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# A new class of composite indicators: The penalized power mean

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ABSTRACT

In this paper we propose a new aggregation method for constructing composite indicators based on a penalization of the power mean. The idea underlying this approach consists in multiplying the power mean by a factor that accounts for the horizontal heterogeneity among indicators while penalizing units with a larger heterogeneity. In line with the minimum loss of information principle, the penalization factor proposed is proven to be linked to the loss of information generated when the indicators are substituted with their power means. As a consequence, the aggregation approach gives rise to the class of penalized power means and the penalized Benefit of the Doubt aggregative approach. Including heterogeneity makes the aggregation approach more suitable for refined rankings. Interestingly, the penalized power mean of order one coincides with the Mazziotta Pareto Index. Some theoretical properties of the penalized power means are proven, thus supporting the Mazziotta Pareto index. An empirical analysis of the Human Development Index in 2019 is presented. Comparisons of the rankings induced by the penalized and non-penalized Benefit of the Doubt and power mean aggregation approaches are shown. There are three main findings: the penalized power means satisfy the properties characterizing weakly monotone aggregation functions; the penalization reduces ranking variations while differentiating units with close means; and the geometric mean provides composite indicators whose ranking is closest to those obtained with power means of different order.

## 1. Introduction

The construction of composite indicators consists in reducing a multidimensional phenomenon to a one-dimensional phenomenon aggregating the multiple dimensions, namely the indicators, into a single indicator called the composite indicator. The resulting composite indicator, although simpler and easier to interpret and understand, is less informative with respect to the vector of the indicators. That is, the aggregation procedure involves a loss of information. Despite the effort of many authors to develop objective measures of information loss (see, among others, Zhou, Ang, and Poh (2006), Zhou and Ang (2009), Zhou, Fan, and Zhou (2010)), choosing an effective measure is a crucial task and depends on the subjective preference of the decision maker.

A good aggregation should conjugate the contrasting twofold objectives of reducing the dimension of the phenomenon under investigation while generating a reasonable loss of information. The majority of aggregation functions are based on minimizing the loss due to replacing the indicators with the aggregated value. Specifically, the power means are found by choosing the Euclidean distance from the vector of indicators transformed through a Box–Cox function as a penalty (loss) function (for more details we refer to Berger and Casella (1992)). In other words, the power mean can be viewed as the least-squares estimate of the vector of indicators in the Box–Cox transformed space. Therefore, in the transformed space, the power mean suffers from the same drawbacks as the arithmetic mean, that is, the compensability and the substitutability. These issues should be considered in the aggregation phase to differentiate units with same power mean value and, as a consequence, to make rankings robust to the aggregation procedure. All these methods share the idea that the importance of an indicator should be related to the level of information it brings along.

The idea of penalizing units in different ways is shared by Mauro, Biggeri, and Maggino (2018) and Biggeri, Clark, Ferrannini, and Mauro (2019) who developed and applied the Multidimensional Synthesis of Indicators (MSI) approach to well-being, aggregating the indicators relative to different units with power means of different order. The order of the power mean is assumed to be a function of the arithmetic mean of the indicators, such that units with a lower indicator of arithmetic mean are associated with a lower order.

Surely, the choice of meaningful aggregative process relies on an appropriate choice of the aggregation and weighting schemes. Moreover, despite the fact that often they are implemented together, the aggregation and the weighting issues attain different and, in some ways, complementary aspects. In fact, the aggregation is related to the choice of a metric that quantifies the relationship or similarity between

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indicators. For example, the choice of the order $p$ in the power mean approach is a metric choice. In fact, the power means of negative order penalize downwards the unbalance among indicators, emphasizing the improvements of the indicators with low values while penalizing those with high values. This is due to the fact that the metrics for negative orders reduce the distances between indicators with high values and increase the distances between indicators with low values. On the other hand, the choice of weights reflects the relative importance/relevance of the indicators to the overall objective of the analysis. This can be done a priori or a posteriori using data-driven methods. In the latter, the contribute of weighting to the construction of composite indicators is constrained to and limited by the choice of the aggregation method.

In the scientific literature, there are many papers addressing weighting methods. Just to mention a few, the papers Karagiannis and Karagiannis (2020), Karagiannis and Karagiannis (2023), Greco, Ishizaka, Tasiou, and Torrisi (2019), Curry and Faulds (1986), Kopalle and Hoffman (1992).

In Karagiannis and Karagiannis (2020), Karagiannis and Karagiannis (2023) several weighting methods based on information theory and completely data-driven are proposed. Specifically, in Karagiannis and Karagiannis (2020) the weighted arithmetic mean approach is used to aggregate the indicators and the Shannon entropy is used to derive a set of common weights. The idea is to assign more importance to the indicators that provide more information and, as a consequence, lower value of uncertainty, measured through the Shannon entropy. In Karagiannis and Karagiannis (2023) the authors determine the weights endogenously using four distance-based methods: the maximizing deviations, the weighted least square deviation from the mean, the weighted least square deviation from the ideal, and the weighted least square dissimilarity.

The evaluation of the uncertainty in the weights for composite indicators is a key ingredient of Greco et al. (2019). The problem is addressed by means of the Stochastic Multiattribute Acceptability Analysis. This analysis allows for the computation of the probability that a unit attains a given ranking position and the probability that a unit is better than another. In particular, following the weighted arithmetic mean aggregation approach, the authors compute, for each unit, the probability distribution of weights, its mean ($\mu$) and standard deviation ($\sigma$). Moreover, they construct the "efficient frontier" in the plane $\sigma - \mu$. The efficient frontier is made by the units that provide the best trade-off between standard deviation and mean.

In Curry and Faulds (1986), Kopalle and Hoffman (1992) the authors investigate the sensitivity to weights of the composite indicators by theoretical and computational points of view. This is done comparing the correlations among ranking associated with composite indicators obtained as arithmetic means with different set of weights. From the analysis it emerges that positively correlated indicators are less sensitive to the choice of weights with respect to negatively correlated indicators. For an extensive and detailed guide to the construction and use of composite indicators we refer to OECD (2008).

In this paper, the focus is the aggregation scheme and a generalization to include weights. Specifically, in line with the works of Mauro et al. (2018), Biggeri et al. (2019), Rogge (2018a) and Rogge (2018b), we propose a new aggregation approach that penalizes the power mean associated with units characterized by larger heterogeneity. The penalization consists in a factor which multiplies the mean and accounts for the horizontal heterogeneity among indicators. We build this factor for each unit, first by computing the power mean of the indicators associated with the unit. Second, we scale the indicators by their power mean. Third, we apply the Box–Cox transformation to the scaled indicators. Finally, we compute the variance of the transformed indicators and the counter image of this variance using the Box–Cox function. The counter image is the resulting penalization factor. This factor is a kind of "variance" in the transformed space and, therefore, can be interpreted as the relative error or loss of information associated with the $i$th unit as the power mean is substituted for the vector of the transformed indicators. Interestingly, the penalized power mean of order one coincides with the Mazziotta Pareto Index (Mazziotta & Pareto, 2016).

Moreover, we implement a Benefit of the Doubt (BoD)-based weighted version of the penalized power means for constructing an Human Development Index (HDI) (for more details, see Rogge (2018a) and Rogge (2018b). The BoD-based weights are considered with the aim of showing that the proposed aggregation is less sensitive to the choice of weights with respect to alternative approaches. This conclusion is motivated by the conjecture that the penalization acts as a unit-dependent system of weights, neutralizing partially the importance attributed by the weights to each indicator, and, as a consequence, originating a more robust composite indicator. It should be noted that the BoD weights are derived solving an optimization problem, and, for this reason, they are not always unique. The non-uniqueness of BoD weights is a crucial task extensively investigated by several authors (see, among others, Cooper, Ruiz, and Sirvent (2007)). Despite their efforts, to the best of our knowledge, there are not standard selection procedures among the multiple optima. Here, the use of BoD approach is only illustrative to study the effects of weights on the penalized power means in comparison with their non-penalized versions. Therefore, although the analysis of the robustness of our approach to different set of weights, associated with multiple optima, deserves attention and further investigation, it falls beyond the scope of paper.

The penalization approach proposed in this paper is fully data-driven and can be effectively applied in many other fields, such as environmental indices (Sadiq, Haji, Cool, & Rodriguez, 2010), fuzzy rule-based systems, pattern recognition, decision-making problems (Khameneh & Kilicman, 2020), and weighted voting systems (Bustince, Jurio, Pradera, Mesiar, & Beliakov, 2013). We also investigate whether it is possible to select the order of the mean to obtain an aggregative approach that provides composite indicators whose induced unit rankings are less "sensitive" to the order $p$ used in the aggregation. The latter is a very delicate issue because recent investments in innovation and green policies have imposed the introduction of "reliable and robust" ranking to make decisions.

Furthermore, the penalization approach also extends to weighted power means, which are therefore applicable to the BoD approach as illustrated in Sections 3 and 4.

There are three main findings: the penalized power means satisfy the properties characterizing weakly monotone aggregation functions; the penalization reduces ranking variations while differentiating units with same means; and the zero-th order (i.e., geometric mean) provides composite indicators whose rankings are the closest to rankings obtained by different power mean approaches, in that the zero-th order mean minimizes the variation of the rankings with a set of order $p$. Interestingly, the penalized power means are power means applied to penalized indicators.

The paper is organized as follows. Section 2 introduces the penalized power means and some properties of this family are proven. Section 3 extends the penalization to weighted power means and Section 4 proposes an empirical analysis of the HDI in 2019, comparing the country rankings of the penalized/non-penalized power mean aggregation to penalized/non-penalized BoD aggregation. Section 5 draws some conclusions. Appendix A provides some auxiliary results necessary to prove that the penalized power means are weakly monotone aggregation functions. Appendix B contains the proofs of the main propositions of Section 2. Appendix C contains the detailed tables of ranking relative to the numerical experiments of Section 4. Finally, Appendix D analyzes the effect of penalization on BoD approach when relative importance constraints on the weights are imposed.

## 2. A new class of composite indicators

Let $I_{i,j}$ be the value of the indicator $j$, $j = 1, 2, \dots, m$, relative to unit $i$, $i = 1, 2, \dots, n$, such that $I_{i,j}$ belongs to the interval $[a, b]$, where $b > a > 0$. Let the superscript $^T$ denote the transpose operator and $\underline{I}_i = [I_{i,1}\, I_{i,2}\, \dots\, I_{i,m}]^T$ the (column) vector of indicators relative to the $i$−th unit, $i = 1, 2, \dots, n$. For $i = 1, 2, \dots, n$, the power mean of order $p$ associated to $\underline{I}_i$ is defined by

$$M_{p,i} = \begin{cases} \left( \dfrac{1}{m} \displaystyle\sum_{j=1}^{m} I_{i,j}^p \right)^{\frac{1}{p}}, & p \neq 0, \\ \left( \displaystyle\prod_{j=1}^{m} I_{i,j} \right)^{\frac{1}{m}}, & p = 0. \end{cases} \tag{1}$$

The arithmetic mean, geometric mean, and harmonic mean are special cases of the power mean for $p = 1$, $p = 0$, and $p = -1$, respectively.

For $i = 1, 2, \dots, n$ the composite indicator $M_{p,i}$ can be read as solution to the following optimization problem (Berger & Casella, 1992):

$$\min_{c>0} F_p(c; \underline{I}_i), \tag{2}$$

where:

$$F_p(c; \underline{I}_i) = \frac{1}{m} \sum_{j=1}^{m} (h_p(I_{i,j}) - h_p(c))^2 \tag{3}$$

is the (information) loss function (or, the penalty function, to use the nomenclature of Calvo and Beliakov (2010)), and $h_p(x)$ is the Box–Cox transformation (Box & Cox, 1964):

$$h_p(x) = \begin{cases} \dfrac{x^p - 1}{p}, & p \neq 0, \\ \ln x, & p = 0. \end{cases} \tag{4}$$

Note that, for any $p$, the function $h_p(x)$, $x > 0$, is strictly increasing and satisfies the condition $h_p(1) = 0$.

The claim mentioned above follows from two simple observations.

First, for $i = 1, 2, \dots, n$, the solution to problem (2) is the constant $c$, such that $h_p(c)$ is the arithmetic mean of values $h_p(I_{i,1})$, $h_p(I_{i,2})$, …, $h_p(I_{i,m})$. The values $h_p(I_{i,j})$, $j = 1, 2, \dots, m$, are interpreted as the values of the statistical (latent) variable $Y_i$ which describes the unit $i$ in the "transformed space". For later convenience, these values are collected in the vector $h_p(\underline{I}_i) = [h_p(I_{i,1})\, h_p(I_{i,2})\, \dots\, h_p(I_{i,m})]^T$.

Second, the arithmetic mean of the values $h_p(\underline{I}_i)$, satisfies

$$M_1(h_p(\underline{I}_i)) = \frac{1}{m} \sum_{j=1}^{m} h_p(I_{i,j}) = h_p(M_{p,i}), \quad i = 1, 2, \dots, n. \tag{5}$$

Eq. (5) allows us to conclude that the optimal value of $c$, the solution to problem (2), is $c = M_{p,i}$.

Therefore, in the $p$-transformed space – the space obtained by transforming the $m$ dimensional vectors via the Box–Cox function of order $p$ – the $p$-order generalized mean acts as the arithmetic mean.

Moreover, for any unit $i$, $i = 1, 2, \dots, n$, we can measure the error (loss of information) caused by substituting the transformed values of the indicators, $h_p(I_{i,j})$, $j = 1, 2, \dots, m$, with $h_p(M_{p,i})$, evaluating the objective function $F_p$ at its optimizer:

$$F_p(M_{p,i}; \underline{I}_i) = \frac{1}{m} \sum_{j=1}^{m} (h_p(I_{i,j}) - h_p(M_{p,i}))^2, \quad i = 1, 2, \dots, n. \tag{6}$$

It is easy to see that for any unit $i$, $i = 1, 2, \dots, n$, this error coincides with the (biased) sample variance of a statistical variable, $Y_i$, whose values are given in the vector $h_p(\underline{I}_i)$. Here and in the rest of paper, we denote this variance with $S^2_{p,i}$.

Therefore, for $i = 1, 2, \dots, n$, the quantity $h_p^{-1}(S^2_{p,i})$ is a measure of the information loss caused by replacing $\underline{I}_i$ with $M_{p,i}$. Note that the size of $h_p^{-1}(S^2_{p,i})$ strongly depends on $M_{p,i}$; hence, variances of the unit $i$ computed with different $p$-orders are not comparable.

We overcome this drawback by scaling the indicators referring to the same unit by a specific criterion that removes the dependence from the power mean. To this end, we consider the vector of scaled indicators, $\underline{\tilde{I}}_i = [\tilde{I}_{i,1}\, \tilde{I}_{i,2}, \dots\, \tilde{I}_{i,m}]^{\top}$, where

$$\tilde{I}_{i,j} = \frac{I_{i,j}}{M_{p,i}}, \quad j = 1, 2, \dots, m, \quad i = 1, 2, \dots, n. \tag{7}$$

Bearing in mind that $h_p(1) = 0$, it follows from the homogeneity property of the $p$-order power means that $M_p(\underline{\tilde{I}}_i) = 1$ and $h_p(M_p(\underline{\tilde{I}}_i)) = h_p(1) = 0$ for $i = 1, 2, \dots, n$. As a consequence, the error (loss of information) caused by substituting the vector of indicators $h_p(\underline{\tilde{I}}_i)$ with $h_p(M_p(\underline{\tilde{I}}_i)) = h_p(1) = 0$ is given by

$$L_{p,i} = \frac{1}{m} \sum_{j=1}^{m} (h_p(\tilde{I}_{i,j}) - h_p(1))^2 = \frac{1}{m} \sum_{j=1}^{m} [h_p(\tilde{I}_{i,j})]^2, \quad i = 1, 2, \dots, n. \tag{8}$$

Note that, independent of the fact that the optimizer of problem (2) for the vector $\underline{I}_i$ is the p-order power mean $M_{p,i}$, scaling the indicator by the optimizer (as done in (7)) makes the optimizer of problem (2) for the resulting scaled vector $\underline{\tilde{I}}_i$ equal to 1, $i = 1, 2, \dots, n$. This means that the corresponding loss of information is equal to the $L^2$ norm of vector $h_p(\underline{\tilde{I}})$ (as in (8)).

However, the choice of considering scaled indicators does not guarantee that the loss of information ranges in a given interval. To overcome this difficulty, we define:

$$\tilde{S}^2_{p,i} := K L_{p,i}, \quad i = 1, 2, \dots, n, \tag{9}$$

where $K > 0$ is a real constant. Commonly, the choice $K$ equal to 1 is strongly recommended except for some circumstances explained in Definition 1.

For $i = 1, 2, \dots, n$, the quantity

$$h_p^{-1}(\tilde{S}^2_{p,i}) = \begin{cases} \left(1 + p\, \tilde{S}^2_{p,i}\right)^{\frac{1}{p}}, & p \neq 0, \\ \exp\left(\tilde{S}^2_{0,i}\right), & p = 0, \end{cases} \tag{10}$$

is independent of the size of $\tilde{M}_{p,i}$. It measures the relative information loss caused by replacing $\underline{I}_i$ with $M_{p,i}$. The higher the value of $h_p^{-1}(\tilde{S}^2_{p,i})$, the greater the loss of information caused by considering $M_{p,i}$ instead of the sub-indicator vector $\underline{I}_i$.

We use $h_p^{-1}(\pm \tilde{S}^2_{p,i})$ to penalize the power mean of order $p$. Specifically, for $i = 1, 2, \dots, n$, the penalized power mean of order $p$ associated with the indicator vector $\underline{I}_i$ is defined by

$$PM^{\pm}_{p,i} = M_{p,i}\, g^{\pm}_{p,i}, \tag{11}$$

where $g^{\pm}_{p,i}$ is the penalization factor:

$$g^{\pm}_{p,i} = h_p^{-1}(\pm \tilde{S}^2_{p,i}) = (1 \pm p\, \tilde{S}^2_{p,i})^{\frac{1}{p}} = \begin{cases} \left(1 \pm p\, \tilde{S}^2_{p,i}\right)^{\frac{1}{p}}, & p \neq 0, \\ \exp\left(\pm \tilde{S}^2_{0,i}\right), & p = 0. \end{cases} \tag{12}$$

The sign in (11), (12) depends on the phenomenon considered. Specifically, if increasing variations of the indicator correspond to positive variations of the phenomenon (positive polarity), we choose the sign −, otherwise (negative polarity) we choose the sign +.

Furthermore, the scaling in (7) ensures that the term $h_p^{-1}(\pm \tilde{S}^2_{p,i})$ penalizes the score of each unit (the $p$-order power mean of the indicators) independent of the value of the power mean itself with a quantity that is directly proportional to the "horizontal variability" of the indicators. The aim of the penalization is to favor the units that, given identical power means, are more balanced among the indicators. This is the idea underlying the "Method of Penalties by Coefficient of

Variation”, introduced by Mazziotta and Pareto (2016), which adjusts the arithmetic mean by a penalization coefficient that, for each unit, is a function of the coefficient of variation defining the Mazziotta Pareto Index (MPI). Indeed, the indicator in (11) when $p = 1$ is the Mazziotta Pareto Index, thereby setting in the new class of penalized power means on firm foundations.

