Existence of heteroclinic and saddle-type solutions for a class of quasilinear problems in whole ℝ2

EXISTENCE OF HETEROCLINIC AND SADDLE TYPE SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS IN WHOLE $\mathbb{R}^2$

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ABSTRACT. In this work, we use variational methods to prove the existence of heteroclinic and saddle type solutions for a class quasilinear elliptic equations of the form

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2,$$

where $\Phi : \mathbb{R} \to [0,+\infty)$ is a N-function, $A : \mathbb{R}^2 \to \mathbb{R}$ is a periodic positive function and $V : \mathbb{R} \to \mathbb{R}$ is modeled on the Ginzburg Landau potential. In particular our main result includes the case of the potential $V(t) = \Phi(|t^2-1|)$, which reduces to the classical double well Ginzburg-Landau potential when $\Phi(t) = |t|^2$, that is, when we are working with the Laplacian operator.

1. INTRODUCTION

The problem of existence and classification of bounded solutions of stationary Allen Cahn type equations

$$-\Delta u + A(x)V'(u) = 0 \quad x \in \mathbb{R}^n,\ u \in \mathbb{R} \tag{E_1}$$

has been widely studied in the last years, providing a rich amount of differently shaped families of solutions. The Allen-Cahn equation was introduced in 1979 by Allen and Cahn in [12] as a model for phase transitions in binary alloys. The standard model of V is the classical double well Ginzburg-Landau potential $V(u) = (u^2-1)^2$. The function u is a phase parameter describing pointwise the state of the material and the global minima of V represent energetically favorite pure phases. Different values of u depict mixed configurations and by transition solutions we mean entire solutions of (E_1) which are asymptotic in different directions to the pure phases of the systems. In the equation (E_1) the presence of the (positive) oscillatory factor $A(x)$ models an inhomogeneous behavior of the system.

When A is a positive constant function (e.g. $A(x) = 1$), a long standing problem is to characterize the set of the solutions $u \in C^2(\mathbb{R}^n)$ of (E_1) satisfying $|u(x)| \le 1$ and $\partial_{x_1} u(x) > 0$. This problem was pointed out by De Giorgi in [25], where he conjectured that, when $n \le 8$ and $V(s) = (s^2-1)^2$, the whole set of these solutions reduces, modulo space roto-translations, to the unique solution $q_+ \in C^2(\mathbb{R})$ of the one dimensional problem:

$$-\ddot{q}(x) + V'(q(x)) = 0, \quad q(0) = 0 \quad \text{and} \quad q(\pm\infty) = \pm 1.$$

The conjecture has been firstly proved in the planar case by Ghoussoub and Gui in [40] even for more general double well potential V. In the case $n = 3$ it has been proved in [15] and, assuming $u(x) \to \pm 1$ as $x_1 \to \pm\infty$, the same rigidity result has been obtained in dimension $n \le 8$ in [53], paper to which we refer also for an extensive bibliography on the argument. Del Pino, Kowalczyk and Wei showed in [28, 29] that the 1-D symmetry of these solutions is

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generally lost when $n \geq 9$. We refer also to [17, 19, 31], where a weaker version of the De Giorgi conjecture, known as Gibbons conjecture, has been obtained for all the dimensions n and in more general settings. These results show that when A is a positive constant and u is a bounded solution of (E_1) satisfying $u(x) \to \pm 1$ as $x_1 \to \pm\infty$ uniformly with respect to $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ then $u(x) = q_+(x_1)$.

This kind of heteroclinic type transition solutions persist when A is not constant. The heteroclinic type problem was first studied by variational methods for more general elliptic equations of the type

$$-\Delta u = g(x, y, u) \quad \text{in} \quad x \in \mathbb{R},\ y \in \Omega,\ u \in \mathbb{R}, \tag{E_2}$$

by Rabinowitz in [47], when Ω is a bounded regular domain on $\mathbb{R}^n$. Assuming the nonlinearity g to be even and periodic in the variable x, Rabinowitz showed the existence of solutions of (E_2) in $\mathbb{R} \times \Omega$ satisfying Dirichlet or Neumann boundary condition on $\partial\Omega$ and being asymptotic as $x \to \pm\infty$ to different minimal solutions $u_\pm$, periodic in the variable x. This result was generalized by Alves in [13] for different conditions on g, including the case in which g is only asymptotically periodic in the variable x. A related variational approach was used to study the heteroclinic type problem for equation (E_1) in the case in which A is periodic in its variable in [3, 48, 49], showing the existence of (minimal) solutions $u(x)$ which are periodic in the variable $(x_2, \dots x_n)$ and such that u is asymptotic to different minima of the potential V as $x_1 \to \pm\infty$. Starting from the existence of this "basic" heteroclinic solutions, these papers show how the presence of a truly oscillatory factor $A(x, y)$ gives generically the existence of complex classes of other heteroclinic type transition solutions in contrast with the above described rigidity results characterizing the autonomous case (see also [11, 18, 50]).

Another kind of transition solutions for (E_1) was introduced by Dang, Fife and Peletier in [24]. In the planar case $n = 2$, when V is an even double well potential and A is a positive constant, they showed by a sub-supersolution method that (E_1) has a unique bounded solution $u \in C^2(\mathbb{R}^2)$ with the same sign as $x_1 x_2$, odd in both the variables x_1 and x_2 and symmetric with respect to the diagonals $x_2 = \pm x_1$. Along any directions not parallel to the coordinate axes the *saddle* solution u is asymptotic to the minima of the potential V representing a phase transition with cross interface. Note that, even if it is related to minimal transition heteroclinic solutions, being asymptotic to q_+ as $x_2 \to +\infty$, it no longer has minimal character (see [44, 54]). Many extensions for Allen-Cahn models have been considered. In the planar case we refer to [8] for a variational study of saddle type solutions with dihedral symmetries of order k (see also [43] for a global variational approach to the saddle problem) and to [30, 41] for a general study regarding k-end solutions. In higher dimension we mention [5, 6, 21, 22, 46] for the equations case and to [2, 7, 42] for the case of systems of autonomous Allen-Cahn equations.

The analogous for saddle solutions for (E_1) in the planar case, when $A \in C(\mathbb{R}^2)$ is positive, even, periodic and symmetric with respect to the plane diagonal $x_2 = x_1$, i,e, when A satisfies

- (A_1) A is a continuous function and $A(x, y) > 0$ for each $(x, y) \in \mathbb{R}^2$,
- (A_2) $A(x, y) = A(-x, y) = A(x, -y)$ for all $(x, y) \in \mathbb{R}^2$,
- (A_3) $A(x, y) = A(x + 1, y) = A(x, y + 1)$ for any $(x, y) \in \mathbb{R}^2$,
- (A_4) $A(x, y) = A(y, x)$ for all $(x, y) \in \mathbb{R}^2$,

has been introduced in [9] where a variational procedure was introduced to find as in the autonomous case a solution u of (E_1) on $\mathbb{R}^2$ which is odd with respect to both its variables, symmetric with respect to the diagonal, strictly positive on the first quadrant and is asymptotic to the minima of V along any directions not parallel to the coordinate axes. Moreover in [9] it is shown that, as $y \to +\infty$ (uniformly w.r.t. $x \in \mathbb{R}$), the solution u is asymptotic to the set of the x-odd minimal heteroclinic type solutions of (E_1) which are periodic in the variable y described above.

In the recent paper [14], motivated by results found in [8], we tackled the problem of existence of saddle solutions for the analogous of Allen Cahn model in the autonomous quasilinear setting. More precisely given an N-function $\Phi : \mathbb{R} \to [0, +\infty)$ of the form

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds \tag{1.1}$$

for a $\phi \in C^1([0, +\infty), [0, +\infty))$ such that:

(ϕ_1) there exist $l, m \in \mathbb{R}$ such that $1 < l \leq m$ and

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \quad \forall t > 0,$$

(ϕ_2) $\phi(t) > 0$ and $(\phi(t)t)' > 0$ for any $t > 0$,

(ϕ_3) ϕ is non-decreasing,

(ϕ_4) there exists $\kappa > 0$ such that

$$\phi(|t|) + \phi'(|t|)|t| \leq \kappa\phi(|t|), \quad \forall t \in \mathbb{R},$$

(ϕ_5) there is $M > 0$ such that $(\phi(t)t)' \geq M\phi(t)$ for all $t > 0$,

and a potential $V \in C^2(\mathbb{R}, \mathbb{R})$ verifying:

(V_1) $V(t) \geq 0$ for all $t \in \mathbb{R}$ and $V(t) = 0 \;\Leftrightarrow\; t = -1, 1$,

(V_2) $V(-t) = V(t)$ for any $t \in \mathbb{R}$,

(V_3) there are $\delta_1 \in (0, 1)$ and $w_1, w_2 > 0$ such that

$$w_1\Phi(|t - 1|) \leq V(t) \leq w_2\Phi(|t - 1|), \quad \forall t \in (1 - \delta_1, 1 + \delta_1),$$

(V_4) there exists $\omega > 0$ such that

$$V'(t) \leq -\omega\phi(|1 - t|)|1 - t|t, \quad \forall t \in [0, 1],$$

(V_5) there is $\delta_0 > 0$ such that V' is increasing on $(1 - \delta_0, 1)$,

(V_6) there are $\gamma > 0$ and $\epsilon > 0$ such that $\tilde{\Phi}(V'(t)) \leq \gamma\Phi(|1 - t|)$ for all $t \in (1 - \epsilon, 1)$

we considered the related quasilinear Allen Cahn model

$$-\Delta_\Phi u + V'(u) = 0 \quad \text{ in } \quad \mathbb{R}^2. \tag{E_3}$$

where $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$. Note that the potential $V(t) = \Phi(|t^2 - 1|)$ satisfies $(V_1) - (V_6)$ and so (E_3) reduces to (E_1) in the case $\Phi(t) = |t|^2$ and $V(t) = (t^2 - 1)^2$.

In [14], we refined and adapted the variational procedure introduced in [9] to show that, like in the Laplacian case, (E_3) admits transition heteroclinic type solutions and, for each integer number $k \geq 2$, a related saddle-type solution with dihedral symmetries of order k.

In recent years, facing the need of a mathematical description of advanced physical problems there has been a growing number of works involving the Φ-laplacian operator Δ_Φ and its theory is by now rather developed. As a first example we may consider the case

$$\Phi(t) = |t|^p, \quad t \in \mathbb{R}, \quad p \in (1, +\infty),$$

which is related to the celebrated p-Laplacian operator that often appears in physical models, for example in Newtonian and non-Newtonian fluids (see [26, 27] and references therein). Motivated by concrete examples of equations arising in fluid mechanics and plasticity theory, Seregin and Fuchs in [34, 35] (see also [33]) were led to the minimization of integrals where appears the logarithmic model

$$\Phi(t) = |t|^p \ln(1 + |t|), \quad t \in \mathbb{R}, \quad p \in [1, +\infty),$$

which is an N-function of the type (1.1). Other model of N-function of the form (1.1) that often arises in a lot of fields of physics and related sciences such as biophysics and chemical reaction design is

$$\Phi(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q, \quad t \in \mathbb{R}, \quad 1 < p < q < +\infty.$$

The differential operator associated with this N-function is known as the (p, q)-Laplacian operator and the prototype for these models can be written in the form

$$u_t = -\Delta_\Phi + f(x, u).$$

In this configuration, the function u generally describes a concentration, Δ_Φ corresponds to the diffusion and $f(x, u)$ is the reaction term that corresponds to source and loss processes. For a quite comprehensive account, the interested reader might start by referring to [16, 32]. Finally, it is worth mentioning that the N-function of the form (1.1)

$$\Phi(t) = (1 + t^2)^\gamma - 1, \quad t \in \mathbb{R}, \quad \gamma > 1,$$

appears in the works [38, 39], where the authors report that studies of quasilinear equations involving the associated operator Δ_Φ are motivated by nonlinear elasticity models. For other examples of N-functions of the type (1.1) and more applications we refer the reader to [33, 36] and the bibliography therein.

In the present paper we continue the study initiated in [14] studying the existence of heteroclinic and related saddle-type weak solutions of the non autonomous version of equation (E_3)

$$-\Delta_\Phi u + A(x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{PDE}$$

where A is a symmetric positive periodic function satisfying $(A_1) - (A_4)$.

As a first step in the present study we use variational methods related to the ones introduced in [9] and [14], to establish the existence of *(minimal) heteroclinic type solutions* of (PDE), i.e. weak solutions $v \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$ which are 1 - periodic in the variable y and such that

$$v(x, y) \to -1 \text{ as } x \to -\infty \text{ and } v(x, y) \to 1 \text{ as } x \to +\infty, \text{ uniformly in } y \in \mathbb{R}.$$

Here we borrow some ideas developed in [9] and [47] to look for minima of the action functional

$$I(u) = \int_{\mathbb{R}} \int_0^1 (\Phi(|\nabla u|) + A(x, y)V(u))\, dydx,$$

on the class

$$E = \left\{ u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R} \times [0, 1]) : \ 0 \le u(x, y) \le 1 \text{ for } x > 0 \text{ and } u \text{ is odd in } x \right\},$$

where $W^{1,\Phi}_{\text{loc}}(\mathbb{R} \times [0, 1])$ denotes the usual Orlicz-Sobolev space. Denoting by K the set of minima of I on E, we show that K is not empty and constituted by (minimal) heteroclinic type solutions of (PDE).

The minimality properties of these heteroclinic type solutions allows us, as a second step, to build up a variational framework inspired to the one introduced in [9] to detect the existence of saddle type solution of (PDE), characterizing their the asymptotic behavior.

More precisely we have the following results:

Theorem 1.1. *Assume (ϕ_1)-(ϕ_2), $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_3) and (A_1)-(A_3). There exists $v \in C^{1,\alpha}_{loc}(\mathbb{R}^2)$, a weak solution of (PDE) that verifies the following:*

(a) $v(x, y) = -v(-x, y)$, for all $(x, y) \in \mathbb{R}^2$;
(b) $v(x, y) = v(x, y + 1)$, for any $(x, y) \in \mathbb{R}^2$;
(c) $0 < v(x, y) < 1$ for each $x > 0$ and $y \in \mathbb{R}$.

Moreover, v is a heteroclinic solution from -1 to 1.

Theorem 1.2. *Assume (ϕ_1)-(ϕ_4), $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_4) and (A_1)-(A_4). There exists $v \in C^{1,\alpha}_{loc}(\mathbb{R}^2)$, a weak solution of (PDE) that verifies the following:*

(a) $v(x, y) > 0$ on the fist quadrant in $\mathbb{R}^2$;
(b) $v(x, y) = -v(-x, y) = -v(x, -y)$ for all $(x, y) \in \mathbb{R}^2$;
(c) $v(x, y) = v(y, x)$ for any $(x, y) \in \mathbb{R}^2$;
(d) There is $u_0 \in K$ such that $\|v - \tau_j u_0\|_{L^\infty(\mathbb{R}\times[j,j+1])} \to 0$ as $j \to +\infty$,

where $\tau_j u_0(x, y) = u_0(x, y - j)$ for all $(x, y) \in \mathbb{R}^2$.

The item (d) of Theorem 1.2 characterizes the asymptotic behavior of v. It guarantees that along directions parallel to the coordinate axes the saddle solution is asymptotic to (rotated of) the minimal heteroclinic set K. This implies that along any direction not parallel to the coordinate axes v is asymptotic at infinity to ± 1 and so, the saddle solution can be seen as a phase transition solution with cross interface.

We point out that Theorems 1.1, 1.2, improve the results in [14] not only in the fact that the function A is allowed to be not constant but also because, unlike in [14], the assumptions (ϕ_5) and (V_5)-(V_6) are not needed. Moreover we note that even though the variational approach is inspired by the one used in [9], many tools used in the classical Laplacian context, such as for example some maximum principles, C^2 regularity, existence and local uniqueness theorems, are no more available in the present framework. The proofs of our results require new estimates based on the Harnack type inequalities found in [55] and on results about $C^{1,\alpha}$ regularity for quasilinear problems as obtained by Liberman in [45].

This paper is organized as follows. In Section 2, we prove Theorem 1.1, while in Section 3 we show some compactness properties. We build up in Section 5 a renormalized minimization procedure inspired by the one used in [9, 10] (see also [8]) that takes into account refined properties studied in Sections 3 and 4, and then the proof of Theorem 1.2 is given. Finally, we write an Appendix A about some facts involving Orlicz–Sobolev spaces for unfamiliar readers with the topic.

