J. Math. Anal. Appl. 550 (2025) 129626

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

On a critical superlinear fractional (p, q)-Kirchhoff equation

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ARTICLE INFO

Article history:
Received 27 January 2025
Available online 29 April 2025
Submitted by J.D. Rossi

Keywords:
(p, q)-Laplacian operator
Critical exponent
Variational methods

ABSTRACT

We study ground state solutions to the following fractional (p, q)-Kirchhoff equation

$$\begin{aligned}&\left(a + b[u]^p_{s,p}\right)(-\Delta)^s_p u + \left(c + d[u]^q_{s,q}\right)(-\Delta)^s_q u + V(x)\left(|u|^{p-2}u + |u|^{q-2}u\right) \\ &= \lambda \mathcal{K}(x) f(u) + \mathcal{Q}(x)|u|^{q^*_s-2}u \quad \text{in } \mathbb{R}^3,\end{aligned}$$

where $a, b, c, d > 0$ are constants, $\lambda > 0$ is a parameter sufficiently large, $s \in (0, 1)$, $1 < p < q$ and $\frac{3}{2} < sq < 3$. Here V is a periodic potential, the weight functions $\mathcal{K}$ and $\mathcal{Q}$ are positive and continuous functions, and f is a subcritical nonlinearity that does not satisfy the Ambrosetti–Rabinowitz condition. By using appropriate variational argument, we prove the existence of ground state solutions for λ large.

1. Introduction

In this paper, we deal with the following class of fractional (p, q)-Kirchhoff equations involving critical nonlinearities:

$$\begin{aligned}&\left(a + b[u]^p_{s,p}\right)(-\Delta)^s_p u + \left(c + d[u]^q_{s,q}\right)(-\Delta)^s_q u + V(x)\left(|u|^{p-2}u + |u|^{q-2}u\right) \\ &= \lambda \mathcal{K}(x) f(u) + \mathcal{Q}(x)|u|^{q^*_s-2}u \quad \text{in } \mathbb{R}^3,\end{aligned} \tag{1.1}$$

where $a, b, c, d > 0$ are constants, $\lambda > 0$ is a parameter large enough, $s \in (0, 1)$, $1 < p < q$, $\frac{3}{2} < sq < 3$ and $q^*_s = \frac{3q}{3-sq}$. The operator $(-\Delta)^s_t$, with $t \in \{p, q\}$, appearing in (1.1) is the fractional t-Laplacian operator which, for every $\varphi \in C^\infty_0(\mathbb{R}^3)$ may be defined, up to normalizing factors depending on N, s and t, by setting

$$(-\Delta)^s_t \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x - y|^{3+st}}\, dy,$$

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https://doi.org/10.1016/j.jmaa.2025.129626
0022-247X/

for all $x \in \mathbb{R}^3$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^3 : |x-y| < \varepsilon\}$. Over the last few years, there has been a growing interest in nonlocal and fractional problems involving the fractional t-Laplacian operator. This interest is due to two key features: its nonlinearity and nonlocal nature. Consequently, various results on the existence, multiplicity, and regularity of solutions to these problems appeared in the literature; see for instance [5,6,18,27].

We start by considering $a = c = 1$ and $b = d = 0$ in (1.1). When $s \to 1$, (1.1) boils down to a (p,q)-Laplacian equation. The study of (p,q)–Laplacian problems stems from the following general reaction–diffusion system

$$u_t = \operatorname{div}(\mathcal{D}(u)\nabla u) + f(x,u) \quad x \in \mathbb{R}^N, t > 0$$

where $\mathcal{D}(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system find applications in biophysics, plasma physics and chemical reaction design (see [12]). In these contexts u stands for a concentration, the term $\operatorname{div}(\mathcal{D}(u)\nabla u)$ corresponds to the diffusion with a diffusion coefficient $\mathcal{D}(u)$, and $f(x,u)$ represents the reaction term which relates to source and loss processes. Some results concerning the existence and multiplicity of solutions can be found in [4,9,15,16,26] and we mention [12,21,25] for problems setting in bounded domain.

We underline that the (p,q)–Laplacian problems were inspired by models arising in nonlinear elasticity and they describe the deformation of an elastic body. More precisely, the (p,q)–Laplacian operator is a special case of the double–phase operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$, where $a(\cdot)$ is a positive bounded function. The analysis of non–autonomous energy functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point was developed by several authors; see for instance [23,24] for more details.

When $s \in (0,1)$ some additional difficulties arise in the (p,q)-Laplacian setting. In fact (p,q)-Laplacian problems involve the sum of two nonlocal nonlinear operators with different scaling properties. The starting point can be addressed to [11] where the authors studied the existence, nonexistence, and multiplicity of solutions to

$$(-\Delta)^s_p u + (-\Delta)^s_q u + a(x)|u|^{p-2}u + b(x)|u|^{q-2}u + \mu(x)|u|^{r-2}u = \lambda h(x)|u|^{m-2}u \quad \text{in } \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, $s \in (0,1)$, $1 < q < p$, $r > 1$ and $sp < N$, and a, b, μ and h are non negative functions that satisfies suitable assumptions. In [2], using suitable variational arguments and concentration compactness lemma, the author proved the existence of a nontrivial non-negative solution to

$$(-\Delta)^s_p u + (-\Delta)^s_q u + |u|^{p-2}u + |u|^{q-2}u = \lambda h(x)f(u) + |u|^{q^*_s-2}u \quad \text{in } \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, h is a bounded perturbation and f is a superlinear continuous function with subcritical growth. The concentration and multiplicity of solutions to a class of fractional (p,q)-Laplacian problems has been obtained in [7] under del Pino–Felmer type conditions on V and assuming that $f \in C(\mathbb{R},\mathbb{R})$ has subcritical growth. In [8] the authors proved the multiplicity result for a class of fractional (p,q)-Laplacian problems with critical nonlinearity in bounded domain.

Now, assume $a = c \neq 0$ and $b = d \neq 0$ in (1.1). When $s \to 1$ and $p = q = 2$, (1.1) reduces to a Kirchhoff equation of the type

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + V(x)u = g(x,u) \quad \text{in } \mathbb{R}^3 \tag{1.2}$$

proposed by Kirchhoff [20] as a generalization of the D'Alembert wave equation for free vibration strings. Problem (1.2) received a great attention after the pioneering paper [22] where the author introduced a functional analysis approach to attack it. Further results can be found in [1,13,17]. Due to the interest toward quasilinear problems and Kirchhoff equations, in [10,19] the authors studied (p, q)–Kirchhoff equations in a bounded domain and in the whole of $\mathbb{R}^3$.

When $s \in (0, 1)$ and $p \neq q$, the only result appeared in the literature is [3] where the author obtained the multiplicity and concentration result for solutions to

$$(1 + [u]_{s,p}^p)(-\Delta)_p^s u + (1 + [u]_{s,q}^q)(-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^N,$$

where the nonlinearity f is a superlinear continuous function with subcritical growth at infinity and the potential V verifies del Pino–Felmer conditions.

As far as we know, there are no existing results in the literature addressing Kirchhoff-type problems driven by the fractional (p, q)-Laplacian operator with periodic potentials. Motivated by this gap and the previously mentioned studies, we investigate the existence of solutions to (1.1) in this work.

In order to state our result, we first introduce the assumptions on the potentials $V, \mathcal{K}$, and $\mathcal{Q}$ and on the nonlinearity f. Throughout the paper the potentials V, $\mathcal{K}$ and $\mathcal{Q}$ satisfy the following conditions:

(h_1) $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous bounded function $\mathbb{Z}^3$-periodic and $V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$,
(h_2) $\mathcal{K} : \mathbb{R}^3 \to \mathbb{R}$ is bounded, $\mathcal{K}(x) > \mathcal{K}_\infty$ for all $x \in \mathbb{R}^3$ and $\lim_{|x|\to\infty} \mathcal{K}(x) = \mathcal{K}_\infty \in (0, \infty)$,
(h_3) $\mathcal{Q} : \mathbb{R}^3 \to \mathbb{R}$ is bounded, $\mathcal{Q}(x) > \mathcal{Q}_\infty$ for all $x \in \mathbb{R}^3$ and $\lim_{|x|\to\infty} \mathcal{Q}(x) = \mathcal{Q}_\infty \in (0, \infty)$,

and $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity such that $f(t) = 0$ for all $t \leq 0$ and verifies the following hypotheses:

(f_1) there exists a constant $c > 0$ such that $|f(t)| \leq c(1 + |t|^{r-1})$ for all $t \in \mathbb{R}$, where $2q < r < q_s^*$,
(f_2) $\displaystyle\lim_{t\to+\infty} \frac{F(t)}{t^{2q}} = +\infty$, where $F(t) = \int_0^t f(\tau)\, d\tau$,
(f_3) $\displaystyle\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = 0$,
(f_4) the map $t \mapsto \dfrac{f(t)}{t^{2q-1}}$ is strictly increasing in $(-\infty, 0)$ and in $(0, +\infty)$.

We emphasize that from (f_1) and (f_3) it follows that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon\, |t|^{p-1} + C_\varepsilon |t|^{r-1}. \tag{1.3}$$

Now, our main result can be stated as follows:

Theorem 1.1. *Suppose that (h_1)–(h_3) and (f_1)–(f_4) hold true. Then there exists $\lambda^* > 0$ such that (1.1) has a ground state solution for all $\lambda > \lambda^*$.*

The strategy of the proof is the following. First we show that the energy functional $\mathcal{I}_\lambda$ associated with (1.1) has a mountain pass geometry (see Lemma 3.1). Invoking a variant of the mountain pass theorem, we produce a Cerami sequence (u_n) for $\mathcal{I}_\lambda$ at the mountain pass value c_λ. We note that, using our assumptions on V, $\mathcal{K}$, $\mathcal{Q}$ and f, we can prove that c_λ coincides with the ground state level c_λ^* (see Lemma 3.3). Due to the presence of the critical exponent, we obtain an upper bound for c_λ when λ is sufficiently large (see Lemma 3.4), and then we use a concentration-compactness argument to deduce the boundedness of (u_n) (see Lemma 3.5). In order to prove that (u_n) weakly converges to a nontrivial critical point u of $\mathcal{I}_\lambda$, we exploit the crucial fact that $c_\lambda < m_\infty$, where m_∞ denotes the ground state energy level of the energy functional associated with the limiting problem related to (1.1). Finally, we conclude that u is a ground state solution to (1.1).

