Check for updates

Received: 28 August 2023 | Revised: 2 October 2023 | Accepted: 12 October 2023

DOI: 10.1002/zamm.202300668

ZAMM

ORIGINAL PAPER

On the static condensation of initially not rectilinear beams

Stefano Lenci[1] | **Sergey Sorokin**[2]

[1]Department of Civil and Building Engineering, and Architecture (DICEA), Polytechnic University of Marche (UNIVPM), Ancona, Italy

[2]Department of Materials and Production, Aalborg University, Denmark

Correspondence
Stefano Lenci, Department of Civil and Building Engineering, and Architecture (DICEA), Polytechnic University of Marche (UNIVPM), via Brecce Bianche 12, 60131, Ancona, Italy.
Email: lenci@univpm.it

Two weakly nonlinear integral equations of motion commonly used in the literature to study the nonlinear dynamics of straight and initially not rectilinear Euler-Bernoulli beams, respectively, are further investigated. Attention is focused on the process known as “static condensation”, which consists of neglecting the axial inertia in the exact, fully nonlinear system of equations of motion to determine the axial displacement as a function of the transversal one. The novelty of the paper relies on showing that, contrarily to expectation and somehow surprisingly, the integral equation for beams with a not rectilinear initial configuration cannot be obtained by the static condensation process starting from the exact, fully nonlinear, equations of motion, apart from a very particular and specific case. On the contrary, it is confirmed the well-known result that for rectilinear beams the integral equation can be obtained by the static condensation. This highlights a major difference between the two integral equations in terms of reliability and allows us a better understanding of the integral equation of rectilinear beams, underlying its stronger mathematical background than the classical counterpart for not straight beams (i.e., its being obtainable from the exact, fully nonlinear, equations of motion via static condensation), which provides it with a “special” behavior and makes it more trustworthy.

1 | INTRODUCTION AND MOTIVATION

In many papers [1], [2] and textbooks [3], [4] the equation

$$\rho A\ddot{w} + EIw^{IV} - \frac{EA}{2L}w'' \int_0^L w'^2 dz = F(z,t)\,, \tag{1}$$

that dates back at least to Mettler [5] and Bolotin [6], has been used to investigate the nonlinear dynamics of initially straight beams in which, due to the boundary conditions (axially restraint, i.e., $u(0,t) = u(L,t) = 0$) and lack of axial-transversal internal resonance, the axial inertia can be neglected, $\ddot{u} \approx 0$, which is known as Kirchhoff assumption [7] or “static” or “kinematic” condensation [8]. Here $u(z,t)$ and $w(z,t)$ are the axial and transversal displacements, respectively, EI the bending stiffness, EA the axial stiffness, ρA the mass per unit length, $F(z,t)$ the external distributed load in the transversal direction and L the length of the beam. Prime denotes derivative with respect to the spatial variable z and dot derivative with the time t. The beam is assumed to be homogeneous, namely EI, EA and ρA are constants.

This is an open access article under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs License, which permits use and distribution in any medium, provided the original work is properly cited, the use is non-commercial and no modifications or adaptations are made.
© 2023 The Authors. *ZAMM - Journal of Applied Mathematics and Mechanics* published by Wiley-VCH GmbH.

Equation (1) is valid only for moderate displacements and has been applied in different fields of engineering, like for example in micromechanics [9], [10]. One of its advantages, and likely the main reason of its popularity, is that when $F(z,t) = f(t)a(z)$, where $a(z) = c_1 \sin(\frac{n\pi z}{L}) + c_2 \cos(\frac{n\pi z}{L})$ is such that $a''(z) = -(\frac{n\pi}{L})^2 a(z)$, and when the boundary conditions are satisfied by $a(z)$ (by properly selecting c_1 and c_2; for example for the hinged-hinged case we have $c_1 = 1$ and $c_2 = 0$), the solution $w(z,t) = b(t)a(z)$ leads *exactly* to the Duffing equation

$$\rho A\ddot{b}(t) + EI\left(\frac{n\pi}{L}\right)^4 b(t) + \frac{EA}{4}\left(\frac{n\pi}{L}\right)^4 \left(c_1^2 + c_2^2\right) b^3(t) = f(t), \tag{2}$$

an ordinary differential equation which is easily solved.

When the initial shape is not straight, and denoted by $w_0(z)$, the counterpart of Equation (1) is

$$\rho A\ddot{w} + EIw^{IV} - \frac{EA}{2L}\left(w'' + w_0''\right)\int_0^L \left(w'^2 + 2w'w_0'\right) dz = F(z,t), \tag{3}$$

which has been used in papers dealing with beams slightly curved (shallow arches) or with unwanted imperfections [11], [12], [13], [14], [15], and again in different fields, including studies of MEMS [16] and internal resonances [17], and also with other "extra" terms like elastic foundation [18] and nonlinear boundary conditions [19]. We refer to [3] and [20] for the details of the derivation of (3), which essentially is based on the assumption that the axial strain in the beam remains approximately constant in space.

To obtain *exactly* an ordinary differential equation from (3) requires that also $w_0(z)$ is proportional to $a(z)$, which is slightly more restrictive but often enough to illustrate the effects of the initial non rectilinearity. In this case, the reduced model has also a quadratic term and is known as Helmholtz-Duffing equation [21]; the quadratic and cubic terms depend on the transverse boundary conditions, while the axial boundary conditions $u(0,t) = u(L,t) = 0$ must always be granted.

One of the reasons for the success and versatility of (3) is that it permits to accurately compute the linear natural frequencies of initially curved beams with small sag; they do not vary significantly from those computed by the exact equations, as shown in the forthcoming Figure 2a.

In the literature also the equivalent version of (3) has been less frequently used [22], [23], [24]

$$\rho A\ddot{y} + EI\left(y^{IV} - w_0^{IV}\right) - \frac{EA}{2L}y''\int_0^L \left(y'^2 - w_0'^2\right) dz = F(z,t), \tag{4}$$

where $y(z,t) = w(z,t) + w_0(z)$.

The large use of (1) and (3) that has been done in the literature naturally calls for a deeper understanding, in particular for their derivation from more general equations in order to have a solid background, and to clearly have in mind potentialities and limits of use of these equations, that we name "integral equations".

