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Letter

An operator methodology for the global dynamic analysis of stochastic nonlinear systems

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ABSTRACT

In a global dynamic analysis, the coexisting attractors and their basins are the main tools to understand the system behavior and safety. However, both basins and attractors can be drastically influenced by uncertainties. The aim of this work is to illustrate a methodology for the global dynamic analysis of nondeterministic dynamical systems with competing attractors. Accordingly, analytical and numerical tools for calculation of nondeterministic global structures, namely attractors and basins, are proposed. First, based on the definition of the Perron-Frobenius, Koopman and Foias linear operators, a global dynamic description through phase-space operators is presented for both deterministic and nondeterministic cases. In this context, the stochastic basins of attraction and attractors' distributions replace the usual basin and attractor concepts. Then, numerical implementation of these concepts is accomplished via an adaptative phase-space discretization strategy based on the classical Ulam method. Sample results of the methodology are presented for a canonical dynamical system.

Various sampling-based methods have been developed for the analysis of stochastic differential equations. Han and Kloeden [1] presented the theory of random ordinary differential equations and their relation to Itô stochastic differential equations and developed numerical methods for their solutions. They highlighted important applications, with a focus on dynamical behavior of biological systems. It is observed that the consideration of noise represents a major difficulty in uncertainty analysis. As probability distributions evolve with time, fixed-point distributions and basin boundaries can change due to uncertainty. Lasota and Mackey [2] studied a variety of mathematical systems generating densities focusing on several aspects of stochastic dynamics. They showed that the evolution of such systems is governed by transfer operators of Markov type and thus linear, positive, and mass conserving. Ulam [3] suggested that these transfer operators could be discretized and distributions approximated by histograms, defining what today is known as the Ulam method. Guder and Kreuzer [4] showed that the Ulam method is equivalent to the well-known generalized cell-mapping developed by Hsu [5].

The Hsu cell-mapping theory [5] constitutes the base for several further developments. Hsu and Chiu developed the hybrid cell-mapping approach [6] for the analysis of systems with stochastic and parametric uncertainties. Later, Sun and Hsu [7] applied the generalized cell mapping method to nonlinear random vibration based upon a short-time Gaussian approximation, showing computational advantages over Monte Carlo simulations. Han and coworkers extended the short-time Gaussian approximation to periodically driven systems [8]. Recently Yue et al. [9] proposed a new compatible cell mapping method and developed new algorithms for refining the phase-space to study the global attractors of nonlinear stochastic systems. A dynamical system under Poisson white noise excitation was studied to demonstrate the effectiveness of the proposed method for the probabilistic response analysis. Yue et al. [10] proposed a new conception of composite cell coordinate systems designed with two distinct scales of cell spaces. Global bifurcations, such as crisis and metamorphosis, of the Rayleigh–Duffing oscillator were studied to demonstrate the efficiency of this method. Comparison between the transfer probability distributions obtained by Yue et al. [11] and the generalized committor functions introduced by Lindner and Hellmann

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[12] shows several similarities. The latter describes how the volume of the generalized basin evolves with time.

Global analysis of dynamical systems, especially those in applied mechanics, rely on Poincare's geometric tools, such as trajectories, attractors, and their basins and invariant manifolds. However, there are difficulties in handling real uncertain systems. An efficient tool explored in recent years has been the use of the Perron-Frobenius, Koopman, and Foias operators: infinite-dimensional, linear operators capable of capturing the full nonlinear dynamics. The use of these operators together with the Ulam method, a method to compute finite-dimensional approximations of these infinite-dimensional operators, has shown to be an efficient tool in nonlinear dynamics [13]. Klus et al. [14] stated that information on dynamical systems can be obtained by analyzing the eigenvalues and eigenfunctions of the Perron-Frobenius operator and its dual, the Koopman operator, associated with a dynamical system. They also compared different numerical approximations of these operators. Dellnitz et al. [15] developed a subdivision strategy with box-covering to compute stable and unstable manifolds, as well as global attractors. The algorithm for the numerical approximation of invariant sets was implemented in the software package GAIO (Global Analysis of Invariant Objects) [16]. Further advances comprised the analysis of dynamical systems with parameter uncertainty [17], invariant sets of infinite-dimensional dynamical systems [18], and a set-oriented path-following method for the computation of parameter-dependent attractors [19]. Koltai and coworkers developed an approach for the computation of the domain of attraction of asymptotically stable continuous-time autonomous [20] and nonautonomous systems [21] based on a set-oriented approximation of the original dynamical system by a Markov jump process. Finally, Klus et al. [22], in a review paper, presented several methods to approximate transfer operators and their eigenvalues, eigenfunctions, and eigenmodes, based on transfer operator theory.