The paper is organized as follows. In section 2 we collect some preliminary results about the fractional Laplacian. In section 3 we prove some properties of the energy functional associated with (1.1). Section 4 is devoted to the limiting problem associated with (1.1). Finally, in section 5 we prove the main result.

2. Notations and some preliminaries

Let $p \in [1, \infty]$ and $A \subset \mathbb{R}^3$. We denote by $\|u\|_{L^p(A)}$ the $L^p(A)$-norm of $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^p(A)$. When $A = \mathbb{R}^3$, the $L^p(\mathbb{R}^3)$-norm of u will be denoted by $\|u\|_p$.

Let $s \in (0, 1)$ and $p \in (1, \infty)$. We define $\mathcal{D}^{s,p}(\mathbb{R}^3)$ as the closure of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$[u]_{s,p}^p = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^p}{|x - y|^{3+sp}} \, dxdy,$$

and we denote by $W^{s,p}(\mathbb{R}^3)$ the set of functions $u \in L^p(\mathbb{R}^3)$ such that $[u]_{s,p} < \infty$ endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^3)} = \left([u]_{s,p}^p + \|u\|_p^p\right)^{\frac{1}{p}} .$$

The following embeddings are well-known (see [5]).

Theorem 2.1. *Let $s \in (0, 1)$ and $p \in [1, \infty)$ be such that $sp < 3$. Then there exists $S_* = S_*(s, p) > 0$ such that*

$$\|u\|_{p_s^*}^p \leq S_*^{-1} [u]_{s,p}^p,$$

for all $u \in \mathcal{D}^{s,p}(\mathbb{R}^3)$. Moreover, $W^{s,p}(\mathbb{R}^3)$ is continuously embedded in $L^\tau(\mathbb{R}^3)$ for any $\tau \in [p, p_s^*]$ and compactly in $L^\tau_{loc}(\mathbb{R}^3)$ for any $\tau \in [1, p_s^*)$.

Let us introduce the space

$$\mathcal{W}_{p,q} = \left\{ u \in \mathcal{D}^{s,p}(\mathbb{R}^3) \cap \mathcal{D}^{s,q}(\mathbb{R}^3) \, : \, \int_{\mathbb{R}^3} V(x) \left(|u|^p + |u|^q\right) \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{W}_{p,q}} = \|u\|_{V,p} + \|u\|_{V,q},$$

where, for all $m > 1$ we set

$$\|u\|_{V,m}^m = [u]_{s,m}^m + \int_{\mathbb{R}^3} V(x) |u|^m \, dx.$$

It is easy to verify that $\mathcal{W}_{p,q}$ is continuously embedded in $L^\tau(\mathbb{R}^3)$ for all $\tau \in [p, q_s^*]$ and compactly embedded in $L^\tau_{loc}(\mathbb{R}^3)$ for all $\tau \in [1, q_s^*)$.

We will often use the following vanishing–Lions type result.

Lemma 2.1. *[5, Lemma 2.2] Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < 3$. Let $r \in [p, p_s^*)$. If (u_n) is a bounded sequence in $W^{s,p}(\mathbb{R}^3)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^r \, dx = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^\tau(\mathbb{R}^3)$ for all $\tau \in (p, p_s^*)$.

Remark 2.1. From assumption (f_4) and $p < q$ it follows that

$$t \in (0,\infty) \mapsto \frac{f(t)}{t^{p-1}} \text{ is strictly increasing.}$$

Now, combining this and (f_3) we have

$$0 = \lim_{t\to 0} \frac{f(t)}{t^{p-1}} = \inf_{s>0} \frac{f(s)}{s^{p-1}},$$

therefore,

$$\frac{f(t)}{t^{p-1}} \geq \inf_{s>0} \frac{f(s)}{s^{p-1}} = 0 \quad \text{ for all } t > 0,$$

hence $f(t) \geq 0$ for all $t \geq 0$. This together with (f_4) yields

$$0 < F(t) = \int_0^t f(\tau)\, d\tau = \int_0^t \frac{f(\tau)}{\tau^{2q-1}} \tau^{2q-1} \, d\tau \leq \frac{f(t)}{t^{2q-1}} \int_0^t \tau^{2q-1}\, d\tau = \frac{tf(t)}{2q},$$

that is

$$tf(t) - 2qF(t) \geq 0 \quad \text{ for all } t > 0. \tag{2.1}$$

Moreover, the map

$$t \mapsto \frac{1}{2q} f(t)t - F(t) \quad \text{ is increasing for } t \geq 0. \tag{2.2}$$

Indeed, taking $0 < t_1 < t_2$,

$$\begin{aligned}
\frac{1}{2q} f(t_2)t_2 - F(t_2) &= \frac{1}{2q} f(t_2)t_2 - F(t_1) - \int_{t_1}^{t_2} f(\tau)\, d\tau \\
&= \frac{1}{2q} f(t_2)t_2 - F(t_1) - \int_{t_1}^{t_2} \frac{f(\tau)}{\tau^{2q-1}} \tau^{2q-1}\, d\tau \\
&> \frac{1}{2q} f(t_2)t_2 - F(t_1) - \frac{f(t_2)}{t_2^{2q-1}} \int_{t_1}^{t_2} \tau^{2q-1}\, d\tau \\
&= \frac{1}{2q} f(t_2)t_2 - F(t_1) - \frac{1}{2q} \frac{f(t_2)}{t_2^{2q-1}} (t_2^{2q} - t_1^{2q})
\end{aligned}$$

$$> \frac{1}{2q} f(t_1)t_1 - F(t_1).$$

3. Functional setting

In order to study (1.1), we look for critical points of the functional $\mathcal{I}_\lambda : \mathcal{W}_{p,q} \to \mathbb{R}$ defined as

$$\mathcal{I}_\lambda(u) = \frac{1}{p}[u]_{s,p}^p \left(a + \frac{b}{2}[u]_{s,p}^p\right) + \frac{1}{q}[u]_{s,q}^q \left(c + \frac{d}{2}[u]_{s,q}^q\right) + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right) dx - \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)F(u)\, dx - \frac{1}{q_s^*}\int_{\mathbb{R}^3} \mathcal{Q}(x)|u|^{q_s^*}\, dx.$$

It is easy to see that $\mathcal{I}_\lambda \in C^1(\mathcal{W}_{p,q}, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle \mathcal{I}_\lambda'(u), \varphi\rangle = \left(a + b[u]_{s,p}^p\right) \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{3+sp}} (\varphi(x)-\varphi(y))\, dxdy \\ + \left(c + d[u]_{s,q}^q\right) \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{3+sq}} (\varphi(x)-\varphi(y))\, dxdy \\ + \int_{\mathbb{R}^3} V(x)\left(|u|^{p-2}u\varphi + |u|^{q-2}u\varphi\right) dx \\ - \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)f(u)\varphi\, dx - \int_{\mathbb{R}^3} \mathcal{Q}(x)|u|^{q_s^*-2}u\varphi\, dx\end{aligned}$$

for any $u, \varphi \in \mathcal{W}_{p,q}$.

The Nehari manifold associated with $\mathcal{I}_\lambda$ is given by

$$\mathcal{N}_\lambda = \{u \in \mathcal{W}_{p,q} \setminus \{0\} : \langle \mathcal{I}_\lambda'(u), u\rangle = 0\}.$$

Let us denote by c_λ^* the ground state level, that is

$$c_\lambda^* := \inf_{u \in \mathcal{N}_\lambda} \mathcal{I}_\lambda(u).$$

First we prove that $\mathcal{I}_\lambda$ possesses a mountain pass geometry.

Lemma 3.1. *The functional $\mathcal{I}_\lambda$ verifies the following properties:*

- (i) $\mathcal{I}_\lambda(0) = 0$,
- (ii) *there exist* $\alpha, \varrho > 0$ *such that* $\mathcal{I}_\lambda(u) \ge \alpha$ *for all* $u \in \mathcal{W}_{p,q}$ *such that* $\|u\|_{\mathcal{W}_{p,q}} = \varrho$,
- (iii) *there exists* $w \in \mathcal{W}_{p,q}$ *such that* $\|w\|_{\mathcal{W}_{p,q}} > \varrho$ *and* $\mathcal{I}_\lambda(w) < 0$.

Proof. Evidently, (i) is true. Now we prove (ii). Using (1.3), (h_1), the boundedness of $\mathcal{K}$ and $\mathcal{Q}$, Theorem 2.1 and choosing $0 < \varepsilon < \frac{V_0}{2\lambda\|K\|_\infty}$, we have

$$\mathcal{I}_\lambda(u) \ge \frac{1}{p}[u]_{s,p}^p \left(a + \frac{b}{2}[u]_{s,p}^p\right) + \frac{1}{q}[u]_{s,q}^q \left(c + \frac{d}{2}[u]_{s,q}^q\right) + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right) dx$$

$$
\begin{aligned}
&- \lambda\|\mathcal{K}\|_\infty \int_{\mathbb{R}^3} \left(\frac{\varepsilon}{p}|u|^p + \frac{C_\varepsilon}{r}|u|^r\right) dx - \frac{1}{q_s^*}\|\mathcal{Q}\|_\infty \int_{\mathbb{R}^3} |u|^{q_s^*}\, dx \\
&\geq \frac{a}{p}[u]_{s,p}^p + \frac{c}{q}[u]_{s,q}^q + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right) dx - \frac{1}{2p}\int_{\mathbb{R}^3} V(x)|u|^p\, dx \\
&- C\|u\|_{\mathcal{W}_{p,q}}^r - C\|u\|_{\mathcal{W}_{p,q}}^{q_s^*} \\
&= \frac{a}{p}[u]_{s,p}^p + \frac{c}{q}[u]_{s,q}^q + \frac{1}{2p}\int_{\mathbb{R}^3} V(x)|u|^p\, dx + \frac{1}{q}\int_{\mathbb{R}^3} V(x)|u|^q\, dx \\
&- C\|u\|_{\mathcal{W}_{p,q}}^r - C\|u\|_{\mathcal{W}_{p,q}}^{q_s^*} \\
&\geq \frac{1}{q}\min\left\{a, c, \frac{1}{2}\right\}\left(\|u\|_{V,p}^p + \|u\|_{V,q}^q\right) - C\|u\|_{\mathcal{W}_{p,q}}^r - C\|u\|_{\mathcal{W}_{p,q}}^{q_s^*},
\end{aligned}
$$

