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Algebraic Geometry. – *Minimal Terracini loci in projective space*, by Edoardo Ballico and Maria Chiara Brambilla, communicated on 8 November 2024.

Abstract. – We characterize the number of points for which there exist non-empty Terracini sets of points in $\mathbb{P}^n$. Then, we study *minimally Terracini* finite sets of points in $\mathbb{P}^n$, and we obtain a complete description, in the case of $\mathbb{P}^3$, when the number of points is less than twice the degree of the linear system.

Keywords. – interpolation problems, minimal Terracini locus, Terracini locus, zero-dimensional schemes.

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1. Introduction

The notion of *Terracini locus* in projective spaces has been recently introduced in [3] and then extended to other projective varieties and investigated in [2, 4, 5, 10]. This property encodes the fact that a set of double points imposes dependent conditions to a linear system; hence, it gives information for interpolation problems over double points in special position.

Moreover, it can be interpreted in terms of special loci contained in higher secant varieties to projective varieties as follows. Recall that the k-th higher secant variety $\sigma_k(X)$ of a projective variety $X \subset \mathbb{P}^N$ is the Zariski closure of the union of all the linear spaces spanned by k independent points of X. The variety X is called k-defective if it has dimension less than the expected one, i.e. $\min(N, k\dim(X) + k - 1)$. By the famous Terracini lemma [13], a variety is k-defective if the tangent spaces to X at k general points span a linear space of dimension less than the expected one. Even when the variety is not k-defective, there may be special sets of points such that the span of the tangent spaces drops dimension. We call *Terracini* such special sets of points. For non-defective varieties, we can see the Terracini sets as the points of the abstract secant variety for which the differential of the map to the secant variety is not injective; see e.g. [3] for more details.

The interest in this subject is also motivated by the connection with the theory of tensors; see e.g. [6, 12] for general reference. In particular, since symmetric tensors can be identified with homogeneous polynomials, the development of geometric methods in projective spaces can give contribution to the study of the rank and decompositions of symmetric tensors.

In this paper, we focus on the case of $\mathbb{P}^n$, and we say that a finite set of points S of $\mathbb{P}^n$ is *Terracini* with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$ if

$$h^0\big(\mathcal{I}_{2S}(d)\big) > 0, \quad h^1\big(\mathcal{I}_{2S}(d)\big) > 0, \quad \text{and} \quad \langle S\rangle = \mathbb{P}^n.$$

We denote by $\mathbb{T}(n, d; x)$ the sets of all subsets $S \subset \mathbb{P}^n$ of cardinality x which are Terracini with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$.

In the language of secant varieties, the first condition means that the secant variety $\sigma_x(X) \subset \mathbb{P}^N$ does not fill the ambient space since $\dim \sigma_x(X) = N - h^0(\mathcal{I}_Z(d))$ (see e.g. [6, Corollary 1]). On the other hand, if $h^0(\mathcal{I}_{2S}(d)) > 0$, then the number $h^1(\mathcal{I}_{2S}(d))$ equals the so-called x*-defect*; that is, $x(n+1) - \dim \sigma_x(X) - 1$ (see Lemma 2.5).

Notice that there are no Terracini sets in $\mathbb{P}^1$; see Lemma 3.3. The first result of this paper characterizes the triples n, d, x such that the Terracini locus is non-empty, as follows.

Theorem 1.1. *Fix positive integers n, d and x.*

(i) *If either $n = 1$ or $d = 2$, then $\mathbb{T}(n, d; x) = \emptyset$ for any x.*

(ii) *$\mathbb{T}(2, 3; x) = \emptyset$ for any x.*

(iii) *If $n \geq 2$, $d \geq 3$ and $(n, d) \neq (2, 3)$, then $\mathbb{T}(n, d; x) \neq \emptyset$ if and only if $x \geq n + \lceil d/2\rceil$.*

In order to make a finer description, it is very useful to study *minimally Terracini loci*. The *minimally Terracini* property has been introduced in [2, Definition 2.2] for any projective variety. A Terracini set of points $S \subset \mathbb{P}^n$ is said to be minimally Terracini with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$ if

$$h^1\big(\mathcal{I}_{2A}(d)\big) = 0 \quad \text{for all } A \subsetneq S.$$

We denote by $\mathbb{T}(n, d; x)'$ the set of all $S \in \mathbb{T}(n, d; x)$ which are minimally Terracini with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$.

In Theorem 3.1, we see that if $S \in S(\mathbb{P}^n, x)$ is minimally Terracini for some $\mathcal{O}_{\mathbb{P}^n}(d)$, then such d is unique and it is the maximal integer t such that $h^1(\mathcal{I}_{2S}(t)) > 0$.

Note that, for fixed n, d, we know that $\mathbb{T}(n, d; x)$ is not empty for infinitely many x, by Theorem 1.1. On the other hand, $\mathbb{T}(n, d; x)' \subseteq \mathbb{T}(n, d; x)$ is not empty only for finitely many x, as proved in Proposition 3.4. In other words, the minimality property is a strong condition which allows us to prove interesting bounds and characterizations of the triples n, d, x for which $\mathbb{T}(n, d; x)'$ is or is not empty.

In Section 4, we investigate the sets of points on rational normal curves and on their degenerations (reducible rational normal curves). In particular, Theorem 4.2 and Proposition 4.7 completely describe the minimal Terracini sets contained in such curves. Since rational normal curves contain elements of $\mathbb{T}(n, d; 1 + \lceil nd/2 \rceil)'$, we may formulate the following conjecture.

CONJECTURE 1.2. *For any* $x \leq \lfloor \frac{nd+1}{2} \rfloor$, *we have* $\mathbb{T}(n, d; x)' = \emptyset$.

Here we prove the conjecture for $\mathbb{P}^2$, Proposition 5.2, and for $\mathbb{P}^3$, Theorem 1.3.

After the easy description of the situation in the plane (see Section 5), we focus on the case of $\mathbb{P}^3$, and we obtain the following three results, which are the main results of this paper.

THEOREM 1.3. *Fix integers* $d \geq 4$ *and* x *such that* $2x \leq 3d + 1$. *Then,* $\mathbb{T}(3, d; x)' = \emptyset$.

THEOREM 1.4. *Fix integers* $d \geq 7$ *and* $x = 1 + \lceil 3d/2 \rceil$. *Then,* $S \in \mathbb{T}(3, d; x)'$ *if and only if* S *is contained in a rational normal curve.*

THEOREM 1.5. *Fix integers* $d \geq 17$ *and* x *such that* $1 + \lceil 3d/2 \rceil < x < 2d$. *Then,* $\mathbb{T}(3, d; x)' = \emptyset$.

The bound in Theorem 1.5 is sharp, as shown in Example 6.2, where $2d$ points lie on an elliptic curve.

Summing up, our results prove that, given $d > 0$ and $x \leq 2d$, the minimal Terracini loci $\mathbb{T}(3, d; x)'$ are empty except for

- either $x = 1 + \lceil 3d/2 \rceil$, and in this case the points lie on a rational normal curve,
- or $x = 2d$, and in this case the points may lie on an elliptic curve.

We call $(0, 1 + \lceil 3d/2 \rceil)$, $(1 + \lceil 3d/2 \rceil, 2d)$ the first two gaps where the minimal Terracini loci are empty.

The situation is completely analogous in $\mathbb{P}^2$, where the first two gaps are $(0, d + 1)$ and $(d + 1, \lfloor 3d/2 \rfloor)$; see Section 5.

We expect that a similar behavior happens also in any dimension $n \geq 2$.

The paper is organized as follows: in Section 2, we present the preliminary results, and in particular we introduce the notion of critical scheme, which is a crucial tool in our proofs. Section 3 contains the first properties of Terracini and minimal Terracini sets and the proof of Theorem 1.1. In Section 4, we characterize the minimally Terracini sets of points on rational normal curves and their degenerations. Section 5 is devoted to the plane, and Section 6 to the case of $\mathbb{P}^3$ and to the proofs of Theorems 1.3, 1.4 and 1.5.

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2. Preliminaries and notation

We work over an algebraically closed field $\mathbb{K}$ of characteristic 0. For any $x \in \mathbb{N}$, let $S(\mathbb{P}^n, x)$ denote the set of all subsets of cardinality x of a projective space $\mathbb{P}^n$. For any set $E \subset \mathbb{P}^n$, let $\langle E \rangle$ denote the linear span of E in $\mathbb{P}^n$.

Remark 2.1. It is well known that the set of configurations of $n+2$ points of $\mathbb{P}^n$ in linear general position is an open orbit for the action of $\mathrm{Aut}(\mathbb{P}^n)$.

Definition 2.2. We denote by $\mathbb{T}_1(n, d; x)$ the set of all $S \in S(\mathbb{P}^n, x)$ such that

- $h^0(\mathcal{I}_{2S}(d)) > 0$ and $h^1(\mathcal{I}_{2S}(d)) > 0$.

We denote by $\mathbb{T}(n, d; x) \subseteq \mathbb{T}_1(n, d; x)$ the set of all $S \in \mathbb{T}_1(n, d; x)$ such that

- $\langle S \rangle = \mathbb{P}^n$.

We call *Terracini locus* the set $\mathbb{T}(n, d; x)$, and we say that a finite set S is *Terracini with respect to* $\mathcal{O}_{\mathbb{P}^n}(d)$ if $S \in \mathbb{T}(n, d; x)$.

Obviously, $\mathbb{T}(n, d; x) = \emptyset$ for all $x \leq n$ since every $S \in \mathbb{T}(n, d; x)$ spans $\mathbb{P}^n$.

We recall from [2, Definition 2.2] the following important definition; it applies to any projective variety, but we write it now only in the case of $\mathbb{P}^n$.

Definition 2.3. A set S is said to be *minimally Terracini with respect to* $\mathcal{O}_{\mathbb{P}^n}(d)$ if it is Terracini and moreover

- $h^1(\mathcal{I}_{2A}(d)) = 0$ for all $A \subsetneq S$.

We denote by $\mathbb{T}(n, d; x)'$ the set of all $S \in \mathbb{T}(n, d; x)$ which are minimally Terracini with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$.

In the next remark, we recall the exceptional cases of the Alexander–Hirschowitz theorem, which are all the cases when any general set of points is minimally Terracini.

Remark 2.4. Assume $(n, d; x) \in \{(2, 4; 5), (3, 4; 9), (4, 4; 14), (4, 3; 7)\}$. Then, by the Alexander–Hirschowitz theorem [1], we know that the Veronese variety $\nu_d(\mathbb{P}^n)$ is x-defective.

Fix a general $S \in S(\mathbb{P}^n, x)$. We have that $h^0(\mathcal{I}_{2S}(d)) > 0$ because the x-secant variety does not fill the ambient space, and $h^1(\mathcal{I}_{2S}(d)) > 0$ because it is defective. Moreover, since $x \geq n+1$, we have $\langle S \rangle = \mathbb{P}^n$ and hence $S \in \mathbb{T}(n, d; x)$.

We prove now that S is minimal. Indeed, since S is general, then any subset $S' \subset S$ of cardinality $y < x$ is general in $S(\mathbb{P}^n, y)$. Since the secant variety $\sigma_y(\nu_d(\mathbb{P}^n))$ is not defective for any $y \leq x-1$, then $h^1(\mathcal{I}_{2S'}(d)) = 0$. Then, we have proved that $S \in \mathbb{T}(n, d; x)'$.

We collect here some preliminary results we will use in the sequel.

LEMMA 2.5. *For any zero-dimensional scheme $Z \subset \mathbb{P}^n$ and any integer $t \geq 0$, we have $h^i(\mathcal{I}_Z(t)) = 0$ for all $i \geq 2$, and*

$$h^0(\mathcal{I}_Z(t)) - h^1(\mathcal{I}_Z(t)) = \binom{n+t}{n} - \deg(Z).$$

PROOF. Since Z is zero-dimensional, we have $h^i(\mathcal{O}_Z(t)) = 0$ for all $i > 0$. Obviously, $h^i(\mathcal{O}_{\mathbb{P}^n}(t)) = 0$ for all $i > 0$. Then, from the exact sequence

$$0 \to \mathcal{I}_Z(t) \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{O}_Z(t) \to 0,$$

we obtain the formulas in the statement. ∎

LEMMA 2.6. *Let $W \subset Z \subset \mathbb{P}^n$ be zero-dimensional schemes and $t \geq 0$. Then, we have*

$$h^0(\mathcal{I}_Z(t)) \leq h^0(\mathcal{I}_W(t)) \quad \textit{and} \quad h^1(\mathcal{I}_W(t)) \leq h^1(\mathcal{I}_Z(t))$$

and

$$h^0(\mathcal{I}_Z(t)) \leq h^0(\mathcal{I}_Z(t+1)) \quad \textit{and} \quad h^1(\mathcal{I}_Z(t+1)) \leq h^1(\mathcal{I}_Z(t)).$$

PROOF. Since $W \subset Z$, then we have the exact sequences

$$0 \to \mathcal{I}_{W,Z}(t) \to \mathcal{O}_W(t) \to \mathcal{O}_Z(t) \to 0$$

and

$$0 \to \mathcal{I}_Z(t) \to \mathcal{I}_W(t) \to \mathcal{I}_{W,Z}(t) \to 0.$$

Since Z is zero-dimensional, then $h^i(\mathcal{I}_{W,Z}(d)) = 0$ for all $i \geq 1$. Then, we get

$$h^0(\mathcal{I}_Z(d)) \leq h^0(\mathcal{I}_W(d)) \quad \text{and} \quad h^1(\mathcal{I}_W(d)) \leq h^1(\mathcal{I}_Z(d)).$$

From the exact sequence

$$0 \to \mathcal{I}_Z(t) \to \mathcal{I}_Z(t+1) \to \mathcal{O}_H(t+1) \to 0,$$

where $H \subset \mathbb{P}^n$ is a hyperplane, it follows that

$$h^0(\mathcal{I}_Z(d)) \leq h^0(\mathcal{I}_Z(d+1)) \quad \text{and} \quad h^1(\mathcal{I}_Z(d+1)) \leq h^1(\mathcal{I}_Z(d)).$$ ∎

LEMMA 2.7. *Given a hyperplane $H \subset \mathbb{P}^n$ and any finite set $S \subset H$, we have*

$$h^1(\mathcal{I}_{2S\cap H,H}(d)) \leq h^1(\mathcal{I}_{2S}(d)) \leq h^1(\mathcal{I}_{2S\cap H,H}(d)) + h^1(\mathcal{I}_S(d-1)).$$

PROOF. From the residual exact sequence with respect to H

$$0 \to \mathcal{I}_S(d-1) \to \mathcal{I}_{2S}(d) \to \mathcal{I}_{2S\cap H,H}(d) \to 0,$$

and by Lemma 2.5, the statement follows. ∎

We recall from [7] the following useful lemma.

LEMMA 2.8 ([7, Lemma 34]). *Let Z be a zero-dimensional scheme in $\mathbb{P}^n$, such that $h^1(\mathcal{I}_Z(d)) > 0$. If $\deg(Z) \leq 2d+1$, then there is a line L such that $\deg(Z\cap L) \geq d+2$. In particular, it follows that $\deg(Z) \geq d+2$.*

We recall the following lemma which we learned from K. Chandler [8, 9].

LEMMA 2.9. *Let W be an integral projective variety, $\mathcal{L}$ a line bundle on W with $h^1(\mathcal{L}) = 0$ and $S \subset W_{\mathrm{reg}}$ a finite collection of points. Then, $h^1(\mathcal{I}_{(2S,W)} \otimes \mathcal{L}) > 0$ if and only if there is a scheme $Z \subset 2S$ such that any connected component of Z has degree ≤ 2 and such that $h^1(\mathcal{I}_Z \otimes \mathcal{L}) > 0$.*

The schemes Z appearing in Lemma 2.9 are a curvilinear subscheme of a collection of double points. More precisely, in the following definition, we introduce the notion of *critical schemes*, which are the crucial tools in our proofs.

DEFINITION 2.10. Given S a collection of x points in $\mathbb{P}^n$, we say that a zero-dimensional scheme Z is d*-critical for* S if

- $Z \subseteq 2S$ and any connected component of Z has degree ≤ 2,
- $h^1(\mathcal{I}_Z(d)) > 0$,
- $h^1(\mathcal{I}_{Z'}(d)) = 0$ for any $Z' \subsetneq Z$.

Note that Lemma 2.9 implies that for every $S \in \mathbb{T}(n,d;x)$, there exists a d-critical scheme for S.

The next lemmas describe the properties of a critical scheme.

