Existence of least energy solutions for a quasilinear Choquard equation

EXISTENCE OF LEAST ENERGY SOLUTIONS FOR A QUASILINEAR CHOQUARD EQUATION

VINCENZO AMBROSIO, GIUSEPPINA AUTUORI AND TERESA ISERNIA

ABSTRACT. The present paper is devoted to the study of the quasilinear Choquard equation driven by the p-Laplacian operator

$$-\Delta_p u + |u|^{p-2}u = (I_\alpha * G(u))G'(u) \quad \text{in } \mathbb{R}^N,$$

where $2 \leq p < N$, I_α denotes the Riesz potential of order $\alpha \in (0, N)$, and $G \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$. Assuming *Berestycki–Lions* type conditions on G, we prove the existence of a least energy solution $u \in W^{1,p}(\mathbb{R}^N)$ by means of variational methods. Moreover, we establish some qualitative properties of u when G is even and non–decreasing.

1. Introduction

In this paper we study existence and qualitative properties of solutions to the following quasilinear Choquard equation

$$-\Delta_p u + |u|^{p-2}u = (I_\alpha * G(u))g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $2 \leq p < N$, $\Delta_p \cdot = \mathrm{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ is the p–Laplacian operator, $g : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity satisfying suitable conditions, $G(t) = \int_0^t g(\tau)d\tau$, $0 < \alpha < N$, and I_α denotes the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha} \cdot \frac{1}{|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

When $p = 2$, we see that (1.1) boils down to the nonlocal elliptic equation

$$-\Delta u + u = (I_\alpha * G(u))\, g(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Choosing $G(u) = \dfrac{|u|^q}{q}$, with $q \in (1, \infty)$, equation (1.2) becomes

$$-\Delta u + u = \left(I_\alpha * \frac{|u|^q}{q}\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

In particular, if $N = 3$ and $\alpha = q = 2$, then (1.3) turns out to be the so called Choquard–Pekar equation

$$-\Delta u + u = \left(I_2 * \frac{|u|^2}{2}\right)u \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

It was introduced in 1954 by S. Pekar [23] to describe the quantum theory of a polaron at rest, and later, in 1976, it appeared in [13] as a model proposed by P. Choquard in the study of

2010 *Mathematics Subject Classification.* 35A15, 35J92, 35B06, 35B38.
Key words and phrases. p-Laplacian operator; Choquard equation; variational methods; qualitative properties.

an electron trapped in its own hole, in a certain approximation to Hartee–Fock theory of one component plasma. Equation (1.4) was also proposed in 1996 by R. Penrose [18] as a model of self–gravitating matter and, in that context, it is known as the nonlinear Schrödinger–Newton equation. Note that, if u solves (1.1), then the function $\psi(t,x) = e^{it}u(x)$ is a solitary wave of the time–dependent Hartee equation

$$i\psi_t + \Delta\psi = -(I_2 * |\psi|^2)\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

and, in this context, (1.1) is also known as the stationary nonlinear Hartee equation. The first existence results for (1.4), via variational methods, are due to H.L. Lieb [13], P.-L. Lions [15] and G.P. Menzala [17]. Later, L. Ma and L. Zhao [16] showed that the positive solutions of (1.3) must be radially symmetric and monotone decreasing about some fixed point, under the assumption that a given set of real numbers, defined in terms of N, α, and q, is nonempty. These existence and symmetry results for (1.3) have been extended by V. Moroz and J. Van Schaftingen [19] for the optimal range of exponents $q \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. Subsequently, in [20] they examined the existence and qualitative properties of least energy solutions for (1.2) whenever F is a general nonlinearity of *Berestycki–Lions* type [7]. In [21] V. Moroz and J. Van Schaftingen used a suited nonlocal penalization argument to investigate the existence of semi-classical solutions to

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}\left(I_\alpha * |u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a small parameter, $V \in \mathcal{C}(\mathbb{R}^N, [0,\infty))$ is an external potential with some restrictions on the decay at infinity and having a local minimum, and $q \geq 2$ belongs to an optimal range of exponents. Inspired by [21], C.O. Alves and M. Yang [4] analyzed the following quasilinear Choquard equation

$$-\varepsilon^p\Delta_p u + V(x)|u|^{p-2}u = \varepsilon^{-\alpha}\left(I_\alpha * H(u)\right)h(u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $p \in (1,N)$, $\alpha \in (N-p,N)$, V is a positive continuous potential with a local minimum, and h is a $\mathcal{C}^1$-nonlinearity verifying the following conditions:

(h_1) there exist $p < \sigma_1 \leq \sigma_2 < \frac{\alpha p}{N-p}$ and $C_0 > 0$ such that

$$|h(t)| \leq C_0(|t|^{\sigma_1-1} + |t|^{\sigma_2-1}) \ \text{ for all } t \in \mathbb{R};$$

(h_2) there exists $\theta > p$ such that $0 < \theta H(t) = \theta\int_0^t h(\tau)\,d\tau \leq 2th(t)$ for all $t > 0$;

(h_3) there exists $\varsigma \in [\frac{\theta}{2}, \frac{\alpha p}{N-p} + \frac{\theta}{2} - p)$ such that

$$h'(t)t^2 - \left(p + \varsigma - \frac{\theta}{2} - 1\right)h(t)t > 0 \ \text{ for all } t > 0.$$

Combining variational methods with Ljusternik–Schnirelmann category theory, the authors studied existence, multiplicity and concentration of positive solutions to (1.5) (see also [2,3] for related results). Observe that, adapting the Nehari manifold argument used in [5], one can obtain the existence of a positive least energy solution to (1.5) with $V = \varepsilon = 1$, by considering nonlinearities $h \in \mathcal{C}(\mathbb{R},\mathbb{R})$ that satisfy (h_1), (h_2) and the following monotonicity condition:

$(h_3)'$ $t \in (0,\infty) \mapsto h(t)t^{1-p/2}$ is increasing.

For a more detailed discussion about the Choquard equation and its variants and generalizations, we refer to [22] and references therein.

Motivated by [2–4,20], the purpose of this paper is to extend the results in [20] for the quasilinear Choquard equation (1.1) when G is a *Berestycki–Lions* type nonlinearity. Throughout the paper, we assume that $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$ satisfies the following assumptions:

(g_1) there exists $C > 0$ such that

$$|t\, g(t)| \le C\left(|t|^{\frac{N+\alpha}{N}\cdot\frac{p}{2}} + |t|^{\frac{N+\alpha}{N-p}\cdot\frac{p}{2}}\right) \qquad \text{for all } t \in \mathbb{R},$$

(g_2)

$$\lim_{t\to 0} \frac{G(t)}{|t|^{\frac{N+\alpha}{N}\cdot\frac{p}{2}}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{G(t)}{|t|^{\frac{N+\alpha}{N-p}\cdot\frac{p}{2}}} = 0,$$

(g_3) there exists $t_0 \in \mathbb{R}\setminus\{0\}$ such that $G(t_0) \neq 0$.

Note that assumptions (g_1)–(g_3) are more general than (h_1)–(h_3) imposed in [4], and when $p = 2$ they coincide with those in [20].

Problem (1.1) has a variational nature. Indeed, the critical points of the energy functional $\mathcal{J} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1), namely,

$$\mathcal{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N} \{|\nabla u|^p + |u|^p\}\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u))\, G(u)\, dx,$$

are weak solutions of (1.1). We recall that $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution to (1.1) if for every $\varphi \in W^{1,p}(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx + \int_{\mathbb{R}^N} |u|^{p-2}u\,\varphi\, dx = \int_{\mathbb{R}^N} (I_\alpha * G(u))\, g(u)\,\varphi\, dx. \tag{1.6}$$

The functional $\mathcal{J}$ is well defined by (g_1), the Sobolev embeddings for $W^{1,p}(\mathbb{R}^N)$ (see [1]), and the following Hardy–Littlewood–Sobolev inequality.

Theorem 1.1. [14, Theorem 4.3] *Let $r, t \in (1,\infty)$ and $\mu \in (0,N)$ with $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{t} = 2$. Let $f \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(r,N,\mu,t) > 0$, independent of f and h, such that*

$$\left|\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\mu}\, dxdy\right| \le C(r,N,\mu,t)\|f\|_r\|h\|_t.$$

We also introduce the Pohožaev functional $\mathcal{P} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathcal{P}(u) = \frac{N-p}{p}\|\nabla u\|_p^p + \frac{N}{p}\|u\|_p^p - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u))\, G(u)\, dx.$$

Thanks to the recent result established in [6], we know that, under suitable restrictions on N, p and α, every weak solution to (1.1) satisfies a Pohožaev type identity.

