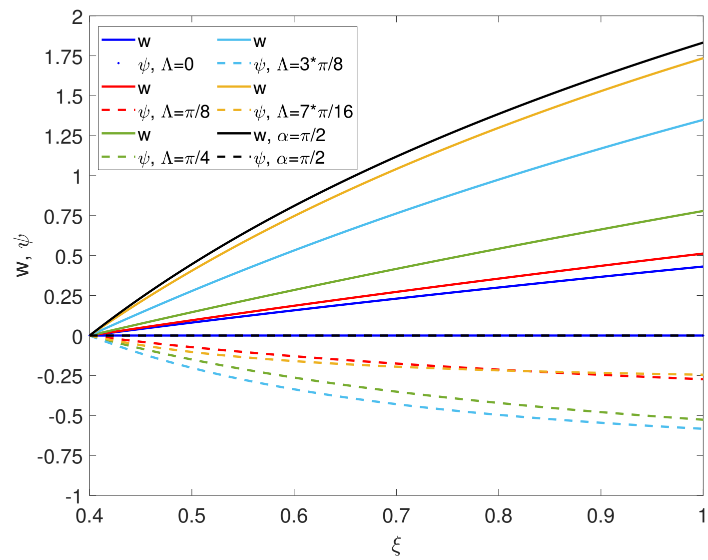

Figure 3: Solutions of (4.19) for $\mu = 2$, $\hat{T} = 2$, $\mu_2 = 0.8393$ and $\xi \in [0.4, 1]$ for several angles Λ.

equations may decouple for special class of materials and specific functional forms of the deformation but not only this happenstance is rare but usually is of minor interest from a mechanical point of view.

The equations we have determined show that for transverse isotropic materials the solutions of boundary-value problems for cylindrical regions are much more complex and rich than in the corresponding case of isotropic materials. The coupling between out-of-plane and in-plane deformations have been already noticed in the framework of linear elasticity and may be exploited to obtain some special *elastic machines* [5], i.e. elastic material that via specific arrangements of the fibers may control how to combine different deformations modes.

As explicit example we have considered the case of a boundary value problem whose solution for isotropic materials is obtained considering an axial shear deformation. In the transverse isotropic case the solution of the same boundary value problem can be obtained, for a generic arrangement of the fibers, only via a helical shear deformation i.e. coupling the axial with an azimuthal shear.

The method we have used may be extended to a general transverse isotropic strain-energy density but we loose the compact form of our computations. Moreover, the equations we have here derived may be extended to elastodynamics.

Many are the direct applications of the results of the present paper. As it has been done in the linear case in [5], now it is possible to imagine *nonlinear elastic machines*. This means that we may use to use the coupling of the various modes of deformation to design new typologies of crankshaft able to perform a conversion between reciprocating motion and rotational motion only using a material and not crankpins. Along the lines of [2] is possible to consider fiber reinforcement to optimize the uncork of a bottle. In the framework of elastodynamics, in the spirit of [3] and [11] the results of the present paper may be used to derive in a rigorous way the Zabolotoskaya equation for transverse isotropic materials.

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