LEMMA 2.11. *Let Z be a zero-dimensional scheme such that $h^1(\mathcal{I}_Z(d)) > 0$ and $h^1(\mathcal{I}_{Z'}(d)) = 0$ for any $Z' \subsetneq Z$. Then, $h^1(\mathcal{I}_Z(d)) = 1$.*

PROOF. Assume $h^1(\mathcal{I}_Z(d)) \geq 2$ and take a subscheme $Z' \subset Z$ such that $\deg(Z') = \deg(Z) - 1$. We have $h^1(\mathcal{I}_{Z'}(d)) \geq h^1(\mathcal{I}_Z(d)) - \deg(Z) + \deg(Z') > 0$. Thus, Z is not critical, a contradiction. ∎

LEMMA 2.12. *Fix $S \in \mathbb{T}(n,d;x)'$ and take Z critical for S. Then, $Z_{\mathrm{red}} = S$.*

PROOF. Assume $S' := Z_{\mathrm{red}} \neq S$. Lemma 2.9 gives $h^1(\mathcal{I}_{2S'}(d)) > 0$. Thus, S does not belong to $\mathbb{T}(n, d; , x)'$, a contradiction. ∎

LEMMA 2.13. *Fix integers $n \geq 2$, $d > t \geq 1$ and $x > 1$. Take $S \in \mathbb{T}(n, d; x)'$ and a critical scheme Z for S. Take $D \in |\mathcal{O}_{\mathbb{P}^n}(t)|$ with $Z \nsubseteq D$. Then, $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(d - t)) > 0$.*

PROOF. Since $Z \nsubseteq D$ and it is critical, then Definition 2.10 gives $h^1(\mathcal{I}_{Z \cap D}(d)) = 0$. Thus, the residual exact sequence with respect to D gives $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(d - t)) > 0$. ∎

3. FIRST RESULTS ON MINIMALLY TERRACINI SETS OF POINTS

We now prove the fact that if $S \in S(\mathbb{P}^n, x)$ is minimally Terracini for some $\mathcal{O}_{\mathbb{P}^n}(d)$, then such d is unique and it is the maximal integer t such that $h^1(I_{2S}(t)) > 0$.

THEOREM 3.1. *Fix $n \geq 2$ and $S \in \mathbb{T}(n, d; x)'$. Then,*

(i) $h^1(\mathcal{I}_{2S}(d + 1)) = 0$,

(ii) $S \notin \mathbb{T}(n, t; x)$ *for any* $t \geq d + 1$,

(iii) $S \notin \mathbb{T}(n, t; x)'$ *for any* $t \leq d - 1$.

PROOF. We now prove (i) by contradiction. Assume $h^1(\mathcal{I}_{2S}(d+1)) > 0$. By Lemma 2.9, there is a $(d + 1)$-critical scheme Z for S. Recall that, in particular, every component of Z has degree ≤ 2. Moreover, by Lemma 2.12, we have $S \subset Z \subset 2S$, whereas from Lemma 2.11, we know that $h^1(\mathcal{I}_Z(d + 1)) = 1$.

Fix $p \in Z_{\mathrm{red}}$ and call $Z(p)$ the connected component of Z supported at p. Set $L := \langle Z(p) \rangle$. Then, L is either a line, or a point $L = Z(p) = \{p\}$.

Let $H \subset \mathbb{P}^n$ be a general hyperplane containing L. Since Z is curvilinear, by generality of H, we can assume that the scheme $Z \cap H$ is equal to the scheme $Z \cap L$. Let us denote $\zeta = Z \cap H = Z \cap L$. We will consider separately two possibilities: $h^1(\mathcal{I}_{\zeta,H}(d + 1)) > 0$ and $h^1(\mathcal{I}_{\zeta,H}(d + 1)) = 0$.

(a) Assume first $h^1(\mathcal{I}_{\zeta,H}(d + 1)) > 0$. Then, L is a line. Since $\zeta \subset L$, then we have the following diagram, whose rows and columns are exact sequences:

$$\begin{array}{ccccc}
\mathcal{I}_{L,H}(d+1) & \longrightarrow & \mathcal{I}_{\zeta,H}(d+1) & \longrightarrow & \mathcal{I}_{\zeta,L}(d+1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I}_{L,H}(d+1) & \longrightarrow & \mathcal{O}_H(d+1) & \longrightarrow & \mathcal{O}_L(d+1) \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{O}_\zeta(d+1) & \longrightarrow & \mathcal{O}_\zeta(d+1).
\end{array}$$

From the diagram, we get $h^1(\mathcal{I}_{\zeta,L}(d + 1)) > 0$, which implies $h^1(\mathcal{I}_\zeta(d + 1)) > 0$.

Since $\langle S\rangle = \mathbb{P}^n$ and $n \geq 2$, the set $S' = S \cap L$ is different from S. Now by Lemma 2.9, we have that $h^1(\mathcal{I}_{2S'}(d+1)) > 0$, and Lemma 2.6 implies that $h^1(\mathcal{I}_{2S'}(d)) > 0$. Hence, we have $S \notin \mathbb{T}(n, d; x)'$, a contradiction.

(b) Now assume $h^1(\mathcal{I}_{\zeta,H}(d+1)) = 0$. In this case, the residual exact sequence with respect to H gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d)) > 0$. Since $\mathrm{Res}_H(Z)_{\mathrm{red}} \subseteq S \setminus \{p\}$, by Lemma 2.9, we have that $h^1(\mathcal{I}_{2(S\setminus\{p\})}(d)) > 0$. This contradicts the minimality of S; that is, $S \notin \mathbb{T}(n, d, x)'$, a contradiction.

Now it is easy to prove (ii). Indeed by using (i) and Lemma 2.6, we get, for any $t \geq d+1$, that $h^1(\mathcal{I}_{2S}(t)) \leq h^1(\mathcal{I}_{2S}(d+1)) = 0$. Hence, $S \notin \mathbb{T}(n, t; x)$.

We prove (iii) by contradiction. Indeed assume $t \leq d-1$ and $S \in \mathbb{T}(n, t; x)'$. But then by (i), we have $h^1(\mathcal{I}_{2S}(t+1)) = 0$. Then, since $t+1 \leq d$ by Lemma 2.6, we have $h^1(\mathcal{I}_{2S}(d)) \geq h^1(\mathcal{I}_{2S}(t+1)) = 0$, which contradicts the assumption $S \in \mathbb{T}(n, d; x)'$. ■

The following result is a kind of *concision* or *autarky* for Terracini loci of Veronese varieties.

Proposition 3.2. *Take a finite set of points $S \subset \mathbb{P}^n$ such that $M := \langle S\rangle \subseteq \mathbb{P}^n$. Then,*

$$h^1\big(M, \mathcal{I}_{2S\cap M,M}(d)\big) > 0 \quad \text{if and only if } h^1\big(\mathcal{I}_{2S}(d)\big) > 0.$$

Proof. By Lemma 2.6, we have $h^1(\mathcal{I}_{2S}(d)) \geq h^1(\mathcal{I}_{2S\cap M}(d))$. Since M is arithmetically Cohen–Macaulay, we get $h^1(M, \mathcal{I}_{2S\cap M,M}(d)) = h^1(\mathcal{I}_{2S\cap M}(d))$. Hence, the *only if* part is obvious.

Now assume $h^1(\mathcal{I}_{2S}(d)) > 0$. Take a hyperplane $H \subset \mathbb{P}^n$ such that $H \supseteq M$ and use induction on $n - \dim M$. It is sufficient to prove that $h^1(H, \mathcal{I}_{2S\cap H,H}(d)) > 0$.

Take a critical scheme Z for S. In order to conclude by Lemma 2.9, it is enough to find a zero-dimensional scheme $W \subset H$ such that $h^1(H, \mathcal{I}_{W,H}(d)) > 0$, $W_{\mathrm{red}} = Z_{\mathrm{red}}$ and for each $p \in Z_{\mathrm{red}}$ the connected components Z_p and W_p of Z and W containing p have the same degree. Fix a general $o \in \mathbb{P}^n \setminus H$. Let $h_o : \mathbb{P}^n \setminus \{o\} \to H$ denote the linear projection from o. Since o is general, o is not contained in one of the finitely many lines spanned by the degree 2 connected components of Z. Since $Z_{\mathrm{red}} \subset H$, o is not contained in a line spanned by 2 points of Z_{red}. Thus, $h_{o|Z}$ is an isomorphism. Set $W := h_o(Z)$. By the semicontinuity theorem for cohomology to prove that $h^1(H, \mathcal{I}_{W,H}(d)) > 0$, it is sufficient to prove that W is a flat limit of a flat family $\{W_c\}_{c\in\mathbb{K}\setminus\{0\}}$ of schemes projectively equivalent to Z. Fix a system $x_0, \ldots, x_n$ of homogeneous coordinates of $\mathbb{P}^n$ such that $H = \{x_0 = 0\}$ and $o = [1 : 0 : \ldots : 0]$. For any $c \in \mathbb{K} \setminus \{0\}$, let h_c denote the automorphism of $\mathbb{P}^n$ defined by the formula $h_c([x_0 : x_1 : \ldots : x_n]) = [cx_0 : x_1 : \ldots : x_n]$. Note that $h_{c|H} : H \to H$ is the identity map. Set $W_c := h_c(W)$. ■

We start now the classification of Terracini and minimal Terracini sets of points in $\mathbb{P}^n$. Obviously, $\mathbb{T}(n, d; x) = \emptyset$ for all $x \leq n$ since every $S \in \mathbb{T}(n, d; x)$ spans $\mathbb{P}^n$.

LEMMA 3.3. $\mathbb{T}(1,d;x) = \mathbb{T}_1(1,d;x) = \emptyset$ *for all* $d > 0$ *and* $x > 0$.

PROOF. Assume by contradiction the existence of $S \in \mathbb{T}_1(1,d;x)$. Then, $h^1(\mathcal{I}_{2S}(d)) > 0$ and hence $2x \geq d+2$ and $h^0(\mathcal{I}_{2S}(d)) > 0$ and hence $2x \leq d+1$, a contradiction. ■

The following proposition shows a key difference between $\mathbb{T}(n,d;x)$ and its subset $\mathbb{T}(n,d;x)'$. In particular for fixed n and d, we have $\mathbb{T}(n,d;x)' \neq \emptyset$ for only finitely many integers x.

PROPOSITION 3.4. *Fix integers* $n \geq 2$ *and* $d \geq 3$. *Set*

$$\rho := \left\lceil \frac{\binom{n+d}{n}+1}{n+1} \right\rceil$$

and then $\mathbb{T}(n,d;x)' = \emptyset$ *for all* $x > \rho$.

PROOF. Let $x > \rho$ and assume by contradiction $S \in \mathbb{T}(n,d;x)'$. Then, $h^0(\mathcal{I}_{2S}(d)) > 0$, and, by Lemma 2.6, $h^0(\mathcal{I}_T(d)) > 0$ for all $T \subseteq S$. Take $T \subset S$ with $\#(T) = x-1 \geq \rho$. Then, we have $h^1(\mathcal{I}_{2T}(d)) > 0$ by Lemma 2.5. Then, S is not minimally Terracini. ■

LEMMA 3.5. $\mathbb{T}(n,2;x) = \emptyset$ *for all* $x > 0$ *and all* $n > 0$.

PROOF. Assume by contradiction that $S \in \mathbb{T}(n,2;x)$. Since $\langle S\rangle = \mathbb{P}^n$, we have $x \geq n+1$.

First assume $x = n+1$. Since $\langle S\rangle = \mathbb{P}^n$, then the points of S are linearly independent. Recall that all the quadrics with the same rank are projectively equivalent. Since a general quadric form in $\mathbb{P}^n$ has rank $n+1$, we have that the $(n+1)$-secant variety to $\nu_2(\mathbb{P}^n)$ fills the ambient space; hence, $h^0(\mathcal{I}_{2S}(2)) = 0$, and this contradicts the fact that S is Terracini.

Now assume $x \geq n+2$. Since $\langle S\rangle = \mathbb{P}^n$, there exists a subset $S' \subset S$ of cardinality $n+1$ and such that $\langle S'\rangle = \mathbb{P}^n$. We just proved that $h^0(\mathcal{I}_{2S'}(2)) = 0$. By Lemma 2.6, we deduce that $h^0(\mathcal{I}_{2S}(2)) = 0$. ■

The following result shows that many elements of $\mathbb{T}_1(n,d;x) \setminus \mathbb{T}(n,d;x)$ are easily produced and not interesting.

LEMMA 3.6. *Fix* $n \geq 2$, $d \geq 2$ *and* $x \geq \lceil d/2\rceil + 1$. *Let* S *be a collection of* x *points on a line* $L \subset \mathbb{P}^n$. *Then,* $S \in \mathbb{T}_1(n,d;x)$.

PROOF. We need to prove that $h^1(\mathcal{I}_{2S}(d)) > 0$ and $h^0(\mathcal{I}_{2S}(d)) > 0$. Fix a hyperplane H containing L. Take $G := 2H$ if $d = 2$ and call G the union of $2H$ and a hypersurface of degree $d-2$ if $d > 2$. Since $S \subset \mathrm{Sing}(G)$, we have $h^0(\mathcal{I}_{2S}(d)) > 0$. Since $\deg(2S \cap L) = 2x \geq d+2$, $h^1(\mathcal{I}_{2S\cap L}(d)) > 0$. Thus, $h^1(\mathcal{I}_{2S}(d)) > 0$, by Lemma 2.6. ■

Lemma 3.7. *For any $x > 0$, we have $\mathbb{T}(3, 3; x)' = \emptyset$.*

Proof. The case $x \leq 4$ will be treated in Proposition 3.10.

Fix now $x \geq 5$ and assume by contradiction that there exists $S \in \mathbb{T}(3, 3; x)'$. If four of the points of S are in a plane H, then S is not minimal. Indeed if A is the union of the four points in the plane, then $h^1(\mathcal{I}_{2A}(3)) = h^1(\mathcal{I}_{2A\cap H,H}(3)) > \deg(2A \cap H) - \binom{2+3}{2} = 12 - 10 = 2$ by Proposition 3.2 and Lemma 2.5.

Therefore, the points of S are in a linearly general position. Consider $S' \subseteq S$ of cardinality 5. The points of S' are in a linearly general position and, by Remark 2.1, they are projectively equivalent to a general set of five points A of $\mathbb{P}^3$.

Since the Veronese variety $\nu_3(\mathbb{P}^3)$ is not defective, by the Alexander–Hirschowitz theorem, we know that $\sigma_4(\nu_3(\mathbb{P}^3))$ fills the ambient space. Hence, $h^0(\mathcal{I}_{2A}(3)) = 0$. Then, $h^0(\mathcal{I}_{2S'}(3)) = 0$, and by Lemma 2.6, we get $h^0(\mathcal{I}_{2S}(3)) = 0$, and this contradicts the fact that S is Terracini. ∎

3.1. Proof of Theorem 1.1

We are now in position to give the proof of Theorem 1.1 which classifies Terracini loci. We start with the following lemma.

Lemma 3.8. *Assume $n \geq 1$ and $d \geq 2$. Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme such that $\deg(Z) \leq d + n + 1$, $h^1(\mathcal{I}_Z(d)) > 0$ and $\langle Z \rangle = \mathbb{P}^n$. Then, there is a line L such that $\deg(L \cap Z) \geq d + 2$ and $\deg(Z) = d + n + 1$.*

Proof. The lemma is trivial for $n = 1$.

We prove the statement by induction on $n \geq 2$. First we assume $n = 2$. Since $\deg(Z) \leq 2d + 1$, there is a line L such that $\deg(Z \cap L) \geq d + 2$, by Lemma 2.8. Clearly, since $\langle Z \rangle = \mathbb{P}^2$, we get $\deg(Z) = d + 3$.

Now assume $n > 2$. Take a hyperplane $H \subset \mathbb{P}^n$ such that $w := \deg(Z \cap H)$ is maximal. Since $\langle Z \rangle = \mathbb{P}^n$, we have $n \leq w < z$ and $\langle Z \cap H \rangle = H$.

If $h^1(\mathcal{I}_{Z\cap H,H}(d)) > 0$, then by induction, we have that there is a line L such that $\deg(L \cap (Z \cap H)) \geq d + 2$ and $\deg(Z \cap H) = d + n$. Hence, it follows that $\deg(L \cap Z) \geq d + 2$ and $\deg(Z) \geq d + n + 1$; hence, $\deg(Z) = d + n + 1$.

Now assume $h^1(\mathcal{I}_{Z\cap H,H}(d)) = 0$, and by the residual exact sequence with respect to H

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d - 1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap H,H}(d) \to 0, \tag{1}$$

we have $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d - 1)) > 0$. By Lemma 2.8, since $\deg(\mathrm{Res}_H(Z)) \leq z - w \leq d + 1 \leq 2d + 1$, we have a line L with $\deg(L \cap \mathrm{Res}_H(Z)) \geq d + 2$. Since $\langle Z \rangle = \mathbb{P}^n$, we must have $\deg(Z) \geq \deg(Z \cap L) + n - 1 \geq d + n + 1$. Hence, the assumption $\deg(Z) \leq d + n + 1$ implies that $\deg(Z) = d + n + 1$ and $\deg(L \cap Z) \geq d + 2$. ■

The following Proposition 3.10 proves the emptiness of the Terracini locus for small number of points. We give first a numerical lemma which will be used in the proof of the proposition.

LEMMA 3.9. *Given $x, y, m, n, d \in \mathbb{N}$, such that $d \geq 3$, $n \geq 3$, $m < n$, $x \geq y + n - m$, $x \leq n + \lceil \frac{d}{2} \rceil - 1$, $y(m+1) \geq \binom{m+d}{m}$, then we have $m = 1$.*

PROOF. Using the assumptions, in particular, $y \leq x - (n - m) \leq x - 1$, we have

$$\binom{m+d}{m} \leq (x-1)(m+1) \leq \left(m + \left\lceil \frac{d}{2} \right\rceil - 2\right)(m+1) \leq \left(m - 1 + \frac{d}{2}\right)(m+1).$$

We prove now by induction on $m \geq 2$ that

$$\binom{m+d}{m} > \left(m - 1 + \frac{d}{2}\right)(m+1). \tag{2}$$

It is easy to check that (2) is true for $m = 2$ and any $d \geq 3$. Now we assume (2) for m and we have, by using the induction hypothesis,

$$\begin{aligned}
\binom{m+1+d}{m+1} &= \binom{m+d}{m+1} + \binom{m+d}{m} = \binom{m+d}{m}\left(\frac{d}{m+1} + 1\right) \\
&> \left(m - 1 + \frac{d}{2}\right)(m+1)\left(\frac{d}{m+1} + 1\right) \\
&= \left(m - 1 + \frac{d}{2}\right)(d + m + 1) \\
&= \left(m - 1 + \frac{d}{2}\right)(m+2) + \left(m - 1 + \frac{d}{2}\right)(d-1) \\
&\geq \left(m + \frac{d}{2}\right)(m+2),
\end{aligned}$$

where the last inequality holds because $(m - 1 + \frac{d}{2})(d-1) \geq (m+2)$ for any $d \geq 3$ and $m \geq 1$.