As already anticipated, (1) can be obtained by neglecting the axial inertia and obtaining the axial displacement as a "static" function of the transversal one (in the sense that it can be achieved by solving an equation that no longer contains the time-dependence), and then substituting this expression in the transversal equation of motion; this approach is known as "static condensation", and provides a clear theoretical base for (1). For example, it tells us that when axial inertia cannot be neglected, that is, when the edges of a beam are movable in the axial direction, (1) is *not* valid, a fact that seems to be not so well understood in the engineering community.

Since (3) is the counterpart of (1) for initially not rectilinear beams, the natural question is: can it also be derived by the static condensation procedure? The main goal of this paper is to show rigorously that in general the answer is no, a fact that makes (3) somehow weaker than (1), and thus less reliable.

To be as general as possible, we apply the static condensation directly to the kinematically exact nonlinear equations, without any a priori approximations that can affect the results. There is quite a lot of literature on kinematically exact equations of motion for straight beams, in papers [25], [26] as well as in textbooks [27], [28], even if we prefer to derive the equations by ourselves to use a specific notation and a given reference configuration that are convenient for us.

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2 | THE EXACT KINEMATIC RELATIONS AND APPROXIMATE STRAIN MEASURES

Let us consider a planar beam, which in this work is identified with its axis and thus it is a curve in the plane. With reference to Figure 1 the following *exact* kinematic relations hold:

initial configuration

$$\begin{gathered}\cos\Phi = \frac{dX}{dS} = \frac{X'}{S'},\ \sin\Phi = \frac{dY}{dS} = \frac{Y'}{S'},\ \tan\Phi = \frac{dY}{dX} = \frac{Y'}{X'},\\ dS = \sqrt{dX^2+dY^2} \to S' = \sqrt{X'^2+Y'^2},\\ K_m = \frac{d\Phi}{dz} = \Phi' = \left[\operatorname{atan}\left(\frac{Y'}{X'}\right)\right]',\ K_g = \frac{d\Phi}{dS} = \frac{d\Phi}{dz}\frac{dz}{dS} = \frac{K_m}{S'},\end{gathered} \tag{5}$$

deformed configuration

$$\begin{gathered}\cos\phi = \frac{dx}{ds} = \frac{x'}{s'},\ \sin\phi = \frac{dy}{ds} = \frac{y'}{s'},\ \tan\phi = \frac{dy}{dx} = \frac{y'}{x'}\\ ds = \sqrt{dx^2+dy^2} \to s' = \sqrt{x'^2+y'^2}\\ k_m = \frac{d\phi}{dz} = \phi' = \left[\operatorname{atan}\left(\frac{y'}{x'}\right)\right]',\ k_g = \frac{d\phi}{ds} = \frac{d\phi}{dz}\frac{dz}{ds} = \frac{k_m}{s'},\end{gathered} \tag{6}$$

deformations

$$\begin{gathered}\varepsilon_z = \frac{ds}{dS} - 1 = \frac{s'}{S'} - 1,\\ k_z = \Delta k_m = k_m - K_m = \phi' - \Phi' = (\phi-\Phi)' = (\Delta\phi)',\\ \Delta k_g = k_g - K_g = \frac{k_m}{s'} - \frac{K_m}{S'} = \frac{k_m}{s'} - (1+\varepsilon_z)\frac{K_m}{s'} = \frac{k_m - K_m - \varepsilon_z K_m}{s'} = \frac{\Delta k_m - \varepsilon_z K_m}{s'},\end{gathered} \tag{7}$$

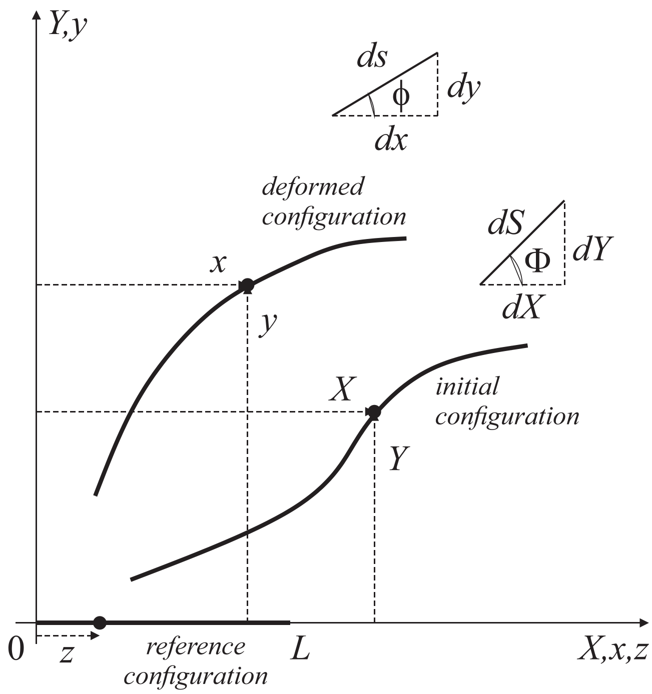

FIGURE 1 Reference, initial and deformed configurations.

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

displacements

$$\begin{aligned} s_x &= x - X, && \text{displacement along } x \\ s_y &= y - Y, && \text{displacement along } y \\ u &= s_x \cos\Phi + s_y \sin\Phi, && \text{axial displacement} \\ w &= -s_x \sin\Phi + s_y \cos\Phi. && \text{transversal displacement} \end{aligned} \tag{8}$$

positions

$$\begin{aligned} x &= X + s_x = X + u\cos\Phi - w\sin\Phi, \\ y &= Y + s_y = Y + u\sin\Phi + w\cos\Phi, \end{aligned} \tag{9}$$

where K_m and k_m are the mechanical curvatures, K_g and k_g the geometrical curvatures [29], [30] and ε_z the axial deformation. All other symbols are implicitly defined in Figure 1.