Theorem 1.2. [6, Theorem 1.2] *Let $p \in [2,\infty)$, $N > p$ and $\alpha \in ((N-2p)_+, N)$. Assume that $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$ fulfills (g_1)-(g_2). Let $u \in W^{1,p}(\mathbb{R}^N)$ be a weak solution to (1.1). Then, $u \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{1,\sigma}_{loc}(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Moreover, u satisfies $\mathcal{P}(u) = 0$, namely, the following Pohožaev identity*

$$\frac{N-p}{p}\|\nabla u\|_p^p + \frac{N}{p}\|u\|_p^p - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u))\, G(u)\, dx = 0. \tag{1.7}$$

In order to state our main results precisely, we give the following definition. Let

$$c_{\mathrm{LE}} = \inf\left\{\mathcal{J}(v) \,:\, v \in W^{1,p}(\mathbb{R}^N) \setminus \{0\},\ \mathcal{J}'(v) = 0\right\}. \tag{1.8}$$

We say that $u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ is a *least energy solution* of (1.1) if $\mathcal{J}'(u) = 0$ and $\mathcal{J}(u) = c_{\mathrm{LE}}$.

Our first result concerns the existence of a least energy solution to (1.1).

Theorem 1.3. *Assume that $2 \le p < N$ and $\alpha \in ((N-2p)_+, N)$. Under assumptions (g_1)–(g_3), problem (1.1) has a least energy solution.*

The proof of Theorem 1.3 rests on variational arguments. Since we are not assuming neither the *Ambrosetti–Rabinowiz* condition (h_2) nor the monotonicity assumption $(h_3)'$, we can not apply the classical Nehari manifold method. To avoid these difficulties, we develop a variational approach inspired by [20]. Denoting by

$$\Gamma = \left\{\gamma \in \mathcal{C}([0,1]; W^{1,p}(\mathbb{R}^N)) \,:\, \gamma(0) = 0 \text{ and } \mathcal{J}(\gamma(1)) < 0\right\}$$

we consider the mountain pass value

$$c_{\mathrm{MP}} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{J}(\gamma(t)). \tag{1.9}$$

We first construct a *Pohožaev–Palais–Smale sequence* $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ at level c_{MP}, that is, a *Palais–Smale sequence* at level c_{MP} and that satisfies asymptotically the *Pohožaev identity* (1.7); see Proposition 2.1. In particular, we have that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R}^N)$; see Lemma 2.1. To study the convergence of $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\mathbb{R}^N)$, we prove an almost everywhere convergence of the gradients of bounded *Palais–Smale sequences*; see Lemma 2.2. Utilizing this result and a concentration-compactness argument, in Proposition 2.2 we show that the sequence $(u_n)_{n\in\mathbb{N}}$ converges, up to translation and extraction of subsequences, in $W^{1,p}(\mathbb{R}^N)$ to a nontrivial solution u to (1.1). Finally, we deduce that this solution is indeed a least energy solution for (1.1), by constructing suitable paths associated with critical points; cf. Proposition 2.3.

When g is odd and does not change sign on $(0,\infty)$, we obtain some qualitative properties of the least energy solutions by means of polarizations.

Theorem 1.4. *Assume that $2 \le p < N$ and $\alpha \in ((N-2p)_+, N)$. If g satisfies (g_1)–(g_3) and, in addition, g is odd and has constant sign on $(0,\infty)$, then every least energy solution of (1.1) has constant sign and it is radially symmetric with respect to some point in $\mathbb{R}^N$ and radially decreasing.*

We point out that the restrictions $p \ge 2$ and $\alpha \in ((N-2p)_+, N)$ in Theorems 1.3 and 1.4 are the same as those in Theorem 1.2 in order to guarantee the use of (1.7).

We conclude observing that Theorems 1.3 and 1.4 improve [2, Theorem 2.2] and [4, Theorem 2.2] because we are considering more general nonlinearities, and extend [20, Theorem 1] and [20, Theorem 4] to the p-Laplacian case.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.3. In Section 3 we provide the proof of Theorem 1.4.

Notations.

- $\mathcal{B}_r(x_0)$ denotes the ball in $\mathbb{R}^N$ centered at $x_0 \in \mathbb{R}^N$ with radius $r > 0$. When $x_0 = 0$, we set $\mathcal{B}_r = \mathcal{B}_r(0)$;

- $\|\cdot\|_{L^p(E)}$ denotes the usual norm in the space $L^p(E)$, with $E \subset \mathbb{R}^N$ measurable set and $1 \leq p \leq \infty$. When $E = \mathbb{R}^N$ we set $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^N)}$;
- $W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space endowed with the norm
$$\|u\|_{1,p} = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx\right)^{\frac{1}{p}};$$
- By $\left(W^{1,p}(\mathbb{R}^N)\right)^*$ we indicate the dual space of $W^{1,p}(\mathbb{R}^N)$ with dual norm $\|\cdot\|_*$.

2. Existence of least energy solutions

In this section we prove the existence of a least energy solution. To reach our goal, we construct a sequence of almost critical points of $\mathcal{J}$ at the level c_{MP} given in (1.9) that fulfills asymptotically (1.7). In this regard, we provide the following definitions. We say that $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ is a Pohožaev–Palais–Smale sequence for $\mathcal{J}$ if

$$\begin{aligned}&(\mathcal{J}(u_n))_{n\in\mathbb{N}} \quad \text{is bounded in } \mathbb{R},\\ &\|\mathcal{J}'(u_n)\|_* \to 0,\\ &\mathcal{P}(u_n) \to 0,\end{aligned}$$

as $n \to \infty$. We say that $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ is a Pohožaev–Palais–Smale sequence at the level $d \in \mathbb{R}$ if

$$\begin{aligned}&\mathcal{J}(u_n) \to d,\\ &\|\mathcal{J}'(u_n)\|_* \to 0,\\ &\mathcal{P}(u_n) \to 0,\end{aligned}$$

as $n \to \infty$.

Proposition 2.1. *There exists a Pohožaev–Palais–Smale sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ at the level c_{MP}.*

Proof. We will break down the proof into three primary steps.
Step 1: $\Gamma \neq \emptyset$.
For this purpose, we aim to find a function $u \in W^{1,p}(\mathbb{R}^N)$ satisfying $\mathcal{J}(u) < 0$. Define $\mathcal{G} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ as follows

$$\mathcal{G}(u) = \int_{\mathbb{R}^N} (I_\alpha * G(u))\, G(u)\, dx.$$

Considering (g_1), we can see that $\mathcal{G}$ exhibits continuity in $L^p(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N)$. Define

$$w(x) = t_0\, \chi_{\mathcal{B}_1}(x),$$

where t_0 is determined by (g_3). Note that

$$\mathcal{G}(w) = G(t_0)^2 \int_{\mathcal{B}_1}\int_{\mathcal{B}_1} I_\alpha(x-y)\, dxdy > 0. \tag{2.1}$$

Hence, employing the density of $W^{1,p}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N)$ and (2.1), there exists a function $v \in W^{1,p}(\mathbb{R}^N)$ satisfying $\mathcal{G}(v) > 0$.

For any $T>0$ and $x\in\mathbb{R}^N$, let us define $u_T(x)=v\left(\frac{x}{T}\right)$. Because of $N-p<N<N+\alpha$, we have that

$$\mathcal{J}(u_T)=\frac{T^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla v|^p\,dx+\frac{T^N}{p}\int_{\mathbb{R}^N}|v|^p\,dx-\frac{T^{N+\alpha}}{2}\mathcal{G}(v)\to-\infty \quad \text{as } T\to\infty.$$

Then, for T large enough, we can find a function $u=u_T$ such that $\mathcal{J}(u)<0$. Consequently, Γ is non-empty. In particular, this leads us to the deduction that c_{MP} is finite.

Step 2: $c_{\mathrm{MP}}>0$.

