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Bifurcations and complex dynamics in a banking duopoly model with macroprudential policy

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ABSTRACT

We consider a banking duopoly model with a macroprudential policy in Indonesia called loan-to-deposit ratio-based reserve requirement (LDR-RR). The objective of the policy is to control the banking loans growth using a LDR-based penalty scheme that requires banks to save more money in the central bank to maintain their liquidity ratio. Following recent studies on banking models, we analyze a piecewise discrete-time model with two banks. We assume that the dynamics of the deposits follows the discrete logistic growth. Moreover, our model has two borders, hence we examine the resulting border-collision bifurcations. From the local stability analysis we find that, according to the parameter values, only the border-collision bifurcation or the flip bifurcation occurs. Finally, we perform several numerical simulations to confirm the stability analysis' results. Our discrete dynamical system offers various possibilities of development for future research perspectives.

1. Introduction

Stability in banking industry is a key point. Many authors faced the problem focusing on different aspects. In particular, Fanti [1] develops a banking model investigating the role of an exogenous capital regulation parameter to the demand of loans; [2,3] extend the model of [1] considering the peculiarity of the Italian banking system. [4] focus on the role of non-performing loans in the stability of the banking system. Finally, Ansori et al. [5] build a monopoly model of banking loan with procyclicality behavior and apply it to monthly data of Indonesian commercial banks before and during the COVID-19 pandemic, to assess the role of the LDR-RR instrument.

Banking systems are different across countries and the policies adopted to stabilize system vary consistently. To this purpose, the role of a model consists in taking into account this heterogeneity through the appropriate modeling scheme. Following the line of the previously cited works, we make use of the theory of Discrete Dynamical Systems to build our model, focusing on the Indonesian banking system. We also stress that the stability of the banking industry can be analyzed by alternative solutions. For example, Claessens et al. [6] focus on the effectiveness and efficacy of macroprudential policies relying on a Generalized Method of Moments panel regression. Belkhir et al. [7] assess the impact of macroprudential policies on banking crises using logit model and bivariate vector autoregressive model. Differently, we make use of Dynamical Systems, in view of the analysis of banking stability. While the formers focus on the effect of the macroprudential policies on different economic variables by using dataset of banks located in several countries, we approach the analysis of stability. Starting from the stylized facts of the macroprudential policy adopted in Indonesia, we attempt to replicate its dynamics with a closed model, that enables us to analyze the consequences of the

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decisions of regulators. It is important to note that one choice does not exclude the other, but they are different approaches which can be combined to get an overall view of the problem.

The focus of this paper is an Indonesian macroprudential instrument known as loan-to-deposit ratio-based reserve requirement (LDR-RR). We take into account a banking model that investigates the stability of the Indonesian banking system considering the effects LDR-RR. This instrument can be used to apply counter-cyclical measures in banking industry by providing disincentive mechanism, when a bank operates outside the preferred operational corridor. At the lower limit of LDR, a requirement of higher RR can push banks to extent more loans, in order to support economic development in a period of economic bust (see [5,8]).

As a result, we build the banking duopoly model on the Indonesian banking balance sheet structure. The balance sheet is simplified here to include only deposits, equity, reserve requirements, liquid assets and loans. Reserve requirements include the LDR-RR instrument implicitly. There are three types of reserve requirements in Indonesia: primary RR, secondary RR, and LDR-RR. The primary RR is a monetary policy instrument for controlling the money supply, whereas the secondary RR is a liquidity instrument.

The LDR-RR policy is outlined in Bank Indonesia Regulation no.12/19/PBI/2010. In particular, the LDR-RR instrument has five parameters: lower bound of LDR, upper bound of LDR, lower disincentive parameter, upper disincentive parameter, and incentive capital adequacy ratio (CAR). It aims at controlling the growth of banking loans in Indonesia. The macroprudential aspects of this regulation has been studied by [8]. In this paper, we concentrate the analysis on the effects of regulation in our banking duopoly model. In particular, we focus on two main parameters of the regulation, namely the lower disincentive parameter and the lower bound of LDR parameter. We find that these two parameters affect the banking loan's stability via flip and border-collision bifurcations. The LDR-RR calculation is performed as follows:

(a) If the bank's LDR is within the target LDR range (i.e. between the lower and upper bounds), the bank's LDR-RR is 0% (zero percent) of rupiah deposits.
(b) If the bank's LDR is less than the lower bound, the bank's LDR-RR is calculated by multiplying the lower disincentive, the difference between the lower bound and the bank's LDR, and deposits in rupiah.
(c) If the bank's LDR exceeds the upper bound and its CAR is less than the incentive CAR, the bank's LDR-RR is calculated by multiplying the upper disincentive, the difference between the bank's LDR and the upper bound, and deposits in rupiah.
(d) If the bank's LDR exceeds the upper bound and its CAR is equal to or greater than the incentive CAR, the bank's LDR-RR is 0% (zero percent) of rupiah deposits.

In other words, the LDR-RR instrument works as a penalty for banks having an LDR that does not meet the regulatory target for a safe measure.

The model results in a two-dimensional discontinuous map with the phase space divided into four different regions. The paper contributes to the existing literature in several ways. First, we conduct a rigorous analysis of all the equilibria of the map and study the local stability of fixed points when they lie in the proper region of definition. Moreover, due to the discontinuous structure of the map, we are able to find conditions on the parameters that cause the emergence of the so called border-collision bifurcations (see [9] for a review of these kind of bifurcations). This enables us to examine further economic scenarios that motivate the usefulness of the LDR-RR instrument. For this purpose, we study possible bifurcation structures of the model and, thereby, establish that it may yield endogenous loan dynamics. Some of the bifurcation structures that we have investigated (in particular, the so called border-collision bifurcations) have received little attention in Economics so far. Second, we concentrate our analysis on two main parameters of the LDR-RR instrument: the lower disincentive parameter (γ_{lb}) and the lower bound of the LDR (δ_{lb}). The reason relies on the fact that these parameters play a key role in periods of economic bust, where the lending activity is more intense than in periods of stability. Third, we find an important economic result which is in line with the analysis of banks in different countries. Indeed, we are able to exclude the case where the market is served by a unique bank, as in [2,3]. Finally, we find several notable economic consequences due to the LDR-RR instrument. In particular, our results indicate that larger values of γ_{lb} and δ_{lb} can improve the intermediation activity of banks and the growth of the whole economy. However, policy-maker should not abuse of these instruments because further increases of these parameters could lead to period of instability in the economic system.