Predicting the systems' response can be challenging in systems with highly intertwining basins or fractal boundaries, even in deterministic systems. In such cases, uncertainty is expected to increase complexity and induce global changes in a system, leading to new dynamic phenomena such as jumping between the competing attractors and global bifurcations as basins' merging and basins' instability.

Schenk-Hoppé [23] and Sharma [24] studied the effect of load noise on the bifurcations of the Duffing-van der Pol oscillator. Later, Xu et al. [25] and He et al. [26] investigated the global behavior of the stochastic bifurcations in Duffing and Duffing–Van der Pol oscillators using the generalized cell mapping method. However, basins obtained by the generalized cell mapping [27] followed the same procedure as in the deterministic case. Thus, the basin definition lacks a proper stochastic description. Agarwal et al. [28] investigated the influence of noise on the frequency responses of softening Duffing oscillators through experimental–numerical analyses, observing the shift in jump locations and showed how noise can destroy the hysteresis observed in the response of a nonlinear oscillator without noise. The global dynamics of Duffing-type oscillators with noise was studied by Cui et al. [29], Agarwal et al. [30], and Cilenti and Balachandran [31]. Still, they used a deterministic view for basins of attraction. Orlando et al. [32] and Silva and Gonçalves [33] investigated the effect of noise on the dynamic integrity of Helmholtz-type oscillators. However, a proper definition of integrity measures for uncertain systems is still an open issue.

This paper proposes tools for the determination of nondeterministic attractors and basins. A global dynamic description in terms of the Perron-Frobenius, Koopman, and Foias phase-space operators is presented for both deterministic and nondeterministic cases. For the numerical implementation of these concepts, which is a crucial point to apply theoretical results to practical examples, adaptative strategies within the set-oriented approach of the Ulam method [13] are used. The definition of basins of attraction by Lindner and Hellmann [12] is considered since it is also based on the transfer operators' theory. A hierarchical discretization of the phase-space is employed, refining basins' boundaries and attractors' distributions, which is a key-point in limiting the computational cost. These tools are applied to a specific example to demonstrate the efficiency of the proposed methodologies in exploring the global behavior of noisy systems.

It is pointed out that, also due to length constraints, this work has to be considered a "methodological" contribution, with the main objective of presenting the operator concept and illustrating it with a simple example. Applying the approach to real cases of practical interest, checking all its properties and performances carefully, using dynamical integrity concepts and tools to assess safety [34], and extending to the case of uncertain parameters, is left for future and already planned works [35].

In a global dynamic analysis, the coexisting attractors and their basins are the main tools to understand the system behavior and safety. Here, instead of an algorithmic-based description, such as grid of starts, Monte Carlo, simple and generalized cell-mappings, etc. [36], a phase-space operator description is proposed for both deterministic and stochastic cases.

The major issue caused by nondeterminism is that the two main global structures, attractors and basins, must be redefined. The pushforward definition of attractors, that is, set of states towards which a system evolves as time increases, is usually considered for deterministic systems. However, this definition is not applicable for nondeterministic systems since attractors are not uniquely located in phase-space. The alternative is to address how attractors are distributed over the phase-space. This interpretation is the most natural under the flow-operator analysis, where asymptotic convergence towards the attractor is understood in a distribution sense. An attractor's distribution is, generally, a positive L^1 function, such as Dirac deltas for fixed points of deterministic systems or complex structures for chaotic or nondeterministic attractors. These functions give information on the attractor's density for each point in phase-space, and their integral gives the probability of the attractor being inside the domain of integration.