Let $u\in W^{1,p}(\mathbb{R}^N)$. Applying Theorem 1.1 with $r=t=\frac{2N}{N+\alpha}$ and $\mu=N-\alpha$, along with (g_1) and Sobolev inequality [1, Theorem 4.31], we can observe that

$$\begin{aligned}\mathcal{G}(u)&\le C\|G(u)\|^2_{\frac{2N}{N+\alpha}}\\&\le C_1\left(\|u\|_p^p+\|u\|_{p^*}^{p^*}\right)^{1+\frac{\alpha}{N}}\\&\le C_2\left[\|u\|_p^{p(1+\frac{\alpha}{N})}+\|u\|_{p^*}^{p^*(1+\frac{\alpha}{N})}\right]\\&\le C_3\left[\|u\|_p^{p(1+\frac{\alpha}{N})}+\|\nabla u\|_p^{p(\frac{N+\alpha}{N-p})}\right].\end{aligned}$$

Hence, there exists $\delta>0$ such that, if $\|u\|_{1,p}\le\delta$, then

$$\mathcal{G}(u)\le\frac{1}{p}\|u\|^p_{1,p}.$$

Therefore, for all $\|u\|_{1,p}\le\delta$, we have

$$\mathcal{J}(u)=\frac{1}{p}\|u\|^p_{1,p}-\frac{1}{2}\mathcal{G}(u)\ge\frac{1}{2p}\|u\|^p_{1,p}.$$

Now, observe that, when $\gamma\in\Gamma$ it results that

$$\|\gamma(1)\|_{1,p}>\delta.$$

Indeed, assuming by contradiction that $\|\gamma(1)\|_{1,p}\le\delta$, we would get

$$\mathcal{J}(\gamma(1))\ge\frac{1}{2p}\|\gamma(1)\|^p_{1,p}\ge 0,$$

and this contradicts the fact that $\gamma\in\Gamma$. Then, since

$$\|\gamma(0)\|_{1,p}=0<\delta<\|\gamma(1)\|_{1,p},$$

we can use the intermediate value theorem to find a value $\bar\tau\in(0,1)$ such that $\|\gamma(\bar\tau)\|_{1,p}=\delta$, and so

$$\frac{\delta^p}{2p}\le\mathcal{J}(\gamma(\bar\tau))\le\sup_{t\in[0,1]}\mathcal{J}(\gamma(t)).$$

From the arbitrariness of $\gamma\in\Gamma$, we can conclude that

$$c_{\mathrm{MP}}\ge\frac{\delta^p}{2p}>0.$$

Step 3: Conclusion.

Inspired by [11], we introduce the map $\Psi:\mathbb{R}\times W^{1,p}(\mathbb{R}^N)\to W^{1,p}(\mathbb{R}^N)$ defined by

$$\Psi(\sigma,v)(x)=v(e^{-\sigma}x),$$

where we endow $\mathbb{R}\times W^{1,p}(\mathbb{R}^N)$ with the norm

$$\|(\sigma,v)\|_{\mathbb{R}\times W^{1,p}(\mathbb{R}^N)} = (|\sigma|^p + \|v\|_{1,p}^p)^{\frac{1}{p}}.$$

We note that, for every $(\sigma,v)\in\mathbb{R}\times W^{1,p}(\mathbb{R}^N)$,

$$\mathcal{J}(\Psi(\sigma,v)) = \frac{e^{(N-p)\sigma}}{p}\int_{\mathbb{R}^N}|\nabla v|^p\,dx + \frac{e^{N\sigma}}{p}\int_{\mathbb{R}^N}|v|^p\,dx - \frac{e^{(N+\alpha)\sigma}}{2}\mathcal{G}(u).$$

By (g_1), we deduce that $\mathcal{J}\circ\Psi$ is Frechét differentiable on $\mathbb{R}\times W^{1,p}(\mathbb{R}^N)$. Define

$$\tilde{\Gamma} = \left\{\tilde{\gamma}\in\mathcal{C}([0,1];\mathbb{R}\times W^{1,p}(\mathbb{R}^N))\,:\,\tilde{\gamma}(0)=(0,0)\text{ and }(\mathcal{J}\circ\Psi)(\tilde{\gamma}(1))<0\right\}.$$

It readily seen that the mountain pass levels of $\mathcal{J}$ and $\mathcal{J}\circ\Psi$ coincide, that is

$$c_{\mathrm{MP}} = \inf_{\gamma\in\tilde{\Gamma}}\sup_{\tau\in[0,1]}(\mathcal{J}\circ\Psi)(\tilde{\gamma}(\tau)).$$

Employing the minimax principle [27, Theorem 2.8], we can find $((\sigma_n,v_n))_{n\in\mathbb{N}}\subset\mathbb{R}\times W^{1,p}(\mathbb{R}^N)$ such that

$$(\mathcal{J}\circ\Psi)(\sigma_n,v_n)\to c_{\mathrm{MP}},$$
$$(\mathcal{J}\circ\Psi)'(\sigma_n,v_n)\to 0\quad\text{ in }(\mathbb{R}\times W^{1,p}(\mathbb{R}^N))^*.$$

Now, observe that, for every $\sigma,h\in\mathbb{R}$ and $v,w\in W^{1,p}(\mathbb{R}^N)$, it holds

$$\begin{aligned}(\mathcal{J}\circ\Psi)'(\sigma,v)(h,w) = {}&\frac{(N-p)h}{p}e^{(N-p)\sigma}\int_{\mathbb{R}^N}|\nabla v|^p\,dx + e^{(N-p)\sigma}\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla w\,dx\\ &+\frac{Nh}{p}e^{N\sigma}\int_{\mathbb{R}^N}|v|^p\,dx + e^{N\sigma}\int_{\mathbb{R}^N}|v|^{p-2}v\,w\,dx\\ &-\frac{(N+\alpha)h}{2}e^{(N+\alpha)\sigma}\int_{\mathbb{R}^N}(I_\alpha*G(v))\,G(v)\,dx\\ &-e^{(N+\alpha)\sigma}\int_{\mathbb{R}^N}(I_\alpha*G(v))\,g(v)w\,dx.\end{aligned}$$

On the other hand, for all $v,w\in W^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned}\mathcal{J}'(\Psi(\sigma,v))(\Psi(\sigma,w)) = {}&e^{(N-p)\sigma}\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla w\,dx + e^{N\sigma}\int_{\mathbb{R}^N}|v|^{p-2}v\,w\,dx\\ &-e^{(N+\alpha)\sigma}\int_{\mathbb{R}^N}(I_\alpha*G(v))\,g(v)w\,dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}(\Psi(\sigma,v))h = {}&\frac{(N-p)h}{p}e^{(N-p)\sigma}\int_{\mathbb{R}^N}|\nabla v|^p\,dx + \frac{Nh}{p}e^{N\sigma}\int_{\mathbb{R}^N}|v|^p\,dx\\ &-\frac{(N+\alpha)h}{2}e^{(N+\alpha)\sigma}\int_{\mathbb{R}^N}(I_\alpha*G(v))\,G(v)\,dx.\end{aligned}$$

Therefore, for every $(h,w)\in\mathbb{R}\times W^{1,p}(\mathbb{R}^N)$,

$$(\mathcal{J}\circ\Psi)'(\sigma_n,v_n)(h,w) = \mathcal{J}'(\Psi(\sigma_n,v_n))(\Psi(\sigma_n,w)) + \mathcal{P}(\Psi(\sigma_n,v_n))h.$$

Then it suffices to take $u_n=\Psi(\sigma_n,v_n)$ to achieve the conclusion. $\square$

Lemma 2.1. *If $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ is a Pohožaev-Palais-Smale sequence of $\mathcal{J}$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Moreover, there exists $u \in W^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence, as $n \to \infty$,*

$$\begin{aligned} &u_n \rightharpoonup u && \text{in } W^{1,p}(\mathbb{R}^N), \\ &u_n \to u && \text{in } L^r_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, p^*), \\ &u_n \to u && \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{2.2}$$

Proof. Let us observe that, for all $n \in \mathbb{N}$,

$$\mathcal{J}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n) = \frac{\alpha+p}{p(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \frac{\alpha}{p(N+\alpha)} \int_{\mathbb{R}^N} |u_n|^p \, dx.$$

Since $(\mathcal{J}(u_n))_{n\in\mathbb{N}}$ is bounded and $\mathcal{P}(u_n) \to 0$, as $n \to \infty$, we readily deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Using that $W^{1,p}(\mathbb{R}^N)$ is reflexive (see [1, Theorem 3.6]) and that $W^{1,p}(\mathbb{R}^N)$ is compactly embedded in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, p^*)$ (see [1, Theorem 6.3]), we can conclude that (2.2) is valid. □

Subsequently, we demonstrate an almost everywhere convergence of the gradients of Pohožaev–Palais–Smale sequences. First we recall a result appeared in [10] (see also [8]).