We now take a deeper look at the behavior of the penalization term $g^-_{p,i}$ (i.e., positive polarity) as a function of the order $p$, for $i = 1, 2, \dots, n$. Bearing in mind that $\tilde{S}^2_{p,i}$ is nonnegative, the penalization factor $(1 - p\,\tilde{S}^2_{p,i})^{\frac{1}{p}}$ is positive only if the inequality $p\,\tilde{S}^2_{p,i} \leq 1$ holds. The latter may or may not hold depending on the sign of $p$. In fact, we only prove that $\tilde{S}^2_{p,i}$ is a positive non-increasing function of $p$. Thus, for $p > 0$ the monotonicity $\tilde{S}^2_{p,i}$ just mentioned, along with the properties $\lim_{p\to+\infty} p\tilde{S}^2_{p,i} = 0$ and $\lim_{p\to 0^+} p\tilde{S}^2_{p,i} = 0$, imply that the inequality $p\,\tilde{S}^2_{p,i} \leq 1$ holds for sufficiently large and small values of $p$.

In contrast to the case $p > 0$, when $p$ is zero or negative, the penalization term is always nonnegative. Furthermore, when $p < 0$ the penalization term, $(1 - p\,\tilde{S}^2_{p,i})^{\frac{1}{p}}$, is smaller than one since the exponent $1/p$ is negative and the base of the power, $1 - p\,\tilde{S}^2_{p,i}$, is greater than one. This implies that no constraints are necessary on the magnitude of $p\tilde{S}^2_{p,i}$ when $p < 0$. In the case $p = 0$, the penalization is less than one for positive polarity $\exp(-\tilde{S}^2_{0,i})$.

For negative polarity of the sub-indicators, the penalization, $(1+p\,\tilde{S}^2_{p,i})^{\frac{1}{p}}$, $p \neq 0$ and $\exp(\tilde{S}^2_{0,i})$, $p = 0$, is larger than one, since higher values of the composite indicator indicate lower ranking positions due to the negative polarity.

We now provide a formal definition of the penalized power mean of order $p$ as a weakly monotone aggregation function (see Proposition 4) and we prove some properties. To simplify the notation, in the rest of this section, where it is not necessary, we drop the subscript $i$.

**Definition 1** (*Penalized Power Mean*). Given a non-empty interval $[a, b] \subseteq (0, +\infty)$ and the vector of indicators $\underline{I} = [I_1, I_2, \dots, I_m]^T \in [a, b]^m$, the penalized power means of order $p$ of $\underline{I}$ are the functions

$$PM^{\pm}_p : [a, b]^m \to [a, b]$$

defined by

$$PM^{\pm}_p(\underline{I}) = M_p(\underline{I})\, g^{\pm}_p(\underline{I}), \tag{13}$$

where

$$M_p(\underline{I}) = \begin{cases} \left(\dfrac{1}{m}\displaystyle\sum_{j=1}^{m} I_j^p\right)^{\frac{1}{p}}, & p \neq 0, \\[2ex] \left(\displaystyle\prod_{j=1}^{m} I_j\right)^{\frac{1}{m}}, & p = 0, \end{cases} \tag{14}$$

is the power mean of order $p$ of $\underline{I}$,

$$g^{\pm}_p(\underline{I}) = (1 \pm p\tilde{S}^2_p(\underline{I}))^{\frac{1}{p}} \tag{15}$$

is the penalization factor associated with $\underline{I}$,

$$\tilde{S}^2_p(\underline{I}) = \frac{K}{m}\sum_{j=1}^{m}\left[h_p\left(\frac{I_j}{M_p(\underline{I})}\right)\right]^2, \tag{16}$$

where $K = 1$ unless a different choice of $K$ is necessary to preserve the range invariance.

We now establish the following elementary features of the penalized means when different units are compared.

**Proposition 1.** *The penalized power mean defined in (11) satisfies the following properties:*

1. $(PM^+_{p,i})^p = (PM^-_{p,i})^p + 2p(M_{p,i})^p\tilde{S}^2_{p,i}$ *for* $p \neq 0$.
2. $PM^+_{0,i} = PM^-_{0,i}\exp\left\{2\,\tilde{S}^2_{0,i}\right\}$.
3. *Given two units $k$ and $h$ ($k \neq h$) with $M_{p,k} = M_{p,h}$, we have*
$$PM^-_{p,k} > PM^-_{p,h} \quad \text{iff} \quad \tilde{S}^2_{p,h} > \tilde{S}^2_{p,k},$$
$$PM^+_{p,k} > PM^+_{p,h} \quad \text{iff} \quad \tilde{S}^2_{p,k} > \tilde{S}^2_{p,h}.$$
4. *Given two units $k$ and $h$ ($k \neq h$) with $M_{p,k} > M_{p,h}$, for $p \neq 0$, we have*
$$PM^-_{p,k} > PM^-_{p,h} \quad \text{iff} \quad M^p_{p,k} - M^p_{p,h} > p\left(M^p_{p,k}\,\tilde{S}^2_{p,k} - M_{p,h}\,\tilde{S}^2_{p,h}\right),$$
$$PM^+_{p,k} > PM^+_{p,h} \quad \text{iff} \quad M^p_{p,k} - M^p_{p,h} > p\left(M^p_{p,h}\,\tilde{S}^2_{p,h} - M^p_{p,k}\,\tilde{S}^2_{p,k}\right).$$
5. *Given two units $k$ and $h$ ($k \neq h$) with $M_{0,k} > M_{0,h}$, we have*
$$PM^-_{0,k} > PM^-_{0,h} \quad \text{iff} \quad \frac{M_{0,k}}{M_{0,h}} > \exp\left\{\tilde{S}^2_{0,k} - \tilde{S}^2_{0,h}\right\},$$
$$PM^+_{0,k} > PM^+_{0,h} \quad \text{iff} \quad \frac{M_{0,k}}{M_{0,h}} > \exp\left\{\tilde{S}^2_{0,h} - \tilde{S}^2_{0,k}\right\}.$$

**Proof.** The proof follows easily from definition (11).

The following results investigate the properties of the penalized mean of order $p$ to investigate whether the term “mean” is used appropriately.

The main result is Proposition 4, where we show that the penalized power means satisfy the properties necessary to be an appropriate aggregative tool for composite indicators.

Before presenting the main proposition, we observe that the loss of information associated with the vector $\underline{I}$, $\tilde{S}^2_p(\underline{I})$, appearing in formula (16) does not increase when all the indicators are translated by the same constant.

**Proposition 2.** *Let $\underline{c} = [c, c, \dots, c]^T \in \mathbb{R}^m$ be the vector with entries equal to a constant $c > 0$. The derivative of $\tilde{S}^2_p(\underline{I} + \underline{c})$ with respect to $c$ is nonpositive, that is, $\tilde{S}^2_p(\underline{I} + \underline{c})$ is a non-increasing function of $c$.*

**Proof.** See Appendix B.

Note that Proposition 2 says that by translating the indicators, we reduce the effect of the penalization. Thus, transformations of the original indicators involving translation affects the penalized means.

We now illustrate the properties of the penalized power mean.

**Proposition 3.** *The penalized power means of order $p$, $PM^{\pm}_p(\underline{I})$ defined in (13) satisfy the following properties:*

1. $PM^{\pm}_p(\underline{c}) = c$, *where* $c > 0$,
2. $\lim_{c\to 0} PM^{\pm}_p(\underline{c}) = 0$, *where* $\underline{c} = [c, c, \dots, c]^T$, *such that* $c > 0$,
3. $PM^+_p(\underline{I}) \geq M_p(\underline{I}) \geq PM^-_p(\underline{I})$,
4. $PM^+_p(\underline{I}) = PM^-_p(\underline{I}) = M_p(\underline{I})$ *iff* $\tilde{S}^2_p(\underline{I}) = 0$,
5. $\lim_{p\to-\infty} PM^{\pm}_p(\underline{I}) = \min_{j=1,2,\dots,m} I_j$,
6. $\lim_{p\to+\infty} PM^{\pm}_p(\underline{I}) = \max_{j=1,2,\dots,m} I_j$,
7. $\lim_{p\to 0} PM^{\pm}_p(\underline{I}) = \exp\left\{\pm\tilde{S}^2_0(\underline{I})\right\}$,
8. $PM^-_p(\underline{I}) \leq \max(\underline{I})$,
9. $\min(\underline{I}) \leq PM^+_p$,
10. $PM^{\pm}_p(c\,\underline{I}) = c\,PM^{\pm}_p(\underline{I})$ *for any $c > 0$ such that $c\underline{I} \in [a, b]^m$,*
11. $PM^-_p(\underline{I} + \underline{c}) \geq PM^-_p(\underline{I})$ *for any $\underline{c} = [c, c, \dots, c]^\top$ such that $c \geq 0$ and $\underline{I} + \underline{c} \in [a, b]^m$,*
12. $PM^+_p(\underline{I} + \underline{c}) \leq PM^+_p(\underline{I})$ *for any $\underline{c} = [c, c, \dots, c]^\top$ such that $c \geq 0$ and $\underline{I} + \underline{c} \in [a, b]^m$,*
13. $a \leq PM^{\pm}_p(\underline{I}) \leq b$.

**Proof.** See Appendix B.

Note that Property 13 guarantees that the penalized power mean are range preserving. Moreover, Properties 5 and 6 in Proposition 3 imply that the penalization has no effect when the power mean of order $-\infty$ (i.e., the minimum function) or $+\infty$ (i.e., the maximum function) is considered. In fact, in the case of positive polarity and negative polarity, the minimum and maximum functions, respectively, already lead to the maximum penalization for unbalanced values of the indicators, so they need no further penalizations. Properties 11 and 12 state that the penalized power means exhibit weakly monotonic behavior (for more details about weakly monotonicity and the definition of aggregation function we refer, respectively, to Wilkin, Beliakov, and Calvo (2014), Grabisch, Marichal, Mesiar, and Pap (2011)).

**Proposition 4.** *The penalized power means (13) are weakly monotone aggregation functions.*

**Proof.** The proof follows easily from Properties 1, 2, 11, 12 and 13 of Proposition 3.

Note that the penalized power means (13) associated with the vector of indicators $\underline{I}$ are power means applied, respectively, to the vector of penalized indicators $\underline{J}^{\pm} = [J_1^{\pm}\; J_2^{\pm}\; \dots\; J_m^{\pm}]^T$, where:

$$J_j^{\pm} = I_j g_p^{\pm}, \quad j = 1, 2, \dots, m.$$

Finally, as mentioned in the Introduction, we prove that the penalized mean of order one is the Mazziotta Pareto Index already used by the Italian central statistical bureau.

**Proposition 5.** *The penalized power mean of order one, $PM_1^{\pm}(\underline{I})$, coincides with the MPI.*

**Proof.** Substituting (4) for $p = 1$ in (16) and bearing in mind that $K = 1$, we have

$$\tilde{S}_1^2(\underline{I}) = \frac{1}{m}\sum_{j=1}^{m}(\tilde{I}_j - 1)^2 = \frac{1}{m}\sum_{j=1}^{m}\left(\frac{I_j}{M_1(\underline{I})} - 1\right)^2 = \frac{\frac{1}{m}\sum_{j=1}^{m}(I_j - M_1(\underline{I}))^2}{M_1^2(\underline{I})} = \frac{S_1^2(\underline{I})}{M_1(\underline{I})^2}, \tag{17}$$

where

$$S_1^2(\underline{I}) = \frac{1}{m}\sum_{j=1}^{m}(I_j - M_1(\underline{I}))^2 \tag{18}$$

is the (biased) sample variance of vector $\underline{I}$.

Substituting (17) into (15) for $p = 1$, we have

$$PM_1^{\pm}(\underline{I}) = M_1(\underline{I})\left(1 \pm \frac{S_1^2(\underline{I})}{M_1(\underline{I})^2}\right), \quad i = 1, 2, \dots, n, \tag{19}$$

that is, the MPI.

## 3. From penalized power means to penalized weighted power means

In this section, we introduce the penalization for the weighted power mean aggregation approach. This extension allows us to apply the penalization to the composite indicators obtained via the "direct" Benefit of the Doubt (BoD) approach illustrated in Cherchye, Moesen, Rogge, and Van Puyenbroeck (2007), Rogge (2018a) and Rogge (2018b). The weights of the BoD composite indicators are country-specific, so these indicators are flexible in capturing differences among the countries. As mentioned in the Introduction, the main focus of the paper is the presentation of the penalized power mean approach and its weighted version. The inclusion of weights is done for studying the sensitivity of the penalized power means to the choice of weights in comparison with their non-penalized versions. To this end, we use the Benefit of the Doubt (BoD) approach to illustrate this study.

For $i = 1, 2, \dots, n$, the weighted power mean of order $p \in \mathbb{R}$ is defined as follows:

$$M_{p,i}^{w} = M_p^{w}(\underline{I}_i) = \begin{cases} \left(\sum_{j=1}^{m} w_{i,p,j} I_{i,j}^{p}\right)^{\frac{1}{p}}, & p \neq 0, \\ \prod_{j=1}^{m} I_{i,j}^{w_{i,0,j}}, & p = 0, \end{cases} \tag{20}$$

where the weights $w_{i,p,j}$ are non-negative and depend on the unit $i$ (country), the order of power mean $p$, and the sub-indicator $j$.

Assuming that the weights sum to one, i.e., $\sum_{j=1}^{m} w_{i,p,j} = 1$, we introduce the "penalized" weighted power mean of order $p$, viewing the pairs $(h(I_{i,1}), w_{i,p,1})$, $(h(I_{i,2}), w_{i,p,2})$, …,$(h(I_{i,m}), w_{i,p,m})$ as the frequency distribution of a statistical variable, $Y_i$, which describes the "$i$th unit".

It is easy to prove that

$$h_p(M_{p,i}^{w}) = M_1^{w}(h_p(\underline{I}_i)) = \sum_{j=1}^{m} w_{i,p,j} h_p(I_{i,j}), \quad i = 1, 2, \dots, n, \tag{21}$$

that is, as in Section 2, in the $p$–transformed space, the $p$-order weighted generalized mean acts as the weighted arithmetic mean.

Hence, arguing as in Section 2, for any unit $i$, we measure the loss of information that originates from replacing the vector of indicators $h_p(\underline{I}_i)$ with $h_p(M_{p,i}^{w})$, evaluating the objective function $F_p^{w}$ at its optimizer:

$$S_{w,p,i}^2 := F_p^{w}(M_{p,i}^{w}; \underline{I}_i) = \sum_{j=1}^{m} w_{i,p,j}(h_p(I_{i,j}) - h_p(M_{p,i}^{w}))^2, \quad i = 1, 2, \dots, n. \tag{22}$$

We can interpret this loss of information in unit $i$ as the variance of the statistical variable $Y_i$.

Finally, we introduce the vector of the scaled indicators as in Eq. (7), i.e., $\underline{\tilde{I}}_i^{w} = [\tilde{I}_{i,1}^{w}, \tilde{I}_{i,2}^{w}, \dots, \tilde{I}_{i,m}^{w}]^T$, with $\tilde{I}_{i,j}^{w} = I_{i,j}/M_{p,i}^{w}$. Proceeding as in Section 2, for $i = 1, 2, \dots, n$, we define the weighted penalized power mean of order $p$ as follows:

$$PM_{w,p,i}^{\pm} = M_{p,i}^{w} g_{w,p,i}^{\pm}, \tag{23}$$

where $g_{w,p,i}^{\pm}$ is the penalization factor:

$$g_{w,p,i}^{\pm} = h_p^{-1}(\pm\tilde{S}_{w,p,i}^2) = (1 \pm p\,\tilde{S}_{w,p,i}^2)^{\frac{1}{p}} = \begin{cases} \left(1 \pm p\,\tilde{S}_{w,p,i}^2\right)^{\frac{1}{p}}, & p \neq 0, \\ \exp\left(\pm\tilde{S}_{w,0,i}^2\right), & p = 0, \end{cases} \tag{24}$$

in which

$$\tilde{S}_{w,p,i}^2 = K\tilde{L}_{w,p,i}\,, \tag{25}$$

and

$$\tilde{L}_{w,p,i} = \sum_{j=1}^{m} w_{i,p,j}(h_p(\tilde{I}_{i,j}^{w}) - h_p(1))^2 = \sum_{j=1}^{m} w_{i,p,j}(h_p(\tilde{I}_{i,j}^{w}))^2\,. \tag{26}$$

In conclusion, formula (23) is the penalized version of the weighted power means.

We note that the definition of penalized weighted power means of order $p$, see Eq. (23), continues to hold when the weights do not sum to one. In this case, we interpret the quantity $\tilde{S}_{w,p,i}^2$ as the loss of information generated by substituting the power $p$ of the sub-indicator $I_{i,j}^{p}$ with the power $p$ of its mean, $(M_{i,p}^{w})^p$, weighted by the importance of the sub-indicators. The loss is zero if and only if the ratio $I_{i,j}^{p}/(M_{i,p}^{w})^p$ equals one that is, $I_{i,j}$ equals $M_{p,i}^{w}$.

In the following section, we apply this generalization to the "direct" BoD composite indicators as proposed in Rogge (2018b).

**Table 1**
Descriptive statistics of the sub-indicators of the HDI (left panel); correlation coefficients of the sub-indicators (right panel).

| Sub-indicator: | $H$ | $E$ | $I$ |
|---|---|---|---|
| Min. | 0.5123 | 0.2506 | 0.3052 |
| 1st Qu. | 0.7292 | 0.5289 | 0.5882 |
| Median | 0.8308 | 0.6833 | 0.7318 |
| Mean | 0.8110 | 0.6592 | 0.7145 |
| 3rd Qu. | 0.8908 | 0.7939 | 0.8590 |
| Max | 0.9985 | 0.9456 | 1.00 |
| st.dev | 0.1136 | 0.1724 | 0.1736 |

| | $H$ | $E$ | $I$ |
|---|---|---|---|
| $H$ | 1 | 0.8176 | 0.8412 |
| $E$ | 0.8176 | 1 | 0.8653 |
| $I$ | 0.8412 | 0.8653 | 1 |

## 4. Empirical analysis

In this section, we investigate whether the ranking induced by power means of order $p$ and the corresponding penalized power means are different and, more interestingly, whether it is possible to choose the order $p$ to minimize the differences in ranking obtained for different order pairs $(p, p')$.

This analysis aims to develop composite indicators that are robust to the order of the power mean used to aggregate the composite indicator itself. In the empirical analysis illustrated in this section $K$ (see Eqs. (9) and (25)) always equals 1 since in our application the penalized power mean and the penalized weighted power mean are always range preserving.

As stressed in Guh, Po, and Lee (2008) and Rogge (2018b), the choice of the order $p$ depends on the objective/attitude of the decision maker, so it is a sort of "behavioral parameter". The parameter $p$ can be chosen a priori based on the expert judgement, or a posteriori based on the indicator values. For example, the a priori choice can be done in a way such that the resulting score favors the improvements of the worst-performing indicators. In this case, as explained in the Introduction, a power mean of negative order should be used. Otherwise, if we are interested in a composite indicator that favors the improvements of the best-performing indicators, power means of positive order do for us. On the other hand, the order of the mean can be chosen a posteriori based on the distribution of the indicator values. For example, if the data is expected to have outliers or extreme values, a higher value of $p$ might be more appropriate to reduce the impact of outliers. Conversely, if the data is highly skewed towards lower values, a lower value of $p$ might be preferred to give more weight to smaller values. Alternatively, if we interpret the Box–Cox transformation as a utility function with Constant Relative Risk Aversion (CRRA), the parameter $p$ can be expressed as $1-\gamma$, where $\gamma$ represents the relative risk aversion coefficient of the decision maker. The loss function measures the loss that a decision maker with CRRA utility associates to substituting the vector of indicators with the composite indicator.

We look for a $p^*$ order which induces a unit ranking as close as possible to the ranking induced by any alternative $p$ order. This $p^*$ order may be considered "fair" since it induces a ranking which is not too sensitive to the power mean order.

We carried out this analysis on the classical and penalized power means as well as on the corresponding weighted means. The weights of the latter are constructed as suggested in Rogge (2018b).