Informally, the basin of attraction is the set of all initial conditions in phase-space that converge to a particular attractor. The basin definition in the nondeterministic case is much more involved. According to Ochs [37], basins become random sets, dependent on the random space sample. Another definition is given by Lindner and Hellmann [12], where basins of attraction are functions g over the phase-space whose assign a probability to each set of initial conditions x of converging to/remaining in an attractor, up to a mean time-horizon. Therefore, basins are time-dependent with this definition, which is suitable for stochastic systems. Finally, both definitions converge to the classical definition of deterministic systems and can be unified in a general formulation.

In the deterministic case, the linear Perron-Frobenius operator is considered to obtain attractors' distribution, while the Koopman operator is considered to obtain the basins of attraction. In the stochastic case, the linear Foias operator is considered to obtain the attractors' distributions, and the mean Koopman operator is considered to obtain the basins of attraction. A description of these operators can be found in Lasota and Mackey [2], who applied them to the analysis of nonlinear dynamical systems.

The operator methodology is implemented by adopting the Ulam method, where the phase-space is discretized into k disjoint cells, with a time-t transfer matrix approximation of Perron-Frobenius, Foias, and Koopman operators. Although intuitive and relatively simple to implement, the Ulam method inserts artificial (due to the discretization) diffusion over the phase-space, blurring the basins' boundaries [12,20, 21]. This can lead to wrong results

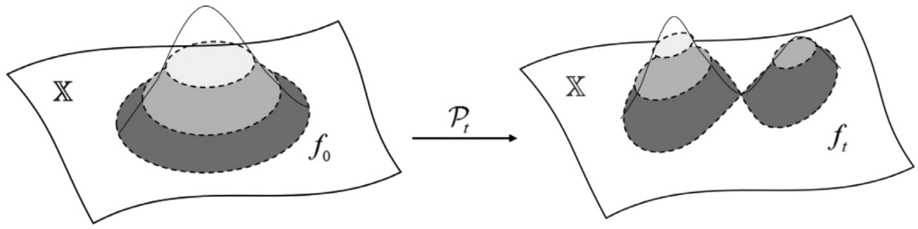

Fig. 1. Evolution of a distribution f under the Perron-Frobenius operator $\mathcal{P}_t$.

in complex dynamical systems with fractal basin boundaries, since uncertain dynamical systems exhibit a natural diffusion. The remedy is to refine the phase-space. However, this increases the computational cost significantly, with possible unimportant regions being refined as well. Thus, adaptative/localized refinement strategies are attractive to mitigate the artificial diffusion to acceptable levels and are computationally advantageous when compared to the total refinement of the phase-space [38,39].

Following Refs. [2,14], the definitions of *flux operators* in the phase-space for deterministic and stochastic cases are presented henceforth.

First, the *deterministic* case is considered. Given dynamical system $(\mathbb{X}, \mathfrak{B}, \mathbb{T}, \varphi_t)$, where $\mathbb{X}$ is a phase-space, $\mathfrak{B}$ is a σ-algebra over $\mathbb{X}$, $\mathbb{T}$ is time, and φ_t is the flow, there is another dynamical system over the space $L^1(\mathbb{X})$, namely the space of distributions f over $\mathbb{X}$,

$$\mathcal{P}_t : L^1(\mathbb{X}) \to L^1(\mathbb{X}),\\ \int_B \mathcal{P}_t[f]\mathrm{d}x = \int_{\varphi_t^{-1}B} f\mathrm{d}x, \qquad \forall B \in \mathfrak{B}, \tag{1}$$

where the function $\varphi_t^{-1}B$ is the preimage of the set $B \in \mathfrak{B}$ given the dynamics φ_t, and $\mathcal{P}_t$ is known as the Perron-Frobenius operator. Stationary solutions f of Eq. (1) are such that $\mathcal{P}_t[f] = f$, and they specify how attractors are localized in the phase-space [2]. For example, fixed points, periodic orbits, and limit-cycle attractors are described by Dirac distributions $\delta_C(.)$ supported on a phase-space set $C \in \mathfrak{B}$ such that

$$\int_A \delta_C(x)\mathrm{d}x = \begin{cases} 0, & C \not\subset A, \\ 1, & C \subseteq A. \end{cases} \tag{2}$$