where we also take into account that $p < q$. Let $u \in \mathcal{W}_{p,q}$ such that $\|u\|_{\mathcal{W}_{p,q}} = \varrho \in (0,1)$. Since $p < q$ and $\|u\|_{V,p} < 1$, we have $\|u\|_{V,p}^p \geq \|u\|_{V,q}^q$. Exploiting

$$
(a_1 + a_2)^\tau \leq 2^{\tau-1}(a_1^\tau + a_2^\tau) \quad \text{for all } a_1, a_2 \geq 0 \text{ and } \tau \geq 1,
$$

we have

$$
\begin{aligned}
\mathcal{I}_\lambda(u) &\geq \frac{1}{q}\min\left\{a, c, \frac{1}{2}\right\}\left(\|u\|_{V,p}^q + \|u\|_{V,q}^q\right) - C\|u\|_{\mathcal{W}_{p,q}}^r - C\|u\|_{\mathcal{W}_{p,q}}^{q_s^*} \\
&\geq \frac{1}{2^{q-1}q}\min\left\{a, c, \frac{1}{2}\right\}(\|u\|_{V,p} + \|u\|_{V,q})^q - C\|u\|_{\mathcal{W}_{p,q}}^r - C\|u\|_{\mathcal{W}_{p,q}}^{q_s^*} \\
&\geq C_1\|u\|_{\mathcal{W}_{p,q}}^q - C\|u\|_{\mathcal{W}_{p,q}}^r - C\|u\|_{\mathcal{W}_{p,q}}^{q_s^*}.
\end{aligned}
$$

Since $2q < r < q_s^*$ we get the thesis.

Finally, we demonstrate (iii). Take $v \in \mathcal{W}_{p,q}$ such that $v \geq 0$, $v \not\equiv 0$, and let $\vartheta \in \mathbb{R}$. Then from (f_2) we deduce that

$$
\begin{aligned}
&\limsup_{\vartheta\to\infty} \frac{\mathcal{I}_\lambda(\vartheta v)}{\|\vartheta v\|_{\mathcal{W}_{p,q}}^{2q}} \\
&\leq \limsup_{\vartheta\to\infty}\left\{\frac{a+1}{p}\frac{1}{\vartheta^{2q-p}\|v\|_{\mathcal{W}_{p,q}}^{2q-p}} + \frac{b}{2p}\frac{1}{\vartheta^{2(q-p)}\|v\|_{\mathcal{W}_{p,q}}^{2(q-p)}}\right. \\
&\quad\left. + \frac{c+1}{q}\frac{1}{\vartheta^q\|v\|_{\mathcal{W}_{p,q}}^q} + \frac{d}{2q} - \lambda\int_{\mathbb{R}^3} \mathcal{K}(x)\frac{F(\vartheta v)}{(\vartheta v)^{2q}}\left(\frac{v}{\|v\|_{\mathcal{W}_{p,q}}}\right)^{2q} dx\right\} \\
&\leq \frac{d}{2q} - \liminf_{\vartheta\to\infty}\lambda\int_{\mathbb{R}^3} \mathcal{K}(x)\frac{F(\vartheta v)}{(\vartheta v)^{2q}}\left(\frac{v}{\|v\|_{\mathcal{W}_{p,q}}}\right)^{2q} dx \leq -\infty.
\end{aligned}
$$

The result follows by taking $w = \vartheta_* v$ for ϑ_* large. □

We now recall a variant of the mountain pass theorem which allows us to find a Cerami sequence.

Theorem 3.1. *[14] Let X be a real Banach space with its dual X^*, and suppose that $I \in C^1(X,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \alpha \leq \inf_{\|x\|=\rho} I(x),$$

for some $\mu < \alpha$, $\rho > 0$ and $e \in X$ with $\|e\| > \rho$. Let $c \geq \alpha$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \,:\, \gamma(0) = 0,\, \gamma(1) = e\}.$$

Then, there exists a Cerami sequence $(x_n) \subset X$ at the level c that is

$$I(x_n) \to c \text{ and } (1 + \|x_n\|)\|I'(x_n)\|_* \to 0$$

as $n \to \infty$.

In view of Lemma 3.1 and Theorem 3.1, we can find a Cerami sequence $(u_n) \subset \mathcal{W}_{p,q}$ for $\mathcal{I}_\lambda$ such that

$$\mathcal{I}_\lambda(u_n) \to c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}_\lambda(\gamma(t)) > 0 \tag{3.1}$$

and

$$(1 + \|u_n\|_{\mathcal{W}_{p,q}})\|\mathcal{I}'_\lambda(u_n)\|_{\mathcal{W}^*_{p,q}} \to 0, \tag{3.2}$$

where $\Gamma = \{\gamma \in C([0,1], \mathcal{W}_{p,q}) \,:\, \gamma(0) = 0,\, \mathcal{I}_\lambda(\gamma(1)) < 0\}$.

Furthermore, we have the following

Lemma 3.2. *For every $u \in \mathcal{W}_{p,q} \setminus \{0\}$ let $\varphi_u : (0, +\infty) \to \mathbb{R}$ be defined by*

$$\varphi_u(\vartheta) = \mathcal{I}_\lambda(\vartheta u).$$

Then, there exists a unique $\vartheta_0 = \vartheta_0(u) > 0$ such that

$$\begin{aligned} &\varphi'_u(\vartheta) > 0 \quad \text{for } \vartheta \in (0, \vartheta_0), \\ &\varphi'_u(\vartheta) < 0 \quad \text{for } \vartheta \in (\vartheta_0, +\infty). \end{aligned}$$

Proof. By Lemma 3.1 we deduce that $\varphi_u(0) = 0$ and

$$\begin{aligned} &\varphi_u(\vartheta) > 0 \quad \text{for } \vartheta \text{ small}, \\ &\varphi_u(\vartheta) < 0 \quad \text{for } \vartheta \text{ large}. \end{aligned}$$

Therefore, using the continuity of φ_u, there exists $\vartheta_0 = \vartheta_0(u) > 0$ which is a global maximum of φ_u with $\vartheta_0 u \in \mathcal{N}_\lambda$. Next we prove the uniqueness of such ϑ_0. We argue by contradiction and we suppose that there are $\vartheta_1, \vartheta_2 > 0$ such that $\vartheta_1 < \vartheta_2$ and $\varphi'_u(\vartheta_1) = \varphi'_u(\vartheta_2) = 0$. Hence,

$$
\begin{aligned}
&\frac{a}{\vartheta_1^{2q-p}}[u]_{s,p}^p + \frac{b}{\vartheta_1^{2q-2p}}[u]_{s,p}^{2p} + \frac{c}{\vartheta_1^q}[u]_{s,q}^q + d[u]_{s,q}^{2q} \\
&\quad + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{\vartheta_1^{2q-p}}|u|^p + \frac{1}{\vartheta_1^q}|u|^q\right) dx - \vartheta_1^{q_s^*-2q}\int_{\mathbb{R}^3} \mathcal{Q}(x)|u|^{q_s^*}\, dx \\
&\quad = \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)\frac{f(\vartheta_1 u)}{(\vartheta_1 u)^{2q-1}} u^{2q}\, dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{a}{\vartheta_2^{2q-p}}[u]_{s,p}^p + \frac{b}{\vartheta_2^{2q-2p}}[u]_{s,p}^{2p} + \frac{c}{\vartheta_2^q}[u]_{s,q}^q + d[u]_{s,q}^{2q} \\
&\quad + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{\vartheta_2^{2q-p}}|u|^p + \frac{1}{\vartheta_2^q}|u|^q\right) dx - \vartheta_2^{q_s^*-2q}\int_{\mathbb{R}^3} \mathcal{Q}(x)|u|^{q_s^*}\, dx \\
&\quad = \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)\frac{f(\vartheta_2 u)}{(\vartheta_2 u)^{2q-1}} u^{2q}\, dx.
\end{aligned}
$$

Subtracting we get

$$
\begin{aligned}
&\left(\frac{1}{\vartheta_1^{2q-p}} - \frac{1}{\vartheta_2^{2q-p}}\right) a[u]_{s,p}^p + \left(\frac{1}{\vartheta_1^{2q-2p}} - \frac{1}{\vartheta_2^{2q-2p}}\right) b[u]_{s,p}^{2p} + \left(\frac{1}{\vartheta_1^q} - \frac{1}{\vartheta_2^q}\right)[u]_{s,q}^q \\
&\quad + \int_{\mathbb{R}^3} V(x)\left\{\left(\frac{1}{\vartheta_1^{2q-p}} - \frac{1}{\vartheta_2^{2q-p}}\right)|u|^p + \left(\frac{1}{\vartheta_1^q} - \frac{1}{\vartheta_2^q}\right)|u|^q\right\} dx \\
&\quad - (\vartheta_1^{q_s^*-2q} - \vartheta_2^{q_s^*-2q})\int_{\mathbb{R}^3} \mathcal{Q}(x)|u|^{q_s^*}\, dx \\
&\quad = \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)\left(\frac{f(\vartheta_1 u)}{(\vartheta_1 u)^{2q-1}} - \frac{f(\vartheta_2 u)}{(\vartheta_2 u)^{2q-1}}\right) u^{2q}\, dx.
\end{aligned}
$$

Taking into account $\vartheta_1 < \vartheta_2$, $\frac{3}{2} < sq$ and (f_4), we get a contradiction. $\square$

Next we define the number

$$
c_\lambda^{**} = \inf_{u \in \mathcal{W}_{p,q}\setminus\{0\}} \max_{t \geq 0} \mathcal{I}_\lambda(tu),
$$

and we prove the following result.