2. Existence of Heteroclinic Solutions

In this section, we show the existence of a heteroclinic solution from -1 to 1 for the quasilinear problem (PDE). To begin with, for $\Omega_0 = \mathbb{R} \times [0, 1]$ let us consider the set

$$E = \left\{u \in W^{1,\Phi}_{\text{loc}}(\Omega_0) : \ 0 \le u(x, y) \le 1 \text{ for } x > 0 \text{ and } u \text{ is odd in } x\right\}.$$

In the sequel, $I : W^{1,\Phi}_{\text{loc}}(\Omega_0) \to \mathbb{R} \cup \{+\infty\}$ designates the functional given by

$$I(u) = \int_{\Omega_0} (\Phi(|\nabla u|) + A(x, y)V(u))\, dydx.$$

An direct computation shows that

$$u_n \rightharpoonup u \text{ in } W^{1,\Phi}_{\text{loc}}(\Omega_0) \Rightarrow I(u) \le \liminf_{n\to+\infty} I(u_n). \tag{2.1}$$

Hereinafter, the expression $u_n \rightharpoonup u$ in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$ means that $u_n \rightharpoonup u$ in $W^{1,\Phi}([L, R] \times [0, 1])$ for every $R, L \in \mathbb{R}$ with $L < R$. Setting

$$\mathcal{L}(u) = \Phi(|\nabla u|) + A(x, y)V(u), \quad u \in W^{1,\Phi}_{\text{loc}}(\Omega_0),$$

it follows from the definitions of Φ, V and A that

$$\mathcal{L}(u) \ge 0, \quad \forall u \in E,$$

and so, the functional I is bounded from below. Now, it is easy to check that the function $\varphi_* : \Omega_0 \to \mathbb{R}$ defined by

$$\varphi_*(x,y) = \begin{cases} 1, & \text{if} \quad x > 1 \quad \text{and} \quad y \in [0,1], \\ x, & \text{if} \quad -1 \leq x \leq 1 \quad \text{and} \quad y \in [0,1], \\ -1, & \text{if} \quad x < -1 \quad \text{and} \quad y \in [0,1] \end{cases} \tag{2.2}$$

belongs to E with $I(\varphi_*) < +\infty$. Therefore, the real number

$$c := \inf_{u \in E} I(u)$$

is well defined.

From now on, for each $x \in \mathbb{R}$ fixed and $u \in E$, we will identify $u(x,\cdot)$ as being a real function in $y \in [0,1]$. For each $y \in [0,1]$ fixed, we will also identify $u(\cdot,y)$ as being a real function in $x \in \mathbb{R}$. Employing Fubini's Theorem, it follows that

$$u(x,\cdot) \in W^{1,\Phi}(0,1) \ \text{ a.e. in } \ x \in \mathbb{R} \ \text{ and } \ u(\cdot,y) \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \ \text{ a.e. in } \ y \in [0,1].$$

Finally, since the functions in E have L^∞-norm less than or equal to 1, without loss of generality, we can make a modification on function V, by assuming that it satisfies the following:

$$V(t) = V(2), \quad \text{for} \quad |t| \geq 2. \tag{2.3}$$

Hereafter, we will denote this new modification of V by itself. Moreover, according to (A_1)-(A_4),

$$0 < \min_{\mathbb{R}^2} A(x,y) \leq A(x,y) \leq \max_{\mathbb{R}^2} A(x,y) < +\infty.$$

In what follows, $\underline{A} = \min_{\mathbb{R}^2} A(x,y)$ and $\overline{A} = \max_{\mathbb{R}^2} A(x,y)$.

Next, we prove an important estimate that will be used often in this paper.

Lemma 2.1. *Let $u \in E$. If $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, then*

$$\int_0^1 \Phi\left(|u(x_2,y) - u(x_1,y)|\right) dy \leq \frac{\xi_1\left(|x_1 - x_2|\right)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi(|u_x|) dx dy,$$

where ξ_1 was given in Lemma A.1.

Proof. First of all note that from Lemma A.4, $u \in W^{1,l}_{\text{loc}}(\Omega_0)$, and hence, by [20, Theorem 8.2],

$$|u(x_2,y) - u(x_1,y)| = \left| \int_{x_1}^{x_2} u_x(x,y) dx \right|.$$

As Φ is even,

$$\Phi\left(|u(x_2,y) - u(x_1,y)|\right) = \Phi\left(\int_{x_1}^{x_2} u_x(x,y) dx \right). \tag{2.4}$$

Invoking Jensen's Inequality given in [52, Theorem 3.3],

$$\Phi\left(\int_{x_1}^{x_2} u_x(x,y) dx \right) \leq \frac{1}{|x_1 - x_2|} \int_{x_1}^{x_2} \Phi((x_2 - x_1) u_x(x,y)) dx, \tag{2.5}$$

then by (2.4) and (2.5),

$$\int_0^1 \Phi\left(|u(x_2,y) - u(x_1,y)|\right) dy \leq \frac{1}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi\left((x_2 - x_1) u_x(x,y)\right) dx dy.$$

According to Lemma A.1,

$$\int_0^1 \Phi\left(|u(x_2,y) - u(x_1,y)|\right) dy \leq \frac{\xi_1(|x_1 - x_2|)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi\left(u_x(x,y)\right) dx dy,$$

and the lemma follows. □

As a consequence of the last lemma, we obtain the following result.

Corollary 2.2. *If $u \in E$ and $I(u) < +\infty$, then:*

a) The function $x \in \mathbb{R} \mapsto u(x, \cdot) \in L^{\Phi}(0,1)$ is uniformly continuous a.e..

b) The function $x \in \mathbb{R} \mapsto \|u(x, \cdot) - 1\|_{L^{\Phi}(0,1)}$ is continuous a.e..

Proof. Let be $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$. Since Φ is an increasing function in $(0, +\infty)$ and $|\partial_x u| \le |\nabla u|$, the Lemma 2.1 ensures that

$$\int_0^1 \Phi\left(|u(x_2, y) - u(x_1, y)|\right) dy \le \frac{\xi_1(|x_1 - x_2|)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi(|\nabla u|) dx dy,$$

and so,

$$\int_0^1 \Phi\left(|u(x_2, y) - u(x_1, y)|\right) dy \le I(u) \max\left\{|x_1 - x_2|^{l-1}, |x_1 - x_2|^{m-1}\right\}.$$

From this, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\int_0^1 \Phi\left(|u(x_2, y) - u(x_1, y)|\right) dy < \epsilon \;\text{ for }\; |x_1 - x_2| < \delta.$$

The last inequality combined with Lemma A.1 gives

$$\xi_0\left(\|u(x_2, \cdot) - u(x_1, \cdot)\|_{L^{\Phi}(0,1)}\right) < \epsilon \;\text{ for }\; |x_1 - x_2| < \delta.$$

Therefore,

$$|x_1 - x_2| < \delta \Rightarrow \|u(x_2, \cdot) - u(x_1, \cdot)\|_{L^{\Phi}(0,1)} < \xi_0^{-1}(\epsilon),$$

finishing the proof of a). The item b) follows from a), because we have the inequality below

$$\left|\|u(x_2, \cdot) - 1\|_{L^{\Phi}(0,1)} - \|u(x_1, \cdot) - 1\|_{L^{\Phi}(0,1)}\right| \le \|u(x_2, \cdot) - u(x_1, \cdot)\|_{L^{\Phi}(0,1)}.$$

This completes the proof. □

Another important consequence of Lemma 2.1 is the following result.

Lemma 2.3. *If $u \in E$ satisfies*

$$\|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \ge r \;\text{ a.e. in } x \in (x_1, x_2) \subset [0, +\infty),$$

for some $r > 0$, then there exists $\mu_r > 0$ independent of x_1 and x_2 satisfying

$$\begin{aligned}\int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx &\ge \frac{|x_2 - x_1|}{2\xi_1\left(|x_2 - x_1|\right)} \int_0^1 \Phi\left(|u(x_2, y) - u(x_1, y)|\right) dy + (2\mu_r)^{\frac{m}{m-1}} |x_2 - x_1| \\ &\ge \mu_r h\left(\int_0^1 \Phi\left(|u(x_2, y) - u(x_1, y)|\right) dy\right),\end{aligned}$$

where $h(t) = \min\left\{t^{\frac{1}{l}}, t^{\frac{1}{m}}\right\}$.

Proof. In what follows, we are going to work with the functional $F : W^{1,\Phi}(0,1) \to \mathbb{R}$ defined by

$$F(v) = \int_0^1 \left(\frac{1}{2}\Phi\left(|v'|\right) + \underline{A}V(v)\right) dy.$$

We claim that for any sequence $(v_n) \subset W^{1,\Phi}(0,1)$ with $0 \le v_n(y) \le 1$ for all $y \in (0,1)$ and $F(v_n) \to 0$ as $n \to +\infty$, we must have

$$\|v_n - 1\|_{W^{1,\Phi}(0,1)} \to 0 \;\text{ as }\; n \to +\infty.$$

Indeed, the limit $F(v_n) \to 0$ gives

$$\int_0^1 \Phi\left(|v_n'|\right) dy \to 0 \quad \text{and} \qquad \int_0^1 V(v_n) dy \to 0, \quad as \quad n \to +\infty.$$

Here we would like point out that by (V_1) and (V_3) that there are $\underline{w}, \overline{w} > 0$ satisfying

$$\underline{w}\Phi(|t-1|) \leq V(t) \leq \overline{w}\Phi(|t-1|), \quad \forall t \in [0,1]. \tag{2.6}$$

In fact, by (V_1) and the fact that $\Phi(t) = 0$ if, and only if $t = 0$, we have that the function $\frac{V(t)}{\Phi(|t-1|)}$ is continuous and strictly positive in $[0, 1-\delta_1]$. Hence, there are $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1\Phi(|t-1|) \leq V(t) \leq \alpha_2\Phi(|t-1|), \quad \forall t \in [0, 1-\delta_1].$$

Now (2.6) follows by taking $\underline{w} = \min\{\alpha_1, w_1\}$ and $\overline{w} = \max\{\alpha_2, w_2\}$, where w_1 and w_2 were given in (V_3). Thus, since $0 \leq v_n(y) \leq 1$ for every $y \in (0,1)$, (2.6) ensures that

$$\int_0^1 \Phi\left(|v_n - 1|\right) dy \leq \frac{1}{\underline{w}} \int_0^1 V(v_n) dy, \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\int_0^1 \Phi\left(|v_n - 1|\right) dy \to 0, \quad as \quad n \to +\infty.$$

The limits above together with the fact that $\Phi \in \Delta_2$ yield

$$\|v_n - 1\|_{W^{1,\Phi}(0,1)} \to 0 \quad \text{as} \quad n \to +\infty,$$

which proves the claim. Thereby, if $v \in W^{1,\Phi}(0,1)$, $0 \leq v \leq 1$ in $(0,1)$ and $\|v-1\|_{W^{1,\Phi}(0,1)} \geq r$, then there exists $\mu_r \in (0, 1/2)$ such that

$$F(v) \geq (2\mu_r)^{\frac{m}{m-1}}.$$

Now, if $u \in E$, we know that $0 \leq u(x,\cdot) \leq 1$ on $(0,1)$ for almost every $x > 0$, and so, if $\|u(x,\cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq r$ a.e. in (x_1, x_2), we must have

$$F\left(u(x,\cdot)\right) \geq (2\mu_r)^{\frac{m}{m-1}} \quad \text{a.e. in} \quad x \in (x_1, x_2),$$

which leads to

$$\begin{aligned}
\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u) dy dx &= \int_{x_1}^{x_2}\int_0^1 \left(\Phi\left(|\nabla u|\right) + A(x,y)V(u)\right) dy dx \\
&\geq \frac{1}{2}\int_{x_1}^{x_2}\int_0^1 \Phi\left(|\partial_x u|\right) dy dx + \int_{x_1}^{x_2}\int_0^1 \left(\frac{1}{2}\Phi\left(|\partial_y u|\right) + \underline{A}V(u)\right) dy dx \\
&\geq \frac{1}{2}\int_{x_1}^{x_2}\int_0^1 \Phi\left(|\partial_x u|\right) dy dx + \int_{x_1}^{x_2} F\left(u(x,\cdot)\right) dx \\
&\geq \frac{1}{2}\int_{x_1}^{x_2}\int_0^1 \Phi\left(|\partial_x u|\right) dy dx + (2\mu_r)^{\frac{m}{m-1}}|x_2 - x_1|.
\end{aligned}$$

Thanks to Lemma 2.1,

$$\begin{aligned}
\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u) dy dx &\geq \frac{1}{2}\frac{|x_1 - x_2|}{\xi_1(|x_1 - x_2|)}\int_0^1 \Phi\left(|u(x_2,y) - u(x_1,y)|\right) dy + (2\mu_r)^{\frac{m}{m-1}}|x_2 - x_1| \\
&\geq \frac{1}{2}\frac{|x_1 - x_2|}{\xi_1(|x_1 - x_2|)}\int_0^1 \Phi\left(|u(x_2,y) - u(x_1,y)|\right) dy + 2^{\frac{m}{m-1}-1}\mu_r^{\frac{m}{m-1}}|x_2 - x_1|.
\end{aligned}$$

Recalling that $\xi_1(|x_2-x_1|) = \max\left\{|x_2-x_1|^l, |x_2-x_1|^m\right\}$, we will consider the cases $\xi_1(|x_2-x_1|) = |x_2-x_1|^m$ and $\xi_1(|x_2-x_1|) = |x_2-x_1|^l$. If $\xi_1(|x_2-x_1|) = |x_2-x_1|^m$,

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \frac{1}{2}\frac{1}{|x_1-x_2|^{m-1}}\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy + 2^{\frac{m}{m-1}-1}\mu_r^{\frac{m}{m-1}}|x_2-x_1|$$

$$\geq \frac{1}{2m}\left[\frac{1}{|x_1-x_2|^{\frac{m-1}{m}}}\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right)^{\frac{1}{m}}\right]^m + \frac{m-1}{2m}\left(2\mu_r|x_2-x_1|^{\frac{m-1}{m}}\right)^{\frac{m}{m-1}}.$$

Using Young's inequality for the conjugate exponents m and $\frac{m}{m-1}$, we find

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \frac{1}{2}\left[\frac{1}{|x_1-x_2|^{\frac{m-1}{m}}}\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right)^{\frac{1}{m}} 2\mu_r|x_2-x_1|^{\frac{m-1}{m}}\right],$$

that is,

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \mu_r\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right)^{\frac{1}{m}}. \tag{2.7}$$

If $\xi_1(|x_1-x_2|) = |x_1-x_2|^l$, a similar argument works to prove that

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \frac{1}{2l}\left[\frac{1}{|x_1-x_2|^{\frac{l-1}{l}}}\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right)^{\frac{1}{l}}\right]^l + (2\mu_r)^{\frac{m}{m-1}}|x_2-x_1|.$$

Now, since $l \leq m$ and $0 < 2\mu_r < 1$, we obtain that $1 < \frac{m}{m-1} \leq \frac{l}{l-1}$ and $(2\mu_r)^{\frac{l}{l-1}} \leq (2\mu_r)^{\frac{m}{m-1}}$. Therefore,

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \frac{1}{2l}\left[\frac{1}{|x_1-x_2|^{\frac{l-1}{l}}}\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right)^{\frac{1}{l}}\right]^l + (2\mu_r)^{\frac{l}{l-1}}|x_2-x_1|.$$

Employing again Young's inequality, we derive

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \mu_r\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right)^{\frac{1}{l}}. \tag{2.8}$$

From (2.7) and (2.8),

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(u)dydx \geq \mu_r h\left(\int_0^1 \Phi\left(|u(x_2,y)-u(x_1,y)|\right)dy\right),$$

where $h(t) = \min\left\{t^{\frac{1}{l}}, t^{\frac{1}{m}}\right\}$, which is precisely the assertion of the lemma. $\square$

The next result characterizes the asymptotic behavior of functions $u \in E$ with $I(u) < +\infty$.

Lemma 2.4. *If $u \in E$ and $I(u) < +\infty$, then*

$$\|u(x,\cdot)-1\|_{L^\Phi(0,1)} \to 0 \quad as \quad x \to +\infty \quad and \quad \|u(x,\cdot)+1\|_{L^\Phi(0,1)} \to 0 \quad as \quad x \to -\infty.$$

Proof. To begin with, we claim that

$$\liminf_{x\to+\infty}\int_0^1 \Phi\left(|u(x,y)-1|\right)dy = 0. \tag{2.9}$$

Indeed, if the limit does not hold, then there are $r > 0$ and $x_1 > 0$ satisfying

$$\int_0^1 \Phi\left(|u(x,y)-1|\right)dy \geq r, \quad \forall x > x_1.$$

So, the properties of Φ together with Lemma A.1 guarantee that

$$\begin{aligned} r &\leq \xi_1\left(\|u(x,\cdot)-1\|_{W^{1,\Phi}(0,1)}\right)\int_0^1 \Phi\left(\frac{|u(x,y)-1|}{\|u(x,\cdot)-1\|_{W^{1,\Phi}(0,1)}}\right)dy \\ &\leq \xi_1\left(\|u(x,\cdot)-1\|_{W^{1,\Phi}(0,1)}\right)\int_0^1 \Phi\left(\frac{|u(x,y)-1|}{\|u(x,\cdot)-1\|_{L^{\Phi}(0,1)}}\right)dy \\ &\leq \xi_1\left(\|u(x,\cdot)-1\|_{W^{1,\Phi}(0,1)}\right), \end{aligned}$$

that is,

$$\|u(x,\cdot)-1\|_{W^{1,\Phi}(0,1)} \geq \xi_1^{-1}(r) := r_1 \text{ for all } x > x_1.$$

The last inequality permits to apply Lemma 2.3 to get $\mu_{r_1} > 0$ satisfying

$$I(u) \geq \int_{x_1}^{x}\int_0^1 \mathcal{L}(u)dydx \geq (2\mu_{r_1})^{\frac{m}{m-1}}(x-x_1).$$

Taking the limit of $x \to +\infty$ we infer that $I(u) = +\infty$, which is absurd, and (2.9) is proved.