Assuming $X(z) = z$ and $Y(z) = w_0(z)$, the following expressions are obtained (remember that $k_z = \Delta k_m = (\Delta\phi)'$)

$$\begin{aligned} \varepsilon_z = &\frac{u'}{\sqrt{1+w_0'^2}} - \frac{w_0''}{\left(1+w_0'^2\right)^{3/2}} w \\ &+ \frac{1}{2}\frac{w'^2}{1+w_0'^2} + \frac{w_0''}{\left(1+w_0'^2\right)^2} u w' + \frac{1}{2}\frac{w_0''^2}{\left(1+w_0'^2\right)^3} u^2 + \cdots, \end{aligned} \tag{10}$$

$$\begin{aligned} \Delta\phi = &\frac{w'}{\sqrt{1+w_0'^2}} + \frac{w_0''}{\left(1+w_0'^2\right)^{3/2}} u \\ &- \frac{1}{1+w_0'^2} u'w' - \frac{w_0''}{\left(1+w_0'^2\right)^2}\left(u'u - w'w\right) + \frac{w_0''^2}{\left(1+w_0'^2\right)^3} uw + \cdots, \end{aligned} \tag{11}$$

where we have considered terms only up to *quadratic* in the unknowns u and w, which are enough for small/moderate values of the displacements and in any case are the first terms providing nonlinearities. The curvature is the mechanical one [29], [30].

When the beam is initially circular, we have $X(z) = -R\cos(\frac{z}{R})$ and $Y(z) = R\sin(\frac{z}{R})$, so that $K_m = K_g = -\frac{1}{R} = -k$, where R is the initial radius of curvature and k the associated curvature; both do *not* depend on z. In this case we have

$$\varepsilon_z = u' + kw + \frac{1}{2}w'^2 - kuw' + \frac{1}{2}k^2u^2 + \cdots, \tag{12}$$

$$\Delta\phi = w' - ku - u'w' + k\left(u'u - w'w\right) + k^2uw + \cdots, \tag{13}$$

where again only terms up to quadratic are kept.

In a quite naïve way, we note that (12)–(13) can be obtained from (10)–(11) by assuming $1 + {w_0'}^2 \approx 1$ and $w_0'' = -k$, so that they are equivalent for almost rectilinear beams that can be approximated by an arch of low sag.

In (10)–(11) and (12)–(13) we have implicitly made the assumption that u and w have the same order of magnitude. This is expected for beams with very large initial curvature (note that up to now no limitations are introduced on the initial shape). However, since the axial stiffness is commonly much larger than the bending stiffness, for beams with moderate initial deviation from the rectilinear configuration it can be assumed that the axial displacement u is at least one order of magnitude smaller than the transversal one.

To support the previous hypothesis, we report in Figure 2b the ratio $r = \frac{w_{max}}{u_{max}}$ between the maximum transversal displacement and the maximum axial displacement for the first three linear normal modes for circular beams with hinged-hinged ends and for varying dimensionless initial curvature $\alpha = kL$. Note that w_{max} and u_{max} are not obtained at the same point of the beam. It is clear that for low values of k we have $r > 10$, a fact that confirms the previous assumption;

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

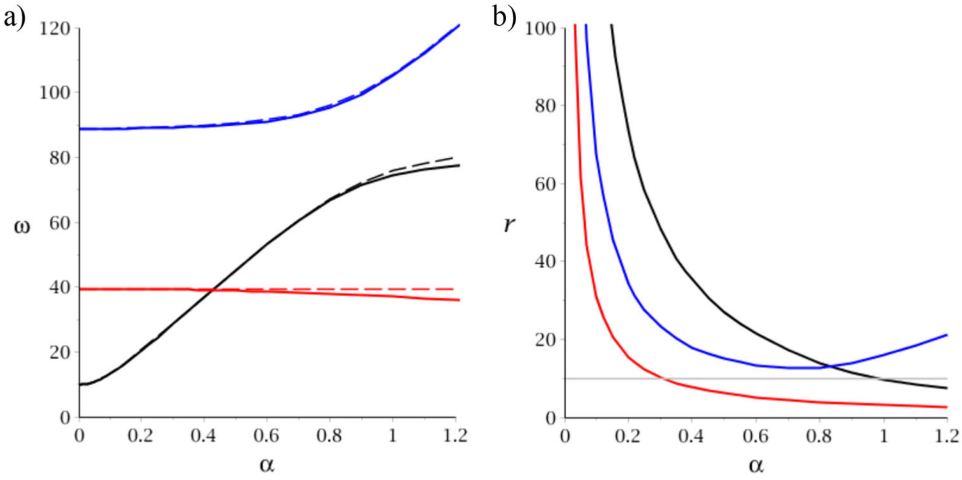

FIGURE 2 (a) The first three dimensionless frequencies $\omega = \bar{\omega}\, L^2\sqrt{\rho A/EI}$ (continuous lines are the exact values, dashed lines are computed by the linearized version of (3): $\rho A\ddot{w} + EIw^{IV} - (EA/L)\, w_0''\int_0^L (w'w_0')\, dz = 0$, where $w_0 = (k/2)z(L-z)$ so that $w_0'' = -k$), and (b) the corresponding ratio $r = \frac{w_{max}}{u_{max}}$ (computed with the exact equations) for a *hinged-hinged circular beam* (whose boundary conditions are $u(0) = u(L) = w(0) = w(L) = w''(0) - ku'(0) = w''(L) - ku'(L) = 0$) with slenderness $\lambda = L\sqrt{EA/EI} = 100$. Black is the first mode (that for $\alpha > 0.41$ becomes the second), red the second mode (that for $\alpha > 0.41$ becomes the first) and blue the third mode. The grey line is the threshold $r = 10$. $\alpha = kL$ is a dimensionless measure of the initial curvature, and practically corresponds to the angle of opening of the arch (in radians).

more precisely, we have that $r > 10$ for $\alpha < 0.31 = 17.8°$ for the second mode, for $\alpha < 0.98 = 56.1°$ for the first mode, while surprisingly for the third mode we have always $r > 10$ in the considered range. Strictly speaking, this is valid only in the linear realm, but it is not difficult to accept that it is still qualitatively acceptable for moderate nonlinearities, as well as for different boundary conditions, of course with different numerical thresholds.