The remainder of the paper is organized as follows. In Section 2 we introduce the main ingredients of the model and we provide the final map. Section 3 is devoted to the study of the fixed points and to the detailed bifurcation analysis. Section 4 outlines the main economic scenarios arising from our analysis through bifurcation diagrams and the structure of the attractors of the model. Section 5 concludes our paper and provides useful suggestions for future developments of the model.

2. The discrete-time dynamical model

In this paper, we consider two banks indexed by $k = 1, 2$. Each bank k at time t has a balance sheet that consists of deposits ($D_{k,t}$), equity ($E_{k,t}$), reserve requirements 'RR' ($R_{k,t}$), loans ($L_{k,t}$) and liquid assets ($A_{k,t}$). The identity of the balance sheet yields total funding (deposits + equity) equals to total financing (RR + liquid assets + loans):

$$L_{k,t} + A_{k,t} + R_{k,t} = D_{k,t} + E_{k,t}.$$

Following [10,11], we assume that the dynamics of deposits follows the discrete logistic growth:

$$D_{k,t+1} = D_{k,t} + \beta_k D_{k,t}\left(1 - \frac{D_{k,t}}{K_k}\right), \quad \beta_k, K_k > 0. \tag{1}$$

[5] argued that in Indonesia deposits displayed an upward trend every month from March 2015 to May 2021. On the one hand, this fact may be interpreted as a linear or an exponential growth of deposits, although such hypotheses are unrealistic. On the other hand, the logistic growth is suitable to model deposits, due to the limitation in the demand of deposits that banks display (this depends on several factors, such as the number of branch offices, the regional coverage of a bank and, the growth of the economy around the regional banks). Thus, there should be a threshold on the growth of the deposits (also known as carrying capacity) that can be accommodated by the logistic model. In fact, it is known that the logistic model exhibits a stable equilibrium which corresponds to the carrying capacity. In this line of reasoning, our duopoly model assumes that deposits converge to the carrying capacity before loans converge to their equilibrium values. This ensures the tractability of the model, since border results in a constant. If this assumption is neglected, borders would depend on the deposits, complicating the analysis of the model.

Each bank must adhere to the capital adequacy ratio (CAR) policy, which requires that its equity to risk weighted asset ratio is greater than a certain percentage set by the regulator. Because RR is a deposit in the central bank, it is risk-free. We say that RR has a risk profile of 0%. We assume that liquid assets have a 0% risk profile, whereas loans have a 100% risk profile. Therefore, we have:

$$\frac{E_{k,t}}{L_{k,t}} \geq \kappa_{car}, \quad 0 < \kappa_{car} < 1,$$

where κ_{car} is the minimum percentage in CAR policy set by the regulator. According to Indonesian banking data (see [5]), the ratio $E_{k,t}/L_{k,t}$ can be assumed to be constant. Thus,

$$E_{k,t} = \kappa_k L_{k,t}, \quad \kappa_{car} \leq \kappa_k < 1. \tag{2}$$

Reserve requirements are sum of primary RR ($R^P_{k,t}$), secondary RR ($R^S_{k,t}$), and LDR-RR ($R^{LDR}_{k,t}$), with:

$$R^P_{k,t} = \rho_p D_{k,t}, \quad 0 < \rho_p < 1, \tag{3}$$

$$R^S_{k,t} = \rho_s D_{k,t}, \quad 0 < \rho_s < 1, \tag{4}$$

$$R^{LDR}_{k,t} = \begin{cases} 0, & \text{if } \delta_{lb} \leq L_{k,t}/D_{k,t} \leq \delta_{ub} \\ \gamma_{lb}\left(\delta_{lb} - \frac{L_{k,t}}{D_{k,t}}\right) D_{k,t}, & \text{if } L_{k,t}/D_{k,t} < \delta_{lb} \\ \gamma_{ub}\left(\frac{L_{k,t}}{D_{k,t}} - \delta_{ub}\right) D_{k,t}, & \text{if } L_{k,t}/D_{k,t} \geq \delta_{ub} \text{ and } E_{k,t}/L_{k,t} < \kappa_{ldr} \\ 0, & \text{if } L_{k,t}/D_{k,t} \geq \delta_{ub} \text{ and } E_{k,t}/L_{k,t} \geq \kappa_{ldr}, \end{cases} \tag{5}$$

where δ_{lb} and δ_{ub} are lower and upper bound of LDR, respectively; γ_{lb} and γ_{ub} are lower and upper disincentive, respectively; and κ_{ldr} is the incentive CAR.

Based on the Indonesian banking data in [5], the banks' CAR is above the value of incentive CAR (κ_{ldr}). Following that work, the third and fourth cases in (5) can be omitted,[1] and therefore:

$$R^{LDR}_{k,t} = \begin{cases} 0, & \text{if } L_{k,t}/D_{k,t} \geq \delta_{lb} \\ \gamma_{lb}\delta_{lb}D_{k,t} - \gamma_{lb}L_{k,t}, & \text{if } L_{k,t}/D_{k,t} < \delta_{lb} \end{cases} \tag{6}$$

Next, the loan dynamics are assumed to follow the gradient adjustment process as in [1,12], whose distribution for the following period is determined by the current loans' marginal profit:

$$L_{k,t+1} = L_{k,t} + \alpha_k L_{k,t}\frac{\partial \pi_{k,t}}{\partial L_{k,t}}, \quad \alpha_k > 0 \tag{7}$$

where α_k is the speed of adjustment.

The profit $\pi_{k,t}$ is calculated by subtracting financing income from funding and operating expenses:

$$\pi_{k,t} = r^L_{k,t}L_{k,t} + r_{A,k}A_{k,t} - r^D_{k,t}D_{k,t} - r_{E,k}E_{k,t} - C_{k,t}, \tag{8}$$

with $r^L_{k,t} = a_k - b_k L_t$ (where $a_k, b_k > 0$ and $L_t = L_{1,t} + L_{2,t}$) is the inverse demand function for loans, $r_{A,k}$ is the constant rate of return of liquid assets, $r^D_{k,t}$ is the deposit interest rate, $r_{E,k}$ is the constant cost of equity, and $C_{k,t} = c_{D,k}D_{k,t} + c_{L,k}L_{k,t}$ (where $0 < c_{D,k}, c_{L,k} < 1$).