As $\Phi \in \Delta_2$, the limit in (2.9) is equivalent to

$$\liminf_{x\to+\infty} \|u(x,\cdot)-1\|_{L^{\Phi}(0,1)} = 0. \tag{2.10}$$

Next we are going to show that

$$\limsup_{x\to+\infty} \|u(x,\cdot)-1\|_{L^{\Phi}(0,1)} = 0. \tag{2.11}$$

To see why, assume by contradiction that $\limsup_{x\to+\infty} \|u(x,\cdot)-1\|_{L^{\Phi}(0,1)} > 0$. Then, there exists $r > 0$ such that

$$\limsup_{x\to+\infty} \|u(x,\cdot)-1\|_{L^{\Phi}(0,1)} > 2r. \tag{2.12}$$

By Corollary 2.2, we can assume that the function $x \in \mathbb{R} \mapsto \|u(x,\cdot)-1\|_{L^{\Phi}(0,1)}$ is continuous in $\mathbb{R}$. So, according to (2.10) and (2.12), there is a sequence of disjoint intervals (σ_i,τ_i) with $0 < \sigma_i < \tau_i < \sigma_{i+1} < \tau_{i+1}$, $i \in \mathbb{N}$, and $\sigma_i \to +\infty$ as $i \to +\infty$ such that for each i,

$$r \leq \|u(x,\cdot)-1\|_{L^{\Phi}(0,1)} \leq 2r \quad \text{for} \quad x \in [\sigma_i,\tau_i]$$

and

$$\|u(\sigma_i,\cdot)-1\|_{L^{\Phi}(0,1)} = r \quad \text{and} \quad \|u(\tau_i,\cdot)-1\|_{L^{\Phi}(0,1)} = 2r.$$

Due to triangular inequality,

$$\|u(\tau_i,\cdot)-u(\sigma_i,\cdot)\|_{L^{\Phi}(0,1)} \geq r \quad \forall i \in \mathbb{N}, \tag{2.13}$$

from where it follows that there exists $\epsilon > 0$ such that

$$\int_0^1 \Phi\left(|u(\tau_i,\cdot)-u(\sigma_i,\cdot)|\right)dy \geq \epsilon, \quad \forall i \in \mathbb{N}. \tag{2.14}$$

In fact, arguing by contradiction, let us suppose that there is a sequence $(i_n) \subset \mathbb{N}$ satisfying

$$\int_0^1 \Phi\left(|u(\tau_{i_n},\cdot)-u(\sigma_{i_n},\cdot)|\right)dy \to 0 \quad \text{as} \quad n \to +\infty.$$

Since $\Phi \in \Delta_2$, the above limit implies that

$$\|u(\tau_{i_n},\cdot)-u(\sigma_{i_n},\cdot)\|_{L^{\Phi}(0,1)} \to 0 \quad \text{as} \quad n \to +\infty,$$

which contradicts (2.13). Consequently, by Lemma 2.3 there exists $\mu_r > 0$ such that

$$I(u) \geq \sum_{i=1}^{+\infty} \int_{\sigma_i}^{\tau_i} \int_0^1 \mathcal{L}(u) dy dx \geq \sum_{i=1}^{+\infty} \mu_r h \left(\int_0^1 \Phi\left(|u(\tau_i, \cdot) - u(\sigma_i, \cdot)| \right) dy \right)$$

that combined with (2.14) provides

$$I(u) \geq \mu_r \sum_{i=1}^{+\infty} h(\epsilon),$$

which is absurd, because $I(u) < +\infty$. Now, the lemma follows from (2.10) and (2.11). □

Our next result is a key point in our approach, because it establishes the existence of heteroclinic solution for a class of problem defined on the strip $\Omega_0 = \mathbb{R} \times [0,1]$, which will be used to prove the existence of heteroclinic solution in whole $\mathbb{R}^2$.

Theorem 2.5. *There exists $u \in E$ such that $I(u) = c$. Moreover, u is a weak solution to the quasilinear elliptic problem*

$$\begin{cases} -\Delta_\Phi u + A(x,y)V'(u) = 0, & in \ \ \Omega_0 \\ \dfrac{\partial u}{\partial \eta}(x,y) = 0, & on \ \ \partial\Omega_0. \end{cases} \tag{P}$$

Proof. Let $(u_n) \subset E$ be a minimizing sequence for I. It is straightforward to check that (u_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$. Then, by a classical diagonal argument, there are a subsequence of (u_n), still denoted by (u_n), and $u \in W^{1,\Phi}_{\text{loc}}(\Omega_0)$ verifying

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,\Phi}_{\text{loc}}(\Omega_0) \quad \text{and} \quad u_n(x,y) \to u(x,y) \ \text{ a.e. in } \ \Omega_0.$$

By the pointwise convergence, it is plain that

$$u(x,y) = -u(-x,y) \ \text{ a.e. in } \Omega_0 \ \text{ and } \ 0 \leq u(x,y) \leq 1 \ \text{ for } \ x \geq 0,$$

from where it follows that $u \in E$. Therefore, from (2.1) we may conclude $I(u) = c$. To complete the proof, it is sufficient to show that

$$\int_{\Omega_0} \left(\phi\left(|\nabla u| \right) \nabla u \nabla \psi + A(x,y)V'(u)\psi \right) dy dx \geq 0,$$

for all $\psi \in X^{1,\Phi}(\Omega_0)$, where

$$X^{1,\Phi}(\Omega_0) = \left\{ w \in W^{1,\Phi}(\Omega_0) \text{ with } w(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0 \right\}. \tag{2.15}$$

Now given $\psi \in X^{1,\Phi}(\Omega_0)$, we can write $\psi(x,y) = \psi_o(x,y) + \psi_e(x,y)$, where

$$\psi_e(x,y) = \frac{\psi(x,y) + \psi(-x,y)}{2} \quad \text{and} \quad \psi_o(x,y) = \frac{\psi(x,y) - \psi(-x,y)}{2}.$$

Note that ψ_o is odd in x and ψ_e is even in x. From this, for $t > 0$ we set

$$\varphi(x,y) = \begin{cases} u(x,y) + t\psi_o(x,y), & \text{if } \ x \geq 0 \text{ and } u(x,y) + t\psi_o(x,y) \geq 0 \\ -u(x,y) - t\psi_o(x,y), & \text{if } \ x \geq 0 \text{ and } u(x,y) + t\psi_o(x,y) \leq 0 \\ -\varphi(-x,y) & \text{if } \ x < 0, \end{cases}$$

from where it follows that φ is odd in the variable x and $\varphi(x,y) \geq 0$ if $x \geq 0$. Moreover, from (V_2), $I(\varphi) = I(u + t\psi_o)$. Next, putting

$$\tilde{\varphi}(x,y) = \max\left\{-1, \min\{1, \varphi(x,y)\}\right\} \quad \text{for } (x,y) \in \Omega_0,$$

a direct computation shows that $\tilde{\varphi} \in E$ with

$$|\nabla \tilde{\varphi}(x,y)| \leq |\nabla(u + t\psi_o)(x,y)|, \quad \forall (x,y) \in \Omega_0.$$

Furthermore, from (V_1)-(V_2),

$$V\left(\tilde{\varphi}(x,y)\right) \leq V\left((u+t\psi_o)(x,y)\right), \quad \forall (x,y) \in \Omega_0.$$

Therefore,

$$I(u+t\psi_o) = I(\varphi) \geq I(\tilde{\varphi}) \geq c = I(u). \tag{2.16}$$

On the other hand, according to (A.2),

$$\Phi\left(|\nabla(u+t\psi)|\right) - \Phi\left(|\nabla(u+t\psi_o)|\right) \geq \phi\left(|\nabla(u+t\psi_o)|\right)\nabla(u+t\psi_o)\nabla(t\psi_e),$$

so

$$\begin{aligned} \int_{\Omega_0} (\Phi(|\nabla(u+t\psi)|) - &\Phi(|\nabla(u+t\psi_o)|))dxdy \\ &\geq \int_{\Omega_0} \phi(|\nabla(u+t\psi_o)|)\left(t\nabla u\nabla\psi_e + t^2\nabla\psi_o\nabla\psi_e\right)dxdy. \end{aligned} \tag{2.17}$$

Since $I(u) = c$ and $\psi \in X^{1,\Phi}(\Omega_0)$, we see that $I(u+t\psi), I(u+t\psi_o) < +\infty$, because for $|x|$ sufficiently large we must have $u(x,y) + t\psi(x,y) = u(x,y)$ and $u(x,y) + t\psi_o(x,y) = u(x,y)$. Thus,

$$\begin{aligned} I(u+t\psi) - I(u+t\psi_o) = &\int_{\Omega_0} \left(\Phi(|\nabla(u+t\psi)|) - \Phi(|\nabla(u+t\psi_o)|)\right)dxdy \\ &+ \int_{\Omega_0} A(x,y)\left(V(u+t\psi) - V(u+t\psi_o)\right)dxdy, \end{aligned}$$

and by (2.17),

$$\begin{aligned} I(u+t\psi) - I(u+t\psi_o) \geq\ &t\int_{\Omega_0} \phi(|\nabla(u+t\psi_o)|)\nabla u\nabla\psi_e dxdy \\ &+ t^2\int_{\Omega_0} \phi(|\nabla(u+t\psi_o)|)\nabla\psi_o\nabla\psi_e dxdy \\ &+ \int_{\Omega_0} A(x,y)\left(V(u+t\psi) - V(u+t\psi_o)\right)dxdy. \end{aligned} \tag{2.18}$$

It is easily seen that the functions $\phi(|\nabla(u+t\psi_o)|)\nabla u\nabla\psi_e$ and $\phi(|\nabla(u+t\psi_o)|)\nabla\psi_o\nabla\psi_e$ are odd in the variable x, and so,

$$\int_{\Omega_0} \phi(|\nabla(u+t\psi_o)|)\nabla u\nabla\psi_e dxdy = \int_{\Omega_0} \phi(|\nabla(u+t\psi_o)|)\nabla\psi_o\nabla\psi_e dxdy = 0. \tag{2.19}$$

Substituting (2.19) into (2.18), we infer that

$$I(u+t\psi) - I(u+t\psi_o) \geq \int_{\Omega_0} A(x,y)\left(V(u+t\psi) - V(u+t\psi_o)\right)dxdy$$

that combines with (2.16) to give

$$I(u+t\psi) - I(u) \geq \int_{\Omega_0} A(x,y)\left(V(u+t\psi) - V(u+t\psi_o)\right)dxdy,$$

and so,

$$
\begin{aligned}
\int_{\Omega_0} (\phi(|\nabla u|)\nabla u \nabla \psi + A(x,y)V'(u)\psi)dxdy &= \lim_{t\to 0^+} \frac{I(u+t\psi)-I(u)}{t} \\
&\geq \lim_{t\to 0^+} \int_{\Omega_0} A(x,y)\frac{V(u+t\psi)-V(u+t\psi_o)}{t}dxdy \\
&\geq \lim_{t\to 0^+} \int_{\Omega_0} A(x,y)\left(\frac{V(u+t\psi)-V(u)}{t} - \frac{V(u+t\psi_o)-V(u)}{t}\right)dxdy \\
&\geq \int_{\Omega_0} A(x,y)V'(u)(\psi-\psi_o)dxdy = \int_{\Omega_0} A(x,y)V'(u)\psi_e dxdy.
\end{aligned} \tag{2.20}
$$

Since the function $A(x,y)V'(u)\psi_e$ is odd in x, it follows that

$$
\int_{\Omega_0} \big(\phi(|\nabla u|)\nabla u\nabla\psi + A(x,y)V'(u)\psi\big)\, dxdy \geq 0, \tag{2.21}
$$

which completes the proof. □

In what follows, let us consider

$$
K = \{u \in E : I(u) = c\}.
$$

Invoking Theorem 2.5, $K \neq \emptyset$ and it consists of critical points of I. In the sequel, for each $u \in K$, we will show that there is a function $v \in K$ depending on u such that

$$
v(x,0) = v(x,1) \text{ for any } x \in \mathbb{R}.
$$

To prove this, we define

$$
E_p = \{w \in E : w(x,0) = w(x,1) \text{ a.e. in } x \in \mathbb{R}\}
$$

and

$$
c_p = \inf_{w\in E_p} I(w).
$$

The next lemma establishes an important relation between c and c_p.

Lemma 2.6. *It holds that $c_p = c$. Moreover, given $u \in K$ there exists $v \in K$, depending on u, such that $v(x,0) = v(x,1)$ for all $x \in \mathbb{R}$.*

Proof. Since $E_p \subset E$, $c \leq c_p$. Now we are going to prove that $c_p \leq c$. To see this, given $w \in E$, we write $I(w) = J_1(w) + J_2(w)$, where

$$
J_1(w) = \int_{\mathbb{R}}\int_0^{1/2} \mathcal{L}(w)dydx \quad \text{and} \quad J_2(w) = \int_{\mathbb{R}}\int_{1/2}^{1} \mathcal{L}(w)dydx.
$$

Let $u \in K$. So, if $J_1(u) \leq J_2(u)$, we consider the function

$$
v(x,y) = \begin{cases} u(x,y), & \text{if } \ 0 \leq y \leq \frac{1}{2}, \\ u(x,1-y), & \text{if } \ \frac{1}{2} \leq y \leq 1 \end{cases}
$$

that belongs to E_p. From (A_2)-(A_3), $J_2(v) = J_1(v) = J_1(u)$, and hence,

$$
I(v) = J_1(v) + J_2(v) = 2J_1(u) \leq J_1(u) + J_2(u) = I(u),
$$

showing that $c_p \leq c$. For that reason, $c_p = c$ and $I(v) = c$ with $v(x,0) = v(x,1)$ for every $x \in \mathbb{R}$. On the other hand, if $J_2(u) \leq J_1(u)$, we consider

$$
\tilde{v}(x,y) = \begin{cases} u(x,1-y), & \text{if } \ 0 \leq y \leq \frac{1}{2} \\ u(x,y), & \text{if } \ \frac{1}{2} \leq y \leq 1. \end{cases}
$$

By a similar argument, $\tilde{v} \in E_p$ and $J_1(\tilde{v}) = J_2(\tilde{v}) = J_2(u)$, from where it follows that $c_p = c$, proving the desired result. □

The Lemma 2.6 shows that the set

$$K_p = \{w \in K : \; w(x,0) = w(x,1) \text{ for all } x \in \mathbb{R}\}$$

is non empty. We would like point out that if $w \in K_p$, then it can extend periodicity on $\mathbb{R}^2$ with period 1. Hereafter, the elements of K_p will be considered extended in whole $\mathbb{R}^2$.

Now, we are ready to prove our main theorem of this section.

Proof of Theorem 1.1.