In detail, we use

$$r^m(p;\underline{I}) = (r_1^m(p;\underline{I}), r_2^m(p;\underline{I}), \dots, r_n^m(p;\underline{I}))$$
$$\text{and} \quad r^{pm}(p;\underline{I}) = (r_1^{pm}(p;\underline{I}), r_2^{pm}(p;\underline{I}), \dots, r_n^{pm}(p;\underline{I}))$$

to denote the ranking of units implied by the composite indicator obtained with the classical and penalized $p$-order means, respectively.

We define the "fair" $p$-value the solution to the following problem:

$$\min_{p\in\mathcal{P}} F(p) := \sum_{p'\in\mathcal{P}} \sum_{j=1}^{n} |r_j^m(p;\underline{I}) - r_j^m(p';\underline{I})|, \tag{27}$$

and a similar problem can be formulated for the penalized mean.

We conduct our study on the freely downloadable data of sub-indicators that define the Human Development Index (HDI) relative to $n = 189$ countries in the year 2019. The data of sub-indicators were downloaded from the UNDP Data Center (https://hdr.undp.org/data-center).

Specifically, the HDI is obtained from three sub-indicators – life expectancy, education, and income – aggregated by the geometric mean function with equal weights (i.e., Eq. (1) with $p = 0$):

$$HDI = (H * E * I)^{1/3},$$

where the health dimension $(H)$ is measured through the life expectancy indicator, the education dimension $(E)$ is the arithmetic mean of the two education indices (mean years of schooling and expected years of schooling), and the gross national income per capita $(I)$ is a proxy that accounts for the standard of living.

Table 1 shows the descriptive statistics of the sub-indicators. We observe that the sub-indicators are positively correlated.

### 4.1. Power mean aggregative approach: unit-independent weights

We compute the composite indicators on HDI data using as an aggregation tool both the power mean and the penalized power mean for different values of the $p$-order. Specifically, we choose $p_l = -M+(2l-1)$, $l = 1, 2, \dots, M$, so the $p$-orders considered are integer values varying from $-M+1$ to $M-1$. That is, in (27) we set $\mathcal{P} = \{p_1, p_2, \dots, p_M\}$. We choose $M = 11$ since in the literature, the order usually varies from $-5$ to 5. Furthermore, we use $CM(p)$ to denote the classical power mean of order $p$ and $PM(p)$ for the corresponding penalized version, while the ranking induced on country $i$, $i = 1, 2, \dots, n$, $(n = 189)$ is denoted with $r_{CM}(i, p)$ and $r_{PM}(i, p)$, respectively.

We first analyze the distribution of the ranking difference $d_i(p) = r_{PM}(i, p) - r_{CM}(i, p)$, $i = 1, 2, \dots, n$, induced by the composite indicators defined by the penalized power mean and the power mean, respectively. Fig. 1 shows the distribution of $d_i(p)$, $i = 1, 2, \dots, n$, for negative values of $p$, $p = -5, -4, -3, -2, -1$, and $p = 0$ (left panel), and for positive $p$, $p = 1, 2, 3, 4, 5$, and $p = 0$ (right panel). The distributions shown in Fig. 1 are not symmetric. This finding is confirmed by the positive skewness of all distributions in Fig. 1, i.e., 1.189, 1.571, 1.598, 1.806, 1.537, 1.047, 0.564, 0.590, 0.285, 0.756, 0.765, respectively for $p$ varying from $-5$ to 5. This implies that the penalized power mean more frequently provides a larger rank than the non-penalized power mean. Finally, the standard deviation of the distributions increases with $p$ (i.e., 1.741, 1.839, 2.174, 2.523, 2.819, 2.932, 2.935, 2.995 2.971, 3.238, 3.219) indicating that ranking differences are less volatile for negative orders.

Second, we solve problem (27) using the power mean and penalized power mean. Fig. 2 shows the graph of the objective function $F$ of Problem (27) when the penalized (blue) and non-penalized (red) power means are used to aggregate the sub-indicators with different set of $p$ ($p = -5, -4, \dots, 4, 5$ at the left panel and $p = -1, 0, \dots, 4, 5$ at the right panel). We see that the minimizer is the middle point of the interval, i.e., $p = 0$ (left panel) and $p = 2$ (right panel), for both aggregative approaches. More interestingly, the objective function $F$ corresponding to penalized means is smaller than that of non-penalized means in both panels. This finding suggests that the ranking differences generated for different values of $p$ reduce when penalized power means are used as an aggregative approach. The penalized geometric mean is the aggregative approach most suitable for reducing ranking variations

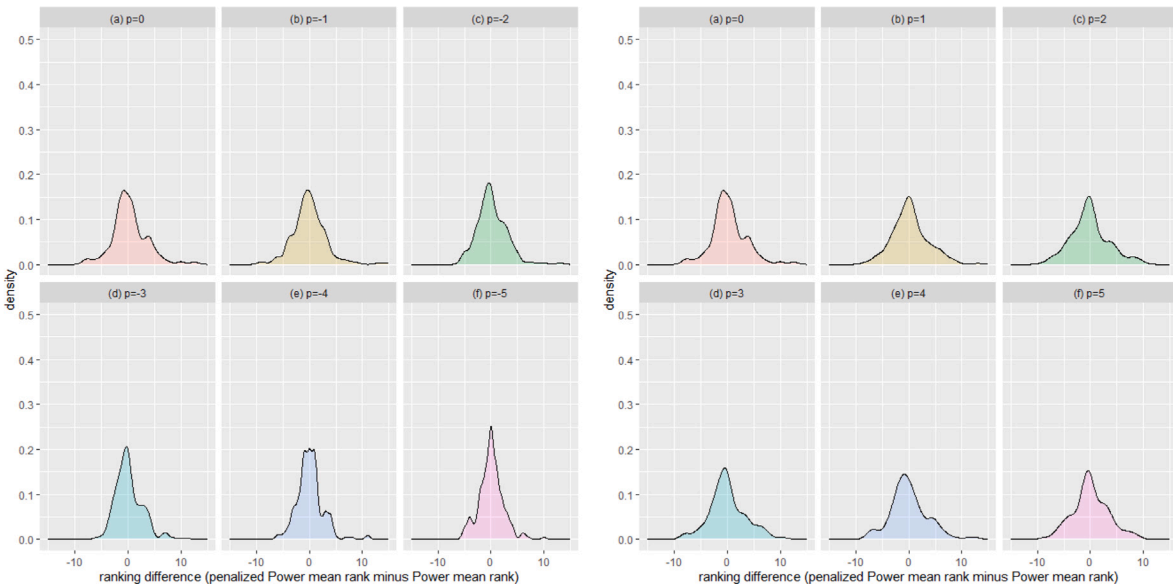


**Fig. 1.** Distribution of ranking difference $d_i(p) = r_{PM}(i,p) - r_{CM}(i,p)$, $i = 1,2,\ldots,n$, (i.e., penalized power mean of order $p$ minus the corresponding power mean of order $p$) for $p = -5,-4,-3,-2,-1,0$ (left panel) and for $p = 0,1,2,3,4,5$ (right panel).

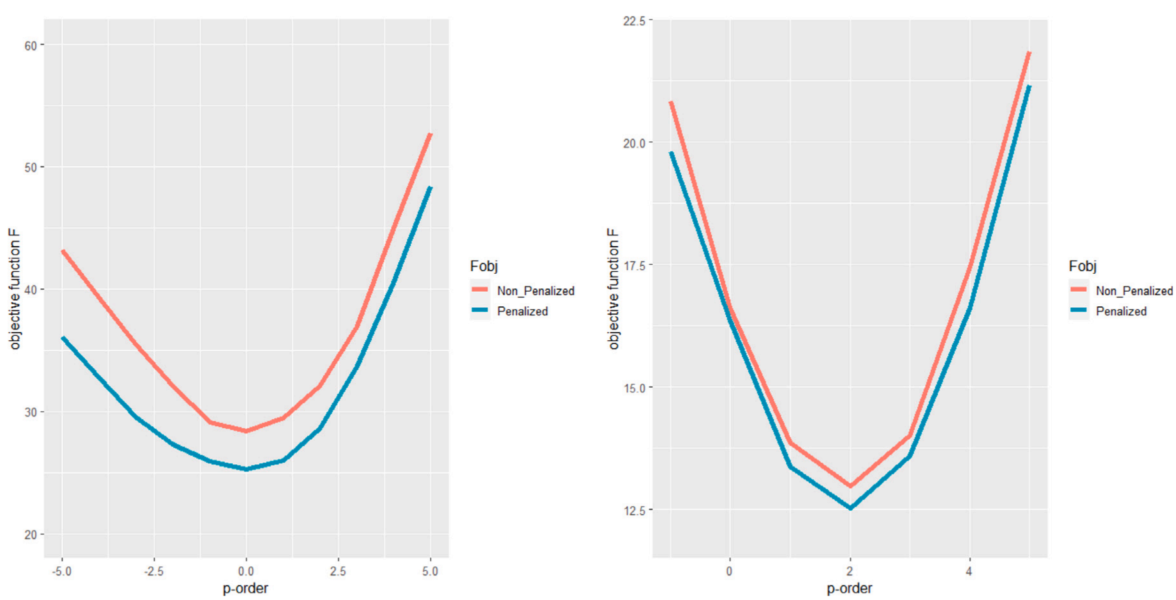


**Fig. 2.** Function $F$ of Problem (27) when the power mean and penalized power mean are used to build the composite indicator and $p$ varies from −5 to 5 (left panel) and from −1 to 5 (right panel).

when the interval of $p$ of the minimization is symmetric with respect to the origin. This is expected, since the classical power means satisfy monotonicity with respect to the order $p$. Indeed, the monotonicity supports the fact that the middle point of the interval, where the minimization is carried out, is the solution to problem (27).

We note that the non-penalized geometric mean corresponds to the aggregative method used for the Human Development Index (HDI).

As Fig. 2 shows, the power means of order $p = -1$ (harmonic mean), $p = 0$ (geometric mean), and $p = 1$ (arithmetic mean) are those with the smallest ranking variations when compared with other (penalized/non-penalized) $p$-order power mean. We focus on these means to analyze countries in the first and last twenty positions of the HDI ranking.

Table 2 shows the countries ranked in the first (top panel) and last (bottom panel) twenty positions according to the HDI (i.e., geometric (rGM) mean), along with the rank of the harmonic (rHM) and arithmetic (rAM) means and the rank of their penalized versions (i.e., columns rPHM, rPGM and rPAM). All the remaining results are available in Appendix C.

Table 2 shows that the penalization refines the ranking of the corresponding non-penalized approach. In fact, looking at the column of the penalized geometric mean in the top panel of Table 2, we see that Ireland is above Switzerland, Iceland is above Hong Kong, the United Kingdom is above Belgium, and Finland is above Singapore. These differences highlighted by the penalization are due to horizontal variability, which seems to be more pronounced in countries with a higher population density.

Looking at the panel at the bottom of Table 2, we notice that, according to the penalized geometric mean, Haiti is above Sudan, Guinea-Bissau is above Congo and Liberia, and, more interestingly, Sierra Leone is four positions above Burkina Faso. This finding does not confirm that the horizontal variability is linked to population density as noted in the first twenty countries.

**Table 2**
The first twenty (top panel) and last twenty (bottom panel) countries in the HDI ranking and the corresponding ranking obtained by the composite indicators aggregated with non-penalized/penalized power mean approaches for $p = -1$ (HM), 0 (GM), 1 (AM).

| Country | Classical power mean aggregation | | | Penalized power mean aggregation | | |
|---|---|---|---|---|---|---|
| | rHM | rGM | rAM | rPHM | rPGM | rPAM |
| Norway | 1 | 1 | 1 | 1 | 1 | 1 |
| Ireland | 2 | 2 | 2 | 2 | 2 | 2 |
| Switzerland | 2 | 2 | 2 | 3 | 3 | 2 |
| Hong Kong | 5 | 4 | 4 | 6 | 6 | 5 |
| Iceland | 4 | 4 | 5 | 4 | 4 | 4 |
| Germany | 5 | 6 | 6 | 5 | 5 | 5 |
| Sweden | 7 | 7 | 7 | 7 | 7 | 7 |
| Australia | 9 | 8 | 8 | 8 | 8 | 9 |
| Netherlands | 7 | 8 | 8 | 8 | 8 | 7 |
| Denmark | 10 | 10 | 10 | 10 | 10 | 10 |
| Finland | 11 | 11 | 12 | 11 | 11 | 11 |
| Singapore | 12 | 11 | 10 | 15 | 12 | 12 |
| Belgium | 13 | 13 | 13 | 13 | 14 | 13 |
| United Kingdom | 13 | 13 | 13 | 12 | 12 | 13 |
| New Zealand | 15 | 15 | 15 | 14 | 15 | 15 |
| Canada | 16 | 16 | 16 | 15 | 16 | 16 |
| United States | 17 | 17 | 17 | 17 | 17 | 17 |
| Austria | 18 | 18 | 18 | 18 | 18 | 18 |
| Israel | 19 | 19 | 21 | 19 | 19 | 19 |
| Liechtenstein | 21 | 19 | 19 | 23 | 23 | 21 |
| Haiti | 167 | 170 | 171 | 167 | 167 | 168 |
| Sudan | 170 | 170 | 167 | 171 | 170 | 170 |
| Gambia | 172 | 172 | 173 | 169 | 169 | 172 |
| Ethiopia | 177 | 173 | 172 | 179 | 179 | 178 |
| Malawi | 175 | 174 | 174 | 177 | 177 | 175 |
| Congo | 174 | 175 | 177 | 173 | 174 | 174 |
| Guinea-Bissau | 173 | 175 | 180 | 170 | 170 | 173 |
| Liberia | 175 | 175 | 175 | 173 | 175 | 177 |
| Guinea | 177 | 178 | 179 | 176 | 175 | 175 |
| Yemen | 179 | 179 | 177 | 181 | 181 | 180 |
| Eritrea | 183 | 180 | 176 | 188 | 186 | 183 |
| Mozambique | 181 | 181 | 182 | 180 | 180 | 181 |
| Burkina Faso | 182 | 182 | 181 | 182 | 182 | 182 |
| Sierra Leone | 180 | 182 | 183 | 175 | 178 | 179 |
| Mali | 186 | 184 | 183 | 186 | 184 | 185 |
| Burundi | 185 | 185 | 185 | 184 | 184 | 186 |
| South Sudan | 184 | 186 | 186 | 183 | 183 | 184 |
| Central African Republic | 187 | 187 | 189 | 185 | 187 | 187 |
| Chad | 188 | 188 | 188 | 187 | 188 | 188 |
| Niger | 189 | 189 | 187 | 189 | 189 | 189 |

The main finding of this section is that the (non-penalized/ penalized) geometric mean ($p = 0$) should be preferred to other power means, not only because it is non-compensative, but also because it solves Problem (27) as a middle point of the power order considered. The second finding is that the penalized mean refines the ranking of the corresponding non-penalized mean. This could be used to investigate the robustness of the HDI ranking.

### 4.2. Benefit of the Doubt aggregative approach: unit-dependent weights

We analyze the effect of penalization on the composite indicators defined by the Benefit of the Doubt (BoD) direct approach (see, Rogge (2018a) and Rogge (2018b)). This strand of the literature on composite indicators explores the methodological issue of weighting the sub-indicators, looking for the proper weights to aggregate the sub-indicators.

In this section, we consider the composite indicators obtained applying the "direct BoD approach" as defined in Rogge (2018a):

$$M^{w}_{p,i} = M^{w}_{p}(\underline{I}_i) = \begin{cases} \left( \sum_{j=1}^{m} \pi_{i,p,j} I^{p}_{i,j} \right)^{\frac{1}{p}}, & p \neq 0, \\ \prod_{j=1}^{m} I^{\pi_{i,0,j}}_{i,j}, & p = 0, \end{cases} \tag{28}$$

where for each unit $i$, $i = 1, 2, \ldots, n$, the unit-dependent weight $\pi_{i,p,j}$, $j = 1, 2, \ldots, m$, with $p \neq 0$ is the solution to the following problem:

$$\max_{\pi_{i,p,j}} \left( \sum_{j=1}^{m} \pi_{i,p,j} I^{p}_{i,j} \right)^{1/p}, \tag{29}$$

subject to the constraints:

$$\left( \sum_{j=1}^{m} \pi_{i,p,j} I^{p}_{c,j} \right)^{1/p} \leq 1, \ \ c = 1, 2, \ldots, n, \tag{30}$$

and

$$\pi_{i,p,j} \geq 0, \quad j = 1, 2, \ldots, m. \tag{31}$$

The ratio of these weights is that each unit (i.e., country, region) tries to maximize its composite indicators (see Eq. (29)), to make the units' composite indicators comparable in magnitude (see Eq. (30)) with non-negative weights (see Eq. (31)).

The case $p = 0$ is obtained numerically by choosing $p$ close to zero (i.e., $p = 0.0001$). Results very close to $p = 0.0001$ are obtained by choosing $p = -0.0001$. The formulation of Problem (29)–(31) for $p = 0$ is still a challenge since the trivial solution $\pi_{p,i,j} = 0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ satisfies Problem (29)–(31) if the values of the sub-indicators fall in the interval $(0, 1]$. This is why, instead of solving problem (29)–(31) with $p = 0$, we prefer solve the problem with $p \neq 0$ small enough.

As in the previous experiment, we consider a grid of $p$-order, $p_l = -M + (2l - 1)$, $l = 1, 2, \ldots, M$, so the $p$-orders considered are integer

**Table 3**
Average values (left panel) and third quartiles (right panel) of BoD weights for different $p$-orders.

| Arithmetic means of BoD weights | | | | Third quartiles of BoD weights | | | |
|---|---|---|---|---|---|---|---|
| p | $\pi_{i,p,1}$ (H) | $\pi_{i,p,2}$ (E) | $\pi_{i,p,3}$ (I) | p | $\pi_{i,p,1}$ (H) | $\pi_{i,p,2}$ (E) | $\pi_{i,p,3}$ (I) |
| −5 | 0.706 | 0.101 | 0.126 | −5 | 0.992 | 0.071 | 0.000 |
| −4 | 0.712 | 0.108 | 0.125 | −4 | 0.994 | 0.082 | 0.000 |
| −3 | 0.715 | 0.116 | 0.126 | −3 | 0.995 | 0.093 | 0.000 |
| −2 | 0.719 | 0.124 | 0.128 | −2 | 0.997 | 0.106 | 0.000 |
| −1 | 0.722 | 0.132 | 0.131 | −1 | 0.998 | 0.121 | 0.000 |
| 0 | 0.721 | 0.150 | 0.129 | 0 | 1.000 | 0.324 | 0.000 |
| 1 | 0.723 | 0.163 | 0.131 | 1 | 1.002 | 0.360 | 0.000 |
| 2 | 0.726 | 0.175 | 0.133 | 2 | 1.003 | 0.399 | 0.000 |
| 3 | 0.730 | 0.189 | 0.134 | 3 | 1.005 | 0.443 | 0.000 |
| 4 | 0.734 | 0.204 | 0.135 | 4 | 1.006 | 0.490 | 0.000 |
| 5 | 0.738 | 0.220 | 0.136 | 5 | 1.008 | 0.543 | 0.000 |

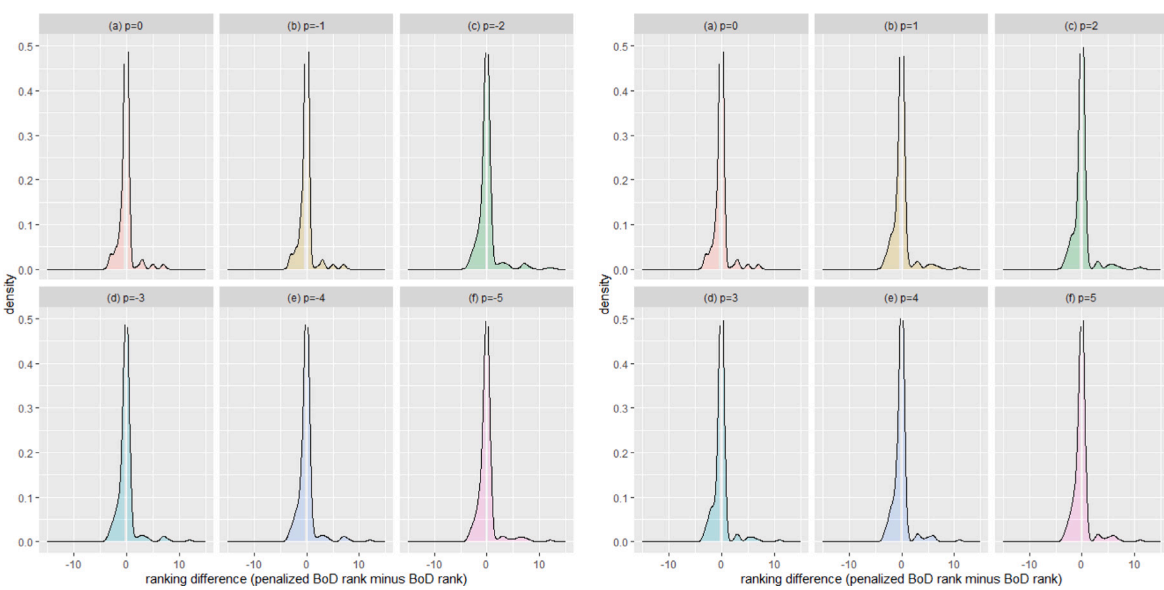


**Fig. 3.** Distribution of ranking difference $d_i(p) = r_{PBoD}(i,p) - r_{BoD}(i,p)$, $i = 1,2,\dots,n$ (i.e., penalized BoD composite indicator of order $p$ minus the corresponding BoD composite indicator of order $p$) for $p = -5,-4,-3,-2,-1,0$ (left panel) $p = 0,1,2,3,4,5$ (right panel).

values varying from $-M+1$ to $M-1$. We choose $M = 11$ since in the literature the $p$-value considered usually varies from −5 to 5.

