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On the Kelvin-Voigt Model in Anisotropic Viscoelasticity

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Abstract

We propose an anisotropic and nonlinear generalization of the Kelvin-Voigt viscoelastic model obtained considering the additive splitting of the Cauchy stress tensor in an elastic and a dissipative part. The former one corresponds to a fiber-reinforced hyperelastic material while the dissipative effect is described by the most general linear transverse-isotropic tensorial function of symmetric part of the velocity gradient. In a such a way we characterize the dissipative contribution via three viscoelastic moduli. We then show, by a detailed analysis of the simple shear quasistatic motion and the corresponding creep phenomena, that this motion may be used to determine experimentally the viscoelastic parameters.

Keywords Fiber-reinforced materials, Kelvin-Voigt viscoelasticity, Simple shear, Creep.

1 Introduction

A basic model of viscoelasticity is due to Kelvin and Voigt and from a rheological point of view it consists of a spring and a damping element that are parallel to each other [1]. Since this model superposes the elastic and the viscous behaviour it is quite simple to generalize it to a nonlinear setting and this has been done for isotropic elastic materials by many authors, see for example [2, 3] and the discussion in [4].

To understand this procedure let us consider the motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, where x_i $(i = 1, 2, 3)$ are the spatial coordinates of the current position of the material point $\mathbf{X}$, whose Cartesian coordinates are X_K, $(K = 1, 2, 3)$, denote the components of the deformation gradient $\boldsymbol{F}$ by $F_{iK} = \partial x_i / \partial X_K$ and the left Cauchy-Green tensor as $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. Moreover, the tensor $\boldsymbol{D}$ is defined as the symmetric part of

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}.$$

A possible generalization of the Kelvin-Voigt model in a nonlinear setting is obtained splitting the Cauchy stress tensor, $\boldsymbol{T}$, in an elastic and a dissipative part, i.e.

$$\boldsymbol{T} = \boldsymbol{T}^e + \boldsymbol{T}^d,$$

where $\boldsymbol{T}^e = \boldsymbol{T}^e(\boldsymbol{B})$ and $\boldsymbol{T}^d = \boldsymbol{T}^d(\boldsymbol{B}, \boldsymbol{D})$. In this framework, we remind the simple but widely used incompressible model

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\frac{\partial \mathcal{W}}{\partial I_1}\boldsymbol{B} - 2\frac{\partial \mathcal{W}}{\partial I_2}\boldsymbol{B}^{-1} + \eta \boldsymbol{D}, \tag{1.1}$$

where $\boldsymbol{I}$ is the identity tensor, $\mathcal{W}(I_1, I_2)$ is the strain-energy density function of $I_1 = \operatorname{trace} \boldsymbol{B}$ and $I_2 = \operatorname{trace} \boldsymbol{B}^{-1}$, p is the unknown Lagrange multiplier associated with the incompressibility constraint $\det \boldsymbol{F} = 1$ and η is the constant viscosity parameter. This model is therefore composed by the classical theory of incompressible isotropic hyperelasticity plus a dissipative part given by the classical incompressible Navier-Stokes constitutive assumption.

In the context of the polymers and soft tissue mechanics, it is often necessary to consider anisotropic models. For example, this is the case of thermoplastic composites [5] or biomechanics of ligaments [6]. It is therefore natural to ask how it is possible to obtain a version of the Kelvin-Voigt model suitable for the nonlinear and anisotropic context.

If we consider a material reinforced with one family of fibers the representation formula for the stress tensor of a material such that $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{m})$, where $\boldsymbol{m} = \boldsymbol{F}\boldsymbol{M}$, being the unit vector $\boldsymbol{M}$ the fiber direction in the reference configuration, can be found in [7]. In this case, to write down the general representation formula it is necessary to consider 16 constitutive functions and 17 invariants. Therefore the general model is of little interest in applications because it is impossible to develop a general methodology to experimentally determine a so high number of constitutive functions.

To make real progress at this point, it is necessary to make simplifying assumptions a priori. For example, in [8] a model to investigate uniaxial tensile deformation tests of porcine aortic valve specimens at various deformation rates is investigated. The dissipative part of the constitutive equation proposed in [8] depends linearly only on two invariants via a dissipation potential. Another simplification is proposed in [9] and in [19]. This last paper is devoted to the study of travelling shear waves in the special model

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{B} + \mu_a(I_4 - 1)\boldsymbol{m} \otimes \boldsymbol{m} + [\eta_1 + \eta_2(I_4 - 1)]\boldsymbol{D}, \tag{1.2}$$

where

$$I_4 = \boldsymbol{m} \cdot \boldsymbol{m},$$

μ and μ_a are two elastic moduli and η_1 and η_2 two viscosity parameters. The elastic part of such model is associated to the strain-energy density of the standard reinforced model

$$2\mathcal{W}(I_1, I_4) = \mu(I_1 - 3) + \frac{1}{2}\mu_a(I_4 - 1)^2. \tag{1.3}$$

These simplifications of the general theory are not dictated by a rational methodology of investigation but only by empirical considerations or computational simplifications. A compromise between the complexity of the general theory in [7] and the specificity of the various particular models as in [8] or [10] may be indicated by the isotropic model (1.1), i.e. by the mix between the classic theory of hyperelasticity and the Navier-Stokes theory.

We remind that the Navier-Stokes theory is obtained requiring "the most general linear isotropic function $\boldsymbol{T}$ of a symmetric second-order tensor $\boldsymbol{D}$" [11]. This requirement in the incompressible case gives the celebrated constitutive equation $\boldsymbol{T} = -p\boldsymbol{I} + \eta\boldsymbol{D}$, but the most general transversely isotropic function $\boldsymbol{T}$ linear with respect to the symmetric second-order tensor $\boldsymbol{D}$ is given by [12]:

$$\boldsymbol{T} = -p\boldsymbol{I} + \eta_T\boldsymbol{D} + \eta_E(\boldsymbol{m}\cdot\boldsymbol{D}\boldsymbol{m})\boldsymbol{m}\otimes\boldsymbol{m} + (\eta_L - \eta_T)(\boldsymbol{m}\otimes\boldsymbol{m}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{m}\otimes\boldsymbol{m}), \tag{1.4}$$

where η_T and η_L are Newtonian viscosities for shear parallel and normal to the fiber direction, respectively, and η_E the measure of the extensional viscosity along the fiber direction. The constitutive equation (1.4) is the Navier-Stokes analogue for transverse isotropic incompressible fluids.

Using this result it is natural to consider an anisotropic Kelvin-Voigt model, that for the sake of algebraic simplicity, we restrict here to a strain-energy density in the form $\mathcal{W}(I_1, I_2, I_4)$, whose constitutive equations is given by

$$\begin{aligned} \boldsymbol{T} = -p\boldsymbol{I} + 2\frac{\partial \mathcal{W}}{\partial I_1}\boldsymbol{B} - 2\frac{\partial \mathcal{W}}{\partial I_2}\boldsymbol{B}^{-1} + 2\frac{\partial \mathcal{W}}{\partial I_4}\boldsymbol{m}\otimes\boldsymbol{m} + \eta_T\boldsymbol{D} \\ + \eta_E(\boldsymbol{m}\cdot\boldsymbol{D}\boldsymbol{m})\boldsymbol{m}\otimes\boldsymbol{m} + (\eta_L - \eta_T)(\boldsymbol{m}\otimes\boldsymbol{m}\ \boldsymbol{D} + \boldsymbol{D}\boldsymbol{m}\otimes\boldsymbol{m}). \end{aligned} \tag{1.5}$$

This model is the transverse isotropic version of (1.1) and is characterized only by three viscous parameters.