The Perron-Frobenius operator is linear, positive ($f \geq 0 \Leftrightarrow \mathcal{P}_t f \geq 0$), and non-expansive ($\|\mathcal{P}_t f\|_{L^1} = \|f\|_{L^1}$), being also a Markov operator [2]. Its adjoint (dual) is the Koopman operator $\mathcal{K}_t$, which defines a dynamical system in $L^\infty(\mathbb{X})$, the space of observables g over $\mathbb{X}$,

$$\mathcal{K}_t : L^\infty(\mathbb{X}) \to L^\infty(\mathbb{X}),\\ \mathcal{K}_t[g] = g(\varphi_t x). \tag{3}$$

Stationary solutions of Eq. (3) are given by $\mathcal{K}_t[g] = g$, governing the basins of attraction distributions over the phase-space [12].

The Perron-Frobenius operator $\mathcal{P}_t$ and the Koopman operator $\mathcal{K}_t$ can be understood as a transport of distributions and observables under the flow φ_t, respectively. Given an initial distribution f_0 in the phase-space, $\mathcal{P}_t$ transports it through the dynamics to a new distribution f_t, as depicted in Fig. 1. If an initial observable g_0 in the phase-space is considered, $\mathcal{K}_t$ drives it through the dynamics to a new observable g_t, as depicted in Fig. 2.

The Perron-Frobenius and Koopman operator are related through a dual property, $\langle \mathcal{P}_t f, g\rangle = \langle f, \mathcal{K}_t g\rangle$, which is expanded as [2]

$$\int_{\mathbb{X}} g\mathcal{P}_t[f]\mathrm{d}x = \int_{\mathbb{X}} \mathcal{K}_t[g]f\mathrm{d}x,\\ \forall f \in L^1(\mathbb{X}), g \in L^\infty(\mathbb{X}). \tag{4}$$

The Foias operator is the global operator in the mean sense for stochastic systems [2]. Its construction starts from the definition of the indicator function of a set $B \in \mathfrak{B}$,

$$\mathrm{id}_B(x) = \begin{cases} 0, & x \notin B, \\ 1, & x \in B, \end{cases} \tag{5}$$

which has the following property for any dynamic system,

$$\mathrm{id}_B(\varphi_t x) = \mathrm{id}_{\varphi_t^{-1}B}(x), \tag{6}$$

allowing rewriting Eq. (1) as

$$\int_B \mathcal{P}_t[f]\mathrm{d}x = \int_{\mathbb{X}} \mathrm{id}_B(\varphi_t x) f\mathrm{d}x. \tag{7}$$

The flow $\varphi_t(\omega)$ depends on a probability space $(\Omega, \mathfrak{F}, \mathbb{P}_\omega)$ for random dynamical systems. By taking the mean of Eq. (7) in $(\Omega, \mathfrak{F}, \mathbb{P}_\omega)$, one obtains

$$\mathbb{E}\left\{\int_B \mathcal{P}_t[f]\mathrm{d}x\right\} = \int_\Omega \left\{\int_{\mathbb{X}} \mathrm{id}_B[\varphi_t(\omega)x]f\mathrm{d}x\right\}\mathbb{P}_\omega(\mathrm{d}\omega), \tag{8}$$

and, by changing the order of integration, the Foias operator is obtained,

$$\int_B \mathcal{F}_t[f]\mathrm{d}x = \int_{\mathbb{X}} \left\{\int_\Omega \mathrm{id}_B[\varphi_t(\omega)x]\mathbb{P}_\omega(\mathrm{d}\omega)\right\} f\mathrm{d}x. \tag{9}$$