Assuming that u is of the second order and that w is of the first order, and still keeping terms only up to the second order, (10)–(11) become

$$\varepsilon_z = -\frac{w_0''}{\left(1+w_0'^2\right)^{\frac{3}{2}}}w + \frac{u'}{\sqrt{1+w_0'^2}} + \frac{1}{2}\frac{w'^2}{1+w_0'^2} + \cdots, \tag{14}$$

$$\Delta\phi = \frac{w'}{\sqrt{1+w_0'^2}} + \frac{w_0''}{\left(1+w_0'^2\right)^{3/2}}u + \frac{w_0''}{\left(1+w_0'^2\right)^2}w'w + \cdots, \tag{15}$$

while (12)–(13) become

$$\varepsilon_z = kw + u' + \frac{1}{2}w'^2 + \cdots, \tag{16}$$

$$\Delta\phi = w' - ku - kw'w + \cdots. \tag{17}$$

For beams such that $w_0'^2 \ll 1$, that is, those with very small initial slopes, the expressions (14)–(15) further simplify:

$$\varepsilon_z = -w_0''w + u' + \frac{1}{2}w'^2 + \cdots, \tag{18}$$

$$\Delta\phi = w' + w_0''u + w_0''w'w + \cdots, \tag{19}$$

while (16)–(17) remain unchanged. Note that only the second derivative of the initial configuration appears in (18)–(19).

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Finally, for straight beams, $w_0(z) = 0$, we have

$$\varepsilon_z = u' + \frac{1}{2}w'^2 + \cdots, \tag{20}$$

$$\Delta\phi = w' + \cdots, \tag{21}$$

where we underline that there are no quadratic terms in the expression of the curvature (that thus is linear), according to the initial symmetry of the beam with respect to the reference configuration. Expressions (20) and (21) are the same used in the Föppl–Kármán nonlinear theory of plates [3].

3 | THE EQUATIONS OF MOTION

Let

$$\begin{aligned} T &= \int_0^L E_T dz,\ E_T = \frac{\rho A}{2}\left(\dot{u}^2 + \dot{w}^2\right), \\ U &= \int_0^L E_U dz,\ E_U = \frac{1}{2}\left(EA\varepsilon_z^2 + EI\kappa_z^2\right), \\ V &= \int_0^L (Qu + Fw)\,dz - Q_0 u(0) + Q_L u(L) - F_0 w(0) + F_L w(L), \end{aligned} \tag{22}$$

be the kinetic energy, potential energy, and work done by the external loads. Q_0, Q_L, F_0, and F_L are the reactions of the boundary constraints, and $u(0)$, $u(L)$, $w(0)$, and $w(L)$ are their known imposed displacements, that commonly vanish. Rotary inertia and shear deformations are neglected, according to the fact that the beam is assumed to be slender as in the Euler-Bernoulli theory. The Lagrangian is

$$\mathcal{L} = \int_{t_0}^{t_1} (T - U + V)\,dt. \tag{23}$$

The equation of motion in the *axial* direction is obtained by the stationarity of $\mathcal{L}$ with respect to u, and is given by

$$\begin{aligned} 0 &= \rho A\ddot{u} - Q + \frac{\partial E_U}{\partial u} - \frac{d}{dz}\frac{\partial E_U}{\partial u'} + \frac{d^2}{dz^2}\frac{\partial E_U}{\partial u''} \\ &= \rho A\ddot{u} - Q + \left(EA\varepsilon_z\frac{\partial \varepsilon_z}{\partial u} + EI\kappa_z\frac{\partial \kappa_z}{\partial u}\right) - \left(EA\varepsilon_z\frac{\partial \varepsilon_z}{\partial u'} + EI\kappa_z\frac{\partial \kappa_z}{\partial u'}\right)' \\ &\quad + \left(EA\varepsilon_z\frac{\partial \varepsilon_z}{\partial u''} + EI\kappa_z\frac{\partial \kappa_z}{\partial u''}\right)''. \end{aligned} \tag{24}$$

The equation of motion in the *transversal* direction is instead obtained by the stationarity of $\mathcal{L}$ with respect to w, and is given by

$$\begin{aligned} 0 &= \rho A\ddot{w} - F + \frac{\partial E_U}{\partial w} - \frac{d}{dz}\frac{\partial E_U}{\partial w'} + \frac{d^2}{dz^2}\frac{\partial E_U}{\partial w''} \\ &= \rho A\ddot{w} - F + \left(EA\varepsilon_z\frac{\partial \varepsilon_z}{\partial w} + EI\kappa_z\frac{\partial \kappa_z}{\partial w}\right) - \left(EA\varepsilon_z\frac{\partial \varepsilon_z}{\partial w'} + EI\kappa_z\frac{\partial \kappa_z}{\partial w'}\right)' \\ &\quad + \left(EA\varepsilon_z\frac{\partial \varepsilon_z}{\partial w''} + EI\kappa_z\frac{\partial \kappa_z}{\partial w''}\right)''. \end{aligned} \tag{25}$$

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The equations of motion strongly depend on the expressions of the deformations ε_z and κ_z that are considered, see (10)–(21), and will lead to different beam models.

The stationarity of $\mathcal{L}$ also gives the boundary conditions, in particular the "essential" or geometric boundary conditions on the displacements u, w and ϕ, and the alternative "natural" or force boundary conditions. In this work only boundary conditions for u are used, see forthcoming Equation (26), while those in the transversal direction are not utilized (apart from the example of Figure 2 where they are reported), and thus they are not discussed.

4 | THE STATIC CONDENSATION

The main idea of the static condensation [7], [8] consists of neglecting the axial inertia term $\rho A\ddot{u}$ in (24). From a mechanical point of view this makes sense only (i) in the absence of external axial load, $Q = 0$, and (ii) when the ends of a beam are fixed in the axial direction,

$$u(0,t) = d_1, \ \ u(L,t) = d_2. \tag{26}$$

Commonly $d_1 = d_2$ $(= 0)$, while the case $d_1 \neq d_2$ corresponds to a pre-stressed initial configuration. These are two important hypotheses that are at the base of the utilized approach and must be properly checked to ascertain the reliability of the final results.

With the previous assumptions, Equation (24) becomes

$$\left(\varepsilon_z \frac{\partial \varepsilon_z}{\partial u} + \rho^2 \kappa_z \frac{\partial \kappa_z}{\partial u}\right) - \left(\varepsilon_z \frac{\partial \varepsilon_z}{\partial u'} + \rho^2 \kappa_z \frac{\partial \kappa_z}{\partial u'}\right)' + \left(\varepsilon_z \frac{\partial \varepsilon_z}{\partial u''} + \rho^2 \kappa_z \frac{\partial \kappa_z}{\partial u''}\right)'' = 0, \tag{27}$$

where $\rho = \sqrt{EI/EA}$ is the radius of gyration of the beam cross-section.