The aim of the present note is to investigate if shearing motions and specifically the classical recovery experiment based on the simple shear motion in the quasistatic setting are sufficient to determine these three viscosity parameters. In such a way we have a clear control of the most general transverse incompressible isotropic viscoelastic model of differential type linear in the second order tensor $\boldsymbol{D}$. The restriction on the hyperelastic part of the stress tensor that we assume for the sake of the simplicity has no impact on our results. Of course, the present study can be extended to more complicated symmetries, as those in [13], and for measurements of the arising constitutive parameters, as, e.g., those discusses in [14].

The plan of the paper is the following. In section 2, we write down the basic equations of the problem and, in section 3, we investigate into details the simple shear motions showing that indeed it is possible to use the recovery experiment to determine the three viscous moduli. The final section 4 is devoted to concluding remarks.

Our aim is to use the same strategy that Ray Ogden and co-workers (see for example [15]) used to investigate anisotropic nonlinear elastic media: we start with a simple model to build up step by step a clear understanding of what to expect in the context of a mechanical behaviour that often defies our induction. For this reason we would like to dedicate this note to Ray, a great gentleman and great scientist, but above all a great friend and esteemed collaborator.

2 Basic Equations

Let us consider the following class of isochoric motions

$$x_1 = X_1 + f(X_3, t), \quad x_2 = X_2 + g(X_3, t), \quad x_3 = X_3, \tag{2.1}$$

such that $I_1 = I_2 = 3 + \gamma^2$, where $\gamma^2 = f_3^2 + g_3^2$, and denote the components of the preferred direction in the reference configuration as

$$\boldsymbol{M} = M_1\boldsymbol{E}_1 + M_2\boldsymbol{E}_2 + M_3\boldsymbol{E}_3.$$

We compute

$$I_4 = 1 + \gamma^2 M_3^2 + 2(f_3 M_1 + g_3 M_2)M_3.$$

However, since:

$$[\boldsymbol{F}] = \begin{bmatrix} 1 & 0 & f_3 \\ 0 & 1 & g_3 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\boldsymbol{F}^{-1}] = \begin{bmatrix} 1 & 0 & -f_3 \\ 0 & 1 & -g_3 \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$[\boldsymbol{D}] = \frac{1}{2} \begin{bmatrix} 0 & 0 & f_{3,t} \\ 0 & 0 & g_{3,t} \\ f_{3,t} & g_{3,t} & 0 \end{bmatrix},$$

i.e.

$$\boldsymbol{m} \cdot \boldsymbol{D}\boldsymbol{m} = \frac{1}{2}(\gamma^2)_t M_3^2 + (f_{3,t}M_1 + g_{3,t}M_2)M_3.$$

The equation of motion in absence of body forces is

$$\rho \boldsymbol{a} = \operatorname{div} \boldsymbol{T},$$

where ρ is the constant density and $\boldsymbol{a}$ the acceleration, and, because the various unknown functions just depend only on X_3 and t, it reduces to

$$\rho f_{tt} = -p_{x_1} + \frac{\partial T_{13}}{\partial x_3}, \quad \rho g_{tt} = -p_{x_2} + \frac{\partial T_{23}}{\partial x_3}, \quad \frac{\partial T_{33}}{\partial x_3} = 0. \tag{2.2}$$

From the first two equations in (2.2) we obtain

$$p(x_1, x_2, x_3, t) = \phi(t)x_1 + \psi(t)x_2 + q(x_3, t),$$

where $\phi(t)$ and $\psi(t)$ are arbitrary functions and $q(x_3, t)$ is obtained by a quadrature from the third equation in (2.2).

Being $x_3 = X_3$, let us introduce the dimensionless variables

$$Z = X_3/L, \quad \tau = t/\mathcal{T}, \quad \hat{f} = f/L, \quad \hat{g} = g/L,$$

and the dimensionless stresses

$$\hat{T}_{13} = T_{13}/\mu, \quad \hat{T}_{23} = T_{23}/\mu,$$

where L is a characteristic length, $\mathcal{T}$ is a characteristic time.

Using the dimensionless variables and the notation $F = \hat{f}_Z$ and $G = \hat{g}_Z$ for the strains, we rewrite the derivative with respect X_3 of the first two equations in (2.2) as

$$\lambda F_{\tau\tau} = \frac{\partial^2 \hat{T}_{13}}{\partial Z^2}, \quad \lambda G_{\tau\tau} = \frac{\partial^2 \hat{T}_{23}}{\partial Z^2}, \tag{2.3}$$

where

$$\lambda = \frac{\rho L^2}{\mu T^2},$$

and

$$\begin{aligned}
\hat{T}^e_{13} &= 2(\hat{\mathcal{W}}_{I_1} + \hat{\mathcal{W}}_{I_2})F + 2\hat{\mathcal{W}}_{I_4}(M_1 + FM_3)M_3, \\
\hat{T}^e_{23} &= 2(\hat{\mathcal{W}}_{I_1} + \hat{\mathcal{W}}_{I_2})G + 2\hat{\mathcal{W}}_{I_4}(M_2 + GM_3)M_3, \\
\hat{T}^d_{13} &= \frac{1}{2}\hat{\eta}_T F_\tau + \hat{\eta}_E \left[\frac{1}{2}(\gamma^2)_\tau M_3^2 + (F_\tau M_1 + G_\tau M_2)M_3\right](M_1 + FM_3)M_3 \\
&\quad + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\left\{F_\tau M_1^2 + G_\tau M_1 M_2 + (F^2 F_\tau + FGG_\tau + F_\tau)M_3^2\right. \\
&\qquad\qquad \left. + [2FF_\tau M_1 + (M_1 G + M_2 F)G_\tau]M_3 \right\}, \\
\hat{T}^d_{23} &= \frac{1}{2}\hat{\eta}_T G_\tau + \hat{\eta}_E \left[\frac{1}{2}(\gamma^2)_\tau M_3^2 + (F_\tau M_1 + G_\tau M_2)M_3\right](M_2 + GM_3)M_3 \\
&\quad + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\left\{G_\tau M_2^2 + F_\tau M_1 M_2 + (G^2 G_\tau + GFF_\tau + G_\tau)M_3^2\right. \\
&\qquad\qquad \left. + [2GG_\tau M_2 + (M_1 G + M_2 F)F_\tau]M_3 \right\}.
\end{aligned}$$

Here $\hat{\mathcal{W}} = \mathcal{W}/\mu$ and

$$\hat{\eta}_L = \frac{\eta_L}{\mu\mathcal{T}}, \quad \hat{\eta}_E = \frac{\eta_E}{\mu\mathcal{T}}, \quad \hat{\eta}_T = \frac{\eta_T}{\mu\mathcal{T}}$$

are dimensionless "viscosities".