We solve Problem (29)–(31) for each value $p_l$, $l = 1,2,\dots,11$ and for each country/unit $i$, $i = 1,2,\dots,n$. We therefore obtain the weights $\pi_{i,p,1}$, $\pi_{i,p,2}$, $\pi_{i,p,3}$ relative, respectively, to the three indicators mentioned above, i.e., Health ($H$), Education ($E$), and Income ($I$). Table 3 shows the average values (left panel) and third quartile (right panel) of the weight distribution $\pi_{i,p_l,j}$, $i = 1,2,\dots,n$ corresponding to the three sub-indicators (i.e. $j = 1,2,3$) and the eleven values of the order $p$.

The left panel of Table 3 shows that the average values and third quartiles of the country-specific weights of the health dimension dominates for education and income for any $p = -5,-4,\dots,4,5$. The mean of the weights associated with health are constant with respect to $p$, while the mean associated with education increases slightly with $p$, and the same behavior can be seen in the third quartile of health and education. Interestingly, the mean of the weights associated with income slightly varies with $p$, but the third quartile is zero as a function of $p$. This implies that only a few units have weights different from zero, indicating that health and education play a more crucial role than income. Furthermore, the right panel of Table 3 shows that education plays a more relevant role when aggregative approaches with positive order $p$ are used.

Recalling that, when a power mean of positive order is used as indicator, the marginal increase in the value of the indicator is much higher when the absolute value of the indicator is large (Rogge, 2018a), we conclude that education especially affects the composite indicators that encourage the improvement of countries with good performance.

As in Section 4.1, we analyze the distribution of the ranking difference $d_i(p) = r_{PBoD}(i,p) - r_{BoD}(i,p)$, $i = 1,2,\dots,n$, induced by the composite indicators defined by the penalized BoD and BoD direct approaches, respectively. Fig. 3 shows the distribution of $d_i(p)$, $i = 1,2,\dots$ for negative values of $p$, $p = -5,-4,-3,-2,-1$ and $p = 0$ (left panel), and for positive $p$, $p = 1,2,3,4,5$ and $p = 0$ (right panel).

The distributions shown in Fig. 3 are not symmetric with negative skewness (i.e., −2.931, −2.792, −2.812, −2.935, −2.826, −2.367, −3.442, −3.324, −3.298, −2.861, −2.876). In line with the power mean (penalized and non-penalized) composite indicators, the penalized BoD provides smaller rankings than those from the non-penalized BoD indicators. A comparison between Fig. 3 and Fig. 1 shows that the standard deviations of the distributions of the BoD ranking differences (penalized minus non-penalized) are smaller than the standard deviations of the distributions of the power mean ranking differences (penalized minus non-penalized). The eleven standard deviations for $p$ varying from −5 to 5 are about halved with respect to those of power mean approach; these are as follows: 1.424, 1.435, 1.429, 1.431, 1.364, 1.311, 1.452, 1.475, 1.501, 1.414, 1.419. This is an expected finding since the BoD weights are country-specific, thereby reducing the effect of the penalization. This finding, together with the results about the quartiles shown in the right panel of Table 3, leads to conclude that the concentration of weights in the dimensions of wealth and education reduces the

**Table 4**
The first twenty (top panel) and the last twenty (bottom panel) countries according to the power means (non penalized/penalized) and BoD direct country-specific weights (non-penalized/penalized) for $p = -1$ (HM), 0 (GM), 1 (AM). The rankings are obtained using three digits after the decimal.

| | "Power-mean ranking" | | | | | | "Benefit of the doubt ranking" | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Country | rHM | rGM | rAM | rPHM | rPGM | rPAM | rHM | rGM | rAM | rPHM | rPGM | PAM |
| Liechtenstein | 21 | 19 | 19 | 23 | 23 | 21 | 1 | 1 | 1 | 1 | 1 | 1 |
| Qatar | 46 | 45 | 39 | 60 | 53 | 46 | 1 | 1 | 1 | 1 | 1 | 1 |
| Ireland | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 5 | 4 | 3 |
| Germany | 5 | 6 | 6 | 5 | 5 | 5 | 1 | 1 | 1 | 8 | 5 | 4 |
| Norway | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 | 5 | 4 |
| Switzerland | 2 | 2 | 2 | 3 | 3 | 2 | 1 | 1 | 1 | 5 | 7 | 4 |
| Singapore | 12 | 11 | 10 | 15 | 12 | 12 | 1 | 1 | 1 | 1 | 7 | 7 |
| Australia | 9 | 8 | 8 | 8 | 8 | 9 | 1 | 1 | 1 | 7 | 7 | 7 |
| Iceland | 4 | 4 | 5 | 4 | 4 | 4 | 1 | 1 | 1 | 8 | 7 | 7 |
| Hong Kong | 5 | 4 | 4 | 6 | 6 | 5 | 1 | 1 | 1 | 1 | 3 | 12 |
| Japan | 21 | 21 | 21 | 22 | 21 | 21 | 11 | 11 | 11 | 11 | 11 | 10 |
| Luxembourg | 24 | 22 | 20 | 24 | 24 | 24 | 11 | 11 | 11 | 11 | 11 | 10 |
| New Zealand | 15 | 15 | 15 | 14 | 15 | 15 | 11 | 11 | 11 | 13 | 13 | 13 |
| Sweden | 7 | 7 | 7 | 7 | 7 | 7 | 14 | 14 | 14 | 13 | 13 | 13 |
| Finland | 11 | 11 | 12 | 11 | 11 | 11 | 15 | 15 | 15 | 15 | 15 | 15 |
| Netherlands | 7 | 8 | 8 | 8 | 8 | 7 | 16 | 16 | 16 | 16 | 16 | 16 |
| United Kingdom | 13 | 13 | 13 | 12 | 12 | 13 | 17 | 17 | 17 | 18 | 18 | 18 |
| United States | 17 | 17 | 17 | 17 | 17 | 17 | 18 | 18 | 18 | 17 | 17 | 17 |
| Denmark | 10 | 10 | 10 | 10 | 10 | 10 | 19 | 19 | 19 | 20 | 20 | 20 |
| Arab Emirates | 32 | 31 | 30 | 34 | 33 | 32 | 20 | 20 | 20 | 19 | 19 | 19 |
| Zimbabwe | 150 | 150 | 152 | 147 | 148 | 149 | 170 | 170 | 170 | 170 | 170 | 170 |
| Gambia | 172 | 172 | 173 | 169 | 169 | 172 | 171 | 171 | 171 | 170 | 170 | 170 |
| Benin | 157 | 158 | 161 | 154 | 155 | 157 | 172 | 172 | 172 | 172 | 172 | 172 |
| Burkina Faso | 182 | 182 | 181 | 182 | 182 | 182 | 173 | 173 | 173 | 173 | 173 | 173 |
| Burundi | 185 | 185 | 185 | 184 | 184 | 186 | 173 | 173 | 173 | 173 | 173 | 173 |
| Guinea | 177 | 178 | 179 | 176 | 175 | 175 | 173 | 173 | 173 | 173 | 173 | 173 |
| Angola | 147 | 148 | 150 | 145 | 145 | 147 | 176 | 176 | 176 | 176 | 176 | 176 |
| Togo | 166 | 167 | 170 | 165 | 165 | 166 | 177 | 177 | 177 | 177 | 177 | 177 |
| Mozambique | 181 | 181 | 182 | 180 | 180 | 181 | 178 | 178 | 178 | 178 | 178 | 178 |
| Congo | 174 | 175 | 177 | 173 | 174 | 174 | 179 | 179 | 179 | 179 | 179 | 179 |
| Cameroon | 151 | 153 | 154 | 148 | 151 | 151 | 180 | 180 | 180 | 180 | 180 | 180 |
| Mali | 186 | 184 | 183 | 186 | 184 | 185 | 181 | 181 | 181 | 181 | 181 | 181 |
| Côte d'Ivoire | 160 | 162 | 162 | 156 | 157 | 160 | 182 | 182 | 182 | 182 | 182 | 182 |
| Guinea-Bissau | 173 | 175 | 180 | 170 | 170 | 173 | 183 | 183 | 183 | 183 | 183 | 183 |
| Nigeria | 158 | 161 | 164 | 153 | 153 | 158 | 184 | 184 | 184 | 184 | 184 | 184 |
| South Sudan | 184 | 186 | 186 | 183 | 183 | 184 | 185 | 185 | 185 | 185 | 185 | 185 |
| Lesotho | 163 | 165 | 168 | 158 | 159 | 162 | 186 | 186 | 186 | 186 | 186 | 186 |
| Sierra Leone | 180 | 182 | 183 | 175 | 178 | 179 | 187 | 187 | 187 | 187 | 187 | 187 |
| Chad | 188 | 188 | 188 | 187 | 188 | 188 | 188 | 188 | 188 | 188 | 188 | 188 |
| Central African Republic | 187 | 187 | 189 | 185 | 187 | 187 | 189 | 189 | 189 | 189 | 189 | 189 |

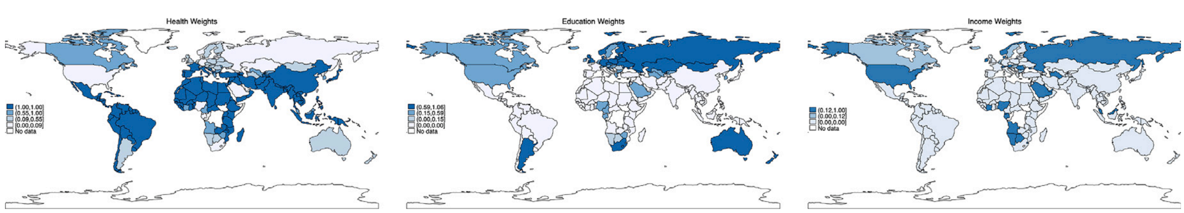


**Fig. 4.** Distribution of the BoD arithmetic mean weights across the globe. Health weights (left map), Education weights (middle map) and Income weights (right map).

variability among indicators and, as a consequence, reduces the effect of penalization.

Table 4 shows the first twenty (top panel) and the last twenty (bottom panel) countries according to the non-penalized arithmetic BoD ranking. Columns two to four contain the rank of non-penalized and penalized harmonic, geometric, and arithmetic means, respectively. Columns five to twelve contain the rank obtained with the penalized/non-penalized harmonic, geometric, and arithmetic BoD procedure. The rankings were obtained using three digits after the decimal.

Table 4 highlights two results. First, there are several units tied for first position in the BoD ranking, while the penalized one refines the ranking. Second, there is a great difference between the power mean and direct BoD approaches mainly based on the fact that the indicator of several countries is constructed only with a weight different from zero. This is the case of Liechtenstein and Qatar, which have zero weights for health and education, while the weights are equal to one for income. To better investigate this point we use Fig. 4, which shows the distribution of the weights relative to health (left), education (middle), and income (right) across the globe. We can see that the weights of the African countries are very concentrated in one dimension, mainly health or, for the richest countries, income. The western countries have more diffused weights (dark gray in the panels of Fig. 4). The BoD and arithmetic mean rankings are rather close for the poorest countries where the values of education and income are very small, and they are

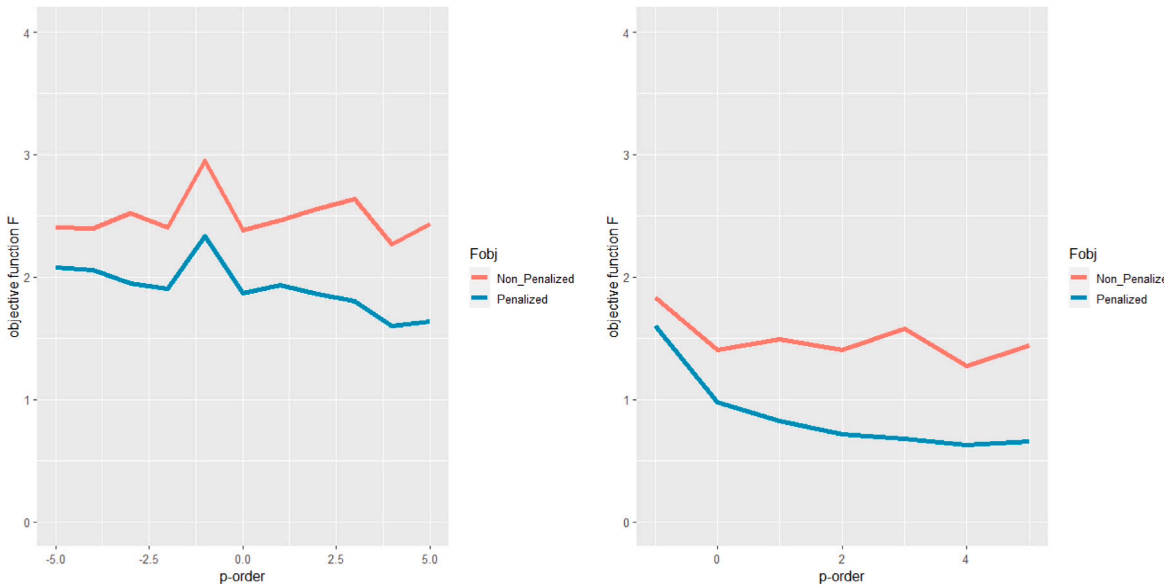


**Fig. 5.** Function $F$ of Problem (27) when the BoD power mean and BoD penalized power mean are used to build the composite indicator and $p$ varies from −5 to 5 (left panel) and from −1 to 5 (right panel).

**Table 5**
Spearman's rank correlation between the power mean and BoD rankings. All correlations are significant with p-values less than 0.001.

| | rHM | rGM | rAM | rPHM | rPGM | rPAM |
|---|---|---|---|---|---|---|
| rBoD_HM | 0.95 | 0.95 | 0.96 | 0.94 | 0.94 | 0.95 |
| rBoD_GM | 0.95 | 0.95 | 0.96 | 0.94 | 0.94 | 0.95 |
| rBoD_AM | 0.95 | 0.95 | 0.96 | 0.94 | 0.94 | 0.95 |
| rBoD_PHM | 0.95 | 0.95 | 0.96 | 0.93 | 0.94 | 0.95 |
| rBoD_PGM | 0.95 | 0.95 | 0.96 | 0.93 | 0.94 | 0.95 |
| rBoD_PAM | 0.95 | 0.95 | 0.96 | 0.93 | 0.94 | 0.95 |

also close for western countries where the weights are spread out. We continue to analyze the relationship between the BoD and power mean approaches using Spearman's rank correlation.

Table 5 shows the values of the rank correlation between the composite indicators obtained with penalized and non-penalized power mean aggregation procedure and the penalized and non-penalized BoD aggregation procedure. We observe that the correlation coefficients reach the largest values for the correlation between the non-penalized arithmetic mean composite indicators and the indicators obtained with the BoD procedure.

We conclude this section by solving problem (27) to investigate whether the geometric mean also plays a crucial role in the BoD aggregation procedure.

Fig. 5 shows the graph of the objective function $F$ of Problem (27) as a function of $p$ when $p$ varies from −5 to 5 (left panel) and from −1 to 5 (right panel) when the BoD approach is used. In contrast to the power mean aggregation, the curves are rather flat with the maximum value achieved by the non-penalized BoD. Interestingly, the penalized mean shows ranking difference magnitudes smaller than those of the non-penalized mean as observed in the case of power means. The results in Fig. 5 show that the ranking differences between the penalized and non-penalized BoD are negligible, especially when compared with the differences observed in Fig. 1. This confirms that the effect of the penalization is reduced by the choice of country-specific weights.

## 5. Conclusions

This paper proposes the penalized power means as an approach to constructing composite indicators that extends and supports the Mazziotta Pareto Index. The penalized mean accounts for the variability across indicators while continuing to satisfy some crucial properties already met by power means. More interestingly, the penalized approach provides composite indicators whose corresponding rankings are less sensitive to the choice of the $p$-order and more refined than the non penalized ones. In fact, the penalization is able to discriminate units taking into account for their horizontal variability. The discriminatory power of the penalized means reduces when unit specific weights are considered. The choice of the parameter $K$ is crucial for unit specific weights and it is should be further investigated. In Appendix D we provide one simple choice. Our empirical analysis shows that the non-penalized/penalized geometric mean is the best choice for reducing ranking variations with respect to order $p$, since $p = 0$ is the "middle point" of the interval $p \in (-\infty, +\infty)$. This result can be explained for the non-penalized power mean by the monotonicity of the power means, and it seems to extend to the penalized power means as well.

Finally, the penalization proposed applies to other aggregation procedures such as the class of Benefit of the Doubt approaches. Extensions to the indirect BoD and Biggeri's approaches deserve further investigation.

Moreover, it could be interesting to investigate how the correlations between indicators and the non-uniqueness issue of BoD weights impact on the ranking. The outcomes resulting from the comparison of the penalized power mean with both the BoD direct and penalized BoD direct approaches may be affected by the positive correlation among the sub-indicators of the HDI. These comparative findings could significantly differ if the sub-indicators exhibit negative or mixed correlations. These points deserve attention and will be the object of future study.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Auxiliary results

Following Grünwald and Páles (2022), we recall the following technical lemma, which is useful for establishing Properties 11 and 12 in Proposition 3. The latter is a simplified version of Corollary 12 in Grünwald and Páles (2022) when $p_1(x) = p_2(x) = \cdots = p_n(x) = p(x) = x^\alpha$ and $q_1(x) = q_2(x) = \cdots = q_n(x) = q(x) = x^\beta$. For the reader's convenience, we outline the proof.