Equation (27) is an ordinary differential equation (in the spatial coordinate z) that, together with the boundary conditions (26), permits to compute the unknown $u(z,t)$, which will depend on $w(z,t)$ and its derivatives. It is a "static" problem since there are no more time derivatives (here time is just a parameter) and this motivates the name "static condensation" adopted for this approach.

While in principle the problem (26)–(27) can be always solved, we are interested in studying the cases in which the solution is "simple" enough to be of interest for engineers, in particular when it is analytical, rather than numerical.

We start by noting that in the most general case (10)–(11) (or (12)–(13)) we were not able to find an "easy" solution, in particular because of the term u^2 in (10) and (12) that makes these equations nonlinear with respect to u. Thus, in the following we consider only moderately not rectilinear beams where the axial displacement is one order of magnitude smaller than the transversal one. From (14) and (15) we obtain

$$\begin{gathered}\frac{\partial \varepsilon_z}{\partial u} = 0, \ \frac{\partial \kappa_z}{\partial u} = \left(\frac{w_0''}{\left(1+{w_0'}^2\right)^{3/2}}\right)' = \frac{w_0'''}{\left(1+{w_0'}^2\right)^{3/2}} - 3\frac{{w_0''}^2 w_0'}{\left(1+{w_0'}^2\right)^{5/2}}, \\ \frac{\partial \varepsilon_z}{\partial u'} = \frac{1}{\sqrt{1+{w_0'}^2}}, \ \frac{\partial \kappa_z}{\partial u'} = \frac{w_0''}{\left(1+{w_0'}^2\right)^{3/2}}, \ \frac{\partial \varepsilon_z}{\partial u''} = 0, \ \frac{\partial \kappa_z}{\partial u''} = 0,\end{gathered} \tag{28}$$

so that (27) becomes, after some simplifications,

$$\left(\frac{\varepsilon_z}{\sqrt{1+{w_0'}^2}}\right)' + \rho^2 \frac{k_z' w_0''}{\left(1+{w_0'}^2\right)^{3/2}} = 0. \tag{29}$$

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Solving (29) with respect to ε_z yields (remember that $k_z = (\Delta\phi)'$)

$$\varepsilon_z = \sqrt{1+w_0'^2}\left\{e_1 - \rho^2 \int_0^z \frac{w_0''}{\left(1+w_0'^2\right)^{3/2}}(\Delta\phi)'' ds\right\}, \tag{30}$$

where e_1 is a constant of integration.

By recalling the expressions (14) and (15) for ε_z and $\Delta\phi$, that contain u' and u, respectively, we note that (30) is an integro-differential equation in the unknown u, that cannot be solved easily (i.e., in closed form), so that for moderate initial deviation from the rectilinear configuration the static condensation fails to provide a *simple* expression for u. This agrees with the fact that (30) shows that ε_z is not constant in space, which is the main hypothesis behind the Equation (3).

For small initial slopes, that is $w_0'^2 \ll 1$, (29) and (30) become

$$\varepsilon_z' + \rho^2 w_0''(\Delta\phi)'' = 0, \tag{31}$$

$$\varepsilon_z = e_1 - \rho^2 \int_0^z w_0''(\Delta\phi)'' ds, \tag{32}$$

and the deformations are now given by (18)–(19). Equation (32) shows that also in this simplified case ε_z is not constant, while (31) is an equation of the kind

$$u'' + \rho^2 w_0''\left(w_0'' u\right)'' = f(z) \tag{33}$$

and does not allow us to have an easy expression for u; thus, also when $w_0'^2 \ll 1$ the static condensation fails.

The next case to be considered is that of an initially circular beam, that is, when the strains are given by (16)–(17). In this case we have (remember that k is constant)

$$\frac{\partial\varepsilon_z}{\partial u} = 0,\ \frac{\partial\kappa_z}{\partial u} = 0,\ \frac{\partial\varepsilon_z}{\partial u'} = 1,\ \frac{\partial\kappa_z}{\partial u'} = -k,\ \frac{\partial\varepsilon_z}{\partial u''} = 0,\ \frac{\partial\kappa_z}{\partial u''} = 0, \tag{34}$$

so that (27) becomes

$$\left(\varepsilon_z - k\rho^2\kappa_z\right)' = 0, \tag{35}$$

namely

$$\varepsilon_z = e_1 + k\rho^2\kappa_z, \tag{36}$$

or

$$\left(1+k^2\rho^2\right) u' = e_1 - kw - \frac{1}{2}w'^2 + k\rho^2 w'' - k^2\rho^2\left(w'w\right)'. \tag{37}$$

Solving (37) gives

$$\left(1+k^2\rho^2\right) u(z) = e_1 z + e_2 + k\rho^2 w' - k^2\rho^2 w' w - \int_0^z \left(kw + \frac{1}{2}w'^2\right) ds, \tag{38}$$

which is easy enough to be considered useful; e_2 is another constant of integration. It is worth observing that kw and w'^2, which are both dimensionless, are of the same (second) order, in agreement with the hypothesis that leads to (14)–(21), namely that the axial displacement u is at least one order of magnitude smaller than the transversal displacement w.

Although (38) is simple enough, it is useful to see if it can be further elaborated. Remembering that $\alpha = kL$ is a dimensionless measure of the initial curvature, that practically corresponds to the angle of opening of the arch (in radians), we

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

have that $k\rho = \alpha\rho/L = \alpha/\lambda$, where $\lambda = L/\rho = L\sqrt{EA/EI}$ is the slenderness. Since we are considering small initial sags, that is small αs, and slender beams, that is large λs, we have that $k^2\rho^2 \ll 1$. Furthermore, $k\rho^2 w' - k^2\rho^2 w' w = k\rho^2 w'(1-kw) \approx k\rho^2 w'$ since kw is of the second order and thus negligible with respect to 1. Thus, (38) becomes

$$u(z) = e_1 z + e_2 + k\rho^2 w' - \int_0^z \left(kw + \frac{1}{2}w'^2\right) ds. \tag{39}$$

From the boundary conditions $u(0) = d_1$ and $u(L) = d_2$ we obtain, respectively,

$$\begin{aligned} e_2 &= d_1 - k\rho^2 w'(0), \\ e_1 L &= (d_2 - d_1) - k\rho^2 \left[w'(L) - w'(0)\right] + \int_0^L \left(kw + \frac{1}{2}w'^2\right) dz, \end{aligned} \tag{40}$$

so that $u(z)$ is fully determined. We also have (higher order terms have been neglected)

$$\varepsilon_z = e_1 + k\rho^2 w'', \tag{41}$$

$$k_z = w'' - k\left(e_1 + w'' w + \frac{w'^2}{2}\right), \tag{42}$$

and observe that even in the "simplest" case (41) the axial deformation ε_z is, again, not constant in space.