It is of fundamental importance to note that

$$\hat{T}^e_{13} = \frac{\partial \hat{\mathcal{W}}}{\partial F}, \quad \hat{T}^e_{23} = \frac{\partial \hat{\mathcal{W}}}{\partial G}. \tag{2.4}$$

When we consider the strain-energy associated with the standard linear solid (1.3) we have that the elastic part of the stress tensor is simplified as

$$\begin{aligned} \hat{T}^e_{13} &= F + \hat{\mu}[\gamma^2 M_3^2 + 2(FM_1 + GM_2)M_3](M_1 + FM_3)M_3, \\ \hat{T}^e_{23} &= G + \hat{\mu}[\gamma^2 M_3^2 + 2(FM_1 + GM_2)M_3](M_2 + GM_3)M_3, \end{aligned} \tag{2.5}$$

where $\hat{\mu} = \mu_a/\mu$.

For the sake of simplicity in the following we will use for the elastic part the standard linear solid (1.3) and this because the elastic part is usually determined by a static experiment and here we wish to focus on the viscoelastic part of the constitutive equation. The use of more general hyperelastic form of the strain-energy has a minor impact on our results.

The system (2.3) is a third order nonlinear highly coupled system of partial differential equations in the unknown $F(Z,\tau)$ and $G(Z,\tau)$. If in (2.3) we set $G \equiv 0$ the system is overdetermined and it is not possible to solve it for any choice of $\boldsymbol{M}$ in the reference configuration. Indeed, when (2.5) is in force, if $G \equiv 0$, we get

$$\begin{aligned} \hat{T}_{23} = \hat{\mu}(F^2 M_3 + 2FM_1)M_2 M_3^2 + \hat{\eta}_E \left(\frac{1}{2}F_\tau^2 M_3 + F_\tau M_1\right) M_2 M_3^2 \\ + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)(F_\tau M_1 + FF_\tau M_3)M_2, \end{aligned}$$

and non trivial solutions of (2.3) are possible if and only if $M_2 \equiv 0$.

It is interesting to consider F and G of order $\mathcal{O}(\epsilon)$ where ϵ is a small parameter and to linearize equations (2.3). In so doing we obtain, always with reference to (2.5), the following set of coupled but linear partial differential equations:

$$\lambda F_{\tau\tau} = \left\{F + \hat{\mu}(FM_1 + GM_2)M_1 M_3^2 + \hat{\eta}_T F_\tau\right\}_{ZZ}$$
$$+ \left\{\left[\hat{\eta}_E M_3^2 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\right](F_\tau M_1 + G_\tau M_2)M_1 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)F_\tau M_3^2\right\}_{ZZ} \tag{2.6}$$

$$\lambda G_{\tau\tau} = \left\{G + \hat{\mu}(FM_1 + GM_2)M_2 M_3^2 + \hat{\eta}_T G_\tau\right\}_{ZZ}$$
$$+ \left\{\left[\hat{\eta}_E M_3^2 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\right](F_\tau M_1 + G_\tau M_2)M_2 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)G_\tau M_3^2\right\}_{ZZ} \tag{2.7}$$

In the next Section we will restrict our attention to the quasistatic limit of equations (2.3), i.e. when $\lambda \to 0$, and therefore we have to solve the system

$$\frac{\partial^2 \hat{T}_{13}}{\partial Z^2} = 0, \quad \frac{\partial^2 \hat{T}_{23}}{\partial Z^2} = 0. \tag{2.8}$$

An important solution of this set of equations is obtained considering $F = \kappa_1(t)$ and $G = \kappa_2(t)$, i.e. taking into account the simple shear motions corresponding to $\hat{f} = \kappa_1 Z$ and $\hat{g} = \kappa_2 Z$ in (2.1).

We point out that the quasistatic limit is meaningful only when a global existence theorem for the solutions of equations (2.3) may be ensured [16] but this is still an open problem also for the general differential viscoelastic model in the isotropic case.

REMARK For a mechanically isolated motion, i.e. no body forces, it must be

$$\frac{d}{dt}\int_{\Omega_t}\frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}dV = -\int_{\Omega_t}\boldsymbol{T}\cdot\boldsymbol{L}dV + \int_{\partial\Omega_t}(\boldsymbol{T}\boldsymbol{n})\cdot\boldsymbol{v}dS,$$

where Ω_t is actual configuration occupied by the body in the Euclidean space, $\boldsymbol{v}$ the velocity vector and $\boldsymbol{T}\cdot\boldsymbol{L} = \text{trace}(\boldsymbol{T}\boldsymbol{L}^T)$. It is therefore natural to require that in our framework it must be

$$T^d_{13}F_\tau + T^d_{23}G_\tau > 0. \tag{2.9}$$

So it is clear that our model is dissipative. We remark that (2.9) is a quadratic form in the variables F_τ and G_τ to which it is possible to associate a symmetric function

$$\mathbb{A} = \mathbb{A}(F, G) \tag{2.10}$$

whose components are functions of F, G and the unit vector $\boldsymbol{M}$. Therefore, it is easy to check the requirement (2.9) .

3 Simple Shear

The simple shear motions are fundamental to investigate the characteristic viscoelastic phenomena of creep and recovering.

Let us focus on the creep experiment. We remind that for an isotropic linear material, if we have the stress history $\hat{T}_{13}(\tau) = \hat{T}_\infty H(\tau)$, where H is the Heaviside step function, it is necessary to consider only $\hat{T}_{13} = \kappa + \hat{\eta}\kappa_\tau/2$, and the creep phenomena is described by the boundary value problem (BVP)

$$\hat{T}_\infty = \kappa + \frac{1}{2}\hat{\eta}\kappa_\tau, \quad \kappa(0) = 0, \quad \lim_{\tau\to\infty}\kappa(\tau) = \kappa_\infty.$$

We point out that in this case it must be $\hat{T}_\infty = \kappa_\infty$ and the solution of our problem is $\kappa(\tau) = \hat{T}_\infty[1 - \exp(-2\tau/\hat{\eta})]$. This experiment is an effective way, once we have characterized by a static test the elastic part of the stress, to measure the viscosity,

3.1 Linear Anisotropic Case

Now let us consider the same problem in the case of the linear but anisotropic situation.

We consider once again the stress history

$$\hat{T}_{13}(\tau) = \hat{T}_\infty H(\tau), \quad \hat{T}_{23} \equiv 0$$

, but now, at least in the general case, equations (2.8) are not decoupled and therefore we have to solve the system

$$\begin{aligned} \hat{T}_\infty = \kappa_1 + 2\hat{\mu}(\kappa_1 M_1 + \kappa_2 M_2) M_1 M_3^2 + \frac{1}{2}\hat{\eta}_T \kappa_{1,\tau} \\ + \left[\hat{\eta}_E M_3^2 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\right](\kappa_{1,\tau} M_1 + \kappa_{2,\tau} M_2) M_1 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\kappa_{1,\tau} M_3^2, \end{aligned} \tag{3.1}$$

$$\begin{aligned} 0 = \kappa_2 + 2\hat{\mu}(\kappa_1 M_1 + \kappa_2 M_2) M_2 M_3^2 + \frac{1}{2}\hat{\eta}_T \kappa_{2,\tau} \\ + \left[\hat{\eta}_E M_3^2 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\right](\kappa_{1,\tau} M_1 + \kappa_{2,\tau} M_2) M_2 + \frac{1}{2}(\hat{\eta}_L - \hat{\eta}_T)\kappa_{2,\tau} M_3^2, \end{aligned} \tag{3.2}$$

subject to $\kappa_1(0) = \kappa_2(0) = 0$, and in this case we compute

$$\lim_{\tau\to\infty} \kappa_1(\tau) = \frac{\hat{T}_\infty(1 + 2\hat{\mu} M_2^2 M_3^2)}{1 + 2\hat{\mu}(M_1^2 + M_2^2) M_3^2}, \lim_{\tau\to\infty} \kappa_2(\tau) = -\frac{\hat{T}_\infty 2\hat{\mu} M 1 M_2 M_3^2}{1 + 2\hat{\mu}(M_1^2 + M_2^2) M_3^2}.$$