Finally, we assume that liquid assets act as the balancing variable. Thus:

$$\begin{aligned} A_{k,t} &= D_{k,t} + E_{k,t} - L_{k,t} - R_{k,t} \\ &= \begin{cases} (1 - [\rho_p + \rho_s])D_{k,t} - (1-\kappa_k)L_{k,t}, & \text{if } L_{k,t}/D_{k,t} \geq \delta_{lb} \\ (1 - [\rho_p + \rho_s])D_{k,t} - (1-\kappa_k)L_{k,t} - \gamma_{lb}\delta_{lb}D_{k,t} + \gamma_{lb}L_{k,t}, & \text{if } L_{k,t}/D_{k,t} < \delta_{lb} \end{cases} \end{aligned} \tag{9}$$

The marginal profit of loans becomes:

$$\frac{\partial \pi_{k,t}}{\partial L_{k,t}} = \begin{cases} a_k - \Lambda_k - b_k L_{-k,t} - 2b_k L_{k,t}, & \text{if } L_{k,t}/D_{k,t} \geq \delta_{lb} \\ a_k - \Lambda_k + r_{A,k}\gamma_{lb} - b_k L_{-k,t} - 2b_k L_{k,t}, & \text{if } L_{k,t}/D_{k,t} < \delta_{lb} \end{cases}$$

where $-k \in \{1,2\}\backslash\{k\}$, $\Lambda_k = r_{A,k}(1-\kappa_k) + \kappa_k r_{E,k} + c_{L,k}$. It is clear that $\Lambda_k > 0$.

[1] Please see [5] for more details

Therefore, we have the final form of loans model as follows:

$$L_{k,t+1} = \begin{cases} L_{k,t} + \alpha_k L_{k,t}\left(a_k - \Lambda_k - b_k L_{-k,t} - 2b_k L_{k,t}\right), & \text{if } L_{k,t}/D_{k,t} \geq \delta_{lb} \\ L_{k,t} + \alpha_k L_{k,t}\left(a_k - \Lambda_k + r_{A,k}\gamma_{lb} - b_k L_{-k,t} - 2b_k L_{k,t}\right), & \text{if } L_{k,t}/D_{k,t} < \delta_{lb} \end{cases} \tag{10}$$

Based on the above model, we derive the final duopoly model as a system of two difference equations with loans as state variables:

$$\begin{cases} L_{1,t+1} = \begin{cases} L_{1,t} + \alpha_1 L_{1,t}\left(a_1 - \Lambda_1 - b_1 L_{2,t} - 2b_1 L_{1,t}\right), & \text{if } L_{1,t}/D_{1,t} \geq \delta_{lb} \\ L_{1,t} + \alpha_1 L_{1,t}\left(a_1 - \Lambda_1 + r_{A,1}\gamma_{lb} - b_1 L_{2,t} - 2b_1 L_{1,t}\right), & \text{if } L_{1,t}/D_{1,t} < \delta_{lb} \end{cases} \\ L_{2,t+1} = \begin{cases} L_{2,t} + \alpha_2 L_{2,t}\left(a_2 - \Lambda_2 - b_2 L_{1,t} - 2b_2 L_{2,t}\right), & \text{if } L_{2,t}/D_{2,t} \geq \delta_{lb} \\ L_{2,t} + \alpha_2 L_{2,t}\left(a_2 - \Lambda_2 + r_{A,2}\gamma_{lb} - b_2 L_{1,t} - 2b_2 L_{2,t}\right), & \text{if } L_{2,t}/D_{2,t} < \delta_{lb} \end{cases} \end{cases} \tag{11}$$

In order to simplify the analysis, in this paper we assume:

$$\begin{aligned} &\alpha_1 = \alpha_2 = \alpha, \quad a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad r_{A,1} = r_{A,2} = r_A, \quad r_{E,1} = r_{E,2} = r_E, \\ &\kappa_1 = \kappa_2 = \kappa, \quad c_{L,1} = c_{L,2} = c, \quad \Lambda_1 = \Lambda_2 = \Lambda = r_A(1-\kappa) + \kappa r_E + c, \\ &\beta_1 = \beta_2 = \beta, \quad K_1 = K_2 = K. \end{aligned}$$

Hence, we have the simplified model as follows:

$$\begin{cases} L_{1,t+1} = \begin{cases} L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t}\right), & \text{if } L_{1,t}/D_{1,t} \geq \delta_{lb} \\ L_{1,t} + \alpha L_{1,t}\left(a - \Lambda + r_A\gamma_{lb} - bL_{2,t} - 2bL_{1,t}\right), & \text{if } L_{1,t}/D_{1,t} < \delta_{lb} \end{cases} \\ L_{2,t+1} = \begin{cases} L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t}\right), & \text{if } L_{2,t}/D_{2,t} \geq \delta_{lb} \\ L_{2,t} + \alpha L_{2,t}\left(a - \Lambda + r_A\gamma_{lb} - bL_{1,t} - 2bL_{2,t}\right), & \text{if } L_{2,t}/D_{2,t} < \delta_{lb} \end{cases} \end{cases} \tag{12}$$

The deposit model in (1) admits two equilibria 0 and K, the former is unstable while the latter is stable for any set of the parameter values. Hence, we can restrict the analysis to the case in which deposit $D_{i,t}$ converges to its carrying capacity K_i and, remembering that $K_1 = K_2 = K$, we immediately obtain the following 2−dimensional map:

$$\begin{cases} L_{1,t+1} = \begin{cases} L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t}\right), & \text{if } L_{1,t} \geq \delta_{lb}K \\ L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t} + r_A\gamma_{lb}\right), & \text{if } L_{1,t} < \delta_{lb}K \end{cases} \\ L_{2,t+1} = \begin{cases} L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t}\right), & \text{if } L_{2,t} \geq \delta_{lb}K \\ L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t} + r_A\gamma_{lb}\right), & \text{if } L_{2,t} < \delta_{lb}K \end{cases} \end{cases} \tag{13}$$

The phase space is divided into four different regions:

$$R_1 = \left\{(L_1, L_2) \in \mathbb{R}^2_+ : L_1 \geq \delta_{lb}K, L_2 \geq \delta_{lb}K\right\}, R_2 = \left\{(L_1, L_2) \in \mathbb{R}^2_+ : L_1 < \delta_{lb}K, L_2 \geq \delta_{lb}K\right\}$$

$$R_3 = \left\{(L_1, L_2) \in \mathbb{R}^2_+ : L_1 < \delta_{lb}K, L_2 < \delta_{lb}K\right\}, R_4 = \left\{(L_1, L_2) \in \mathbb{R}^2_+ : L_1 \geq \delta_{lb}K, L_2 < \delta_{lb}K\right\}$$

which are separated by the set $R := \left\{(L_1, L_2) \in \mathbb{R}^2_+ : L_1 = \delta_{lb}K\right\} \cup \left\{(L_1, L_2) \in \mathbb{R}^2_+ : L_2 = \delta_{lb}K\right\}$, hence the border is defined by the vertical line $L_1 = \delta_{lb}K$ and the horizontal line $L_2 = \delta_{lb}K$.