The notation can be further simplified. Considering the subset of the probability space $\Omega_x(B)$ for which the dynamical system is in B under the flow $\varphi_t(\omega)$ [17],

$$\Omega_x(B) = \{\omega \in \Omega : \varphi_t(\omega)x \in B\}, \tag{10}$$

the final Foias operator is written as

$$\mathcal{F}_t : L^1(\mathbb{X}) \to L^1(\mathbb{X}),\\ \int_B \mathcal{F}_t[f]\mathrm{d}x = \int_{\mathbb{X}} \left\{\int_{\Omega_x(B)} \mathbb{P}_\omega(\mathrm{d}\omega)\right\} f\mathrm{d}x. \tag{11}$$

The Foias operator $\mathcal{F}_t$ is a mean flux of distributions f over the phase-space $\mathbb{X}$ given the probability space $(\Omega, \mathfrak{F}, \mathbb{P}_\omega)$. That is, it governs the mean evolution of distributions f according to the underlying random dynamical system $\varphi_t(\omega)$. Finally, the adjoint operator is obtained by setting

$$\mathcal{K}_t : L^\infty(\mathbb{X}) \to L^\infty(\mathbb{X}),\\ \mathcal{K}_t[g] = \int_\Omega g(\varphi_t(\omega)x)\mathbb{P}_\omega(\mathrm{d}\omega), \tag{12}$$

which is a mean Koopman operator over the probability space $(\Omega, \mathfrak{F}, \mathbb{P}_\omega)$. By inspection of expressions (11) and (12), a duality relation is identified,

$$\int_{\mathbb{X}} g\mathcal{F}_t[f]\mathrm{d}x = \int_{\mathbb{X}} \mathcal{K}_t[g]f\mathrm{d}x,\\ \forall f \in L^1(\mathbb{X}), g \in L^\infty(\mathbb{X}). \tag{13}$$

The *discretization* of transfer operators $\mathrm{P_t}$ and $\mathrm{F_t}$ is given by the Ulam method [23], equivalent to the generalized cell-mapping [4]. Let $\mathbb{B} = \{b_1, b_2, \ldots, b_k\}$ be a disjoint partition of the phase-space $\mathbb{X}$ into k sets. Additionally, consider the subspace $\Delta_h \subset L^1(\mathbb{X})$

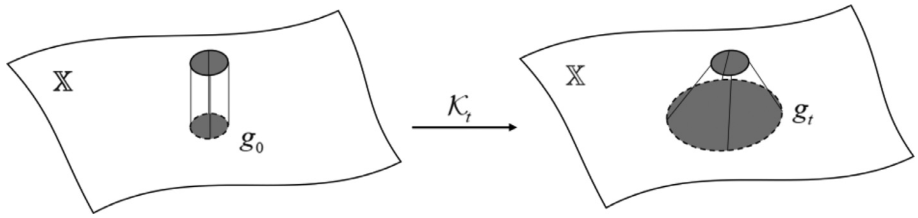

Fig. 2. Evolution of an observable g under the Koopman operator $\mathcal{K}_t$.

spanned by the normalized indicator functions of $\mathbb{B}$, i.e., with basis $\{\mathbf{1}_1, \mathbf{1}_2, \dots, \mathbf{1}_k\}$, where $\mathbf{1}_i = \mathrm{id}_{b_i}/m(b_i)$, $m(b_i)$ is the measure in $\mathbb{X}$ and h is the characteristic size of the partition. With these definitions, a projection operator Q_h is defined such that a distribution $f \in L^1(\mathbb{X})$ is projected onto the subspace Δ_h, that is,

$$Q_h : L^1(\mathbb{X}) \to \Delta_h,$$
$$Q_h f = \sum_{i=1}^{k} \mathbf{1}_i \int_{b_i} f \mathrm{d}x. \tag{14}$$