Hence, since we have proved (2) for any $m \geq 2$, we conclude that $m = 1$. ■

PROPOSITION 3.10. *Assume $n, d \geq 2$ and fix an integer x such that*

$$x \leq n + \left\lceil \frac{d}{2} \right\rceil - 1.$$

Then, $\mathbb{T}(n, d; x) = \emptyset$.

Proof. The case $d = 2$ is true by Lemma 3.5; hence, we can assume $d \geq 3$.

Assume $n = 2$. Assume by contradiction that $S \in \mathbb{T}(2, d; x)$. Let Z be a critical scheme for S; then, we have $\deg(Z) \leq 2x \leq d + 3$. Hence, by Lemma 3.8, there exists a line L such that $\deg(Z \cap L) \geq d + 2$ and hence $x > \#(S \cap L) \geq \lceil d/2 \rceil + 1$, a contradiction.

Assume $n \geq 3$ and use now induction on n. By contradiction, assume $S \in \mathbb{T}(n, d; x)$. Let $S' \subseteq S$ be the minimal subset such that $h^1(\mathcal{I}_{2S'}(d)) > 0$. Set $y := \#S'$, $M := \langle S' \rangle$ and $m := \dim M$. Proposition 3.2 gives $h^1(\mathcal{I}_{2S' \cap M, M}(d)) > 0$. Notice that Lemmas 2.9 and 2.8 imply that $2y \geq d + 2$.

(I) If $m < n$, then we consider two cases.

(a) If $h^0(M, \mathcal{I}_{2S' \cap M}(d)) > 0$, then we have $y \geq m + \lceil d/2 \rceil$ by the induction assumption. Then, $x \geq y + (n - m) = n + \lceil d/2 \rceil$, a contradiction.

(b) If $h^0(M, \mathcal{I}_{2S' \cap M}(d)) = 0$, then $y(m + 1) \geq \binom{m+d}{m}$. Since S spans $\mathbb{P}^n$, then $x \geq y + (n - m)$. Hence, by Lemma 3.9, we get $m = 1$. Then, M is a line and in this case we have again a contradiction because, since $2y \geq d + 2$, we have

$$x \geq y + (n - 1) \geq \frac{d + 2}{2} + n - 1 = n + \frac{d}{2}.$$

(II) Thus, we may assume $m = n$. Let $H \subset \mathbb{P}^n$ be any hyperplane such that H is spanned by $S' \cap H$. Let $S'' = S' \cap H$. Then, $n \leq \#(S'') < y$. Since $\mathrm{Res}_H(2S'') = S''$, we have the exact sequence:

$$0 \to \mathcal{I}_{S''}(d - 1) \to \mathcal{I}_{2S''}(d) \to \mathcal{I}_{2S'' \cap H, H}(d) \to 0. \tag{3}$$

The minimality of S' and Proposition 3.2 give $h^1(H, \mathcal{I}_{2S'' \cap H, H}(d)) = 0$.

(a) Now, if $h^1(\mathcal{I}_{S''}(d - 1)) > 0$, then

  (a.1) either $\#(S'') \geq n + d$, which gives a contradiction with $x \leq n + \lceil \frac{d}{2} \rceil - 1$;

  (a.2) or $\#(S'') \leq n + d - 1$. In the latter case, Lemma 3.8 applied to $S'' \subset H$ gives $\#(S'') = n + d - 1$, which also contradicts $x \leq n + \lceil d/2 \rceil - 1$.

(b) Hence, we may assume $h^1(\mathcal{I}_{S''}(d - 1)) = 0$. From the exact sequence (3), we get $h^1(\mathcal{I}_{2S''}(d)) = 0$.

We consider now the residual exact sequence with respect to the quadric hypersurface $2H$:

$$0 \to \mathcal{I}_{S' \setminus S''}(d - 2) \to \mathcal{I}_{2S'}(d) \to \mathcal{I}_{2S'', 2H}(d) \to 0,$$

where $\mathrm{Res}_{2H}(2S') = S' \setminus S''$.

Since the quadric hypersurface $2H$ in $\mathbb{P}^n$ is arithmetically Cohen–Macaulay, we get $h^1(\mathcal{I}_{2S'', 2H}(d)) = 0$, which implies $h^1(\mathcal{I}_{S' \setminus S''}(d - 2)) > 0$. Then, by Lemma 2.8, we have $\deg(S' \setminus S'') \geq d$. But since $\deg(S' \setminus S'') = \#(S' \setminus S'') \leq y - n \leq \lceil d/2 \rceil - 1$, we have a contradiction, since $\lceil d/2 \rceil - 1 < d$ for all $d \geq 2$. ■

We now give the proof of the main result of this section.

PROOF OF THEOREM 1.1. Part (i) is true by Lemmas 3.3 and 3.5.

We prove now part (ii). Assume $n = 2$ and $d = 3$. A singular plane cubic C with at least 3 singular points is either the union of 3 lines or a triple line or the union of a double line and another line. Thus, if $\mathrm{Sing}(C)$ spans $\mathbb{P}^2$, then $\#\,\mathrm{Sing}(C) = 3$ and $\mathrm{Sing}(C)$ is projectively equivalent to any configuration of 3 non-collinear points. Hence, $\mathbb{T}(2, 3; x) = \emptyset$ for all $x \geq 4$. Thus, we have proved (ii) because clearly $\mathbb{T}(2, 3; 3) = \emptyset$.

For part (iii), assume that $n \geq 2$, $d \geq 3$ and $(n, d) \neq (2, 3)$. By Proposition 3.10, we have that if $x < n + \lceil d/2 \rceil$, then $\mathbb{T}(n, d; x) = \emptyset$. Hence, it is enough to prove that $\mathbb{T}(n, d; x) \neq \emptyset$ for $x \geq n + \lceil d/2 \rceil$.

We now analyze three different cases separately.

(I) Consider first the case $n = 2$ and $d \geq 4$. We assume $x \geq \lceil d/2 \rceil + 2$. Let L, M, N be three distinct lines and $G := (d-2)L \cup M \cup N$. Take as S the union of the point $M \cap N$ and $x - 1$ points on $L \setminus (M \cup N)$. Since $S \subset \mathrm{Sing}(G)$, then $h^0(\mathcal{I}_{2S}(d)) > 0$. Furthermore, we claim that $h^1(\mathcal{I}_{2S}(d)) > 0$. Indeed L contains at least $\lceil d/2 \rceil + 1$ points of L; hence, $\deg(2S \cap L) \geq d + 2$ and by Lemma 2.7 we have $h^1(\mathcal{I}_{2S}(d)) \geq h^1(\mathcal{I}_{2S \cap L, L}(d)) > 0$. Summing up, since $\langle S \rangle = \mathbb{P}^2$, we get $S \in \mathbb{T}(2, d; x)$, i.e., $\mathbb{T}(2, d; x) \neq \emptyset$.

(II) Now assume $n \geq 3$, $d = 3$ and $x \geq n + 2$. Fix hyperplanes H, K, U of $\mathbb{P}^n$ such that $\dim H \cap K \cap U = n - 3$. Since $H \cap K$ and $H \cap U$ are 2 different codimension 1 subspaces of H, their union spans H.

Let S be the union of n general points in $(H \cap K)$, one point in $(H \cap U) \setminus (H \cap K \cap U)$ and a point in $(K \cap U) \setminus (H \cap K \cap U)$. Then, $\langle S \rangle = \mathbb{P}^n$, $h^0(\mathcal{I}_{2S}(3)) \neq 0$ and it is easy to show (by induction on n) that $h^1(\mathcal{I}_{2S}(3)) \neq 0$. Hence, by Lemma 2.6, for any configuration S' of points such that $S \subset S' \subset \mathrm{Sing}(L \cup M \cup N)$, we have $S' \in \mathbb{T}(n, d; \#(S'))$. In consequence, $\mathbb{T}(n, 3; x) \neq \emptyset$ for all $x \geq n + 2$ and $n \geq 3$.

(III) Now assume $n \geq 3$, $d \geq 4$ and $x \geq n + \lceil d/2 \rceil$. As before, fix hyperplanes H, K, U with $\dim(H \cap K \cap U) = n - 3$ and take a line $L \subset H$ and set

$$G := (d-2)H \cup K \cup U.$$

Consider a collection E of $x - n + 1$ points on the line L. Since $\#E \geq \lceil d/2 \rceil + 1$, by Lemma 3.6, we have $h^1(\mathcal{I}_{2E}(d)) > 0$. Let $A \subset H$ be a collection of $n - 2$ general points. Note that $\langle E \cup A \rangle = H$. Take as S the union of $A \cup E$ and a point of $(U \cap K) \setminus (H \cap K \cap U)$. Obviously, S spans $\mathbb{P}^n$ and $h^1(\mathcal{I}_{2S}(d)) > 0$ by Lemma 2.6. Moreover, $h^0(\mathcal{I}_{2S}(d)) > 0$ by construction, and in consequence, $S \in \mathbb{T}(n, d; x) \neq \emptyset$. ■

Notice that the set of points $S \in \mathbb{T}(3,3;5)$ produced in the previous proof is not minimally Terracini because 4 points belong to a plane. Indeed by Lemma 3.7, we already know that $\mathbb{T}(3,3;5)' = \emptyset$.

4. Rational normal curves

We start now to analyze the set of points lying on a rational normal curve. For each $n > 1$, we denote by $\mathcal{C}_n$ the set of all rational normal curves of $\mathbb{P}^n$.

LEMMA 4.1. *Fix integers $n \geq 2$, $d \geq 4$ and $x \leq \lceil nd/2 \rceil$. Take a rational normal curve $C \in \mathcal{C}_n$ and let $S \subset C$ be a collection of x points on C. Then, $h^1(\mathcal{I}_{2S}(d)) = 0$.*

PROOF. Assume by contradiction that $h^1(\mathcal{I}_{2S}(d)) > 0$. By Lemma 2.9, there exists a d-critical scheme Z for S. Since C is scheme-theoretically cut-out by quadrics, there is $Q \in |\mathcal{I}_C(2)|$ such that $Q \cap Z = C \cap Z := \zeta$ and we have

$$0 \to \mathcal{I}_{C,Q}(d) \to \mathcal{I}_{\zeta,Q}(d) \to \mathcal{I}_{\zeta,C}(d) \to 0.$$

Since $\deg(Z) \leq 2x \leq nd + 1$, we have $h^1(\mathcal{I}_{\zeta,C}(d)) = 0$, and since C is projectively normal, we get $h^1(\mathcal{I}_{\zeta,Q}(d)) = 0$. Thus, the residual exact sequence with respect to Q and the fact that $h^1(\mathcal{I}_Z(d)) > 0$ give $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) > 0$.

Since $\mathrm{Res}_Q(Z) \subseteq S \subset C$, we have

$$0 \to \mathcal{I}_C(d-2) \to \mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2) \to \mathcal{I}_{\mathrm{Res}_Q(Z),C}(d-2) \to 0.$$

We have $h^1(\mathcal{I}_C(d-2)) = 0$ because C is projectively normal.

Note that $\deg(\mathrm{Res}_Q(Z)) < n(d-2) + 2$; indeed

$$\deg\left(\mathrm{Res}_Q(Z)\right) \leq x \leq \left\lceil \frac{nd}{2} \right\rceil \leq n(d-2) + 1,$$

where the last inequality is true for $d \geq 4$. Then, we have

$$h^1\left(\mathcal{I}_{\mathrm{Res}_Q(Z),C}(d-2)\right) = h^1\left(\mathcal{O}_{\mathbb{P}^1}\left(n(d-2) - \deg\left(\mathrm{Res}_Q(Z)\right)\right)\right) = 0,$$

and we have a contradiction with $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) > 0$. ■

THEOREM 4.2. *Fix integers $n \geq 2$, $d \geq 3$ and assume $(n,d) \neq (2,3)$. Given a rational normal curve $C \in \mathcal{C}_n$ and a collection $S \subset C$ of x points on the curve. Then,*

(i) *if $n \geq 3$, $d \geq 4$ and $x \geq 1 + \lceil nd/2 \rceil$, then $S \in \mathbb{T}(n,d;x)$;*

(ii) *if $n \geq 4$, $d = 3$ and $x = 1 + \lceil nd/2 \rceil$, then $S \in \mathbb{T}(n,d;x)$;*

(iii) *if $n \geq 2$, $d \geq 4$ and $x = 1 + \lceil nd/2 \rceil$, then $S \in \mathbb{T}(n,d;x)'$.*

PROOF. By the exact sequence

$$0 \to \mathcal{I}_{C\cup 2S}(d) \to \mathcal{I}_{2S}(d) \to \mathcal{I}_{2S\cap C,C}(d) \to 0$$

since $h^1(\mathcal{I}_{2S\cap C,C}(d)) = h^1(\mathcal{O}_{\mathbb{P}^1}(nd-2x)) > 0$ whenever $x \geq 1 + \lceil nd/2 \rceil$, we have $h^1(\mathcal{I}_{2S}(d)) > 0$. Since $x \geq n+1$ and C is a rational normal curve, then $\langle S \rangle = \mathbb{P}^n$.

If $n \geq 3$, then $h^0(\mathcal{I}_C(2)) \geq 2$; hence, C is contained in a quadric hypersurface. Thus, if $d \geq 4$, we have $h^0(\mathcal{I}_{2S}(d)) > 0$. Hence, $S \in \mathbb{T}(n,d;x)$ and we have proved (i).

Assume now $x = 1 + \lceil nd/2 \rceil$. Fix a collection A of x general points on C and note that by generality, $h^0(\mathcal{I}_{2S}(d)) \geq h^0(\mathcal{I}_{2A}(d))$.

Hence, assuming $d = 3$, we have

$$h^0\big(\mathcal{I}_{2S}(3)\big) \geq \binom{n+3}{3} - (n+1)x > 0,$$

where the last inequality is true for any $n \geq 5$. If $n = 4$ and $x = 7$, we have

$$h^0\big(\mathcal{I}_{2S}(3)\big) \geq h^0\big(\mathcal{I}_{2A}(3)\big) = 1,$$

by the Alexander–Hirschowitz theorem. In consequence, $S \in \mathbb{T}(n, 3 : x)$, which ends the proof of (ii).

Now assume $n = 2$ and $x = d+1$. We have $h^0(\mathcal{I}_{2S}(d)) \geq \binom{d+2}{2} - 3(d+1) > 0$, for $d \geq 5$. If $d = 4$ and $x = 5$, then $h^0(\mathcal{I}_{2S}(4)) \geq h^0(\mathcal{I}_{2A}(4)) = 1$, again by the Alexander–Hirschowitz theorem. Hence, $S \in \mathbb{T}(n,d;x)$ for $n = 2$ and $d \geq 4$.

In order to complete the proof of (iii), we need to prove the minimality of S, and this follows by Lemma 4.1. ■

REMARK 4.3. Recall that by Theorem 1.1, we know that $\mathbb{T}(2,3;x) = \emptyset$ for all $x > 0$. Moreover, in the proof of Lemma 3.7, we have seen that a set of $x \geq 5$ points in a linearly general position in $\mathbb{P}^3$ is not Terracini. Hence, if S is a collection of $x \geq 5$ points on a rational normal cubic curve, we have $S \notin \mathbb{T}(3,3;5)$.

4.1. Degenerations of rational normal curves

We introduce now the notion of *reducible rational normal curves.*

DEFINITION 4.4. A reduced, connected and reducible curve $T \subset \mathbb{P}^n$, for $n \geq 2$, such that $\deg(T) = n$ and $\langle T \rangle = \mathbb{P}^n$ is called *reducible rational normal curve*.

Of course, in $\mathbb{P}^2$, a reducible rational normal curve is a reducible conic.

Since T is connected, there is an ordering $T_1, \dots, T_s$ of the irreducible component such that each $T[i] := T_1 \cup \dots \cup T_i$, $1 \leq i \leq s$, is connected. We say that each such ordering of the irreducible components of T is a *good ordering*.

Set $n_i := \deg(T_i)$. Note that $n = n_1 + \cdots + n_s$ and $\dim\langle T_i\rangle \le n_i$ with equality if and only if T_i is a rational normal curve in its linear span. For $i = 1, \dots, s-1$, we have the following Mayer–Vietoris exact sequence

$$0 \to \mathcal{O}_{T[i+1]}(t) \to \mathcal{O}_{T[i]}(t) \oplus \mathcal{O}_{T_{i+1}}(t) \to \mathcal{O}_{T[i]\cap T_{i+1}}(t) \to 0, \tag{4}$$

in which $T[i] \cap T_{i+1}$ is the scheme-theoretic intersection. Since $T[i+1]$ is connected, $\deg(T[i] \cap T[i+1]) > 0$. Thus, (4) gives $\dim\langle T[i+1]\rangle \le \dim\langle T[i]\rangle + n_i$ with equality if and only if $\deg(T[i] \cap T[i+1]) = 1$, T_{i+1} is a rational normal curve in its linear span and $\langle T[i]\rangle \cap \langle T_{i+1}\rangle$ is the point $T[i] \cap T_{i+1}$.

