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Research Article

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Boundary value problems for integro-differential and singular higher-order differential equations

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Abstract: We investigate third-order strongly nonlinear differential equations of the type

$$(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)), \quad \text{a.e. on } [0, T],$$

where Φ is a strictly increasing homeomorphism, and the non-negative function k may vanish on a set of measure zero. Using the upper and lower solution method, we prove existence results for some boundary value problems associated with the aforementioned equation. Moreover, we also consider second-order integro-differential equations like

$$(\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), \quad \text{a.e. on } [0, T],$$

for which we provide existence results for various types of boundary conditions, including periodic, Sturm-Liouville, and Neumann-type conditions.

Keywords: boundary value problems, nonlinear differential operators, Φ-Laplacian operator, singular equation, Nagumo condition

MSC 2020: 34B15, 34B24, 34C25, 34L30

1 Introduction

Nonlinear differential equations governed by general differential operators, such as the so-called Φ-Laplacian (a generalization of the classical r-Laplacian $\Phi(y) \coloneqq y\,|y|^{r-2}$, with $r > 1$), are widely studied in view of many applications. Indeed, such operators are involved in various models, e.g., in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, and theory of capillary surfaces. In this framework, the theory concerning second-order differential problems has been widely investigated (see, e.g., [1–8] and references therein).

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On the other hand, boundary value problems (BVPs) associated with higher-order equations arise in various applicative contexts. In particular, third-order nonlinear differential equations are involved in some models in fluid mechanics or electromagnetic frameworks (see, e.g., [9,10] and the more recent articles [11,12] for results on Blasius equation, [13] in regard to Falkner-Skan equation, and the recent articles [14–19]). Nevertheless, to the best of our knowledge, a general theory concerning BVPs for third-order nonlinear differential equations has not been developed yet.

The first aim of this article is to provide a general approach to deal with fully nonlinear third-order differential equations, possibly singular, governed by a generic Φ-Laplacian operator.

More in detail, we are concerned with third-order equations of the type

$$(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)), \quad \text{a.e. on } I \coloneqq [0, T], \tag{1}$$

where f is a Carathéodory function, $\Phi : \mathbb{R} \to \mathbb{R}$ is a generic strictly increasing homeomorphism (not necessarily satisfying $\Phi(0) = 0$), and k is a non-negative measurable function that can vanish in a subset of I of zero measure; consequently, equation (1) can be *singular*.

Assuming $1/k$ to be in $L^p(I)$, for some $p > 1$, it is therefore natural to look for solutions of (1) in $W^{2,p}(I)$. A similar framework can be found in the recent articles [4,5].

In this context, we consider the following BVP:

$$(\mathrm{P})\begin{cases}(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)), & \text{a.e. on } [0, T],\\ u(0) = a, \quad u'(0) = b, \quad u'(T) = c, \end{cases} \tag{2}$$

where $a, b, c \in \mathbb{R}$ are given, for which we provide a general existence result (Theorem 3.3), under rather mild conditions: the existence of a pair of lower and upper solutions, the validity of a weak form of the Wintner-Nagumo condition, and the monotonicity of the function $f(t, \cdot, y, z)$.

We also provide some classes of examples illustrating our main result, which can be applied to a wide class of highly nonlinear equations. For instance, the following singular third-order BVP

$$\begin{cases}(\sqrt{t(1-t)}\,u''(t))' = (e^{u'(t)} - u(t)^3)|u''(t)|^{\alpha}, & \text{a.e. on } [0, 1],\\ u(0) = 0, \quad u'(0) = b, \ \ u'(1) = c, \end{cases} \tag{3}$$

admits solutions whatever $b, c \in \mathbb{R}$ may be, provided that $\alpha < 3/2$ (Example 5.3).

This approach can also be successfully employed to deal with second-order integro-differential equations, a subject of an increasing interest, in view of the applications, such as in some ecological and epidemiological models [20,21].

Indeed, the third-order BVP (P) is equivalent to the following Dirichlet-type second-order integro-differential problem (where for simplicity, we put $a = 0$):

$$\begin{cases}(\Phi(k(t)v'(t)))' = f\left(t, \displaystyle\int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), & \text{a.e. on } [0, T],\\ v(0) = b, \quad v(T) = c. \end{cases} \tag{4}$$

So, the existence result stated for problem (P) can be reformulated as an existence result for Problem (4), allowing us to treat different nonlinear boundary conditions including periodic, Sturm-Liouville, and Neumann-type conditions (Section 4).

As in the aforementioned situation, our existence result can be applied to a large class of nonlinear singular integro-differential equations. As an example, we are able to treat the following singular integro-differential problem:

$$\begin{cases}(\sinh(\sqrt{t}\,v'(t)))' = v(t) - \arctan\left(\displaystyle\int_0^t v(s)\mathrm{d}s\right) + e^{v'(t)}v''(t), & \text{a.e. on } [0, T],\\ v(0) = b, \quad v(T) = c, \end{cases}$$

showing that it admits solutions for any choice of $b, c \in \mathbb{R}$ (Example 5.4).

In order to achieve our existence result, we adopt a topological technique, introducing an auxiliary general functional problem to which we apply a fixed-point result (Section 2). Then, by a refined truncation argument and the method of upper and lower solutions we show that the general abstract result can be applied to problem (P) (Section 3). Section 4 is devoted to integro-differential equations, and finally, in Section 5, we present various examples.

2 Auxiliary functional problem

In this section, we introduce a general functional Dirichlet problem in order to obtain sufficient conditions for the existence of solutions. The next section is devoted to find suitable concrete assumptions for problem (P) in such a way that it can be included in this general abstract setting.

In order to do this, we first introduce the closed, convex subspace of $W^{2,p}(I)$ defined as

$$\mathcal{W}_a(I) = \{u \in W^{2,p}(I) : u(0) = a\}.$$

Of course, looking for a solution $u \in W^{2,p}(I)$ of (2) is equivalent to look for a solution $u \in \mathcal{W}_a(I)$ of the Dirichlet-type problem:

$$(\mathrm{P}_a)\begin{cases}(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)), & \text{a.e. on } I,\\ u'(0) = b, \quad u'(T) = c.\end{cases} \tag{5}$$

Moreover, problem (P_a) can be framed in the following wide class of functional BVPs

$$\begin{cases}(\Phi(k(t)u''(t)))' = F_u(t), & \text{a.e. on } I,\\ u'(0) = b, \quad u'(T) = c,\end{cases} \tag{6}$$

where $F : \mathcal{W}_a(I) \to L^1(I)$, $u \mapsto F_u$, is a functional operator.

The aim of this section is to prove the existence of a solution to (6), i.e. a function $u \in \mathcal{W}_a(I)$, with $u'(0) = b$, $u'(T) = c$, such that

$$\Phi \circ (k \cdot u'') \in W^{1,1}(I) \quad \text{and} \quad (\Phi(k(t)u''(t)))' = F_u(t), \quad \text{a.e. on } I.$$

We remark that if $u \in W^{2,p}(I)$ is such that $\Phi \circ (k \cdot u'') \in W^{1,1}(I)$, the continuity of Φ^{-1} implies $ku'' \in C(I)$, meaning that ku'' has a continuous extension on the whole interval I.

In what follows, we will assume that $k : I \to \mathbb{R}$ is a measurable function satisfying

$$k(t) > 0, \quad \text{for a.e. } t \in I, \quad \text{and} \quad \frac{1}{k} \in L^p(I), \quad p > 1. \tag{7}$$

For the sake of brevity, in the sequel, we denote

$$k_p := \left\|\frac{1}{k}\right\|_{L^p} \quad \text{and} \quad k_1 := \left\|\frac{1}{k}\right\|_{L^1}.$$

From now on, we assume that the operator F appearing in the right-hand side of equation in (6) is continuous and there exists a function $\eta \in L^1(I)$ such that

$$|F_u(t)| \le \eta(t), \quad \text{a.e. on } I, \ \text{for every } u \in \mathcal{W}_a(I). \tag{8}$$

Moreover, let us introduce the integral operator $\mathcal{H} : \mathcal{W}_a(I) \to C(I)$ defined by

$$\mathcal{H}_u(t) = \int_0^t F_u(s)\mathrm{d}s, \quad t \in I. \tag{9}$$

Note that by assumption (8), also the operator $\mathcal{H}$ is continuous in $\mathcal{W}_a(I)$, and we have

$$|\mathcal{H}_u(t)| \le \|\eta\|_{L^1(I)}, \quad \text{for every } u \in \mathcal{W}_a(I), \quad t \in I. \tag{10}$$

The following lemma will be used in the next existence result.