Let $v \in K_p$. Then $i)$ and $ii)$ are immediate. According to the proof of Theorem 2.5,

$$\int_{\Omega_0} \big(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi\big)\, dydx = 0 \quad \forall \psi \in X^{1,\Phi}(\Omega_0).$$

In the sequel, we fix $\Omega_1 = \mathbb{R} \times [1,2]$,

$$E_1 = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\Omega_1) : w(x,y) = -w(-x,y), \; x \in \mathbb{R}, \text{ and } 0 \leq w(x,y) \leq 1 \text{ for } x > 0 \right\},$$

the functional $I^1 : W^{1,\Phi}_{\text{loc}}(\Omega_1) \to \mathbb{R} \cup \{+\infty\}$ given by

$$I^1(w) = \int_{\Omega_1} \mathcal{L}(w)dydx,$$

and the real number $c^1 = \inf\limits_{w \in E_1} I^1(w)$. It is easily seen that $c = c^1$ and

$$\int_{\Omega_1} \big(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi\big)\, dydx = 0,$$

for each $\psi \in X^{1,\Phi}(\Omega_1)$, where

$$X^{1,\Phi}(\Omega_1) = \left\{ u \in W^{1,\Phi}(\Omega_1) \text{ with } u(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0 \right\}. \tag{2.22}$$

From this, a straightforward computation ensures that

$$\int_{\mathbb{R}\times[0,2]} \big(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi\big)\, dydx = 0,$$

for any $\psi \in X^{1,\Phi}(\mathbb{R} \times [0,2])$, where

$$X^{1,\Phi}(\mathbb{R}\times[0,2]) = \left\{ u \in W^{1,\Phi}(\mathbb{R} \times [0,2]) \text{ with } u(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0 \right\}. \tag{2.23}$$

A similar argument works to prove that

$$\int_{\mathbb{R}\times[l,k]} \big(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi\big)\, dydx = 0,$$

for all $l, k \in \mathbb{Z}$ with $l < k$ e for any $\psi \in X^{1,\Phi}(\mathbb{R} \times [l,k])$ where

$$X^{1,\Phi}(\mathbb{R} \times [l,k]) = \left\{ u \in W^{1,\Phi}(\mathbb{R} \times [l,k]) \text{ with } u(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0 \right\}. \tag{2.24}$$

So, since k and l are arbitrary, we get

$$\int_{\mathbb{R}^2} \big(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi\big)\, dydx = 0,$$

for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$. By [45, Theorem 1.7] there exist $\alpha > 0$ and $M > 0$ such that $v \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2, \mathbb{R})$ with $\|v\|_{C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)} \leq M$. Next, we will show now that v is a heteroclinic solution from -1 to 1. To do this, given $n \in \mathbb{N}$, we set

$$v_n(x,y) = v(x+n,y), \quad \forall (x,y) \in [0,1] \times [0,1].$$

Thereby, (v_n) is bounded in $C^{1,\alpha}([0,1]\times[0,1])$, and so there exists $v_0 \in C^1([0,1]\times[0,1])$ and a subsequence (v_{n_j}) of (v_n) such that $v_{n_j} \to v_0$ in $C^1([0,1]\times[0,1])$. In particular, for $x \in [0,1]$ fixed, $v_{n_j}(x,\cdot) \to v_0(x,\cdot)$ as $j \to +\infty$ uniformly in $y \in [0,1]$. According to Lemma 2.4, $v_{n_j}(x,\cdot) \to 1$ in $L^{\Phi}(0,1)$ as $j \to +\infty$. Passing to a subsequence if necessary, $v_{n_j}(x,y) \to 1$ for almost every $y \in [0,1]$, and hence, $v_0(x,y) = 1$ in $[0,1]\times[0,1]$. Thus, $v_{n_j}(x,y) \to 1$ as $j \to +\infty$ uniformly in $y \in [0,1]$, and consequently, $v(x,y) \to 1$ as $x \to +\infty$ uniformly in $y \in [0,1]$. Since v is 1-periodic in the variable y and odd in the variable x, we conclude

$$v(x,y) \to -1 \ \text{ as } \ x \to -\infty \quad \text{and} \quad v(x,y) \to 1 \ \text{ as } \ x \to +\infty, \quad \text{uniformly in } \ y \in \mathbb{R}.$$

Finally, adapting the same arguments explored in reference [14, Lemma 3.9], we conclude that $0 < v(x,y) < 1$ for all $x > 0$ and $y \in \mathbb{R}$, and the proof is complete. □

If $u \in K$, then we can extend u by periodicity on $\mathbb{R}^2$ with period 2 in y satisfying the equation (PDE). Indeed, defining the function

$$\tilde{u}(x,y) = \begin{cases} u(x,y), & \text{if} \quad (x,y) \in \mathbb{R}\times[0,1], \\ u(x,2-y), & \text{if} \quad (x,y) \in \mathbb{R}\times[1,2], \end{cases}$$

we have that

$$\tilde{u}(x,0) = \tilde{u}(x,2) \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \eta}(x,0) = 0 = \frac{\partial \tilde{u}}{\partial \eta}(x,2).$$

Now, we extend $\tilde{u}$ by periodicity to whole $\mathbb{R}^2$ by setting $\overline{u} : \mathbb{R}^2 \to \mathbb{R}$ by $\overline{u} = \tilde{u}$ in $\mathbb{R}\times[0,2]$ and $\overline{u}(x,y) = \tilde{u}(x,y-2k)$, where $y \in \mathbb{R}$ and $k \in \mathbb{Z}$ is the only integer such that $0 \le y - 2k < 2$. From now on, without loss of generality, we can assume that $u \in K$ is a periodic function with period 2 in the variable y.

Arguing as in the proof of Theorem 1.1, we have the following result.

Theorem 2.7. *Assume (ϕ_1)-(ϕ_2), (V_1)-(V_3) and (A_1)-(A_3). If $u \in K$, then u is a weak solution of (PDE) in $C^{1,\alpha}_{loc}(\mathbb{R}^2,\mathbb{R})$, for some positive α, that verifies the following:*

i) $u(x,y) = -u(-x,y)$, for all $(x,y) \in \mathbb{R}^2$,
ii) $u(x,y) = u(x,y+2)$, for each $(x,y) \in \mathbb{R}^2$,
iii) $0 < u(x,y) < 1$ for any $x > 0$ and $y \in \mathbb{R}$.

Moreover, u is a heteroclinic solution from -1 to 1, i.e.

$$u(x,y) \to -1 \ \text{ as } \ x \to -\infty \quad \text{and} \quad u(x,y) \to 1 \ \text{ as } \ x \to +\infty, \quad \text{uniformly in } \ y \in \mathbb{R}.$$

Remark 2.1. *If $\Phi(t) = \frac{|t|^2}{2}$, the operator Δ_Φ is the Laplacian operator, and in this case, using a local unique theorem for elliptic equations it is possible to prove that Theorems 1.1 and 2.7 are essentially the same, because every 2-periodic solution of (PDE) is exactly 1-periodic solution, for more details see [9, Lemma 2.4] or [47, Proposition 2.18]. Here, since we are working with a large class of operator we were not able to prove that these theorems are equal.*

Remark 2.2. *Here we would like to point out that Theorems 1.1 and 2.7 are valid for the p-Laplacian operator with $1 < p < +\infty$.*

3. Compacteness properties of I

In this section, for our purposes, we need to better characterize the compactness properties of I. For this to happen, given $L \in (0,+\infty]$ we set $\Omega_{0,L} = (-L,L)\times[0,1]$ and

$$I_{0,L}(w) = \int\int_{\Omega_{0,L}} \mathcal{L}(w)dydx \text{ for } w \in W^{1,\Phi}(\Omega_{0,L}).$$

Note that $\Omega_{0,+\infty} = \Omega_0$, $I_{0,+\infty} = I$ and $I_{0,L}$ is also well defined on E being weakly lower semicontinuous with respect to the $W^{1,\Phi}(\Omega_{0,L})$ topology. Moreover, given $u \in E$, we can identify $u|_{\Omega_{0,L}}$ with u itself, and so if $0 < L_1 < L_2$, we have

$$I_{0,L_1}(u) \leq I_{0,L_2}(u) \leq I(u), \quad \forall u \in E.$$

From now on, given $\delta \in (0,1)$, we set

$$\lambda_\delta = 2^{m+1}\delta^l + \overline{A} \max_{|s-1|\leq \Lambda\delta} V(s) \quad \text{and} \quad l_\delta = \frac{c+1}{(2\mu_\delta)^{\frac{m}{m-1}}}, \tag{3.1}$$

where $\Lambda > 0$ and $\mu_\delta > 0$ were given in (A.1) and Lemma 2.3 respectively.

The next lemma is crucial to prove a compactness result involving the functional I, see Lemma 3.6 for more details.

Lemma 3.1. *There exists $\delta_0 \in (0, \frac{\delta_1}{2})$ such that, for any $\delta \in (0, \delta_0)$, if $u \in E$, $L \in (l_\delta + 1, +\infty]$ and $I_{0,L}(u) \leq c + \lambda_\delta$, then the following hold:*

(i) There exists $x_+ \in (0, l_\delta)$ verifying

$$\|u(x_+, \cdot) - 1\|_{W^{1,\Phi}(0,1)} < \delta.$$

(ii) For x_+ given in (i) we have

$$\int_{x_+}^{L}\int_0^1 (\Phi(|\nabla u|) + A(x,y)V(u))\, dydx \leq \frac{3}{2}\lambda_\delta.$$

(iii) For each $x \in (x_+, L)$,

$$\|u(x,\cdot) - 1\|_{L^\Phi(0,1)} \leq \delta_1.$$

Proof. First note that $\lambda_\delta \to 0$ as $\delta \to 0$. Then we can fix $\delta_0 \in (0, \delta_1/2)$ satisfying

$$\lambda_\delta < \min\left\{1, \frac{2}{3}\mu_{\frac{\delta_1}{2}}\left(\frac{\delta_1}{2}\right)^{\frac{m}{l}}\right\}, \quad \forall \delta \in (0, \delta_0), \tag{3.2}$$

where $\delta_1 > 0$ was defined in (V_3) and $\mu_{\frac{\delta_1}{2}}$ given in Lemma 2.3 in correspondence to $r = \frac{\delta_1}{2}$. Let $u \in E$, $L \in (l_\delta + 1, +\infty]$ and $\delta \in (0, \delta_0)$ with $I_{0,L}(u) \leq c + \lambda_\delta$. Assuming that (i) is false, we deduce

$$\|u(x,\cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq \delta, \quad \forall x \in (0, l_\delta).$$

According to Lemma 2.3, there exists $\mu_\delta > 0$ such that

$$I_{0,L}(u) \geq \int_0^{l_\delta}\int_0^1 \mathcal{L}(u)dydx \geq (2\mu_\delta)^{\frac{m}{m-1}} l_\delta = c + 1 > c + \lambda_\delta,$$

which is a contradiction. Therefore, there is $x_+ \in (0, l_\delta)$ checking item (i).

To prove (ii), let us consider

$$\tilde{u}(x,y) = \begin{cases} u(x,y), & \text{if } 0 \leq x \leq x_+ \text{ and } y \in [0,1], \\ (x - x_+) + (x_+ + 1 - x)u(x_+, y), & \text{if } x_+ \leq x \leq x_+ + 1 \text{ and } y \in [0,1], \\ 1, & \text{if } x_+ + 1 \leq x \text{ and } y \in [0,1], \\ -\tilde{u}(-x, y), & \text{if } x < 0 \text{ and } y \in [0,1]. \end{cases}$$

Thereby, $\tilde{u} \in E$ and $c \leq I(\tilde{u}) = I_{0,x_+ + 1}(\tilde{u})$. Moreover,

$$\partial_x \tilde{u}(x,y) = 1 - u(x_+, y) \text{ and } \partial_y \tilde{u}(x,y) = (x_+ + 1 - x)\partial_y u(x_+, y) \text{ in } (x_+, x_+ + 1) \times [0,1].$$

Using Lemma A.1 and the fact that Φ is incresing on $(0, +\infty)$, it is possible to show that

$$\Phi(|\nabla \tilde{u}|) \leq 2^m \Phi(|1 - u(x_+, y)|) + 2^m \Phi(|\partial_y u(x_+, y)|) \text{ in } (x_+, x_+ + 1) \times [0,1],$$

from where it follows that

$$\begin{aligned}\int_{x_+}^{x_++1}\int_0^1 \mathcal{L}(\tilde{u})dydx \le 2^m \int_{x_+}^{x_++1}\int_0^1 (\Phi(|1-u(x_+,y)|)+\Phi(|\partial_y u(x_+,y)|))dydx \\ + \int_{x_+}^{x_++1}\int_0^1 A(x,y)V(\tilde{u})dydx.\end{aligned} \tag{3.3}$$

Applying again Lemma A.1,

$$\begin{aligned}\int_0^1 \Phi(|u(x_+,y)-1|)dy &= \int_0^1 \Phi\left(\frac{|u(x_+,y)-1|}{\|u(x_+,\cdot)-1\|_{L^\Phi(0,1)}}\|u(x_+,\cdot)-1\|_{L^\Phi(0,1)}\right)dy \\ &\le \xi_1\left(\|u(x_+,\cdot)-1\|_{L^\Phi(0,1)}\right) \le \xi_1(\delta)=\delta^l.\end{aligned} \tag{3.4}$$

A similar argument works to prove that

$$\int_0^1 \Phi(|\partial_y u(x_+,y)|)dy \le \delta^l. \tag{3.5}$$

Gathering (3.3) with (3.4) and (3.5), we obtain

$$\int_{x_+}^{x_++1}\int_0^1 \mathcal{L}(\tilde{u})dydx \le 2^{m+1}\delta^l + \overline{A}\int_{x_+}^{x_++1}\int_0^1 V(\tilde{u})dydx.$$

By item (i) and (A.1),

$$\|\tilde{u}(x,\cdot)-1\|_{L^\infty(0,1)} \le \Lambda\delta \quad \forall x\in(x_+,x_++1),$$

and hence

$$\int_{x_+}^{x_++1}\int_0^1 \mathcal{L}(\tilde{u})dydx \le 2^{m+1}\delta^l + \overline{A}\max_{|s-1|\le\Lambda\delta} V(s)=\lambda_\delta. \tag{3.6}$$

Now, since

$$I_{0,L}(\tilde{u}) = I_{0,x_+}(u)+2\int_{x_+}^{L}\int_0^1 \mathcal{L}(\tilde{u})dydx = I_{0,L}(u)+2\int_{x_+}^{L}\int_0^1 \mathcal{L}(\tilde{u})dydx - 2\int_{x_+}^{L}\int_0^1 \mathcal{L}(u)dydx,$$

and $c \le I_{0,L}(\tilde{u})$ follows from (3.6) that

$$\int_{x_+}^{L}\int_0^1 \mathcal{L}(u)dydx \le \frac{3}{2}\lambda_\delta,$$

which proves (ii).

Finally, if (iii) does not hold, we should find $\theta\in(x_+,L)$ satisfying

$$\|u(\theta,\cdot)-1\|_{L^\Phi(0,1)} > \delta_1.$$

Recalling that by (i),

$$\|u(x_+,\cdot)-1\|_{L^\Phi(0,1)} < \frac{\delta_1}{2},$$

the Corollary 2.2 together with Intermediate Value Theorem guarantees the existence of $\sigma\in(x_+,\theta)$ such that

$$\|u(\theta,\cdot)-u(\sigma,\cdot)\|_{L^\Phi(0,1)} \ge \frac{\delta_1}{2} \ \text{ and } \ \|u(x,\cdot)-1\|_{L^\Phi(0,1)} \ge \frac{\delta_1}{2}, \quad \forall x\in(\sigma,\theta).$$

Invoking Lemma 2.3,

$$\int_{x_+}^{L}\int_0^1 \mathcal{L}(u)dydx \ge \mu_{\frac{\delta_1}{2}}h\left(\int_0^1 \Phi(|u(\theta,y)-u(\sigma,y)|)dy\right).$$

On the other hand, from Lemma A.1,

$$\begin{aligned}\int_0^1 \Phi(|u(\theta,y)-u(\sigma,y)|)dy &\geq \xi_0\left(\|u(\theta,\cdot)-u(\sigma,\cdot)\|_{L^\Phi(0,1)}\right)\int_0^1 \Phi\left(\frac{|u(\theta,y)-u(\sigma,y)|}{\|u(\theta,\cdot)-u(\sigma,\cdot)\|_{L^\Phi(0,1)}}\right)dy\\ &\geq \xi_0\left(\|u(\theta,\cdot)-u(\sigma,\cdot)\|_{L^\Phi(0,1)}\right)\geq \xi_0\left(\frac{\delta_1}{2}\right)=\left(\frac{\delta_1}{2}\right)^m.\end{aligned}$$

Hence, by definition of function h we get the inequality below

$$\int_{x_+}^L\int_0^1 \mathcal{L}(u)dydx \geq \mu_{\frac{\delta_1}{2}}\left(\frac{\delta_1}{2}\right)^{\frac{m}{l}}$$

that combines with (ii) to give

$$\mu_{\frac{\delta_1}{2}}\left(\frac{\delta_1}{2}\right)^{\frac{m}{l}} \leq \frac{3}{2}\lambda_\delta,$$

which contradicts (3.2), and the lemma follows. □

From Lemma 3.1, we obtain in particular the following result.