At the interior of each region of the phase space smooth maps apply. Every map is different from the other along the set R, in other terms System (13) defines a piecewise-smooth discontinuous map, i.e. it is discontinuous across the boundary.

Summing up, System (13) is composed by the following maps:

1. System 1, $\forall (L_1, L_2) \in R_1$

$$S_1 = \begin{cases} L_{1,t+1} = L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t}\right) \\ L_{2,t+1} = L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t}\right) \end{cases} \tag{14}$$

2. System 2, $\forall (L_1, L_2) \in R_2$

$$S_2 = \begin{cases} L_{1,t+1} = L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t} + r_A\gamma_{lb}\right) \\ L_{2,t+1} = L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t}\right) \end{cases} \tag{15}$$

3. System 3, $\forall (L_1, L_2) \in R_3$

$$S_3 = \begin{cases} L_{1,t+1} = L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t} + r_A\gamma_{lb}\right) \\ L_{2,t+1} = L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t} + r_A\gamma_{lb}\right) \end{cases} \tag{16}$$

4. System 4, $\forall (L_1, L_2) \in R_4$

$$S_4 = \begin{cases} L_{1,t+1} = L_{1,t} + \alpha L_{1,t}\left(a - \Lambda - bL_{2,t} - 2bL_{1,t}\right) \\ L_{2,t+1} = L_{2,t} + \alpha L_{2,t}\left(a - \Lambda - bL_{1,t} - 2bL_{2,t} + r_A\gamma_{lb}\right) \end{cases} \tag{17}$$

In the following section, we analyze the final map, taking into account that inside the single region R_i smooth bifurcation theory in discrete-time applies but, being our system piecewise-smooth discontinuous, also intersections between invariant sets and the discontinuity boundary may arise. These type of bifurcations involves the qualitative theory of non-smooth dynamical system.

3. Stability and bifurcations

We start the local analysis by considering the interior of each region R_i ($i = 1, 2, 3, 4$), where the standard theory of smooth maps applies.

3.1. Equilibrium points and Jacobian matrices

Equilibria of our map are calculated by setting $L_{k,t+1} = L_{k,t}$ for $k = 1, 2$ and $\forall t$.

In the following we analyze equilibria of Systems (14)–(17) and we remember that a fixed point is *real* or *actual* if it is inside the corresponding region of definition, it is *virtual* if it lies inside another region, while we have a *boundary* fixed point when it belongs to the border R (see [13] for all generalities of piecewise-smooth systems and [14] for a detailed analysis of virtual fixed points).

As a preliminary result, we note that the trivial economic equilibrium $(0, 0)$ is a fixed point for all the Systems 1–4 but it is a real fixed point only for System 3 defined by (16), otherwise (for Systems 1,2 and 4) the origin is a virtual fixed point.

In what follows we denote by $\mathbb{E}_{ij} = (L_1^*, L_2^*)$ the real equilibria of System i ($i = 1, 2, 3, 4$), where the index j numbers them, in other words the first index (i) refers to the region R_i of definition, while the second index (j) numbers the fixed points in each region.

System 1. There is a unique equilibrium given by:

$$\mathbb{E}_{11} = \left(\frac{a-\Lambda}{3b}, \frac{a-\Lambda}{3b}\right)$$

which is a real fixed point iff $\frac{a-\Lambda}{3b} > K\delta_{lb}$ (it is instead a virtual fixed point iff $\frac{a-\Lambda}{3b} < K\delta_{lb}$ and a boundary fixed point iff $\frac{a-\Lambda}{3b} = K\delta_{lb}$).

The Jacobian matrix for the real fixed point $\mathbb{E} = (L_1^*, L_2^*)$ is:

$$J = \begin{bmatrix} 1+\alpha(a-\Lambda)-\alpha b L_2^* - 4\alpha b L_1^* & -\alpha b L_1^* \\ -\alpha b L_2^* & 1+\alpha(a-\Lambda)-\alpha b L_1^* - 4\alpha b L_2^* \end{bmatrix}. \tag{18}$$

System 2. We get the following equilibria:

$$\mathbb{E}_{21} = \left(0, \frac{a-\Lambda}{2b}\right) \quad \text{and} \quad \mathbb{E}_{22} = \left(\frac{a-\Lambda+2r_A\gamma_{lb}}{3b}, \frac{a-\Lambda-r_A\gamma_{lb}}{3b}\right).$$

The equilibrium point $\mathbb{E}_{22}$ cannot be real since $L_1^\star > L_2^\star$ and it cannot belong to the sub-region R_2. Differently, $\mathbb{E}_{21}$ is a real fixed point iff $\frac{a-\Lambda}{2b} > K\delta_{lb}$. The Jacobian matrix is:

$$J = \begin{bmatrix} 1+\alpha(a-\Lambda+r_A\gamma_{lb})-\alpha b L_2^* - 4\alpha b L_1^* & -\alpha b L_1^* \\ -\alpha b L_2^* & 1+\alpha(a-\Lambda)-\alpha b L_1^* - 4\alpha b L_2^* \end{bmatrix}. \tag{19}$$

System 3. We get the following equilibria:

$$\mathbb{E}_{31} = (0,0), \quad \mathbb{E}_{32} = \left(\frac{a-\Lambda+r_A\gamma_{lb}}{2b}, 0\right), \quad \mathbb{E}_{33} = \left(0, \frac{a-\Lambda+r_A\gamma_{lb}}{2b}\right),$$
$$\mathbb{E}_{34} = \left(\frac{a-\Lambda+r_A\gamma_{lb}}{3b}, \frac{a-\Lambda+r_A\gamma_{lb}}{3b}\right),$$

they are real fixed points if $L_1^* < K\delta_{lb}$ and $L_2^* < K\delta_{lb}$ and the Jacobian matrix is:

$$J = \begin{bmatrix} 1+\alpha(a-\Lambda+r_A\gamma_{lb})-\alpha b L_2^* - 4\alpha b L_1^* & -\alpha b L_1^* \\ -\alpha b L_2^* & 1+\alpha(a-\Lambda+r_A\gamma_{lb})-\alpha b L_1^* - 4\alpha b L_2^* \end{bmatrix}. \tag{20}$$

System 4. The equilibria are:

$$\mathbb{E}_{41} = \left(\frac{a-\Lambda}{2b}, 0\right) \quad \text{and} \quad \mathbb{E}_{42} = \left(\frac{a-\Lambda-r_A\gamma_{lb}}{3b}, \frac{a-\Lambda+2r_A\gamma_{lb}}{3b}\right).$$

The equilibrium point $\mathbb{E}_{42}$ cannot be real since $L_1^\star < L_2^\star$ and it cannot belong to the sub-region R_4. Differently, $\mathbb{E}_{41}$ is a real fixed point iff $\frac{a-\Lambda}{2b} > K\delta_{lb}$.