Lemma 3.3. *$c_\lambda = c_\lambda^* = c_\lambda^{**}$ for every $\lambda > 0$.*

Proof. It follows from Lemma 3.2 that $c_\lambda^* = c_\lambda^{**}$. Next we prove that $c_\lambda \leq c_\lambda^{**}$. Proceeding as in Lemma 3.1-(iii), we can find $\vartheta^* > 0$ sufficiently large, such that, for any $u \in \mathcal{W}_{p,q} \setminus \{0\}$ it holds $\varphi_u(\vartheta^*) = \mathcal{I}_\lambda(\vartheta^* u) < 0$. Now, let us consider the path $\eta : [0,1] \to \mathcal{W}_{p,q}$ as $\eta(t) = t\vartheta^* u$. Evidently, $\eta(0) = 0$ and $\mathcal{I}_\lambda(\eta(1)) < 0$, so $\eta \in \Gamma$. Consequently, $c_\lambda \leq c_\lambda^{**}$. Finally, we prove that $c_\lambda^* \leq c_\lambda$. The manifold $\mathcal{N}_\lambda$ separates $\mathcal{W}_{p,q}$ into two components containing a small ball around the origin. Take u in this component and $0 \leq \vartheta \leq \vartheta_0$, where ϑ_0 is given in Lemma 3.2. Then we have

$$\begin{aligned}
\varphi_u'(\vartheta) &= \vartheta^{2q-1}\Bigg\{ \frac{a}{\vartheta^{2q-p}}[u]_{s,p}^p + \frac{b}{\vartheta^{2q-2p}}[u]_{s,p}^{2p} + \frac{c}{\vartheta^q}[u]_{s,q}^q + d[u]_{s,q}^{2q} \\
&\quad + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{\vartheta^{2q-p}}|u|^p + \frac{1}{\vartheta^q}|u|^q\right) dx - \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)\frac{f(\vartheta u)}{(\vartheta u)^{2q-1}}u^{2q}\, dx \\
&\quad - \int_{\mathbb{R}^3} \vartheta^{q_s^*-2q}\mathcal{Q}(x)|u|^{q_s^*}\, dx \Bigg\} \\
&\geq \vartheta^{2q-1}\Bigg\{ \frac{a}{\vartheta_0^{2q-p}}[u]_{s,p}^p + \frac{b}{\vartheta_0^{2q-2p}}[u]_{s,p}^{2p} + \frac{c}{\vartheta_0^q}[u]_{s,q}^q + d[u]_{s,q}^{2q} \\
&\quad + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{\vartheta_0^{2q-p}}|u|^p + \frac{1}{\vartheta_0^q}|u|^q\right) dx - \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)\frac{f(\vartheta_0 u)}{(\vartheta_0 u)^{2q-1}}u^{2q}\, dx \\
&\quad - \int_{\mathbb{R}^3} \vartheta_0^{q_s^*-2q}\mathcal{Q}(x)|u|^{q_s^*}\, dx \Bigg\} \\
&= \frac{\vartheta^{2q-1}}{\vartheta_0^{2q}}\langle \mathcal{I}_\lambda'(\vartheta_0 u), \vartheta_0 u\rangle = 0.
\end{aligned}$$

Hence $\varphi_u'(\vartheta) \geq 0$ for all $0 \leq \vartheta \leq \vartheta_0$, and so $\mathcal{I}_\lambda(u) \geq 0$. Thus, every path in Γ has to cross the Nehari manifold $\mathcal{N}_\lambda$, and $c_\lambda^* \leq c_\lambda$. □

Since we are dealing with the critical setting, we now prove a useful upper estimate for the mountain pass level, which will be helpful later.

Lemma 3.4. *It holds* $\lim_{\lambda\to+\infty} c_\lambda = 0$. *In particular, there exists a constant* $\lambda^* > 0$ *such that*

$$c_\lambda < \frac{q_s^* - q}{qq_s^*}\mathfrak{c}^{\frac{q_s^*}{q_s^*-q}} S_*^{\frac{q_s^*}{q_s^*-q}} \|\mathcal{Q}\|_\infty^{-\frac{q}{q_s^*-q}} \text{ for any } \lambda > \lambda^*,$$

where $\mathfrak{c} = \min\{c, 1\}$.

Proof. Let $\lambda_0 > 0$ and $w \in \mathcal{W}_{p,q}$ given by Lemma 3.1-(*iii*). By Lemma 3.2 there exists $\vartheta_\lambda > 0$, with $\lambda \geq \lambda_0$, such that

$$\max_{\vartheta\geq 0} \mathcal{I}_\lambda(\vartheta w) = \mathcal{I}_\lambda(\vartheta_\lambda w).$$

Hence, for all $\lambda \geq \lambda_0$, it holds $\langle \mathcal{I}_\lambda'(\vartheta_\lambda w), \vartheta_\lambda w\rangle = 0$. Let us consider the set $\Theta = \{\lambda \geq \lambda_0 : \vartheta_\lambda \geq 1\}$. Then, recalling that V is bounded, for all $\lambda \in \Theta$ we have

$$\begin{aligned}
& a\vartheta_\lambda^p[w]_{s,p}^p + b\vartheta_\lambda^{2p}[w]_{s,p}^{2p} + c\vartheta_\lambda^q[w]_{s,q}^q + d\vartheta_\lambda^{2q}[w]_{s,q}^{2q} \\
&\quad + \|V\|_\infty(\vartheta_\lambda^p\|w\|_p^p + \vartheta_\lambda^q\|w\|_q^q) \\
&\qquad \geq \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)f(\vartheta_\lambda w)\vartheta_\lambda w\, dx + \vartheta_\lambda^{q_s^*}\mathcal{Q}_\infty\|w\|_{q_s^*}^{q_s^*}.
\end{aligned} \tag{3.3}$$

From (3.3) we derive

$$
\begin{aligned}
&a\vartheta_\lambda^p[w]_{s,p}^p + b\vartheta_\lambda^{2p}[w]_{s,p}^{2p} + c\vartheta_\lambda^q[w]_{s,q}^q + d\vartheta_\lambda^{2q}[w]_{s,q}^{2q} \\
&\quad + \|V\|_\infty(\vartheta_\lambda^p\|w\|_p^p + \vartheta_\lambda^q\|w\|_q^q) \\
&\quad\quad \geq \vartheta_\lambda^{q_s^*}\mathcal{Q}_\infty\|w\|_{q_s^*}^{q_s^*},
\end{aligned}
$$

which combined with the fact that $q_s^* > 2q > 2p$ implies that $(\vartheta_\lambda)_\lambda$ is a bounded sequence. So there exists a sequence $\lambda_n \to \infty$ such that $\vartheta_{\lambda_n} \to \mathfrak{t} \geq 0$ as $n \to \infty$. Next we aim to prove that $\mathfrak{t} = 0$. With this aim, suppose by contradiction that $\mathfrak{t} > 0$. Furthermore, from (3.3), we get

$$
\begin{aligned}
&a\vartheta_{\lambda_n}^p[w]_{s,p}^p + b\vartheta_{\lambda_n}^{2p}[w]_{s,p}^{2p} + c\vartheta_{\lambda_n}^q[w]_{s,q}^q + d\vartheta_{\lambda_n}^{2q}[w]_{s,q}^{2q} \\
&\quad + \|V\|_\infty(\vartheta_{\lambda_n}^p\|w\|_p^p + \vartheta_{\lambda_n}^q\|w\|_q^q) \\
&\quad\quad \geq \lambda_n\mathcal{K}_\infty\int_{\mathbb{R}^3} f(\vartheta_{\lambda_n}w)\vartheta_{\lambda_n}w\,dx.
\end{aligned}
$$

Taking the limit as $n \to \infty$ and using (2.1) and (f_2), we obtain

$$
\begin{aligned}
&a\mathfrak{t}^{p-2q}[w]_{s,p}^p + b\mathfrak{t}^{2p-2q}[w]_{s,p}^{2p} + d[w]_{s,q}^{2q} \\
&\quad \geq \liminf_{n\to\infty}\lambda_n\mathcal{K}_\infty\int_{\mathbb{R}^3}\frac{f(\vartheta_{\lambda_n}w)}{(\vartheta_{\lambda_n}w)^{2q-1}}w^{2q}\,dx = +\infty,
\end{aligned}
$$

which gives a contradiction. Hence, $\mathfrak{t} = 0$. Moreover, there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$ one has

$$
\begin{aligned}
0 < c_\lambda &\leq \max_{\vartheta\in[0,1]}\mathcal{I}_\lambda(\vartheta w) = \mathcal{I}_\lambda(\vartheta_\lambda w) \\
&\leq \frac{1}{p}\left(a + \frac{b}{2}[\vartheta_\lambda w]_{s,p}^p\right)[\vartheta_\lambda w]_{s,p}^p + \frac{1}{q}\left(c + \frac{d}{2}[\vartheta_\lambda w]_{s,q}^q\right)[\vartheta_\lambda w]_{s,q}^q \\
&\quad + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{p}|\vartheta_\lambda w|^p + \frac{1}{q}|\vartheta_\lambda w|^q\right)dx \to 0 \text{ as } \lambda \to \infty.
\end{aligned}
$$

The proof is complete. □

Next we prove the boundedness of Cerami sequences.