Lemma 2.1. *Assume that conditions* (7) *and* (8) *hold. Then, for every* $u \in \mathcal{W}_a(I)$, *there exists a unique constant* $I_u \in \mathbb{R}$ *such that*

$$\int_0^T \frac{1}{k(t)} \Phi^{-1}(I_u + \mathcal{H}_u(t))\mathrm{d}t = c - b. \tag{11}$$

Moreover, the following estimate holds:

$$|I_u| \le \left|\Phi\left(\frac{c-b}{k_1}\right)\right| + \|\eta\|_{L^1(I)}, \quad \text{for every } u \in \mathcal{W}_a(I). \tag{12}$$

Proof. By (10), for every $\xi \in \mathbb{R}$, $u \in \mathcal{W}_a(I)$, and $t \in I$, we have

$$\xi - \|\eta\|_{L^1(I)} \le \xi + \mathcal{H}_u(t) \le \xi + \|\eta\|_{L^1(I)}.$$

Then, since Φ^{-1} is strictly increasing and k is positive, we obtain

$$\Phi^{-1}(\xi - \|\eta\|_{L^1(I)}) \int_0^T \frac{1}{k(t)}\mathrm{d}t \le \int_0^T \frac{1}{k(t)} \Phi^{-1}(\xi + \mathcal{H}_u(t))\mathrm{d}t \le \Phi^{-1}(\xi + \|\eta\|_{L^1(I)}) \int_0^T \frac{1}{k(t)}\mathrm{d}t.$$

So, the function $\varphi_u : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_u(\xi) := \int_0^T \frac{1}{k(t)} \Phi^{-1}(\xi + \mathcal{H}_u(t))\mathrm{d}t$$

is well defined and continuous by Lebesgue's dominated convergence theorem. Moreover, since Φ^{-1} is strictly increasing and k is positive, also φ_u is strictly increasing and

$$\lim_{\xi \to -\infty} \varphi_u(\xi) = -\infty, \quad \lim_{\xi \to +\infty} \varphi_u(\xi) = +\infty,$$

implying that φ_u is a homeomorphism. Therefore, for every $u \in \mathcal{W}_a(I)$, there exists a unique $I_u \in \mathbb{R}$ such that

$$\int_0^T \frac{1}{k(t)} \Phi^{-1}(I_u + \mathcal{H}_u(t))\mathrm{d}t = c - b.$$

Now, by the weighted mean value theorem, we obtain that for every $u \in \mathcal{W}_a(I)$, there exists a value $t_u \in I$ such that

$$c - b = \int_0^T \frac{1}{k(t)} \Phi^{-1}(I_u + \mathcal{H}_u(t))\mathrm{d}t = \Phi^{-1}(I_u + \mathcal{H}_u(t_u)) \int_0^T \frac{1}{k(t)}\mathrm{d}t.$$

Thus, $\Phi^{-1}(I_u + \mathcal{H}_u(t_u)) = (c-b)/k_1$ so that

$$I_u + \mathcal{H}_u(t_u) = \Phi\left(\frac{c-b}{k_1}\right).$$

Hence, estimate (12) directly follows from (10). □

Now, we have all the tools to prove the following existence result for the functional Problem (6).

Theorem 2.2. *Let* $F : \mathcal{W}_a(I) \to L^1(I)$ *be a continuous operator satisfying* (8) *and assume that condition* (7) *holds. Then, problem* (6) *admits a solution.*

Proof. Let $G : \mathcal{W}_a(I) \to \mathcal{W}_a(I)$ be the operator defined by

$$G_u(t) \coloneqq a + bt + \int_0^t\int_0^s \frac{1}{k(\tau)}\Phi^{-1}(I_u + \mathcal{H}_u(\tau))\mathrm{d}\tau\mathrm{d}s, \quad t \in I.$$

First of all, we note that G is well defined. Indeed, given $u \in \mathcal{W}_a(I)$, we have

$$G_u(0) = a, \quad G_u'(t) = b + \int_0^t \frac{1}{k(s)}\Phi^{-1}(I_u + \mathcal{H}_u(s))\mathrm{d}s, \quad \text{a.e. on } I,$$

$$G_u''(t) = \frac{1}{k(t)}\Phi^{-1}(I_u + \mathcal{H}_u(t)), \quad \text{a.e. on } I.$$

Thus, thanks to (7) and (10), we deduce that $G_u', G_u'' \in L^p(I)$. Hence, we obtain $G_u \in \mathcal{W}_a(I)$. Observe, owing to (11), that $u \in \mathcal{W}_a(I)$ is a solution of Problem (6) if and only if it is a fixed point of the operator G.

- **Claim 1**: G **is continuous.**
  Let $u_1, u_2 \in \mathcal{W}_a(I)$ be fixed. By (11), we obtain

$$\int_0^T \frac{1}{k(t)}[\Phi^{-1}(I_{u_1} + \mathcal{H}_{u_1}(t)) - \Phi^{-1}(I_{u_2} + \mathcal{H}_{u_2}(t))]\mathrm{d}t = 0.$$

Hence, by the weighted mean value theorem there exists $\hat{t} \in I$ such that

$$\Phi^{-1}(I_{u_1} + \mathcal{H}_{u_1}(\hat{t})) - \Phi^{-1}(I_{u_2} + \mathcal{H}_{u_2}(\hat{t})) = 0.$$

Since Φ^{-1} is strictly increasing, it follows

$$I_{u_1} + \mathcal{H}_{u_1}(\hat{t}) = I_{u_2} + \mathcal{H}_{u_2}(\hat{t}).$$

This implies that

$$|I_{u_1} - I_{u_2}| = |\mathcal{H}_{u_1}(\hat{t}) - \mathcal{H}_{u_2}(\hat{t})| \le \|\mathcal{H}_{u_1} - \mathcal{H}_{u_2}\|_{C(I)}.$$

Moreover, since for any $t \in I$, we have

$$|\mathcal{H}_{u_1}(t) - \mathcal{H}_{u_2}(t)| \le \int_0^t |F_{u_1}(s) - F_{u_2}(s)|\mathrm{d}s \le \|F_{u_1} - F_{u_2}\|_{L^1(I)},$$

we conclude that

$$|I_{u_1} - I_{u_2}| \le \|\mathcal{H}_{u_1} - \mathcal{H}_{u_2}\|_{C(I)} \le \|F_{u_1} - F_{u_2}\|_{L^1(I)}. \tag{13}$$

Now, by (10) and (12), for every $t \in I$ and for every $u \in \mathcal{W}_a(I)$, we obtain

$$|I_u + \mathcal{H}_u(t)| \le |I_u| + |\mathcal{H}_u(t)| \le \left|\Phi\left(\frac{c-b}{k_1}\right)\right| + 2\|\eta\|_{L^1}. \tag{14}$$

Note that Φ^{-1} is uniformly continuous on any compact interval of $\mathbb{R}$. Therefore, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any interval $[r_1, r_2] \subset \mathbb{R}$ with $|r_1 - r_2| < \delta$ and

$$|r_1|, |r_2| \le \left|\Phi\left(\frac{c-b}{k_1}\right)\right| + 2\|\eta\|_{L^1(I)}$$

by the uniform continuity of Φ^{-1} on $[r_1, r_2]$, we obtain

$$|\Phi^{-1}(r_1) - \Phi^{-1}(r_2)| < \min\left\{\frac{\varepsilon}{3k_p}, \frac{\varepsilon}{3k_1T^{\frac{1}{p}}}, \frac{\varepsilon}{3k_1T^{1+\frac{1}{p}}}\right\} \coloneqq \rho_\varepsilon. \tag{15}$$

Now, let $(u_n)_n$ be a sequence in $\mathcal{W}_a(I)$ converging to $u \in \mathcal{W}_a(I)$ in $W^{2,1}(I)$ as $n \to +\infty$. First of all, we note that $(F_{u_n})_n$ converges to F_u in $L^1(I)$, since the operator F is continuous. Moreover, by (13), also $(I_{u_n})_n$ converges to

I_u. Thus, if $\varepsilon > 0$ is fixed and $\delta = \delta(\varepsilon) > 0$ as before, there exists $\bar{n} = \bar{n}(\varepsilon)$ such that, for $n \geq \bar{n}$, $\|F_{u_n} - F_u\|_{L^1(I)} < \delta/2$. Consequently, for $n \geq \bar{n}$ and $t \in I$, by (13), we obtain

$$|I_{u_n} + \mathcal{H}_{u_n}(t) - I_u - \mathcal{H}_u(t)| \leq 2\|F_{u_n} - F_u\|_{L^1(I)} < \delta.$$

So, for $n \geq \bar{n}$ and a.e. $t \in I$, by (14) and (15), we obtain

$$\begin{aligned}|G''_{u_n}(t) - G''_u(t)| &= \left|\frac{1}{k(t)}[\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t)) - \Phi^{-1}(I_u + \mathcal{H}_u(t))]\right| \\ &\leq \frac{1}{k(t)}|\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t)) - \Phi^{-1}(I_u + \mathcal{H}_u(t))| \\ &< \frac{1}{k(t)}\rho_\varepsilon \leq \frac{1}{k(t)}\frac{\varepsilon}{3k_p},\end{aligned}$$

implying

$$\|G''_{u_n} - G''_u\|_{L^p(I)} < \frac{\varepsilon}{3}.$$

Moreover, for $n \geq \bar{n}$ and a.e. $t \in I$, by (13) and (15), we obtain

$$\begin{aligned}|G'_{u_n}(t) - G'_u(t)| &= \left|\int_0^t \frac{1}{k(s)}[\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(s)) - \Phi^{-1}(I_u + \mathcal{H}_u(s))]\mathrm{d}s\right| \\ &\leq \int_0^t \frac{1}{k(s)}|\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(s)) - \Phi^{-1}(I_u + \mathcal{H}_u(s))|\mathrm{d}s \\ &< \rho_\varepsilon \int_0^t \frac{1}{k(s)}\mathrm{d}s \leq k_1 \frac{\varepsilon}{3k_1 T^{\frac{1}{p}}} \leq \frac{\varepsilon}{3T^{\frac{1}{p}}}.\end{aligned}$$

Thus, we have

$$\|G'_{u_n} - G'_u\|_{L^p(I)} < \frac{\varepsilon}{3}.$$

Finally, for $n \geq \bar{n}$ and $t \in I$, again by (15), we have

$$\begin{aligned}|G_{u_n}(t) - G_u(t)| &= \left|\int_0^t\int_0^s \frac{1}{k(\tau)}[\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(\tau)) - \Phi^{-1}(I_u + \mathcal{H}_u(\tau))]\mathrm{d}\tau\mathrm{d}s\right| \\ &\leq \int_0^t\int_0^s \frac{1}{k(\tau)}|\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(\tau)) - \Phi^{-1}(I_u + \mathcal{H}_u(\tau))|\mathrm{d}\tau\mathrm{d}s \\ &< \rho_\varepsilon \int_0^t\int_0^s \frac{1}{k(\tau)}\mathrm{d}\tau\mathrm{d}s \leq Tk_1 \frac{\varepsilon}{3k_1 T^{1+\frac{1}{p}}} = \frac{\varepsilon}{3T^{\frac{1}{p}}},\end{aligned}$$

implying that

$$\|G_{u_n} - G_u\|_{L^p(I)} < \frac{\varepsilon}{3}.$$

Summarizing, we proved that, for any $\varepsilon > 0$, there exists $\bar{n} = \bar{n}(\varepsilon)$ such that, for $n \geq \bar{n}$,

$$\|G_{u_n} - G_u\|_{W^{2,p}(I)} = \|G_{u_n} - G_u\|_{L^p(I)} + \|G'_{u_n} - G'_u\|_{L^p(I)} + \|G''_{u_n} - G''_u\|_{L^p(I)} < \varepsilon.$$

That is, the operator G is continuous.