Theorem 2.1. [10, Theorem 1] *Let $\Omega \subset \mathbb{R}^N$ be an open set and $p \in (1, \infty)$. For $\varepsilon > 0$ and $t \in \mathbb{R}$, put*

$$S_\varepsilon(t) = \begin{cases} t & \text{if } |t| \le \varepsilon, \\ \varepsilon \dfrac{t}{|t|} & \text{if } |t| \ge \varepsilon, \end{cases} \quad \text{and} \quad t^k = S_k(t) \text{ for } k \ge 1.$$

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $W^{1,p}_{loc}(\Omega)$. Then,

- *There is a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, and a function $u \in W^{1,p}_{loc}(\Omega)$ such that $u_n \to u$ a.e. in Ω, as $n \to \infty$.*
- *If, furthermore, we assume that for all $\varphi \in \mathcal{C}^\infty_c(\Omega)$ and for all $k \ge k_0$:*

  $$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla(\varphi S_\varepsilon(u_n - u^k)) \, dx \le o_\varepsilon(1),$$

  where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, then there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that

  $$\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N.$$

Lemma 2.2. *Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ be a Pohožaev–Palais–Smale sequence of $\mathcal{J}$. Then there exists $u \in W^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence, as $n \to \infty$,*

$$\begin{aligned} &\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N, \\ &|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } \left(L^{\frac{p}{p-1}}(\mathbb{R}^N)\right)^N. \end{aligned}$$

Proof. By Lemma 2.1, we know that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and that, up to a subsequence, $u_n \to u$ a.e. in $\mathbb{R}^N$, for some $u \in W^{1,p}(\mathbb{R}^N)$. Fix $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$. Then we see that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(\varphi S_\varepsilon(u_n-u^k))\,dx \\
&= -\int_{\mathbb{R}^N} |u_n|^{p-2}u_n\varphi S_\varepsilon(u_n-u^k)\,dx + \int_{\mathbb{R}^N} (I_\alpha * G(u_n))g(u_n))\,(\varphi S_\varepsilon(u_n-u^k))\,dx \\
&\quad + \langle \mathcal{J}'(u_n), \varphi S_\varepsilon(u_n-u^k)\rangle \\
&= (I)_{n,\varepsilon,k} + (II)_{n,\varepsilon,k} + (III)_{n,\varepsilon,k}.
\end{aligned} \tag{2.3}
$$

Now, first note that

$$
|S_\varepsilon(t)| \le \varepsilon \quad \text{for all } \varepsilon > 0 \text{ and all } t\in\mathbb{R}. \tag{2.4}
$$

Exploiting the Hölder inequality, the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N)$ and (2.4), we have, for all $n\in\mathbb{N}$, $\varepsilon>0$ and $k\ge 1$,

$$
\begin{aligned}
|(I)_{n,\varepsilon,k}| &\le \int_{\mathbb{R}^N} |u_n|^{p-1}|\varphi S_\varepsilon(u_n-u^k)|\,dx \\
&\le \varepsilon \left(\int_{\mathbb{R}^N} |u_n|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |\varphi|^p\,dx\right)^{\frac{1}{p}} \le C_1\,\varepsilon\,.
\end{aligned} \tag{2.5}
$$

To estimate $(II)_{n,\varepsilon,k}$, we utilize Theorem 1.1, the growth assumption (g_1), condition (2.4), the Hölder inequality, and the fact that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^N)\cap L^{p^*}(\mathbb{R}^N)$, to obtain that, for all $n\in\mathbb{N}$, $\varepsilon>0$ and $k\ge 1$,

$$
\begin{aligned}
|(II)_{n,\varepsilon,k}| &\le C_2 \left(\int_{\mathbb{R}^N} |G(u_n)|^{\frac{2N}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |g(u_n)\varphi S_\varepsilon(u_n-u^k)|^{\frac{2N}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{2N}} \\
&\le C_3\,\varepsilon \left(\|u_n\|_p^p + \|u_n\|_{p^*}^{p^*}\right)^{\frac{N+\alpha}{2N}} \Bigg[\int_{\mathbb{R}^N} |u_n|^{\frac{(N+\alpha)p-2N}{N+\alpha}}|\varphi|^{\frac{2N}{N+\alpha}}\,dx \\
&\qquad + \int_{\mathbb{R}^N} |u_n|^{\frac{((N+\alpha)p-2(N-p))N}{(N-p)(N+\alpha)}}|\varphi|^{\frac{2N}{N+\alpha}}\,dx\Bigg]^{\frac{N+\alpha}{2N}} \\
&\le C_4\,\varepsilon \Bigg[\left(\int_{\mathbb{R}^N} |u_n|^p\,dx\right)^{\frac{(N+\alpha)p-2N}{p(N+\alpha)}} \left(\int_{\mathbb{R}^N} |\varphi|^p\,dx\right)^{\frac{2N}{p(N+\alpha)}} \\
&\qquad + \left(\int_{\mathbb{R}^N} |u_n|^{p^*}\,dx\right)^{\frac{(N+\alpha)p-2(N-p)}{p(N+\alpha)}} \left(\int_{\mathbb{R}^N} |\varphi|^{p^*}\,dx\right)^{\frac{2(N-p)}{p(N+\alpha)}}\Bigg]^{\frac{N+\alpha}{2N}} \le C_5\,\varepsilon\,.
\end{aligned} \tag{2.6}
$$

Concerning $(III)_{n,\varepsilon,k}$, since $\|\mathcal{J}'(u_n)\|_* \to 0$ and $\|\varphi S_\varepsilon(u_n-u^k)\|_{1,p}$ is bounded independently of ε, n, and k, we derive from

$$
|(III)_{n,\varepsilon,k}| \le \|\mathcal{J}'(u_n)\|_* \|\varphi S_\varepsilon(u_n-u^k)\|_{1,p}
$$

that, as $n\to\infty$,

$$
(III)_{n,\varepsilon,k} \to 0 \;\; \text{uniformly with respect to } \varepsilon>0 \text{ and } k>0. \tag{2.7}
$$

Combining (2.3)–(2.7), we deduce that

$$
\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(\varphi S_\varepsilon(u_n-u^k))\,dx \le C_6\,\varepsilon\,.
$$

Then, invoking Theorem 2.1, there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Due to the boundedness of $(|\nabla u_n|^{p-2}\nabla u_n)_{n\in\mathbb{N}}$ in $(L^{\frac{p}{p-1}}(\mathbb{R}^N))^N$, we obtain that, up to a subsequence, $|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u$ in $(L^{\frac{p}{p-1}}(\mathbb{R}^N))^N$. This concludes the proof of Lemma 2.2. □

Let's now examine the convergence of Pohožaev–Palais–Smale sequences.

Proposition 2.2. *Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ be a Pohožaev–Palais–Smale sequence of $\mathcal{J}$. Then,*
(1) *either, up to a subsequence, $u_n \to 0$ in $W^{1,p}(\mathbb{R}^N)$,*
(2) *or there exists $u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ such that $\mathcal{J}'(u)=0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that, $u_n(\cdot - x_n) \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ as $n\to\infty$, up to a subsequence.*