Lemma 3.2. *For all $\epsilon>0$ there are $\overline{\lambda}_\epsilon>0$ and $\bar{l}_\epsilon>0$ such that if $u\in E$ and $I(u)\leq c+\overline{\lambda}_\epsilon$, then $u-1\in W^{1,\Phi}\left((\bar{l}_\epsilon,+\infty)\times(0,1)\right)$ and*

$$\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\left(\Phi(|u-1|)+\Phi(|\nabla u|)\right)dydx\leq\epsilon.$$

Proof. By definition of λ_δ, see (3.1), we know that $\lambda_\delta\to 0$ as $\delta\to 0$. Thereby, given $\epsilon>0$ we can choose $\delta_0\in(0,\delta_1/2)$ satisfying

$$\frac{3}{2}\lambda_\delta\leq\frac{\epsilon}{\max\left\{1,\frac{1}{\underline{A}\ \underline{w}}\right\}},\quad \forall\delta\in(0,\delta_0),$$

where $\underline{w}$ was given in (2.6). Denoting $\overline{\lambda}_\epsilon=\lambda_\delta$, $\bar{l}_\epsilon=l_\delta$ and $L=+\infty$, it follows from Lemma 3.1 that

$$\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\left(\Phi(|\nabla u|)+A(x,y)V(u)\right)dydx\leq\frac{3}{2}\lambda_\delta\leq\frac{\epsilon}{\max\left\{1,\frac{1}{\underline{A}\ \underline{w}}\right\}}. \tag{3.7}$$

According to (2.6),

(3.8)
$$\begin{aligned}\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\left(\Phi(|u-1|)+\Phi(|\nabla u|)\right)dydx &\leq \int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\Phi(|\nabla u|)dydx+\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\frac{1}{\underline{w}}V(u)dydx\\ &\leq \int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\Phi(|\nabla u|)dydx+\frac{1}{\underline{w}\ \underline{A}}\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 A(x,y)V(u)dydx\\ &\leq \max\left\{1,\frac{1}{\underline{A}\ \underline{w}}\right\}\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\left(\Phi(|\nabla u|)+A(x,y)V(u)\right)dydx.\end{aligned}$$

From (3.7) and (3.8), $u-1\in W^{1,\Phi}\left((\bar{l}_\epsilon,+\infty)\times(0,1)\right)$ with

$$\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1\left(\Phi(|u-1|)+\Phi(|\nabla u|)\right)dydx\leq\epsilon,$$

and this is precisely the assertion of the lemma. □

In order to continue our analysis, we will fix the following set

$$\tilde{E} = \left\{ w \in W^{1,\Phi}_{\mathrm{loc}}(\Omega_0) : w \text{ is odd in } x \text{ and } w - 1 \in W^{1,\Phi}([0,+\infty) \times [0,1]) \right\}$$

and the real number

$$\tilde{c} = \inf_{w \in \tilde{E}} I(w).$$

It is very important to point out that $\tilde{E} \neq \emptyset$, because the function φ_* given in (2.2) belongs to $\tilde{E}$. Moreover, it is easy to check that if $w \in \tilde{E}$, then $w + 1 \in W^{1,\Phi}((-\infty, 0] \times [0,1])$, and that if $w_1, w_2 \in \tilde{E}$, then $w_1 - w_2 \in W^{1,\Phi}(\Omega_0)$. Have this in mind, we are able to define on $\tilde{E}$ the metric $\rho : \tilde{E} \times \tilde{E} \to [0, +\infty)$ given by

$$\rho(w_1, w_2) = \|w_1 - w_2\|_{W^{1,\Phi}(\Omega_0)}.$$

A direct computation guarantees that $(\tilde{E}, \rho)$ is a complete metric space.

The next lemma shows that the numbers c and $\tilde{c}$ are equal.

Lemma 3.3. *It holds that $\tilde{c} = c$. Moreover, if $(u_n) \subset E$ and $I(u_n) \to c$, then there exists $n_0 \in \mathbb{N}$ such that $u_n \in \tilde{E}$ for any $n \geq n_0$. Therefore, (u_n) is a minimizing sequence for I on $\tilde{E}$.*

Proof. Let $(u_n) \subset E$ be a sequence with $I(u_n) \to c$. Thus, given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ verifying $I(u_n) \leq c + \epsilon$ for any $n \geq n_0$. By Lemma 3.2, there exists $\bar{l}_\epsilon > 0$ such that $u_n - 1 \in W^{1,\Phi}((\bar{l}_\epsilon, +\infty) \times [0,1])$ for all $n \geq n_0$. Hence,

$$u_n - 1 \in W^{1,\Phi}([0,+\infty) \times [0,1]), \quad \forall n \geq n_0.$$

From this, $(u_n) \subset \tilde{E}$ and

$$\tilde{c} \leq I(u_n) = c + o_n(1), \quad \forall n \geq n_0.$$

Taking the limit of $n \to +\infty$, we get $\tilde{c} \leq c$. Now, let us consider $(v_n) \subset \tilde{E}$ with $I(v_n) \to \tilde{c}$ and

$$\overline{v}_n(x,y) = \begin{cases} 1, & \text{if} \quad v_n(x,y) \geq 1 \\ v_n(x,y), & \text{if} \quad -1 \leq v_n(x,y) \leq 1 \\ -1, & \text{if} \quad v_n(x,y) \leq -1. \end{cases}$$

From the properties of Φ, V and $\overline{v}_n$, $I(\overline{v}_n) \leq I(v_n)$ for every $n \in \mathbb{N}$. Setting

$$\tilde{v}_n(x,y) = \begin{cases} \overline{v}_n(x,y), & \text{if} \quad \overline{v}_n \geq 0 \text{ and } x > 0 \\ -\overline{v}_n(x,y), & \text{if} \quad \overline{v}_n \leq 0 \text{ and } x > 0 \\ -\overline{v}_n(-x,y), & \text{if} \quad x \leq 0, \end{cases}$$

it is easy to see that $(\tilde{v}_n) \subset E$ and $I(\tilde{v}_n) = I(\overline{v}_n)$ for each $n \in \mathbb{N}$. Therefore,

$$c \leq I(\tilde{v}_n) = I(\overline{v}_n) \leq I(v_n) = \tilde{c} + o_n(1).$$

Taking the limit of $n \to +\infty$ we obtain $c \leq \tilde{c}$, from where it follows that $c = \tilde{c}$. Finally, if $(u_n) \subset E$ and $I(u_n) \to c$, then we already know that there is $n_0 \in \mathbb{N}$ such that $u_n \in \tilde{E}$ for $n \geq n_0$, and as $c = \tilde{c}$, we deduce that (u_n) is a minimizing sequence for I on $\tilde{E}$. □

In the sequel, we say that a sequence (u_n) is a $(PS)_d$ sequence for I, with $d \in \mathbb{R}$, if $(u_n) \subset \tilde{E}$ such that

$$I(u_n) \to d \quad \text{and} \quad \|I'(u_n)\|_* \to 0 \quad \text{as} \quad n \to +\infty,$$

where

$$\|I'(w)\|_* = \sup \left\{ I'(w)\psi : \psi \in X^{1,\Phi}(\Omega_0) \text{ and } \|\psi\|_{W^{1,\Phi}(\Omega_0)} \leq 1 \right\}.$$

Lemma 3.4. *If $(u_n) \subset E$ and $I(u_n) \to c$, then there exists a sequence $(w_n) \subset \tilde{E}$ such that (w_n) is a $(PS)_c$ sequence for I and*

$$\|u_n - w_n\|_{W^{1,\Phi}(\Omega_0)} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Proof. Let $(u_n) \subset E$ with $I(u_n) \to c$. As $(\tilde{E}, \rho)$ is a complete metric space, we can employ the Ekeland's Variational Principle to find a sequence $(w_n) \subset \tilde{E}$ satisfying:

(a) $I(w_n) \leq I(u_n)$ for any $n \in \mathbb{N}$,
(b) $\rho(w_n, u_n) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$,
(c) $I(w_n) - I(w) < \frac{1}{n}\|w_n - w\|_{W^{1,\Phi}(\Omega_0)}$ for each $w \in \tilde{E}$ with $w \neq w_n$.

Now, given $\psi \in X^{1,\Phi}(\Omega_0)$ we can write $\psi = \psi_o + \psi_e$, where ψ_o is odd in the variable x and ψ_e is ever in x. It is easily seen that $w_n + t\psi_o \in \tilde{E}$ for all $n \in \mathbb{N}$ and $t > 0$. From (c),

$$\begin{aligned} I(w_n + t\psi) - I(w_n) &= I(w_n + t\psi) - I(w_n + t\psi_o) + I(w_n + t\psi_o) - I(w_n) \\ &\geq I(w_n + t\psi) - I(w_n + t\psi_o) - \frac{1}{n}\|t\psi_o\|_{W^{1,\Phi}(\Omega_0)}, \end{aligned}$$

or equivalently,

$$\frac{I(w_n + t\psi) - I(w_n)}{t} \geq \frac{I(w_n + t\psi) - I(w_n + t\psi_o)}{t} - \frac{1}{n}\|\psi_o\|_{W^{1,\Phi}(\Omega_0)}.$$

Arguing as in the proof of Theorem 2.5, we find

$$I'(w_n)\psi \geq -\frac{1}{n}\|\psi_o\|_{W^{1,\Phi}(\Omega_0)}. \tag{3.9}$$

Here we would like point out that the same arguments found in [14, Lemma 4.6] work to show that

$$\|\psi_o\|_{W^{1,\Phi}(\Omega_0)} \leq \|\psi\|_{W^{1,\Phi}(\Omega_0)}. \tag{3.10}$$

From (3.9)-(3.10) and replacing ψ by $-\psi$, we get

$$|I'(w_n)\psi| \leq \frac{1}{n}\|\psi\|_{W^{1,\Phi}(\Omega_0)}.$$

Thereby,

$$\|I'(w_n)\|_* \to 0 \quad \text{as} \quad n \to +\infty.$$

Finally, from Lemma 3.3 and (a),

$$c = \tilde{c} \leq I(w_n) \leq I(u_n) = c + o_n(1),$$

showing that $I(w_n) \to c$. Therefore, (w_n) is a $(PS)_c$ sequence for I, and the lemma is proved. □

From now on, we consider $(u_n) \subset E$ and $(w_n) \subset \tilde{E}$ as in the last lemma. So, (w_n) is also bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$. Indeed, for each $L > 0$ the Lemma 3.4 ensures that

$$\|w_n\|_{W^{1,\Phi}(\Omega_{0,L})} \leq \|w_n - u_n\|_{W^{1,\Phi}(\Omega_{0,L})} + \|u_n\|_{W^{1,\Phi}(\Omega_{0,L})} \leq \frac{1}{n} + \|u_n\|_{W^{1,\Phi}(\Omega_{0,L})}.$$

Since (u_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$, it follows that (w_n) also is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$. Then, for some subsequence, there is $u_0 \in W^{1,\Phi}_{\text{loc}}(\Omega_0)$ verifying

$$w_n \rightharpoonup u_0 \quad \text{in} \quad W^{1,\Phi}_{\text{loc}}(\Omega_0), \tag{3.11}$$

$$w_n \to u_0 \quad \text{in} \quad L^{\Phi}_{\text{loc}}(\Omega_0), \tag{3.12}$$

$$w_n \to u_0 \quad \text{in} \quad L^1_{\text{loc}}(\Omega_0) \tag{3.13}$$

and

$$w_n(x,y) \to u_0(x,y) \ \text{ a.e. in } \ \Omega_0. \tag{3.14}$$

Lemma 3.5. *There exists a subsequence of (w_n), still denoted by itself, such that*

$$\nabla w_n(x,y) \to \nabla u_0(x,y) \ \textit{a.e. in } \Omega_0.$$

Proof. Given $L > 0$, let us consider $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfying

$$0 \le \psi \le 1, \quad \psi \equiv 1 \ \text{ in } \ \Omega_{0,L} \ \text{ and } \ \text{supp}(\psi) \subset \Omega_{0,L+1}.$$

From (ϕ_1)-(ϕ_2), it is possible to show that

$$\langle \phi(|z_1|)z_1 - \phi(|z_2|)z_2, z_1 - z_2 \rangle > 0, \quad \forall z_1, z_2 \in \mathbb{R}^2, \ z_1 \ne z_2. \tag{3.15}$$

Thereby,

(3.16)

$$\begin{aligned} 0 &\le \int_{\Omega_{0,L}} (\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0)dydx \\ &\le \int_{\Omega_{0,L+1}} \psi(\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0)dydx \\ &\le \int_{\Omega_{0,L+1}} \psi\phi(|\nabla w_n|)\nabla w_n(\nabla w_n - \nabla u_0)dydx - \int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0(\nabla w_n - \nabla u_0)dydx. \end{aligned}$$

Setting the linear functional $f : W^{1,\Phi}(\Omega_{0,L+1}) \to \mathbb{R}$ given by

$$f(v) = \int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0 \nabla v dydx,$$

we have that it is continuous, because $\phi(|\nabla u_0|)\nabla u_0 \in L^{\tilde{\Phi}}(\Omega_{0,L+1})$ via Lemma A.3, and so, by Hölder's inequality

$$\left| \int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0 \nabla v dydx \right| \le 2\|\phi(|\nabla u_0|)\nabla u_0\|_{L^{\tilde{\Phi}}(\Omega_{0,L+1})} \|v\|_{W^{1,\Phi}(\Omega_{0,L+1})},$$

for all $v \in W^{1,\Phi}(\Omega_{0,L+1})$. Therefore, (3.11) asserts that $f(w_n - u_0) \to 0$, or equivalently,

$$\int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0(\nabla w_n - \nabla u_0)dydx \to 0. \tag{3.17}$$

Using again the Lemma A.3 and the boundedness of (w_n) in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$, there is $C > 0$ such that

$$\int_{\Omega_{0,L+1}} \tilde{\Phi}(\phi(|\nabla w_n|)\nabla w_n)dydx \le C, \quad \forall n \in \mathbb{N},$$

implying that $(\phi(|\nabla w_n|)\nabla w_n)$ is bounded in $L^{\tilde{\Phi}}(\Omega_{0,L+1})$. So, by (3.12) and Hölder's inequality,

$$\int_{\Omega_{0,L+1}} (w_n - u_0)\phi(|\nabla w_n|)\nabla w_n \nabla\psi dydx \to 0. \tag{3.18}$$

Now, considering the sequence (ψw_n) we have that $(\psi w_n) \subset W^{1,\Phi}(\Omega_0)$, because ψ has compact support, and by (3.14), passing to a subsequence if necessary, we can assume that

$$\psi w_n \rightharpoonup \psi u_0 \ \text{ in } \ W^{1,\Phi}(\Omega_{0,L+1}) \quad \text{and} \quad \psi w_n \to \psi u_0 \quad \text{a.e.} \quad \Omega_0.$$

Consequently,

$$A(x,y)V'(w_n(x,y))(\psi(x,y)w_n(x,y)-\psi(x,y)u_0(x,y))\to 0 \text{ a.e. in } \Omega_{0,L+1}.$$

From (2.3) and (3.13), there exist $h\in L^1(\Omega_{0,L+1})$ and $\alpha>0$ such that, along a subsequence,

$$\big|A(x,y)V'(w_n)(\psi w_n-\psi u_0)\big|\le \alpha\overline{A}|\psi|(h+|u_0|)\in L^1(\Omega_{0,L+1}).$$

Applying the Lebesgue's Dominated Convergence Theorem we obtain

$$\int_{\Omega_{0,L+1}} A(x,y)V'(w_n)(\psi w_n-\psi u_0)dydx\to 0. \tag{3.19}$$

Finally, we would like point out that

$$I'(w_n)(\psi w_n-\psi u_0)\to 0. \tag{3.20}$$

In fact, just note that

$$\big|I'(w_n)(\psi w_n-\psi u_0)\big|\le \|I'(w_n)\|_*\|\psi w_n-\psi u_0\|_{W^{1,\Phi}(\Omega_0)},$$

$(\psi w_n)\subset X^{1,\Phi}(\Omega_0)$ is a bounded sequence in $W^{1,\Phi}(\Omega_0)$ and (w_n) is a $(PS)_c$ sequence for I. Recalling that

$$\begin{aligned}I'(w_n)(\psi w_n-\psi u_0)=\int_{\Omega_{0,L+1}}&\phi(|\nabla w_n|)\nabla w_n\nabla(\psi w_n-\psi u_0)dydx\\&+\int_{\Omega_{0,L+1}}A(x,y)V'(w_n)(\psi w_n-\psi u_0)dydx,\end{aligned}$$

from where it follows by (3.19) and (3.20) that

$$\int_{\Omega_{0,L+1}}\phi(|\nabla w_n|)\nabla w_n\nabla(\psi w_n-\psi u_0)dydx\to 0. \tag{3.21}$$

Since $\nabla(\psi w_n-\psi u_0)=\psi\nabla w_n+w_n\nabla\psi-\psi\nabla u_0-u_0\nabla\psi$, we also have

$$\begin{aligned}\int_{\Omega_{0,L+1}}\psi\phi(|\nabla w_n|)\nabla w_n(\nabla w_n-\nabla u_0)dydx=&\int_{\Omega_{0,L+1}}\phi(|\nabla w_n|)\nabla w_n\nabla(\psi w_n-\psi u_0)dydx\\&-\int_{\Omega_{0,L+1}}(w_n-u_0)\phi(|\nabla w_n|)\nabla w_n\nabla\psi dydx.\end{aligned} \tag{3.22}$$

From (3.18), (3.21) and (3.22),

$$\int_{\Omega_{0,L+1}}\psi\phi(|\nabla w_n|)\nabla w_n(\nabla w_n-\nabla u_0)dydx\to 0. \tag{3.23}$$

Finally, from (3.17), (3.23) and (3.16),

$$\int_{\Omega_{0,L}}(\phi(|\nabla w_n|)\nabla w_n-\phi(|\nabla u_0|)\nabla u_0)(\nabla w_n-\nabla u_0)dydx\to 0.$$

This limit combined with (3.15) leads to, along a subsequence,

$$\langle\phi(|\nabla w_n|)\nabla w_n-\phi(|\nabla u_0|)\nabla u_0,\nabla w_n-\nabla u_0\rangle\to 0 \ \text{ a.e. in } \ \Omega_{0,L}.$$

Applying a result found in Dal Maso and Murat [23], we infer that

$$\nabla w_n(x,y)\to\nabla u_0(x,y) \text{ a.e. in } \Omega_{0,L}.$$

As $L>0$ is arbitrary, there exists a subsequence of (w_n), still denoted by itself, such that

$$\nabla w_n(x,y)\to\nabla u_0(x,y) \text{ almost everywhere in } \Omega_0,$$

finishing the proof of the lemma. □

The next lemma establishes the strong convergence for minimizing sequences of I on E.