- **Claim 2**: G **is bounded.**
  By (14) and the continuity of Φ^{-1}, there exists a constant C such that

$$|\Phi^{-1}(I_u + \mathcal{H}_u(t))| \leq C \quad \forall t \in I, \quad \forall u \in \mathcal{W}_a(I). \tag{16}$$

Thus, for every $u \in \mathcal{W}_a(I)$ and a.e. $t \in I$, we obtain

$$|G_u''(t)| = \frac{1}{k(t)}|\Phi^{-1}(I_u + \mathcal{H}_u(t))| \le \frac{C}{k(t)}, \tag{17}$$

implying that

$$\|G_u''\|_{L^p(I)} \le Ck_p.$$

Moreover, we have

$$\begin{aligned} |G_u'(t)| &= \left| b + \int_0^t \frac{1}{k(s)}\Phi^{-1}(I_u + \mathcal{H}_u(s))\mathrm{d}s \right| \\ &\le |b| + \int_0^t \frac{1}{k(s)}|\Phi^{-1}(I_u + \mathcal{H}_u(s))|\mathrm{d}s \le |b| + Ck_1, \end{aligned} \tag{18}$$

implying $\|G_u'\|_p \le (|b| + Ck_1)T^{\frac{1}{p}}$ for every $u \in \mathcal{W}_a(I)$. Finally, for every $u \in \mathcal{W}_a(I)$ and $t \in I$, we have

$$|G_u(t)| = \left| a + bt + \int_0^t\int_0^s \frac{1}{k(\tau)}\Phi^{-1}(I_u + \mathcal{H}_u(\tau))\mathrm{d}s\mathrm{d}\tau \right| \le |a| + |b|T + Ck_1T.$$

Consequently,

$$\|G_u\|_{L^p(I)} \le (|a| + |b|T + Ck_1T)T^{1/p}.$$

Summarizing,

$$\begin{aligned} \|G_u\|_{W^{2,p}(I)} &= \|G_u\|_{L^p(I)} + \|G_u'\|_{L^p(I)} + \|G_u''\|_{L^p(I)} \\ &\le Ck_p + (|b| + Ck_1)T^{\frac{1}{p}} + (|a| + |b|T + Ck_1T)T^{1/p}. \end{aligned}$$

- **Claim 3: G is a compact operator.**

Let us fix a bounded set $D \subset \mathcal{W}_a(I)$. Our aim is to show that $G(D)$ is relatively compact, i.e., for any sequence $(u_n)_n \subset D$, the sequence $(G_{u_n})_n$ admits a converging subsequence in $\mathcal{W}_a(I)$.

First of all, we show that $(G_{u_n}'')_n$ admits a converging subsequence in $L^p(I)$. Indeed, by (18), for every $s, t \in I$ and $n \in \mathbb{N}$, we have

$$\left| \int_s^t G_{u_n}''(\tau)\mathrm{d}\tau \right| \le |G_{u_n}'(t)| + |G_{u_n}'(s)| \le 2|b| + 2Ck_1.$$

Moreover, by (17), for every $n \in \mathbb{N}$ and a.e. $t \in I$, we have

$$|G_u''(t)| \le \frac{C}{k(t)}.$$

Since by (7), we have $1/k \in L^p(I)$, we obtain that $((G_{u_n})'')_n$ is uniformly integrable. Hence, if we prove

$$\lim_{h\searrow 0} \int_0^{T-h} |G_{u_n}''(t + h) - G_{u_n}''(t)|^p\mathrm{d}t = 0, \ \text{ uniformly in } \ n, \tag{19}$$

we can apply the characterization of relatively compact sets in L^p given by [22, Theorem 2.3.6] and infer the relative compactness of the sequence $(G_{u_n}'')_n$.

In order to prove (19), we fix $\varepsilon > 0$ and we observe that, since $1/k \in L^p(I)$, there is $\rho_1 = \rho_1(\varepsilon) > 0$ such that, for $0 < h < \rho_1$,

$$\int_0^{T-h} \left| \frac{1}{k(t + h)} - \frac{1}{k(t)} \right|^p \mathrm{d}t < \frac{\varepsilon}{(2C)^p}, \tag{20}$$

where C is the constant appearing in (16). The uniform continuity of Φ^{-1} on any compact interval of $\mathbb{R}$ implies the existence of $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$|\Phi^{-1}(r_1) - \Phi^{-1}(r_2)| < \frac{\varepsilon^{\frac{1}{p}}}{2k_p}, \tag{21}$$

for every r_1, r_2 with $|r_1 - r_2| < \delta_1$ and $|r_1|, |r_2| \leq \left|\Phi\left(\frac{b-a}{k_1}\right)\right| + 2\|\eta\|_{L^1(I)}$, where η is the L^1 function in (8). Furthermore, there exists $\rho_2 = \rho_2(\varepsilon) > 0$ such that

$$\left|\int_{\theta_1}^{\theta_2} \eta(t)\mathrm{d}t\right| < \delta_1, \quad \text{for every } \theta_1, \theta_2 \in I \text{ with } |\theta_1 - \theta_2| < \rho_2.$$

Thus, by (8) and (9), for every $n \in \mathbb{N}$, we have

$$|\mathcal{H}_{u_n}(\theta_1) - \mathcal{H}_{u_n}(\theta_2)| = \left|\int_{\theta_1}^{\theta_2} F_{u_n}(t)\mathrm{d}t\right| < \left|\int_{\theta_1}^{\theta_2} \eta(t)\mathrm{d}t\right| < \delta_1.$$

Consequently, from (10), (12), and (21), we obtain

$$|\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(\theta_1)) - \Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(\theta_2))| < \frac{\varepsilon^{\frac{1}{p}}}{2k_p}, \tag{22}$$

for every $n \in \mathbb{N}$ and every $\theta_1, \theta_2 \in I$ with $|\theta_1 - \theta_2| < \rho_2$.

Now, let $t \in I$ and $h > 0$ be fixed such that $t + h \in I$. By (16), we have

$$\begin{aligned}|G''_{u_n}(t+h) - G''_{u_n}(t)| &= \left|\left(\frac{1}{k(t+h)} - \frac{1}{k(t)}\right)\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t)) \right.\\ &\quad \left. + \frac{1}{k(t+h)}[\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t+h)) - \Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t))]\right| \\ &\leq C\left|\frac{1}{k(t+h)} - \frac{1}{k(t)}\right| + \frac{1}{k(t+h)}|\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t+h)) - \Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t))|.\end{aligned}$$

Therefore, by the convexity of the function $\tau \mapsto |\tau|^p$, we obtain

$$\begin{aligned}&\int_0^{T-h} |G''_{u_n}(t+h) - G''_{u_n}(t)|^p \mathrm{d}t \\ &\leq 2^{p-1}C^p \int_0^{T-h} \left|\frac{1}{k(t+h)} - \frac{1}{k(t)}\right|^p \mathrm{d}t \\ &\quad + 2^{p-1}\int_0^{T-h} \frac{1}{(k(t+h))^p}|\Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t+h)) - \Phi^{-1}(I_{u_n} + \mathcal{H}_{u_n}(t))|^p \mathrm{d}t.\end{aligned}$$

Now, let $0 < h < \rho$ with $\rho = \min\{\rho_1, \rho_2\}$. From estimates (20) and (22) with $\theta_1 \coloneqq t + h$ and $\theta_2 \coloneqq t$, it follows that

$$\int_0^{T-h} |G''_{u_n}(t+h) - G''_{u_n}(t)|^p \mathrm{d}t < \varepsilon,$$

for every $n \in \mathbb{N}$, and this implies that condition (19) holds. Hence, the sequence $(G''_{u_n})_n$ verifies the assumptions of [22, Theorem 2.3.6], so there exists a subsequence, denoted again $(G''_{u_n}(t))_n$, converging in L^p to a certain $y_0 \in L^p(I)$.

Second, our aim is to show that $(G'_{u_n})_n$ admits a converging subsequence in $L^p(I)$. In order to do this, we introduce

$$z(t) \coloneqq b + \int_0^t y_0(s)\mathrm{d}s.$$

By Hölder's inequality, we have

$$|G'_{u_n}(t) - z(t)| = \left| \int_0^t (G''_{u_n}(s) - y_0(s))\mathrm{d}s \right| \le T^{\frac{1}{p'}} \|G''_{u_n} - y_0\|_{L^p(I)},$$

where p' is the conjugate exponent of p. Therefore, $G'_{u_n}(t) \to z(t)$ uniformly in I, implying that $G'_{u_n} \to z$ in $L^p(I)$. Hence, taking into account that $z'(t) = y_0(t)$ a.e., we conclude that $(G'_{u_n})_n$ converges to z in $W^{1,p}(I)$.