When $M_2 \equiv 0$ a solution of such equations is given with $\kappa_2 \equiv 0$ and

$$\kappa_1(\tau) = \frac{\hat{T}_\infty}{1 + 2\hat{\mu} M_1^2 M_3^2} \left[1 - \exp(-\alpha\tau)\right], \tag{3.3}$$

where, since $M_1^2 + M_3^2 = 1$,

$$\alpha = \frac{2(1 + 2\hat{\mu} M_1^2 M_3^2)}{\hat{\eta}_L + 2\hat{\eta}_E M_1^2 M_3^2}, \tag{3.4}$$

and

$$\lim_{\tau \to \infty} \kappa_1(\tau) = \frac{\hat{T}_\infty}{1 + 2\hat{\mu} M_1^2 M_3^2}.$$

With the representation $M_1 = \cos\theta,\ M_3 = \sin\theta$ (it is sufficient to consider $\theta \in [-\pi/2, \pi/2]$), we realize that the asymptotic amount of shear is maximum if $\theta = -\pi/2, 0, \pi/2$ and minimum for $\theta = \pm\pi/4$ and a similar behaviour characterizes the recovery speed.

The equations (3.1) and (3.2) decouple also in the case $M_1 = 0$ for which we obtain the solution

$$\kappa_1(\tau) = \hat{T}_\infty \left(1 - \exp\{-2[\hat{\eta}_T + (\hat{\eta}_T - \hat{\eta}_L) M_3^2]\tau\}\right). \tag{3.5}$$

The general solution of the first order system of the linear ordinary differential equations (3.1) and (3.2) is easy to obtain and reads as follows:

$$\kappa_1(\tau) = \frac{\hat{T}_\infty}{\beta} \left\{(M_1^2 + M_2^2)(1 + 2\hat{\mu} M_2^2 M_3^2) - [1 + 2\hat{\mu}(M_1^2 + M_2^2) M_3^2] M_2^2 e^{-\alpha_1 \tau} - M_1^2 e^{-\alpha_2 \tau}\right\}, \tag{3.6}$$

$$\kappa_2(\tau) = M_1 M_2 \frac{\hat{T}_\infty}{\beta} \left\{[1 + 2\hat{\mu}(M_1^2 + M_2^2) M_3^2] e^{-\alpha_1 \tau} - 2\hat{\mu}(M_1^2 + M_2^2) M_3^2 - e^{-\alpha_2 \tau}\right\},$$

where

$$\alpha_1 = \frac{2}{(\hat{\eta}_L - \hat{\eta}_T)M_3^2 + \hat{\eta}_T}, \quad \alpha_2 = \frac{2[1 + 2\hat{\mu}(M_1^2 + M_2^2)M_3^2]}{\hat{\eta}_L + 2\hat{\eta}_E(M_1^2 + M_2^2)M_3^2},$$

$$\beta = (M_1^2 + M_2^2)[1 + 2\hat{\mu}\left(M_1^2 + M_2^2\right)M_3^2].$$

Clearly $\alpha_1 > 0$ and $\alpha_2 > 0$.

Using the representation $M_1 = \cos(\phi)\cos(\theta)$, $M_2 = \sin(\phi)$, $M_3 = \cos(\phi)\sin(\theta)$, the solutions (3.6) are plotted in Fig. 1 for different values of the angle ϕ.

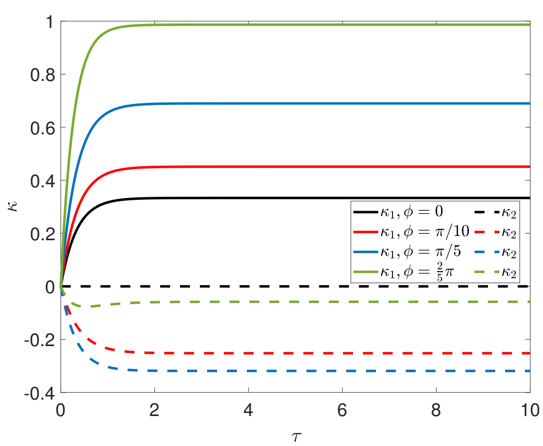

Figure 1: Plot of κ_1 and κ_2 for different values of ϕ and $\theta = \pi/4$. The other parameters are set as follows: $\hat{T}_\infty = 1$, $\hat{\mu} = 4$, $\eta_T = 0.5$, $\eta_E = 2$, $\eta_L = 1$.

In Fig. 2 we show in more detail the non monotonic behaviour of κ_2 and its time derivative $\kappa_{2,\tau}$ in the case $\theta = \pi/4$ and $\phi = 2\pi/5$ when $\hat{T}_\infty = 1$, $\hat{\mu} = 4$, $\eta_T = 0.5$, $\eta_E = 2$, $\eta_L = 1$. This is possible because for

$$\tau^* = -\log\left(\frac{\alpha_2[1 + 2\hat{\mu}(M_1^2 + M_2^2)M_3^2]}{\alpha_1(\alpha_2 - \alpha_1)}\right)$$

we have that $d\kappa_2(\tau^*)/dt = 0$ if $\alpha_2 > \alpha_1$. On the other hand, it is easy to check that $\kappa_1(\tau)$ is an increasing monotonic function for any fiber direction and any value of the parameters.

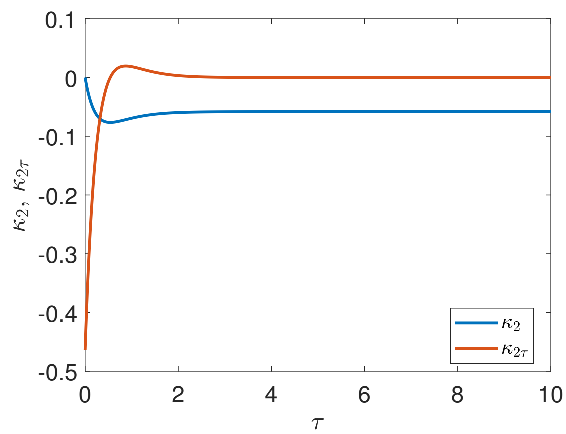

Figure 2: Plot of κ_2 and $\kappa_{2\tau}$ for $\theta = \pi/4$ and $\phi = 2\pi/5$. ($\hat{T}_\infty = 1$, $\hat{\mu} = 4$, $\eta_T = 0.5$, $\eta_E = 2$, $\eta_L = 1$).

These solutions are interesting because they show in a simple direct way that, when $M_2 \neq 0$ despite the fact that we are applying a shear stress only along X_1, we shear the material with respect to both directions in the X_1-X_2 plane. The fact that in anisotropic materials the load diffuses in any possible direction via the fibers arrangement has been already discussed in [17] and [18].

The solutions (3.3), (3.5) and the general solution of the system (3.1) and (3.2) show that by considering appropriate angles between the direction of the fiber and the shear strain we have sufficient information to determine experimentally the various viscosity coefficients.