Proof. Suppose that condition (1) is not satisfied, meaning
$$\liminf_{n\to\infty} \|u_n\|_{1,p}^p > 0. \tag{2.8}$$
Let us prove that, for every $q\in(p,p^*)$,
$$\liminf_{n\to\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^q(\mathcal{B}_1(x_0))} > 0. \tag{2.9}$$
We argue indirectly. So, suppose that, for some $q_0\in(p,p^*)$,
$$\liminf_{n\to\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^{q_0}(\mathcal{B}_1(x_0))} = 0. \tag{2.10}$$
Thanks to the continuity of G and (g_2), fixed $\varepsilon>0$ we can find $C_\varepsilon>0$ such that
$$|G(t)|^{\frac{2N}{N+\alpha}} \le \varepsilon\left(|t|^p + |t|^{p^*}\right) + C_\varepsilon |t|^{q_0} \qquad \text{for all } t\in\mathbb{R}. \tag{2.11}$$
Now we note that given $q\in[p,p^*)$, we have, for every $u\in W^{1,p}(\mathbb{R}^N)$,
$$\|u\|_q^q \le C\left(\sup_{x_0\in\mathbb{R}^N} \|u\|_{L^q(\mathcal{B}_1(x_0))}^q\right)^{1-\frac{p}{q}} \|u\|_{1,p}^p. \tag{2.12}$$
Indeed, fixed $x_0\in\mathbb{R}^N$, it follows from the Hölder inequality and the Sobolev embedding $W^{1,p}(\mathcal{B}_1(x_0)) \subset L^{p^*}(\mathcal{B}_1(x_0))$ (see [1, Theorem 4.12]) that
$$\begin{aligned}
\|u\|_{L^q(\mathcal{B}_1(x_0))}^q &\le \left(\int_{\mathcal{B}_1(x_0)} |u|^{\frac{N(q-p)}{p}}\,dx\right)^{\frac{p}{N}} \left(\int_{\mathcal{B}_1(x_0)} |u|^{p^*}\,dx\right)^{\frac{p}{p^*}} \\
&\le C_1 \|u\|_{L^q(\mathcal{B}_1(x_0))}^{q-p} |\mathcal{B}_1|^{\frac{pq-N(q-p)}{Nq}} \|u\|_{W^{1,p}(\mathcal{B}_1(x_0))}^p \\
&\le C_1 |\mathcal{B}_1|^{\frac{pq-N(q-p)}{Nq}} \left(\sup_{x_0\in\mathbb{R}^N} \|u\|_{L^q(\mathcal{B}_1(x_0))}^q\right)^{1-\frac{p}{q}} \|u\|_{W^{1,p}(\mathcal{B}_1(x_0))}^p,
\end{aligned}$$
where C_1 is independent of x_0. Covering $\mathbb{R}^N$ by balls with radius 1 in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we deduce that (2.12) holds.

Then, using (2.11), Lemma 2.1, (2.12), and (2.10), we deduce that

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\mathbb{R}^N}|G(u_n)|^{\frac{2N}{N+\alpha}}\,dx\\
&\le \liminf_{n\to\infty}\left[\varepsilon\left(\int_{\mathbb{R}^N}|u_n|^p\,dx+\int_{\mathbb{R}^N}|u_n|^{p^*}\,dx\right)+C_\varepsilon\int_{\mathbb{R}^N}|u_n|^{q_0}\,dx\right]\\
&\le C\,\varepsilon+C'_\varepsilon\liminf_{n\to\infty}\left(\sup_{x_0\in\mathbb{R}^N}\int_{\mathcal{B}_1(x_0)}|u_n|^{q_0}\,dx\right)^{1-\frac{p}{q}}\\
&\le C\,\varepsilon\,.
\end{aligned}\tag{2.13}$$

By (2.13) and considering that $\varepsilon>0$ is arbitrary, we obtain

$$\liminf_{n\to\infty}\|G(u_n)\|_{\frac{2N}{N+\alpha}}=0.\tag{2.14}$$

On the other hand, exploiting the definition of $\mathcal{P}(u_n)$, (2.8) and the fact that $\mathcal{P}(u_n)\to 0$, as $n\to\infty$, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * G(u_n))\,G(u_n)\,dx\\
&=\liminf_{n\to\infty}\left\{\frac{2}{p}\cdot\frac{N-p}{N+\alpha}\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx+\frac{2}{p}\cdot\frac{N}{N+\alpha}\int_{\mathbb{R}^N}|u_n|^p\,dx-\frac{2}{N+\alpha}\mathcal{P}(u_n)\right\}>0.
\end{aligned}\tag{2.15}$$

Putting together (2.15), Theorem 1.1 and (2.14), we get

$$0<\liminf_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * G(u_n))\,G(u_n)\,dx\le C'\liminf_{n\to\infty}\|G(u_n)\|^2_{\frac{2N}{N+\alpha}}=0,$$

which is a contradiction. Therefore, (2.9) is valid. Consequently, up to a translation, we may suppose that there exists a $q\in(p,p^*)$ such that

$$\liminf_{n\to\infty}\|u_n\|_{L^q(\mathcal{B}_1)}>0.\tag{2.16}$$

Using Lemma 2.1, we can find $u\in W^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence, (2.2) holds. Clearly, (2.2) and (2.16) ensure that $u\not\equiv 0$. Hereafter, we show that $\mathcal{J}'(u)=0$. By (2.2), (g_1), the continuity of g, and the dominated convergence theorem, we see that

$$g(u_n)\to g(u)\ \text{in}\ L^r_{loc}(\mathbb{R}^N)\ \text{ for all } r\in\left[1,\frac{2Np}{(N+\alpha)p-2(N-p)}\right).\tag{2.17}$$

Due to the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N)\cap L^{p^*}(\mathbb{R}^N)$ and assumption (g_1), we can see that $(G(u_n))_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. From the continuity of G, it follows that $G(u_n)\to G(u)$ a.e. in $\mathbb{R}^N$. So, $G(u_n)\rightharpoonup G(u)$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Since Theorem 1.1 implies that the Riesz potential is a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we obtain that

$$I_\alpha * G(u_n)\rightharpoonup I_\alpha * G(u)\ \text{ in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).\tag{2.18}$$

Putting together (2.17) and (2.18), and observing that

$$\frac{N-\alpha}{2N}+\frac{(N+\alpha)p-2(N-p)}{2Np}=\frac{Np-N+p}{Np},$$

we get

$$(I_\alpha * G(u_n))g(u_n) \rightharpoonup (I_\alpha * G(u))g(u) \text{ in } L^r(\mathbb{R}^N) \text{ for all } r \in \left[1, \frac{Np}{Np-N+p}\right). \tag{2.19}$$

Exploiting Lemma 2.2 and (2.19), we find that, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla\varphi + |u|^{p-2}u\,\varphi\right) dx - \int_{\mathbb{R}^N} (I_\alpha * G(u))\, g(u)\,\varphi\, dx \\ &= \lim_{n\to\infty} \left[\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi + |u_n|^{p-2}u_n\,\varphi\right) dx - \int_{\mathbb{R}^N} (I_\alpha * G(u_n))\, g(u_n)\,\varphi\, dx\right] = 0.\end{aligned}$$

The assertion follows by the density of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$ (see [1, Corollary 3.23]). □

In passing from weak solutions to least energy solutions we utilize the following Proposition 2.3 to construct an optimal trajectory. This result follows from the Pohožaev identity (1.7) and permits to associate to any variational solution to (1.1) a suitable path of Γ. The proof is inspired by an argument found in [12, Lemma 2.1].

Proposition 2.3. *Let $u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ be a solution of (1.1). Then, there exists a $t_0 > 1$ and a path $\gamma \in \mathcal{C}\left([0,1]; W^{1,p}(\mathbb{R}^N)\right)$ such that*

$$\begin{aligned}&\gamma(0) = 0,\\ &\gamma(1/t_0) = u,\\ &\mathcal{J}(\gamma(t)) < \mathcal{J}(u) \quad \text{for all } t \in [0,1]\setminus\{1/t_0\},\\ &\mathcal{J}(\gamma(1)) < 0.\end{aligned}$$

Proof. Let us define the path $\phi : [0,\infty) \to W^{1,p}(\mathbb{R}^N)$ by setting

$$\phi(t)(x) = \begin{cases} u\left(\dfrac{x}{t}\right) & \text{if } t > 0,\\ 0 & \text{if } t = 0.\end{cases}$$