Finally, we are left with the proof that $(G_{u_n})_n$ admits a converging subsequence in $L^p(I)$. Thus, we introduce

$$w(t) \coloneqq a + \int_0^t z(\tau)\mathrm{d}\tau = a + bt + \int_0^t \int_0^s y_0(\tau)\mathrm{d}\tau.$$

Once again, by applying Hölder's inequality, we obtain

$$\begin{aligned} |G_{u_n}(t) - w(t)| &= \left| \int_0^t (G'_{u_n}(s) - z(s))\mathrm{d}s \right| \\ &\le \left| \int_0^t \int_0^s (G''_{u_n}(\tau) - y_0(\tau))\mathrm{d}s\mathrm{d}\tau \right| \\ &\le T^{1+\frac{1}{p'}} \|G''_{u_n} - y_0\|_{L^p(I)}. \end{aligned}$$

This shows that $G(D)$ is relatively compact in the set $\mathcal{W}_a(I)$.

Taking into account Claims 1–3, we are able to apply the Schauder fixed point theorem and prove the existence of a fixed point for the operator G. This concludes the proof. □

3 Main result

This section is devoted to state and prove our main existence result of solutions for problem (P). First of all, let us precisely clarify the meaning of *solution*.

Definition 3.1. A function $u \in \mathcal{W}_a(I)$ is said to be a *solution* of (5) if it satisfies $\Phi \circ (k \cdot u'') \in W^{1,1}(I)$, $(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t))$ for a.e. $t \in I$, $u'(0) = b$ and $u'(T) = c$.

As we already observed in the previous section, the condition $\Phi \circ (k \cdot u'') \in W^{1,1}(I)$ implies that the product function $k \cdot u''$ has a continuous extension on the whole interval I.

Our technique is based on a suitable combination of fixed point results and lower and upper solutions method. So, let us now give the definition of lower and upper solution of equation (1) appearing in Problem (5).

Definition 3.2. A function $\alpha \in \mathcal{W}_a(I)$ is said to be a *lower solution* of equation (1) if it satisfies $\Phi \circ (k \cdot \alpha'') \in W^{1,1}(I)$ and

$$(\Phi(k(t)\alpha''(t)))' \ge f(t, \alpha(t), \alpha'(t), \alpha''(t)),$$

for a.e. $t \in I$. Analogously, a function $\beta \in \mathcal{W}_a(I)$ is said to be an *upper solution* of equation (1) if it satisfies $\Phi \circ (k \cdot \beta'') \in W^{1,1}(I)$ and

$$(\Phi(k(t)\beta''(t)))' \le f(t, \beta(t), \beta'(t), \beta''(t)),$$

for a.e. $t \in I$.

A pair α and β of lower and upper solutions is said to be *well ordered* if $\alpha'(t) \le \beta'(t)$ for every $t \in I$.

Theorem 3.3. *Let us suppose that the following conditions are satisfied:*

(H1) *there exist a lower solution $\alpha \in \mathcal{W}_a(I)$ and an upper solution $\beta \in \mathcal{W}_a(I)$ of equation* (1), *which are well ordered on I*;

(H2) *$f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, monotone decreasing with respect to the second variable, i.e.,*

$$f(t, x_1, y, z) \geq f(t, x_2, y, z), \quad \text{for a.e. } t \in I,$$

for every $x_1, x_2, y, z \in \mathbb{R}$ such that $x_1 \leq x_2$;

(H3) *for every $R > 0$ and for every non-negative function $\gamma \in L^p(I)$, there exists a non-negative function $h = h_{R,\gamma} \in L^1(I)$ such that*

$$|f(t, x, y, z(t))| \leq h(t), \quad \text{for a.e. } t \in I,$$

for every $x, y \in \mathbb{R}$ such that $|x|, |y| \leq R$ and every $z \in L^p(I)$ such that $|z(t)| \leq \gamma(t)$ for a.e. $t \in I$;

(H4) *there exist a constant $H > 0$, a non-negative function $\nu \in L^q(I)$, with $1 < q \leq \infty$, a non-negative function $\ell \in L^1(I)$, and a measurable function $\psi : (0, +\infty) \to (0, +\infty)$ satisfying*

$$\frac{1}{\psi} \in L^1_{\text{loc}}(0, +\infty) \quad \text{and} \quad \int^{\infty} \frac{1}{\psi(s)} \mathrm{d}s = +\infty,$$

such that

$$|f(t, x, y, z)| \leq \psi(|\Phi(k(t)z)|)\left(\ell(t) + \nu(t)|z|^{\frac{q-1}{q}}\right), \quad \text{for a.e. } t \in I,$$

for every $x, y, z \in \mathbb{R}$ such that $x \in [\alpha(t), \beta(t)]$, $y \in [\alpha'(t), \beta'(t)]$, and $|z| \geq H$, where $\frac{q-1}{q} = 1$ if $q = +\infty$.

Then, for every $b, c \in \mathbb{R}$ such that $\alpha'(0) \leq b \leq \beta'(0)$ and $\alpha'(T) \leq c \leq \beta'(T)$ problem (5) *admits a solution $u \in \mathcal{W}_a(I)$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \text{and} \quad \alpha'(t) \leq u'(t) \leq \beta'(t) \quad \forall t \in I.$$

Proof. By assumption (H1), there exists a well ordered pair of lower and upper solutions α and β of equation (1), i.e., $\alpha'(t) \leq \beta'(t)$ for every $t \in I$ and since $\alpha(0) = \beta(0)$, we also have $\alpha(t) \leq \beta(t)$ for every $t \in I$.

Choose $M > 0$ such that

$$\|\alpha\|_{L^\infty(I)}, \|\beta\|_{L^\infty(I)}, \|\alpha'\|_{L^\infty(I)}, \|\beta'\|_{L^\infty(I)} \leq M. \tag{23}$$

Since Φ is a strictly increasing homeomorphism, there exists $N > 0$ such that

$$\Phi(N) > 0, \quad \Phi(-N) < 0, \quad \text{and} \quad N > \max\left\{H, \frac{2M}{T}\right\} \cdot \|k\|_{L^\infty(I)}. \tag{24}$$

Furthermore, by Definition 3.2 and taking (H4) into account, we can fix $L = L(H, M) \geq N > 0$ such that $\|k\alpha''\|_{L^\infty(I)}, \|k\beta''\|_{L^\infty(I)} \leq L$ and

$$\min\left\{\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi} \mathrm{d}s, \int_{-\Phi(-N)}^{-\Phi(-L)} \frac{1}{\psi} \mathrm{d}s\right\} > \|l\|_{L^1(I)} + \|\nu\|_{L^q(I)} \cdot (2M)^{\frac{q-1}{q}}. \tag{25}$$

Finally, we introduce the function $\gamma_0 \in L^p(I)$ defined by

$$\gamma_0(t) := \frac{L}{k(t)}, \quad \text{for a.e. } t \in I. \tag{26}$$

Now, following the notation of [8, Appendix A], given a pair of functions $\xi, \zeta \in L^1(I)$ satisfying the ordering relation $\xi(t) \leq \zeta(t)$ a.e. $t \in I$, we introduce the truncating operator

$$\mathcal{T}^{\xi,\zeta} : L^1(I) \to L^1(I),$$

defined by

$$\mathcal{T}^{\xi,\zeta}_x(t) = \max\{\xi(t), \min\{x(t), \zeta(t)\}\}, \quad \text{for a.e. } t \in I.$$

By [8, Lemma A.1], the following statements hold true:

$$\begin{aligned}
&\bullet\ |\mathcal{T}^{\xi,\zeta}_x(t) - \mathcal{T}^{\xi,\zeta}_y(t)| \le |x(t) - y(t)|,\ \ \forall x, y \in L^1(I) \text{ and a.e. } t \in I;\\
&\bullet\ \text{if } \xi, \zeta \in W^{1,1}(I), \text{ then } \mathcal{T}^{\xi,\zeta}(W^{1,1}(I)) \subseteq W^{1,1}(I);\\
&\bullet\ \text{if } \xi, \zeta \in W^{1,1}(I), \text{ then } \mathcal{T}^{\xi,\zeta} \text{ is continuous from } W^{1,1}(I) \text{ into itself.}
\end{aligned} \tag{27}$$

From now on, given $u \in W^{2,p}(I)$, we denote

$$\mathcal{D}_{u'}(t) := \mathcal{T}^{-\gamma_0,\gamma_0}_{(\mathcal{T}^{\alpha',\beta'}_{u'})'}(t), \quad \text{for a.e. } t \in I, \tag{28}$$

where γ_0 is the function defined in (26). Observe that definition (28) is well posed since, by (27), the map $\mathcal{T}^{\alpha',\beta'}_{u'}$ belongs to L^1 whenever $u \in W^{2,p}(I)$. Furthermore, we introduce the operator

$$\mathcal{F} : \mathcal{W}_a(I) \to L^1(I), \quad u \mapsto \mathcal{F}_u,$$

defined by

$$\mathcal{F}_u(t) := f(t, \mathcal{T}^{\alpha,\beta}_u(t), \mathcal{T}^{\alpha',\beta'}_{u'}(t), \mathcal{D}_{u'}(t)) + \arctan(u'(t) - \mathcal{T}^{\alpha',\beta'}_{u'}(t)),$$

for a.e. $t \in I$. Actually, in Step 1 we will show that the definition of $\mathcal{F}$ is well posed. Finally, we consider the following auxiliary problem:

$$(\mathrm{P}_\tau) \quad \begin{cases} (\Phi(k(t)u''(t)))' = \mathcal{F}_u(t), & \text{a.e. on } I,\\ u'(0) = b, u'(T) = c. \end{cases} \tag{29}$$

STEP 1. Existence of a solution to the auxiliary problem (P_τ). The aim of this step is to show that $\mathcal{F}$ satisfies the assumptions of Theorem 2.2, i.e., $\mathcal{F}$ is continuous and bounded by a summable function.