3.2 Nonlinear Anisotropic Case

The nonlinear case is clearly more complex. In the isotropic case this problem have been considered into details in [19]. Here we analyze first the situation when the system (2.8) decouples to point out the possible "pathologies" of the anisotropic case and then we give a condition to avoid such pathologies for a generic arrangement of the fibers.

3.3 case $M_2 \equiv 0$

To decouple equations (2.8) we set $M_2 \equiv 0$ and therefore $\kappa_2(\tau) \equiv 0$.

To simplify the notation we choose the characteristic time $\mathcal{T} \equiv \eta_L/\mu$ and therefore $\hat{\eta}_L \equiv 1$. In this situation we get

$$\hat{T}^e_{13} = \kappa_1 + \hat{\mu}(\sin\theta\kappa_1^2 + 2\cos\theta\kappa_1)(\cos\theta + \sin\theta\kappa_1)\sin^2\theta,$$

and

$$\hat{T}^d_{13} = \Lambda(\kappa_1, M_1, M_3)\kappa_{1,\tau},$$

where

$$\Lambda \equiv \left\{\frac{1}{2} + \hat{\eta}_E(M_1 + M_3\kappa_1)^2 M_3^2 + \frac{1}{2}(1 - \hat{\eta}_T)(2M_1 + M_3\kappa_1)M_3\kappa_1\right\}.$$

We point out that because of the requirement (2.9) it must be $\Lambda > 0$. Moreover, if $\hat{\eta}_T = 1$ and $\hat{\eta}_E = 0$, we obtain $\Lambda \equiv 1/2$.

The BVP to solve for the recovery phenomena is given by

$$\kappa_{1,\tau} = \frac{\hat{T}_\infty - \hat{T}^e_{13}}{\Lambda}, \quad \kappa_1(0) = 0, \quad \lim_{\tau\to\infty}\kappa_1(\tau) = \kappa_{1,\infty}. \tag{3.7}$$

Here $\kappa_{1,\infty}$ is the solution of the equation

$$\hat{T}_\infty = \hat{T}^e_{13}(\kappa_{1,\infty}). \tag{3.8}$$

We remind, without loss of generality, that we choose $\hat{T}_\infty > 0$.

The differential equation in this BVP problem may be solved by quadrature as

$$\int \frac{\Lambda}{\hat{T}_\infty - \hat{T}^e_{13}} d\kappa_1 = \tau + \text{const.} \tag{3.9}$$

The following theorem is fundamental for our analysis.

Theorem: If $\hat{\mathcal{W}}$ is an analytic function of the elastic invariants and

$$\frac{d\hat{T}^e_{13}}{d\kappa_1} \equiv \frac{d^2\hat{\mathcal{W}}}{d\kappa_1^2} > 0, \qquad \lim_{\kappa_1 \to \infty} \hat{T}^e_{13} = \infty, \tag{3.10}$$

the solution $\kappa_1(\tau)$ of the BVP (3.7) exists, it is unique and regular.

The hypothesis in (3.10) ensures that the algebraic solution $\kappa_{1,\infty} > 0$ of (3.8) exists for any choice of $\hat{T}_\infty > 0$ and that this algebraic solution is unique. Therefore, being $\Lambda > 0$, it is possible to write down the solution of the BVP (3.7) using (3.9) as

$$\int_0^{\kappa_1} \frac{\Lambda(s)}{\hat{T}_\infty - \hat{T}^e_{13}(s)} ds = \tau. \tag{3.11}$$

Indeed for $\kappa_1 \in [0, \kappa_{1,\infty}[$ the LHS of (3.11) is an improper integral and $\kappa_1(\tau)$ defined by (3.11) is a regular monotone increasing function such that $\kappa_1(0) = 0$ and $\lim_{\tau\to\infty} \kappa_1 = \kappa_{1,\infty}$. $\square$

The problem is that, given a strain-energy density, the conditions (3.10) may be not guaranteed for all the directions $\boldsymbol{M}$ of the fibers. For example, for the standard linear solid, being

$$\frac{d\hat{T}^e_{13}}{d\kappa_1} = 3\hat{\mu}\sin^4\theta\kappa_1^2 + 6\hat{\mu}\sin^3\theta\cos\theta\kappa_1 + 1 + 2\hat{\mu}\sin^2\theta\cos^2\theta,$$

the roots of

$$d\hat{T}^e_{13}/d\kappa_1 = 0 \tag{3.12}$$

are given by

$$\kappa_1^{\pm} = -\frac{\cos\theta}{\sin\theta} \pm \frac{\sqrt{3}\sqrt{\Delta}}{3\sin^2\theta}, \tag{3.13}$$

where

$$\Delta = \cos^2\theta \sin^2\theta - \frac{1}{\hat{\mu}},$$

and the roots in (3.13) are real so that the monotony of $\hat{T}^e_{13}(\kappa_1)$ is not ensured.

For this reason it is fundamental to consider the sign of Δ for $\theta \in [-\pi/2, \pi/2]$ (this range span all the fibers directions in the X_1-X_3 plane). In Fig. 2 we plot $\Delta = 0$ and we divide the plane θ-$\hat{\mu}$ in four regions. When $\theta = 0$ (the red line in Fig. 3), it is $\hat{T}^e_{13} \equiv \kappa_1$: this is a special situation of no interest.

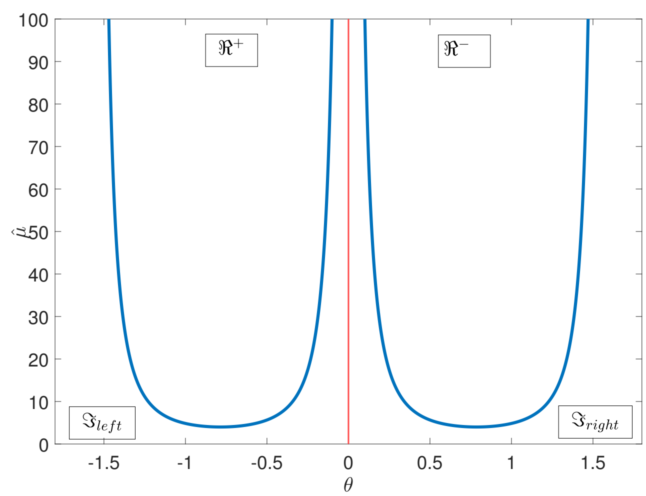

Figure 3: Plot of $\Delta = 0$ for $\theta \in [-\pi/2, \pi/2]$.

On the other hand, when we are in the regions $\Im_{left}$ and $\Im_{right}$ it is $\Delta < 0$

and no real roots of (3.12) exist, therefore the conditions (3.10) are fulfilled. For $\theta \in [0, \pi/2]$ real roots in the region $\Re^-$ exist but they are negative therefore when $\hat{T}_\infty > 0$ they have no impact on the existence and regularity of the solutions.

The problems arise only if we are in the region $\Re^+$ and to answer about what is going on in this eventuality we consider the special value $\theta = -\pi/4$. In this case

$$\kappa_1^\pm = 1 \pm \frac{\sqrt{3\hat{\mu}(\hat{\mu}-4)}}{3\hat{\mu}}, \tag{3.14}$$

and it is clear that we enter the region $\Re^+$ when $\hat{\mu} = 4$; for this reason in Fig. 4 we plot the shearing stress $\hat{T}^e_{13}$ for $\hat{\mu} = 2, 4, 8$.