It is easy to check that $Q_h \mathbf{1}_i = \mathbf{1}_i$, for all $1 \le i \le k$. A projected distribution over Δ_h is generically denominated as $Q_h f = f_h$. Following Ref. [40], the projection of $\mathcal{P}_t$ is defined from the composition of Q_h and $\mathcal{P}_t$ over a distribution $f \in L^1(\mathbb{X})$. The resulting projected operator is $Q_h \mathcal{P}_t = P_h$, that is,

$$P_h : \Delta_h \to \Delta_h,$$
$$f_h P_h = \sum_{i,j=1}^{k} f_i p_{ij} \mathbf{1}_j, \tag{15}$$

where f_i and p_{ij} are

$$f_i = \int_{b_i} f \mathrm{d}x,$$
$$p_{ij} = \frac{m\left(b_i \cap \varphi_t^{-1} b_j\right)}{m(b_i)}. \tag{16}$$

The elements p_{ij} constitute a row-stochastic matrix, while f_i are elements of a row-vector with sum equal to one. We say that the continuous operator $\mathcal{P}_t$ has a matrix representation $p_{ij} \in [0;1]^{k \times k}$ of its projection $P_h \in \Delta_h \times \Delta_h$ [12]. Similarly, the continuous distribution f has a vector representation $f_i \in [0;1]^k$ of its projection $f_h \in \Delta_h$. Therefore, the continuous transport of distributions under the Perron-Frobenius operator $\mathcal{P}_t[f]$, Eq. (16), is represented by a row-vector matrix multiplication $f_i p_{ij}$. Finally, the vector representation $f_i \in [0;1]^k$ of stationary distributions is a solution of the fixed-space eigenvalue problem

$$f_i p_{ij} = f_i \delta_{ij}, \tag{17}$$

where δ_{ij} is the Kronecker delta. It corresponds to the classical attractor definition, as stated in the previous section.

In turn, considering the same subspace of normalized indicator functions $\Delta_h \subset L^1(\mathbb{X})$ and the projection operator $Q_h : L^1(\mathbb{X}) \to \Delta_h$, the projected Foias operator $Q_h \mathcal{F}_t = F_h$ is

$$F_h : \Delta_h \to \Delta_h,$$
$$f_h F_h = \sum_{i,j=1}^{k} f_i p_{ij}^s \mathbf{1}_j, \tag{18}$$

where f_i and p_{ij}^s are

$$f_i = \int_{b_i} f \mathrm{d}x,$$
$$p_{ij}^s = \frac{1}{m(b_i)} \int_{b_i} \left\{ \int_{\Omega_x(b_j)} \mathbb{P}_\omega(\mathrm{d}\omega) \right\} \mathrm{d}x, \tag{19}$$
$$\Omega_x\left(b_j\right) = \left\{\omega \in \Omega : \varphi_t(\omega)x \in b_j\right\}.$$

and the fixed-space eigenvalue problem is

$$f_i p_{ij}^s = f_i \delta_{ij}. \tag{20}$$

Given that the Foias operator governs the mean flow of random dynamical systems, Eq. (18) approximates it. Discretized mean distributions also evolve according to it.

The matrix representation of the projected Koopman operators K_h for both deterministic and stochastic cases is given by the transpose of p_{ij}^s, calculated according to Eqs. (16) or (19), respectively. K_h encodes the basin of attraction in its fixed space, that is, the case $K_h g_h = g_h$ is a basin of an attractor. This relation in matrix representation is given by the left-multiplication$p_{ij}^s g_j = g_i$, whereas the attractor space in matrix representation is given by the right-multiplication $f_i p_{ij}^s = f_j$.

The Hénon map with added noise is considered for application of the preceding formulation. It is described by

$$x_{n+1} = 1 - \mu x_n^2 + y_n + \sigma \xi_n,$$
$$y_{n+1} = J x_n + \sigma \rho_n, \tag{21}$$

where x_n, y_n are state variables, μ, J are control parameters, σ is the noise amplitude, and ξ_n, ρ_n are noise sources. This system can present complex behavior depending on the parameter choice, as demonstrated in Ref. [41]. For $(\mu, J) = (1, -0.9)$, this system displays a period 1 and a period 3 attractor.