**Lemma 1.** *Let $I \subset \mathbb{R}_+$, $n \geq 2$, $x_i \in I$, $i = 1, 2, \dots, n$. Let $\alpha, \beta \in \mathbb{R}$ such that $\beta \geq \alpha > 0$, then*

$$\frac{\sum_{i=1}^n x_i^{\beta-1}}{\sum_{i=1}^n x_i^{\beta}} \leq \frac{\sum_{i=1}^n x_i^{\alpha-1}}{\sum_{i=1}^n x_i^{\alpha}}. \tag{32}$$

**Proof of Lemma 1.**
Let $\delta = \beta - \alpha \geq 0$. It is simple to prove that the following inequality holds:

$$t^\delta \left(1 - \frac{1}{t}\right) \geq \left(1 - \frac{1}{t}\right), \quad \forall t > 0. \tag{33}$$

For $x_1, x_2, \dots, x_n, y \in I$, choosing $t = x_i/y$ in (33), inequality (33) becomes

$$\left(\frac{x_i}{y}\right)^{\beta-\alpha} \left(1 - \frac{y}{x_i}\right) \geq \left(1 - \frac{y}{x_i}\right), \tag{34}$$

which also reads

$$\left(\frac{x_i}{y}\right)^{\beta} \left(1 - \frac{y}{x_i}\right) \geq \left(\frac{x_i}{y}\right)^{\alpha} \left(1 - \frac{y}{x_i}\right), \tag{35}$$

that is,

$$\left(\frac{1}{y}\right)^{\beta} \left(x_i^\beta - y x_i^{\beta-1}\right) \geq \left(\frac{1}{y}\right)^{\alpha} \left(x_i^\alpha - x_i^{\alpha-1}\right). \tag{36}$$

Summing (36) in $i \in \{1, 2, \dots, n\}$, we obtain

$$\left(\frac{1}{y}\right)^{\beta} \left(\sum_{i=1}^n x_i^\beta - y \sum_{i=1}^n x_i^{\beta-1}\right) \geq \left(\frac{1}{y}\right)^{\alpha} \left(\sum_{i=1}^n x_i^\alpha - \sum_{i=1}^n x_i^{\alpha-1}\right). \tag{37}$$

Choosing $y = \frac{\sum_{i=1}^n x_i^\alpha}{\sum_{i=1}^n x_i^{\alpha-1}}$ in the right side of (37) and simplifying $\left(\frac{1}{y}\right)^\beta$ yield

$$\left(\sum_{i=1}^n x_i^{\beta-1}\right)\left(\frac{\sum_{i=1}^n x_i^\beta}{\sum_{i=1}^n x_i^{\beta-1}} - y\right) \geq 0. \tag{38}$$

Thus, bearing in mind that $y$ equals $\frac{\sum_{i=1}^n x_i^\alpha}{\sum_{i=1}^n x_i^{\alpha-1}}$, the inequality (38) reads

$$\left(\sum_{i=1}^n x_i^{\beta-1}\right)\left(\frac{\sum_{i=1}^n x_i^\beta}{\sum_{i=1}^n x_i^{\beta-1}} - \frac{\sum_{i=1}^n x_i^\alpha}{\sum_{i=1}^n x_i^{\alpha-1}}\right) \geq 0. \tag{39}$$

This concludes the proof. □

## Appendix B. Proofs of Proposition 2 and Proposition 3

*Proof of Proposition 2*

The derivative of $S_p^2(\underline{I} + \underline{c})$ with respect to $c$ is

$$\frac{\partial \tilde{S}_p^2(\underline{I} + \underline{c})}{\partial c} = \frac{2K}{p} \frac{\left[\frac{1}{m}\sum_{j=1}^m (I_j + c)^{2p}\right]}{\left[\frac{1}{m}\sum_j^m (I_j + c)^p\right]^3} \times \left[\frac{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{2p-1}}{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{2p}} - \frac{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{p-1}}{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{p}}\right], \tag{40}$$

which also reads

$$\frac{\partial \tilde{S}_p^2(\underline{I} + \underline{c})}{\partial c} = \frac{2K}{p} \frac{\left[\frac{1}{m}\sum_{j=1}^m (I_j + c)^{2p}\right]}{\left[\frac{1}{m}\sum_j^m (I_j + c)^p\right]^3} H_p(c), \tag{41}$$

where $H_p(c)$ is given by

$$H_p(c) = \left[\frac{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{2p-1}}{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{2p}} - \frac{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{p-1}}{\frac{1}{m}\sum_{j=1}^m (I_j + c)^{p}}\right]. \tag{42}$$

The sign of $\frac{\partial \tilde{S}_p^2(\underline{I}+\underline{c})}{\partial c}$ depends on $\frac{H_p(c)}{p}$. Applying Lemma 1 in Appendix A with $\beta = 2p$ and $\alpha = p$ when $p > 0$, we obtain $H_p(c) \leq 0$ and $\frac{H_p(c)}{p} \leq 0$. Choosing $\alpha = 2p$, $\beta = p$, when $p < 0$, we have $H_p(c) \geq 0$ and $\frac{H_p(c)}{p} \leq 0$. By virtue of Eq. (41), this implies that $\tilde{S}_p^2(\underline{I} + \underline{c})$ is a decreasing function of $c$.

This concludes the proof. □

*Proof of Proposition 3*

Observing that $M_p(\underline{c}) = c$ and $\tilde{S}_p^2(\underline{c}) = 0\ \forall p$, we have $PM_p^{\pm}(\underline{c}) = c$.

Since $\lim_{c\to 0} M_p(\underline{c}) = 0$ and $\lim_{c\to 0} \tilde{S}_p^2(\underline{c}) = 0$, Property 2 follows directly from (16).

Properties 3 and 4 follow easily, observing that $g_p^-(\underline{I}) \leq 1$, $g_p^+(\underline{I}) \geq 1$ and $g_p^{\pm} = 1$ if and only if $\tilde{S}_p^2(\underline{I}) = 0$.

The quantity $\tilde{S}_p^2(\underline{I})$ in (16) can be rewritten as follows:

$$\tilde{S}_p^2(\underline{I}) = \frac{K}{p^2} \frac{1}{m} \sum_{j=1}^m \left(\left(\frac{I_j}{M_p(\underline{I})}\right)^p - 1\right)^2. \tag{43}$$

Taking the limit of (43) for $p \to -\infty$ and recalling that $M_p(\underline{I}) \underset{p\to-\infty}{\longrightarrow} \min(I_1, I_2, \dots, I_m) \leq I_j$, $j = 1, 2, \dots, m$, we have

$$\lim_{p\to-\infty} p\, \tilde{S}_p^2(\underline{I}) = \lim_{p\to-\infty} \frac{1}{p} = 0^-. \tag{44}$$

Substituting (44) into (15) we prove Property 5.

Analogously, taking the limit of (43) for $p \to +\infty$ and recalling that $M_p(\underline{I}) \underset{p\to+\infty}{\longrightarrow} \max(I_1, I_2, \dots, I_m) \geq I_j$, $j = 1, 2, \dots, m$, we obtain

$$\lim_{p\to+\infty} p\, \tilde{S}_p^2(\underline{I}) = \lim_{p\to+\infty} \frac{1}{p} = 0^+. \tag{45}$$

Substituting (45) into (15), we obtain Property 6.

Taking the limit for $p \to 0$ of $g_p^{\pm}(\underline{I})$, we obtain

$$\begin{aligned}\lim_{p\to 0} g_p^{\pm}(\underline{I}) &= \lim_{p\to 0} \left(1 \pm p\tilde{S}_p^2(\underline{I})\right)^{\frac{1}{p}} = \lim_{p\to 0} \exp\left\{\frac{\ln\left(1 \pm p\tilde{S}_p^2(\underline{I})\right)}{p}\right\} \\ &= \lim_{p\to 0} \exp\left\{\tilde{S}_p^2(\underline{I}) \frac{\ln\left(1 \pm p\tilde{S}_p^2(\underline{I})\right)}{p\tilde{S}_p^2(\underline{I})}\right\} \\ &= \exp\left\{\lim_{p\to 0} \tilde{S}_p^2(\underline{I}) \frac{\ln\left(1 \pm p\tilde{S}_p^2(\underline{I})\right)}{p\tilde{S}_p^2(\underline{I})}\right\}\end{aligned}$$

$$= \exp\left\{\tilde{S}_0^2(\underline{I})\right\} \exp\left\{\lim_{p\to 0} \frac{\ln\left(1 \pm p\tilde{S}_p^2(\underline{I})\right)}{p\tilde{S}_p^2(\underline{I})}\right\}, \tag{46}$$

and using the L'Hôpital's rule, we obtain

$$\lim_{p\to 0} g_p^{\pm}(\underline{I}) = \exp\left\{\pm\tilde{S}_0^2(\underline{I})\right\}. \tag{47}$$

This concludes the proof of Property 7.

Property 8 follows from Properties 3 and 6.

Property 9 follows from Properties 3 and 5.

Property 10 follows from the homogeneity property of the power means, observing that $g_p^{\pm}(c\,\underline{I}) = g_p^{\pm}(\underline{I})$.

To prove Property 11 and 12, it is enough to prove that the penalization factor $g_p^-(\underline{I} + \underline{c}) \geq g_p^-(\underline{I})$ and $g_p^+(\underline{I} + \underline{c}) \leq g_p^+(\underline{I})$, for any $\underline{c} \geq \underline{0}$.

Let $G_p^{\pm}(c) = g_p^{\pm}(\underline{I} + \underline{c})$. The derivative of $G_p^{\pm}$ with respect to $c$ is

$$\frac{dG_p^{\pm}(c)}{dc} = \pm(G_p^{\pm})^{1-p} \frac{\partial \tilde{S}_p^2(\underline{I} + \underline{c})}{\partial c}. \tag{48}$$

The derivative of $S_p^2(\underline{I} + \underline{c})$ with respect to $c$ is non-positive as proven in Proposition 2. This concludes the proof of Properties 11 and 12.

Property 8 implies that $PM_p^-(\underline{I}) \leq b$ for all choice of $K$ in Definition 1, moreover, Property 1 implies that $PM_p^-(\underline{a}) = a$, therefore it is always possible to find $K > 0$ such that $PM_p^-(\underline{I}) \geq a$ for all $\underline{I} \in [a,b]^m$. On the other hand, Property 9 implies that $PM_p^+(\underline{I}) \geq a$ for all choice of $K$ in Definition 1, moreover, Property 1 implies that $PM_p^+(\underline{b}) = b$, therefore it is always possible to find $K > 0$ such that $PM_p^+(\underline{I}) \leq b$ for all $\underline{I} \in [a,b]^m$.

This concludes the proof of Property 13 and the proof of the Proposition. □

## Appendix C. Detailed ranking corresponding to the composite indicators of Sections 4.1 and 4.2

Table 6 and Table 7 show the country rankings according to harmonic (HM), geometric (GM), and arithmetic (AM) means and their penalized versions denoted with PHM, PGM, and PAM. The prefix $r$ means rank.

## Appendix D. Benefit of the Doubt aggregative approach with relative importance constraints

In this Appendix we analyze the effect of penalization on the composite indicators defined by the Benefit of the Doubt (BoD) direct approach (see, Rogge (2018a) and Rogge (2018b)) when relative importance constraints on the weights are imposed.

The composite indicators obtained applying the "direct BoD approach" with relative importance constraints are given by (28), where, for each unit $i$, $i = 1,2,\ldots,n$, the weights $\pi_{i,p,j}$, $j = 1,2,\ldots,m$, with $p \neq 0$ are determined as solution of optimization problem (29)–(31) with the addition of the following constraints (see Rogge (2018a)):

$$\frac{\pi_{i,p,j}\, I_{i,j}^p}{\sum_{s=1}^m \pi_{c,p,s} I_{c,s}^p} \geq 0.1,\ \ c = 1,2,\ldots,n,\ j = 1,2,\ldots,m. \tag{49}$$

The constraints (49) guarantee that each indicator has a relative contribution of at least 10%.

The case $p = 0$ is obtained numerically by choosing $p$ close to zero (i.e., $p = 0.0001$). Results very close to $p = 0.0001$ are obtained by choosing $p = -0.0001$.

As done in Section 4, we solve Problem (29)–(31), (49) for each value $p_l = -M + (2l-1)$, $l = 1,2,\ldots,M$, with $M = 11$, and for each country/unit $i$, $i = 1,2,\ldots,n$. Here, the penalized power means are computed rescaling the quantities $\tilde{L}_{w,p,i}$, with $w = \pi$ and $p = p_l$, in (26) by choosing $K$ as follows:

$$K = \frac{0.01}{\max\limits_{\substack{i=1,2,\ldots,n,\\ l=1,2,\ldots,M}} \tilde{L}_{\pi,p_l,i}}. \tag{50}$$

This rescaling is necessary to make the loss of information $\tilde{S}_{w,p,i}^2$, with $w = \pi$ and $p = p_l$, comparable across different units $i$, $i = 1,2,\ldots,n$, and orders $p_l$, $l = 1,2,\ldots,M$. Note that, with the choice (50), the maximum value attained by the quantities $\tilde{S}_{w,p,i}^2$, with $w = \pi$ and $p = p_l$, is equal to 0.01. This is in line with the standardization process made by Mazziotta and Pareto, where the variance of the indicators is to 0.01 times its mean. Note that with this choice of $K$ the penalized power means are range preserving.

Table 8 shows the average values (left panel) and third quartile (right panel) of the weight distribution $\pi_{i,p_l,j}$, $i = 1,2,\ldots,n$ corresponding to the three sub-indicators (i.e. $j = 1,2,3$) and the eleven values of the order $p$. We recall that $\pi_{i,p,1}$, $\pi_{i,p,2}$, $\pi_{i,p,3}$ are, respectively, the weights relative to the three indicators Health ($H$), Education ($E$), and Income ($I$).

Comparing Table 8 with Table 3 we can observe that, although the average values and third quartiles of the country-specific weights of the health dimension continue to dominate for education and income for any $p = -5,-4,\ldots,4,5$, this dominance is weaker. Therefore, we can conclude that health and education play a more crucial role than income. Differently from the quartiles shown in Table 3, the third quartiles of income, although smaller than those of health and education, are not zero. This is the consequence of the relative importance constraints.

As done in Section 4, we analyze the distribution of the ranking difference $d_i(p) = r_{PBoD}(i,p) - r_{BoD}(i,p)$, $i = 1,2,\ldots,n$, induced by the composite indicators defined by the penalized BoD and BoD direct approaches with relative importance constraints, respectively. Fig. 6 shows the distribution of $d_i(p)$, $i = 1,2,\ldots$ for negative values of $p$, $p = -5,-4,-3,-2,-1$ and $p = 0$ (left panel), and for positive $p$, $p = 1,2,3,4,5$ and $p = 0$ (right panel).

The distributions shown in Fig. 6 are not symmetric with negative skewness except for $p = -3$ (i.e., −0.066, −0.948, 0.728, −0.681, −0.273, −0.195, −1.737, −0.435, −1.773, −2.206, −1.830, respectively, for $p$ varying from −5 to 5). A comparison between Figs. 3 and 6 shows that the standard deviations of the distributions of the BoD ranking differences (penalized minus non-penalized) with relative importance constraints are smaller and more variable with respect to those of BoD approach for all values of $p$. The values of the standard deviations are as follows: 0.365, 0.505, 0.461, 0.417, 0.623, 0.696, 0.999, 0.768, 0.988, 1.085, 1.229. This means that the use of relative importance constraints contributes further to reduce the effect of penalization.

We conclude investigating the role of the geometric mean in the BoD aggregation procedure with relative importance constraints. To this end, we show in Fig. 7 the graph of the objective function $F$ of Problem (27) as a function of $p$ when $p$ varies from −5 to 5 (left panel) and from −1 to 5 (right panel) when the BoD approach with relative importance constraints is used. Note that the addition of relative importance constraints to BoD approach makes the graph of function $F$ more similar in size and shape to the function $F$ associated with the power mean approach (see Fig. 2). Moreover, analogously to the power mean aggregation, the minimizer for both the functions shown in Fig. 7 is the middle point of the interval, i.e., $p = 0$ (left panel) and $p = 2$ (right panel).

Table 9 shows the first twenty (top panel) and the last twenty (bottom panel) countries according to the non penalized and penalized BoD with relative importance constraints.

To investigate the role of relative importance constraints, we compare the rankings obtained with the BoD without (Table 4) and with (Table 9) relative importance constraints. Note that, the rankings of Table 4 are obtained choosing $K = 1$. Nevertheless, the selection of $K$, as outlined in (50), yields identical rankings to those presented in Table 4. The comparison indicates that incorporating relative importance constraints into the BoD diminishes variations in rankings. This is likely due to the fact that the introduction of constraints reduces the admissible set of weights, and, as consequence, weakens the refining property of the penalization approach. This effect is more evident for $p = 0$ and $p = -1$.

**Table 6**
Country rankings, listed in alphabetical order, obtained with the power means (non penalized/penalized) and BoD direct country-specific weights (non-penalized/penalized). The method is "min" and three digits after the decimal are used to obtain the ranking.