Claim 1: There exists a non-negative function $\eta \in L^1(I)$ such that $|\mathcal{F}_u(t)| \le \eta(t)$ for a.e. $t \in I$ and for every $u \in \mathcal{W}_a(I)$.

By the choice of M in (23) and the definition of the truncating operators, for every $t \in I$ and $u \in \mathcal{W}_a(I)$, we obtain

$$-M \le \mathcal{T}^{\alpha,\beta}_u(t) \le M, \quad -M \le \mathcal{T}^{\alpha',\beta'}_{u'}(t) \le M.$$

Moreover, from the definition of $\mathcal{D}$ (see (28)), for every $u \in \mathcal{W}_a(I)$ and a.e. $t \in I$, we have

$$|\mathcal{D}_{u'}(t)| \le \gamma_0(t).$$

Hence, by assumption (H3), there exists a non-negative function $h = h(M, \gamma_0) \in L^1(I)$ such that

$$|\mathcal{F}_u(t)| \le |f(t, \mathcal{T}^{\alpha,\beta}_u(t), \mathcal{T}^{\alpha',\beta'}_{u'}(t), \mathcal{D}_{u'}(t))| + \frac{\pi}{2} \le h(t) + \frac{\pi}{2} := \eta(t), \tag{30}$$

for every $u \in \mathcal{W}_a(I)$ and for a.e. $t \in I$. Since $h \in L^1(I)$, also $\eta \in L^1(I)$; hence, we conclude that $\mathcal{F}_u \in L^1(I)$ for every $u \in \mathcal{W}_a(I)$. This shows that $\mathcal{F}_u$ is well defined and satisfies the boundedness assumption of Theorem 2.2 and the claim is proved.

Claim 2: $\mathcal{F}$ is continuous from $\mathcal{W}_a(I) \subset W^{2,p}(I)$ into $L^1(I)$.

Given a sequence $(u_n)_n \in \mathcal{W}_a(I)$ converging to $u \in \mathcal{W}_a(I)$ in $W^{2,p}(I)$, our aim is to show that $\mathcal{F}_{u_n}(t) \to \mathcal{F}_u(t)$ in $L^1(I)$, up to a subsequence. First, by possibly passing to a subsequence, we have

$$u_n \to u, \quad u_n' \to u' \quad \text{in } W^{1,1}(I), \quad \text{and} \quad u_n'' \to u'' \quad \text{in } L^1(I).$$

Then by [8, Lemma A.1], we obtain

$$\mathcal{T}^{\alpha,\beta}_{u_n} \to \mathcal{T}^{\alpha,\beta}_u \quad \text{in } W^{1,1}(I), \quad \text{and} \quad \mathcal{T}^{\alpha',\beta'}_{u_n'} \to \mathcal{T}^{\alpha',\beta'}_{u'}, \quad \text{in } W^{1,1}(I),$$

which, up to a subsequence, also implies that

$$(\mathcal{T}^{\alpha',\beta'}_{u_n'})' \to (\mathcal{T}^{\alpha',\beta'}_{u'})' \quad \text{in } L^1(I).$$

Then,

$$(\mathcal{T}^{\alpha',\beta'}_{u_n'})'(t) \to (\mathcal{T}^{\alpha',\beta'}_{u'})'(t), \quad \text{for a.e. } t \in I.$$

Thus, combining this convergence with [8, Lemma A.1], we have

$$\mathcal{D}_{u_n'}(t) \to \mathcal{D}_{u'}(t), \quad \text{for a.e. } t \in I.$$

By the previous convergence relations, since f is a Carathéodory function, we then obtain

$$\begin{aligned}\lim_{n\to+\infty} \mathcal{F}_{u_n}(t) &= \lim_{n\to+\infty} [f(t, \mathcal{T}^{\alpha,\beta}_{u_n}(t), \mathcal{T}^{\alpha',\beta'}_{u_n'}(t), \mathcal{D}_{u_n'}(t)) + \arctan(u_n'(t) - \mathcal{T}^{\alpha',\beta'}_{u_n'}(t))]\\ &= \mathcal{F}_u(t), \quad \text{for a.e. } t \in I.\end{aligned}$$

By combining this pointwise result with a standard dominated convergence argument based on (30), we conclude that $\mathcal{F}_{u_n} \to \mathcal{F}_u$ in $L^1(I)$ as $n \to +\infty$, which is the desired result.

As a consequence of Claims 1 and 2, we can apply Theorem 2.2 obtaining the existence of a solution $u \in \mathcal{W}_a(I)$ to the auxiliary Problem (29). This completes the proof of Step 1.

STEP 2. Any solution u to (29) is also a solution to (5).

Claim 1: $\alpha'(t) \le u'(t) \le \beta'(t)$ and so $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$.

Let $u \in \mathcal{W}_a(I)$ be a solution to (29). We show that $\alpha'(t) \le u'(t)$ for every $t \in I$ (the proof that $u'(t) \le \beta'(t)$ is analogous). In order to do this, we proceed by contradiction and assume that there exists a $\underline{t} \in I$ such that $u'(\underline{t}) - \alpha'(\underline{t}) < 0$. Since u is a solution to (29) and α is a lower solution to (5), we obtain

$$u'(0) = b \ge \alpha'(0) \quad \text{and} \quad u'(T) = c \ge \alpha'(T).$$

This implies $u'(0) - \alpha'(0) \ge 0$ and $u'(T) - \alpha'(T) \ge 0$. As a consequence, it is possible to find two points $t_1, t_2 \in I$, with $t_1 < t_2$, such that $u'(t_i) - \alpha'(t_i) = 0$ for $i = 1, 2$, and $u'(t) - \alpha'(t) < 0$ for all $t \in (t_1, t_2)$. So, $\mathcal{T}^{\alpha',\beta'}_{u'} \equiv \alpha'$ on (t_1, t_2) and consequently (see (28))

$$\mathcal{D}_{u'}(t) = \mathcal{T}^{-\gamma_0,\gamma_0}_{\alpha''}(t) = \alpha''(t) \quad \text{for a.e. } t \in (t_1, t_2).$$

Hence, by the monotonicity assumption (H2), for a.e. $t \in (t_1, t_2)$, we obtain

$$\begin{aligned}(\Phi(k(t)u''(t)))' &= f(t, \mathcal{T}^{\alpha,\beta}_{u}(t), \mathcal{T}^{\alpha',\beta'}_{u'}(t), \mathcal{D}_{u'}(t)) + \arctan(u'(t) - \mathcal{T}^{\alpha',\beta'}_{u'}(t))\\ &= f(t, \mathcal{T}^{\alpha,\beta}_{u}(t), \alpha'(t), \alpha''(t)) + \arctan(u'(t) - \alpha'(t))\\ &< f(t, \mathcal{T}^{\alpha,\beta}_{u}(t), \alpha'(t), \alpha''(t))\\ &\le f(t, \alpha(t), \alpha'(t), \alpha''(t)) \le (\Phi(k(t)\alpha''(t)))',\end{aligned}$$

from which, we obtain

$$(\Phi(k(t)u''(t)))' < (\Phi(k(t)\alpha''(t)))'. \tag{31}$$

Then, integrating (31) on (t_1, t_2), we obtain

$$\Phi(k(t_2)u''(t_2)) - \Phi(k(t_2)\alpha''(t_2)) < \Phi(k(t_1)u''(t_1)) - \Phi(k(t_1)\alpha''(t_1)). \tag{32}$$

On the other hand, since $u'(t_1) = \alpha'(t_1)$ and $u'(t) < \alpha'(t)$ in (t_1, t_2), recalling that $k(t) > 0$ for a.e. $t \in I$, we have

$$k(t_1)u''(t_1) \le k(t_1)\alpha''(t_1),$$

and similarly, we have

$$k(t_2)u''(t_2) \ge k(t_2)\alpha''(t_2).$$

Hence,

$$\Phi(k(t_1)u''(t_1)) - \Phi(k(t_1)\alpha''(t_1)) \le 0 \quad \text{and} \quad \Phi(k(t_2)u''(t_2)) - \Phi(k(t_2)\alpha''(t_2)) \ge 0.$$

This is in contradiction with (32). Thus, $u'(t) - \alpha'(t) \ge 0$ for every $t \in I$. By adapting this argument, one is also able to prove that $u'(t) - \beta'(t) \le 0$ for every $t \in I$. Hence, $\alpha'(t) \le u'(t) \le \beta'(t)$ for every $t \in I$. So, since $u(0) = \alpha(0) = \beta(0)$, it immediately follows $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$.

Claim 2: $|u(t)|, |u'(t)| \leq M$ for every $t \in I$.

By Claim 1, we have $\alpha(t) \leq u(t) \leq \beta(t)$ and $\alpha'(t) \leq u'(t) \leq \beta'(t)$ for every $t \in I$. Then, by the definition of M in (23), for every $t \in I$ we obtain

$$-M \leq \alpha(t) \leq u(t) \leq \beta(t) \leq M, \quad \text{and} \quad -M \leq \alpha'(t) \leq u'(t) \leq \beta'(t) \leq M.$$

Claim 3: $\min_{t\in I}|k(t)u''(t)| \leq N$, where $N > 0$ is chosen as in (24).