The proposed formulation is used together with an adaptative phase-space refinement procedure developed in Refs. [38,39]. The adaptative refinement allows the correct identification of basins' boundaries and attractors' densities, necessary to understand the effects of noise on the global dynamics. The phase-space is taken as the plane $\mathbb{X} = [-1, 1.5] \times [-1.5, 1]$, initially subdivided into 2^{10} disjoint sets (cells). Then, attractors' distributions and basins' boundaries are subdivided up to 8 times, 4 in each dimension. The cell structure is organized as a binary tree, similar to previous developments [16].

Figure 3 presents the results for the deterministic case, while Figs. 4 and 5 present the results for stochastic cases. Complex basin structures are observed, with the period 3 attractor's star-shaped basin surrounding the period 1 attractor's basin, for both deterministic and stochastic cases, Fig. 3a, 3b, 4a, 4b, 5a, and 5b, while an escape region surrounds both basins, Figs. 3c, 4c, and 5c. The color bars from black to yellow or white to black define the probability of convergence to a depicted attractor or escape, given as solution of

$$\left[\delta_{ij} - (1-\varepsilon) p_{ij}^s\right] g_j(\varepsilon) = \varepsilon \, \mathrm{id}_A, \tag{22}$$

for each attractor or escape cell A for a distant time-horizon $1/\varepsilon = 10^8$ [12]. It is expected that, in the deterministic case, g_j converges

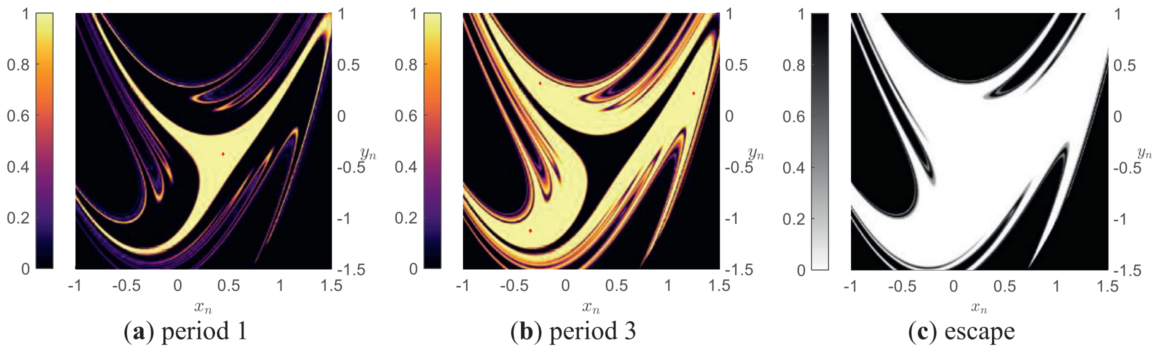

Fig. 3. Hénon global dynamics for $\mu = 1$, J = -0.9, $\sigma = 0$ (deterministic). Color bars: basins of attraction.