Lemma 3.6. *Let $(u_n) \subset E$ with $I(u_n) \to c$. Then, there exists $u_0 \in K$ such that, along a subsequence,*

$$\|u_n - u_0\|_{W^{1,\Phi}(\Omega_0)} \to 0.$$

Proof. Invoking Lemma 3.4 there is a sequence $(w_n) \subset \tilde{E}$ with $I(w_n) \to c$ and

$$\|u_n - w_n\|_{W^{1,\Phi}(\Omega_0)} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.24}$$

Hence there exits $u_0 \in W^{1,\Phi}_{\text{loc}}(\Omega_0)$ satisfying (3.11)-(3.14). Moreover,

$$\|u_n - u_0\|_{L^\Phi(\Omega_{0,L})} \leq \frac{1}{n} + \|w_n - u_0\|_{L^\Phi(\Omega_{0,L})}, \quad \forall L > 0. \tag{3.25}$$

Thereby, by (3.12), u_0 is the punctual limit of (u_n), $u_0 \in E$ and $I(u_0) = c$, that is, $u_0 \in K$. Now, arguing as in [14, Lemma 4.9],

$$\|\nabla w_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \to 0.$$

From (3.24),

$$\|\nabla u_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \leq \frac{1}{n} + \|\nabla w_n - \nabla u_0\|_{L^\Phi(\Omega_0)},$$

implying that

$$\|\nabla u_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \to 0. \tag{3.26}$$

Finally, according to Lemma 3.2, given $\epsilon > 0$, there are $l_\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{l_\epsilon}^{+\infty} \int_0^1 \Phi(|u_0 - 1|)dydx \leq \frac{\epsilon}{2^m} \quad \text{and} \quad \int_{l_\epsilon}^{+\infty} \int_0^1 \Phi(|u_n - 1|)dydx \leq \frac{\epsilon}{2^m}, \quad \forall n \geq n_0.$$

So, it is easy to see that

$$\int_{l_\epsilon}^{+\infty} \int_0^1 \Phi(|u_n - u_0|)dydx \leq 2^{m-1} \int_{l_\epsilon}^{+\infty} \int_0^1 (\Phi(|u_n - 1|) + \Phi(|u_0 - 1|))dydx \leq \epsilon, \quad \forall n \geq n_0. \tag{3.27}$$

As $\Phi \in \Delta_2$, (3.25) together with (3.27) gives

$$\|u_n - u_0\|_{L^\Phi(\Omega_0)} \to 0. \tag{3.28}$$

Now, the lemma follows from (3.26) and (3.28). □

4. The Approximating Functionals

In the sequel, given $j \in \mathbb{N} \cup \{0\}$, let us define the sets

$$\Omega_j = \mathbb{R} \times [j, j+1] \quad \text{and} \quad T_j = \{(x, y) \in \Omega_j : \ |x| \leq y\}.$$

Associated with sets above, we consider

$$E_j = \left\{w \in W^{1,\Phi}(T_j) : 0 \leq w(x, y) \leq 1 \text{ for } x > 0 \text{ and } w \text{ is odd in } x\right\},$$

and the functional $I_j : W^{1,\Phi}(T_j) \to \mathbb{R} \cup \{+\infty\}$ given by

$$I_j(w) = \iint_{T_j} \mathcal{L}(w)dydx.$$

By a direct computation, we see that I_j is lower semicontinuous with respect to the weak topology of $W^{1,\Phi}(T_j)$ and bounded from below. Moreover, since $I_j(0) < +\infty$ the real number

$$c_j := \inf_{w \in E_j} I_j(w)$$

is well defined. For each $j \in \mathbb{N} \cup \{0\}$ let us also consider

$$K_j = \{w \in E_j : I_j(w) = c_j\}.$$

Arguing as in the proof of Lemma 2.5, it is possible to prove the following result.

Lemma 4.1. *For every $j \in \mathbb{N} \cup \{0\}$, $K_j \neq \emptyset$. Moreover, if $u_j \in K_j$, then u_j is a weak solution in $C^{1,\alpha}(T_j)$, for some $\alpha > 0$, of*

$$-\Delta_\Phi u_j + A(x,y)V'(u_j) = 0 \quad \text{in} \quad T_j,$$

with $0 < u_j(x,y) < 1$ for $x > 0$,

$$\partial_y u_j(x,j) = 0 \text{ for } |x| < j \quad \text{and} \quad \partial_y u_j(x,j+1) = 0 \text{ for } |x| < j+1.$$

As immediate consequence of the last lemma is the corollary below.

Corollary 4.2. *For all $j \in \mathbb{N} \cup \{0\}$ we have $c_j \leq c_{j+1} < c$.*

Proof. Invoking Lemma 4.1, for each $j \geq 0$ there exists $u_{j+1} \in K_{j+1}$. Now, considering the function

$$\overline{u}_j(x,y) = u_{j+1}(x,y+1) \quad \text{for } (x,y) \in T_j,$$

we see that $\overline{u}_j \in E_j$ and

$$c_j \leq I_j(\overline{u}_j) \leq I_{j+1}(u_{j+1}) = c_{j+1}.$$

Finally, from Theorem 1.1, there exists $v : \mathbb{R}^2 \to \mathbb{R}$ such that $v \in E$ with $I(v) = c$ and v is 1-periodic in the variable y. So, $v \in E_j$ for any $j \in \mathbb{N} \cup \{0\}$ and

$$c_j \leq I_j(v) < I(v) = c, \quad \forall j \in \mathbb{N} \cup \{0\},$$

showing the desired result. $\square$

If $j > 1$ and $u_j \in K_j$, then arguing as in the end of Section 3, u_j have an extension 2-periodic v_j in $(-j,j) \times \mathbb{R}$, i.e., there exists $v_j : (-j,j) \times \mathbb{R} \to \mathbb{R}$ that is 2-periodic in the variable y such that

$$v_j = u_j \quad \text{in} \quad (-j,j) \times (j,j+1).$$

Moreover, v_j is a weak solution in $C^{1,\alpha}_{\text{loc}}((-j,j) \times \mathbb{R}, \mathbb{R})$, for some positive α, of the equation

$$-\Delta_\Phi v_j + A(x,y)V'(v_j) = 0 \quad \text{in} \quad (-j,j) \times \mathbb{R}.$$

An direct computation shows that

$$\int_{-j}^{j} \int_{j}^{j+1} \mathcal{L}(u_j)dydx = \int_{-j}^{j} \int_{0}^{1} \mathcal{L}(v_j)dydx. \tag{4.1}$$

From now on, given $u_j \in K_j$, with $j > 1$, let's fix v_j as above. Then, we have the following result.

Lemma 4.3. *There exists $L > 0$ such that for $j > L + \frac{1}{4}$, if $u_j \in K_j$ we must have*

$$|u_j(x,y) - 1| \leq \delta_1, \quad \forall (x,y) \in T_j \quad \text{with} \quad x \in \left(L, j - \frac{1}{4}\right),$$

where δ_1 was given in (V_3).

Proof. Arguing by contradiction, assume that there is a sequence of indices $(j_n) \subset (0, +\infty)$ with $j_n \to +\infty$ such that for each j_n there exists $u_{j_n} \in K_{j_n}$ and points

$$(x_n, y_n) \in \left(0, j_n - \frac{1}{4}\right) \times (j_n, j_n + 1)$$

with $x_n \to +\infty$ satisfying

$$1 - \delta_1 > u_{j_n}(x_n, y_n) > 0. \tag{4.2}$$

Given $j > 1$, we fix the rectangles

$$Q_j = \left(-j + \frac{1}{8}, j - \frac{1}{8}\right) \times (j - 1, j + 2) \quad \text{and} \quad \tilde{Q}_j = \left(-j + \frac{1}{4}, j - \frac{1}{4}\right) \times (j, j + 1).$$

Now, taking $\eta_0 \in (0, \frac{1}{32})$ and $(x, y) \in \tilde{Q}_j$, it is clear that

$$B_{\eta_0}(x, y) \subset B_{2\eta_0}(x, y) \subset Q_j.$$

Defining the operator

$$B(x, y) = A(x, y)V'(v_j(x, y)) \text{ for } (x, y) \in Q_j,$$

there exists $\Lambda_1 > 0$ such that $|B(x, y)| \leq \Lambda_1$ for every $(x, y) \in Q_j$. So, since v_j is a weak solution of the equation

$$\Delta_\Phi w + B(x, y) = 0 \ \text{ in } \ Q_j$$

with $\|v_j\|_{L^\infty(Q_j)} \leq 1$, it follows from [45, Theorem 1.7] that there is $C > 0$ such that

$$\|v_j\|_{C^1(\tilde{Q}_j)} \leq C, \quad \forall j \in \mathbb{N}, \tag{4.3}$$

and so,

$$\|v_j\|_{C^1(B_{\eta_0}(x,y))} \leq C, \quad \forall (x, y) \in \tilde{Q}_j.$$

From this, taking $\eta < \eta_0$ such that $C\eta < \delta_1/2$ and invoking the Mean Value Theorem, we arrive at

$$|v_{j_n}(x, y) - v_{j_n}(x_n, y_n)| \leq C\eta < \frac{\delta_1}{2}, \quad \forall (x, y) \in B_\eta(x_n, y_n) \ \text{ and } \ \forall n \in \mathbb{N}. \tag{4.4}$$

Thereby, from (4.2) and (4.4),

$$|1 - u_{j_n}(x, y)| \geq \frac{\delta_1}{2}, \quad \forall (x, y) \in B_\eta(x_n, y_n) \cap \tilde{Q}_{j_n},$$

leading to

$$\|1 - u_{j_n}(x, \cdot)\|_{L^\infty(j_n, j_n+1)} > \frac{\delta_1}{2}, \quad \forall x \in (x_n - \eta/2, x_n).$$

As the constant of embedding $W^{1,\Phi}(j_n, j_n + 1) \hookrightarrow L^\infty(j_n, j_n + 1)$ are independent of $n \in \mathbb{N}$, because such constants depend only on the length of the intervals $(j_n, j_n + 1)$, then there exists $r > 0$ such that

$$\|1 - u_{j_n}(x, \cdot)\|_{W^{1,\Phi}(j_n, j_n+1)} \geq r, \quad \forall x \in (x_n - \eta/2, x_n).$$

Now, setting

$$\tilde{u}_{j_n}(x, y) = u_{j_n}(x, y + j_n), \quad \text{for} \quad (x, y) \in (-j_n, j_n) \times (0, 1),$$

we obtain

$$\|1 - \tilde{u}_{j_n}(x, \cdot)\|_{W^{1,\Phi}(0,1)} \geq r, \quad \forall x \in (x_n - \eta/2, x_n).$$

From Lemma 2.3, there exists $\mu_r > 0$ satisfying

$$\int_{x_n - \eta/2}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \geq (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2}, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

On the other hand, for each $n \in \mathbb{N}$ it is well known that

$$I_{0,j_n}(\tilde{u}_{j_n}) = \int_{-j_n}^{j_n}\int_0^1 \mathcal{L}(\tilde{u}_{j_n})dydx = \int_{-j_n}^{j_n}\int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx \leq I(u_{j_n}) = c_{j_n} < c.$$

Using the fact that $j_n \to +\infty$, it follows from the Lemma 3.1 that there are $x_+ > 0$ and $n_0 \in \mathbb{N}$ satisfying

$$\int_{x_+}^{j_n}\int_0^1 \mathcal{L}(\tilde{u}_{j_n})dydx < \frac{3}{2}\lambda_\delta, \quad \forall n \geq n_0.$$

Next, we take λ_δ arbitrarily small of such way that

$$\int_{x_+}^{j_n}\int_0^1 \mathcal{L}(\tilde{u}_{j_n})dydx < (2\mu_r)^{\frac{m}{m-1}}\frac{\eta}{2}, \quad \forall n \geq n_0.$$

Therefore, as $x_n \to +\infty$, increasing n_0 if necessary, we find

$$\int_{x_n-\eta/2}^{x_n}\int_0^1 \mathcal{L}(\tilde{u}_{j_n})dydx \leq \int_{x_+}^{x_n}\int_0^1 \mathcal{L}(\tilde{u}_{j_n})dydx \leq \int_{x_+}^{j_n}\int_0^1 \mathcal{L}(\tilde{u}_{j_n})dydx < (2\mu_r)^{\frac{m}{m-1}}\frac{\eta}{2},$$

for any $n \geq n_0$, which contradicts (4.5), and the proof is over. □

In what follows, our goal is to get an estimate from above of the exponential type for $c - c_L$. In order to do that, we fix the real function

$$\zeta(x) = \delta_1 \frac{\cosh\left(a\left(x - \frac{j-\frac{1}{4}+L}{2}\right)\right)}{\cosh\left(a\frac{j-\frac{1}{4}-L}{2}\right)}, \quad x \in \mathbb{R},$$

where $L > 0$ was given in the Lemma 4.3 for some constant $a > 0$ that will chose later. A simple computation provides $\zeta''(x) = a^2\zeta(x)$ for all $x \in \mathbb{R}$, which together with (ϕ_4) permit to use the same idea found in [14] to show that

$$\left(\phi(|\zeta'(x)|)\zeta'(x)\right)' \leq \kappa a^2 \phi(|\zeta'(x)|)\zeta(x), \quad \forall x \in \mathbb{R}.$$

Since $|\zeta'(x)| \leq a\zeta(x)$ for each $x \in \mathbb{R}$, taking $a < 1$ and using (ϕ_3), we get $\phi(|\zeta'(x)|) \leq \phi(\zeta(x))$ for every $x \in \mathbb{R}$, and so,

$$-\left(\phi(|\zeta'(x)|)\zeta'(x)\right)' + \kappa a^2\phi(\zeta(x))\zeta(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

Therefore, if we define $w(x,y) = \zeta(x)$ for each $(x,y) \in \mathbb{R}^2$, then

$$-\Delta_\Phi w + \kappa a^2 \phi(w)w \geq 0 \text{ in } \mathbb{R}^2. \tag{4.6}$$

Now, fixing $u_j \in K_j$ satisfying Lemma 4.3 and setting the function

$$\nu(x,y) = 1 - v_j(x,y), \quad (x,y) \in (-j,j) \times \mathbb{R},$$

it follows from Lemma 4.3 that $0 < v_j(x,y) < 1$ for any $x \in (0,j)$, and so, since v_j is a periodic function in the variable y and continuous, there exists $b_j > 0$ verifying

$$0 < b_j \leq v_j(x,y) < 1, \quad \forall (x,y) \in \left[L, j - \frac{1}{4}\right] \times \mathbb{R}.$$

According to (V_4),

$$V'(v_j) \leq -\omega b_j \phi(\nu)(\nu) \quad \text{in} \quad \left(L, j - \frac{1}{4}\right) \times \mathbb{R}. \tag{4.7}$$

In what follows, we take $a > 0$ sufficiently small such that $\kappa a^2 < \underline{A} b_j \omega$.

Claim 4.4. *Let* $j_0 \in \mathbb{N}$ *and* $\psi \in X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0))$ *with* $\psi \geq 0$*, where*

$$X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0)) = \left\{ u \in W^{1,\Phi}(\mathbb{R} \times (-j_0, j_0)) \text{ with } u(x,y) = 0 \text{ for } x \notin \left(L, j - \frac{1}{4}\right) \right\},$$

then

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} \left(\phi(|\nabla \nu|)\nabla \nu \nabla \psi + \kappa a^2 \phi(\nu)\nu\psi\right) dydx \leq 0.$$

In fact, from (4.7) it may be concluded that

$$\begin{aligned}
\int_{\mathbb{R}} \int_{-j_0}^{j_0} (\phi(|\nabla \nu|)\nabla \nu \nabla \psi + \kappa a^2 \phi(\nu)\nu\psi) dydx &= \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (-\phi(|\nabla v_j|)\nabla v_j \nabla \psi + \kappa a^2 \phi(\nu)\nu\psi) dydx \\
&= \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (A(x,y)V'(v_j)\psi + \kappa a^2 \phi(\nu)\nu\psi) dydx \\
&\leq \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (A(x,y)V'(v_j)\psi + \omega A(x,y) b_j \phi(\nu)\nu\psi) dydx \\
&\leq \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (A(x,y)V'(v_j)\psi - A(x,y)V'(v_j)\psi) dydx = 0,
\end{aligned}$$

proving the Claim 4.4.