Since $n = n_1 + \cdots + n_s$, by induction on i, we get $p_a(T) = 0$ and each T_i is a rational normal curve in its linear span. Using (4) and induction on t, we also get $h^1(\mathcal{O}_T(t)) = 0$ and $h^0(\mathcal{O}_T(t)) = nt + 1$ for all $t \ge 0$, and that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_T(t))$ is surjective; i.e., T is arithmetically Cohen–Macaulay. In the same way, we see that each $T[i]$ is arithmetically Cohen–Macaulay in its linear span.

Recall that each T_i is smooth. For any $p \in T_i$, let $L_i(p)$ denote the tangent line of T_i at (p). Take $p \in \mathrm{Sing}(T)$ and let $T_{i_1}, \dots, T_{i_k}$, $k \ge 2$, be the irreducible components of T passing through p. Since $n = n_1 + \cdots + n_s$ and $p_a(T) = 0$, the k lines $L_{i_1}(p), \dots, L_{i_k}(p)$ through p span a k-dimensional linear space (such a singularity is often called a seminormal or a weakly normal curve singularity).

An irreducible component T_i of T is said to be a *final component* if $\#(T_i \cap \mathrm{Sing}(T)) = 1$. Since $s \ge 2$, T has at least 2 final components (e.g. T_1 and T_s for any good ordering of the irreducible components of T), but it may have many final components (e.g. for some T with $s \ge 3$, we may have

$$\#\big(T_i \cap \mathrm{Sing}(T)\big) = 1 \quad \text{for all } i \ge 2$$

and there is one T, unique up to a projective transformation, formed by n lines through the same point).

Remark 4.5. Take a (reducible) rational normal curve $T \subset \mathbb{P}^n$. Since $h^1(\mathcal{O}_T) = 0$, the exact sequence

$$0 \to \mathcal{I}_T \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_T \to 0$$

gives $h^2(\mathcal{I}_T) = 0$. Since $h^1(\mathcal{I}_T(1)) = 0$, the Castelnuovo–Mumford lemma implies that the homogeneous ideal of T is generated by quadrics. Thus, T is scheme-theoretically cut out by quadrics.

Lemma 4.6. *Fix $n \ge 2$, $d \ge 4$. Let T be a reducible rational normal curve in $\mathbb{P}^n$ and $S \in S(\mathbb{P}^n, x)$ such that $S \subset T_{\mathrm{reg}}$ and $\langle S\rangle = \mathbb{P}^n$. If $2x \ge dn + 2$, then $S \in \mathbb{T}(n, d; x)$.*

PROOF. Since $h^0(\mathcal{I}_T(2)) = \binom{n}{2}$, we have that $h^0(\mathcal{I}_{2S}(d)) > 0$ if $d \geq 4$.

Set $Z := 2S \cap T$. Since $S \cap \mathrm{Sing}(T) = \emptyset$, $\deg(Z) = 2x$ and Z is a Cartier divisor of T. Since $h^0(\mathcal{O}_T(d)) = nd + 1$, then $h^1(\mathcal{I}_{Z,T}(d)) \geq 1$. Hence, $h^1(\mathcal{I}_Z(d)) \geq 1$ since T is arithmetically Cohen–Macaulay, and $S \in \mathbb{T}_1(n, d; x)$. Finally, since by assumption $\langle S \rangle = \mathbb{P}^n$, we conclude that $S \in \mathbb{T}(n, d; x)$. ∎

PROPOSITION 4.7. *Assume $n \geq 2$ and $d \geq 5$ and set*

$$x = 1 + \left\lceil \frac{nd}{2} \right\rceil.$$

Fix a reducible rational normal curve $T = T_1 \cup \cdots \cup T_s \subset \mathbb{P}^n$, $s \geq 2$. Assume the existence of $S \in \mathbb{T}(n, d; x)'$ such that $S \subset T$. Set $n_i := \deg(T_i)$ and $x_i := \#(S \cap T_i)$. Then,

(i) $S \subset T_{\mathrm{reg}}$;

(ii) *n is even and d is odd;*

(iii) *every final component T_i of T has n_i odd and $2x_i = n_i d + 1$.*

PROOF. Set $W := 2S \cap T$. Note that $x_1 + \cdots + x_s \geq x$ and that $x_1 + \cdots + x_s = x$ if and only if $S \subset T_{\mathrm{reg}}$. We have $n = n_1 + \cdots + n_s$, $2x = nd + 2$ if nd is even and $2x = nd + 3$ if n and d are odd. Obviously, $s \leq d$ and hence $s - 1 < x$.

STEP 1. We prove first of all that, for any i,

$$2x_i \leq n_i d + 1. \tag{5}$$

Assume, by contradiction, that there exists i such that $2x_i > n_i d + 1$ and set $S' = S \cap T_i$. Note that $h^1(\mathcal{I}_{2S'}(d)) = h^1(\mathcal{I}_{2S',T_i}(d))$ since T_i is arithmetically Cohen–Macaulay. Then, since $h^0(\mathcal{O}_{T_i}(d)) = n_i d + 1$ and $\deg(2S') \geq n_i d + 2$, we have $h^1(\mathcal{I}_{2S'}(d)) > 0$ and hence $S \notin \mathbb{T}(n, d; x)'$, a contradiction.

STEP 2. We prove now (i) by contradiction. Set $S_1 := S \cap \mathrm{Sing}(T)$ and $S_2 := S \setminus S_1$. Since T has at most $s - 1$ singular points, $S_2 \neq \emptyset$. We assume by contradiction that $S_1 \neq \emptyset$.

For each $o \in \mathrm{Sing}(T)$, let $m(o)$ denote the number of irreducible components of T passing through o. We saw that T has Zariski tangent of dimension $m(o)$ and hence the connected component $W(o)$ of W supported at the point o has degree $m(o) + 1$. Thus, denoting $w = \deg(W)$, we have

$$w = 2\#(S_2) + \sum_{o \in S_1} \big(m(o) + 1\big) \geq 2x + \#(S_1). \tag{6}$$

If $\deg(W) \geq nd + 4$, then fix $u \in S_2$ and set $S' := S \setminus \{u\}$. Note that

$$h^1\big(\mathcal{I}_{2S'}(d)\big) = h^1\big(\mathcal{I}_{2S',T}(d)\big)$$

since T is arithmetically Cohen–Macaulay. Then, since $h^0(\mathcal{O}_T(d)) = nd + 1$ and $\deg(2S') \geq w - 2 \geq nd + 2$, we have $h^1(\mathcal{I}_{2S'}(d)) > 0$ and hence $S \notin \mathbb{T}(n,d;x)'$, a contradiction.

Then, we can assume

$$\deg(W) \leq nd + 3. \tag{7}$$

(a) Assume first nd even. Hence, we have $2x = nd + 2$. Then, it follows that $\#S_1 = 1$, say $S_1 = \{u\}$, and T is nodal at u. Since $p_a(T) = 0$, T is connected, the irreducible components of T are smooth and T is nodal at u, $T \setminus \{u\}$ has 2 connected components. Call T' and T'' the closures in $\mathbb{P}^n$ of the two connected components of $T \setminus \{u\}$. Note that $\deg(W) = \deg(W \cap T') + \deg(W \cap T'')$ and $n = \dim\langle T'\rangle + \dim\langle T''\rangle$, either $\deg(W \cap T') \geq \dim\langle T'\rangle + 2$ or $\deg(W \cap T'') \geq \dim\langle T''\rangle + 2$. Thus, $S \notin \mathbb{T}(n,d;x)'$, and we have a contradiction. We have proved (i) in this case.

(b) Now assume d odd and n odd. Then, $2x = nd + 3$, and by using (6) and (7), we get $S_1 = \emptyset$. We have proved (i) in this case.

STEP 3. Since $d \geq 5$, a good ordering of the irreducible components of T and $s - 1$ Mayer–Vietoris exact sequences give $h^1(\mathcal{I}_S(d-2)) = 0$. Let Z be a critical scheme for S; that is, $h^1(\mathcal{I}_Z(d)) > 0$. Since $h^1(\mathcal{I}_T(1)) = 0$ and $h^2(\mathcal{I}_T) = h^2(\mathcal{O}_T(1)) = 0$, the Castelnuovo–Mumford lemma gives that $\mathcal{I}_T(2)$ is globally generated. Since $\mathcal{I}_T(2)$ is globally generated and every connected component of Z has degree ≤ 2, $Q \cap Z = T \cap Z$ for a general $Q \in |\mathcal{I}_T(2)|$. Since $\mathrm{Res}_Q(Z) \subseteq S$ and $h^1(\mathcal{I}_S(d-2)) = 0$ and Q is arithmetically Cohen–Macaulay, the residual exact sequence with respect to Q gives $h^1(\mathcal{I}_{Z\cap Q}(d)) = 0$ and hence $Z \subset T$. Thus, $Z \subseteq W$. Since T is arithmetically Cohen–Macaulay, we get $h^1(\mathcal{I}_{Z,T}(d)) > 0$ and hence

$$h^1\big(\mathcal{I}_{W,T}(d)\big) > 0. \tag{8}$$

STEP 4. We prove now (ii). Recall that, since $S \subset T_{\mathrm{reg}}$, we have $x_1 + \cdots + x_s = x$ and $n_1 + \cdots + n_s = n$.

Assume by contradiction that d is even. Recall the inequality (5) from Step 1. If d is even, $2x_i \leq n_i d + 1$ is equivalent to $2x_i \leq n_i d$, and this implies $2x \leq nd$ which contradicts the assumption $2x = nd + 2$. We have proved that d is odd.

From now on, we assume d odd. Recall (5), and in particular, $2x_i \leq n_i d + 1$ for all odd n_i and $2x_i \leq n_i d$ for all even n_i.

Now assume n odd by contradiction. In particular, since $2x = nd + 3$, by (5) we have $s \geq 3$, and there are at least three odd n_i with $2x_i = n_i d + 1$. Let T' be a minimal connected subcurve of T such that $\deg(T' \cap W) \geq 2 + d \dim(\langle T' \rangle)$. Since $2x_i \leq n_i d + 1$ for all i, by (5), and each subcurve T'' of T has at least one final component (a final component of T'', not necessarily of T), the minimality of T' gives $\deg(T' \cap W) = 2 + d \dim(\langle T' \rangle)$. It follows that $S \cap T' \in \mathbb{T}(n, d; x)$ and, since $S \cap T' \subsetneq S$, we conclude that $S \notin \mathbb{T}(n, d; x)'$, a contradiction.

Then, we have proved (ii).

Step 5. We finally prove (iii). We know that d is odd and n is even by (ii).

Let T_i any final component of T. Let Y be the union of all other components of T. Since T_i is a final component, Y is connected. Then, $\deg(Y) = \dim\langle Y \rangle$, and hence Y is a, possibly reducible, rational normal curve in $\langle Y \rangle$, $\langle Y \rangle \cap \langle T_i \rangle$ is a point, p, and $\{p\}$ is the scheme-theoretic intersection of T_i and Y. We proved in Step 2 that $p \notin S$. Since $\langle S \rangle = \mathbb{P}^n$ and $p \notin S$, then $\langle S \cap T_i \rangle = \langle T_i \rangle$ and $\langle S \cap Y \rangle = \langle Y \rangle$ and in particular $S \cap T_i \neq \emptyset$ and $S \cap Y \neq \emptyset$. Since S is minimal and T is arithmetically Cohen–Macaulay, $h^1(\mathcal{I}_{Z \cap T_i}(d)) = h^1(\mathcal{I}_{Z \cap Y, T}(d)) = 0$. The following Mayer–Vietoris type sequence on T

$$0 \to \mathcal{I}_{W,T}(d) \to \mathcal{I}_{W \cap T_i, T_i}(d) \oplus \mathcal{I}_{W \cap Y, Y}(d) \to \mathcal{O}_p(d) \to 0 \tag{9}$$

is exact because $p \notin S$. We proved that $h^1(\mathcal{I}_{Z \cap T_i, T_i}(d)) = h^1(\mathcal{I}_{Z \cap Y, Y}(d)) = 0$.

Assume by contradiction that n_i is even. Then, we have $2x_i \leq n_i d$. The restriction map

$$H^0(\mathcal{I}_{W \cap T_i, T_i}(d)) \to H^0(\mathcal{O}_p(d))$$

is surjective because $T_i \cong \mathbb{P}^1$ and $\deg(W \cap T_i) \leq \deg(\mathcal{O}_{T_i}(d))$. Thus, (9) gives $h^1(\mathcal{I}_{W,T}(d)) = 0$, a contradiction with (8). Then, we have proved that n_i is even for every final component T_i of T. Hence, we also have $2x_i = n_i d + 1$ and this concludes the proof. ∎

5. Minimally Terracini finite sets in the plane

In this section, we focus on the case of the plane. We deduce from [11] the following result, which we will need in the sequel.

Remark 5.1. Fix positive integers d, z such that $z \leq 3d$. Let $Z \subset \mathbb{P}^2$ be a zero-dimensional scheme, $Z \neq \emptyset$. If $\deg(Z) = z$ and d is the maximal integer t such that $h^1(I_Z(t)) > 0$, then either there is line L such that $\deg(L \cap Z) \geq d + 2$ or there is a conic such that $\deg(Z \cap D) \geq 2d + 2$ or $z = 3d$ and Z is the complete intersection of a plane cubic and a degree d plane curve (see [11, Remarque (i), p. 116]).

PROPOSITION 5.2. *Fix integers $x > 0$ and $d \geq 4$.*

(a) *If $x \leq d$, then $\mathbb{T}(2, d; x)' = \emptyset$.*

(b) *Let $S \in S(\mathbb{P}^2, d+1)$. Then, $S \in \mathbb{T}(2, d, d+1)'$ if and only if S is contained in a reduced conic D. Moreover, if $D = R \cup L$ is reducible (with L and R lines), then d is odd, $\#(S \cap R) = \#(S \cap L) = (d+1)/2$ and $S \cap R \cap L = \emptyset$.*

(c) *Assume $d \geq 5$. Then, $\mathbb{T}(2, d; x)' = \emptyset$ for all x such that $d + 2 \leq x < 3d/2$.*

PROOF. We prove (a) by contradiction. Assume $x \leq d$ and consider $S \in \mathbb{T}(2, d; x)'$. Let Z be a critical scheme for S. We have $\deg(Z) \leq 2x$ and d is the maximal integer such that $h^1(\mathcal{I}_Z(d)) > 0$ by Theorem 3.1. Then, $\deg(Z) \leq 2d$ and, by Lemma 2.8, there is a line L such that $\deg(Z \cap L) \geq d + 2$. Thus, $h^1(\mathcal{I}_{Z \cap L}(d)) > 0$. Since $\langle S \rangle = \mathbb{P}^2$, S is not minimal, a contradiction.

The *if* implication of part (b) follows from Theorem 4.2 (iii).

We prove now the other implication of (b). Take $S \in \mathbb{T}(2, d; d+1)'$ and let Z be a critical scheme for S. By Lemma 2.12, $Z_{\text{red}} = S$. Assume that S is not contained in a reduced conic. Since $\langle S \rangle = \mathbb{P}^2$, S is not contained in a double line; therefore, S is not contained in a conic. Hence, Remark 5.1 implies that there is a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap Z) \geq d + 2$ and hence $h^1(\mathcal{I}_{Z \cap L}(d)) > 0$. Hence, S is not minimal. Finally, Proposition 4.7 gives the last part of (b).

We prove finally (c) by contradiction. Assume $d + 2 \leq x < 3d/2$ and let $S \in \mathbb{T}(2, d; x)$ with Z critical for S. Since S is minimal, $\#(S \cap L) \leq (d+1)/2$ for all lines L and $\#(S \cap D) \leq 2d + 1$ for each conic. Since Z is critical, $\deg(Z \cap L) \leq d + 1$ for each line L and $\deg(D \cap Z) \leq 2d + 1$ for any conic D. Thus, since $\deg(Z) \leq 3d - 1$, by Remark 5.1, we have $h^1(\mathcal{I}_Z(d)) = 0$, a contradiction. ∎

Just above the range covered by Proposition 5.2, we have the following examples.

EXAMPLE 5.3. Assume $d = 2k$, for $k \in \mathbb{N}$, $d \geq 6$, and take $x := 3k$. Let $C \subset \mathbb{P}^2$ be a smooth plane cubic and T a smooth plane curve of degree k. Take as S the complete intersection $C \cap T$. Set $Z := C \cap 2T = 2S \cap C$. Since $\deg(Z) = 3d$ and $h^0(\mathcal{O}_C(d)) = 3d$, then

$$h^1(\mathcal{I}_{Z,C}(d)) = h^0(\mathcal{I}_{Z,C}(d)) = 1.$$

Since $h^0(\mathcal{O}_C(d-3)) = 3d - 9 \geq 3k = \#S$, we get $h^1(\mathcal{I}_{S,C}(d-3)) = 0$. Since C is arithmetically normal, $h^1(\mathcal{I}_S(d-3)) = 0$. Thus, the residual exact sequence with respect to C gives

$$h^1(\mathcal{I}_{2S}(d)) = h^1(\mathcal{I}_{Z,C}(d)) = 1.$$

We also get $h^1(\mathcal{I}_{2S' \cap C, C}(d)) = 0$ for all $S' \subsetneq S$ since $\deg(2S' \cap C) \leq 3d - 2$. Thus, $S \in \mathbb{T}(2, d; 3d/2)'$.