The Jacobian matrix is:

$$J = \begin{bmatrix} 1+\alpha(a-\Lambda)-\alpha b L_2^* - 4\alpha b L_1^* & -\alpha b L_1^* \\ -\alpha b L_2^* & 1+\alpha(a-\Lambda+r_A\gamma_{lb})-\alpha b L_1^* - 4\alpha b L_2^* \end{bmatrix}. \tag{21}$$

Notice that in all the previous cases the economic meaning of equilibria is needed, in other words we have to ensure their positiveness. To this end, we underline that condition

$$a > \Lambda + r_A\gamma_{lb} \tag{22}$$

guarantees $\mathbb{E} > (0,0)$ for every fixed point $\mathbb{E} \neq (0,0)$.

It is important to underline that systems composing our Map (13) have the same set-up and then, the same functional form as in [2]. In that work, authors proved the existence of the compact global attractor contained in a positively invariant interval of $\mathbb{R}^2_+$. Additionally, as it is usual in this kind of models, the study of the long-run behavior herewith carried out enables us to rely on conditions on parameter sets such that dynamics are bounded in $\mathbb{R}^2_+$.

Let us go to move to the local stability analysis of all the real equilibria, for which the theory of smooth systems applies.

3.2. Local stability of real fixed points

The local stability of every interior equilibrium point can be analyzed using Jury's conditions as follows. The Jacobian matrices J at equilibrium points are evaluated as in the previous subsection. Then the characteristic polynomial is given by:

$$P(\lambda) = \lambda^2 + \mathrm{tr}(J)\lambda + \det(J). \tag{23}$$

The equilibrium is locally stable if the following stability conditions hold:

$$\begin{cases} F := 1 + \mathrm{tr}(J) + \det(J) > 0 \\ T := 1 - \mathrm{tr}(J) + \det(J) > 0, \\ NS := 1 - \det(J) > 0 \end{cases} \tag{24}$$

where $\mathrm{tr}(J)$ and $\det(J)$ denote trace and determinant of J, respectively.

In this paper we study the LDR-RR in (6), or in other words, we study its dynamics when the most interesting parameters $(\gamma_{lb}, \delta_{lb}, \kappa_{ldr})$ vary. The following theorems state the local stability of each equilibrium, and the stability conditions are written in order to focus on the LDR-RR's parameters, if it is possible.

Theorem 1 (*System 1*). *Let* $\frac{a-\Lambda}{3b} > K\delta_{lb}$, *then equilibrium* $\mathbb{E}_{11} = \left(\frac{a-\Lambda}{3b}, \frac{a-\Lambda}{3b}\right)$ *is locally asymptotically stable if* $\alpha < \frac{2}{a-\Lambda}$.

Proof. Conditions on the parameters guarantee that $\mathbb{E}_{11}$ lies inside the region R_1, i.e. it is actual. Consequently. we can consider the Jacobian matrix (18) evaluated at $\mathbb{E}_{11}$:

$$J_{11} = \begin{bmatrix} 1 - \frac{2}{3}\alpha(a-\Lambda) & -\frac{1}{3}\alpha(a-\Lambda) \\ -\frac{1}{3}\alpha(a-\Lambda) & 1 - \frac{2}{3}\alpha(a-\Lambda) \end{bmatrix},$$

and $\mathrm{tr}(J_{11}) = -\frac{4}{3}\alpha(a-\Lambda) + 2$, and $\det(J_{11}) = \frac{1}{3}\alpha^2(a-\Lambda)^2 - \frac{4}{3}\alpha(a-\Lambda) + 1$.

Then,

$$F = \frac{1}{3}\alpha^2(a-\Lambda)^2 - \frac{8}{3}\alpha(a-\Lambda) + 4 > 0 \quad \text{if} \quad \alpha < \frac{2}{a-\Lambda} \text{ or } \alpha > \frac{6}{a-\Lambda},$$
$$T = \frac{1}{3}\alpha^2(a-\Lambda)^2 > 0,$$
$$NS = -\frac{1}{3}\alpha^2(a-\Lambda)^2 + \frac{4}{3}\alpha(a-\Lambda) > 0 \quad \text{if} \quad \alpha < \frac{4}{a-\Lambda}.$$

Therefore, $\mathbb{E}_{11}$ is locally stable if $\alpha < \frac{2}{a-\Lambda}$. □

Theorem 2 (*System 2*). *Let* $\frac{a-\Lambda}{2b} > K\delta_{lb}$, *then equilibrium* $\mathbb{E}_{21} = \left(0, \frac{a-\Lambda}{2b}\right)$ *is unstable.*

Proof. Again, conditions on the parameters are needed for the admissibility of equilibria. Since the proof is similar to the next Theorem 4 (System 4), we remind to it. □

Theorem 3 (*System 3*). $\mathbb{E}_{31} = (0,0)$ *is always a real equilibrium and it is a locally unstable node for any set of the parameter values. For* $\frac{a-\Lambda+r_A\gamma_{lb}}{2b} < K\delta_{lb}$, $\mathbb{E}_{32} = \left(\frac{a-\Lambda+r_A\gamma_{lb}}{2b}, 0\right)$ *and* $\mathbb{E}_{33} = \left(0, \frac{a-\Lambda+r_A\gamma_{lb}}{2b}\right)$ *are locally unstable. Meanwhile, for* $\frac{a-\Lambda+r_A\gamma_{lb}}{3b} < K\delta_{lb}$ *equilibrium* $\mathbb{E}_{34} = \left(\frac{a-\Lambda+r_A\gamma_{lb}}{3b}, \frac{a-\Lambda+r_A\gamma_{lb}}{3b}\right)$ *is locally asymptotically stable if* $\gamma_{lb} < \frac{1}{\alpha r_A}[2 - \alpha(a-\Lambda)]$.