On the other hand, the definitions of ν and w together with Lemma 4.3 ensure that

$$\nu(x,y) \leq w(x,y) \quad \text{on} \quad \left\{L, j - \frac{1}{4}\right\} \times \mathbb{R}. \tag{4.8}$$

Lemma 4.5. *It holds that* $\nu(x,y) \leq w(x,y)$ *in* $(L, j - 1/4) \times \mathbb{R}$.

Proof. Suppose by contradiction that the lemma is false. Then, we can find $(x_1, y_1) \in (L, j - 1/4) \times \mathbb{R}$ such that $\nu(x_1, y_1) > w(x_1, y_1)$. Let $j_0 \in \mathbb{N}$ such that $(x_1, y_1) \in (L, j - 1/4) \times (-j_0, j_0)$. Now, from (4.8) the function $\psi_* : \mathbb{R} \times (-j_0, j_0) \to \mathbb{R}$ given by

$$\psi_*(x,y) = \begin{cases} (\nu - w)^+(x,y), & \text{if } \ x \in (L, j - 1/4) \\ 0, & \text{if } \ x \notin (L, j - 1/4) \end{cases}$$

is well defined. Moreover, $\psi_* \in X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0))$ and ψ_* is a nonnegative continuous. Therefore, according to Claim 4.4 and (4.6),

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} \left(\phi(|\nabla w|)\nabla w \nabla \psi_* + \kappa a^2 \phi(w) w \psi_*\right) dydx \geq 0$$

and

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} \left(\phi(|\nabla \nu|)\nabla \nu \nabla \psi_* + \kappa a^2 \phi(\nu) \nu \psi_*\right) dydx \leq 0,$$

which leads to

$$\iint_P \left((\phi(|\nabla \nu|)\nabla \nu - \phi(|\nabla w|)\nabla w)\nabla(\nu - w) + \kappa a^2 (\phi(\nu)\nu - \phi(w)w)(\nu - w)\right) dydx \leq 0,$$

where $P = \{(x,y) \in \mathbb{R} \times (-j_0, j_0) : \nu(x,y) \geq w(x,y)\}$. From (3.15), $\nu(x,y) \leq w(x,y)$ for all $(x,y) \in (L, j - 1/4) \times (-j_0, j_0)$, which is impossible. □

Now, we are ready to prove an exponential estimate from above to $c - c_j$.

Lemma 4.6. *There are $\theta_1, \theta_2 > 0$ such that*

$$0 < c - c_j \le \theta_1 e^{-\theta_2 j}, \quad \forall j \in \mathbb{N} \cup \{0\}.$$

In particular, $c_j \to c$ as $j \to +\infty$.

Proof. First of all, we note that by Lemma 4.5,

$$|v_j(x,y) - 1| \le \delta_1 \frac{\cosh\left(a\left(x - \frac{j-\frac{1}{4}+L}{2}\right)\right)}{\cosh\left(a\frac{j-\frac{1}{4}-L}{2}\right)}, \quad \forall (x,y) \in \left(L, j - \frac{1}{4}\right) \times \mathbb{R}.$$

Choosing $x_+ = \frac{j-\frac{1}{4}+L}{2}$, we have that

$$|v_j(x_+,y) - 1| \le \frac{\delta_1}{\cosh\left(a\frac{j-\frac{1}{4}-L}{2}\right)} \quad \forall y \in \mathbb{R},$$

which implies

$$|v_j(x_+,y) - 1| \le 2\delta_1 e^{-\frac{a}{2}(j-\frac{1}{4}-L)} := \rho_j \quad \text{and} \quad \Phi(|v_j(x_+,y)-1|) \le \Phi(\rho_j) \quad \forall y \in \mathbb{R}. \tag{4.9}$$

In the sequel, we fix j sufficiently large such that $x_+ + \rho_j \le j$ and

$$\tilde{v}_j(x,y) = \begin{cases} v_j(x,y), & \text{if} \quad 0 \le x \le x_+ & \text{and} \quad y \in \mathbb{R} \\ v_j(x_+,y) + \frac{1}{\rho_j}(x - x_+)(1 - v_j(x_+,y)), & \text{if} \quad x_+ \le x \le x_+ + \rho_j & \text{and} \quad y \in \mathbb{R} \\ 1, & \text{if} \quad x_+ + \rho_j \le x & \text{and} \quad y \in \mathbb{R} \\ -\tilde{v}_j(-x,y), & \text{if} \quad x \le 0 & \text{and} \quad y \in \mathbb{R}. \end{cases}$$

Hereafter, let us identify $\tilde{v}_j|_{\Omega_0}$ with the $\tilde{v}_j$ itself, and consequently $\tilde{v} \in E$ and $c \le I(\tilde{v})$. Now let us take a look at some important estimates for the end of the proof.

Claim 4.7. *$|\partial_x \tilde{v}_j| \le 1$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$.*

Indeed, note that $\partial_x \tilde{v}_j(x,y) = \frac{1}{\rho_j}(1 - v_j(x_+,y))$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$. From (4.9),

$$|\partial_x \tilde{v}_j(x,y)| \le \frac{1}{\rho_j}|1 - v_j(x_+,y)| \le 1, \quad \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Claim 4.8. *$|\partial_y \tilde{v}_j| \le 2C$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$, where $C > 0$ was given in (4.3).*

By definition of $\tilde{v}_j$, $|\partial_y \tilde{v}_j(x,y)| \le 2|\partial_y v_j(x_+,y)|$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$. Now, the definition of v_j combined with (4.3) leads to

$$|\partial_y \tilde{v}_j(x,y)| \le 2C \quad \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Claim 4.9. *$A(x,y)V(\tilde{v}_j) \le \overline{A}\,\overline{w}\Phi(\rho_j)$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$.*

From (2.6),

$$A(x,y)V(\tilde{v}_j(x,y)) \le \overline{A}\,\overline{w}\Phi(|\tilde{v}_j(x,y) - 1|) \quad \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Now, the definition of $\tilde{v}_j$ together with (4.9) yields

$$A(x,y)V(\tilde{v}_j(x,y)) \le \overline{A}\,\overline{w}\Phi(|v_j(x_+,y) - 1|) \le \overline{A}\,\overline{w}\Phi(\rho_j) \quad \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R},$$

proving the Claim 4.9.

According to Claims 4.7, 4.8 and 4.9,

$$\begin{aligned} \int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j)\,dy\,dx &\le \int_{x_+}^{x_+ + \rho_j} \int_0^1 \left(2^m \Phi(|\partial_x \tilde{v}_j|) + 2^m \Phi(|\partial_y \tilde{v}_j|) + A(x,y)V(\tilde{v}_j)\right) dy\,dx \\ &\le 2^m \Phi(1)\rho_j + 2^m \Phi(2C)\rho_j + \overline{A}\,\overline{w}\Phi(\rho_j)\rho_j. \end{aligned}$$

Now, since $\rho_j \to 0$ as $j \to +\infty$, there is a constant $\tilde{M} > 0$, independent of j and $\tilde{v}_j$ such that

$$\int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \leq \tilde{M} \rho_j,$$

and so, by (4.1),

$$\begin{aligned} c \leq I(\tilde{v}_j) = \int_{-x_+ - \rho_j}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx &\leq \int_{-j}^{j} \int_0^1 \mathcal{L}(v_j) dy dx + 2 \int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \\ &\leq \int_{-j}^{j} \int_{j+1}^{j} \mathcal{L}(u_j) dy dx + 2\tilde{M}\rho_j \leq I_j(u_j) + 2\tilde{M}\rho_j = c_j + 2\tilde{M}\rho_j, \end{aligned}$$

that is,

$$0 < c - c_j \leq 4\tilde{M}\delta_1 e^{-\frac{a}{2}(j - \frac{1}{4} - L)},$$

for j sufficiently large. Therefore, it is possible to find real numbers $\theta_1, \theta_2 > 0$ satisfying precisely the assertion of the lemma. □

Next, we establish further compactness property concerning the functionals I_{j_n}.

Lemma 4.10. *Let $j_n \to +\infty$ and $u_{j_n} \in E_{j_n}$ such that $I_{j_n}(u_{j_n}) - c_{j_n} \to 0$ as $n \to +\infty$. Then, there exists $u_0 \in K$ verifying*

$$\|u_{j_n} - \tau_{j_n} u_0\|_{W^{1,\Phi}(T_{j_n})} \to 0 \text{ as } n \to +\infty,$$

where $\tau_j u_0(x, y) = u_0(x, y - j)$ for all $j \in \mathbb{N}$.

Proof. Setting

$$w_{j_n}(x, y) = u_{j_n}(x, y + j_n), \text{ for } (x, y) \in (-j_n, j_n) \times [0, 1],$$

it is easily seen that $I_{0,j_n}(w_{j_n}) \leq I_{j_n}(u_{j_n})$. Since $c_{j_n} < c$ for all $n \in \mathbb{N}$ and $I_{j_n}(u_{j_n}) = c_{j_n} + o_n(1)$,

$$I_{0,j_n}(w_{j_n}) < c + o_n(1), \quad \forall n \in \mathbb{N}. \tag{4.10}$$

We claim that for each $n \in \mathbb{N}$ there exists $x_{+,n} \in (\frac{j_n}{2}, j_n)$ satisfying

$$\alpha_n := \|w_{j_n}(x_{+,n}, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \to 0 \text{ as } n \to +\infty.$$

Indeed, if the claim is not true, then there is $r > 0$ such that, for some subsequence,

$$\|w_{j_n}(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq r, \ \forall x \in (\frac{j_n}{2}, j_n) \text{ and } \forall n \in \mathbb{N}.$$

Invoking Lemma 2.3, there exists $\mu_r > 0$ verifying

$$I_{0,j_n}(w_{j_n}) \geq \int_{\frac{j_n}{2}}^{j_n} \int_0^1 \mathcal{L}(w_{j_n}) dy dx \geq (2\mu_r)^{\frac{m}{m-1}} \frac{j_n}{2}.$$

Taking j_n sufficiently large we have $I_{0,j_n}(w_{j_n}) > c + o_n(1)$, contrary to (4.10), and the claim is proved. Without loss of generality, we can assume that $\alpha_n > 0$ for any $n \in \mathbb{N}$, and so we define the function $\tilde{w}_{j_n} : \Omega_0 \to \mathbb{R}$ by

$$\tilde{w}_{j_n}(x, y) = \begin{cases} w_{j_n}(x, y), & \text{if } 0 \leq x \leq x_{+,n} \\ w_{j_n}(x_{+,n}, y) + \frac{1}{\alpha_n}(x - x_{+,n})(1 - w_{j_n}(x_{+,n}, y)), & \text{if } x_{+,n} \leq x \leq x_{+,n} + \alpha_n \\ 1, & \text{if } x_{+,n} + \alpha_n \leq x \\ -\tilde{w}_{j_n}(-x, y), & \text{if } x \leq 0. \end{cases}$$

Thus, $\tilde{w}_{j_n} \in E$ and

$$c \leq I(\tilde{w}_{j_n}) = I_{0,x_{+,n}}(w_{j_n}) + 2 \int_{x_{+,n}}^{x_{+,n} + \alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n}) dy dx. \tag{4.11}$$

On the other hand, from (A.1),

$$|\partial_x \tilde{w}_{j_n}| \leq \Lambda \ in \ (x_{+,n}, x_{+,n} + \alpha_n) \times (0,1), \quad \forall n \in \mathbb{N}. \tag{4.12}$$

Indeed, using (A.1), for each $(x,y) \in (x_{+,n}, x_{+,n} + \alpha_n) \times (0,1)$ we have

$$|\partial_x \tilde{w}_{j_n}(x,y)| = \frac{1}{\alpha_n}|1 - w_{j_n}(x_{+,n}, y)| \leq \frac{1}{\alpha_n}\|1 - w_{j_n}(x_{+,n}, \cdot)\|_{L^\infty(0,1)} \leq \Lambda, \quad \forall n \in \mathbb{N}.$$

Moreover, an easy computation shows that

$$|\partial_y \tilde{w}_{j_n}(x,y)| \leq 2|\partial_y w_{j_n}(x_{+,n}, y)|, \quad \forall (x,y) \in (x_{+,n}, x_{+,n} + \alpha_n) \times (0,1). \tag{4.13}$$

Now, since $\alpha_n \to 0$ we can take n sufficiently large such that $\alpha_n < 1$, and for such values of n, the convexity of Φ ensures that

$$\begin{aligned} \int_0^1 \Phi(|\partial_y w_{j_n}(x_{+,n}, y)|)dy &= \int_0^1 \Phi\left(\|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)} \frac{|\partial_y w_{j_n}(x_{+,n}, y)|}{\|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)}}\right) dy \\ &\leq \|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)} \int_0^1 \Phi\left(\frac{|\partial_y w_{j_n}(x_{+,n}, y)|}{\|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)}}\right) dy \leq \alpha_n, \end{aligned}$$

that is,

$$\int_0^1 \Phi(|\partial_y w_{j_n}(x_{+,n}, y)|)dy \leq \alpha_n. \tag{4.14}$$

A similar argument works to prove that $A(x,y)V(\tilde{w}_{j_n}) \leq \overline{A}\overline{w}\Phi(|1 - w_{j_n}(x_{+,n}, y)|)$ in $(x_{+,n}, x_{+,n} + \alpha_n) \times (0,1)$ and

$$\int_0^1 \Phi(|1 - w_{j_n}(x_{+,n}, y)|)dy \leq \alpha_n. \tag{4.15}$$

Therefore, we conclude from (4.12)-(4.15) that

$$\int_{x_{+,n}}^{x_{+,n}+\alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n})dydx \to 0 \quad \text{as} \quad n \to +\infty. \tag{4.16}$$

According to (4.10), (4.11) and (4.16), $I(\tilde{w}_{j_n}) \to c$. From Lemma 3.6, there exists $u_0 \in K$ such that, along a subsequence,

$$\|\tilde{w}_{j_n} - u_0\|_{W^{1,\Phi}(\Omega_0)} \to 0.$$

As $\tilde{w}_{j_n}(x,y) = u_{j_n}(x, y + j_n)$ for $|x| \leq x_{+,n}$ and $y \in [0,1]$, we deduce

$$\|u_{j_n} - \tau_{j_n} u_0\|_{W^{1,\Phi}([-x_{+,n}, x_{+,n}] \times [j_n, j_n+1])} \to 0 \quad \text{as } n \to +\infty. \tag{4.17}$$

By definition of $\tilde{w}_{j_n}$,

$$I(\tilde{w}_{j_n}) = \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx + 2\int_{-x_{+,n}}^{x_{+,n}+\alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n})dydx$$

that combines with (4.16) to provide

$$\int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx \to c. \tag{4.18}$$

Setting $R_{+,n} = T_{j_n} \setminus ([-x_{+,n}, x_{+,n}] \times [j_n, j_n + 1])$, we have

$$\iint_{R_{+,n}} \mathcal{L}(u_{j_n})dydx = I_{j_n}(u_{j_n}) - \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx.$$

Now, the estimate $I_{j_n}(u_{j_n}) = c_{j_n} + o_n(1)$ together with (4.18) ensures that

$$\iint_{R_{+,n}} \mathcal{L}(u_{j_n})dydx \to 0. \tag{4.19}$$

On the other hand, by (2.6),

(4.20)

$$\begin{aligned}\iint_{R_{+,n}} (\Phi(|\nabla u_{j_n}|) + \Phi(|u_{j_n} - 1|))dydx &\le \iint_{R_{+,n}} \left(\Phi(|\nabla u_{j_n}|) + \frac{1}{\underline{w}\ \underline{A}} A(x,y)V(u_{j_n})\right) dydx \\ &\le \max\left\{1, \frac{1}{\underline{w}\ \underline{A}}\right\} \iint_{R_{+,n}} \mathcal{L}(u_{j_n})dydx.\end{aligned}$$

This combined with (4.19) leads to

$$\|u_{j_n} - 1\|_{W^{1,\Phi}(R_{+,n})} \to 0. \tag{4.21}$$

Finally, by Lemma 3.2, we also have that $\Phi(|\nabla u_0|), \Phi(|u_0 - 1|) \in L^1(\Omega_0)$, and so,

$$\iint_{R_{+,n}} \Phi(|\nabla \tau_{j_n} u_0|)dydx \to 0 \text{ and } \iint_{R_{+,n}} \Phi(|\tau_{j_n} u_0 - 1|)dydx \to 0.$$

As $\Phi \in \Delta_2$, these limits guarantee that

$$\|\tau_{j_n} u_0 - 1\|_{W^{1,\Phi}(R_{+,n})} \to 0. \tag{4.22}$$

Now the lemma follows from (4.21), (4.22) and (4.17). □

5. Saddle-Type Solutions

In this last section we collect the results obtained above to prove Theorem 1.2. To this aim, let us consider

$$\Gamma = \bigcup_{j=0}^{\infty} T_j \quad \text{and} \quad \Gamma_k = \Gamma \cap \{y < k\} \quad \text{for each } k \in \mathbb{N}.$$

Setting

$$E_\infty = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\Gamma) : \ 0 \le w(x,y) \le 1 \text{ for } x \ge 0 \text{ and } w \text{ is odd in } x \right\},$$

we infer that if $w \in E_\infty$ then $w|_{T_j} \in E_j$ for every $j \in \mathbb{N} \cup \{0\}$. Hereafter, let us identify $w|_{T_j}$ with w itself. With everything, we may define the functional $J : W^{1,\Phi}_{\text{loc}}(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ by

$$J(w) = \sum_{j=0}^{\infty} (I_j(w) - c_j).$$

Clearly, J is bounded from below on E_∞. Here, we would like point out that there exists $u \in E_\infty$ such that $J(u) < +\infty$. Indeed, from Theorem 1.1, there exists a function $u_* : \mathbb{R}^2 \to \mathbb{R}$ such that $u_* \in E_\infty$ with $I(u_*) = c$. Invoking Lemma 4.6,

$$I_j(u_*) - c_j \le I(u_*) - c_j = c - c_j \le \theta_1 e^{-\theta_2 j}, \quad \forall j \in \mathbb{N} \cup \{0\}.$$

Thus,

$$J(u_*) = \sum_{j=0}^{\infty} (I_j(u_*) - c_j) \le \theta_1 \sum_{j=0}^{\infty} e^{-\theta_2 j} < +\infty,$$

and the real number

$$d_\infty := \inf_{w \in E_\infty} J(w)$$

is well defined.