Lemma 3.5. *Every sequence $(u_n) \subset \mathcal{W}_{p,q}$ satisfying (3.1) and (3.2) is bounded in $\mathcal{W}_{p,q}$ for $\lambda > \lambda^*$. In particular, there exists $u \in \mathcal{W}_{p,q}$ such that*

$$
\begin{aligned}
&u_n \rightharpoonup u \text{ in } \mathcal{W}_{p,q}, \\
&u_n \to u \text{ in } L^\tau_{loc}(\mathbb{R}^3) \text{ for all } \tau \in [1, q_s^*), \\
&u_n \to u \text{ a.e. in } \mathbb{R}^3.
\end{aligned}
\tag{3.4}
$$

Proof. By means of (3.1) and (3.2), using (2.1) and $2q < q_s^*$ we have

$$
\begin{aligned}
c_\lambda + o_n(1) &= \mathcal{I}_\lambda(u_n) - \frac{1}{2q}\langle\mathcal{I}'_\lambda(u_n), u_n\rangle \\
&= \left(\frac{1}{p} - \frac{1}{2q}\right)\left\{a[u_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|u_n|^p\,dx\right\} + \left(\frac{1}{2p} - \frac{1}{2q}\right)b[u_n]_{s,p}^{2p}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2q}\left\{c[u_n]_{s,q}^q+\int\limits_{\mathbb{R}^3} V(x)|u_n|^q\,dx\right\}\\
&-\lambda\int\limits_{\mathbb{R}^3}\mathcal{K}(x)\left(F(u_n)-\frac{1}{2q}f(u_n)u_n\right)dx-\left(\frac{1}{q_s^*}-\frac{1}{2q}\right)\int\limits_{\mathbb{R}^3}\mathcal{Q}(x)|u_n|^{q_s^*}\,dx\\
\geq&\left(\frac{1}{p}-\frac{1}{2q}\right)\min\{a,1\}\left\{[u_n]_{s,p}^p+\int\limits_{\mathbb{R}^3}V(x)|u_n|^p\,dx\right\}\\
=&\left(\frac{1}{p}-\frac{1}{2q}\right)\min\{a,1\}\|u_n\|_{V,p}^p,
\end{aligned}
$$

that is $\|u_n\|_{V,p}$ is bounded. Next we prove that $\|u_n\|_{V,q}$ is bounded. Suppose by contradiction that $\|u_n\|_{V,q}\to+\infty$ as $n\to\infty$ and set $v_n=u_n\|u_n\|_{V,q}^{-1}$. Then, $\|v_n\|_{V,q}=1$ and we have the following alternatives:

(i) $\lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^3}\int\limits_{B_R(y)}|v_n|^q\,dx=0$ for all $R>0$,

(ii) there are $\ell>0$ and $R<+\infty$ such that $\lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^3}\int\limits_{B_R(y)}|v_n|^q\,dx=\ell$.

Suppose that (i) is true. Then we can apply Lemma 2.1 to the sequence (v_n) to deduce that $v_n\to 0$ in $L^\tau(\mathbb{R}^3)$ for any $\tau\in(q,q_s^*)$. Take $t_0=\mathfrak{c}^{\frac{1}{q_s^*-q}}S_*^{\frac{q_s^*}{q(q_s^*-q)}}\|\mathcal{Q}\|_\infty^{-\frac{1}{q_s^*-q}}$, where $\mathfrak{c}$ is defined in Lemma 3.4. Since $\|u_n\|_{V,q}\to\infty$ we have that $t_0\|u_n\|_{V,q}^{-1}\in(0,1)$ for n large. Hence,

$$
\begin{aligned}
\mathcal{I}_\lambda(t_0v_n)\geq&\frac{t_0^p}{p}\left(a[v_n]_{s,p}^p+\int\limits_{\mathbb{R}^3}V(x)|u|^p\,dx\right)+\frac{t_0^q}{q}\left(c[v_n]_{s,q}^q+\int\limits_{\mathbb{R}^3}V(x)|u|^q\,dx\right)\\
&-\lambda\int\limits_{\mathbb{R}^3}\mathcal{K}(x)F(t_0v_n)\,dx-\frac{t_0^{q_s^*}}{q_s^*}\int\limits_{\mathbb{R}^3}\mathcal{Q}(x)|v_n|^{q_s^*}\,dx\\
\geq&\frac{t_0^q}{q}\mathfrak{c}-\lambda\int\limits_{\mathbb{R}^3}\mathcal{K}(x)F(t_0v_n)\,dx-\frac{t_0^{q_s^*}\|\mathcal{Q}\|_\infty}{q_s^*S_*^{\frac{q_s^*}{q}}}\\
\geq&\frac{q_s^*-q}{qq_s^*}\mathfrak{c}^{\frac{q_s^*}{q_s^*-q}}S_*^{\frac{q_s^*}{q_s^*-q}}\|\mathcal{Q}\|_\infty^{-\frac{q}{q_s^*-q}}-\lambda\int\limits_{\mathbb{R}^3}\mathcal{K}(x)F(t_0v_n)\,dx.
\end{aligned}
\tag{3.5}
$$

By Lemma 3.2 there exists $t_n\in(0,1)$ such that

$$
\mathcal{I}_\lambda(t_nu_n)=\max_{t\in[0,1]}\mathcal{I}_\lambda(tu_n)\quad\text{and}\quad t_nu_n\in\mathcal{N}_\lambda.\tag{3.6}
$$

Therefore, using the definition of v_n together with (3.6), (2.2), $t_n\in(0,1)$ and $\frac{3}{2}<sq<3$, we get

$$
\begin{aligned}
\mathcal{I}_\lambda(t_0v_n)\leq\max_{t\in[0,1]}\mathcal{I}_\lambda(tu_n)&=\mathcal{I}_\lambda(t_nu_n)=\mathcal{I}_\lambda(t_nu_n)-\frac{1}{2q}\langle\mathcal{I}_\lambda'(t_nu_n),t_nu_n\rangle\\
&\leq\mathcal{I}_\lambda(u_n)-\frac{1}{2q}\langle\mathcal{I}_\lambda'(u_n),u_n\rangle=c_\lambda+o_n(1).
\end{aligned}
\tag{3.7}
$$

Let us point out that from (1.3) we deduce that for any $\lambda > 0$

$$\lim_{n\to\infty} \lambda \int_{\mathbb{R}^3} \mathcal{K}(x)F(t_0 v_n)\, dx = 0. \tag{3.8}$$

Putting together (3.5), (3.7) and (3.8), we get

$$c_\lambda + o_n(1) \geq \frac{q_s^* - q}{q q_s^*} \mathfrak{c}^{\frac{q_s^*}{q_s^*-q}} S_*^{\frac{q_s^*}{q_s^*-q}} \|\mathcal{Q}\|_\infty^{-\frac{q}{q_s^*-q}}$$

which leads to a contradiction thanks to Lemma 3.4. Hence (i) is not fulfilled.

Suppose that (ii) is verified. Then, we can find $(y_n) \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |v_n|^q\, dx = \frac{\ell}{2} > 0.$$

Assume that there exists $\rho > R$ and $(\bar{y}_n) \subset \mathbb{Z}^3$ such that

$$\int_{B_\rho(\bar{y}_n)} |v_n|^q\, dx \geq \frac{\ell}{2} > 0. \tag{3.9}$$

Denote by $\tilde{v}_n(\cdot) = v_n(\cdot + \bar{y}_n)$. Then, $\tilde{v}_n$ is bounded in $\mathcal{W}_{p,q}$, indeed by (h_1) we have that $0 < V_0 \leq V(x) \leq V_1 < +\infty$,

$$\begin{aligned}
\|\tilde{v}_n\|_{V,p}^p &= [\tilde{v}_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|\tilde{v}_n|^p\, dx \leq [\tilde{v}_n]_{s,p}^p + V_1 \int_{\mathbb{R}^3} |\tilde{v}_n|^p\, dx \\
&= [v_n]_{s,p}^p + V_1 \int_{\mathbb{R}^3} |v_n|^p\, dx \leq \frac{V_1}{V_0}\left([v_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|v_n|^p\, dx\right) \\
&\leq \frac{V_1}{V_0}\|v_n\|_{V,p}^p \leq C\frac{V_1}{V_0},
\end{aligned}$$

where we also used $\|v_n\|_{V,p} \leq C$ (due to the boundedness of $\|u_n\|_{V,p}$ and $\|u_n\|_{V,q} \to \infty$), and similarly

$$\|\tilde{v}_n\|_{V,q}^q \leq \frac{V_1}{V_0}\|v_n\|_{V,q}^q = \frac{V_1}{V_0}.$$

Passing to a subsequence we have

$$\tilde{v}_n \rightharpoonup \tilde{v} \text{ in } \mathcal{W}_{p,q}.$$

Now, from the definition of $\tilde{v}_n$ and (3.9) we have

$$\int_{B_\rho} |\tilde{v}_n|^q\, dx = \int_{B_\rho(\bar{y}_n)} |v_n|^q\, dx \geq \frac{\ell}{2} > 0,$$

so $\tilde{v} \neq 0$. Set $\tilde{u}_n = \tilde{v}_n \|u_n\|_{V,q}$. Since $\tilde{v} \neq 0$ we have that $|\{x \in \mathbb{R}^3 : \tilde{v}(x) \neq 0\}| > 0$ and $|\tilde{u}_n| \to +\infty$ as $n \to +\infty$. Taking into account that $\|u_n\|_{V,p}$ is bounded we have

$$\frac{\mathcal{I}_\lambda(u_n)}{\|u_n\|_{V,q}^{2q}} \leq \frac{C}{\|u_n\|_{V,q}^{2q}} + \frac{c+1}{q}\frac{1}{\|u_n\|_{V,q}^{q}} + \frac{d}{2q} - \lambda\int\limits_{\mathbb{R}^3} \mathcal{K}(x)\frac{F(u_n)}{\|u_n\|_{V,q}^{2q}}\,dx.$$

Let us observe that on one hand we have $\mathcal{I}_\lambda(u_n) = c_\lambda + o_n(1)$, and on the other hand utilizing (h_1), (h_2) and (f_2) we have

$$\begin{aligned}\liminf_{n\to+\infty}\int\limits_{\mathbb{R}^3} \mathcal{K}(x)\frac{F(u_n)}{\|u_n\|_{V,q}^{2q}}\,dx &\geq \mathcal{K}_\infty \liminf_{n\to\infty}\int\limits_{\mathbb{R}^3}\frac{F(\tilde{u}_n)}{\|\tilde{u}_n\|_{V,q}^{2q}}\,dx \\ &= \mathcal{K}_\infty \liminf_{n\to\infty}\int\limits_{\{x\in\mathbb{R}^3\,:\,\tilde{v}(x)\neq 0\}}\frac{F(\tilde{u}_n)}{|\tilde{u}_n|^{2q}}|\tilde{v}_n|^{2q}\,dx = +\infty.\end{aligned}$$

Putting all together we get a contradiction. Therefore $\|u_n\|_{V,q}$ is bounded. The proof is now complete. □

4. The limiting problem

In this section we consider the limiting problem related to (1.1), that is

$$\begin{aligned}&\left(a + b[u]_{s,p}^p\right)(-\Delta)_p^s u + \left(c + d[u]_{s,q}^q\right)(-\Delta)_q^s u + V(x)\left(|u|^{p-2}u + |u|^{q-2}u\right) \\ &\quad = \lambda\mathcal{K}_\infty f(u) + \mathcal{Q}_\infty |u|^{q_s^*-2}u \quad \text{in } \mathbb{R}^3.\end{aligned} \tag{4.1}$$