For $\hat{\mu} = 2$ we are in the region $\Im_{left}$ and the elastic stress is a monotone increasing function of κ_1. For $\hat{\mu} = 4$ we are on the curve $\Delta = 0$ border of the regions $\Im_{left}$ and $\Re^+$ and we have that the roots $\kappa_1^+ = \kappa_1^-$ are coincident, i.e we have a double root $\kappa_1^\pm = 1$ (see the red vertical dotted line in Fig. 4); the $\hat{T}^e_{13}(\kappa_1)$ is once again a monotonic increasing function of its argument.

For the last choice, i.e. $\hat{\mu} = 8$, the real roots are (see the green vertical dotted lines in Fig. 4)

$$\kappa_1^- \approx 0.592, \quad \kappa_1^+ \approx 1.408,$$

therefore we note a non monotonic behaviour and our theorem it is no more valid.

To understand how it is possible to solve the BVP (3.7) in the non-monotonic case it is interesting to analyze in more details the *borderline* case $\hat{\mu} = 4$. In Fig. 5 we plot the solution of our problem for $\hat{T}_\infty = 0.9$ and $\hat{T}_\infty = 1$. In this last case,

$$1 - \hat{T}^e_{13} = 0$$

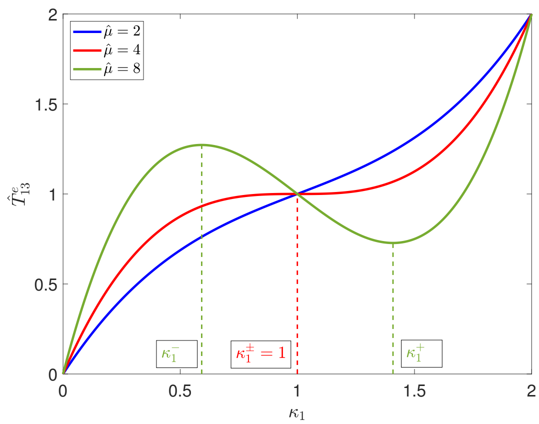

Figure 4: Plot of $\hat{T}^e_{13}(\kappa_1)$ for various values of $\hat{\mu}$.

is characterized by a unique algebraic solution $\kappa_{1,\infty} = 1$ but this solution is a double root. It is therefore possible to apply our theorem but we point out that when $\hat{T}_\infty = 1$ the improper integral in the solution (3.11) diverges as $(\kappa_1 - \kappa_{1,\infty})^{-2}$ and not as usual like $(\kappa_1 - \kappa_{1,\infty})^{-1}$. For this reason the time we need to "substantially" approach the asymptotic value is much longer (see Fig. 5).

The next step is to increase the applied shearing stress to a value $\hat{T}_\infty > 1$. For example, in Fig. 6 we consider $\hat{T}_\infty = 1.1$. In this case the solution $\kappa_1(\tau)$ is quite interesting because it is still regular and compatible with our intuition of the creep phenomena but around the value of $\kappa_1 = 1$ the solution seems to be linear: observe in Fig. 6 that the derivative of κ_1 in a range centered in $\kappa_1 = 1$ is almost flat.

The previous solution is helpful to understand what happens to the creep solution when we contradict the first of the requirements in (3.10) as in the case $\theta = -\pi/4$ and $\hat{\mu} = 8$.

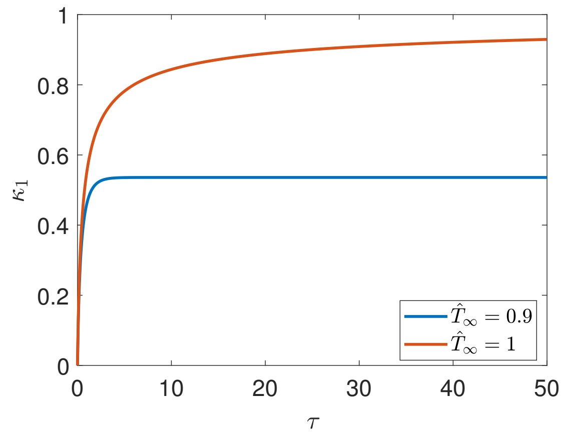

Figure 5: Plot of the creep solution $\kappa_1(\tau)$ for $\hat{T}_\infty = 0.9, 1$, $\Lambda = 1/2$ and $\hat{\mu} = 4$.

In Fig. 7 we plot $\hat{T}^e_{13}$ for $\hat{\mu} = 8$ and the corresponding Maxwell line (red line in the plot) obtained, as usual, via the equal areas rule. It is well known that the Maxwell criterion plays a fundamental role in the static case because it is connected with the emergence and development of kink surfaces in the context of a variety of boundary value problems for anisotropic elastic materials. Kink surfaces are surfaces across which the deformation gradient is discontinuous [20].

In Fig. 7 we note that the Maxwell line intersects $\hat{T}^e_{13}(\kappa_1)$ in κ_1^* along the first stable branch, in κ_1^{**} along the second stable branch, and in $\kappa_1 = 1$ along the metastable branch. Therefore if we consider $\hat{T}_\infty$ under the Maxwell line or just on the Maxwell line we have no problems, our Theorem applies and the creep solution is the standard one.

If we consider $\hat{T}_\infty$ above the Maxwell line two are the possibilities. If during the creep experiment we are controlling the displacement (therefore

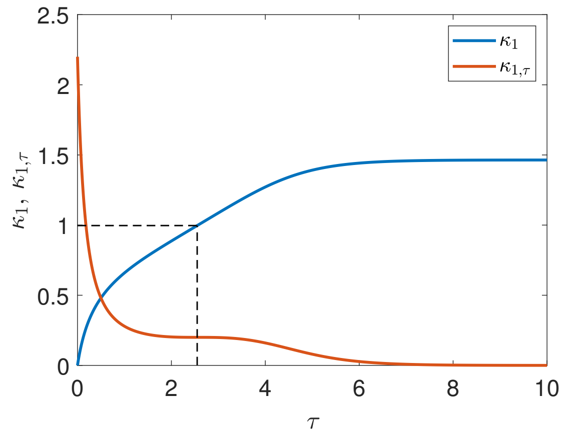

Figure 6: Plot of the creep solution $\kappa_1(\tau)$ and its temporal derivative $\kappa_{1,\tau}$ for $\hat{T}_\infty = 1.1$ and $\hat{\mu} = 4$.

we are using a hard device in the sense of [21]), we start from $\kappa_1(0) = 0$ and, to reach $\kappa_{1,\infty}$, we first have to move along first branch of $\hat{T}^e_{13}$ up to k_1^*. When we have reached this amount of shear, we move on the Maxwell line up to κ_1^{**} where we drop the Maxwell line to continue on the second branch of the stress/strain curve toward the asymptotic value of the amount of shear. The solution in this situation is reported in Fig. 8. We point out that this solution is german of the solution we have already analyzed in Fig. 6: indeed in the range $[k_1^*, k_1^{**}]$, centered in $k_1 = 1$, now the derivative is really constant.

It is important to note that in the non monotonic situation the solution is no more analytic.