| | "Power-mean ranking" | | | | | | "Benefit of the doubt ranking" | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Country | rHM | rGM (rHDI) | rAM | rPHM | rPGM | rPAM | rHM | rGM | rAM | rPHM | rPGM | PAM |
| Afghanistan | 167 | 169 | 169 | 168 | 168 | 168 | 157 | 157 | 157 | 157 | 157 | 157 |
| Albania | 69 | 69 | 67 | 68 | 68 | 69 | 55 | 55 | 55 | 52 | 52 | 52 |
| Algeria | 91 | 91 | 91 | 93 | 92 | 91 | 72 | 72 | 72 | 71 | 71 | 71 |
| Andorra | 37 | 36 | 36 | 44 | 40 | 37 | 30 | 29 | 29 | 29 | 29 | 29 |
| Angola | 147 | 148 | 150 | 145 | 145 | 147 | 176 | 176 | 176 | 176 | 176 | 176 |
| Antigua and Barbuda | 80 | 78 | 74 | 82 | 80 | 80 | 68 | 68 | 68 | 68 | 68 | 68 |
| Argentina | 45 | 46 | 47 | 44 | 45 | 45 | 46 | 46 | 46 | 47 | 47 | 47 |
| Armenia | 77 | 81 | 81 | 73 | 75 | 78 | 90 | 90 | 90 | 90 | 90 | 90 |
| Australia | 9 | 8 | 8 | 8 | 8 | 9 | 1 | 1 | 1 | 7 | 7 | 7 |
| Austria | 18 | 18 | 18 | 18 | 18 | 18 | 29 | 29 | 29 | 31 | 31 | 31 |
| Azerbaijan | 86 | 88 | 89 | 85 | 85 | 86 | 108 | 108 | 108 | 108 | 108 | 108 |
| Bahamas | 57 | 57 | 59 | 57 | 57 | 57 | 67 | 67 | 67 | 67 | 67 | 67 |
| Bahrain | 42 | 41 | 41 | 43 | 42 | 41 | 45 | 45 | 45 | 45 | 45 | 45 |
| Bangladesh | 133 | 133 | 133 | 136 | 133 | 133 | 113 | 113 | 113 | 113 | 113 | 113 |
| Barbados | 57 | 59 | 59 | 58 | 59 | 59 | 47 | 47 | 47 | 46 | 46 | 46 |
| Belarus | 53 | 53 | 53 | 51 | 51 | 53 | 61 | 61 | 61 | 62 | 62 | 62 |
| Belgium | 13 | 13 | 13 | 13 | 14 | 13 | 25 | 25 | 25 | 25 | 27 | 27 |
| Belize | 112 | 111 | 107 | 113 | 114 | 112 | 95 | 95 | 95 | 95 | 95 | 95 |
| Benin | 157 | 158 | 161 | 154 | 155 | 157 | 172 | 172 | 172 | 172 | 172 | 172 |
| Bhutan | 130 | 129 | 129 | 132 | 131 | 129 | 121 | 121 | 121 | 120 | 120 | 120 |
| Bolivia (Plurinational State of) | 107 | 107 | 109 | 103 | 105 | 107 | 124 | 124 | 124 | 124 | 124 | 124 |
| Bosnia and Herzegovina | 76 | 73 | 74 | 75 | 75 | 76 | 64 | 64 | 64 | 64 | 64 | 64 |
| Botswana | 96 | 100 | 101 | 96 | 96 | 96 | 131 | 131 | 131 | 132 | 132 | 132 |
| Brazil | 84 | 84 | 84 | 83 | 83 | 84 | 83 | 83 | 83 | 83 | 83 | 83 |
| Brunei Darussalam | 47 | 47 | 46 | 53 | 51 | 47 | 27 | 27 | 27 | 25 | 25 | 25 |
| Bulgaria | 55 | 56 | 58 | 52 | 54 | 55 | 83 | 83 | 83 | 85 | 85 | 85 |
| Burkina Faso | 182 | 182 | 181 | 182 | 182 | 182 | 173 | 173 | 173 | 173 | 173 | 173 |
| Burundi | 185 | 185 | 185 | 184 | 184 | 186 | 173 | 173 | 173 | 173 | 173 | 173 |
| Cabo Verde | 126 | 126 | 127 | 126 | 126 | 126 | 108 | 108 | 108 | 108 | 108 | 108 |
| Cambodia | 144 | 144 | 143 | 146 | 146 | 144 | 134 | 134 | 134 | 134 | 134 | 134 |
| Cameroon | 151 | 153 | 154 | 148 | 151 | 151 | 180 | 180 | 180 | 180 | 180 | 180 |
| Canada | 16 | 16 | 16 | 15 | 16 | 16 | 22 | 22 | 21 | 22 | 22 | 22 |
| Central African Republic | 187 | 187 | 189 | 185 | 187 | 187 | 189 | 189 | 189 | 189 | 189 | 189 |
| Chad | 188 | 188 | 188 | 187 | 188 | 188 | 188 | 188 | 188 | 188 | 188 | 188 |
| Chile | 42 | 43 | 44 | 41 | 42 | 43 | 44 | 44 | 44 | 43 | 43 | 43 |
| China | 85 | 85 | 85 | 87 | 86 | 85 | 72 | 72 | 72 | 71 | 71 | 71 |
| Colombia | 83 | 83 | 83 | 84 | 83 | 83 | 65 | 65 | 65 | 65 | 65 | 65 |
| Comoros | 153 | 156 | 156 | 152 | 152 | 152 | 161 | 161 | 161 | 161 | 161 | 161 |
| Congo | 149 | 149 | 151 | 149 | 149 | 149 | 158 | 158 | 158 | 158 | 158 | 158 |
| Congo (Democratic Republic of the) | 174 | 175 | 177 | 173 | 174 | 174 | 179 | 179 | 179 | 179 | 179 | 179 |
| Costa Rica | 63 | 63 | 62 | 65 | 64 | 63 | 43 | 43 | 43 | 42 | 42 | 42 |
| Croatia | 44 | 44 | 45 | 40 | 42 | 44 | 52 | 52 | 52 | 55 | 55 | 55 |
| Cuba | 73 | 70 | 70 | 78 | 77 | 73 | 51 | 51 | 51 | 51 | 51 | 51 |
| Cyprus | 32 | 33 | 33 | 31 | 31 | 32 | 39 | 39 | 39 | 39 | 39 | 39 |
| Czechia | 26 | 27 | 27 | 25 | 26 | 26 | 36 | 36 | 36 | 36 | 36 | 36 |
| Côte d'Ivoire | 160 | 162 | 162 | 156 | 157 | 160 | 182 | 182 | 182 | 182 | 182 | 182 |
| Denmark | 10 | 10 | 10 | 10 | 10 | 10 | 19 | 19 | 19 | 20 | 20 | 20 |
| Djibouti | 169 | 166 | 159 | 178 | 172 | 167 | 147 | 147 | 147 | 147 | 147 | 147 |
| Dominica | 96 | 93 | 94 | 99 | 99 | 98 | 58 | 58 | 58 | 57 | 57 | 57 |
| Dominican Republic | 89 | 88 | 88 | 90 | 89 | 89 | 101 | 101 | 101 | 101 | 101 | 101 |
| Ecuador | 86 | 86 | 86 | 86 | 87 | 86 | 68 | 68 | 68 | 68 | 68 | 68 |
| Egypt | 115 | 116 | 115 | 116 | 116 | 115 | 120 | 120 | 120 | 119 | 119 | 119 |
| El Salvador | 124 | 124 | 123 | 124 | 124 | 124 | 106 | 106 | 106 | 105 | 105 | 105 |
| Equatorial Guinea | 146 | 145 | 145 | 149 | 147 | 145 | 141 | 141 | 141 | 141 | 141 | 141 |
| Eritrea | 183 | 180 | 176 | 188 | 186 | 183 | 152 | 152 | 152 | 152 | 152 | 152 |
| Estonia | 29 | 29 | 30 | 28 | 28 | 29 | 38 | 38 | 38 | 38 | 38 | 38 |
| Eswatini (Kingdom of) | 138 | 138 | 142 | 133 | 136 | 138 | 168 | 168 | 168 | 168 | 168 | 168 |
| Ethiopia | 177 | 173 | 172 | 179 | 179 | 178 | 151 | 151 | 151 | 151 | 151 | 151 |
| Fiji | 92 | 93 | 96 | 91 | 91 | 92 | 115 | 115 | 115 | 115 | 115 | 115 |
| Finland | 11 | 11 | 12 | 11 | 11 | 11 | 15 | 15 | 15 | 15 | 15 | 15 |
| France | 27 | 26 | 26 | 27 | 27 | 27 | 30 | 29 | 29 | 29 | 29 | 29 |
| Gabon | 117 | 119 | 119 | 114 | 115 | 116 | 140 | 140 | 140 | 140 | 140 | 140 |
| Gambia | 172 | 172 | 173 | 169 | 169 | 172 | 171 | 171 | 171 | 170 | 170 | 170 |
| Georgia | 57 | 61 | 63 | 56 | 57 | 57 | 48 | 48 | 48 | 48 | 48 | 48 |
| Germany | 5 | 6 | 6 | 5 | 5 | 5 | 1 | 1 | 1 | 8 | 5 | 4 |
| Ghana | 137 | 138 | 140 | 133 | 136 | 137 | 163 | 163 | 163 | 163 | 163 | 163 |
| Greece | 31 | 31 | 32 | 30 | 30 | 31 | 34 | 34 | 34 | 34 | 34 | 34 |
| Grenada | 71 | 74 | 78 | 69 | 70 | 71 | 104 | 104 | 104 | 104 | 104 | 104 |
| Guatemala | 127 | 127 | 125 | 130 | 129 | 127 | 99 | 99 | 99 | 99 | 99 | 99 |
| Guinea | 177 | 178 | 179 | 176 | 175 | 175 | 173 | 173 | 173 | 173 | 173 | 173 |
| Guinea-Bissau | 173 | 175 | 180 | 170 | 170 | 173 | 183 | 183 | 183 | 183 | 183 | 183 |

(continued on next page)

**Table 6** (*continued*).

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Guyana | 121 | 122 | 122 | 121 | 121 | 121 | 133 | 133 | 133 | 133 | 133 | 133 |
| Haiti | 167 | 170 | 171 | 167 | 167 | 168 | 165 | 165 | 165 | 165 | 165 | 165 |
| Honduras | 134 | 132 | 131 | 138 | 138 | 134 | 87 | 87 | 87 | 87 | 87 | 87 |
| Hong Kong | 5 | 4 | 4 | 6 | 6 | 5 | 1 | 1 | 1 | 1 | 3 | 12 |
| Hungary | 40 | 40 | 41 | 38 | 39 | 40 | 57 | 57 | 57 | 58 | 58 | 58 |
| Iceland | 4 | 4 | 5 | 4 | 4 | 4 | 1 | 1 | 1 | 8 | 7 | 7 |
| India | 130 | 130 | 130 | 129 | 130 | 131 | 135 | 135 | 135 | 135 | 135 | 135 |
| Indonesia | 107 | 107 | 107 | 107 | 105 | 107 | 122 | 122 | 122 | 122 | 122 | 122 |
| Iran (Islamic Republic of) | 70 | 71 | 71 | 71 | 71 | 70 | 75 | 75 | 75 | 74 | 74 | 74 |
| Iraq | 123 | 123 | 123 | 123 | 123 | 123 | 129 | 129 | 129 | 129 | 129 | 129 |
| Ireland | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 5 | 4 | 3 |
| Israel | 19 | 19 | 21 | 19 | 19 | 19 | 21 | 21 | 21 | 22 | 22 | 22 |
| Italy | 30 | 29 | 29 | 32 | 31 | 30 | 25 | 25 | 25 | 22 | 22 | 22 |
| Jamaica | 100 | 101 | 100 | 97 | 98 | 100 | 96 | 96 | 96 | 96 | 96 | 96 |
| Japan | 21 | 21 | 21 | 22 | 21 | 21 | 11 | 11 | 11 | 11 | 11 | 10 |
| Jordan | 101 | 102 | 102 | 100 | 101 | 102 | 96 | 96 | 96 | 96 | 96 | 96 |
| Kazakhstan | 51 | 51 | 51 | 47 | 49 | 51 | 68 | 68 | 68 | 73 | 73 | 73 |
| Kenya | 142 | 143 | 143 | 139 | 140 | 142 | 150 | 150 | 150 | 150 | 150 | 150 |
| Kiribati | 132 | 134 | 134 | 131 | 132 | 132 | 141 | 141 | 141 | 141 | 141 | 141 |
| Korea (Republic of) | 23 | 24 | 24 | 21 | 21 | 23 | 28 | 28 | 28 | 28 | 28 | 28 |
| Kuwait | 67 | 64 | 56 | 75 | 69 | 66 | 32 | 32 | 32 | 32 | 32 | 32 |
| Kyrgyzstan | 120 | 120 | 120 | 120 | 120 | 120 | 117 | 117 | 117 | 117 | 117 | 117 |
| Lao People's Democratic Republic | 139 | 137 | 137 | 141 | 139 | 139 | 143 | 143 | 143 | 143 | 143 | 143 |
| Latvia | 36 | 37 | 38 | 36 | 36 | 36 | 41 | 41 | 41 | 41 | 41 | 41 |
| Lebanon | 96 | 92 | 90 | 102 | 100 | 96 | 49 | 49 | 49 | 48 | 48 | 48 |
| Lesotho | 163 | 165 | 168 | 158 | 159 | 162 | 186 | 186 | 186 | 186 | 186 | 186 |
| Liberia | 175 | 175 | 175 | 173 | 175 | 177 | 163 | 163 | 163 | 163 | 163 | 163 |
| Libya | 105 | 104 | 104 | 109 | 105 | 105 | 111 | 111 | 111 | 111 | 111 | 111 |
| Liechtenstein | 21 | 19 | 19 | 23 | 23 | 21 | 1 | 1 | 1 | 1 | 1 | 1 |
| Lithuania | 34 | 34 | 34 | 33 | 34 | 34 | 37 | 37 | 37 | 37 | 37 | 37 |
| Luxembourg | 24 | 22 | 20 | 24 | 24 | 24 | 11 | 11 | 11 | 11 | 11 | 10 |
| Madagascar | 165 | 163 | 162 | 166 | 166 | 165 | 149 | 149 | 149 | 149 | 149 | 149 |
| Malawi | 175 | 174 | 174 | 177 | 177 | 175 | 161 | 161 | 161 | 161 | 161 | 161 |
| Malaysia | 60 | 62 | 63 | 59 | 60 | 62 | 81 | 81 | 81 | 80 | 80 | 80 |
| Maldives | 101 | 95 | 91 | 114 | 109 | 101 | 49 | 49 | 49 | 48 | 48 | 48 |
| Mali | 186 | 184 | 183 | 186 | 184 | 185 | 181 | 181 | 181 | 181 | 181 | 181 |
| Malta | 28 | 28 | 28 | 29 | 28 | 28 | 32 | 32 | 32 | 32 | 32 | 32 |
| Marshall Islands | 119 | 117 | 115 | 119 | 119 | 118 | 101 | 101 | 101 | 101 | 101 | 101 |
| Mauritania | 161 | 157 | 156 | 163 | 162 | 161 | 156 | 156 | 156 | 156 | 156 | 156 |
| Mauritius | 65 | 66 | 66 | 63 | 64 | 65 | 87 | 87 | 87 | 87 | 87 | 87 |
| Mexico | 75 | 74 | 76 | 73 | 74 | 75 | 92 | 92 | 92 | 91 | 91 | 91 |
| Micronesia (Federated States of) | 136 | 136 | 136 | 133 | 134 | 136 | 143 | 143 | 143 | 143 | 143 | 143 |
| Moldova (Republic of) | 90 | 90 | 93 | 87 | 90 | 90 | 119 | 119 | 119 | 120 | 120 | 120 |
| Mongolia | 93 | 99 | 99 | 92 | 93 | 93 | 124 | 124 | 124 | 125 | 125 | 125 |
| Montenegro | 48 | 48 | 48 | 46 | 46 | 48 | 61 | 63 | 63 | 62 | 63 | 63 |
| Morocco | 122 | 121 | 121 | 122 | 122 | 122 | 75 | 75 | 75 | 74 | 74 | 74 |
| Mozambique | 181 | 181 | 182 | 180 | 180 | 181 | 178 | 178 | 178 | 178 | 178 | 178 |
| Myanmar | 148 | 147 | 146 | 151 | 149 | 148 | 147 | 147 | 147 | 147 | 147 | 147 |
| Namibia | 129 | 131 | 132 | 127 | 127 | 129 | 158 | 158 | 158 | 160 | 160 | 160 |
| Nepal | 143 | 142 | 140 | 143 | 143 | 143 | 128 | 128 | 128 | 128 | 128 | 128 |
| Netherlands | 7 | 8 | 8 | 8 | 8 | 7 | 16 | 16 | 16 | 16 | 16 | 16 |
| New Zealand | 15 | 15 | 15 | 14 | 15 | 15 | 11 | 11 | 11 | 13 | 13 | 13 |
| Nicaragua | 127 | 128 | 128 | 128 | 128 | 128 | 96 | 96 | 96 | 96 | 96 | 96 |
| Niger | 189 | 189 | 187 | 189 | 189 | 189 | 169 | 169 | 169 | 169 | 169 | 169 |
| Nigeria | 158 | 161 | 164 | 153 | 153 | 158 | 184 | 184 | 184 | 184 | 184 | 184 |
| North Macedonia | 82 | 82 | 82 | 79 | 80 | 82 | 85 | 85 | 85 | 84 | 84 | 84 |
| Norway | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 | 5 | 4 |
| Oman | 60 | 59 | 59 | 62 | 60 | 59 | 59 | 59 | 59 | 59 | 59 | 59 |
| Pakistan | 156 | 154 | 152 | 161 | 159 | 154 | 146 | 146 | 146 | 146 | 146 | 146 |
| Palau | 50 | 50 | 50 | 47 | 47 | 50 | 52 | 52 | 52 | 53 | 53 | 53 |
| Palestine | 115 | 115 | 114 | 117 | 117 | 116 | 103 | 103 | 103 | 103 | 103 | 103 |
| Panama | 60 | 57 | 55 | 63 | 62 | 59 | 56 | 56 | 56 | 55 | 55 | 55 |
| Papua New Guinea | 154 | 155 | 154 | 155 | 154 | 152 | 160 | 160 | 160 | 159 | 159 | 159 |
| Paraguay | 103 | 103 | 103 | 101 | 102 | 103 | 99 | 99 | 99 | 99 | 99 | 99 |
| Peru | 79 | 79 | 78 | 77 | 77 | 79 | 75 | 75 | 75 | 74 | 74 | 74 |
| Philippines | 107 | 109 | 111 | 103 | 105 | 107 | 126 | 126 | 126 | 126 | 126 | 126 |
| Poland | 35 | 35 | 35 | 34 | 35 | 35 | 40 | 40 | 40 | 40 | 40 | 40 |
| Portugal | 37 | 38 | 37 | 38 | 38 | 38 | 35 | 35 | 35 | 35 | 35 | 35 |
| Qatar | 46 | 45 | 39 | 60 | 53 | 46 | 1 | 1 | 1 | 1 | 1 | 1 |
| Romania | 49 | 49 | 49 | 49 | 47 | 49 | 79 | 79 | 79 | 79 | 79 | 79 |
| Russian Federation | 52 | 52 | 52 | 49 | 50 | 52 | 74 | 74 | 74 | 78 | 78 | 78 |
| Rwanda | 162 | 160 | 158 | 162 | 163 | 163 | 139 | 139 | 139 | 139 | 139 | 139 |
| Saint Kitts and Nevis | 77 | 74 | 73 | 79 | 79 | 76 | 89 | 89 | 89 | 89 | 89 | 89 |
| Saint Lucia | 86 | 86 | 86 | 87 | 87 | 86 | 82 | 82 | 82 | 82 | 82 | 82 |
| Saint Vincent and the Grenadines | 93 | 98 | 98 | 95 | 95 | 93 | 114 | 114 | 114 | 114 | 114 | 114 |
| Samoa | 111 | 111 | 109 | 112 | 112 | 111 | 106 | 106 | 106 | 105 | 105 | 105 |

(*continued on next page*)