The second step of the static condensation consists of inserting the function u so far obtained in the transversal equation of motion (25) so that a single equation in the unknown function w is obtained. Still dealing with an initially circular beam, from (16) and (17) we get

$$\frac{\partial\varepsilon_z}{\partial w} = k,\ \frac{\partial\kappa_z}{\partial w} = -kw'',\ \frac{\partial\varepsilon_z}{\partial w'} = w',\ \frac{\partial\kappa_z}{\partial w'} = -2kw',\ \frac{\partial\varepsilon_z}{\partial w''} = 0,\ \frac{\partial\kappa_z}{\partial w''} = 1 - kw. \tag{43}$$

After long computations these give

$$\rho A\ddot{w} + EIw^{IV} + EAe_1\left(k - w''\right) - kEI\left[3w''^2 + 4w'w''' + 2w^{IV}w\right] = F, \tag{44}$$

where use is made of the fact that k is small and so terms k^2, k^3, etc. have been disregarded. Inserting the expression (40) of e_1, and still keeping only first order terms with respect to k, yields

$$\begin{aligned} &\rho A\ddot{w} + EIw^{IV} - \frac{EA}{L}w''\left[(d_2 - d_1) + \frac{1}{2}\int_0^L w'^2 dz\right] \\ &+ k\frac{EA}{L}\left[(d_2 - d_1) + \frac{1}{2}\int_0^L w'^2 dz - w''\int_0^L w dz\right] \\ &- kEI\left[w''\left(\frac{w'(0) - w'(L)}{L}\right) + 3w''^2 + 4w'w''' + 2w^{IV}w\right] = F, \end{aligned} \tag{45}$$

or, in the case without pre-stress, $d_2 = d_1$,

$$\begin{aligned} &\rho A\ddot{w} + EIw^{IV} - \frac{EA}{2L}w''\int_0^L w'^2 dz \\ &+k\frac{EA}{L}\left(\frac{1}{2}\int_0^L w'^2 dz - w''\int_0^L w dz\right) \\ &-kEI\left[w''\left(\frac{w'(0) - w'(L)}{L}\right) + 3w''^2 + 4w'w''' + 2w^{IV}w\right] = F. \end{aligned} \tag{46}$$

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We note that when $k = 0$, that is for initially rectilinear beams, (46) corresponds to (1), while for $k \neq 0$ the Equation (46) does *not* corresponds to (3). It is *not* equal to (4), too.

It is thus proved that (3) cannot be derived by the static condensation, not even in the simplest case of the initially shallow slender arch. This shows some kind of singularity, or "discontinuity": Equation (3) for any $w_0(z) \neq 0$ does not derive from the static condensation, while its limit for $w_0(z) \to 0$ does.

Equation (46) is a new one and, to the best of the authors' knowledge, has not been reported in the literature so far, even if similar nonlinear terms are reported in Sect. 4.5.2 of [4], but referring to an initially straight beam, thus in a different case. Since it comes from the firm theoretical background illustrated in this work, we believe it is trustable (of course when it is reasonable to neglect axial inertia). A deeper investigation of (46) requires studying the effect of neglecting the nonlinear terms multiplying the bending stiffness EI because of the slenderness of the beam (see the next section). Furthermore, it is also interesting to carefully compare (46) with (3) and with the kinematically exact equations. These, together with other possible further developments, are out of the scope of this work and are left for future investigations.

Here we observe only that the single mode Galerkin equation obtained by (3) is given by

$$\rho A\ddot{b}(t) + EI\left(\frac{n\pi}{L}\right)^4 b(t) + g(n)\ k\, b(t)^2 + \frac{EA}{4}\left(\frac{n\pi}{L}\right)^4 \left(c_1^2 + c_2^2\right) b^3(t) = f(t)$$
$$g(n) = \begin{cases} -c_2 n^2\pi^2 \dfrac{EA}{L^2}, \text{ for } \cos(n\pi) = 1, \\[2ex] c_1 3n\pi\ \dfrac{EA}{L^2}, \text{ for } \cos(n\pi) = -1, \end{cases} \tag{47}$$

where $w(z,t) = b(t)a(z)$, $a(z) = c_1 \sin(\frac{n\pi z}{L}) + c_2 \cos(\frac{n\pi z}{L})$ and k^2 and higher order terms have been neglected, while that obtained by (46) is given by

$$\rho A\ddot{b}(t) + EI\left(\frac{n\pi}{L}\right)^4 b(t) + g(n)\ k\, b(t)^2 + \frac{EA}{4}\left(\frac{n\pi}{L}\right)^4 \left(c_1^2 + c_2^2\right) b^3(t) = f(t),$$
$$g(n) = \begin{cases} 0, \text{ for } \cos(n\pi) = 1, \\[2ex] c_1\ n\pi\left[\dfrac{EA}{L^2}\left(\dfrac{2n^2\pi^2}{\lambda^2} + 3\right) - \dfrac{4EI}{L^4}\ n^2\pi^2 \dfrac{2c_1^2 + 5c_2^2}{c_1^2 + c_2^2}\right], \text{ for } \cos(n\pi) = -1. \end{cases} \tag{48}$$

The linear and cubic terms are the same as (2) (to which both reduce for the rectilinear case $k = 0$), but the quadratic coefficients are very different from each other, a fact that calls for further investigations, too.