On the other hand, in a soft device [21], it is possible also to consider that the amount of shear $\kappa_1(\tau)$ snaps through between the two stable branches as soon as we arrive in κ_1^*. In this situation the discontinuity of the function $\kappa_1(\tau)$ is more evident. A similar phenomena has been investigated in the

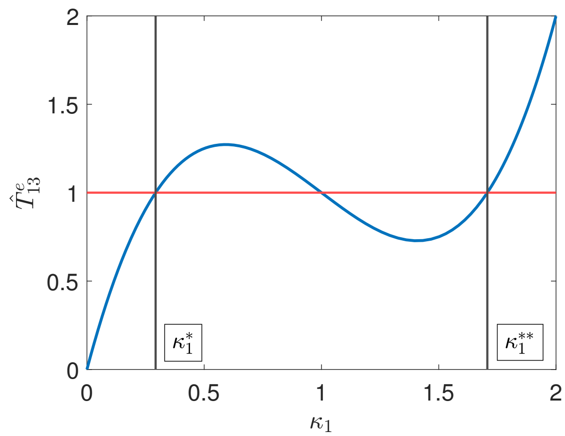

Figure 7: Plot of the Maxwell line for $\hat{T}^e_{13}(\kappa_1)$ when $\hat{\mu} = 8$.

static case in [20, 22].

This example shows what is going on when our theorem cannot be applied. It is clear that in an experiment designed to measure the viscosity moduli it is reasonable to avoid these, theoretically interesting, but, in some sense, "exotic" phenomena and therefore to choose angles between the shear direction and the fibers away from the region $\Re^+$.

3.4 General fiber direction

Having analyzed in detail the reasons that can mathematically lead pathologies, it is now possible to treat the general case in such a way as to avoid these problems.

When $M_2 \neq 0$ the equations (2.8) are not decoupled and it is necessary to require two steps for the solution of the corresponding BVP.

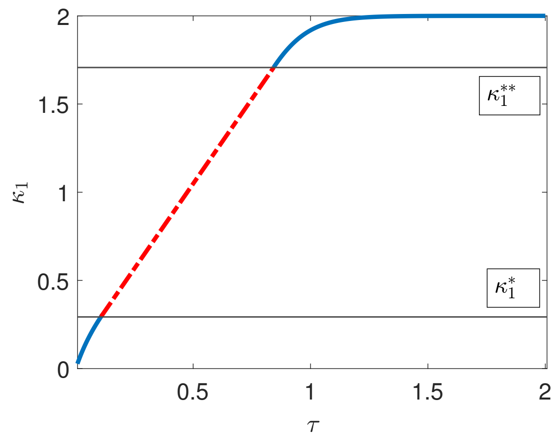

Figure 8: Solution of (3.7) when $\hat{\mu} = 8$ and $\hat{T}_\infty = 2$.

First, we have to ensure the uniqueness of the solution of the system

$$\hat{T}_{1,\infty} = \hat{T}^e_{13}(\kappa_{1,\infty}, \kappa_{2,\infty}), \quad \hat{T}_{2,\infty} = \hat{T}^e_{23}(\kappa_{1,\infty}, \kappa_{2,\infty}).$$

Using the relationship (2.4) this uniqueness is guaranteed by assuming the positive definiteness of the Hessian matrix of the strain energy function $\hat{\mathcal{W}}(\cdot,\cdot)$, namely,

$$\frac{\partial^2 \hat{\mathcal{W}}(\kappa_1,\kappa_2)}{\partial \kappa_1^2}\eta^2 + 2\frac{\partial^2 \hat{\mathcal{W}}(\kappa_1,\kappa_2)}{\partial \kappa_1 \partial \kappa_2}\eta\xi + \frac{\partial^2 \hat{\mathcal{W}}(\kappa_1,\kappa_2)}{\partial \kappa_2^2}\xi^2 > 0, \quad \forall(\eta,\xi) \neq (0,0). \tag{3.15}$$

Second, we have to rewrite the (2.8) as

$$\mathbb{A}(\kappa_1,\kappa_2)\begin{bmatrix}\kappa_{1,\tau}\\ \kappa_{2,\tau}\end{bmatrix} = \begin{bmatrix}\hat{T}_{1,\infty} - \hat{T}^e_{13}(\kappa_1,\kappa_2)\\ \hat{T}_{2,\infty} - \hat{T}^e_{13}(\kappa_1,\kappa_2)\end{bmatrix}. \tag{3.16}$$

Since the matrix $\mathbb{A}$, the same we have in (2.10), is positive definite the inverse $\mathbb{A}^{-1}$ exists and therefore it is possible to rewrite the system (3.16) in normal form

$$\kappa_{1,\tau} = \mathcal{E}_1(\kappa_1, \kappa_2), \quad \kappa_{2,\tau} = \mathcal{E}_2(\kappa_1, \kappa_2). \tag{3.17}$$

The functions $\mathcal{E}_1$, $\mathcal{E}_2$ under our hypothesis are clearly analytic when $(\kappa_1, \kappa_2) \in [0, \kappa_{1,\infty}] \times [0, \kappa_{2,\infty}]$ and standard theorems [23] about systems of ordinary differential equations ensure that the initial value problem composed by (3.17) and the initial conditions $\kappa_1(0) = 0, \kappa_2(0) = 0$ admits a unique regular solution. This solution is the one with the right asymptotic behaviour requested by the creep phenomena.

Therefore, under the hypotheses (2.9) and (3.15), we may use, as we have done in the linear case, the creep experiment to characterize the viscosity moduli in the nonlinear context in a direct and safe way, and clearly the method we have here outlined may be applied to any general strain energy density $\mathcal{W}(I_1, I_2, I_4)$. When the condition (3.15) fails we have shown in the previous section that it is possible to obtain in a more complex way the creep solution. This situation does not preclude useful information for the determination of viscous parameters but complicates the mathematical representation of the experiment in an unnecessary way

The solution of the non linear system (2.3) is shown in Fig. 9, for which the same parameters of the linear case considered in Fig. 1 are used. We point out that the linear and nonlinear case are qualitatively similar.

4 Concluding Remarks

We have presented a nonlinear model of viscoelasticity that is the strict analogue for the case of a transversely isotropic solid of the basic Kelvin-

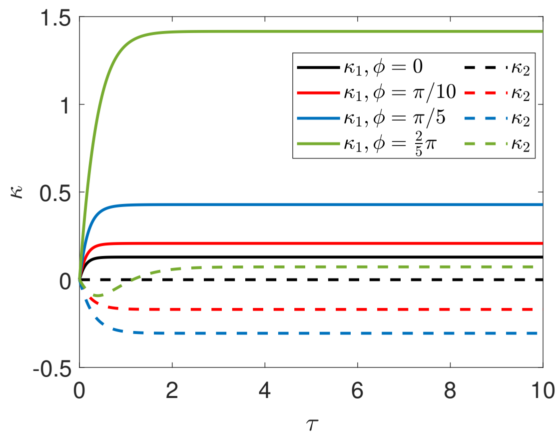

Figure 9: Plot of κ_1 and κ_2 for different values of θ and $\phi = \pi/4$ in the nonlinear case. We are using the same values for the parameters we have used in Fig. 1, i.e. $\hat{T}_\infty = 1$, $\hat{\mu} = 4$, $\eta_T = 0.5$, $\eta_E = 2$, $\eta_L = 1$.

Voigt isotropic model (1.1). Moreover, we have shown that the classical recovery experiment in a simple shear quasistatic deformation is sufficient to experimentally measure the three viscous parameters that characterize the model (1.5) we have proposed.