By contradiction, we assume that

$$k(t)u''(t) > N, \quad \text{for every } t \in I.$$

By integrating on I both sides of the previous inequality, by (24) we have

$$\begin{aligned} NT &< \int_0^T k(t)u''(t)\mathrm{d}t \\ &\leq \|k\|_{L^\infty(I)} \int_0^T u''(t)\mathrm{d}t = \|k\|_{L^\infty(I)}(c-b) \\ &\leq 2M \cdot \|k\|_{L^\infty(I)} < NT, \end{aligned}$$

which is a contradiction. By arguing in the same way, we are able to complete the proof by showing that also $k(t)u''(t) < -N$ for every $t \in I$ leads to a contradiction. Hence, $\min_{t\in I}|k(t)u''(t)| \leq N$.

Claim 4: $|k(t)u''(t)| \leq L$ for every $t \in I$, where $L > N$ is chosen as in (25).

Once again, we proceed by contradiction, and we assume that there exists $\tau \in I$ such that $|k(\tau)u''(\tau)| > L$. Let us consider the case

$$k(\tau)u''(\tau) > L > 0.$$

Since $k \cdot u''$ is a continuous function, by Claim 3 there exist two points $t_1, t_2 \in I$, with $t_1 < t_2$ to fix ideas, such that $k(t_1)u''(t_1) = N$, $k(t_2)u''(t_2) = L$ and $N < k(t)u''(t) < L$ for every $t \in (t_1, t_2) \subseteq I$.

Hence, by (24), we obtain

$$L > k(t)u''(t) > N > H\|k\|_{L^\infty(I)}, \quad \text{for every } t \in (t_1, t_2). \tag{33}$$

That is, by the definition of γ_0 (see (26)),

$$H < u''(t) < \frac{L}{k(t)} = \gamma_0(t) \quad \text{for a.e. } t \in (t_1, t_2). \tag{34}$$

Combining this estimate with Claim 1 and the definition of $\mathcal{D}$ (see (28)), we deduce $\mathcal{D}_{u'} \equiv u''$ on (t_1, t_2). Consequently, since u is a solution to (29), by (34) assumption (H4) implies that, for a.e. $t \in (t_1, t_2)$,

$$|(\Phi(k(t)u''(t)))'| = |f(t, u(t), u'(t), u''(t))| \leq \psi(|\Phi(k(t)u''(t))|)\cdot(\ell(t) + \nu(t)|u''(t)|^{\frac{q-1}{q}}).$$

Moreover, by (33), since Φ is a strictly increasing function, we have

$$\Phi(k(t)u''(t)) > \Phi(N) > 0, \quad \text{for every } t \in (t_1, t_2).$$

Recalling that $\Phi \circ (ku'') \in W^{1,1}(I)$, we then obtain

$$\begin{aligned} \int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(s)}\mathrm{d}s &= \int_{\Phi(k(t_1)u''(t_1))}^{\Phi(k(t_2)u''(t_2))} \frac{1}{\psi(s)}\mathrm{d}s = \int_{t_1}^{t_2} \frac{(\Phi(k(t)u''(t)))'}{\psi(\Phi(k(t)u''(t)))}\mathrm{d}t \\ &\leq \int_{t_1}^{t_2} (\ell(t) + \nu(t)(u''(t)^{\frac{q-1}{q}}))\mathrm{d}t \\ &\leq \|\ell\|_{L^1(I)} + \|\nu\|_{L^q(I)}(u'(t_2) - u'(t_1))^{\frac{q-1}{q}} \quad \text{(by Hölder inequality)} \\ &\leq \|\ell\|_{L^1(I)} + \|\nu\|_{L^q(I)}(2M)^{\frac{q-1}{q}} \qquad \text{(by Claim 2)}, \end{aligned}$$

which contradicts our choice (25) of L. Hence, $k(t)u''(t) \le L$ for every $t \in I$. By a similar argument, we are also able to conclude that $k(t)u''(t) \ge -L$ for every $t \in I$. The claim is then proved.

Claim 5: $\mathcal{D}_{u'} \equiv u''$ a.e. on I.

By Claim 4 and (26), we obtain

$$|u''(t)| = \frac{|k(t)u''(t)|}{k(t)} \le \frac{L}{k(t)} = \gamma_0(t), \quad \text{for a.e. } t \in I.$$

Therefore, from (28), we conclude $\mathcal{D}_{u'} \equiv u''$ a.e. on I.

Combining the results established in Claims 1–5, we are able to complete the proof of this step. Indeed, by Claims 1 and 5, we obtain $\mathcal{T}_u^{\alpha,\beta} \equiv u$, $\mathcal{T}_{u'}^{\alpha',\beta'} \equiv u'$ on I and $\mathcal{D}_{u'} \equiv u''$ a.e. on I. Collecting all these facts and considering that u is a solution of (29), for a.e. $t \in I$ we obtain

$$(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)).$$

Hence, u is a solution to (5). □

4 Existence results for integro-differential problems

As we pointed out in Section 1, by the change of variable $u' = v$, setting for simplicity $a = 0$, the BVP (P) is in fact equivalent to (4).

The maps $\Phi : \mathbb{R} \to \mathbb{R}$, $k : I \to \mathbb{R}$, $f : I \times \mathbb{R}^3 \to \mathbb{R}$ verify the same properties as mentioned earlier. In particular, $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function such that the monotonicity condition (H2) holds (Theorem 3.3).

We look for solutions of (4) in the following subset of $W^{1,p}(I)$:

$$\mathcal{V} = \{u \in W^{1,p}(I) : k \cdot u' \in C(I), \Phi \circ (k \cdot u') \in W^{1,1}(I)\}.$$

By a *solution* of Problem (4), we mean a function $v \in \mathcal{V}$, satisfying $v(0) = b$, $v(T) = c$ and such that $(\Phi(k(t)v'(t)))' = f(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t))$ a.e. on I.

Moreover, a function $\sigma \in \mathcal{V}$ is called a *lower* [resp. *upper*] solution of the integro-differential equation

$$(\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), \quad \text{a.e. on } I, \tag{35}$$

appearing in Problem (4), if

$$(\Phi(k(t)\sigma'(t)))' \ge \ [\le]\ f\left(t, \int_0^t \sigma(s)\mathrm{d}s, \sigma(t), \sigma'(t)\right), \quad \text{a.e. on } I.$$

By adapting the assumptions to the second-order case, we are able to state the following existence result for (4) as a consequence of Theorem 3.3.

Note that this result is actually more general than [4, Theorem 3.1], since here the right-hand side f also depends on the integral function of v.

Theorem 4.1. *Assume that conditions (H2) and (H3) in Theorem* 3.3 *hold. Moreover, suppose that the following additional conditions are satisfied*:

(*H*1*) *there exists a well ordered pair of lower and upper solutions* $\sigma, \tau \in \mathcal{V}$ of (35): i.e. $\sigma(t) \le \tau(t)$, for every $t \in I$.

(*H*4*) *there exist a constant* $H > 0$, *a function* $\nu \in L^q_+(I)$ *for some* $1 < q \le \infty$, *a non-negative function* $\ell \in L^1(I)$, *and a function* $\psi : (0, \infty) \to (0, \infty)$, *with* $1/\psi \in L^1_{\mathrm{loc}}(0, \infty)$ *and* $\int^{+\infty} \frac{1}{\psi(s)}\mathrm{d}s = +\infty$, *such that*

$$|f(t, x, y, z)| \le \psi(|\Phi(k(t)z)|) \cdot \left(\ell(t) + \nu(t)|z|^{\frac{q-1}{q}}\right),$$

for a.e. $t \in I$, *all* $x \in [\int_0^t \sigma(s)\mathrm{d}s, \int_0^t \tau(s)\mathrm{d}s]$, $y \in [\sigma(t), \tau(t)]$ *and all* z *with* $|z| > H$, *where* $\frac{q-1}{q} = 1$ *if* $q = \infty$.

Then, for every b, c such that $\sigma(0) \le b \le \tau(0)$, $\sigma(T) \le c \le \tau(T)$, problem (4) *has a solution $v_{b,c} \in \mathcal{V}$ such that $\sigma(t) \le v_{b,c}(t) \le \tau(t)$ for every $t \in I$.*

Remark 4.2. As in [4, Theorem 3.1], one can prove the following further claim: for every $M > 0$ there exists a constant $L = L(M, H, \nu, \ell, \psi)$, such that if $\|\sigma\|_{C(I)} \le M$, $\|\tau\|_{C(I)} \le M$, $\|k \cdot \sigma'\|_{C(I)} \le L$, and $\|k \cdot \tau'\|_{C(I)} \le L$, then the solution v as in the statement of Theorem 4.1 verifies

$$\|v\|_{C(I)} \le M \quad \text{and} \quad \|k \cdot v'\|_{C(I)} \le L. \tag{36}$$

4.1 General nonlinear boundary conditions

Let us now show how from Theorem 4.1, one can obtain existence results also for various boundary conditions, more general than Dirichlet ones. The following results generalize those in [4, Section 4], in the spirit of [1].

First of all, we state a compactness-type result for the solutions of Dirichlet problems in the framework of integro-differential equations. We omit the proof, since it can be carried out just as in [4, Lemma 4.1], since that proof does not depend on the variables of the right-hand side f.

Proposition 4.3. *Let $\sigma, \tau \in \mathcal{V}$ be a well ordered pair of lower and upper solutions of* (35).