5 | A NAÏVE CONSIDERATION

Let us rescale the spatial variable as $z = \zeta L, 0 \leq \zeta \leq 1$, the time as $t = L^2\sqrt{\frac{\rho A}{EI}}\,\tau$ and the displacement as $w = WL$. Note that this corresponds to the use the length of the beam as a length scale, for both spatial coordinate and displacement. Equation (46) becomes (now prime is derivative with respect to ζ and dot with respect to τ, while we remember that $\alpha = kL$ and $\lambda = L\sqrt{\frac{EA}{EI}}$ is the slenderness)

$$\begin{aligned} &\ddot{W} + W^{IV} - \frac{\lambda^2}{2} W'' \int_0^1 W'^2 d\zeta \\ &+ \alpha\lambda^2 \left(\frac{1}{2}\int_0^1 W'^2 d\zeta - W''\int_0^1 W d\zeta\right) \\ &- \alpha\left[W''\left(W'(0) - W'(1)\right) + 3W''^2 + 4W'W''' + 2W^{IV}W\right] = \frac{FL^3}{EI}. \end{aligned} \tag{49}$$

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In (49) there are two quadratic terms, one proportional to $\alpha\lambda^2$ and the other to α. If the beam is very slender, λ is very large and thus the second quadratic term can be neglected with respect to the first one:

$$\ddot{W} + W^{IV} - \frac{\lambda^2}{2} W'' \int_0^1 W'^2 d\zeta + \alpha\lambda^2 \left(\frac{1}{2} \int_0^1 W'^2 d\zeta - W'' \int_0^1 W d\zeta \right) = \frac{FL^3}{EI}. \tag{50}$$

Now it is worth to rewrite (3) in the following equivalent form, obtained by integrating by parts,

$$\rho A\ddot{w} + EIw^{IV} - \frac{EA}{2L} \left(w'' + w_0'' \right) \int_0^L \left(w'^2 - 2ww_0'' \right) dz - \frac{EA}{L} \left(w'' + w_0'' \right) \left(w\, w_0' \Big|_0^L \right) = F, \tag{51}$$

Making the very naïve assumption $k = -w_0''$ (which is correct only when w_0'' is constant and the initial curvature is small), assuming $w(0) = w(L) = 0$ (or $w_0'(0) = w_0'(L) = 0$) and neglecting the terms proportional to k^2, gives

$$\rho A\ddot{w} + EIw^{IV} - \frac{EA}{2L} w'' \int_0^L w'^2 dz - k\frac{EA}{L} w'' \int_0^L w dz + k\frac{EA}{2L} \int_0^L w'^2 dz = F. \tag{52}$$

Introducing the dimensionless variable as done at the beginning of this section it is possible to see that (52) corresponds to (50). It thus seems that also (3) can be obtained as the static condensation, but only if (i) w_0'' is constant, (ii) $w(0) = w(L) = 0$ (or $w_0'(0) = w_0'(L) = 0$) and (iii) the beam is slender. These are too restrictive hypotheses, and thus we keep the general conclusion of the previous section that (3) cannot be derived from the static condensation. We report the developments of this section just as a curiosity, and for completeness of the analysis.

6 | CONCLUSIONS

In this paper, the possibility to derive by static condensation the well-known and well-used integral equations commonly used to describe the nonlinear dynamics of Euler-Bernoulli beams has been investigated in detail.

With the exact kinematic relations for planar beams having arbitrary initial configurations as the point of departure, the exact strain measures have been initially obtained, with special reference to the case of circular beams, where the curvature is constant and large. They have been successively approximated by showing that for moderate deviations from the rectilinear configuration, the axial displacement is at least one order of magnitude smaller than the transversal one. Finally, the case of small initial curvature has been detailed, too.

The two (axial and transversal) equations of motion have been obtained by the Lagrangian approach. The first one is carefully investigated under the hypothesis that the axial inertia can be neglected (static condensation), which is plausible only if there are no loads in the axial direction and if the edges of a beam are immovable in the axial direction, a fact that seems to be not so well understood in the literature. These main hypotheses are essential to obtain the axial displacement u as a function of the transversal one w and of its spatial derivatives.

With the goal of having useful results for engineers, attention is focused on looking for simple (i.e., analytical) function $u(w, w')$. All expressions of strain previously introduced, from the exact to the more approximate ones, have been considered, and it has been shown that only in the case of initially circular beams, with constant curvature k, it is possible to find the simple enough function (38), that simplifies to (39) for small initial curvatures. This latter expression has been inserted in the second (transversal) equation of motion, and the integral equation (45) is obtained. It is never reported in the literature (to the best of authors' knowledge).

It has been shown that in the elementary case of an initially straight beam, the classical Equation (1) has been recovered, as expected since it is well-known that it can be derived by the static condensation. On the contrary, it has been proved that it does *not* correspond to the classical integral Equation (3) for not rectilinear beams, not even for shallow arches (i.e., constant initial curvature), apart from a very specific case that cannot be generalized. This is the main finding of this work and proves that the (3) is less trustable than its counterpart for straight beams.

The present work is focused on Euler-Bernoulli beams. A worthy development consists of applying the same ideas also to shallow cables, continuing what has already been done in [8], to other beams models (e.g., Timoshenko) as well as to plates and shells. These require not trivial elaborations, that are left for future work.

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

ACKNOWLEDGMENTS

The work of S.L. has been done as a part of his belonging to the Italian "Gruppo Nazionale per Fisica Matematica," and as a part of the DICEA "Dipartimento di Eccellenza" project 2023–2027.

REFERENCES

[1] Lenci S., Rega G.: Dimension reduction of homoclinic orbits of buckled beams via the nonlinear normal modes technique. Int. J. Non Linear Mech. 42(4), 515–528 (2007) https://doi.org/10.1016/j.ijnonlinmec.2007.02.004

[2] Nayfeh A.H., Emam S.A.: Exact solutions and stability of the postbuckling configurations of beams. Nonlinear Dyn. 54(5), 395–408 (2008) https://doi.org/10.1007/s11071-008-9338-2

[3] Villaggio P.: Mathematical Models for Elastic Structures, Cambridge University Press, Pisa. (1997) ISBN-13: 978–0521573245

[4] Nayfeh A.H., Pai P.F.: Linear and Nonlinear Structural Mechanics, Wiley Interscience, New York, (2004) ISBN: 9780471593560

[5] Mettler E.: Dynamic buckling. In: Flügge, W. (ed.) Handbook of Engineering Mechanics, McGraw-Hill, New York, (1962) (Chapter 62). ISBN-13: 978–0070213920

[6] Bolotin V.V.: The Dynamic Stability of Elastic Systems, Holden-Day, San Francisco. (1964) ISBN-13: 978–1114366091

[7] Kauderer H.: Nichtlineare mechanik, Springer, Berlin, (1958). ISBN: 978-3-642-92734-8 (in German)

[8] Srinil N., Rega G.: The effects of kinematic condensation on internally resonant forced vibrations of shallow horizontal cables. Int. J. Non-Linear Mech. 42, 180–195 (2007) https://doi.org/10.1016/j.ijnonlinmec.2006.09.005

[9] Song X., Liu H: Free Vibration of Bistable Clamped-Clamped Beams: Modeling and Experiment. ASME J. Vibr. Acoustics 143, 044501 (2020) https://doi.org/10.1115/1.4048553

[10] Younis M.I.: MEMS Linear and Nonlinear Statics and Dynamics, Springer, Binghamton. (2011) ISBN 9781441960191.