In what follows, if $(u_n) \subset W^{1,\Phi}_{\text{loc}}(\Gamma)$ and $u \in W^{1,\Phi}_{\text{loc}}(\Gamma)$, we write $u_n \rightharpoonup u$ in $W^{1,\Phi}_{\text{loc}}(\Gamma)$ to denote that $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Omega)$ for any Ω relatively compact in Γ. Here we would like point out that the same arguments found in [14, Lemma 6.2] work to show that

$$u_n \rightharpoonup u \text{ in } W^{1,\Phi}_{\text{loc}}(\Gamma) \Rightarrow J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

From this, we are ready to show the following result.

Lemma 5.1. *There exists $\overline{u} \in E_\infty$ such that $J(\overline{u}) = d_\infty$.*

Proof. Let $(w_n) \subset E_\infty$ be a minimizing sequence for J. Then there is $M > 0$ satisfying $J(w_n) \leq M$ for every $n \in \mathbb{N}$. Thereby, for each $k \in \mathbb{N}$ fixed,

$$\iint_{\Gamma_k} \Phi(|\nabla w_n|)dydx \leq \iint_{\Gamma_k} \mathcal{L}(w_n)dydx \leq \sum_{j=0}^{k} I_j(w_n) \leq J(w_n) + \sum_{j=0}^{k} c_j \leq M + (k+1)c$$

that together with $\|w_n\|_{L^\infty(\Gamma)} \leq 1$ ensures that (w_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Gamma)$. By a classical diagonal argument, for some subsequence, there exists $\overline{u} \in W^{1,\Phi}_{\text{loc}}(\Gamma)$ such that

$$w_n \rightharpoonup \overline{u} \text{ in } W^{1,\Phi}_{\text{loc}}(\Gamma) \quad \text{and} \quad w_n(x,y) \to \overline{u}(x,y) \quad \text{a.e. in } \Gamma.$$

Next, by pointwise convergence, $\overline{u}(x,y) = -\overline{u}(-x,y)$ for almost every $(x,y) \in \Gamma$ and $0 \leq \overline{u}(x,y) \leq 1$ for almost every $(x,y) \in \Gamma$ with $x \geq 0$, that is, $\overline{u} \in E_\infty$. Moreover, $J(\overline{u}) = d_\infty$, which completes the proof. $\square$

Setting

$$K_\infty = \{w \in E_\infty : J(w) = d_\infty\},$$

we have by the previous lemma that $K_\infty \neq \emptyset$. Repeating the arguments used in the proof of Theorem 2.5, it is possible to prove the following result.

Lemma 5.2. *If $\overline{u} \in K_\infty$, then for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with ψ compact support in $\mathbb{R}^2$ we have*

$$\iint_{\Gamma} \left(\phi(|\nabla\overline{u}|)\nabla\overline{u}\nabla\psi + A(x,y)V'(\overline{u})\psi\right) dydx = 0.$$

As a consequence of Lemma 5.2, if $\overline{u} \in K_\infty$ then $\overline{u}$ is weak solution of

$$-\Delta_\Phi w + A(x,y)V'(w) = 0 \ \text{ in } \ \Gamma.$$

Elliptic regularity theory yields that $\overline{u}$ is a solution in $C^{1,\alpha}_{\text{loc}}(\Gamma)$, for some $\alpha > 0$. Furthermore, arguing as in the proof of Theorem 1.1 we also have that

$$0 < \overline{u}(x,y) < 1 \text{ for } (x,y) \in \Gamma \text{ with } x > 0.$$

Finally, we can now prove our main result.

Proof of Theorem 1.2.

The existence of saddle-type solution v will be done via a recursive reflection of the function $\overline{u} : \Gamma \to \mathbb{R}$ given by Lemma 5.1. First of all, let us consider the rotation matrix

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that is, $T(x,y) = (y,-x)$ for any $(x,y) \in \mathbb{R}^2$. Setting $\Gamma^0 = \Gamma$, we designate $\Gamma^i = T^i(\Gamma)$ for $i = 0,1,2,3$, i.e., Γ^i is the $i\frac{\pi}{2}$-rotated de Γ. Consequently,

$$\mathbb{R}^2 = \bigcup_{i=0}^{3} \Gamma^i, \quad T^{-i}(\Gamma^i) = \Gamma, \quad \text{and} \quad int(\Gamma^i) \cap int(\Gamma^j) = \emptyset \ \text{ for } \ i \neq j.$$

Finally, we define the function $v : \mathbb{R}^2 \to \mathbb{R}$ by

$$v(x,y) = (-1)^i \overline{u}\left(T^{-i}(x,y)\right), \quad \forall (x,y) \in \Gamma^i.$$

Note that $v|_{\Gamma^i}$ is the reflection of $v|_{\Gamma^{i-1}}$ with respect to the axis separating Γ^{i-1} from Γ^i, for any $i = 1,2,3$. From the properties of the reflection operator, $v \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}^2)$. Now, we note that if $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$, then $\psi \circ T^i \in W^{1,\Phi}(\mathbb{R}^2)$ and has compact support in $\mathbb{R}^2$, because T^i is a linear operator. Moreover, from (A_4),

$$A\left(T^i(x,y)\right) = A(x,y), \quad \forall (x,y) \in \mathbb{R}^2.$$

Thus, invoking Lemma 5.2,

$$\begin{aligned}\int_{\Gamma^i} &(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)dydx \\ &= (-1)^i \int_{\Gamma} \left(\phi(|\nabla \overline{u}|)\nabla \overline{u}\nabla(\psi \circ T^i) + A(x,y)V'(\overline{u})(\psi \circ T^i)\right)dydx = 0.\end{aligned}$$

Therefore, for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$,

$$\begin{aligned}\int_{\mathbb{R}^2} &(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)dydx \\ &= \sum_{i=0}^{3} \int_{\Gamma^i} \left(\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi\right)dydx = 0.\end{aligned}$$

Furthermore, by regularity arguments, v is a weak solution of equation (PDE) in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$, for some $\alpha > 0$. A direct computation shows that v checks the conditions (a)-(c) of Theorem 1.2. To complete the proof, we are going to prove that v satisfies item (d). Since $J(v) = d_\infty < +\infty$, we must have $I_j(v) - c_j \to 0$ as $j \to +\infty$. By Lemma 4.10, there is $u_0 \in K$ such that

$$\|v - \tau_j u_0\|_{W^{1,\Phi}(T_j)} \to 0 \quad \text{as} \quad j \to +\infty. \tag{5.1}$$

Now, we claim that

$$\|v - \tau_j u_0\|_{L^\infty(T_j)} \to 0 \quad \text{as} \quad j \to +\infty. \tag{5.2}$$

In fact, assume by contradiction that there exits $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there are $j_n > n$ and $(x_n, y_n) \in T_{j_n}$ satisfying

$$|v(x_n,y_n) - \tau_j u_0(x_n,y_n)| \geq 3\epsilon_0.$$

From Mean Value Theorem, there is $\theta > 0$ sufficiently small such that

$$|\tau_{j_n} u_0(x,y) - \tau_{j_n} u_0(x_n,y_n)| \leq \epsilon_0, \quad \forall (x,y) \in B_\theta(x_n,y_n) \cap T_{j_n}$$

and

$$|v(x,y) - v(x_n,y_n)| \leq \epsilon_0, \quad \forall (x,y) \in B_\theta(x_n,y_n) \cap T_{j_n}.$$

Consequently,

$$\iint_{T_{j_n}} \Phi(|v - \tau_{j_n} u_0|)dydx \geq \Phi(\epsilon_0)\,|B_\theta(x_n,y_n) \cap T_{j_n}| \geq \beta_0, \quad \forall n \in \mathbb{N},$$

for some $\beta_0 > 0$. As $\Phi \in \Delta_2$, there is $r > 0$ such that

$$\|v - \tau_{j_n} u_0\|_{L^\Phi(T_{j_n})} \geq r, \quad \forall n \in \mathbb{N},$$

which contradicts (5.1). Thereby, from (5.2), given $\epsilon > 0$ there is $j_0 > 0$ such that

$$|v(x,y) - \tau_j u_0(x,y)| < \frac{\epsilon}{2}, \ \forall (x,y) \in T_j \text{ and } \forall j > j_0.$$

On the other hand, since $u_0(x,y) \to 1$ as $x \to +\infty$ uniformly in $y \in [0,1]$ we may take j_0 sufficiently large satisfying

$$|\tau_j u_0(x,y) - 1| < \frac{\epsilon}{2}, \ \forall (x,y) \in T_j \text{ with } x > j_0 \text{ and } j \geq 0.$$

Therefore,

$$|v(x,y) - 1| < \epsilon, \quad \forall x > j_0 \text{ and } y > j_0.$$

A similar argument works to prove that

$$|v(x,y) + 1| < \epsilon, \quad \forall x < -j_0 \text{ and } y > j_0.$$

Gathering these estimates together with (5.2) we conclude the proof the theorem. □

The above proof suggests the following behavior of the solution v.

Corollary 5.3. *Let v be given as in Theorem 1.2. Then, the following hold:*

(a) $v(x,y) \to 1$ as $x \to +\infty$ and $y \to +\infty$,
(b) $v(x,y) \to -1$ as $x \to -\infty$ and $y \to +\infty$,
(c) $v(x,y) \to -1$ as $x \to +\infty$ and $y \to -\infty$,
(d) $v(x,y) \to 1$ as $x \to -\infty$ and $y \to -\infty$.

Appendix A. Basic Results About Orlicz-Sobolev Spaces

Here we give a brief review of Orlicz-Sobolev spaces. The reader can find more details in [1,51]. We recall that a continuous function $\Phi : \mathbb{R} \to [0,+\infty)$ is a **N-function** if:

i) Φ is convex,
ii) $\Phi(t) = 0 \Leftrightarrow t = 0$,
iii) Φ is even,
iv) $\displaystyle\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ and $\displaystyle\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty$.

Moreover, we say that a N-function Φ verifies the **Δ_2-condition** ($\Phi \in \Delta_2$ for short) if there are constants $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq t_0. \tag{Δ_2}$$

Below are some examples of N-functions that satisfy (Δ_2) with $t_0 = 0$:

(a) $\Phi_1(t) = \dfrac{|t|^p}{p}$ with $1 < p < +\infty$,
(b) $\Phi_2(t) = \dfrac{|t|^p}{p} + \dfrac{|t|^q}{q}$ for $1 < p < q < +\infty$,
(c) $\Phi_3(t) = (1 + |t|)\ln(1 + |t|) - |t|$,
(d) $\Phi_4(t) = (1 + t^2)^\gamma - 1$ with $\gamma > 1$,
(e) $\Phi_5(t) = \int_0^t s^{1-\gamma}(\sinh^{-1} s)^\beta ds$ with $0 \leq \gamma < 1$ and $\beta > 0$.

An N-function that does not satisfy (Δ_2) is $\Phi(t) = (e^{t^2} - 1)/2$.

If Ω is an open set of $\mathbb{R}^N$ ($N \geq 1$) and Φ is a N-function, the Orlicz space associated with Φ is defined by

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm given by

$$\|u\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

When $\Phi \in \Delta_2$,

$$L^{\Phi}(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi(|u|)dx < +\infty \right\} \text{ and } \int_\Omega \Phi\left(\frac{|u|}{\|u\|_{L^\Phi(\Omega)}}\right) dx = 1.$$

The corresponding Orlicz-Sobolev space is defined as the Banach space

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} = u_{x_i} \in L^\Phi(\Omega),\ i = 1, ..., N \right\},$$

endowed with the norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|\nabla u\|_{L^\Phi(\Omega)} + \|u\|_{L^\Phi(\Omega)}.$$

The complementary function $\tilde{\Phi}$ associated with Φ is defined by Legendre's transformation

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\} \text{ for } s \geq 0.$$

Moreover, $\tilde{\Phi}$ is an N-function and the functions Φ and $\tilde{\Phi}$ are complementary each other. From inequality,

$$st \leq \Phi(t) + \tilde{\Phi}(s), \quad \forall s, t \geq 0, \quad \text{(Young type inequality)}$$

an immediate consequence is the Hölder type inequality

$$\int_\Omega |uv|dx \leq 2\|u\|_{L^\Phi(\Omega)}\|v\|_{L^{\tilde{\Phi}}(\Omega)}, \text{ for all } u \in L^\Phi(\Omega) \text{ and } v \in L^{\tilde{\Phi}}(\Omega).$$

If Φ and $\tilde{\Phi}$ satisfy the Δ_2-condition, then the spaces $L^\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are reflexive and separable. Under the Δ_2-condition,

$$u_n \to u \text{ in } L^\Phi(\Omega) \Leftrightarrow \int_\Omega \Phi(|u_n - u|)dx \to 0$$

and

$$u_n \to u \text{ in } W^{1,\Phi}(\Omega) \Leftrightarrow \int_\Omega \Phi(|u_n - u|)dx \to 0 \text{ and } \int_\Omega \Phi\left(|\nabla u_n - \nabla u|\right) dx \to 0.$$

As is mentioned in [14, 37, 38], we have the next four lemmas.

Lemma A.1. *Let Φ be a N-function of the form (1.1) satisfying (ϕ_1)-(ϕ_2). Set*

$$\xi_0(t) = \min\left\{t^l, t^m\right\} \quad \textit{and} \quad \xi_1(t) = \max\left\{t^l, t^m\right\}, \quad \forall t \geq 0.$$

Then Φ satisfies

$$\xi_0(t)\Phi(s) \leq \Phi(st) \leq \xi_1(t)\Phi(s), \quad \forall s, t \geq 0$$

and

$$\xi_0\left(\|u\|_{L^\Phi(\Omega)}\right) \leq \int_\Omega \Phi(u)dx \leq \xi_1\left(\|u\|_{L^\Phi(\Omega)}\right), \quad \forall u \in L^\Phi(\Omega).$$

Lemma A.2. *If Φ is a N-function of the form (1.1) satisfying (ϕ_1)-(ϕ_2), then $\Phi, \tilde{\Phi} \in \Delta_2$.*

Lemma A.3. *If Φ is a N-function of the form (1.1) satisfying (ϕ_1)-(ϕ_2), then*

$$\tilde{\Phi}\left(\phi(t)t\right) \leq \Phi(2t), \quad \forall t \geq 0.$$

Lemma A.4. *Let Φ be a N-function of the form (1.1) satisfying (ϕ_1)-(ϕ_2). If Ω is a bounded domain in $\mathbb{R}^N$, then*

a) $L^\Phi(\Omega) \hookrightarrow L^l(\Omega)$,
b) $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,l}(\Omega)$.

It is well known that $W^{1,l}(0,1) \hookrightarrow L^\infty(0,1)$ (see for instance [20, Corollary 9.14]). By Lemma A.4 -b),

$$W^{1,\Phi}(0,1) \hookrightarrow L^\infty(0,1).$$

From now on, $\Lambda > 0$ is a constant satisfying

$$\|u\|_{L^\infty(0,1)} \leq \Lambda \|u\|_{W^{1,\Phi}(0,1)} \quad \forall u \in W^{1,\Phi}(0,1). \tag{A.1}$$

To end this section, assuming that the N-function Φ is C^1 we get

$$\Phi(|w|) - \Phi(|z|) \geq \Phi'(|z|)\frac{z}{|z|}.(w-z), \quad \forall w, z \in \mathbb{R}^N,\ z \neq 0, \tag{A.2}$$

where $z.w$ denotes the usual inner product in $\mathbb{R}^N$.

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