EXAMPLE 5.4. Take d odd, $d \geq 7$, and set $x := (3d+1)/2$. Let $C \subset \mathbb{P}^2$ a smooth plane cubic. Take $S \subset C$ such that $\#S = (3d+1)/2$. By assumption, S is a Cartier divisor of C. Since $p_a(C) = 1$ and $\deg(\mathcal{O}_C(d-3)) = 3d-9 > \#S$, then $h^1(C, \mathcal{I}_{S,C}(d-3)) = 0$. Since C is arithmetically normal, $h^1(\mathcal{I}_S(d-3)) = 0$. Thus, the residual exact sequence with respect to C gives $h^1(\mathcal{I}_{2S}(d)) = h^1(\mathcal{I}_{2S\cap C,C}(d))$. Since $p_a(C) = 1$, we get

$$h^1\big(\mathcal{I}_{2S\cap C,C}(d)\big) = 1.$$

We also have $h^1(\mathcal{I}_{2S'\cap C,C}(d)) = 0$ for all $S' \subsetneq S$ since $\deg(2S' \cap C)ł \leq 3d-1$; hence, $S \in \mathbb{T}(2, d; (3d+1)/2)'$.

6. Minimally Terracini finite sets in $\mathbb{P}^3$

Now we consider the case of finite sets of points in $\mathbb{P}^3$. The following proposition extends Remark 5.1 to the case of schemes of $\mathbb{P}^3$.

PROPOSITION 6.1. *Fix a positive integer d. Let $Z \subset \mathbb{P}^3$ be a zero-dimensional scheme such that $\langle Z\rangle = \mathbb{P}^3$, its connected components have degree ≤ 2 and $z := \deg(Z) \leq 3d+1$. We have*

$$h^1\big(\mathcal{I}_Z(d)\big) > 0$$

if and only if one of the following cases occur:

(i) *there is a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap Z) \geq d+2$;*

(ii) *there is a conic D such that $\deg(D \cap Z) \geq 2d+2$;*

(iii) *there is a plane cubic T such that $\deg(T \cap Z) = 3d$ and $T \cap Z$ is the complete intersection of T and a degree d plane curve.*

PROOF. Set $S := Z_{\mathrm{red}}$.

Since the *if* part is trivial, we only need to prove the *only if* part.

We use induction on d. The case $d = 1$ is obvious since conditions $\deg(Z) \leq 4$ and $\langle Z\rangle = \mathbb{P}^3$ imply that Z is linearly independent and hence $h^1(\mathcal{I}_Z(1)) = 0$.

Assume $d \geq 2$ and that the proposition is true for lower degrees. If there is a plane H such that $h^1(\mathcal{I}_{Z\cap H}(d)) > 0$, then we may use Remark 5.1 and we conclude.

Now we assume that

$$h^1\big(\mathcal{I}_{Z\cap H}(d)\big) = 0 \quad \text{for any plane } H \subset \mathbb{P}^3. \tag{10}$$

Take a plane $H \subset \mathbb{P}^3$ such that $w := \deg(Z \cap H)$ is maximal. Since $\langle Z\rangle = \mathbb{P}^3$, then we have $z \geq 4$, and $w \geq 3$, and hence $\deg(\mathrm{Res}_H(Z)) = z - w \leq 3(d-1)+1$. Since $h^1(\mathcal{I}_{Z\cap H}(d)) = 0$ by (10), then the residual exact sequence with respect to H gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. The inductive assumption applied to the scheme $\mathrm{Res}_H(Z)$ implies that we are in one of the following cases:

(i′) either there is a line R such that $\deg(R \cap \mathrm{Res}_H(Z)) \geq d+1$,

(ii′) or there is a conic E such that $\deg(E \cap \mathrm{Res}_H(Z)) \geq 2d$,

(iii′) or there is a plane cubic C such that $\deg(C \cap \mathrm{Res}_H(Z)) = 3d-3$ and $\mathrm{Res}_H(Z) \cap C$ is the complete intersection of C and a degree $d-1$ plane curve.

We analyze separately the three cases in the following three steps (a), (b), (c).

STEP (a). Assume first that we are in case (iii′). Since $\deg(\mathrm{Res}_H(Z) \cap C) = 3d-3$, then $z - w = \deg(\mathrm{Res}_H(Z)) \geq 3d-3$. On the other hand, since $\mathrm{Res}_H(Z) \cap C$ is contained in a plane, we also have $w \geq 3d-3$ and hence $z \geq 6d-6$. Now since $z \leq 3d+1$, we get $d = 2$.

Since $d = 2$, we have $z \leq 7$. Moreover, since $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1)) > 0$, then the scheme $\mathrm{Res}_H(Z)$ is linearly dependent and so we have $w \geq \deg(\mathrm{Res}_H(Z))$, by the maximality assumption on w. So we have $z - w \leq 3 = 2(d-1)+1$, and by Lemma 2.8, it follows that there is a line J such that $\deg(J \cap \mathrm{Res}_H(Z)) = (d-1)+2 = 3$.

Take now a plane $M \supset J$ such that $w' := \deg(M \cap Z)$ is maximal. Since $\dim|\mathcal{I}_J(1)| = 1$, we have $w' \geq 4$. We get $w = w' = 4$ and $z = 7$. Taking M instead of H and repeating the argument above, we have $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(1)) > 0$, and again by Lemma 2.8, it follows that there exists a line K such that $\deg(K \cap \mathrm{Res}_M(Z)) = 3$; hence, $\mathrm{Res}_M(Z) \subset K$.

If $\deg(K \cap Z) = 4$ or $\deg(J \cap Z) = 4$, then we are in case (i) and the theorem is proved.

Now we exclude the remaining case which is

$$\deg(Z \cap K) = \deg(Z \cap J) = 3. \tag{11}$$

Assume by contradiction (11) and consider separately the following three possibilities: either $J \cap K \neq \emptyset$ and $J \neq K$, or $K \cap J = \emptyset$, or $J = K$.

(a1) Assume that $J \cap K \neq \emptyset$ and $J \neq K$. Recall that any connected component of Z has degree ≤ 2, and clearly we have $\deg(J \cap K) = 1$. Hence, the plane spanned by $J \cup K$ gives $w \geq \deg(J \cap Z) + \deg(J \cap K) - 1 = 5$, a contradiction with $w = 4$.

(a2) Assume $K \cap J = \emptyset$. Since $\deg(Z) = 7$ and $h^1(\mathcal{I}_Z(2)) > 0$, by Lemma 2.5, we have $\dim|\mathcal{I}_Z(2)| = h^0(\mathcal{I}_Z(2)) - 1 \geq 3$. Take a general $Q \in |\mathcal{I}_Z(2)|$. The theorem of Bézout and the assumptions (11) imply that $J \cup K \subset Q$. Since $J \cap K = \emptyset$, Q is not an irreducible quadric cone or double points. Moreover, since Q is general, then Q is not the union of a plane containing J and a plane containing K. Thus, Q is a smooth quadric. Since $J \cap K = \emptyset$, then J and K are contained in the same ruling of Q, say $J, K \in |\mathcal{O}_Q(1,0)|$. We have $h^1(Q, \mathcal{I}_{Z,Q}(2,2)) = h^1(\mathcal{I}_Z(2)) > 0$.

Note that, by using (11), we have

$$h^1\big(K, \mathcal{I}_{Z \cap K, K}(2)\big) = h^1\big(J, \mathcal{I}_{Z \cap J, J}(2)\big) = 0.$$

Since $\deg(Z) = 7$, then the degree of $\mathrm{Res}_{J \cup K}(Z)$ is 1, and hence it follows that

$h^1(Q, \mathcal{I}_{\mathrm{Res}_{J\cup K}(Z),Q}(0,2)) = 0$. Now, taking the cohomology of the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{J\cup K}(Z),Q}(0,2) \to \mathcal{I}_{Z,Q}(2,2) \to \mathcal{I}_{(Z\cap J)\cup(Z\cap K),Q}(2,2) \to 0,$$

we obtain $h^1(Q, \mathcal{I}_{Z,Q}(2,2)) = 0$, which is a contradiction.

(a3) Assume finally that $J = K$. Recall that all the connected components of Z have degree ≤ 2 and $S = Z_{\mathrm{red}}$. From (11) we deduce the following facts: $\#(S \cap J) = 3$, each connected component of Z supported at J has degree 2 and none of them is contained in J. Moreover, since $\deg(Z) = 7$, we have that $S \setminus (S \cap J)$ is a simple point p. Let H_1 be a plane containing J and not containing p. Set $Q_1 := 2H_1$ and consider the residual exact sequence with respect to Q_1,

$$0 \to \mathcal{I}_{\mathrm{Res}_{Q_1}(Z)} \to \mathcal{I}_Z(2) \to \mathcal{I}_{Z\cap Q_1,Q_1}(2) \to 0.$$

Since $J \subset \mathrm{Sing}(Q_1)$ and each connected component of Z has degree ≤ 2, we have $Z_1 := Z \cap Q_1 = Z \setminus \{p\}$ and $\mathrm{Res}_{Q_1}(Z) = \{p\}$. Hence, we have $h^1(\mathcal{I}_{\mathrm{Res}_{Q_1}(Z)}) = h^1(\mathcal{I}_p) = 0$. It follows from the exact sequence that $h^1(\mathcal{I}_{Z_1,Q_1}(2)) \ge h^1(\mathcal{I}_Z(2)) > 0$ and hence $h^1(\mathcal{I}_{Z_1}(2)) > 0$.

Fix now $p_1 \in S \setminus \{p\}$ and let A be the connected component of Z_1 supported at p_1. Take a plane U containing $A \cup J$. Since $w = 4$, by maximality, we have

$$\deg(U \cap Z_1) \le \deg(U \cap Z) \le 4,$$

and hence $\deg(U \cap Z_1) = 4$. Since $\deg(\mathrm{Res}_U(Z_1)) = 2$, then $h^1(\mathcal{I}_{\mathrm{Res}_U(Z_1)}(1)) = 0$. Thus, taking the cohomology of the residual exact sequence with respect to U,

$$0 \to \mathcal{I}_{\mathrm{Res}_U(Z_1)}(1) \to \mathcal{I}_{Z_1}(2) \to \mathcal{I}_{Z_1\cap U,U}(2) \to 0,$$

we obtain $h^1(\mathcal{I}_{Z_1\cap U,U}(2)) \ge h^1(\mathcal{I}_{Z_1}(2)) > 0$. This implies, by Lemma 2.6, that there is a plane U such that $h^1(\mathcal{I}_{Z\cap U}(2)) > 0$, and this contradicts our assumption (10).

Step (b). Assume now that we are in case (ii′). Since there is a conic E such that $\deg(E \cap \mathrm{Res}_H(Z)) \ge 2d$, we get $w \ge 2d$ and $z - w \ge 2d$. It follows that $z \ge 4d$, which contradicts the assumptions $z \le 3d + 1$ and $d \ge 2$.

Step (c). Assume finally that we are in case (i′); i.e., assume that there is a line R such that $\deg(R \cap \mathrm{Res}_H(Z)) \ge d + 1$. If $\deg(R \cap Z) \ge d + 2$, then we may take $L = R$ and we are in case (i) and the theorem is proved.

Now we assume that $\deg(R \cap Z) = d + 1$ and we will prove that either we are again in case (i), or we have a contradiction.

Since $\deg(R \cap Z) = d + 1$, then we have $R \cap Z = R \cap \mathrm{Res}_H(Z)$. By the maximality assumption on H, we also know that $w = \deg(Z \cap H) \ge d + 1$.

Take a general plane $M \supset R$ and consider the scheme $X := Z \cap (H \cup M)$. Since $\deg(M \cap \mathrm{Res}_H(Z)) \geq \deg(R \cap \mathrm{Res}_H(Z)) = d + 1$, we have

$$\deg(X) \geq w + d + 1 \geq 2d + 2.$$

Hence, the hypothesis $\deg(Z) \leq 3d + 1$ implies that $\deg(\mathrm{Res}_{H \cup M}(Z)) \leq d - 1$. Then, by Lemma 2.8, we get $h^1(\mathcal{I}_{\mathrm{Res}_{H\cup M}(Z)}(d-2)) = 0$.

The residual exact sequence of Z with respect to $H \cup M$,

$$0 \to \mathcal{I}_{\mathrm{Res}_{H\cup M}(Z)}(d-2) \to \mathcal{I}_Z(d) \to \mathcal{I}_{X, H\cup M}(d) \to 0,$$

gives $h^1(\mathcal{I}_X(d)) = h^1(\mathcal{I}_{X,H\cup M}(d)) \geq h^1(\mathcal{I}_Z(d)) > 0$.

Since $h^1(\mathcal{I}_{Z\cap M}(d)) = 0$, by assumption (10), then we have also $h^1(\mathcal{I}_{X\cap M}(d)) = 0$, by Lemma 2.6. The residual exact sequence of X with respect to M,

$$0 \to \mathcal{I}_{\mathrm{Res}_M(X)}(d-1) \to \mathcal{I}_X(d) \to \mathcal{I}_{X\cap M, M}(d) \to 0,$$

gives $h^1(\mathcal{I}_{\mathrm{Res}_M(X)}(d-1)) > 0$.

We consider now separately the two following cases: either the line R is contained in H, or it is not contained.

(c1) Assume $H \supset R$. Recall that $S = Z_{\mathrm{red}}$. Since each connected component of Z has degree ≤ 2, we deduce the following facts: $\#(S \cap R) = d + 1$, each connected component of Z supported at a point of $S \cap R$ has degree 2 and no connected component of Z is contained in R.

Take general planes $H_1, H_2 \in |\mathcal{I}_R(1)|$. Since $R = \mathrm{Sing}(H_1 \cup H_2)$ and H_1, H_2 are general, $Z' = Z \cap (H_1 \cup H_2)$ is the union of the connected components of Z which are supported at a point of $S \cap R$. Since $\deg(\mathrm{Res}_{H_1\cup H_2}(Z)) \leq 3d + 1 - 2(d+1) = d - 1$, by Lemma 2.8, we have $h^1(\mathcal{I}_{\mathrm{Res}_{H_1\cup H_2}(Z)}(d-2)) = 0$. Then, the residual exact sequence of Z with respect to $H_1 \cup H_2$,

$$0 \to \mathcal{I}_{\mathrm{Res}_{H_1\cup H_2}(Z)}(d-2) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z', H_1\cup H_2}(d) \to 0,$$

gives $h^1(\mathcal{I}_{Z'}(d)) = h^1(\mathcal{I}_{Z', H_1\cup H_2}(d)) \geq h^1(\mathcal{I}_Z(d)) > 0$.

Take a connected component A of Z'. Since $\deg(A) = 2$ and $\deg(A \cap R) = 1$, there is a unique plane H_3 containing $A \cup R$. Since $h^1(\mathcal{I}_{Z\cap H_3}(d)) = 0$ by assumption (10), we have $h^1(\mathcal{I}_{Z'\cap H_3}(d)) = 0$ by Lemma 2.6. Since $\deg(\mathrm{Res}_{H_3}(Z')) \leq d$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{H_3}(Z')}(d-1)) = 0$ by Lemma 2.8. Thus, the residual exact sequence of Z' with respect to H_3,

$$0 \to \mathcal{I}_{\mathrm{Res}_{H_3}(Z')}(d-1) \to \mathcal{I}_{Z'}(d) \to \mathcal{I}_{Z'\cap H_3, H_3}(d) \to 0,$$

gives $h^1(\mathcal{I}_{Z'}(d)) = 0$, a contradiction.

(c2) We assume now $H \not\supseteq R$. Thus, H contains at most one point of $S \cap R$. For any $p \in R \cap S$, let A_p denote the connected component of Z supported at p.

Since M is general and $S \cap R$ is finite, $M \not\supseteq A_p$ for any $p \in S \cap R$. Recall that $X = Z \cap (H \cup M)$. Thus, if $S \cap H \cap R = \emptyset$, then we have that $X = (Z \cap H) \cup (R \cap S)$ (as schemes), while if $R \cap H \cap S = \{p\}$, then X is the union of A_p, the points $(S \setminus \{p\}) \cap R$ and the scheme $(Z \cap H) \setminus \{p\}$.

Since $\deg(X) \geq 2d + 2 > d + 1 = \deg(Z \cap R)$, there is a plane $U \supset R$ such that $\deg(X \cap U) \geq d + 2$. If $p \in S \cap R$ with $\deg(A_p) = 2$, we take as U the plane spanned by $R \cup A_p$.

We have

$$\deg\big(\mathrm{Res}_U(X)\big) = \deg(X) - \deg(X \cap U) \leq 3d + 1 - (d + 2) = 2(d - 1) + 1.$$

By Lemma 2.8, there is a line J such that $\deg(\mathrm{Res}_U(X) \cap J) \geq d + 1$.

Since by construction we know that $\mathrm{Res}_U(Z) \cap R = \emptyset$, then $J \neq R$.

If $\deg(J \cap Z) \geq d + 2$, we take $L = J$ and we are in case (i) and the theorem is proved. Thus, we may assume $\deg(J \cap Z) = d + 1$ and we will find a contradiction.