For all $t > 0$, we have

$$\|\phi(t)\|_{1,p}^p = t^{N-p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + t^N \int_{\mathbb{R}^N} |u|^p\, dx,$$

and so $\phi \in \mathcal{C}([0,\infty); W^{1,p}(\mathbb{R}^N))$. Now, we compute $\mathcal{J}\circ\phi$:

$$\begin{aligned}\mathcal{J}(\phi(t)) &= \frac{t^{N-p}}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \frac{t^N}{p}\int_{\mathbb{R}^N} |u|^p\, dx - \frac{t^{N+\alpha}}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u))\, G(u)\, dx \\ &= \left(\frac{t^{N-p}}{p} - \frac{(N-p)t^{N+\alpha}}{p(N+\alpha)}\right)\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \left(\frac{t^N}{p} - \frac{Nt^{N+\alpha}}{p(N+\alpha)}\right)\int_{\mathbb{R}^N} |u|^p\, dx,\end{aligned}$$

where in the last equality we used $\mathcal{P}(u) = 0$. Since

$$\frac{d}{dt}(\mathcal{J}(\phi(t))) = t^{N+\alpha-1}\left[\left(\frac{1}{t^{\alpha+p}} - 1\right)\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \left(\frac{1}{t^\alpha} - 1\right)\frac{N}{p}\int_{\mathbb{R}^N} |u|^p\, dx\right],$$

we deduce that

$$\frac{d}{dt}(\mathcal{J}(\phi(t))) = 0 \quad \text{if and only if } t = 1,$$
$$\frac{d}{dt}(\mathcal{J}(\phi(t))) > 0 \quad \text{if and only if } 0 < t < 1,$$
$$\frac{d}{dt}(\mathcal{J}(\phi(t))) < 0 \quad \text{if and only if } t > 1.$$

Thus, $\mathcal{J} \circ \phi$ attains its global maximum at $t = 1$. Finally,

$$\lim_{t\to\infty} \mathcal{J}(\phi(t)) = -\infty,$$

so there exists $t_0 > 1$ such that $\mathcal{J}(\phi(t_0)) < 0$. Taking the path $\gamma(t)(x) = \phi(t_0 t)(x)$ for all $t \in [0, 1]$ and $x \in \mathbb{R}^N$, we obtain the assertion. ☐

Next we give the proof of Theorem 1.3.

Proof of Theorem 1.3. From Propositions 2.1 and 2.2, we can deduce the existence of a Pohožaev–Palais–Smale sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ of $\mathcal{J}$ at level c_{MP}. Moreover, after passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$, $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, for some $u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ which is a weak solution to (1.1). Using Theorem 1.2, $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, Fatou's lemma, and the fact that $\mathcal{P}(u_n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
\mathcal{J}(u) &= \mathcal{J}(u) - \frac{1}{N+\alpha}\mathcal{P}(u) \\
&= \frac{\alpha+p}{p(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^p\,dx + \frac{\alpha}{p(N+\alpha)} \int_{\mathbb{R}^N} |u|^p\,dx \\
&\le \liminf_{n\to\infty} \left(\frac{\alpha+p}{p(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx + \frac{\alpha}{p(N+\alpha)} \int_{\mathbb{R}^N} |u_n|^p\,dx \right) \\
&= \liminf_{n\to\infty} \left(\mathcal{J}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n) \right) \\
&= \liminf_{n\to\infty} \mathcal{J}(u_n) = c_{\mathrm{MP}}.
\end{aligned} \tag{2.20}$$

As u is a nontrivial weak solution to (1.1), we can infer from the definition of c_{LE} provided in (1.8) and relation (2.20), that the following holds

$$c_{\mathrm{LE}} \le \mathcal{J}(u) \le c_{\mathrm{MP}}. \tag{2.21}$$

Now, let $v \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be an arbitrary weak solution to (1.1) with $\mathcal{J}(v) \le \mathcal{J}(u)$. If we lift v to a path as in Proposition 2.3, then the definition of mountain pass level ensures that

$$\mathcal{J}(v) \ge c_{\mathrm{MP}},$$

which together with (2.21) gives

$$\mathcal{J}(u) \ge \mathcal{J}(v) \ge c_{\mathrm{MP}} \ge \mathcal{J}(u) \ge c_{\mathrm{LE}}.$$

Accordingly, $\mathcal{J}(v) = \mathcal{J}(u) = c_{\mathrm{MP}} = c_{\mathrm{LE}}$. The proof of the theorem is now complete. ☐

Lastly, we present a valuable result that enables us to deduce the compactness, with translation invariance, of the set of least energy solutions

$$\mathcal{S}_{\mathrm{LE}} = \left\{ u \in W^{1,p}(\mathbb{R}^N) \,:\, \mathcal{J}(u) = c_{\mathrm{LE}} \text{ and } \mathcal{J}'(u) = 0 \right\}.$$

Corollary 2.1. *Suppose that the assumptions of Proposition 2.2 are satisfied. If*
$$\liminf_{n\to\infty}\|u_n\|_{1,p}^p>0 \quad \text{and} \quad \limsup_{n\to\infty}\mathcal{J}(u_n)\le c_{\rm LE},$$
then we can find $u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ such that $\mathcal{J}'(u)=0$, and $(x_n)_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that, up to a subsequence, $u_n(\cdot-x_n)\to u$ in $W^{1,p}(\mathbb{R}^N)$ as $n\to\infty$.

Proof. According to Proposition 2.2, up to a subsequence and translations, we may suppose that, as $n\to\infty$, $u_n\rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$, $u_n\to u$ and $\nabla u_n\to\nabla u$ a.e. in $\mathbb{R}^N$, for some $u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ that satisfies (1.1). Then, as in the proof of Theorem 1.3, we see that
$$\begin{aligned}
c_{\rm LE}&\le \mathcal{J}(u)-\frac{1}{N+\alpha}\mathcal{P}(u)\\
&=\frac{\alpha+p}{p(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\frac{\alpha}{p(N+\alpha)}\int_{\mathbb{R}^N}|u|^p\,dx\\
&\le\limsup_{n\to\infty}\left(\frac{\alpha+p}{p(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx+\frac{\alpha}{p(N+\alpha)}\int_{\mathbb{R}^N}|u_n|^p\,dx\right)\\
&=\limsup_{n\to\infty}\left(\mathcal{J}(u_n)-\frac{1}{N+\alpha}\mathcal{P}(u_n)\right)\\
&=\limsup_{n\to\infty}\mathcal{J}(u_n)\le c_{\rm LE},
\end{aligned}$$
and thus
$$\begin{aligned}
&\frac{\alpha+p}{p(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\frac{\alpha}{p(N+\alpha)}\int_{\mathbb{R}^N}|u|^p\,dx\\
&=\limsup_{n\to\infty}\left(\frac{\alpha+p}{p(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx+\frac{\alpha}{p(N+\alpha)}\int_{\mathbb{R}^N}|u_n|^p\,dx\right).
\end{aligned}$$
Using that $W^{1,p}(\mathbb{R}^N)$ is uniformly convex for all $p\in(1,\infty)$ (see [1, Theorem 3.6]), and invoking [9, Proposition 3.32], we deduce that $u_n\to u$ in $W^{1,p}(\mathbb{R}^N)$. □

3. Qualitative properties of least energy solutions

In this section we establish sign and symmetry properties of the least energy solutions of (1.1). We first prove an auxiliary result.

Lemma 3.1. *Let $\gamma\in\Gamma$ and $t_*\in(0,1)$ such that*
$$\mathcal{J}(\gamma(t))<\mathcal{J}(\gamma(t_*))=c_{\rm MP}\quad \text{for all } t\in[0,1]\setminus\{t_*\}. \tag{3.1}$$
Then $\mathcal{J}'(\gamma(t_*))=0$.

Proof. Arguing indirectly, assume that $\mathcal{J}'(\gamma(t_*))\neq0$. By continuity, we can select $\varepsilon,\delta>0$ such that
$$\inf\{\|\mathcal{J}'(v)\|_*\,:\,\|v-\gamma(t_*)\|_{1,p}\le\delta\}>\frac{8\,\varepsilon}{\delta}.$$
Using the deformation lemma [27, Lemma 2.3] with $X=W^{1,p}(\mathbb{R}^N)$, $S=\{\gamma(t_*)\}$ and $c=c_{\rm MP}$, we can find $\eta\in\mathcal{C}([0,1]\times W^{1,p}(\mathbb{R}^N);W^{1,p}(\mathbb{R}^N))$ such that

(*i*) $\eta(t,u)=u$ if $t=0$ or if $u\notin\mathcal{J}^{-1}([c_{\rm MP}-2\,\varepsilon,c_{\rm MP}+2\,\varepsilon])\cap S_{2\delta}$,
(*ii*) $\eta(1,\mathcal{J}^{c_{\rm MP}+\varepsilon}\cap S)\subset\mathcal{J}^{c_{\rm MP}-\varepsilon}$,
(*iii*) $\eta(t,\cdot)$ is an homeomorphism of $W^{1,p}(\mathbb{R}^N)$, for all $t\in[0,1]$,