Then, for every pair of sequences $(b_n)_n$ and $(c_n)_n$ of real numbers satisfying $b_n \in [\sigma(0), \tau(0)]$ and $c_n \in [\sigma(T), \tau(T)]$ for every $n \in \mathbb{N}$, and for every sequence $(v_n)_n$ of solutions of problem

$$\begin{cases} (\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), & \text{a.e. on } I, \\ v(0) = b_n, \quad v(T) = c_n, \end{cases}$$

such that $\sigma(t) \le v_n(t) \le \tau(t)$ for every $n \in \mathbb{N}$ and $t \in I$, and satisfying the estimate (36) *of Remark* 4.2 *for some $M > 0$, there exists a subsequence $(v_{n_j})_j$ such that, for every $t \in I$,*

$$v_{n_j}(t) \to v_0(t), \quad k(t)v'_{n_j}(t) \to k(t)v'(t), \qquad \text{as } j \to \infty,$$

for some solution v_0 of equation (35).

In order to handle various types of boundary conditions, it is convenient to introduce the following general problem:

$$\begin{cases} (\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), & \text{a.e. on } I, \\ g(u(0), u(T), k(0)u'(0), k(T)u'(T)) = 0, \\ u(T) = h(u(0)), \end{cases} \tag{37}$$

where $g : \mathbb{R}^4 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions. Note that here we deal with "weighted" boundary conditions, i.e., involving $k(0)v'(0)$, $k(T)v'(T)$ since we are looking for solutions in the set $\mathcal{V}$, i.e., functions $v \in W^{1,p}(I)$ with $k \cdot v' \in C(I)$.

The next existence result holds (cf. [4, Theorem 4.3]). We omit the proof since it can be carried out as in [23, Theorem 3], by applying the previous compactness result Proposition 4.3 instead of [23, Lemma 1].

Theorem 4.4. *Let σ and τ be a well ordered pair of lower and upper solutions of* (35) *such that*

$$\begin{cases} g(\sigma(0), \sigma(T), k(0)\sigma'(0), k(T)\sigma'(T)) \ge 0, \\ \sigma(T) = h(\sigma(0)), \end{cases}$$

$$\begin{cases} g(\tau(0), \tau(T), k(0)\tau'(0), k(T)\tau'(T)) \le 0, \\ \tau(T) = h(\tau(0)). \end{cases}$$

Assume that conditions (*H*2), (*H*3), *and* (*H*4*) *are satisfied. Moreover, suppose that h is increasing and*

$$g(u, v, \cdot, z) \text{ is increasing and } g(u, v, w, \cdot) \text{ is decreasing}. \tag{38}$$

Then, problem (37) *has a solution* $v \in \mathcal{V}$ *such that* $\sigma(t) \le v(t) \le \tau(t)$ *for every* $t \in I$ *and*

$$\|v\|_{C(I)} \le M \quad \text{and} \quad \|k \cdot v'\|_{C(I)} \le L, \tag{39}$$

where M and L are as in Remark 4.2.

The general boundary conditions considered in Problem (37) include, as a particular case, periodic-type boundary conditions, i.e.,

$$\begin{cases} (\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), \quad \text{a.e. on } I, \\ v(0) = v(T), \quad k(0)v'(0) = k(T)v'(T). \end{cases} \tag{40}$$

The next result is an immediate consequence of Theorem 4.4.

Corollary 4.5. *Let σ and τ be a well ordered pair of lower and upper solutions of* (35) *such that*

$$\begin{cases} \sigma(0) = \sigma(T), \\ k(0)\sigma'(0) \ge k(T)\sigma'(T), \end{cases} \quad \text{and} \quad \begin{cases} \tau(0) = \tau(T), \\ k(0)\tau'(0) \le k(T)\tau'(T). \end{cases}$$

Assume that conditions (*H*2), (*H*3), *and* (*H*4*) *are satisfied. Then, problem* (40) *has a solution* $v \in \mathcal{V}$ *such that* $\sigma(t) \le v(t) \le \tau(t)$ *for every* $t \in I$.

Finally, we now consider a different kind of BVP which includes, as particular cases, both Sturm-Liouville and Neumann-type boundary conditions:

$$\begin{cases} (\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\mathrm{d}s, v(t), v'(t)\right), \quad \text{a.e. } t \in I, \\ P(v(0), k(0)v'(0)) = 0, \quad Q(v(T), k(T)v'(T)) = 0, \end{cases} \tag{41}$$

where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. The following result holds (cf. [4, Theorem 4.5]).

Theorem 4.6. *Let σ and τ be a well ordered pair of lower and upper solutions of* (35) *such that*

$$\begin{cases} P(\sigma(0), k(0)\sigma'(0)) \ge 0, \\ Q(\sigma(T), k(T)\sigma'(T)) \ge 0, \end{cases} \quad \text{and} \quad \begin{cases} P(\tau(0), k(0)\tau'(0)) \le 0, \\ Q(\tau(T), k(T)\tau'(T)) \le 0. \end{cases}$$

Assume that conditions $(H1^*)$, $(H3)$, *and* $(H4^*)$ *are satisfied. Moreover, assume that for every* $s \in \mathbb{R}$, *we have*

$$P(s, \cdot) \text{ is increasing and } Q(s, \cdot) \text{ is decreasing}.$$

Then, problem (41) *has a solution* $v \in X$ *such that* $\sigma(t) \le v(t) \le \tau(t)$ *for every* $t \in I$.

Sketch of the proof. The proof follows by using the compactness-type result Proposition 4.3 similar to [23, Theorem 5] with small modifications. □

5 Some examples

Let us now provide some examples illustrating our main results.

We start from a quite general class of nonlinearities f such that $f(t,x,y,\cdot)$ is linear. This allows the choice $q=1$ and ψ constant in the Wintner-Nagumo-type condition (H4).

Example 5.1. Consider the following BVP:

$$\begin{cases}(\Phi(k(t)u''(t)))' = \mu(t)[\rho(t,u(t)) + u'(t)] + g(u'(t))u''(t), & \text{a.e. on } [0,T],\\ u(0)=a, \quad u'(0)=b, \quad u'(T)=c,\end{cases} \tag{42}$$

where $\Phi:\mathbb{R}\to\mathbb{R}$ is a strictly increasing homeomorphism with $\Phi(0)=0$; $k:I\to\mathbb{R}$ is a.e. positive with $1/k\in L^p(I)$ for some $p>1$; $\mu:I\to\mathbb{R}$ belongs to $L^1(I)$ with $\mu(t)\geq 0$ for a.a. $t\in I$; $\rho:I\times\mathbb{R}\to\mathbb{R}$ is continuous, and finally, also $g:\mathbb{R}\to\mathbb{R}$ is continuous.

Assume that the function ρ is monotone decreasing with respect to the second variable and globally bounded, i.e. suppose that

$$\rho(t,x_1)\geq\rho(t,x_2), \quad \text{for any } t\in I \text{ and any } x_1,x_2\in\mathbb{R} \text{ with } x_1\leq x_2; \tag{43}$$

$$\text{there exists } L_\rho>0 \text{ such that } |\rho(t,x)|\leq L_\rho, \quad \text{for all } (t,x)\in I\times\mathbb{R}. \tag{44}$$

Let us prove that Problem (42) admits solutions whatever $b,c\in\mathbb{R}$ may be. Put

$$f(t,x,y,z) := \mu(t)(\rho(t,x)+y) + g(y)z.$$

Observe that f is a Carathéodory function, and by (43), assumption (H2) is satisfied.

Moreover, if we take $K:=\max\{L_\rho,|b|,|c|\}$, where b and c are the boundary data, then the linear functions $\alpha(t):=a-Kt$ and $\beta(t):=a+Kt$ are a pair of well ordered lower and upper solutions of the equation appearing in (42).

Let us show that also (H3) is fulfilled. Indeed, let $R>0$ and $\gamma\in L^p_+(I)$ be fixed. Define $M_R:=\max_{y\in[-R,R]}|g(y)|$. By (44), we have

$$|f(t,x,y,z(t))|\leq\mu(t)(K+R)+M_R\gamma(t) =: h_{R,\gamma}(t),$$

whenever $x\in\mathbb{R}$, $|y|\leq R$ and $z\in L^p(I)$ with $|z(t)|\leq\gamma(t)$ for a.e. $t\in I$. Since $h_{R,\gamma}\in L^1_+(I)$, we have that (H3) holds.

Finally, for all $y\in[-K,K]$, we have

$$|f(t,x,y,z)|\leq 2K\mu(t)+M_K|z|,$$

where $M_K:=\max_{y\in[-K,K]}|g(y)|$. Consequently, the Nagumo-Wintner assumption (H4) holds with the choice

$$H:=1, \quad \psi(s):=1, \quad \ell(t):=2K\mu(t), \quad \nu(t):=M_K, \quad q=+\infty.$$

Hence, Theorem 3.3 applies yielding the existence of a solution of Problem (42).

We now provide a similar example in which we drop the global boundedness of the nonlinearity $f(t,x,y,z)$ with respect to x (condition (44)).

Example 5.2. Consider the following BVP:

$$\begin{cases}(\Phi(k(t)u''(t)))' = \mu(t)[\rho(u(t)) + u'(t)] + g(u'(t))u''(t), & \text{a.e. on } [0,T],\\ u(0)=0, \quad u'(0)=b, \quad u'(T)=c,\end{cases} \tag{45}$$

where Φ, k, μ, and g are as in Example 5.1, while $\rho:\mathbb{R}\to\mathbb{R}$ is continuous, decreasing, and such that

$$|\rho(x)|\leq\frac{|x|}{T}, \quad \text{for all } x\in\mathbb{R}. \tag{46}$$

We show that Problem (45) admits solutions whatever $b, c \in \mathbb{R}$ may be. In fact, set

$$f(t,x,y,z) \coloneqq \mu(t)(\rho(x)+y)+g(y)z.$$

Then, f is a Carathéodory function, and (H2) is satisfied.