Let us now consider the functional $\mathcal{I}_\lambda^\infty : \mathcal{W}_{p,q} \to \mathbb{R}$ associated with (4.1)

$$\begin{aligned}\mathcal{I}_\lambda^\infty(u) &= \frac{1}{p}[u]_{s,p}^p\left(a + \frac{b}{2}[u]_{s,p}^p\right) + \frac{1}{q}[u]_{s,q}^q\left(c + \frac{d}{2}[u]_{s,q}^q\right) \\ &\quad + \int\limits_{\mathbb{R}^3} V(x)\left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right)dx - \lambda\mathcal{K}_\infty\int\limits_{\mathbb{R}^3} F(u)\,dx - \frac{\mathcal{Q}_\infty}{q_s^*}\int\limits_{\mathbb{R}^3}|u|^{q_s^*}\,dx.\end{aligned}$$

It is easy to see that $\mathcal{I}_\lambda^\infty \in C^1(\mathcal{W}_{p,q}, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle(\mathcal{I}_\lambda^\infty)'(u), \varphi\rangle &= \left(a + b[u]_{s,p}^p\right)\iint\limits_{\mathbb{R}^6}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{3+sp}}(\varphi(x)-\varphi(y))\,dxdy \\ &\quad + \left(c + d[u]_{s,q}^q\right)\iint\limits_{\mathbb{R}^6}\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{3+sq}}(\varphi(x)-\varphi(y))\,dxdy \\ &\quad + \int\limits_{\mathbb{R}^3} V(x)\left(|u|^{p-2}u\varphi + |u|^{q-2}u\varphi\right)dx \\ &\quad - \lambda\mathcal{K}_\infty\int\limits_{\mathbb{R}^3} f(u)\varphi\,dx - \mathcal{Q}_\infty\int\limits_{\mathbb{R}^3}|u|^{q_s^*-2}u\varphi\,dx\end{aligned}$$

for any $u, \varphi \in \mathcal{W}_{p,q}$.

We define the Nehari manifold associated with $\mathcal{I}_\lambda^\infty$ as

$$\mathcal{N}_\lambda^\infty = \{u \in \mathcal{W}_{p,q}\setminus\{0\} \,:\, \langle(\mathcal{I}_\lambda^\infty)'(u), u\rangle = 0\}.$$

Let us denote by m_∞^* the ground state energy, that is

$$m_\infty^* = \inf_{u \in \mathcal{N}_\lambda^\infty} \mathcal{I}_\lambda^\infty(u).$$

Arguing as in the previous section it is easy to prove that $\mathcal{I}_\lambda^\infty$ has a mountain pass geometry.

Lemma 4.1. *The functional $\mathcal{I}_\lambda^\infty$ verifies the following properties:*

(i) *there exist $\alpha, \rho > 0$ such that $\mathcal{I}_\lambda^\infty(u) \geq \alpha$ with $\|u\|_{\mathcal{W}_{p,q}} = \rho$,*
(ii) *there exists $w \in \mathcal{W}_{p,q}$ with $\|w\|_{\mathcal{W}_{p,q}} > \rho$ such that $\mathcal{I}_\lambda^\infty(w) < 0$.*

Thanks to Lemma 4.1 and Theorem 3.1, $\mathcal{I}_\lambda^\infty$ admits a Cerami sequence $(u_n) \subset \mathcal{W}_{p,q}$ at the level

$$m_\infty = \inf_{\gamma \in \Gamma_\infty} \max_{t \in [0,1]} \mathcal{I}_\lambda^\infty(\gamma(t))$$

where $\Gamma_\infty = \{\gamma \in C([0,1], \mathcal{W}_{p,q}) : \gamma(0) = 0, \mathcal{I}_\lambda^\infty(\gamma(1)) < 0\}$.

We stress that from (h_2) and (h_3) it follows that $\mathcal{I}_\lambda(u) < \mathcal{I}_\lambda^\infty(u)$ for any $u \in \mathcal{W}_{p,q} \setminus \{0\}$, and so

$$c_\lambda < m_\infty. \tag{4.2}$$

Furthermore, we have the following characterization

$$m_\infty = m_\infty^* = m_\infty^{**} = \inf_{u \in \mathcal{W}_{p,q} \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\lambda^\infty(tu).$$

Now, we are in the position to prove the main result of this section. For completeness, we recall that a critical point $u \neq 0$ of $\mathcal{I}_\lambda^\infty$ satisfying $\mathcal{I}_\lambda^\infty(u) = \inf_{\mathcal{N}_\lambda^\infty} \mathcal{I}_\lambda^\infty$ is called a ground state solution to (4.1).

Theorem 4.1. *Under the assumptions (h_1)–(h_3) and (f_1)–(f_4), (4.1) admits a ground state solution for $\lambda > 0$ sufficiently large.*

Proof. Let (u_n) be a Cerami sequence for $\mathcal{I}_\lambda^\infty$ at the level m_∞. Proceeding as in the proof of Lemma 3.5, we deduce that (u_n) is bounded in $\mathcal{W}_{p,q}$ for λ sufficiently large. Moreover, we may assume that (3.4) hold. Furthermore, we deduce that the vanishing cannot occur for $\|u_n\|_q^q$. Therefore there exist a sequence $(y_n) \subset \mathbb{Z}^3$ and constants $\ell_1, \rho_1 > 0$ such that

$$\int\limits_{B_{\rho_1}(y_n)} |u_n|^q \, dx \geq \ell_1 > 0. \tag{4.3}$$

Denote by $w_n(\cdot) = u_n(\cdot + y_n)$. Due to the invariance by translations of $\mathbb{R}^3$ and thanks to assumption (h_1), it is clear that $\|w_n\|_{V,p} = \|u_n\|_{V,p}$ and $\|w_n\|_{V,q} = \|u_n\|_{V,q}$, so (w_n) is bounded in $\mathcal{W}_{p,q}$ and there exists w such that

$$\begin{aligned} &w_n \rightharpoonup w \text{ in } \mathcal{W}_{p,q}, \\ &w_n \to w \text{ in } L^\tau_{loc}(\mathbb{R}^N) \text{ for all } \tau \in [1, q_s^*), \\ &w_n \to w \text{ a.e. in } \mathbb{R}^3. \end{aligned} \tag{4.4}$$

This together with (4.3) implies that $w \not\equiv 0$.

In what follows we prove that $(\mathcal{I}_\lambda^\infty)'(w) = 0$. Fix $\varphi \in C_c^\infty(\mathbb{R}^3)$. Note that (4.4) and (2.1) imply that (w_n) is bounded in $L^p(\mathbb{R}^3) \cap L^{q_s^*}(\mathbb{R}^3)$. Let $t \in \{p, q\}$. Consider the sequence

$$\psi_n(x,y)=\frac{|w_n(x)-w_n(y)|^{t-2}(w_n(x)-w_n(y))}{|x-y|^{\frac{3+st}{t'}}}$$

where $t'=\frac{t}{t-1}$. It is easy to check that (ψ_n) is a bounded sequence in $L^{t'}(\mathbb{R}^6)$, with $\psi_n\to\psi$ a.e. in $\mathbb{R}^6$ where

$$\psi(x,y)=\frac{|w(x)-w(y)|^{t-2}(w(x)-w(y))}{|x-y|^{\frac{3+st}{t'}}}.$$

Since $L^{t'}(\mathbb{R}^6)$ is a reflexive space, there exists a subsequence, still denoted by (h_n), such that $h_n\rightharpoonup h$ in $L^{t'}(\mathbb{R}^6)$, that is, for all $g\in L^t(\mathbb{R}^6)$ it holds

$$\iint_{\mathbb{R}^6}\psi_n(x,y)g(x,y)\,dxdy\to\iint_{\mathbb{R}^6}\psi(x,y)g(x,y)\,dxdy.$$

Now, since

$$g(x,y)=\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{3+st}{t}}}\in L^t(\mathbb{R}^6),$$

we deduce

$$\begin{aligned}&\iint_{\mathbb{R}^6}\frac{|w_n(x)-w_n(y)|^{t-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{3-st}}\,dxdy\\&\quad\to\iint_{\mathbb{R}^6}\frac{|w(x)-w(y)|^{t-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{3-st}}\,dxdy.\end{aligned}\tag{4.5}$$

It is clear that

$$\int_{\mathbb{R}^3}V(x)|w_n|^{t-2}w_n\varphi\,dx\to\int_{\mathbb{R}^3}V(x)|w|^{t-2}w\varphi\,dx\qquad\text{for } t\in\{p,q\},$$

$$\int_{\mathbb{R}^3}f(w_n)\varphi\,dx\to\int_{\mathbb{R}^3}f(w)\varphi\,dx.$$

Considering that also (w_n) is a Cerami sequence, we know that $\langle(\mathcal{I}_\lambda^\infty)'(w_n),\varphi\rangle=o_n(1)$, i.e.