Once again, we have made clear that in continuum mechanics an accurate rheological model of the complex mechanical behaviour of materials cannot be done without laying its foundations on a robust understanding of hyperelasticity. Indeed, the possibility of using experimental tests based on simple shear motions depends substantially on the behaviour of the elastic part of the Cauchy stress tensor. This fact supports Alan Wineman's point of view [24]: *An education in nonlinear elasticity means that one is also educated, though perhaps unaware of it, in the foundations of several additional subjects.*

It is believed that the model proposed here can be a starting point for a more rigorous analysis of the nonlinear viscoelasticity of anisotropic materials. In fact, in the current literature, the choice of the numerous constitutive functions that appear in the completely general theory is made empirically and arbitrarily. In our model, however, one can still aspire to have control of the various dissipative parts of the Cauchy stress tensor and it is possible to have a clear understanding of the possibility of certain simplifications of (1.5). For example, when we study thermoplastic composites it is usual to consider the reinforcing fibers as inextensible [5]. In this framework it is therefore possible to set the extensional viscosity $\eta_E \equiv 0$. However, when some soft tissues as ligaments are considered, the viscoelastic characteristic of the fibers is more fundamental than the one of the matrix and therefore we may hypothesize $\eta_L \equiv \eta_T \equiv 0$ [25]. Our results can also be generalised with some computational effort to the case of more complex strain-energy density functions able to take into account the case of materials with statistically oriented reinforcements as in [14].

Other opportunities for model (1.5) may be obtained considering the viscoelastic parameters as functions of the various invariants. In so doing we may be able to capture for example strain rate effects in the viscosities. However, one must be very careful when considering these generalizations because the good mathematical position of these models even in the quasistatic case can be quite complex [26]. Finally, it must be remembered that the Kelvin-Voigt model cannot describe the stress-relaxation phenomena typical of many polymeric models but it is not hard to use (1.5) to generalize to an anisotropic framework a model like the one proposed in [27].

For all these reasons we think that the model proposed here has a great potential to be applied in various contexts, where it is necessary to describe the mechanical behaviour of anisotropic polymeric materials or soft tissues.

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References

[1] Wineman, A. S., & Rajagopal, K. R. (2000). Mechanical response of polymers: an introduction. Cambridge university press.

[2] Antman, S. S.,& Malek-Madani, R. (1988). Travelling waves in nonlinearly viscoelastic media and shock structure in elastic media. Quarterly of applied mathematics, 46(1), 77-93.

[3] Destrade, M., & Saccomandi, G. (2005). Finite amplitude elastic waves propagating in compressible solids. Physical Review E, 72(1), 016620.

[4] Destrade, M., Saccomandi, G., & Vianello, M. (2013). Proper formulation of viscous dissipation for nonlinear waves in solids. The Journal of the Acoustical Society of America, 133(3), 1255-1259.

[5] McGuinness, G. B., & Ó Brádaigh, C. M. (1997). Development of rheological models for forming flows and picture-frame shear testing of fabric reinforced thermoplastic sheets. Journal of Non-Newtonian Fluid Mechanics, 73(1-2), 1-28.

[6] De Vita, R., & Slaughter, W. S. (2006). A structural constitutive model for the strain rate-dependent behavior of anterior cruciate ligaments. International Journal of Solids and Structures, 43(6), 1561-1570.

[7] Merodio, J. (2006). On constitutive equations for fiber-reinforced nonlinearly viscoelastic solids. Mechanics Research Communications, 33(6), 764-770.

[8] Anssari-Benam, A., Bucchi, A., Screen, H. R., & Evans, S. L. (2017). A transverse isotropic viscoelastic constitutive model for aortic valve tissue. Royal Society Open Science, 4(1), 160585.

[9] O'Callaghan, L., O'Reilly, O. M., & Zhornitskaya, L. (2011). On azimuthal shear waves in a transversely isotropic viscoelastic mixture: Application to diffuse axonal injury. Mathematics and mechanics of solids, 16(6), 625-636.

[10] Destrade, M., & Saccomandi, G. (2007). Creep, recovery, and waves in a nonlinear fiber-reinforced viscoelastic solid. SIAM Journal on Applied Mathematics, 68(1), 80-97.

[11] Truesdell, C., & Toupin, R. (1960). The classical field theories. In Principles of classical mechanics and field theory/Prinzipien der Klassischen Mechanik und Feldtheorie (pp. 226-858). Springer, Berlin, Heidelberg.

[12] Spencer, A. J. M. (2004). Some results in the theory of non-Newtonian transversely isotropic fluids. Journal of non-newtonian fluid mechanics, 119(1-3), 83-90.

[13] Gilormini, P., & Chinesta, F. "Viscous drag and rod orientation kinematics in anorthotropic fluid." Journal of Non-Newtonian Fluid Mechanics 270 (2019): 96-103.

[14] Grillo, A., Wittum, G., Tomic, A., Federico, S. "Remodelling instatistically oriented fibre-reinforced materials and biological tissues." Mathematics and Mechanics ofSolids 20, no. 9 (2015): 1107-1129.

[15] Merodio, J., & Ogden, R. W. (2002). Material instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation. Archives of Mechanics, 54(5-6), 525-552.

[16] Saccomandi, G., & Vergori, L. (2016). Large time approximation for shearing motions. SIAM Journal on Applied Mathematics, 76(5), 1964-1983.

[17] Coco, M., & Saccomandi, G. (2022). Superposing plane strain on anti-plane shear deformations in a special class of fiber-reinforced incompress-

ible hyperelastic materials. International Journal of Solids and Structures, 256, 111994.

[18] Grine, F., Saccomandi, G., & Arfaoui, M. (2020). Elastic machines: a non standard use of the axial shear of linear transversely isotropic elastic cylinders. International Journal of Solids and Structures, 185, 57-64.

[19] Pucci, E., & Saccomandi, G. (2010). On a special class of nonlinear viscoelastic solids. Mathematics and mechanics of solids, 15(8), 803-811.

[20] Baek, S., & Pence, T. J. (2010). Emergence and disappearance of load induced fiber kinking surfaces in transversely isotropic hyperelastic materials. Zeitschrift fuer angewandte Mathematik und Physik, 61(4), 745-772.

[21] Ericksen, J. L. (1975). Equilibrium of bars. Journal of elasticity, 5, 191-201.

[22] Merodio, J., Saccomandi, G., & Sgura, I. (2007). The rectilinear shear of fiber-reinforced incompressible non-linearly elastic solids. International Journal of Non-Linear Mechanics, 42(2), 342-354.

[23] Amann, H. (1990). Ordinary differential equations, volume 13 of de Gruyter Studies in Mathematics.

[24] Wineman, A. S. (2011). Nonlinear elasticity and its role in continuum theories. Continuum Mechanics. EOLSS Publication.

[25] Sasaki, N., & Odajima, S. (1996). Stress-strain curve and Young's modulus of a collagen molecule as determined by the X-ray diffraction technique. Journal of biomechanics, 29(5), 655-658.

[26] Pucci, E., & Saccomandi, G. (2012). On the nonlinear theory of viscoelasticity of differential type. Mathematics and mechanics of solids, 17(6), 624-630.

[27] Filograna, L., Racioppi, M., Saccomandi, G., & Sgura, I. (2009). A simple model of nonlinear viscoelasticity taking into account stress relaxation. Acta mechanica, 204(1-2), 21.