**Table 6** (*continued*).

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sao Tome and Principe | 135 | 135 | 135 | 136 | 134 | 135 | 131 | 131 | 131 | 131 | 131 | 131 |
| Saudi Arabia | 41 | 41 | 41 | 41 | 41 | 41 | 42 | 42 | 42 | 43 | 43 | 43 |
| Senegal | 171 | 168 | 166 | 172 | 172 | 171 | 143 | 143 | 143 | 143 | 143 | 143 |
| Serbia | 64 | 64 | 65 | 60 | 63 | 63 | 80 | 80 | 80 | 80 | 80 | 80 |
| Seychelles | 66 | 67 | 67 | 66 | 66 | 67 | 91 | 91 | 91 | 92 | 92 | 92 |
| Sierra Leone | 180 | 182 | 183 | 175 | 178 | 179 | 187 | 187 | 187 | 187 | 187 | 187 |
| Singapore | 12 | 11 | 10 | 15 | 12 | 12 | 1 | 1 | 1 | 1 | 7 | 7 |
| Slovakia | 39 | 39 | 39 | 37 | 37 | 39 | 52 | 52 | 52 | 53 | 53 | 53 |
| Slovenia | 20 | 22 | 23 | 19 | 19 | 20 | 24 | 24 | 24 | 25 | 25 | 25 |
| Solomon Islands | 152 | 152 | 149 | 159 | 158 | 156 | 108 | 108 | 108 | 108 | 108 | 108 |
| South Africa | 113 | 114 | 118 | 109 | 111 | 113 | 136 | 136 | 136 | 136 | 136 | 136 |
| South Sudan | 184 | 186 | 186 | 183 | 183 | 184 | 185 | 185 | 185 | 185 | 185 | 185 |
| Spain | 25 | 25 | 25 | 26 | 25 | 25 | 23 | 23 | 23 | 21 | 21 | 21 |
| Sri Lanka | 71 | 72 | 71 | 72 | 73 | 71 | 68 | 68 | 68 | 68 | 68 | 68 |
| Sudan | 170 | 170 | 167 | 171 | 170 | 170 | 155 | 155 | 155 | 155 | 155 | 155 |
| Suriname | 93 | 95 | 97 | 94 | 94 | 93 | 122 | 122 | 122 | 122 | 122 | 122 |
| Sweden | 7 | 7 | 7 | 7 | 7 | 7 | 14 | 14 | 14 | 13 | 13 | 13 |
| Switzerland | 2 | 2 | 2 | 3 | 3 | 2 | 1 | 1 | 1 | 5 | 7 | 4 |
| Syrian Arab Republic | 154 | 151 | 147 | 160 | 161 | 155 | 112 | 112 | 112 | 112 | 112 | 112 |
| Tajikistan | 125 | 125 | 126 | 125 | 125 | 125 | 127 | 127 | 127 | 127 | 127 | 127 |
| Tanzania | 164 | 163 | 165 | 164 | 164 | 164 | 154 | 154 | 154 | 154 | 154 | 154 |
| Thailand | 81 | 79 | 76 | 81 | 80 | 81 | 66 | 66 | 66 | 66 | 66 | 66 |
| Timor-Leste | 141 | 141 | 139 | 142 | 142 | 141 | 137 | 137 | 137 | 136 | 136 | 136 |
| Togo | 166 | 167 | 170 | 165 | 165 | 166 | 177 | 177 | 177 | 177 | 177 | 177 |
| Tonga | 104 | 104 | 105 | 103 | 103 | 104 | 105 | 105 | 105 | 107 | 107 | 107 |
| Trinidad and Tobago | 67 | 67 | 69 | 66 | 66 | 68 | 94 | 94 | 94 | 94 | 94 | 94 |
| Tunisia | 96 | 95 | 95 | 97 | 97 | 98 | 75 | 75 | 75 | 74 | 74 | 74 |
| Turkey | 54 | 54 | 53 | 55 | 55 | 54 | 61 | 61 | 61 | 61 | 61 | 61 |
| Turkmenistan | 110 | 110 | 112 | 107 | 109 | 110 | 138 | 138 | 138 | 138 | 138 | 138 |
| Uganda | 159 | 158 | 160 | 156 | 156 | 159 | 167 | 167 | 167 | 167 | 167 | 167 |
| Ukraine | 73 | 74 | 78 | 70 | 71 | 73 | 92 | 92 | 92 | 92 | 92 | 92 |
| United Arab Emirates | 32 | 31 | 30 | 34 | 33 | 32 | 20 | 20 | 20 | 19 | 19 | 19 |
| United Kingdom | 13 | 13 | 13 | 12 | 12 | 13 | 17 | 17 | 17 | 18 | 18 | 18 |
| United States | 17 | 17 | 17 | 17 | 17 | 17 | 18 | 18 | 18 | 17 | 17 | 17 |
| Uruguay | 56 | 55 | 56 | 54 | 56 | 56 | 59 | 59 | 59 | 59 | 59 | 59 |
| Uzbekistan | 106 | 106 | 106 | 103 | 104 | 106 | 115 | 115 | 115 | 116 | 116 | 116 |
| Vanuatu | 140 | 140 | 138 | 140 | 140 | 140 | 130 | 130 | 130 | 130 | 130 | 130 |
| Venezuela | 113 | 113 | 113 | 111 | 113 | 113 | 118 | 118 | 118 | 117 | 117 | 117 |
| Viet Nam | 118 | 117 | 115 | 118 | 118 | 118 | 86 | 86 | 86 | 86 | 86 | 86 |
| Yemen | 179 | 179 | 177 | 181 | 181 | 180 | 153 | 153 | 153 | 153 | 153 | 153 |
| Zambia | 145 | 146 | 148 | 143 | 144 | 145 | 166 | 166 | 166 | 166 | 166 | 166 |
| Zimbabwe | 150 | 150 | 152 | 147 | 148 | 149 | 170 | 170 | 170 | 170 | 170 | 170 |

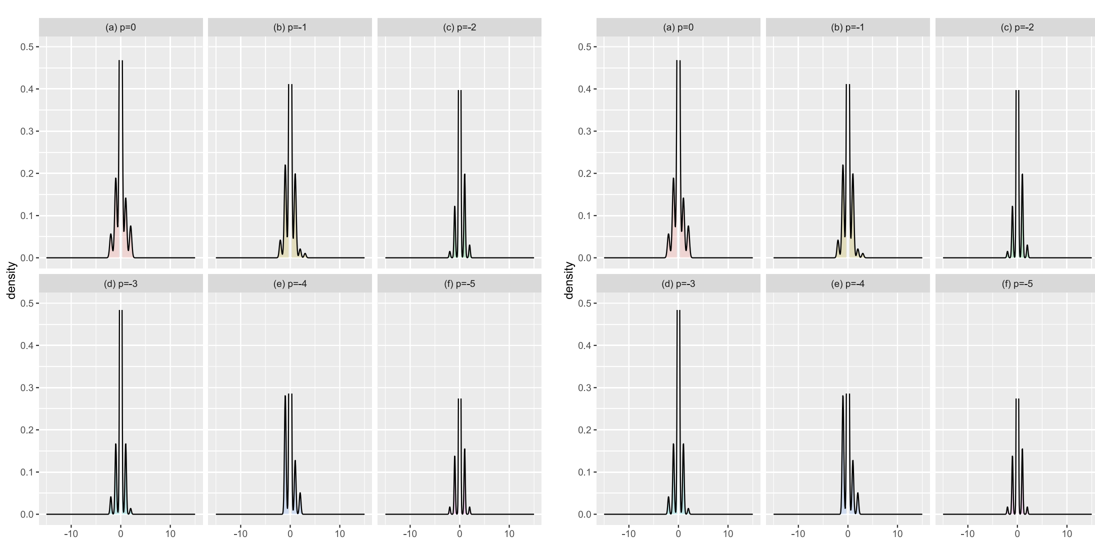


**Fig. 6.** Distribution of ranking difference $d_i(p) = r_{PBoD}(i,p) - r_{BoD}(i,p)$, $i = 1, 2, \dots, n$ (i.e., penalized BoD composite indicator of order $p$ minus the corresponding BoD composite indicator of order $p$ with relative importance constraints) for $p = -5, -4, -3, -2, -1, 0$ (left panel) $p = 0, 1, 2, 3, 4, 5$ (right panel).

**Table 7**
Country rankings obtained with the power means (non penalized/penalized) and BoD direct country-specific weights (non-penalized/penalized). The method is "min" and three digits after the decimal are used to obtain the ranking. The countries are ordered by HDI (rGM) rank.

| | "Power-mean ranking" | | | | | | "Benefit of the doubt ranking" | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Country | rHM | rGM (rHDI) | rAM | rPHM | rPGM | rPAM | rHM | rGM | rAM | rPHM | rPGM | PAM |
| Norway | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 | 5 | 4 |
| Ireland | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 5 | 4 | 3 |
| Switzerland | 2 | 2 | 2 | 3 | 3 | 2 | 1 | 1 | 1 | 5 | 7 | 4 |
| Hong Kong | 5 | 4 | 4 | 6 | 6 | 5 | 1 | 1 | 1 | 1 | 3 | 12 |
| Iceland | 4 | 4 | 5 | 4 | 4 | 4 | 1 | 1 | 1 | 8 | 7 | 7 |
| Germany | 5 | 6 | 6 | 5 | 5 | 5 | 1 | 1 | 1 | 8 | 5 | 4 |
| Sweden | 7 | 7 | 7 | 7 | 7 | 7 | 14 | 14 | 14 | 13 | 13 | 13 |
| Australia | 9 | 8 | 8 | 8 | 8 | 9 | 1 | 1 | 1 | 7 | 7 | 7 |
| Netherlands | 7 | 8 | 8 | 8 | 8 | 7 | 16 | 16 | 16 | 16 | 16 | 16 |
| Denmark | 10 | 10 | 10 | 10 | 10 | 10 | 19 | 19 | 19 | 20 | 20 | 20 |
| Finland | 11 | 11 | 12 | 11 | 11 | 11 | 15 | 15 | 15 | 15 | 15 | 15 |
| Singapore | 12 | 11 | 10 | 15 | 12 | 12 | 1 | 1 | 1 | 1 | 7 | 7 |
| Belgium | 13 | 13 | 13 | 13 | 14 | 13 | 25 | 25 | 25 | 25 | 27 | 27 |
| United Kingdom | 13 | 13 | 13 | 12 | 12 | 13 | 17 | 17 | 17 | 18 | 18 | 18 |
| New Zealand | 15 | 15 | 15 | 14 | 15 | 15 | 11 | 11 | 11 | 13 | 13 | 13 |
| Canada | 16 | 16 | 16 | 15 | 16 | 16 | 22 | 22 | 21 | 22 | 22 | 22 |
| United States | 17 | 17 | 17 | 17 | 17 | 17 | 18 | 18 | 18 | 17 | 17 | 17 |
| Austria | 18 | 18 | 18 | 18 | 18 | 18 | 29 | 29 | 29 | 31 | 31 | 31 |
| Israel | 19 | 19 | 21 | 19 | 19 | 19 | 21 | 21 | 21 | 22 | 22 | 22 |
| Liechtenstein | 21 | 19 | 19 | 23 | 23 | 21 | 1 | 1 | 1 | 1 | 1 | 1 |
| Japan | 21 | 21 | 21 | 22 | 21 | 21 | 11 | 11 | 11 | 11 | 11 | 10 |
| Luxembourg | 24 | 22 | 20 | 24 | 24 | 24 | 11 | 11 | 11 | 11 | 11 | 10 |
| Slovenia | 20 | 22 | 23 | 19 | 19 | 20 | 24 | 24 | 24 | 25 | 25 | 25 |
| Korea (Republic of) | 23 | 24 | 24 | 21 | 21 | 23 | 28 | 28 | 28 | 28 | 28 | 28 |
| Spain | 25 | 25 | 25 | 26 | 25 | 25 | 23 | 23 | 23 | 21 | 21 | 21 |
| France | 27 | 26 | 26 | 27 | 27 | 27 | 30 | 29 | 29 | 29 | 29 | 29 |
| Czechia | 26 | 27 | 27 | 25 | 26 | 26 | 36 | 36 | 36 | 36 | 36 | 36 |
| Malta | 28 | 28 | 28 | 29 | 28 | 28 | 32 | 32 | 32 | 32 | 32 | 32 |
| Estonia | 29 | 29 | 30 | 28 | 28 | 29 | 38 | 38 | 38 | 38 | 38 | 38 |
| Italy | 30 | 29 | 29 | 32 | 31 | 30 | 25 | 25 | 25 | 22 | 22 | 22 |
| Greece | 31 | 31 | 32 | 30 | 30 | 31 | 34 | 34 | 34 | 34 | 34 | 34 |
| United Arab Emirates | 32 | 31 | 30 | 34 | 33 | 32 | 20 | 20 | 20 | 19 | 19 | 19 |
| Cyprus | 32 | 33 | 33 | 31 | 31 | 32 | 39 | 39 | 39 | 39 | 39 | 39 |
| Lithuania | 34 | 34 | 34 | 33 | 34 | 34 | 37 | 37 | 37 | 37 | 37 | 37 |
| Poland | 35 | 35 | 35 | 34 | 35 | 35 | 40 | 40 | 40 | 40 | 40 | 40 |
| Andorra | 37 | 36 | 36 | 44 | 40 | 37 | 30 | 29 | 29 | 29 | 29 | 29 |
| Latvia | 36 | 37 | 38 | 36 | 36 | 36 | 41 | 41 | 41 | 41 | 41 | 41 |
| Portugal | 37 | 38 | 37 | 38 | 38 | 38 | 35 | 35 | 35 | 35 | 35 | 35 |
| Slovakia | 39 | 39 | 39 | 37 | 37 | 39 | 52 | 52 | 52 | 53 | 53 | 53 |
| Hungary | 40 | 40 | 41 | 38 | 39 | 40 | 57 | 57 | 57 | 58 | 58 | 58 |
| Bahrain | 42 | 41 | 41 | 43 | 42 | 41 | 45 | 45 | 45 | 45 | 45 | 45 |
| Saudi Arabia | 41 | 41 | 41 | 41 | 41 | 41 | 42 | 42 | 42 | 43 | 43 | 43 |
| Chile | 42 | 43 | 44 | 41 | 42 | 43 | 44 | 44 | 44 | 43 | 43 | 43 |
| Croatia | 44 | 44 | 45 | 40 | 42 | 44 | 52 | 52 | 52 | 55 | 55 | 55 |
| Qatar | 46 | 45 | 39 | 60 | 53 | 46 | 1 | 1 | 1 | 1 | 1 | 1 |
| Argentina | 45 | 46 | 47 | 44 | 45 | 45 | 46 | 46 | 46 | 47 | 47 | 47 |
| Brunei Darussalam | 47 | 47 | 46 | 53 | 51 | 47 | 27 | 27 | 27 | 25 | 25 | 25 |
| Montenegro | 48 | 48 | 48 | 46 | 46 | 48 | 61 | 63 | 63 | 62 | 63 | 63 |
| Romania | 49 | 49 | 49 | 49 | 47 | 49 | 79 | 79 | 79 | 79 | 79 | 79 |
| Palau | 50 | 50 | 50 | 47 | 47 | 50 | 52 | 52 | 52 | 53 | 53 | 53 |
| Kazakhstan | 51 | 51 | 51 | 47 | 49 | 51 | 68 | 68 | 68 | 73 | 73 | 73 |
| Russian Federation | 52 | 52 | 52 | 49 | 50 | 52 | 74 | 74 | 74 | 78 | 78 | 78 |
| Belarus | 53 | 53 | 53 | 51 | 51 | 53 | 61 | 61 | 61 | 62 | 62 | 62 |
| Turkey | 54 | 54 | 53 | 55 | 55 | 54 | 61 | 61 | 61 | 61 | 61 | 61 |
| Uruguay | 56 | 55 | 56 | 54 | 56 | 56 | 59 | 59 | 59 | 59 | 59 | 59 |
| Bulgaria | 55 | 56 | 58 | 52 | 54 | 55 | 83 | 83 | 83 | 85 | 85 | 85 |
| Bahamas | 57 | 57 | 59 | 57 | 57 | 57 | 67 | 67 | 67 | 67 | 67 | 67 |
| Panama | 60 | 57 | 55 | 63 | 62 | 59 | 56 | 56 | 56 | 55 | 55 | 55 |
| Barbados | 57 | 59 | 59 | 58 | 59 | 59 | 47 | 47 | 47 | 46 | 46 | 46 |
| Oman | 60 | 59 | 59 | 62 | 60 | 59 | 59 | 59 | 59 | 59 | 59 | 59 |
| Georgia | 57 | 61 | 63 | 56 | 57 | 57 | 48 | 48 | 48 | 48 | 48 | 48 |
| Malaysia | 60 | 62 | 63 | 59 | 60 | 62 | 81 | 81 | 81 | 80 | 80 | 80 |
| Costa Rica | 63 | 63 | 62 | 65 | 64 | 63 | 43 | 43 | 43 | 42 | 42 | 42 |
| Kuwait | 67 | 64 | 56 | 75 | 69 | 66 | 32 | 32 | 32 | 32 | 32 | 32 |
| Serbia | 64 | 64 | 65 | 60 | 63 | 63 | 80 | 80 | 80 | 80 | 80 | 80 |
| Mauritius | 65 | 66 | 66 | 63 | 64 | 65 | 87 | 87 | 87 | 87 | 87 | 87 |
| Seychelles | 66 | 67 | 67 | 66 | 66 | 67 | 91 | 91 | 91 | 92 | 92 | 92 |
| Trinidad and Tobago | 67 | 67 | 69 | 66 | 66 | 68 | 94 | 94 | 94 | 94 | 94 | 94 |
| Albania | 69 | 69 | 67 | 68 | 68 | 69 | 55 | 55 | 55 | 52 | 52 | 52 |
| Cuba | 73 | 70 | 70 | 78 | 77 | 73 | 51 | 51 | 51 | 51 | 51 | 51 |

(continued on next page)