(*iv*) $\|\eta(t,u)-u\|_{1,p}\le\delta$ for all $u\in W^{1,p}(\mathbb{R}^N)$ and $t\in[0,1]$,
(*v*) $\mathcal{J}(\eta(\cdot,u))$ is non increasing for all $u\in W^{1,p}(\mathbb{R}^N)$,
(*vi*) $\mathcal{J}(\eta(t,u))<c_{\mathrm{MP}}$ for all $u\in\mathcal{J}^{c_{\mathrm{MP}}}\cap S_\delta$ and $t\in(0,1]$,

where $\mathcal{J}^d(u)=\{u\in W^{1,p}(\mathbb{R}^N):\mathcal{J}(u)\le d\}$ and $S_\delta=\{u\in W^{1,p}(\mathbb{R}^N):\|u\|_{1,p}=\delta\}$ for $d\in\mathbb{R}$ and $\delta>0$. Define $\psi(t)=\eta(1,\gamma(t))$ for all $t\in[0,1]$. Using $\gamma\in\Gamma$ and (i), we see that

$$\psi(0)=\eta(1,\gamma(0))=\eta(1,0)=0,$$

and thanks to (i) and (v), we get

$$\mathcal{J}(\psi(1))=\mathcal{J}(\eta(1,\gamma(1)))\le\mathcal{J}(\eta(0,\gamma(1)))=\mathcal{J}(\gamma(1))<0.$$

Therefore, $\psi\in\Gamma$. Now, in view of (v), (i), and (3.1), we have that

$$\mathcal{J}(\psi(t))\le\mathcal{J}(\eta(0,\gamma(t)))=\mathcal{J}(\gamma(t))<c_{\mathrm{MP}}\quad\text{ for all } t\in[0,1]\setminus\{t_*\},$$

On the other hand, (ii) implies

$$\mathcal{J}(\psi(t_*))\le c_{\mathrm{MP}}-\varepsilon<c_{\mathrm{MP}}.$$

As a result,

$$\sup_{t\in[0,1]}\mathcal{J}(\psi(t))<c_{\mathrm{MP}},$$

and this is in contrast with the definition of c_{MP}. The proof of the lemma is now complete. □

Let us now show that if g is odd and has constant sign in $(0,\infty)$, then also least energy solutions have constant sign.

Proposition 3.1. *If g is odd and does not change sign on $(0,\infty)$, then every least energy solution $u\in W^{1,p}(\mathbb{R}^N)$ of (1.1) has constant sign.*

Proof. Suppose that $g\ge0$ on $(0,\infty)$. Let $u\in W^{1,p}(\mathbb{R}^N)$ be a least energy solution of (1.1). By Proposition 2.3, we can find a $t_0>1$ and a path $\gamma\in\Gamma$ such that

$$\begin{aligned}&\gamma(1/t_0)=u,\\&\mathcal{J}(\gamma(t))<\mathcal{J}(u)\quad\text{ for all } t\in[0,1]\setminus\{1/t_0\}.\end{aligned}\tag{3.2}$$

Due to the fact that G is even, for each $v\in W^{1,p}(\mathbb{R}^N)$, it holds

$$\mathcal{J}(|v|)=\mathcal{J}(v).\tag{3.3}$$

Hence, (3.3) and (3.2) give

$$\mathcal{J}(|\gamma(t)|)=\mathcal{J}(\gamma(t))<\mathcal{J}(u)=\mathcal{J}(|\gamma(1/t_0)|)\quad\text{ for all } t\in[0,1]\setminus\{1/t_0\}.$$

This together with $\mathcal{J}(|\gamma(1/t_0)|)=\mathcal{J}(|u|)=\mathcal{J}(u)=c_{\mathrm{MP}}$ and Lemma 3.1 implies that $\mathcal{J}'(|u|)=0$. Consequently, $|u|$ satisfies

$$-\Delta_p|u|+|u|^{p-1}=(I_\alpha*G(|u|))g(|u|)\quad\text{ in }\mathbb{R}^N.$$

From Theorem 1.2, we know that $|u|$ is continuous and bounded in $\mathbb{R}^N$. An application of the Harnack inequality [25, Theorem 1.2] implies that $|u|>0$ in $\mathbb{R}^N$ and so u has constant sign. □

Next we prove symmetry properties of least energy solutions by means of a polarization argument. We first recall some elements of the theory of polarization (see [28]). Assume that $\mathcal{H} \subset \mathbb{R}^N$ is a closed half–space and that $\sigma_{\mathcal{H}}$ is the reflection with respect to $\partial\mathcal{H}$. The polarization $u^{\mathcal{H}}$ of $u : \mathbb{R}^N \to \mathbb{R}$ is defined for $x \in \mathbb{R}^N$ by

$$u^{\mathcal{H}}(x) = \begin{cases} \max\{u(x), u(\sigma_{\mathcal{H}}(x))\} & \text{if } x \in \mathcal{H}, \\ \min\{u(x), u(\sigma_{\mathcal{H}}(x))\} & \text{if } x \in \mathbb{R}^N \setminus \mathcal{H}. \end{cases}$$

We will use the next fundamental results.

Proposition 3.2. [28, Propositions 8.3.7 and 8.3.12] *Let $1 \le p < \infty$ and $u \in W^{1,p}(\mathbb{R}^N)$. Then $u^{\mathcal{H}} \in W^{1,p}(\mathbb{R}^N)$ and it holds*

$$\|\nabla u^{\mathcal{H}}\|_p = \|\nabla u\|_p \text{ and } \|u^{\mathcal{H}}\|_p = \|u\|_p.$$

Lemma 3.2. [19, Lemma 5.3] *Let $\alpha \in (0, N)$, $u \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $\mathcal{H} \subset \mathbb{R}^N$ be a closed half–space. If $u \ge 0$ then*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)u(y)}{|x-y|^{N-\alpha}}\, dxdy \le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^{\mathcal{H}}(x)u^{\mathcal{H}}(y)}{|x-y|^{N-\alpha}}\, dxdy,$$

with equality if and only if $u^{\mathcal{H}} = u$ or $u^{\mathcal{H}} = u \circ \sigma_{\mathcal{H}}$.

Lemma 3.3. [19, Lemma 5.4] *Let $s \in [1, \infty)$ and $u \in L^s(\mathbb{R}^N)$ be such that $u \ge 0$. Then there exist $x_0 \in \mathbb{R}^N$ and a nonincreasing function $v : (0, \infty) \to \mathbb{R}$ such that $u(x) = v(|x - x_0|)$ for a.e. $x \in \mathbb{R}^N$ if and only if $u^{\mathcal{H}} = u$ or $u^{\mathcal{H}} = u \circ \sigma_{\mathcal{H}}$ for every closed half–space $\mathcal{H} \subset \mathbb{R}^N$.*

Now, we are in the position to prove that least energy solutions are radial.

Proposition 3.3. *If g is odd and does not change sign in $(0, \infty)$, then every least energy solution $u \in W^{1,p}(\mathbb{R}^N)$ of (1.1) is radially symmetric about a point.*

Proof. Without loss of generality, we may assume that $g \ge 0$ on $(0, \infty)$. Let $u \in W^{1,p}(\mathbb{R}^N)$ be a least energy solution of (1.1). Take a closed half–space $\mathcal{H} \subset \mathbb{R}^N$. By Proposition 3.1, we may suppose that $u > 0$. By virtue of Proposition 2.3, there exist a $t_0 > 1$ and a path $\gamma \in \Gamma$ such that

$$\begin{aligned} &\gamma(t) \ge 0 \quad \text{for all } t \in [0,1], \\ &\gamma(1/t_0) = u, \\ &\mathcal{J}(\gamma(t)) < \mathcal{J}(u) \quad \text{for all } t \in [0,1] \setminus \{1/t_0\}. \end{aligned} \tag{3.4}$$

Consider $\gamma^{\mathcal{H}} : [0,1] \to W^{1,p}(\mathbb{R}^N)$ given by $\gamma^{\mathcal{H}}(t) = (\gamma(t))^{\mathcal{H}}$. According to [26, Corollary 2.40], we know that $\gamma^{\mathcal{H}} \in \mathcal{C}([0,1]; W^{1,p}(\mathbb{R}^N))$. Note that, since G is nondecreasing,

$$G\left(u^{\mathcal{H}}\right) = G(u)^{\mathcal{H}}, \tag{3.5}$$

and therefore, exploiting Lemma 3.2 and Proposition 3.2, we have that

$$\mathcal{J}\left(\gamma^{\mathcal{H}}(t)\right) \le \mathcal{J}(\gamma(t)) \quad \text{for all } t \in [0,1]. \tag{3.6}$$