If $J \cap R \neq \emptyset$, the plane N spanned by $J \cup R$ proves that $w \geq \deg(N \cap X) \geq 2d + 2$ and hence $\deg(\mathrm{Res}_H(Z)) = z - w \leq d - 1 < \deg(Z \cap R) - 1$, which is impossible since $\deg(Z \cap R \cap H) \leq 1$.

Now assume $J \cap R = \emptyset$. Fix a general $Q \in |\mathcal{I}_{J \cup R}(2)|$. Since any 2 pairs of 2 skew lines are projectively equivalent, Q is smooth. Since $\mathcal{I}_{J \cup R}(2)$ is globally generated, Q is general, each connected component of Z has degree at most 2 and Z is finite, $Z \cap Q = Z \cap (J \cup R)$ (as schemes). Since $\deg(\mathrm{Res}_Q(Z)) \leq 3d + 1 - 2d - 2 = d - 1$, we have by Lemma 2.8 that $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d - 2)) = 0$.

Hence, the residual exact sequence with respect to Q,

$$0 \to \mathcal{I}_{\mathrm{Res}_Q(Z)}(d - 2) \to \mathcal{I}_Z(d) \to \mathcal{I}_{(Z \cap J) \cup (Z \cap R), Q}(d) \to 0,$$

gives $h^1(\mathcal{I}_{(Z \cap J) \cup (Z \cap R)}(d)) = h^1(\mathcal{I}_{(Z \cap J) \cup (Z \cap R), Q}(d)) > 0$.

Taking a plane N_1 containing the line J and exactly one point of $R \cap S$, we get $\deg(\mathrm{Res}_{N_1}(Z \cap J) \cup (Z \cap R)) \leq 2d + 2 - (d + 2) = d$; hence, by Lemma 2.8, we have

$$h^1\big(\mathcal{I}_{\mathrm{Res}_{N_1}(Z \cap J) \cup (Z \cap R)}(d - 1)\big) = 0;$$

on the other hand, by assumption (10), we know that $h^1(\mathcal{I}_{N_1 \cap Z}(d)) = 0$ and by Lemma 2.6 we get $h^1(\mathcal{I}_{N_1 \cap ((Z \cap J) \cup (Z \cap R))}(d)) = 0$.

Hence, from the following residual exact sequence,

$$\begin{aligned} 0 \to \mathcal{I}_{\mathrm{Res}_{N_1}(Z \cap J) \cup (Z \cap R)}(d - 1) &\to \mathcal{I}_{(Z \cap J) \cup (Z \cap R)}(d) \\ &\to \mathcal{I}_{N_1 \cap ((Z \cap J) \cup (Z \cap R)), N_1}(d) \to 0, \end{aligned}$$

we obtain $h^1(\mathcal{I}_{(Z \cap J) \cup (Z \cap R)}(d)) = 0$, which is a contradiction. This ends the proof. ■

Notice that if $z \leq 3d$, case (iii) of the previous theorem never occurs since $\langle Z\rangle = \mathbb{P}^3$.

Thanks to Proposition 6.1, we can easily prove Theorem 1.3 which states the emptiness of the minimal Terracini loci $\mathbb{T}(3,d;x)'$ for $0 < 2x \leq 3d+1$.

Proof of Theorem 1.3. Consider $S \in \mathbb{T}(3,d;x)'$ and let Z be a critical scheme for S. By Lemma 2.12, we know that $Z_{\mathrm{red}} = S$ and hence $\langle Z\rangle = \mathbb{P}^3$. Since $\deg(Z) \leq 2x \leq 3d+1$, we can apply Proposition 6.1.

In any of the three cases, there is a plane H and a subset $S' = S \cap H$ which contradicts the minimality of S. ∎

Now we will prove Theorem 1.4, which characterizes the elements of $\mathbb{T}(3,d;1+\lceil 3d/2\rceil)'$, i.e. the sets of minimal cardinality which are minimal Terracini with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$ in $\mathbb{P}^3$. Notice that one implication follows from Theorem 4.2 (iii). By Proposition 4.7, we also know that if S is contained in a reducible rational normal curve, then $S \notin \mathbb{T}(3,d;1+\lceil 3d/2\rceil)'$.

Proof of Theorem 1.4. We only need to prove that any $S \in \mathbb{T}(3,d;1+\lceil 3d/2\rceil)'$ is contained in a rational normal curve.

Given $d \geq 7$ and $x = 1+\lceil 3d/2\rceil$, we set $\varepsilon := 1$ if d is even and $\varepsilon := 0$ if d is odd. Given $S \in \mathbb{T}(3,d;x)'$, let Z be a critical scheme for S and $z := \deg(Z)$. Recall that $Z_{\mathrm{red}} = S$ and $z \leq 3d+3-\varepsilon$.

Take a quadric $Q \in |\mathcal{O}_{\mathbb{P}^3}(2)|$ such that $w := \deg(Z \cap Q)$ is maximal.

Step (a). In this step we want to prove that $Z \subset Q$. Assume by contradiction that $Z \nsubseteq Q$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, $h^0(\mathcal{I}_A(2)) > 0$ for every zero-dimensional scheme $A \subset \mathbb{P}^3$ such that $\deg(A) \leq 9$. Thus, $w \geq 9$. By the minimality of S, we also have $h^1(\mathcal{I}_{Z\cap Q}(d)) = 0$; hence, $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) > 0$.

Since $\deg(\mathrm{Res}_Q(Z)) \leq z - w \leq 3(d-2)-\varepsilon$, then Proposition 6.1 implies that we are in one of the following cases:

(i) there is a line L such that $\deg(\mathrm{Res}_Q(Z)\cap L) \geq d$,

(ii) there is a plane conic D such that $\deg(\mathrm{Res}_Q(Z)\cap D) \geq 2d-2$,

(iii) $\varepsilon = 0$, $z = 3d+3$, $w = 9$ and $\mathrm{Res}_Q(Z)$ is the complete intersection of a plane cubic and a plane curve of degree $d-2$.

(a1) First we exclude cases (ii) and (iii). Indeed, in both cases (ii) and (iii), there is a plane U such that $\deg(U\cap Z) \geq \deg(U \cap \mathrm{Res}_Q(Z)) \geq 2d-2$. Since $h^0(\mathcal{I}_U(2)) = 4$, we have $w \geq \deg(U\cap Z)+3 \geq 2d+1$ and hence we have $\deg(\mathrm{Res}_Q(Z)) = z-w \leq 3d+3-(2d+1) < 2d-2$, which is a contradiction.

(a2) We assume now that we are in case (i); i.e., there is a line L such that

$$\deg\left(L \cap \mathrm{Res}_Q(Z)\right) \geq d.$$

Note that, since $Z \not\subseteq Q$, there is a plane H such that $L \subset H$ and $\deg(H \cap Z) \geq \deg(Z \cap R) + 1 \geq d + 1$. We have $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$, by the minimality of S, and $\deg(\mathrm{Res}_H(Z)) \leq 3d+3-d-1 = 2d+2 < 3(d-1)$. By applying Proposition 6.1 to $\mathrm{Res}_H(Z)$, we are in one of the following cases:

(1) there is a line R such that $\deg(R \cap \mathrm{Res}_H(Z)) \geq d+1$;

(2) there is a conic D such that $\deg(D \cap \mathrm{Res}_H(Z)) \geq 2d$.

Now we consider separately these two possibilities (i1) and (i2).

(a2.1) Assume we are in case (1); that is, assume the existence of a line R such that $\deg(R \cap \mathrm{Res}_H(Z)) \geq d+1$. The minimality of S gives $\deg(R \cap Z) = d+1$ and $R \cap Z = R \cap \mathrm{Res}_H(Z)$.

Now we study the following cases: either $R = L$, or $R \neq L$ and $R \cap L \neq \emptyset$, or $R \cap L = \emptyset$.

(a2.1.1) First assume $R = L \subset H$. Since Z is critical, every connected component of $\mathrm{Res}_H(Z)$ supported at a point of R is a simple point. Thus, we get $\#(S \cap R) \geq d+1$. Thus, $h^1(\mathcal{I}_{2(S\cap R)}(d)) = h^1(\mathcal{I}_{2(S\cap R),R}(d)) > 0$, contradicting the minimality of S.

(a2.1.2) Now assume $R \neq L$ and $R \cap L \neq \emptyset$. Consider the plane $M = \langle L \cup R \rangle$. Since $\deg(L \cap R) = 1$, then $\deg(Z \cap M) \geq 2d$. Since $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) > 0$ and $\deg(\mathrm{Res}_M(Z)) \leq d+3$, there is a line E such that $\deg(E \cap \mathrm{Res}_M(Z)) \geq d+1$. As above we get $E \neq L$ and $E \neq R$. Take $Q' \in |\mathcal{I}_{E\cup L\cup R}(2)|$. Since $Z \not\subseteq Q$ and w is maximal, we have $Z \not\subseteq Q'$. Hence, $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(Z)}(d-2)) > 0$ and, by Lemma 2.8, we have $\deg(\mathrm{Res}_{Q'}(Z)) \geq d-1$. Hence, $z \geq (d-1) + \deg(Z \cap (L \cup R \cup E)) = (d-1) + (2d+d+1-3) = 4d-3$, a contradiction since $d \geq 7$.

(a2.1.3) Now assume $R \cap L = \emptyset$. Take $Q'' \in |\mathcal{I}_{R\cup L}(2)|$ such that $\deg(Z \cap Q'')$ is maximal. The maximality of w gives $Z \not\subseteq Q''$. Thus, $h^1(\mathcal{I}_{\mathrm{Res}_{Q''}(Z)}(d-2)) > 0$ and $\deg(\mathrm{Res}_{Q''}) \leq 3d+3-(d+1+d) = d+2 \leq 2(d-2)+1$. Hence, there is a line F such that $\deg(F \cap \mathrm{Res}_{Q''}(Z)) \geq d$. We conclude as in case (a2.1.2), using L, R and F instead of L, R and E.

(a2.2) Assume that we are in case (2); that is, there exists a conic D such that $\deg(\mathrm{Res}_H(Z)) \geq 2d$, and call $\langle D \rangle$ the plane spanned by D. Since Z is minimally Terracini, $h^1(\mathcal{I}_{\langle D\rangle \cap Z}(d)) = 0$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_{\langle D\rangle}(Z)}(d-1)) > 0$. Since

$$\deg\left(\mathrm{Res}_{\langle D\rangle}(Z)\right) \leq d+3-\varepsilon,$$

Lemma 2.8 gives the existence of a line R such that $\deg(R \cap \mathrm{Res}_{\langle D\rangle}(Z)) \geq d+1$. Thus, the minimality of S implies $\deg(J \cap Z) = d+1$. The steps (a2.1.1), (a2.1.2) and (a2.1.3) work verbatim taking J instead of R.

Step (b). In Step (a), we proved that $Z \subset Q$; hence, we have $|\mathcal{I}_Z(2)| \neq \emptyset$. In this step we prove that every quadric in $|\mathcal{I}_Z(2)|$ is integral.

Assume by contradiction that Z is contained in a quadric which is either not reduced, or reducible. We consider separately the following two cases.

(b1) Assume first $Z \subset 2H$ where H is a plane. Thus, since $S \subset Z$, we would have $S \subset H$, contradicting our definition of Terracini set.

(b2) Assume now $Z \subset H \cup M$ where H and M are planes and $H \neq M$. With no loss of generality we may assume $\deg(Z \cap H) \geq \deg(Z \cap M)$. The minimality of S gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_H(Z)) \leq \lfloor z/2 \rfloor < 2(d-1)+1$, then Lemma 2.8 implies that there is a line L such that $\deg(L \cap \mathrm{Res}_H(Z)) \geq d+1$.

Let N be a general plane containing L. Since $\deg(\mathrm{Res}_{H\cup N}(Z)) \leq z-d-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{H\cup N}(Z)}(d-2)) = 0$, again by Lemma 2.8. The minimality of S gives $Z \subset H \cup N$. Taking different planes N and N' containing L, we get $S \subset (H \cup N) \cap (H \cup N') = H \cup L$. The minimality of S implies $2\#(L \cap S) \leq d+1$; i.e., $\#(S \cap L) \leq \lfloor (d+1)/2 \rfloor$. Since $\deg(\mathrm{Res}_H(Z) \cap L) = d+1$, we get d odd, $H \cap Z \cap L = \emptyset$ and $\mathrm{Res}_H(Z) \subset L$. Since $\#(S \cap R) > 1$ and $H \cap Z \cap L = \emptyset$, $L \nsubseteq H$.

Recall that N is a general plane containing L. Again by the minimality of S, we have $h^1(\mathcal{I}_{\mathrm{Res}_N(Z)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_N(Z)) \leq 3d+3-d-1$, then Proposition 6.1 implies that

(I) either there is a line R such that $\deg(R \cap \mathrm{Res}_N(Z)) \geq d+1$,

(II) or there is a conic D with $\deg(D \cap \mathrm{Res}_N(Z)) \geq 2d$.

We analyze separately the two cases and we will show a contradiction in both cases.

(b2.1) Assume first the existence of a conic D as in case (II).

Since $|\mathcal{I}_D(2)|$ is globally generated and each connected component of Z has degree at most 2, then $Q_1 \cap Z = D \cap Z$ for a general $Q_1 \in |\mathcal{I}_D(2)|$. Since $\deg(\mathrm{Res}_{N\cup Q_1}(Z)) \leq z-(d+1)-2d \leq 2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{N\cup Q_1}(Z)}(d-3)) = 0$. The minimality of S gives $Z \subset Q_1 \cup N$. Since $N \cap Z \cap H = \emptyset$, and $Q_1 \cap Z = D \cap Z$ and $Z \cap N = Z \cap L$, we get $Z \subset D \cup L$.

By the minimality of S, we have that $\#(S \cap D) \leq d$ on the conic and $2\#(S \cap L) \leq d+2$ on the line, which implies $\#(S \cap L) \leq \lfloor (d+1)/2 \rfloor$. Then, we would have

$$1 + \left\lfloor \frac{3d}{2} \right\rfloor = x \leq d + \left\lfloor \frac{d+1}{2} \right\rfloor$$

which is false.

(b2.2) Assume now the existence of a line R as in case (I).

Since S is minimal, then $\deg(Z \cap R) = \deg(Z \cap L) = d+1$. Since $L \nsubseteq H$, and $S \subset H \cup L$, we have $R \neq L$. Since $H \cap Z \cap S = \emptyset$, we have $R \cap L \cap S = \emptyset$. Thus, $\deg(Z \cap (R \cup L)) = 2d+2$. Since $\langle S \rangle = \mathbb{P}^3$ and Z is minimal, $R \cup L$ is not a conic; i.e., $R \cap L = \emptyset$.

Take a general $Q' \in |\mathcal{I}_{R\cup L}(2)|$. Since $\mathcal{I}_{R\cup L}(2)$ is globally generated, then Q' is smooth and $Z \cap Q' = Z \cap (R \cup L)$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}}(d-2)) > 0$ and $\deg(\mathrm{Res}_{Q'}(Z)) \le d+1$, Lemma 2.8 implies that there is a line E such that $\deg(E \cap Z) \ge d$. Since d is odd, we get $\#(E \cap S) = (d+1)/2$. Since $Z \subset H \cup L$ and $L \subset H$, then we get $L \cap E \neq \emptyset$. The conic $L \cup E$ contradicts the minimality of S.

STEP (c). In Steps (a) and (b), we proved that Z is contained in a quadric Q and that each quadric containing Z is integral. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, for any degree 8 scheme $W \subset Z$, we have $h^0(\mathcal{I}_W(2)) \ge 2$. Thus, there is quadric $T \subset \mathbb{P}^3$ such that $\deg(T \cap Z) \ge 8 - \varepsilon$ and $T \neq Q$. In this step we prove that $Z \subset T$.

Assume by contradiction that $Z \not\subseteq T$. Since $\deg(\mathrm{Res}_T(Z)) \le 3(d-2)+1$, the residual exact sequence of T gives $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(d-2)) > 0$. First assume

$$\deg\left(\mathrm{Res}_T(Z)\right) = 3d-5$$

and that $\langle \mathrm{Res}_T(Z)\rangle$ is contained in a plane M. Since Q is irreducible, $Q \cap M$ is a conic containing at least $\lceil (3d-5)/2\rceil$ points of S, contradicting the minimality of S.

Since $\mathrm{Res}_T(Z)$ is not a scheme of degree $3d-5$ contained in a plane, then Proposition 6.1 implies that we have the following cases:

(α) either there is a line L_1 such that $\deg(L_1 \cap \mathrm{Res}_T(Z)) \ge d$,

(β) or there is a conic D_1 such that $\deg(D_1 \cap \mathrm{Res}_T(Z)) \ge 2d-2$,

(γ) or there is a plane cubic C_1 such that $\deg(C_1 \cap \mathrm{Res}_T(Z)) \ge 3d-6$.

Now we analyze separately the three cases and we will get to a contradiction in any case.

(c1) Assume first the existence of the plane cubic C_1 as in case (γ).

Since Z is contained in an integral quadric Q, then we have $\langle C_1\rangle \not\subset Q$. Then, $\deg(C_1 \cap Q) \le 6$ and this gives a contradiction because $6 < 3d-6$.

(c2) Assume now the existence of the conic D_1 as in case (β).