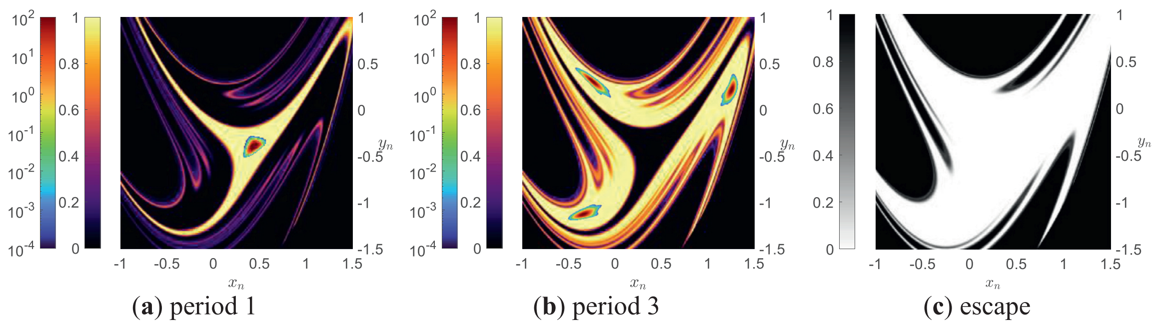

Fig. 4. Hénon global dynamics for $\mu = 1$, J = -0.9, σ = 0.004 (stochastic). Color bars: attractors' densities and stochastic basins of attraction.

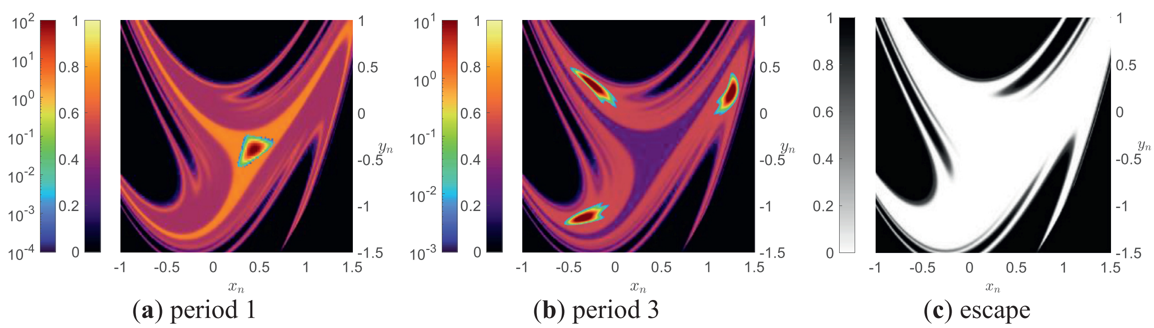

Fig. 5. Hénon global dynamics for $\mu = 1$, J = -0.9, σ = 0.006 (stochastic). Color bars: attractors' densities and stochastic basins of attraction.

to a step function over the basin for $\varepsilon = 0$ and infinity resolution. For stochastic cases, the boundaries are time-dependent and blurred for large noise values. The attractors' distributions for the deterministic case converge to Dirac distributions, here marked as red dots. More general distributions represent the stochastic cases, depicted through the color bar from blue to red, spreading over the phase-space. This diffusion of both basins and attractors is a consequence of the noise inclusion [39]. The last stochastic case, Fig. 5 with σ = 0.006, demonstrates a degeneration of the basins for large noise values. The basins no longer have regions with 100% certainty of convergence, evolving to any of the two attractors with a given probability. It is worth noting that the vanishing escape probability of the deterministic case persists in the whole stochastic regime because no escape tongues enter the two attractors' deterministic region, Fig. 3a and 3b. Attractors' distributions are also affected, spreading over larger regions in phase-space.

An operator methodology for the global analysis of stochastic dynamical systems has been proposed and illustrated with a simple example. The classical Ulam method was modified to accommodate random dynamical systems. When applied to such systems, the modified method results in a discretization of the Foias transfer operator, which governs the flow map in the mean sense. The discretization scheme is the most natural method within the uncertainty framework, having already been applied to many random dynamical systems. Finally, a Hénon map with two attractors, a period 1 and a period 3, and an escape solution, has been dealt with as an example. Three cases have been studied, one deterministic and two stochastic, revealing the complex basin structure. The noise causes diffusion of both attractors' densities and basins' boundaries. Large noise amplitudes break the separation between the two attractors, with basins no longer presenting regions with 100% of certainty of convergence to a given attractor.

Future developments will focus on a real case of practical interest, using dynamical integrity concepts and extending to the case of uncertain parameters [35].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The raw data required to reproduce these findings are available on request. The processed data required to reproduce these findings are available on request.

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