$$\begin{aligned}\lim_{n\to\infty}\Bigg\{&(a+b[w_n]_{s,p}^p)\iint_{\mathbb{R}^6}\frac{|w_n(x)-w_n(y)|^{p-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{3-sp}}\,dxdy\\&+(c+d[w_n]_{s,q}^q)\iint_{\mathbb{R}^6}\frac{|w_n(x)-w_n(y)|^{q-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{3-sq}}\,dxdy\\&+\int_{\mathbb{R}^3}V(x)|w_n|^{p-2}w_n\varphi\,dx+\int_{\mathbb{R}^3}V(x)|w_n|^{q-2}w_n\varphi\,dx\\&-\lambda\mathcal{K}_\infty\int_{\mathbb{R}^3}f(w_n)\varphi\,dx-\mathcal{Q}_\infty\int_{\mathbb{R}^3}|w_n|^{q_s^*-2}w_n\varphi\,dx\Bigg\}=0.\end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\begin{aligned}
&(a+b\mathfrak{L}_p)\iint\limits_{\mathbb{R}^6} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{3-sp}}\,dxdy \\
&\quad + (c+d\mathfrak{L}_q)\iint\limits_{\mathbb{R}^6} \frac{|w(x)-w(y)|^{q-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{3-sq}}\,dxdy \\
&\quad + \int\limits_{\mathbb{R}^3} V(x)|w|^{p-2}w\varphi\,dx + \int\limits_{\mathbb{R}^3} V(x)|w|^{q-2}w\varphi\,dx \\
&\quad - \lambda\mathcal{K}_\infty \int\limits_{\mathbb{R}^3} f(w)\varphi\,dx - \mathcal{Q}_\infty \int\limits_{\mathbb{R}^3} |w|^{q_s^*-2}w\varphi\,dx = 0,
\end{aligned} \tag{4.6}$$

where

$$\mathfrak{L}_t := \lim_{n\to\infty} [w_n]_{s,t}^t.$$

Thanks to the density of $C_c^\infty(\mathbb{R}^3)$ in $\mathcal{W}_{p,q}$, we deduce that (4.6) holds for all $\varphi \in \mathcal{W}_{p,q}$. Now we show that $\mathfrak{L}_t = [w]_{s,t}^t$. Note that, by Fatou's lemma, we get

$$[w]_{s,t}^t \le \liminf_{n\to\infty} [w_n]_{s,t}^t = \mathfrak{L}_t.$$

Suppose by contradiction that $[w]_{s,t}^t < \mathfrak{L}_t$. Using this fact and (4.6) we have

$$\begin{aligned}
&(a+b[w]_{s,p}^p)[w]_{s,p}^p + (c+d[w]_{s,q}^q)[w]_{s,q}^q \\
&\quad + \int\limits_{\mathbb{R}^3} V(x)(|w|^p+|w|^q)\,dx - \lambda\mathcal{K}_\infty \int\limits_{\mathbb{R}^3} f(w)w\,dx - \mathcal{Q}_\infty \int\limits_{\mathbb{R}^3} |w|^{q_s^*}\,dx \\
&< (a+b\mathfrak{L}_p)[w]_{s,p}^p + (c+d\mathfrak{L}_q)[w]_{s,q}^q \\
&\quad + \int\limits_{\mathbb{R}^3} V(x)(|w|^p+|w|^q)\,dx - \lambda\mathcal{K}_\infty \int\limits_{\mathbb{R}^3} f(w)w\,dx - \mathcal{Q}_\infty \int\limits_{\mathbb{R}^3} |w|^{q_s^*}\,dx = 0.
\end{aligned}$$

Thus, $\langle (\mathcal{I}_\lambda^\infty)'(w), w\rangle < 0$. From the assumptions we have $\langle (\mathcal{I}_\lambda^\infty)'(\theta_0 w), \theta_0 w\rangle > 0$ for some $0 < \theta_0 << 1$. Therefore, there is $\theta \in (\theta_0, 1)$ such that $\langle (\mathcal{I}_\lambda^\infty)'(\theta w), \theta w\rangle = 0$. Hence by Fatou's lemma, (2.2), $\theta < 1$, $p < q$ and $\frac{3}{2} < sq < 3$ (which in particular ensures that $2q < q_s^*$) we have

$$\begin{aligned}
m_\infty &\le \mathcal{I}_\lambda^\infty(\theta w) = \mathcal{I}_\lambda^\infty(\theta w) - \frac{1}{2q}\langle (\mathcal{I}_\lambda^\infty)'(\theta w), \theta w\rangle \\
&= \left(\frac{1}{p} - \frac{1}{2q}\right)\theta^p \left(a[w]_{s,p}^p + \int\limits_{\mathbb{R}^3} V(x)|w|^p\,dx\right) + \left(\frac{1}{2p} - \frac{1}{2q}\right)\theta^{2p} b[w]_{s,p}^{2p} \\
&\quad + \frac{\theta^q}{2q}\left(c[w]_{s,q}^q + \int\limits_{\mathbb{R}^3} V(x)|w|^q\,dx\right) \\
&\quad + \lambda\mathcal{K}_\infty \int\limits_{\mathbb{R}^3} \left(\frac{1}{2q} f(\theta w)\theta w - F(\theta w)\right)\,dx - \left(\frac{1}{q_s^*} - \frac{1}{2q}\right)\mathcal{Q}_\infty \theta^{q_s^*} \|w\|_{q_s^*}^{q_s^*}
\end{aligned}$$

$$\leq \liminf_{n\to\infty} \left\{ \left(\frac{1}{p} - \frac{1}{2q}\right) \theta^p \left(a[w_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|w_n|^p \, dx \right) + \left(\frac{1}{2p} - \frac{1}{2q}\right) \theta^{2p} b[w_n]_{s,p}^{2p} \right.$$
$$+ \frac{\theta^q}{2q} \left(c[w_n]_{s,q}^q + \int_{\mathbb{R}^3} V(x)|w_n|^q \, dx \right)$$
$$\left. + \lambda \mathcal{K}_\infty \int_{\mathbb{R}^3} \left(\frac{1}{2q} f(\theta w_n)\theta w_n - F(\theta w_n) \right) dx - \left(\frac{1}{q_s^*} - \frac{1}{2q} \right) \mathcal{Q}_\infty \theta^{q_s^*} \|w_n\|_{q_s^*}^{q_s^*} \right\}$$
$$< \liminf_{n\to\infty} \left\{ \left(\frac{1}{p} - \frac{1}{2q}\right) \left(a[w_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|w_n|^p \, dx \right) + \left(\frac{1}{2p} - \frac{1}{2q}\right) b[w_n]_{s,p}^{2p} \right.$$
$$+ \frac{1}{2q} \left(c[w_n]_{s,q}^q + \int_{\mathbb{R}^3} V(x)|w_n|^q \, dx \right)$$
$$\left. + \lambda \mathcal{K}_\infty \int_{\mathbb{R}^3} \left(\frac{1}{2q} f(w_n) w_n - F(w_n) \right) dx - \left(\frac{1}{q_s^*} - \frac{1}{2q} \right) \mathcal{Q}_\infty \|w_n\|_{q_s^*}^{q_s^*} \right\}$$
$$= \liminf_{n\to\infty} \left(\mathcal{I}_\lambda^\infty(w_n) - \frac{1}{2q} \langle (\mathcal{I}_\lambda^\infty)'(w_n), w_n \rangle \right) = m_\infty,$$

which leads to a contradiction. This shows that

$$[w_n]_{s,t}^t \to \mathfrak{L}_t = [w]_{s,t}^t \quad \text{as } n \to \infty.$$

Hence, $\langle (\mathcal{I}_\lambda^\infty)'(w), w \rangle = 0$. Since $w \not\equiv 0$ we have that $w \in \mathcal{N}_\infty$ and so $\mathcal{I}_\lambda^\infty(w) \geq m_\infty$. Now, proceeding as before with $\theta = 1$, we obtain

$$\mathcal{I}_\lambda^\infty(w) = \mathcal{I}_\lambda^\infty(w) - \frac{1}{2q} \langle (\mathcal{I}_\lambda^\infty)'(w), w \rangle \leq \liminf_{n\to\infty} \left[\mathcal{I}_\lambda^\infty(w_n) - \frac{1}{2q} \langle (\mathcal{I}_\lambda^\infty)'(w_n), w_n \rangle \right] = m_\infty.$$

Therefore, $\mathcal{I}_\lambda^\infty(w) = m_\infty$. This completes the proof. □

5. Proof of Theorem 1.1

Let (u_n) be a sequence that verifies (3.1) and (3.2). Then, by Lemma 3.5 (u_n) is bounded in $\mathcal{W}_{p,q}$ for $\lambda > \lambda^*$ and, passing to a subsequence, there exists $u \in \mathcal{W}_{p,q}$ such that (3.4) hold true. In particular, by Lemma 3.5 we deduce that the vanishing cannot occur for $\|u_n\|_q^q$. Therefore there exist a sequence $(z_n) \subset \mathbb{Z}^3$ and constants $\ell_2, \rho_2 > 0$ such that

$$\int_{B_{\rho_2}(z_n)} |u_n|^q \, dx \geq \ell_2 > 0. \tag{5.1}$$

Next we prove that $u \not\equiv 0$. In fact, once proved this fact, we can argue as in the proof of Theorem 4.1 to conclude that u is a ground state solution to (1.1). In order to ensure that $u \not\equiv 0$, we show that (z_n) is bounded in $\mathbb{Z}^3$. Assume by contradiction that, up to a subsequence, $|z_n| \to \infty$ as $n \to \infty$. Denote by $\tilde{u}_n(\cdot) = u_n(\cdot + z_n)$. Due to (h_1) and the invariance by translations of $\mathbb{R}^3$, we have that $\|\tilde{u}_n\|_{V,p} = \|u_n\|_{V,p}$

and $\|\tilde{u}_n\|_{V,q} = \|u_n\|_{V,q}$, so $(\tilde{u}_n)$ is bounded in $\mathcal{W}_{p,q}$ and there exists $\tilde{u}$ such that (3.4) are verified. Utilizing again the invariance by translations of $\mathbb{R}^3$ and assumption (h_1), for all $\varphi \in \mathcal{W}_{p,q}$ we have

$$\begin{aligned}
\langle (\mathcal{I}_\lambda^\infty)'(\tilde{u}), \varphi\rangle &= \langle (\mathcal{I}_\lambda^\infty)'(\tilde{u}_n), \varphi\rangle + o_n(1) \\
&= \left(a + b[\tilde{u}_n]_{s,p}^p\right) \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))}{|x-y|^{3+sp}} (\varphi(x) - \varphi(y))\, dxdy \\
&\quad + \left(c + d[\tilde{u}_n]_{s,q}^q\right) \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{q-2}(\tilde{u}_n(x) - \tilde{u}_n(y))}{|x-y|^{3+sq}} (\varphi(x) - \varphi(y))\, dxdy \\
&\quad + \int_{\mathbb{R}^3} V(x) \left(|\tilde{u}_n|^{p-2}\tilde{u}_n\varphi + |\tilde{u}_n|^{q-2}\tilde{u}_n\varphi\right) dx \\
&\quad - \lambda\mathcal{K}_\infty \int_{\mathbb{R}^3} f(\tilde{u}_n)\varphi\, dx - \mathcal{Q}_\infty \int_{\mathbb{R}^3} |\tilde{u}_n|^{q_s^*-2}\tilde{u}_n\varphi\, dx \\
&= \langle \mathcal{I}_\lambda'(u_n), \tilde{\varphi}_n\rangle + \lambda \int_{\mathbb{R}^3} (\mathcal{K}(x) - \mathcal{K}_\infty)\, f(u_n)\tilde{\varphi}_n\, dx \\
&\quad + \int_{\mathbb{R}^3} (\mathcal{Q}(x) - \mathcal{Q}_\infty)\, |u_n|^{q_s^*-2}u_n\tilde{\varphi}_n\, dx + o_n(1)
\end{aligned}$$