The scheme $\mathrm{Res}_{\langle D_1\rangle}(Z)$ has degree $\le d+5-\varepsilon$ and $h^1(\mathcal{I}_{\mathrm{Res}_{\langle D_1\rangle}(Z)}(d-1)) > 0$ because Z is critical. Thus, by Lemma 2.8, there is a line L_2 such that

$$\deg\left(\mathrm{Res}_{\langle D_1\rangle}(Z) \cap L_2\right) \ge d+1.$$

Take a general plane $M \supset L_2$. We have $\deg(\mathrm{Res}_{M\cup\langle D_1\rangle}(Z)) \le 4-\varepsilon$. The minimality of S gives $Z \subset M \cup \langle D_1\rangle$. Then, we proved that Z is contained in a reducible quadric, which is impossible by step (b).

(c3) Assume finally the existence of the line L_1 as in case (α).

Bézout's theorem gives $L_1 \subset Q$. Take a general plane $U \supset L_1$. Since each connected component of Z has degree ≤ 2, then $L_1 \cap Z = U \cap Z$. Since $\deg(\mathrm{Res}_U(Z)) \le 2d+3-\varepsilon$ and $d \ge 6$, by Proposition 6.1, it follows that either there is a line L_3 such that $\deg(\mathrm{Res}_U(Z) \cap L_3) \ge d+1$, or there is a conic D_3 such that $\deg(D_3 \cap \mathrm{Res}_U(Z)) \ge 2d$.

We can again exclude the existence of D_3 following the same argument used in step (c2).

Now assume that there exists L_3 such that $\deg(\mathrm{Res}_U(Z) \cap L_3) \geq d+1$. In this case we have

$$\#\big(S \cap (L_1 \cup L_3)\big) \geq \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{d+1}{2} \right\rceil = d+1;$$

we also get that d is odd. Since S is minimal, then $L_1 \cap L_3 = \emptyset$. Thus, the integral quadric Q is not a cone; i.e., Q is smooth.

Then, following the same argument used in step (a2.1.3), we get a contradiction (note that both steps (a2.1.2) and (a2.1.3) do not use the assumption $Z \not\subseteq Q$ made in step (a)).

STEP (d). By the previous steps, we know that Z is contained in no reducible quadric and in infinitely many integral quadrics. Moreover, every quadric containing a degree $8-\varepsilon$ subscheme of Z contains Z.

Let Q be a general element of $|\mathcal{I}_Z(2)|$.

Since in every pencil of quadrics at least one is singular, we can assume that T is a quadric cone containing Z. Since Q is general, we may take T such that $T \neq Q$. Call o its vertex. Every line L such that $\deg(L \cap Z) \geq 3$ is contained in T and any union of 2 lines of T is a reducible conic because they contain o.

Set $E := Q \cap T$ as a scheme-theoretic intersection. Since $Z \subset T$ and $Z \subset Q$, then $Z \subset E$. Since E is the complete intersection of 2 quadric surfaces, the adjunction formula gives $\omega_E \cong \mathcal{O}_E$. The Koszul complex of the equations of Q and T gives $h^0(\mathcal{O}_E) = 1$. Hence, by duality, we have $h^1(\mathcal{O}_E) = 1$.

First assume E integral; i.e., E is an irreducible quartic curve. Since the rank 1 torsion-free sheaf $\mathcal{I}_{Z,E}(d)$ has degree $4d - \deg(Z) > 0$, then $h^1(E, \mathcal{I}_{Z,E}(d)) = 0$. Since E is arithmetically Cohen–Macaulay, $h^1(\mathcal{I}_Z(d)) = 0$, which is a contradiction.

Then, we may assume that E is not integral. If E is not reduced, it may have multiple components, but no embedded point. If $E_{\mathrm{red}} \neq E$, then E_{red} is a reduced curve of degree ≤ 3 containing S. Since $h^0(\mathcal{O}_E) = 1$, E_{red} is connected; hence, Proposition 4.7 gives a contradiction.

Thus, the curve $E = E_{\mathrm{red}}$ is reduced and reducible. Each irreducible component of E is either a line, or a smooth conic, or a rational normal curve.

First assume $E = E_1 \cup E_2$ with E_1 and E_2 reduced conics. Since Z is critical and S is minimal, then $h^1(\mathcal{I}_{\mathrm{Res}_{\langle E_i \rangle}(Z)}(d-1)) > 0$ for $i = 1, 2$, and hence we have $\deg(Z \cap E_1) + \deg(Z \cap E_2) - \deg(Z \cap E_1 \cap E_2) \geq (2d+2) + (2d+2) - 4 = 4d$, which contradicts the assumption $z \leq 3d+3$, since $d \geq 4$.

Thus, E has at most one smooth conic among its irreducible components and it is not formed by 4 lines through o. Hence, there is a connected degree three curve $C \subset E$, which is either a rational normal curve, or a reducible rational normal curve.

We consider now the following two cases: either $Z \nsubseteq C$, or $Z \subseteq C$.

(d1) First we assume that $Z \nsubseteq C$. Since $\mathcal{I}_C(2)$ is globally generated and every connected component of Z has degree ≤ 2, for a general $Q' \in |\mathcal{I}_C(2)|$, we have $Q' \cap Z = C \cap Z$. Hence, it follows that $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(Z)}(d-2)) > 0$. We write $E = L_4 \cup C$ with L_4 a line. We have $\mathrm{Res}_{Q'}(Z) \subset L_4$ and $\deg(\mathrm{Res}_{Q'}(Z)) \geq d$. Take a general plane $M \supset L_4$.

Since $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) > 0$ by minimality of S, and $\deg(\mathrm{Res}_M(Z)) \leq 2d + 3 - \varepsilon \leq 3d$, then by Proposition 6.1, we have that

(d1.1) either there is a line $L_5 \subset C$ such that $\deg(L_5 \cap \mathrm{Res}_M(Z)) \geq d+1$, and this is impossible because E would be a union of 2 reduced conics;

(d1.2) or there is a conic D_4 such that $\deg(\mathrm{Res}_M(Z) \cap D_4) \geq 2d$, and also in this case E would be a union of 2 reduced conics. In both cases we find a contradiction and this completes the case $Z \nsubseteq C$.

(d2) Now we assume $Z \subset C$. By Proposition 4.7, we obtain that C is a rational normal curve and this ends the proof of the theorem. ∎

We are going finally to prove our last main result, which is Theorem 1.5. We point out that the bound in Theorem 1.5 is sharp, as shown in the following example, which implies that $\mathbb{T}(3, d; 2d)' \neq \emptyset$ for all $d \geq 5$.

Example 6.2. Take $d \geq 5$. Let $C \subset \mathbb{P}^3$ be a smooth linearly normal elliptic curve. Let $\mathcal{L}$ be a line bundle on C such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(d)$. Since $\deg(\mathcal{L}) = 2d$ and C has genus 1, $\mathcal{L}$ is very ample.

Fix any $S \subset |\mathcal{L}|$ formed by $2d$ points. We will show that $S \in \mathbb{T}(3, d; 2d)'$. Obviously, $\langle S \rangle = \mathbb{P}^3$. Since $2S \cap C \in |\mathcal{O}_C(d)|$, we have $h^i(\mathcal{I}_{2S\cap C, C}(d)) = 1$, $i = 0, 1$.

The curve C is the smooth complete intersection of 2 quadric surfaces, say $C = Q \cap Q'$. Clearly, Q and Q' are smooth at each point of S and $\mathrm{Res}_Q(2S) = S$ and $\mathrm{Res}_{Q'}(2S \cap Q) = S$; hence, the residual exact sequence with respect to Q in $\mathbb{P}^3$ and of C in Q gives

$$0 \to \mathcal{I}_S(d-2) \to \mathcal{I}_{2S}(d) \to \mathcal{I}_{2S\cap Q, Q}(d) \to 0, \tag{12}$$

$$0 \to \mathcal{I}_{S,Q}(d-2) \to \mathcal{I}_{2S\cap Q, Q}(d) \to \mathcal{I}_{2S\cap C, C}(d) \to 0. \tag{13}$$

Since $d \geq 5$, we have

$$\#S = 2d < 4d - 8 = \deg\big(\mathcal{O}_C(d-2)\big).$$

Thus, $h^1(\mathcal{I}_{S,C}(d-2)) = 0$. Since C is arithmetically Cohen–Macaulay, we have $h^1(\mathcal{I}_S(d-2)) = 0$, and hence $h^1(\mathcal{I}_{S,Q}(d-2)) = 0$. Using (13) and (12), we get $h^1(\mathcal{I}_{2S}(d)) = 1$ and $h^0(\mathcal{I}_{2S}(d)) \geq 1$.

Take now $S' \subsetneq S$. Since $\deg(2S' \cap C) < 4d$, we have $h^1(\mathcal{I}_{2S' \cap C, C}(d)) = 0$. Moreover, $h^1(Q, \mathcal{I}_{S',Q}(d-2)) = 0$, by Lemma 2.6. Hence, using again (13) and (12) (with S' instead of S), we get $h^1(\mathcal{I}_{2S'}(d)) = 0$.

Thus, $S \in \mathbb{T}(3, d; 2d)'$.

From the previous example, we can deduce the following remark.

Remark 6.3. Fix integers $x < 2d$. Let $E \subset \mathbb{P}^3$ be an integral complete intersection of two quadric surfaces. Let S be a collection of x points on E; then, $h^1(\mathcal{I}_{2S}(d)) = 0$.

The following technical lemma generalizes Remark 6.3 to reducible quartic curves satisfying further suitable conditions.

Lemma 6.4. *Fix $d \geq 5$. Let $T \subset \mathbb{P}^3$ be a reduced curve with* $\deg(T) \leq 4$ *and such that any irreducible component of T is a line or a conic or a rational normal cubic. Assume also that no plane contains a subcurve of T of degree ≥ 3. Let $S \subset T$ be a collection of points such that $\#(S) \leq 2d - 1$ and*

- *$\#(S \cap L) \leq \lceil d/2 \rceil$ for any line $L \subseteq T$,*
- *$\#(S \cap C) \leq d$ for any conic $C \subseteq T$,*
- *$\#(S \cap D) \leq (3d + 1)/2$ for any rational normal cubic $D \subseteq T$.*

Let $Z \subset T$ be a zero-dimensional scheme such that $Z_{\mathrm{red}} = S$, any connected component of Z has degree ≤ 2, Z is contained in an integral quadric surface and Z is not contained in any reducible quadric. Then, $h^1(\mathcal{I}_Z(d)) = 0$.

Proof. Since $h^1(\mathcal{I}_T(t)) = 0$ for all $t \geq 5$, it is sufficient to prove that $h^1(\mathcal{I}_{Z,T}(d)) = 0$. We already analyzed all cases with $\deg(T) \leq 3$ and T connected. Thus, we may assume that T is connected and $\deg(T) = 4$.

Consider a good ordering $T_1, \dots, T_s$ of the irreducible components of T and set $Y = T_1 \cup \cdots \cup T_{s-1}$. The components T_1 and T_s are final components, and for every final component T_i of T, there is a good ordering with T_i as its first component. Thus, changing if necessary the good ordering, we may assume $\deg(T_1) \geq \deg(T_s)$. Thus, $\deg(T_s) \leq 2$ and $\deg(T_s) = 2$ if and only if $s = 2$ and $\deg(T_1) = 2$. This case is excluded because T would be contained in a reducible quadric.

Hence, $\deg(T_1) \geq \deg(T_s) = 1$. Set $E := T_s \cap Y$ (scheme-theoretic intersection). Since T contains no plane subcurves of degree ≥ 3, then we can assume, up to choosing a good ordering, that $\deg(T_s \cap Y) \leq 2$. Set $e := \#(S \cap E)$ and $z := \deg(Z) \leq 2(\#S)$. Note that $\#S = \#(S \cap T_s) + \#(S \cap Y) - e$. We have the following Mayer–Vietoris type sequence on T:

$$0 \to \mathcal{I}_{Z,T}(d) \to \mathcal{I}_{Z \cap T_s, T_s}(d) \oplus \mathcal{I}_{Z \cap Y, Y}(d) \to \mathcal{I}_{Z \cap E, E}(d) \to 0. \tag{14}$$

(a) Assume $\#(S \cap T_s) \leq \lceil d/2 \rceil - 1$. Thus, $h^1(\mathcal{I}_{E\cup(Z\cap T_s),T_s}(d)) = 0$ since

$$\deg\big(E \cup (Z \cap T_s)\big) \leq 2 + 2\left(\left\lceil \frac{d}{2} \right\rceil - 1\right).$$

Then, the restriction map $H^0(\mathcal{I}_{Z\cap T_s,T_s}(d)) \to H^0(\mathcal{I}_{Z\cap E,E}(d))$ is surjective. Thus, the exact sequence (14) gives $h^1(\mathcal{I}_{Z,T}(d)) = 0$ and we conclude.

(b) Assume $\#(S \cap T_s) = \lceil d/2 \rceil$. If $S \cap Y \cap T_s = \emptyset$, then we have $\mathcal{I}_{Z\cap E,E}(d) = \mathcal{O}_E(d)$ and we conclude as in step (a). Thus, from now on we assume $S \cap T_s \cap Y \neq \emptyset$. Let M be a plane containing L_s such that $\deg(Z \cap M)$ is maximal.

(b1) Assume that M contains another irreducible component, T_i, of T. Since T contains no planar subcurve of degree ≥ 3, $\deg(T_i) = 1$ and T_i is unique in M. Since $T_s \cup T_i$ is a conic, $\#(S \cap (T_s \cup T_i)) \leq d$. The closure A of $T \setminus (T_s \cup T_i)$ is either a reduced conic or the union of 2 disjoint lines. The first case is excluded because T is not contained in a reducible quadric. Now assume that A is the union of 2 disjoint lines, say $A = L \cup R$. The lines L and R are the final components of T. By step (a), we may assume $\#(S \cap L) = \#(S \cap R) = \lceil d/2 \rceil$. Thus, $L \cap T_s = L \cap R = \emptyset$. Let Q be the unique quadric containing $L \cup R \cup T_s$. Since $L \cap R = \emptyset$, Q is a smooth quadric. Changing if necessary the names of the 2 rulings of Q, we may assume $L \cup R \cup T_s \in |\mathcal{O}_Q(3,0)|$. Since T_i meets each connected component of $L \cup R \cup T_s$, Bézout's theorem gives $T_i \subset Q$ and $T_i \in |\mathcal{O}_Q(0,1)|$. Let $Z' \subset Q$ be the residual of Z with respect to the divisor $L \cup R \cup T_s$. It is sufficient to prove that $h^1(Q, \mathcal{I}_{Z'}(d-3,d)) = 0$. Since $T_i \cup T_s$ is a reducible conic, $\#(S \cap T_i \cup T_s) \leq d$ and hence $\#(S \cap T_i) \leq d - \lceil d/2 \rceil$ with strict inequality if $S \cap T_i \cap T_s \neq \emptyset$. Thus, $\deg(Z') \leq 4d - 2 - 6\lceil d/2 \rceil \leq d - 2$ and hence $h^1(\mathcal{I}_{Z',Q}(d-3,d)) = 0$, and we conclude that $h^1(\mathcal{I}_{Z,T}(d)) = 0$.

(b2) Assume that T_s is the unique connected component of T contained in M. Thus, $\deg(Y \cap (M \setminus T_s)) \leq 3$. Hence, $h^1(\mathcal{I}_{Z\cap M}(d)) = 0$. By the residual exact sequence with respect to M, it is sufficient to prove that $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) = 0$. Assume by contradiction that $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) > 0$. Since $\deg(M \cap Z) > \deg(Z \cap T_s)$, we have $\deg(\mathrm{Res}_M(Z)) \leq 4d - 2 - 2\lceil d/2 \rceil - 1 \leq 3(d-1)$. Since T contains no plane curve of degree ≥ 3, Proposition 6.1 gives that either there is a line L_1 such that $\deg(L_1 \cap \mathrm{Res}_M(Z)) \geq d+1$ or there is a conic D_1 such that $\deg(D_1 \cap \mathrm{Res}_M(Z)) \geq 2d$.

(b2.1) Assume first the existence of the line L_1. Since $\#(S \cap L_1) \leq \lceil d/2 \rceil$, we get d odd and $\deg(Z \cap L_1) = d+1$. Since $\#(S \cap J) \leq d$ for all conics $J \subset T$ and d is odd, $L_1 \cap T_s = \emptyset$. Let A_1 denote the closure of $T \setminus (L_1 \cup T_s)$. Either A_1 is a reduced conic or it is the union of 2 disjoint lines. We have $\#(S \cap (T \setminus (T_s \cup L_1))) \leq d-2$. There is an integral quadric Q containing $T_s \cup L_1$ and at least one point of $S \cap (T \setminus (T_s \cup R_1))$ for each component of A_1. Thus, $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) = 0$. Thus, it is sufficient to prove that $h^1(\mathcal{I}_{Z\cap Q,Q}(d)) = 0$. Since $L_1 \cap T_s = \emptyset$, Q is a smooth quadric. We get $h^1(\mathcal{I}_{Z\cap Q,Q}(d)) = 0$, unless Q contains another irreducible component of T.

First assume $A_1 \subset Q$. Since Q is a smooth quadric, we get (for a suitable choice of the 2 rulings of Q) that either $T \in |\mathcal{O}_Q(4,0)|$ (excluded, because T is reduced and connected) or $T \in |\mathcal{O}_Q(3,1)|$ or $T \in |\mathcal{O}_Q(2,2)|$, which are also excluded. Now assume that Q only contains one component, R, of A_1. Write $A_1 = R \cup R_2$ and $A_2 := L_1 \cup T_s \cup R$. Either $A_2 \in |\mathcal{O}_Q(3,0)|$ or $A_2 \in |\mathcal{O}_Q(2,1)|$. In both cases we get $h^1(Q, \mathcal{I}_{Z\cap A_2,Q}(d,d)) = 0$. To conclude the proof we need to consider $R_2 \cap Z \cap Q$. We have $\deg(R_2 \cap Z \cap Q) \leq 4$ and hence $h^1(Q, \mathcal{I}_{Z\cap Q,Q}(d,d)) = 0$.