In particular, $\gamma^{\mathcal{H}} \in \Gamma$, and so, from the definition of c_{MP},

$$\max_{t \in [0,1]} \mathcal{J}\left(\gamma^{\mathcal{H}}(t)\right) \ge c_{\mathrm{MP}}.$$

Combining (3.6) and (3.4), we find

$$\mathcal{J}\left(\gamma^{\mathcal{H}}(t)\right) \leq \mathcal{J}(\gamma(t)) < \mathcal{J}(u) = c_{\mathrm{MP}} \quad \text{for all } t \in [0,1] \setminus \{1/t_0\}.$$

Therefore,

$$\mathcal{J}\left(u^{\mathcal{H}}\right) = \mathcal{J}\left(\gamma^{\mathcal{H}}(1/t_0)\right) = c_{\mathrm{MP}} = \mathcal{J}(\gamma(1/t_0)) = \mathcal{J}(u).$$

In particular, in view of Proposition 3.2, Lemma 3.2 and (3.5), we have that either $G(u)^{\mathcal{H}} = G(u)$ or $G\left(u^{\mathcal{H}}\right) = G(u \circ \sigma_{\mathcal{H}})$.

Suppose first that $G(u)^{\mathcal{H}} = G(u)$. Recalling that G is nondecreasing, it results that

$$\int_{u(\sigma_{\mathcal{H}}(x))}^{u(x)} g(\tau)\, d\tau = G(u(x)) - G(u(\sigma_{\mathcal{H}}(x))) = G(u(x))^{\mathcal{H}} - G(u(\sigma_{\mathcal{H}}(x))) \geq 0 \quad \text{for all } x \in \mathcal{H}.$$

This implies that either $u(\sigma_{\mathcal{H}}(x)) \leq u(x)$ or $g = 0$ in $[u(x), u(\sigma_{\mathcal{H}}(x))]$. Consequently,

$$g\left(u^{\mathcal{H}}\right) = g(u) \text{ on } \mathbb{R}^N. \tag{3.7}$$

Since $\mathcal{J}\left(\gamma^{\mathcal{H}}(t)\right) < c_{\mathrm{MP}}$ for all $t \in [0,1] \setminus \{1/t_0\}$, we can apply Lemma 3.1 to $\gamma^{\mathcal{H}}$ to deduce that $\mathcal{J}'\left(u^{\mathcal{H}}\right) = 0$. Therefore, for all $\varphi \in W^{1,p}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u^{\mathcal{H}}|^{p-2} \nabla u^{\mathcal{H}} \nabla \varphi\, dx + \int_{\mathbb{R}^N} |u^{\mathcal{H}}|^{p-2} u^{\mathcal{H}}\, \varphi\, dx = \int_{\mathbb{R}^N} (I_\alpha * G(u))\, g(u)\, \varphi\, dx, \tag{3.8}$$

where we have used $G\left(u^{\mathcal{H}}\right) = G(u)$ and (3.7). Subtracting (3.8) by (1.6) and using $\varphi = u - u^{\mathcal{H}}$ as test function, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u - |\nabla u^{\mathcal{H}}|^{p-2}\nabla u^{\mathcal{H}}\right)\left(\nabla u - \nabla u^{\mathcal{H}}\right)\, dx \\ &\qquad + \int_{\mathbb{R}^N} \left(|u|^{p-2}u - |u^{\mathcal{H}}|^{p-2}u^{\mathcal{H}}\right)\left(u - u^{\mathcal{H}}\right)\, dx = 0, \end{aligned}$$

which combined with the Simon inequality [24, formula (2.2)]

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq C_p|x-y|^p \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } p \geq 2,$$

yields $C_p\|u - u^{\mathcal{H}}\|_{1,p}^p \leq 0$. Hence, $u = u^{\mathcal{H}}$.

In the case in which $G\left(u^{\mathcal{H}}\right) = G(u \circ \sigma_{\mathcal{H}})$, we can proceed in a similar way to deduce that $u^{\mathcal{H}} = u \circ \sigma_{\mathcal{H}}$. Due to the arbitrariness of $\mathcal{H}$, it follows from Lemma 3.3 that u is radially symmetric and radially decreasing. □

Proof of Theorem 1.4. This is a direct consequence of Propositions 3.1 and 3.3. □

Funding information. The authors are partially supported by INdAM-GNAMPA Project CUP E53C23001670001.

Data availability statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest. Authors state no conflict of interest.

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev spaces* Pure Appl. Math. (Amst.), 140, Elsevier/Academic Press, Amsterdam, 2003, xiv+305 pp.

[2] C.O. Alves and M. Yang, *Multiplicity and concentration of solutions for a quasilinear Choquard equation*, J. Math. Phys. **55** (2014), no. 6, 061502, 21 pp.

[3] C.O. Alves and M. Yang, *Existence of semiclassical ground state solutions for a generalized Choquard equation*, J. Differential Equations **257** (2014), no. 11, 4133–4164.

[4] C.O. Alves and M. Yang, *Investigating the multiplicity and concentration behaviour of solutions for a quasi–linear Choquard equation via the penalization method*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), no. 1, 23–58.

[5] V. Ambrosio, *On the multiplicity and concentration of positive solutions for a p-fractional Choquard equation in* $\mathbb{R}^N$, Comput. Math. Appl. **78** (2019), no. 8, 2593–2617.

[6] V. Ambrosio, *Regularity and Pohožaev identity for the Choquard equation involving the p–Laplacian operator*, Appl. Math. Lett. **145** (2023), Paper No. 108742, 9 pp.

[7] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.

[8] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. **19** (1992), no. 6, 581–597.

[9] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equation*, Universitext, Springer, New York, 2011. xiv+599 pp.

[10] A. El Hamidi and J.M. Rakotoson, *Compactness and quasilinear problems with critical exponents*, Differential Integral Equations **18** (2005), no. 11, 1201–1220.

[11] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28** (1997), no. 10, 1633–1659.

[12] L. Jeanjean and K. Tanaka, *A remark on least energy solutions in* $\mathbb{R}^N$, Proc. Amer. Math. Soc. **131** (2003) 2399–2408.

[13] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. **57** (1976/77) 93–105.

[14] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997. xxii+346 pp.

[15] P.-L.Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), 1063–1072.

[16] L. Ma and L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), no. 2, 455–467.

[17] G.P. Menzala, *On regular solutions of a nonlinear equation of Choquard's type*, Proc. Roy. Soc. Edinburgh Sect. A **86** (1980), no. 3-4, 291–301.

[18] I.M. Moroz, R. Penrose, P. Tod, *Spherically–symmetric solutions of the Schrödinger–Newton equations*, Topology of the Universe Conference (Cleveland, OH, 1997), Classical Quantum Gravity **15** (1998), no.9, 2733–2742.

[19] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184.

[20] V. Moroz and J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6557–6579.

[21] V. Moroz and J. Van Schaftingen, *Semi-classical states for the Choquard equation*, Calc. Var. Partial Differential Equations **52** (2015), no. 1-2, 199–235.

[22] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), no. 1, 773–813.

[23] S. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, p.2. Akademie Verlag, Berlin (1954).

[24] J. Simon, *Régularité de la solution d'une équation non linéaire dans* $\mathbb{R}^N$, Journées d'Analyse Non Linéaire (Proc. Conf., Besançon, 1977), pp. 205–227, Lecture Notes in Math., 665, Springer, Berlin, 1978.

[25] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[26] J. Van Schaftingen, *Approximation of symmetrizations and symmetry of critical points*, Topol. Methods Nonlinear Anal. **28** (2006), no. 1, 61–85.
[27] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996. x+162 pp.
[28] M. Willem, *Functional analysis. Fundamentals and applications*, Cornerstones. Birkhäuser/Springer, New York, 2013. xiv+213 pp.

Vincenzo Ambrosio, Giuseppina Autuori and Teresa Isernia
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
via brecce bianche 12, 60131 Ancona, Italy
Email address: v.ambrosio@staff.univpm.it, g.autuori@staff.univpm.it, t.isernia@staff.univpm.it