Moreover, if we take $K \coloneqq \max\{|b|,|c|\}$, where b and c are the boundary data, then the linear functions $\alpha(t) \coloneqq -Kt$ and $\beta(t) \coloneqq Kt$ are a pair of well ordered lower and upper solutions of the equation appearing in (45). In fact,

$$\rho(\alpha(t)) + \alpha'(t) = \rho(-Kt) - K \le 0, \quad \text{for a.e. } t \in I,$$

in view of (46). An analogous remark holds for β.

We proceed as in Example 5.1: to see that (H3) holds, we fix $R > 0$ and $\gamma \in L^p_+(I)$, and, we put $M_R \coloneqq \max_{y\in[-R,R]}|g(y)|$. By (46), we have

$$|f(t,x,y,z(t))| \le \mu(t)\left(\frac{R}{T}+R\right) + M_R\gamma(t) \eqqcolon h_{R,\gamma}(t),$$

whenever $|x| \le R$, $|y| \le R$ and $z \in L^p(I)$ with $|z(t)| \le \gamma(t)$ for a.e. $t \in I$, where $h_{R,\gamma} \in L^1_+(I)$. Thus, (H3) is verified.

Finally, for all $x \in [-KT, KT]$ and $y \in [-K,K]$, we have

$$|f(t,x,y,z)| \le 2K\mu(t) + M_K|z|,$$

where $M_K \coloneqq \max_{y\in[-K,K]}|g(y)|$. Consequently, the Nagumo-Wintner assumption (H4) holds with the choice

$$H \coloneqq 1, \quad \psi(s) \coloneqq 1, \quad \ell(t) \coloneqq 2K\mu(t), \quad \nu(t) \coloneqq M_K, \quad q = +\infty.$$

Hence, Theorem 3.3 applies yielding the existence of a solution of Problem (45).

In the next example, the nonlinearity $f(t,x,y,z)$ has a product structure and may exhibit a superlinear growth with respect to the last variable.

Example 5.3. Consider the following BVP:

$$\begin{cases} (\Phi_r(k(t)u''(t)))' = \mu(t)g(u(t),u'(t))|u''(t)|^\alpha, & \text{a.e. on } [0,T], \\ u(0) = a, \quad u'(0) = b, \quad u'(T) = c, \end{cases} \tag{47}$$

where $\Phi_r : \mathbb{R} \to \mathbb{R}$ is the r-Laplacian, i.e., $\Phi_r(\xi) \coloneqq \xi\,|\xi|^{r-2}$, with $r > 1$; $k : I \to \mathbb{R}$ is a.e. positive with $1/k \in L^p(I)$ for some $p > 1$; $\mu : I \to \mathbb{R}$ belongs to $L^\beta(I)$ for some β with $1 < \beta \le \infty$; moreover, $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and monotone decreasing with respect to the first variable, i.e.,

$$g(x_1,y) \ge g(x_2,y), \quad \text{for any } y \in \mathbb{R} \text{ and any } x_1, x_2 \in \mathbb{R} \text{ with } x_1 \le x_2. \tag{48}$$

Finally, $\alpha > 0$ is fixed and such that

$$\alpha \le 1 - \frac{1}{\beta} + (r-1)\left(1-\frac{1}{p}\right). \tag{49}$$

Moreover, assume that

$$\frac{1}{\beta} + \frac{r-1}{p} < 1. \tag{50}$$

We claim that Problem (47) admits solutions for every choice of $b, c \in \mathbb{R}$. Indeed, let $f(t,x,y,z) \coloneqq \mu(t)g(x,y)|z|^\alpha$. The function f is Carathéodory, and from (48), we have that assumption (H2) is satisfied.

Moreover, if we take $K \coloneqq \max\{|b|,|c|\}$, where b and c are the boundary data, then the linear functions $\alpha(t) \coloneqq a - Kt$ and $\beta(t) \coloneqq a + Kt$ are a pair of well ordered lower and upper solutions of the equation appearing in (47).

Now, for every $R > 0$, define $M_R \coloneqq \max_{(x,y)\in[-R,R]^2}|g(x,y)|$. Then, for any $\gamma \in L^p_+(I)$, we have

$$|f(t,x,y,z(t))| \le |\mu(t)|M_R(\gamma(t))^\alpha \eqqcolon h_{R,\gamma}(t),$$

whenever $|x| \le R$, $|y| \le R$ and $z \in L^1(I)$ with $|z(t)| \le \gamma(t)$ for a.e. $t \in I$. Moreover, from the inequalities in (49) and (50), it follows

$$\alpha - \left(1 - \frac{1}{\beta}\right)p \le 1 - \frac{1}{\beta} + (r-1)\left(1 - \frac{1}{p}\right) - \left(1 - \frac{1}{\beta}\right)p = (r-1)\frac{p-1}{p} - \left(1 - \frac{1}{\beta}\right)(p-1) < 0,$$

so $\frac{p}{\alpha} > \frac{\beta}{\beta - 1}$, implying that $\gamma^\alpha \in L^{\frac{\beta}{\beta-1}}(I)$. Hence, by Hölder's inequality, the function $h_{R,\gamma}$ belongs to $L^1(I)$ and condition (H3) is fulfilled as well.

In order to show the validity of (H4), define

$$M_K := \max\{g(x,y) : x \in [a - KT, a + KT], y \in [-K, K]\},$$

and choose

$$H := 1, \quad \psi(s) := s, \quad \ell(t) := 0, \quad \text{and} \quad \nu(t) := \frac{M_K|\mu(t)|}{(k(t))^{r-1}}.$$

Put $q := \frac{\beta p}{p + \beta(r-1)}$. By (50), we obtain $q > 1$, and the summability assumptions on μ and k imply that

$$\mu^q \in L^{\frac{p+\beta(r-1)}{p}}(I) \quad \text{and} \quad \left(\frac{1}{k^{r-1}}\right)^q \in L^{\frac{p+\beta(r-1)}{\beta(r-1)}}(I).$$

Then, by applying again Hölder's inequality, we deduce that $\nu^q \in L^1(I)$, i.e., $\nu \in L^q(I)$. Furthermore, (49) implies

$$\alpha \le (r-1) + 1 - \frac{1}{\beta} - \frac{r-1}{p} = (r-1) + \frac{q-1}{q}.$$

Consequently, whenever $x \in [a - KT, a + KT]$, $y \in [-K, K]$, and $|z| > 1$, we obtain

$$|f(t,x,y,z)| \le M_K|\mu(t)|\cdot|z|^\alpha \le M_K|\mu(t)||z|^{r-1}\cdot|z|^{\frac{q-1}{q}} = \psi(|\Phi(k(t)z)|)\cdot \nu(t)|z|^{\frac{q-1}{q}},$$

and this shows that condition (H4) is satisfied.

So, Theorem 3.3 applies and we conclude that there exists a solution for any choice of $b, c \in \mathbb{R}$, as claimed.

Observe that Problem (3) in Section 1 is a particular case of (47). Indeed, in (3), we have $r = 2$, $\mu(t) \equiv 1$, so that $\mu \in L^{1/\varepsilon}([0,1])$ for any $\varepsilon > 0$, and finally, $1/k \in L^{(2-\varepsilon)}([0,1])$ for any $\varepsilon > 0$. Now, if $\alpha < 3/2$, then for ε sufficiently small, we have

$$\alpha < 2 - \varepsilon - \frac{1}{2 - \varepsilon} \quad \text{and} \quad \varepsilon + \frac{1}{2 - \varepsilon} < 1$$

so that both inequalities (49) and (50) hold. Hence, Problem (3) admits solutions whatever $b, c \in \mathbb{R}$ may be, as we stated in Section 1.

We explicitly point out that, with the procedure illustrated in Section 4, it is possible to modify the previous examples and to provide a new class of examples.

Example 5.4. Consider the following second-order integro-differential Dirichlet problem:

$$\begin{cases} (\Phi(k(t)v'(t)))' = \mu(t)\left[\rho\left(t, \int_0^t v(s)\mathrm{d}s\right) + v(t)\right] + g(v(t))v'(t), & \text{a.e. on } [0,T], \\ v(0) = b, \quad v(T) = c, \end{cases} \tag{51}$$

where the functions Φ, k, μ, ρ, and g are as in Example 5.1.

In particular, the function ρ is monotone decreasing in the second variable and bounded, i.e., conditions (43) and (44) hold.

Then, one can show that for every fixed $b, c \in \mathbb{R}$, Theorem 4.1 applies yielding the existence of a solution of Problem (51). Note that Problem (1) in Section 1 can be framed in this class of example.

We stress that, if ρ does not depend on t, we can avoid the boundedness condition provided that (46) holds, as in Example 5.2.

Example 5.5. Consider the following second-order integro-differential Dirichlet problem:

$$\begin{cases} (\Phi_r(k(t)v'(t)))' = \mu(t)g\left(\int_0^t v(s)\mathrm{d}s,\, v(t)\right)|v'(t)|^\alpha, & \text{a.e. on } [0,T], \\ v(0) = b, \quad v(T) = c, \end{cases} \tag{52}$$

where $\Phi_r : \mathbb{R} \to \mathbb{R}$ is the r-Laplacian, $\alpha > 0$ is fixed, and the functions k, μ, and g are as in Example 5.3.

As in Example 5.3, assume that conditions (48), (49), and (50) are satisfied. Then, one can prove that Problem (52) admits solutions for every choice of $b, c \in \mathbb{R}$.

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