Proof. In order to make the reading easier, pose $q = \alpha(a - \Lambda + r_A\gamma_{lb})$. From (22), we have $q > 0$. For the case of $\mathbb{E}_{31}$, we have the Jacobian matrix

$$J_{31} = \begin{bmatrix} 1+q & 0 \\ 0 & 1+q \end{bmatrix}.$$

Since $NS = 1 - \det(J_{31}) = -q\,[q+2] < 0$, then $\mathbb{E}_{31}$ is unstable when it is real, for the parameter values defined in the theorem. The eigenvalues are: $k_1 = k_2 = 1 + q = 1 + \alpha(a - \Lambda + r_A\gamma_{lb}) > 1$ and the fixed point is a local unstable node.

Analogously, for the given parameter values $\mathbb{E}_{32}$ is real and the Jacobian matrix is:

$$J_{32} = \begin{bmatrix} 1-q & -\frac{1}{2}q \\ 0 & 1+\frac{1}{2}q \end{bmatrix}.$$

Since $T = 1 - \mathrm{tr}(J_{32}) + \det(J_{32}) = -\frac{1}{2}q^2 < 0$, then $\mathbb{E}_{32}$ is unstable. The eigenvalues are: $k_1 = 1 - q$ and $k_2 = 1 + \frac{1}{2}q > 1$, hence it is a locally unstable node for $q > 2$ or a saddle node for $q < 2$.[2]

The case of $\mathbb{E}_{33}$ is similar to $\mathbb{E}_{32}$. Thus, $\mathbb{E}_{33}$ is also locally unstable.

Now, for the case of $\mathbb{E}_{34}$, we have the Jacobian matrix when it is a real fixed point for the defined parameter values:

$$J_{34} = \begin{bmatrix} 1 - \frac{2}{3}q & -\frac{1}{3}q \\ -\frac{1}{3}q & 1 - \frac{2}{3}q \end{bmatrix},$$

with $\mathrm{tr}(J_{34}) = 2 - \frac{4}{3}q$ and $\det(J_{34}) = 1 - \frac{4}{3}q + \frac{1}{3}q^2$.

Then, we have

$$F = 4 - \frac{8}{3}q + \frac{1}{3}q^2 > 0 \quad \text{if} \quad \gamma_{Ib} < \frac{1}{\alpha r_A}[2 - \alpha(a - \Lambda)] \text{ or } \gamma_{Ib} > \frac{1}{\alpha r_A}[6 - \alpha(a - \Lambda)],$$

$$T = \frac{1}{3}q^2 > 0,$$

$$NS = \frac{4}{3}q - \frac{1}{3}q^2 > 0 \quad \text{if} \quad \gamma_{Ib} < \frac{1}{\alpha r_A}[4 - \alpha(a - \Lambda)].$$

Therefore, $\mathbb{E}_{34}$ is locally stable if $\gamma_{Ib} < \frac{1}{\alpha r_A}[2 - \alpha(a - \Lambda)]$. □

Theorem 4 (*System 4*). *Let $\frac{a-\Lambda}{2b} > K\delta_{Ib}$, then equilibrium $\mathbb{E}_{41} = \left(\frac{a-\Lambda}{2b}, 0\right)$ is locally unstable.*

Proof. The Jacobian matrix (18) evaluated at $\mathbb{E}_{41}$ becomes:

$$J_{41} = \begin{bmatrix} 1 - \alpha(a - \Lambda) & -\frac{1}{2}\alpha(a - \Lambda) \\ 0 & 1 + \frac{1}{2}\alpha(a - \Lambda) + \alpha r_A \gamma_{Ib} \end{bmatrix},$$

and $\mathrm{tr}(J_{41}) = -\frac{1}{2}\alpha(a - \Lambda) + \alpha r_A \gamma_{Ib} + 2$, and $\det(J_{41}) = -\frac{1}{2}\alpha^2(a - \Lambda)^2 - \frac{1}{2}\alpha(a - \Lambda) - [\alpha(a - \Lambda) - 1]\alpha r_A \gamma_{Ib} + 1$.

Notice that

$$T = -\left[\frac{1}{2}\alpha^2(a - \Lambda)^2 + \alpha^2(a - \Lambda) r_A \gamma_{Ib}\right] < 0.$$

Thus, $\mathbb{E}_{41}$ is locally unstable. The eigenvalues are: $k_1 = 1 - \alpha(a - \Lambda)$ and $k_2 = 1 + \frac{1}{2}\alpha(a - \Lambda) + \alpha r_A \gamma_{Ib}$, hence it can be an unstable node or an unstable saddle node, according to parameter values. □

Starting from the local stability analysis of steady states, in the following subsection we investigate the bifurcations owned by the system. More precisely, previous theorems enable us to study the standard local bifurcations, for which we can make use of the smooth bifurcation theory. But our model is able to exhibit also another type of bifurcations, belonging to the class of discontinuity-induced bifurcations. In particular, we will investigate border-collision bifurcations, which are related to the contact of an invariant set with the border separating the regions of different definition of the map. We will see that some fixed points collide with the border separating the different regions R_i ($i = 1, 2, 3, 4$), at some critical parameter values.

3.3. Bifurcation analysis

We start the analysis by investigating qualitative changes causing the loss of stability of the real locally stable fixed points, i.e. ‘smooth bifurcations’.

We recall that in (24) the necessary conditions are: $F = 0$ (and $T, NS > 0$) for the flip bifurcation, $T = 0$ (and $F, NS > 0$) for the fold bifurcation and $NS = 0$ (with $F, T > 0$) for the Neimark–Sacker bifurcation.

According to Theorems 1–4, the real locally stable steady states are $\mathbb{E}_{11}$, $\mathbb{E}_{34}$ and T is positive for all of them. Hence, a preliminary result is found: the fold bifurcation never happens.

Moreover, based on the analysis performed in proofs, we find that a second result does hold: the flip bifurcation occurs first before the possible Neimark–Sacker bifurcation. The following corollary details these results.