[11] Qiao W., Guo T., Kang H., Zhao Y.: Softening–hardening transition in nonlinear structures with an initial curvature: a refined asymptotic analysis. Nonlinear Dyn. 107, 357–374 (2022) https://doi.org/10.1007/s11071-021-07013-x

[12] Huang J.L., Su R.K.L., Lee Y.Y, Chen S.H.: Nonlinear vibration of a curved beam under uniform base harmonic excitation with quadratic and cubic nonlinearities. J. Sound Vibration 330, 5151–5164 (2011) https://doi.org/10.1016/j.jsv.2011.05.023

[13] Rehfield L.W.: Nonlinear flexural oscillations of shallow arches. AIAA J. 12, 91–93 (1974) https://doi.org/10.2514/3.49158

[14] Lakrad F., Chtouki A., Belhaq M.: Nonlinear vibrations of a shallow arch under a low frequency and a resonant harmonic excitations. Meccanica 51, 2577–2587 (2016) https://doi.org/10.1007/s11012-016-0470-7

[15] Krylov S., Ilic B.R., Schreiber D., Seretensky S., Craighead H.: The pull-in behavior of electrostatically actuated bistable microstructures. J Micromech Microeng 18, 055026 (2008) https://doi.org/10.1088/0960-1317/18/5/055026

[16] Alkharabsheh S.A., Younis M.I.: Dynamics of MEMS arches of flexible supports. J. Microelectromech. Syst. 22(1), 216–224 (2013) https://doi.org/10.1109/JMEMS.2012.2226926

[17] Yi Z., Stanciulescu I.: Nonlinear normal modes of a shallow arch with elastic constraints for two-to-one internal resonances. Nonlinear Dyn. 83, 1577–1600 (2016) https://doi.org/10.1007/s11071-015-2432-3

[18] Oz H.R., Pakdemirli M., Ozkaya E., Yilmaz M.: Non-linear vibrations of slightly curved beam resting on a non-linear elastic foundation. J. Sound Vibration 212(2), 295–309 (1998) https://doi.org/10.1006/jsvi.1997.1428

[19] Ye S.-Q., Mao X.-Y., Ding H., Ji J.-C., Chen L.-Q.: Nonlinear vibrations of a slightly curved beam with nonlinear boundary conditions. Int. J. Mech. Sci. 168, 105294 (2020) https://doi.org/10.1016/j.ijmecsci.2019.105294

[20] Kalinin V.S.: On forced nonlinear flexural vibrations of prismatic rods. Trudy Vyssh. V.-M. Inzh. Uchil. 36, 102–110 (1960) (in Russian)

[21] Lenci S., Rega G.: Global optimal control and system-dependent solutions in the hardening Helmholtz-Duffing oscillator. Chaos, Sol. & Fract. 21(5), 1031–1046 (2004) https://doi.org/10.1016/S0960-0779(03)00387-4

[22] Kessler Y., Liberzon A., Krylov S.: On sampling rate limits in bistable microbeam sensors. J. Microelectromecha. Syst. 30(7), 980–989 (2021) https://doi.org/10.1109/JMEMS.2021.3117838

[23] Chen J.-S., Yang C.-H.: Experiment and theory on the nonlinear vibration of a shallow arch under harmonic excitation at the end. J. Appl. Mech. 74(7), 1061–1070 (2007) https://doi.org/10.1115/1.2165231

[24] Medina L., Gilat R., Krylov S.: Experimental investigation of the snap-through buckling ofelectrostatically actuated initially curved pre-stressed micro beams. Sens. Actuators A 220, 323–332 (2014) https://doi.org/10.1016/j.sna.2014.10.016

[25] Lenci S., Clementi F., Rega G.: A comprehensive analysis of hardening/softening behaviour of shearable planar beams with whatever axial boundary constraint. Meccanica 51(11), 2589–2606 (2016) https://doi.org/10.1007/s11012-016-0374-6

[26] Lenci S., Rega G.: Nonlinear free vibrations of planar elastic beams: a unified treatment of geometrical and mechanical effects. Proc. IUTAM 19, 35–42 (2016) https://doi.org/10.1016/j.piutam.2016.03.007

[27] Lacarbonara W.: Nonlinear Structural Mechanics: Theory, Dynamical Phenomena and Modeling, Springer, Rome, (2013) ISBN-13: 978–1441912756

[28] Luongo A., Zulli D.: Mathematical Models of Beams and Cables, Wiley-ISTE, L'Aquila, (2013) ISBN-13: 978–1848214217

[29] Babilio E., Lenci S.: On the notion of curvature and its mechanical meaning in a geometrically exact plane beam theory. Int. J. Mechanical Science 128–129, 277–293 (2017) https://doi.org/10.1016/j.ijmecsci.2017.03.031

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[30] Babilio E., Lenci S.: Consequences of different definitions of bending curvature on nonlinear dynamics of beams. Procedia Eng., X International Conference on Structural Dynamics, EURODYN 2017. 199, 1411–1416 (2017) https://doi.org/10.1016/j.proeng.2017.09.382

How to cite this article: Lenci, S., Sorokin, S.: On the static condensation of initially not rectilinear beams. Z Angew Math Mech. e202300668 (2023). https://doi.org/10.1002/zamm.202300668

15214001, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/zamm.202300668 by University Polit Delle Marche-Ancona Ctr Ateneo, Wiley Online Library on [26/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License