(b2.2) Assume the existence of the conic D_1. Since $\#(S \cap D_1) \leq d$, we get $\#(S \cap D_1) = d$ and hence $\deg(Z \cap D_1) = 2d$. By step (b1), we may assume that if D_1 is reducible, then none of its component contains $\lceil d/2 \rceil$ points of S. We get $T = D_1 \cup R \cup T_s$ with R a line and $\#(S \cap (T \setminus D_1 \cup T_s)) \leq d - 1 - \lceil d/2 \rceil$. If R is a final component of T, then we use step (a) and that $\#(R \cap S) < \lceil d/2 \rceil$. Now assume that R is not a final component of T. Assume for the moment $T_s \cap D_1 \neq \emptyset$. Since T contains no degree 3 planar subcurve, $D_1 \cup T_s$ is a reducible rational normal curve and we may find a quadric Q_1 containing $D_1 \cup T_s$, but not R. To conclude in this case we need $\deg(\mathrm{Res}_{Q_1}(Z)) \leq d - 1$. We have $\#(S \cap R) \leq 2d - 1 - d - \lceil d/2 \rceil$, and we can conclude. Now assume $D_1 \cap T_s = \emptyset$. Since T is connected, R meet T_s and D_1 at a different point. In this case T is contained in the reducible quadric $\langle R \cup T_s \rangle \cup \langle D_1 \rangle$, a contradiction. ■

We give now the proof of Theorem 1.5, which states that $\mathbb{T}(3, d; x)'$ is empty if $1 + \lceil 3\,d/2 \rceil < x < 2d$.

Proof of Theorem 1.5. Assume by contradiction the existence of $S \in \mathbb{T}(3, d; x)'$ and fix a critical scheme Z of S. Set $z := \deg(Z) \leq 4d - 2$.

Set $Z_0 = Z$. For any $i > 0$, let Q_i be a quadric surface such that $z_i := \deg(Z_{i-1} \cap Q_i)$ is maximal and set $Z_i := \mathrm{Res}_{Q_i}(Z_{i-1})$. The sequence $\{z_i\}_{i\geq 1}$ is weakly decreasing. Let e be the maximal i such that $z_i \neq 0$. Then, $z = z_1 + \cdots + z_e$ and $Z_e = \emptyset$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, $z_i \geq 9$ for all $i < e$, hence we have $e \leq (4d + 6)/9$, for $z \leq 4d - 2$. By Lemma 2.13, since Z is critical and $S \in \mathbb{T}(3, d; x)'$, we have

$$h^1\big(\mathcal{I}_{Z_{e-1}}(d - 2e + 2)\big) > 0.$$

(I) Assume first $e \geq 2$; i.e., Z is not contained in any quadric surface. Since

$$h^1\big(\mathcal{I}_{Z_{e-1}}(d - 2e + 2)\big) > 0,$$

then Proposition 6.1 implies that either $z_e \geq 3(d - 2e + 2) + 1$ or there is a line L such that $\deg(Z_{e-1} \cap L) \geq d - 2e + 4$ or there is a plane conic D such that $\deg(Z_{e-1} \cap D) \geq 2d - 4e + 6$.

(I.a) First assume $z_e \geq 3(d-2e+2)+1$. Since the sequence z_i is weakly decreasing, we get $z \geq e(3d-6e+6)$. It is easy to check that $e(3d-6e+6) > 4d-2$ for any $d \geq 13$ and $2 \leq e \leq (4d+6)/9$. This contradicts our hypothesis.

(I.b) Now assume the existence of a plane conic D such that $\deg(Z_{e-1} \cap D) \geq 2d-4e+6$. Since $h^0(\mathcal{I}_D(2)) = 5$, we get $z_i \geq (2d-4e+6)+4$ for all $i < e$. Thus, $z \geq e(2d-4e+10)-4$. It is easy to check that $e(2d-4e+10)-4 > 4d-2$ for any $2 \leq e \leq (4d+6)/9$, and this gives again a contradiction.

(I.c) Finally, assume the existence of a line L such that $\deg(Z_{e-1} \cap L) \geq d-2e+4$. Since $h^0(\mathcal{I}_L(2)) = 7$, we have $z_i \geq (d-2e+4)+6$ for all $i < e$. Hence, $z \leq e(d-2e+10)-6$. It is easy to check that $e(2d-4e+11)-6 > 4d-2$ for any $4 \leq e \leq (4d+6)/9$. Hence, we get $e \in \{2, 3\}$.

Let H be a general plane containing L. Since each connected component of Z has degree ≤ 2, we may assume $Z \cap L = Z \cap H$.

(I.c1) First assume $e = 3$. Since $z_1 \geq z_2 \geq z_3 \geq d-2$ and $z_1 + z_2 \geq \lceil 2z/3 \rceil$, we have $\deg(\mathrm{Res}_{Q_1 \cup Q_2 \cup H}(Z)) \leq z - \lceil 2z/3 \rceil - (d-2) = \lfloor z/3 \rfloor - d + 2 < d-3 = (d-5)+2$, since $d \geq 7$. Since S is minimally Terracini, we get $Z \subset Q_1 \cup Q_2 \cup L$. Since $e > 2$ and H is contained in a quadric surface, $Z \nsubseteq Q_1 \cup H$. Since S is minimally Terracini, $h^1(\mathcal{I}_{\mathrm{Res}_{Q_1 \cup H}(Z)}(d-3)) > 0$. We have

$$\begin{aligned}\deg\left(\mathrm{Res}_{Q_1 \cup H}(Z)\right) &\leq (z - z_1) - (d-2) \leq z - \left\lceil \frac{z}{3} \right\rceil - d + 2 \\ &\leq \frac{5d+2}{3} \leq 2(d-3)+1, \quad \text{for } d \geq 17.\end{aligned}$$

Hence, there is a line R such that $\deg(R \cap \mathrm{Res}_{Q_1 \cup H}(Z)) \geq d-1$. Taking a general plane containing R and taking again the residual, we get $Z \subset Q_1 \cup L \cup R$. But since $h^0(\mathcal{I}_{R \cup L}(2)) > 0$ and $e \leq 2$, we have a contradiction.

(I.c2) Now assume $e = 2$ and hence $z_1 \geq \lceil z/2 \rceil$. We have $\deg(\mathrm{Res}_H(Z)) \leq z - d$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$.

First assume $\langle \mathrm{Res}_H(Z) \rangle = \mathbb{P}^3$. Since $z - d \leq 3(d-1)+1$, Proposition 6.1 implies that either there is a plane cubic T_3 with $T_3 \cap \mathrm{Res}_H(Z)$ the complete intersection of T_3 and a degree $d-1$ plane curve or there is a conic T_2 such that $\deg(T_2 \cap \mathrm{Res}_H(Z)) \geq 2d$ or there is a line T_1 such that $\deg(T_1 \cap \mathrm{Res}_H(Z)) \geq d+1$.

First assume the existence of T_3. Since $\deg(\mathrm{Res}_{H \cup \langle T_3 \rangle}(Z)) \leq 1$, by minimality of S we get $Z \subset H \cup \langle T_3 \rangle$, contradicting the assumption $e > 1$.

Assume the existence of T_2. Since $\deg(\mathrm{Res}_{H \cup \langle T_2 \rangle}(Z)) \leq z - 3d \leq d-1$, we get $Z \subset H \cup \langle T_2 \rangle$, again a contradiction.

Now assume the existence of T_1. Take a general quadric $U \in |\mathcal{I}_{L \cup T_1}(2)|$. Since $\deg(\mathrm{Res}_U(Z)) \leq z - 2d - 1 \leq 2(d-2)+1$, by Lemma 2.8, there is a line R_1 such that

$$\deg\left(R_1 \cap \mathrm{Res}_U(Z)\right) \geq d.$$

Take a general $U' \in |\mathcal{I}_{L\cup T_1\cup R_1}(2)|$. Since $\deg(\mathrm{Res}_{U'}(Z)) \le z-3d-1 < d$ and S is minimally Terracini, $Z \subset U'$, contradicting the assumption $e > 1$.

Now assume $\dim\langle \mathrm{Res}_H(Z)\rangle \le 2$. The only new case is if $\deg(\mathrm{Res}_H(Z)) = 3d-2$ and $\mathrm{Res}_H(Z)$ is contained in a plane cubic C. Since $\deg(\mathrm{Res}_{\langle C\rangle}(Z)) \le d$, S is not minimally Terracini.

(II) Assume now $e = 1$; that is, Z is contained in a quadric Q.

If Q is reducible, we argue as in step (b) of the proof of Theorem 1.4 and we get a contradiction. So we can assume that Z is not contained in any reducible quadric. In particular, Q is irreducible and reduced.

Set $W_0 := Z$. Take $D_1 \in |\mathcal{O}_Q(2)|$, such that $w_1 = \deg(W_0 \cap D_1)$ is maximal, and set $W_1 := \mathrm{Res}_{D_1}(W_0)$. For $i \ge 2$, we iterate the construction: choose divisors $D_i \in |\mathcal{O}_Q(2)|$ such that $w_i := \deg(W_{i-1} \cap D_i)$ is maximal and set $W_i := \mathrm{Res}_{D_i}(W_{i-1})$. The sequence $\{w_i\}_{i\ge 1}$ is weakly decreasing. Let $c \ge 1$ be the maximal i such that $w_i \ne 0$; i.e., $W_c = \emptyset$ and $z = w_1 + \cdots + w_c$.

By Lemma 2.13, since Z is critical for S minimal, we have

$$h^1\big(\mathcal{I}_{W_{c-1}}(d-2c+2)\big) > 0.$$

Since $\dim|\mathcal{O}_Q(2)| = 8$, if $w_i \le 7$, then $w_{i+1} = 0$ and $W_{i+1} = \emptyset$. Thus, $w_i \ge 8$ for $1 \le i < c$; hence, we get $c \le \frac{4d+5}{8}$ since $z \le 4d-2$.

(II.a) If $c = 1$, then we have $Z \subset D_1 = Q \cap Q'$ where Q' is an integral quadric. Hence, D_1 is a complete intersection of two quadrics. If D_1 is integral, then by Remark 6.3, we have $h^1(\mathcal{I}_Z(d)) = 0$, a contradiction. If D_1 is reducible, we have again a contradiction by Lemma 6.4 and by the minimality of S.

(II.b) Now we assume $c = \lceil d/2\rceil$. Hence, either d is even and $h^1(\mathcal{I}_{W_{c-1}}(2)) > 0$, or d is odd and $h^1(\mathcal{I}_{W_{c-1}}(1)) > 0$.

First assume d odd and $c = \lceil d/2\rceil$. Then, we have $8(\lceil d/2\rceil - 1) + \deg(W_{c-1}) \le 4d-2$, and then $\deg(W_{c-1}) \le 2$, which is a contradiction.

Now assume d even and $c = d/2$. Since $8(d/2-1) + \deg(W_{c-1}) \le 4d-2$, then $\deg(W_{c-1}) \le 6$. Thus, either there is a line L such that $\deg(W_{c-1} \cap L) \ge 4$ or $\deg(W_{c-1}) = 6$ and W_{c-1} is contained in a conic D.

First assume the existence of the line L such that $\deg((W_{c-1}) \cap L) \ge 4$. Bézout's theorem implies $L \subset Q$. Since $h^0(\mathcal{I}_{L,Q}(2)) = 6$, the maximality of the integer w_{c-1} implies $w_{c-1} \ge w_c + 5 \ge 9$. Thus, $4d-2 \ge (d/2-1)9+4$, a contradiction, since $d \ge 7$.

Now assume $\deg(W_{c-1}) = 6$ and that W_{c-1} is contained in a conic D. If D is reducible, we may assume that no irreducible component J of D satisfied

$$\deg(J \cap W_{c-1}) \ge 4.$$

With these assumptions, Bézout's theorem implies $D \subset Q$. Since $h^0(\mathcal{I}_{D,Q}(2)) = 4$, the maximality of the integer w_{c-1} gives $w_{c-1} \geq w_c + 3 = 9$, which leads again to a contradiction.

(II.c) Now we may assume $2 \leq c < d/2$.

Assume for the moment $w_c \geq 3(d-2c+2)$. Since the sequence $\{w_i\}$ is weakly decreasing, $4d-2 \geq z \geq 3c(d-2c+2)$. Since $c < d/2$, we get $c = 1$, a contradiction.

Now assume $w_c < 3(d-2c+2)$. By applying Proposition 6.1, we know that either there is a conic D such that $\deg(D \cap W_{c-1}) \geq 2(d-2c+2)+2 = 2d-4c+6$, or there is a line L such that $\deg(L \cap W_{c-1}) \geq d-2c+4$.

(II.c1) In the first case, since $h^0(\mathcal{I}_{D,Q}(2)) = 4$, we have $w_i \geq (2d-4c+6)+3$ for all $i < c$. Hence, $z \geq c(2d-4c+9)-3$. Since $z \leq 4d-2$, then we have again a contradiction.

(II.c2) Assume now the existence of L. Since $h^0(\mathcal{I}_{L,Q}(2)) = 6$, we get

$$w_i \geq (d-2c+4)+5 \quad \text{for all } i < c.$$

Thus, $z \geq c(d-2c+9)-5$. It is easy to check that $2 \leq c \leq 3$; hence, $\deg(L \cap Z) \geq d-2$.

Take a quadric $U \in |\mathcal{O}_Q(2)|$ containing L and such that $\deg(Z \cap U)$ is maximal. Since $h^0(\mathcal{I}_{L,Q}(2)) = 6$, we have $\deg(L \cap U) \geq (d-2)+5 = d+3$. Thus, $\deg(\mathrm{Res}_U(Z)) \leq 4d-2-d-3 = 3(d-2)+1$. By Proposition 6.1, either there is a plane cubic E such that $\deg(E \cap \mathrm{Res}_U(Z)) \geq 3(d-2)$ or there is a conic F such that $\deg(\mathrm{Res}_U(Z) \cap F) \geq 2d-2$ or there is a line R such that $\deg(\mathrm{Res}_U(Z) \cap R) \geq d$. In all cases (since $d \geq 5$), Bézout's theorem implies that R, F and E are contained in Q (or at least all the components supporting Z). Since Q is an integral quadric, we exclude the plane cubic E.

(II.c2.1) Assume the existence of a conic F. Even if Q is not assumed to be smooth, F is a plane section of Q and $F \cup L$ is a reducible rational normal curve.

Thus, $Z \nsubseteq F \cup L$.

Since $\mathcal{I}_{F\cup L}(2)$ is globally generated, a general $Q' \in |\mathcal{I}_{F\cup L}(2)|$ has

$$Q' \cap Z = (F \cup L) \cap Z,$$

and hence $\mathrm{Res}_{Q'}(Z) \neq \emptyset$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(Z)}(d-2)) > 0$ and $\deg(\mathrm{Res}_{Q'}(Z)) \leq 4d-2-3d+4$ and $d \geq 7$, there is a line R' such that $\deg(\mathrm{Res}_{Q'}(Z) \cap R') \geq d$. Since $\mathcal{I}_{F\cup L\cup R'}(t)$ is globally generated for, say, $t = 4$, we get $Z \subset F \cup L \cup R'$. Hence, we conclude by Lemma 6.4.

(II.c2.2) Assume finally the existence of the line R. Since each connected component of Z has degree ≤ 2 and no line contains $d-2$ points of S, $R \neq L$.

(II.c2.2.1) First assume $R \cap L \neq \emptyset$. Thus,

$$H := \langle R \cup L \rangle$$

is a plane. Since $\deg(\operatorname{Res}_H(Z)) \le 4d-2-2d+2$ and $h^1(\mathcal{I}_{\operatorname{Res}_H(Z)}(d-1)) > 0$, either $\deg(\operatorname{Res}_H(Z)) = 2d$ and $\operatorname{Res}_H(Z)$ is contained in a conic F_1 or there is a line R_1 such that $\deg(R_1 \cap \operatorname{Res}_H(Z)) \ge 2$. In the first case we get $Z \subset L \cup R \cup F_1$, and we conclude by Lemma 6.4.

(II.c2.2.2) Now assume $R \cap L = \emptyset$. Take a general $Q_1 \in |\mathcal{I}_{R\cup L}(2)|$. Thus, $Q_1 \cap Z = (R \cup L) \cap Z$. We get $h^1(\mathcal{I}_{\operatorname{Res}_{Q_1}(Z)}(d-2)) > 0$ with $\deg(\operatorname{Res}_{Q_1}(Z)) \le 2d$. We get that either there is a conic F_2 with $\deg(F_2 \cap \operatorname{Res}_{Q_1}(Z)) \ge 2d-2$ or a line R_2 such that $\deg(R_2 \cap \operatorname{Res}_{Q_1}(Z)) \ge d$. If F_2 exist, we get $Z \subset R \cup L \cup F_2$ and we use Lemma 6.4. If R_2 exists, we take a general $U_1 \in |\mathcal{I}_{R\cup L\cup R_2}(3)|$ and get that Z is contained in the union of 4 lines. Hence, we conclude again by Lemma 6.4. ∎

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