Corollary 1. *Consider the locally stable real equilibria as in Theorems 1–4 for which stability conditions $F, NS, T > 0$ hold, where F, NS, T are defined by (24). Hence, T is always positive while the sign of F becomes negative first before NS when the corresponding parameter values vary, meaning that real fixed points lose stability via period-doubling bifurcation. More precisely:*

- *Equilibrium $\mathbb{E}_{11}$ undergoes to the flip bifurcation at $\alpha = \frac{2}{a-\Lambda}$,*
- *Equilibrium $\mathbb{E}_{34}$ undergoes to the flip bifurcation at $\gamma_{Ib} = \gamma_{Ib}^F$ (where $\gamma_{Ib}^F = \frac{1}{\alpha r_A}[2 - \alpha(a - \Lambda)]$).*

Proof. The proof directly comes from the analysis of the signs of F, NS, T made in the proofs of Theorems 1–4. □

[2] Mathematically speaking, a saddle point is unstable.

As it is well-documented before, our duopoly model in (12) has borders, namely $L_1 = K\delta_{lb}$ and $L_2 = K\delta_{lb}$, which separate the state phase into four regions. When we let some key parameters to vary, equilibrium loans may collide with the border. If this happens, i.e. $L^* = K\delta_{lb}$, it is called border-collision bifurcation. Generally speaking, border-collision bifurcations occur when a trajectory collides with the boundary separating different regions of definition for the system. These phenomena are well-known for piecewise-smooth maps which are continuous (see, among others, [13,15,16]), while they are not completely understood for maps which are discontinuous on the border. Nevertheless, piecewise-smooth and discontinuous maps find large applicability in explaining bifurcations in Economics and Finance and, for this reason, they became more popular in last years (e.g. [17–20]). From this non-canonical bifurcations very rich dynamics arise.

Now, we would like to study the LDR-RR's parameters associated to this phenomenon.

To this end, let us go to consider the locally stable steady state $\mathbb{E}_{11} = \left(\frac{a-\Lambda}{3b}, \frac{a-\Lambda}{3b}\right)$. The equilibrium is interior to its region of definition (i.e. it is a real fixed point) if $L_1^* > K\delta_{lb}$ and $L_2^* > K\delta_{lb}$. In other words, the lower bound of LDR has to satisfy:

$$\delta_{lb} < \frac{a-\Lambda}{3bK}.$$

The border-collision bifurcation of $\mathbb{E}_{11}$ happens when $\delta_{lb} = \delta_{lb}^B$, where

$$\delta_{lb}^B = \frac{a-\Lambda}{3bK}.$$

The fixed point $\mathbb{E}_{34} = \left(\frac{a-\Lambda+r_A\gamma_{lb}}{3b}, \frac{a-\Lambda+r_A\gamma_{lb}}{3b}\right)$ has to satisfy $L_1^* < K\delta_{lb}$ and $L_2^* < K\delta_{lb}$, that is

$$\gamma_{lb} < \frac{1}{r_A}(3bK\delta_{lb} - (a-\Lambda)).$$

The border-collision bifurcation of $\mathbb{E}_{34}$ happens when $\gamma_{lb} = \gamma_{lb}^B$, in this case:

$$\gamma_{lb}^B = \frac{1}{r_A}(3bK\delta_{lb} - (a-\Lambda)).$$

In terms of the lower bound of LDR parameter:

$$\delta_{lb} > \frac{a-\Lambda+r_A\gamma_{lb}}{3bK}.$$

The border-collision bifurcation happens when $\delta_{lb} = \delta_{lb}^B$, where:

$$\delta_{lb}^B = \frac{a-\Lambda+r_A\gamma_{lb}}{3bK}.$$

From the analysis of bifurcations of the real locally stable steady states, we obtain an expected result for the lower disincentive parameter γ_{lb}, as resumed in the following remark.

Remark 1. There are two possibilities for the loss of stability of the real equilibria, which take place at the critical parameter values as above:

1. if $\gamma_{lb}^B < \gamma_{lb}^F$ then the border-collision bifurcation occurs, due to their collision with the border of definition of the map,
2. if $\gamma_{lb}^B > \gamma_{lb}^F$ then they lose stability via period doubling bifurcation.

Thanks to our analysis we also know the critical parameter values, corresponding to these bifurcations for real fixed points.

3.4. Complex dynamics

In order to analyze thoroughly the possibility of complex dynamics arising, we consider the invariant sets of system. In particular, we observe that the diagonal is a positively invariant set, since $S_1(L,L) = (L',L')$ and $S_3(L,L) = (L',L')$. In other terms, system acts from the main diagonal into itself. Thanks to this fact, we can focus on the dynamics embedded by system on the diagonal, which can be studied through the following restriction:

$$L_{t+1} = \begin{cases} L_t + \alpha L_t\left(a-\Lambda-3bL_t\right), & if \quad L_t \geq \delta_{lb}K \\ L_t + \alpha L_t\left(a-\Lambda-3bL_t + r_A\gamma_{lb}\right), & if \quad L_t < \delta_{lb}K \end{cases} \tag{25}$$

which can be rewritten as:

$$L_{t+1} = f(L_t) = AL_t^2 + BL_t, \quad A = -3\alpha b < 0, \quad B = \begin{cases} B_1 = 1 + \alpha\left(a-\Lambda+r_A\gamma_{lb}\right), & if \quad L_t < \delta_{lb}K \\ B_2 = 1 + \alpha\left(a-\Lambda\right), & if \quad L_t \geq \delta_{lb}K \end{cases}. \tag{26}$$

As a consequence, the dynamics on the main diagonal are governed by a discontinuous one-dimensional map. Functions in (26) are topologically conjugate to the logistic map $z' = \mu z(1-z)$, then the dynamics generated by each map in (26) are completely known and can be obtained from those of the logistic map.

Let us go to study the dynamics of map taking into account the economic meaning of parameters.

We start the analysis by proving the existence of a closed and positively invariant interval, as described in the following.

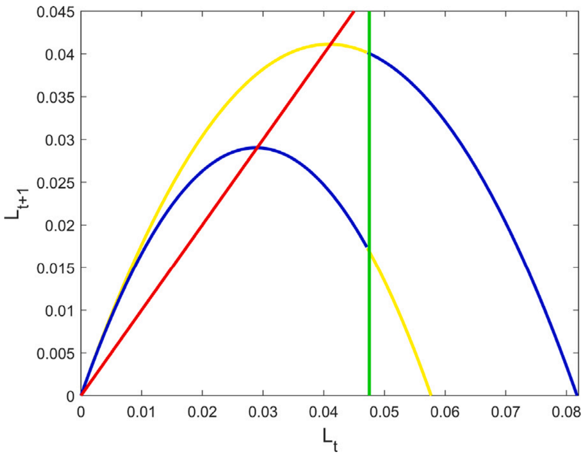

Fig. 1. Graph of the map f when the conditions (i) $\delta_{lb}K > \frac{a-\Lambda+r_A\gamma_{lb}}{3b}$ and (ii) $f'(\mathbb{E}_{34}) = 2 - B_1 < 0$ hold.