where we denoted by $\tilde{\varphi}_n(\cdot) = \varphi(\cdot - z_n)$. Now, if we prove that

$$\int_{\mathbb{R}^3} (\mathcal{K}(x) - \mathcal{K}_\infty)\, f(u_n)\tilde{\varphi}_n\, dx = o_n(1) \tag{5.2}$$

and

$$\int_{\mathbb{R}^3} (\mathcal{Q}(x) - \mathcal{Q}_\infty)\, |u_n|^{q_s^*-2}u_n\tilde{\varphi}_n\, dx = o_n(1), \tag{5.3}$$

we conclude that $(\mathcal{I}_\lambda^\infty)'(\tilde{u}) = 0$. We only prove (5.2) since (5.3) can be obtained in a similar way. From (h_2) we have that for all $\delta > 0$ there exists $R_\delta > 0$ such that $|\mathcal{K}(x) - \mathcal{K}_\infty| \leq \delta$ for all $|x| \geq R_\delta$. Since $|z_n| \to +\infty$, there exists $\nu_\delta \in \mathbb{N}$ such that $|z_n| \geq 2R_\delta$ for all $n \geq \nu_\delta$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\mathcal{K}(x) - \mathcal{K}_\infty)\, f(u_n)\tilde{\varphi}_n\, dx \\
&= \int_{B_{R_\delta}} (\mathcal{K}(x) - \mathcal{K}_\infty)\, f(u_n)\tilde{\varphi}_n\, dx + \int_{\mathbb{R}^3 \setminus B_{R_\delta}} (\mathcal{K}(x) - \mathcal{K}_\infty)\, f(u_n)\tilde{\varphi}_n\, dx.
\end{aligned} \tag{5.4}$$

Now, utilizing the boundedness of $\mathcal{K}$, (1.3) and the Hölder inequality we find

$$\left| \int_{B_{R_\delta}} (\mathcal{K}(x) - \mathcal{K}_\infty)\, f(u_n)\tilde{\varphi}_n\, dx \right| \leq \|\mathcal{K}\|_{L^\infty(\mathbb{R}^3)} \int_{B_{R_\delta}} |f(u_n)||\varphi(x - z_n)|\, dx$$

$$\le C\|\mathcal{K}\|_{L^\infty(\mathbb{R}^3)}\left\{\|u_n\|^{p-1}_{L^p(\mathbb{R}^3)}\left(\int_{B_{R_\delta}}|\varphi(x-z_n)|^p\,dx\right)^{\frac{1}{p}}\right.$$

$$\left.+\|u_n\|^{r-1}_{L^r(\mathbb{R}^3)}\left(\int_{B_{R_\delta}}|\varphi(x-z_n)|^r\,dx\right)^{\frac{1}{r}}\right\}$$

$$\le C\left\{\left(\int_{B_{R_\delta}}|\varphi(x-z_n)|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{B_{R_\delta}}|\varphi(x-z_n)|^r\,dx\right)^{\frac{1}{r}}\right\}$$

$$= C\left\{\left(\int_{B_{R_\delta}(-z_n)}|\varphi(x)|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{B_{R_\delta}(-z_n)}|\varphi(x)|^r\,dx\right)^{\frac{1}{r}}\right\}.$$

Fix $n \ge \nu_\delta$. Then, if $|x+z_n| < R_\delta$ then, thanks to $|z_n| \ge 2R_\delta$ for all $n \ge \nu_\delta$, we have

$$|x| \ge |z_n| - |x+z_n| \ge |z_n| - R_\delta \ge \frac{|z_n|}{2}.$$

Now, fixed $\beta \in \{p, r\}$ we have

$$\int_{B_{R_\delta}(-z_n)}|\varphi(x)|^\beta\,dx \le \int_{\mathbb{R}^3\setminus B_{\frac{|z_n|}{2}}}|\varphi|^\beta\,dx \quad \forall n \ge \nu_\delta, \tag{5.5}$$

from which

$$\left|\int_{B_{R_\delta}}\left(\mathcal{K}(x)-\mathcal{K}_\infty\right)f(u_n)\tilde{\varphi}_n\,dx\right| \le C\left\{\left(\int_{\mathbb{R}^3\setminus B_{\frac{|z_n|}{2}}}|\varphi|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^3\setminus B_{\frac{|z_n|}{2}}}|\varphi|^r\,dx\right)^{\frac{1}{r}}\right\} \quad \forall n \ge \nu_\delta. \tag{5.6}$$

For what concern the second integral in (5.4), combining $|\mathcal{K}(x)-\mathcal{K}_\infty| \le \delta$ for all $|x| \ge R_\delta$, (1.3) and Hölder inequality we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3\setminus B_{R_\delta}}\left(\mathcal{K}(x)-\mathcal{K}_\infty\right)f(u_n)\tilde{\varphi}_n\,dx\right| &\le \delta\int_{\mathbb{R}^3}|f(u_n)||\varphi(x-z_n)|\,dx\\ &\le \delta\left\{\|u_n\|^{p-1}_{L^p(\mathbb{R}^3)}\|\varphi(\cdot-z_n)\|_{L^p(\mathbb{R}^3)}+\|u_n\|^{r-1}_{L^r(\mathbb{R}^3)}\|\varphi(\cdot-z_n)\|_{L^r(\mathbb{R}^3)}\right\}\\ &\le C\delta \quad \forall n \in \mathbb{N}.\end{aligned} \tag{5.7}$$

Putting together (5.4), (5.6) and (5.7) we get

$$\left|\int_{\mathbb{R}^3} (\mathcal{K}(x)-\mathcal{K}_\infty) f(u_n)\tilde{\varphi}_n\,dx\right| \le C\left\{\left(\int_{\mathbb{R}^3\setminus B_{\frac{|z_n|}{2}}} |\varphi|^p\,dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^3\setminus B_{\frac{|z_n|}{2}}} |\varphi|^r\,dx\right)^{\frac{1}{r}}\right\} + C\delta \quad \forall n\ge \nu_\delta,$$

from which

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3} (\mathcal{K}(x)-\mathcal{K}_\infty) f(u_n)\tilde{\varphi}_n\,dx\right| \le C\delta,$$

and taking the limit as $\delta\to 0$ we obtain the assertion.

Using $(\mathcal{I}_\lambda^\infty)'(\tilde{u})=0$, assumptions (h_1), (h_2) and (h_3) and Fatou's lemma we obtain

$$\begin{aligned}
m_\infty &\le \mathcal{I}_\lambda^\infty(\tilde{u}) = \mathcal{I}_\lambda^\infty(\tilde{u}) - \frac{1}{2q}\langle (\mathcal{I}_\lambda^\infty)'(\tilde{u}),\tilde{u}\rangle \\
&= \left(\frac{1}{p}-\frac{1}{2q}\right)\left(a[\tilde{u}]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|\tilde{u}|^p\,dx\right) + \left(\frac{1}{2p}-\frac{1}{2q}\right) b[\tilde{u}]_{s,p}^{2p} \\
&\quad + \frac{1}{2q}\left(c[\tilde{u}]_{s,q}^q + \int_{\mathbb{R}^3} V(x)|\tilde{u}|^q\,dx\right) \\
&\quad + \lambda\mathcal{K}_\infty \int_{\mathbb{R}^3}\left(\frac{1}{2q}f(\tilde{u})\tilde{u} - F(\tilde{u})\right)dx - \left(\frac{1}{q_s^*}-\frac{1}{2q}\right)\mathcal{Q}_\infty \|\tilde{u}\|_{q_s^*}^{q_s^*} \\
&\le \liminf_{n\to\infty}\left\{\left(\frac{1}{p}-\frac{1}{2q}\right)\left(a[\tilde{u}_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|\tilde{u}_n|^p\,dx\right) + \left(\frac{1}{2p}-\frac{1}{2q}\right) b[\tilde{u}_n]_{s,p}^{2p}\right. \\
&\quad + \frac{1}{2q}\left(c[\tilde{u}_n]_{s,q}^q + \int_{\mathbb{R}^3} V(x)|\tilde{u}_n|^q\,dx\right) \\
&\quad \left. + \lambda\mathcal{K}_\infty \int_{\mathbb{R}^3}\left(\frac{1}{2q}f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n)\right)dx - \left(\frac{1}{q_s^*}-\frac{1}{2q}\right)\mathcal{Q}_\infty \|\tilde{u}_n\|_{q_s^*}^{q_s^*}\right\} \\
&= \liminf_{n\to\infty}\left\{\left(\frac{1}{p}-\frac{1}{2q}\right)\left(a[\tilde{u}_n]_{s,p}^p + \int_{\mathbb{R}^3} V(x)|\tilde{u}_n|^p\,dx\right) + \left(\frac{1}{2p}-\frac{1}{2q}\right) b[\tilde{u}_n]_{s,p}^{2p}\right. \\
&\quad + \frac{1}{2q}\left(c[\tilde{u}_n]_{s,q}^q + \int_{\mathbb{R}^3} V(x)|\tilde{u}_n|^q\,dx\right) \\
&\quad \left. + \lambda \int_{\mathbb{R}^3}\mathcal{K}(x)\left(\frac{1}{2q}f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n)\right)dx - \left(\frac{1}{q_s^*}-\frac{1}{2q}\right)\int_{\mathbb{R}^3}\mathcal{Q}(x)|\tilde{u}_n|^{q_s^*}\,dx\right\} \\
&= \liminf_{n\to\infty}\left(\mathcal{I}_\lambda(u_n) - \frac{1}{2q}\langle \mathcal{I}_\lambda'(u_n),u_n\rangle\right) = c_\lambda,
\end{aligned}$$

which contradicts (4.2). Therefore $u\not\equiv 0$, and the proof is now complete.

Acknowledgments

The author is partially supported by INdAM-GNAMPA Project CUP E5324001950001.

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