**Table 7** (*continued*).

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Iran (Islamic Republic of) | 70 | 71 | 71 | 71 | 71 | 70 | 75 | 75 | 75 | 74 | 74 | 74 |
| Sri Lanka | 71 | 72 | 71 | 72 | 73 | 71 | 68 | 68 | 68 | 68 | 68 | 68 |
| Bosnia and Herzegovina | 76 | 73 | 74 | 75 | 75 | 76 | 64 | 64 | 64 | 64 | 64 | 64 |
| Grenada | 71 | 74 | 78 | 69 | 70 | 71 | 104 | 104 | 104 | 104 | 104 | 104 |
| Mexico | 75 | 74 | 76 | 73 | 74 | 75 | 92 | 92 | 92 | 91 | 91 | 91 |
| Saint Kitts and Nevis | 77 | 74 | 73 | 79 | 79 | 76 | 89 | 89 | 89 | 89 | 89 | 89 |
| Ukraine | 73 | 74 | 78 | 70 | 71 | 73 | 92 | 92 | 92 | 92 | 92 | 92 |
| Antigua and Barbuda | 80 | 78 | 74 | 82 | 80 | 80 | 68 | 68 | 68 | 68 | 68 | 68 |
| Peru | 79 | 79 | 78 | 77 | 77 | 79 | 75 | 75 | 75 | 74 | 74 | 74 |
| Thailand | 81 | 79 | 76 | 81 | 80 | 81 | 66 | 66 | 66 | 66 | 66 | 66 |
| Armenia | 77 | 81 | 81 | 73 | 75 | 78 | 90 | 90 | 90 | 90 | 90 | 90 |
| North Macedonia | 82 | 82 | 82 | 79 | 80 | 82 | 85 | 85 | 85 | 84 | 84 | 84 |
| Colombia | 83 | 83 | 83 | 84 | 83 | 83 | 65 | 65 | 65 | 65 | 65 | 65 |
| Brazil | 84 | 84 | 84 | 83 | 83 | 84 | 83 | 83 | 83 | 83 | 83 | 83 |
| China | 85 | 85 | 85 | 87 | 86 | 85 | 72 | 72 | 72 | 71 | 71 | 71 |
| Ecuador | 86 | 86 | 86 | 86 | 87 | 86 | 68 | 68 | 68 | 68 | 68 | 68 |
| Saint Lucia | 86 | 86 | 86 | 87 | 87 | 86 | 82 | 82 | 82 | 82 | 82 | 82 |
| Azerbaijan | 86 | 88 | 89 | 85 | 85 | 86 | 108 | 108 | 108 | 108 | 108 | 108 |
| Dominican Republic | 89 | 88 | 88 | 90 | 89 | 89 | 101 | 101 | 101 | 101 | 101 | 101 |
| Moldova (Republic of) | 90 | 90 | 93 | 87 | 90 | 90 | 119 | 119 | 119 | 120 | 120 | 120 |
| Algeria | 91 | 91 | 91 | 93 | 92 | 91 | 72 | 72 | 72 | 71 | 71 | 71 |
| Lebanon | 96 | 92 | 90 | 102 | 100 | 96 | 49 | 49 | 49 | 48 | 48 | 48 |
| Dominica | 96 | 93 | 94 | 99 | 99 | 98 | 58 | 58 | 58 | 57 | 57 | 57 |
| Fiji | 92 | 93 | 96 | 91 | 91 | 92 | 115 | 115 | 115 | 115 | 115 | 115 |
| Maldives | 101 | 95 | 91 | 114 | 109 | 101 | 49 | 49 | 49 | 48 | 48 | 48 |
| Suriname | 93 | 95 | 97 | 94 | 94 | 93 | 122 | 122 | 122 | 122 | 122 | 122 |
| Tunisia | 96 | 95 | 95 | 97 | 97 | 98 | 75 | 75 | 75 | 74 | 74 | 74 |
| Saint Vincent and the Grenadines | 93 | 98 | 98 | 95 | 95 | 93 | 114 | 114 | 114 | 114 | 114 | 114 |
| Mongolia | 93 | 99 | 99 | 92 | 93 | 93 | 124 | 124 | 124 | 125 | 125 | 125 |
| Botswana | 96 | 100 | 101 | 96 | 96 | 96 | 131 | 131 | 131 | 132 | 132 | 132 |
| Jamaica | 100 | 101 | 100 | 97 | 98 | 100 | 96 | 96 | 96 | 96 | 96 | 96 |
| Jordan | 101 | 102 | 102 | 100 | 101 | 102 | 96 | 96 | 96 | 96 | 96 | 96 |
| Paraguay | 103 | 103 | 103 | 101 | 102 | 103 | 99 | 99 | 99 | 99 | 99 | 99 |
| Libya | 105 | 104 | 104 | 109 | 105 | 105 | 111 | 111 | 111 | 111 | 111 | 111 |
| Tonga | 104 | 104 | 105 | 103 | 103 | 104 | 105 | 105 | 105 | 107 | 107 | 107 |
| Uzbekistan | 106 | 106 | 106 | 103 | 104 | 106 | 115 | 115 | 115 | 116 | 116 | 116 |
| Bolivia (Plurinational State of) | 107 | 107 | 109 | 103 | 105 | 107 | 124 | 124 | 124 | 124 | 124 | 124 |
| Indonesia | 107 | 107 | 107 | 107 | 105 | 107 | 122 | 122 | 122 | 122 | 122 | 122 |
| Philippines | 107 | 109 | 111 | 103 | 105 | 107 | 126 | 126 | 126 | 126 | 126 | 126 |
| Turkmenistan | 110 | 110 | 112 | 107 | 109 | 110 | 138 | 138 | 138 | 138 | 138 | 138 |
| Belize | 112 | 111 | 107 | 113 | 114 | 112 | 95 | 95 | 95 | 95 | 95 | 95 |
| Samoa | 111 | 111 | 109 | 112 | 112 | 111 | 106 | 106 | 106 | 105 | 105 | 105 |
| Venezuela (Bolivarian Republic of) | 113 | 113 | 113 | 111 | 113 | 113 | 118 | 118 | 118 | 117 | 117 | 117 |
| South Africa | 113 | 114 | 118 | 109 | 111 | 113 | 136 | 136 | 136 | 136 | 136 | 136 |
| Palestine | 115 | 115 | 114 | 117 | 117 | 116 | 103 | 103 | 103 | 103 | 103 | 103 |
| Egypt | 115 | 116 | 115 | 116 | 116 | 115 | 120 | 120 | 120 | 119 | 119 | 119 |
| Marshall Islands | 119 | 117 | 115 | 119 | 119 | 118 | 101 | 101 | 101 | 101 | 101 | 101 |
| Viet Nam | 118 | 117 | 115 | 118 | 118 | 118 | 86 | 86 | 86 | 86 | 86 | 86 |
| Gabon | 117 | 119 | 119 | 114 | 115 | 116 | 140 | 140 | 140 | 140 | 140 | 140 |
| Kyrgyzstan | 120 | 120 | 120 | 120 | 120 | 120 | 117 | 117 | 117 | 117 | 117 | 117 |
| Morocco | 122 | 121 | 121 | 122 | 122 | 122 | 75 | 75 | 75 | 74 | 74 | 74 |
| Guyana | 121 | 122 | 122 | 121 | 121 | 121 | 133 | 133 | 133 | 133 | 133 | 133 |
| Iraq | 123 | 123 | 123 | 123 | 123 | 123 | 129 | 129 | 129 | 129 | 129 | 129 |
| El Salvador | 124 | 124 | 123 | 124 | 124 | 124 | 106 | 106 | 106 | 105 | 105 | 105 |
| Tajikistan | 125 | 125 | 126 | 125 | 125 | 125 | 127 | 127 | 127 | 127 | 127 | 127 |
| Cabo Verde | 126 | 126 | 127 | 126 | 126 | 126 | 108 | 108 | 108 | 108 | 108 | 108 |
| Guatemala | 127 | 127 | 125 | 130 | 129 | 127 | 99 | 99 | 99 | 99 | 99 | 99 |
| Nicaragua | 127 | 128 | 128 | 128 | 128 | 128 | 96 | 96 | 96 | 96 | 96 | 96 |
| Bhutan | 130 | 129 | 129 | 132 | 131 | 129 | 121 | 121 | 121 | 120 | 120 | 120 |
| India | 130 | 130 | 130 | 129 | 130 | 131 | 135 | 135 | 135 | 135 | 135 | 135 |
| Namibia | 129 | 131 | 132 | 127 | 127 | 129 | 158 | 158 | 158 | 160 | 160 | 160 |
| Honduras | 134 | 132 | 131 | 138 | 138 | 134 | 87 | 87 | 87 | 87 | 87 | 87 |
| Bangladesh | 133 | 133 | 133 | 136 | 133 | 133 | 113 | 113 | 113 | 113 | 113 | 113 |
| Kiribati | 132 | 134 | 134 | 131 | 132 | 132 | 141 | 141 | 141 | 141 | 141 | 141 |
| Sao Tome and Principe | 135 | 135 | 135 | 136 | 134 | 135 | 131 | 131 | 131 | 131 | 131 | 131 |
| Micronesia (Federated States of) | 136 | 136 | 136 | 133 | 134 | 136 | 143 | 143 | 143 | 143 | 143 | 143 |
| Lao People's Democratic Republic | 139 | 137 | 137 | 141 | 139 | 139 | 143 | 143 | 143 | 143 | 143 | 143 |
| Eswatini (Kingdom of) | 138 | 138 | 142 | 133 | 136 | 138 | 168 | 168 | 168 | 168 | 168 | 168 |
| Ghana | 137 | 138 | 140 | 133 | 136 | 137 | 163 | 163 | 163 | 163 | 163 | 163 |
| Vanuatu | 140 | 140 | 138 | 140 | 140 | 140 | 130 | 130 | 130 | 130 | 130 | 130 |
| Timor-Leste | 141 | 141 | 139 | 142 | 142 | 141 | 137 | 137 | 137 | 136 | 136 | 136 |
| Nepal | 143 | 142 | 140 | 143 | 143 | 143 | 128 | 128 | 128 | 128 | 128 | 128 |
| Kenya | 142 | 143 | 143 | 139 | 140 | 142 | 150 | 150 | 150 | 150 | 150 | 150 |
| Cambodia | 144 | 144 | 143 | 146 | 146 | 144 | 134 | 134 | 134 | 134 | 134 | 134 |
| Equatorial Guinea | 146 | 145 | 145 | 149 | 147 | 145 | 141 | 141 | 141 | 141 | 141 | 141 |

(*continued on next page*)

**Table 7** (*continued*).

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Zambia | 145 | 146 | 148 | 143 | 144 | 145 | 166 | 166 | 166 | 166 | 166 | 166 |
| Myanmar | 148 | 147 | 146 | 151 | 149 | 148 | 147 | 147 | 147 | 147 | 147 | 147 |
| Angola | 147 | 148 | 150 | 145 | 145 | 147 | 176 | 176 | 176 | 176 | 176 | 176 |
| Congo | 149 | 149 | 151 | 149 | 149 | 149 | 158 | 158 | 158 | 158 | 158 | 158 |
| Zimbabwe | 150 | 150 | 152 | 147 | 148 | 149 | 170 | 170 | 170 | 170 | 170 | 170 |
| Syrian Arab Republic | 154 | 151 | 147 | 160 | 161 | 155 | 112 | 112 | 112 | 112 | 112 | 112 |
| Solomon Islands | 152 | 152 | 149 | 159 | 158 | 156 | 108 | 108 | 108 | 108 | 108 | 108 |
| Cameroon | 151 | 153 | 154 | 148 | 151 | 151 | 180 | 180 | 180 | 180 | 180 | 180 |
| Pakistan | 156 | 154 | 152 | 161 | 159 | 154 | 146 | 146 | 146 | 146 | 146 | 146 |
| Papua New Guinea | 154 | 155 | 154 | 155 | 154 | 152 | 160 | 160 | 160 | 159 | 159 | 159 |
| Comoros | 153 | 156 | 156 | 152 | 152 | 152 | 161 | 161 | 161 | 161 | 161 | 161 |
| Mauritania | 161 | 157 | 156 | 163 | 162 | 161 | 156 | 156 | 156 | 156 | 156 | 156 |
| Benin | 157 | 158 | 161 | 154 | 155 | 157 | 172 | 172 | 172 | 172 | 172 | 172 |
| Uganda | 159 | 158 | 160 | 156 | 156 | 159 | 167 | 167 | 167 | 167 | 167 | 167 |
| Rwanda | 162 | 160 | 158 | 162 | 163 | 163 | 139 | 139 | 139 | 139 | 139 | 139 |
| Nigeria | 158 | 161 | 164 | 153 | 153 | 158 | 184 | 184 | 184 | 184 | 184 | 184 |
| Côte d'Ivoire | 160 | 162 | 162 | 156 | 157 | 160 | 182 | 182 | 182 | 182 | 182 | 182 |
| Madagascar | 165 | 163 | 162 | 166 | 166 | 165 | 149 | 149 | 149 | 149 | 149 | 149 |
| Tanzania (United Republic of) | 164 | 163 | 165 | 164 | 164 | 164 | 154 | 154 | 154 | 154 | 154 | 154 |
| Lesotho | 163 | 165 | 168 | 158 | 159 | 162 | 186 | 186 | 186 | 186 | 186 | 186 |
| Djibouti | 169 | 166 | 159 | 178 | 172 | 167 | 147 | 147 | 147 | 147 | 147 | 147 |
| Togo | 166 | 167 | 170 | 165 | 165 | 166 | 177 | 177 | 177 | 177 | 177 | 177 |
| Senegal | 171 | 168 | 166 | 172 | 172 | 171 | 143 | 143 | 143 | 143 | 143 | 143 |
| Afghanistan | 167 | 169 | 169 | 168 | 168 | 168 | 157 | 157 | 157 | 157 | 157 | 157 |
| Haiti | 167 | 170 | 171 | 167 | 167 | 168 | 165 | 165 | 165 | 165 | 165 | 165 |
| Sudan | 170 | 170 | 167 | 171 | 170 | 170 | 155 | 155 | 155 | 155 | 155 | 155 |
| Gambia | 172 | 172 | 173 | 169 | 169 | 172 | 171 | 171 | 171 | 170 | 170 | 170 |
| Ethiopia | 177 | 173 | 172 | 179 | 179 | 178 | 151 | 151 | 151 | 151 | 151 | 151 |
| Malawi | 175 | 174 | 174 | 177 | 177 | 175 | 161 | 161 | 161 | 161 | 161 | 161 |
| Congo (Democratic Republic of the) | 174 | 175 | 177 | 173 | 174 | 174 | 179 | 179 | 179 | 179 | 179 | 179 |
| Guinea-Bissau | 173 | 175 | 180 | 170 | 170 | 173 | 183 | 183 | 183 | 183 | 183 | 183 |
| Liberia | 175 | 175 | 175 | 173 | 175 | 177 | 163 | 163 | 163 | 163 | 163 | 163 |
| Guinea | 177 | 178 | 179 | 176 | 175 | 175 | 173 | 173 | 173 | 173 | 173 | 173 |
| Yemen | 179 | 179 | 177 | 181 | 181 | 180 | 153 | 153 | 153 | 153 | 153 | 153 |
| Eritrea | 183 | 180 | 176 | 188 | 186 | 183 | 152 | 152 | 152 | 152 | 152 | 152 |
| Mozambique | 181 | 181 | 182 | 180 | 180 | 181 | 178 | 178 | 178 | 178 | 178 | 178 |
| Burkina Faso | 182 | 182 | 181 | 182 | 182 | 182 | 173 | 173 | 173 | 173 | 173 | 173 |
| Sierra Leone | 180 | 182 | 183 | 175 | 178 | 179 | 187 | 187 | 187 | 187 | 187 | 187 |
| Mali | 186 | 184 | 183 | 186 | 184 | 185 | 181 | 181 | 181 | 181 | 181 | 181 |
| Burundi | 185 | 185 | 185 | 184 | 184 | 186 | 173 | 173 | 173 | 173 | 173 | 173 |
| South Sudan | 184 | 186 | 186 | 183 | 183 | 184 | 185 | 185 | 185 | 185 | 185 | 185 |
| Central African Republic | 187 | 187 | 189 | 185 | 187 | 187 | 189 | 189 | 189 | 189 | 189 | 189 |
| Chad | 188 | 188 | 188 | 187 | 188 | 188 | 188 | 188 | 188 | 188 | 188 | 188 |
| Niger | 189 | 189 | 187 | 189 | 189 | 189 | 169 | 169 | 169 | 169 | 169 | 169 |

**Table 8**
Average values (left panel) and third quartiles (right panel) of BoD weights for different *p*-orders with relative importance constraints.

| Arithmetic means of BoD weights | | | |
|---|---|---|---|
| p | $\pi_{i,p,1}$ (H) | $\pi_{i,p,2}$ (E) | $\pi_{i,p,3}$ (I) |
| −5 | 0.536 | 0.156 | 0.172 |
| −4 | 0.551 | 0.164 | 0.177 |
| −3 | 0.565 | 0.172 | 0.183 |
| −2 | 0.582 | 0.178 | 0.187 |
| −1 | 0.596 | 0.187 | 0.190 |
| 0 | 0.604 | 0.197 | 0.200 |
| 1 | 0.617 | 0.207 | 0.202 |
| 2 | 0.628 | 0.219 | 0.205 |
| 3 | 0.635 | 0.237 | 0.208 |
| 4 | 0.646 | 0.253 | 0.209 |
| 5 | 0.657 | 0.270 | 0.210 |

| Third quartiles of BoD weights | | | |
|---|---|---|---|
| p | $\pi_{i,p,1}$ (H) | $\pi_{i,p,2}$ (E) | $\pi_{i,p,3}$ (I) |
| −5 | 0.720 | 0.200 | 0.090 |
| −4 | 0.738 | 0.224 | 0.092 |
| −3 | 0.755 | 0.251 | 0.094 |
| −2 | 0.771 | 0.130 | 0.096 |
| −1 | 0.786 | 0.142 | 0.098 |
| 0 | 0.800 | 0.155 | 0.100 |
| 1 | 0.813 | 0.170 | 0.106 |
| 2 | 0.825 | 0.187 | 0.112 |
| 3 | 0.836 | 0.205 | 0.118 |
| 4 | 0.847 | 0.436 | 0.125 |
| 5 | 0.857 | 0.554 | 0.132 |

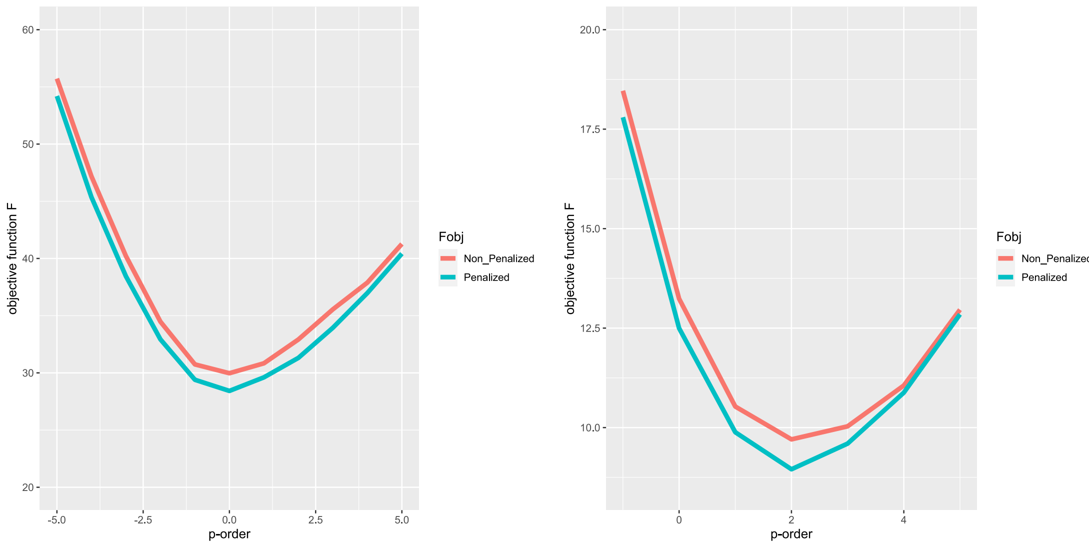


**Fig. 7.** Function $F$ of Problem (27) when the BoD power mean and BoD penalized power mean with relative importance constraints are used to build the composite indicator and $p$ varies from −5 to 5 (left panel) and from −1 to 5 (right panel).

**Table 9**
The first twenty (top panel) and the last twenty (bottom panel) countries according to the BoD direct country-specific weights with relative importance constraints (non-penalized/penalized) for $p = -1$ (HM), 0 (GM), 1 (AM). The rankings are obtained using three digits after the decimal.

| Country | "BoD ranking" | | | | | |
|---|---|---|---|---|---|---|
| | rHM | rGM | rAM | rPHM | rPGM | rPAM |
| Hong Kong | 1 | 1 | 1 | 1 | 1 | 1 |
| Ireland | 1 | 1 | 1 | 1 | 1 | 1 |
| Norway | 1 | 1 | 1 | 1 | 1 | 1 |
| Singapore | 1 | 1 | 1 | 1 | 1 | 1 |
| Switzerland | 1 | 1 | 1 | 1 | 1 | 1 |
| Australia | 1 | 1 | 1 | 1 | 1 | 6 |
| Germany | 1 | 1 | 1 | 1 | 1 | 6 |
| Iceland | 1 | 1 | 1 | 1 | 1 | 6 |
| Sweden | 9 | 9 | 9 | 9 | 9 | 9 |
| Liechtenstein | 9 | 9 | 9 | 10 | 9 | 9 |
| Luxembourg | 14 | 11 | 11 | 14 | 11 | 11 |
| Finland | 11 | 11 | 12 | 11 | 11 | 11 |
| Netherlands | 11 | 11 | 12 | 11 | 11 | 13 |
| New Zealand | 11 | 11 | 12 | 11 | 11 | 13 |
| Japan | 15 | 15 | 15 | 15 | 15 | 15 |
| United Kingdom | 15 | 16 | 16 | 15 | 16 | 16 |
| Denmark | 17 | 17 | 17 | 17 | 16 | 16 |
| United States | 18 | 18 | 18 | 18 | 18 | 18 |
| Canada | 19 | 19 | 19 | 19 | 19 | 19 |
| Belgium | 20 | 20 | 20 | 20 | 20 | 20 |
| Angola | 164 | 168 | 170 | 163 | 168 | 170 |
| Benin | 167 | 170 | 171 | 167 | 170 | 171 |
| Gambia | 173 | 172 | 172 | 173 | 172 | 172 |
| Togo | 173 | 174 | 173 | 172 | 174 | 173 |
| Cameroon | 171 | 173 | 174 | 170 | 172 | 174 |
| Guinea | 176 | 175 | 175 | 177 | 175 | 175 |
| Burkina Faso | 179 | 178 | 176 | 180 | 179 | 176 |
| Congo | 178 | 176 | 177 | 178 | 177 | 177 |
| Burundi | 181 | 181 | 178 | 183 | 181 | 178 |
| Niger | 185 | 182 | 178 | 185 | 182 | 179 |
| Mozambique | 179 | 180 | 180 | 179 | 180 | 181 |
| Côte d'Ivoire | 175 | 176 | 181 | 175 | 176 | 179 |
| Nigeria | 176 | 179 | 182 | 176 | 178 | 182 |
| Mali | 184 | 185 | 183 | 184 | 185 | 183 |
| Guinea-Bissau | 182 | 183 | 184 | 182 | 183 | 183 |
| Lesotho | 182 | 184 | 185 | 181 | 184 | 185 |
| South Sudan | 186 | 186 | 186 | 186 | 186 | 186 |
| Sierra Leone | 187 | 187 | 187 | 187 | 187 | 187 |
| Chad | 188 | 188 | 188 | 188 | 188 | 188 |
| Central African Republic | 189 | 189 | 189 | 189 | 189 | 189 |

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