Proposition 1. *Let* $\delta_{lb}K \leq min\left\{-\frac{B_2}{A}, -\frac{B_1}{2A}\right\}$*, hence the closed interval* $J = \left[0, -\frac{B_2}{A}\right]$ *is positively invariant.*

Proof. In order to prove the statement, we show that $L_t \in \left[0, -\frac{B_2}{A}\right]$ implies $L_{t+1} \in \left[0, -\frac{B_2}{A}\right]$ for all t. To this end, we consider:

$$f(L) = \begin{cases} f_1(L) = AL^2 + B_1 L, & if \quad L < \delta_{lb}K \\ f_2(L) = AL^2 + B_2 L, & if \quad L \geq \delta_{lb}K \end{cases}.$$

Being $B_2 < B_1$, then $f(L) \geq 0$ $\forall L \in J$. Moreover, $\forall L \in J$, $f_2(L) \leq f_1(L)$ and $f_1(L) \leq f_1(K\delta_{lb})$ under the assumption $\delta_{lb}K \leq -\frac{B_1}{2A}$ since $-\frac{B_1}{2A}$ is the maximum point of f_1. □

According to the previous proposition, every initial condition belonging to J generates bounded trajectories converging to an attractor included into the interval.

Now, we observe that the fixed points are: $\mathbb{E}_{11} = \frac{a-\Lambda}{2b}$, $\mathbb{E}_{31} = 0$ and $\mathbb{E}_{34} = \frac{a-\Lambda+r_A\gamma_{lb}}{3b}$, which do exist being $f'(0) > 1$ for both the cases $B = B_1$ and $B = B_2$. Nevertheless, $\mathbb{E}_{31}$ and $\mathbb{E}_{34}$ cannot be admissible simultaneously, being $B_2 < B_1$.

Let us go to explore the case: (i) $\delta_{lb}K > \frac{a-\Lambda+r_A\gamma_{lb}}{3b}$ (so that $\mathbb{E}_{34}$ is a real fixed point) and (ii) $f'(\mathbb{E}_{34}) = 2 - B_1 < 0$. For making the reading easier, in Fig. 1 we represent the graph of the map f when these assumptions hold.

Observe that, in this case, for given values of the parameter $A = -3\alpha b$ such that $-\frac{B_1-1}{A} < \delta_{lb}K$, dynamics are increasingly complex if the parameter $B_1 = \frac{a-\Lambda+r_A\gamma_{lb}}{3b}$ is big enough. In fact, the real fixed point $\mathbb{E}_{34}$ is locally stable for $B_1 < 3$. As B_1 increases, the steady state loses stability at $B_1 = 3$ and a stable two-cycle appears, if B_1 still increases other period doubling bifurcations occur. This scenario is depicted in Fig. 2, where we let the parameter γ_{lb} to vary. We observe the typical cascade of period-doubling bifurcations when the parameter in our interest increases.

Observe that, when $-\frac{B_1-1}{A} = \delta_{lb}K$ the fixed point collides with the border and, consequently the border-collision bifurcation of the fixed point happens. In Fig. 3 is shown the 1−dimensional bifurcation diagram when we let the lower bound of LDR δ_{lb} to vary. Unlike Fig. 2, now it is clear a sharp transition from stability to complex dynamics when δ_{lb} increases, which characterizes border-collision bifurcations. Moreover, Fig. 4 shows the 2−dimensional bifurcation diagram in the plane $(\delta_{lb}, \gamma_{lb})$ that highlights the complex structure of the border-collision bifurcation. From an economic point of view, an excessive increase of the lower bound of the LDR can force the instability of the banking system, this is independent of γ_{lb}.

Notice that, when the real fixed point is $\mathbb{E}_{11}$ (i.e. $-\frac{B_2-1}{A} < \delta_{lb}K$), similar results hold. In fact, for increasing values of B_2, the equilibrium loses stability via period doubling bifurcation and dynamics become more and more complex. Moreover, as well as $\mathbb{E}_{34}$, the fixed point $\mathbb{E}_{11}$ undergoes a border-collision bifurcation at $\frac{a-\Lambda}{2b} = \delta_{lb}K$.

4. Analysis of economic scenarios

The aim of this section is to stress the obtained analytical results. Indeed, given the discontinuous nature of the Dynamical System (13), we will show that our model enables us to explore a large set of economic scenarios due to the occurrence of border-collision bifurcations in addition to the bifurcations emerging in smooth systems. We complement our analysis emphasizing differences and similarities with respect to the results outlined in the Italian banking system by [2,3].

A first feature of our model is that it has many fixed points depending on the region under consideration. In Table 1, we sum-up the real equilibria $\mathbb{E}_{ij} = (L_1^*, L_2^*)$ of System i $(i = 1, 2, 3, 4)$, where the index j numbers them.

As we can observe, depending on the region of definition, the number of fixed points varies from one to four. Thanks to Condition (22), we are able to exclude the case where the market is served by a unique bank. This result on Indonesian banks

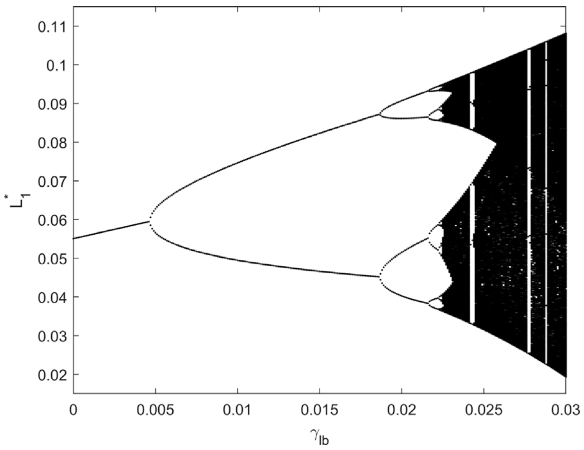

Fig. 2. 1−dimensional bifurcation diagram w.r.t. γ_{lb} for $a = 0.1331$, $b = 0.028$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 400$, $\beta = 0.1$, $K = 0.105$, $\delta_{lb} = 0.78$.

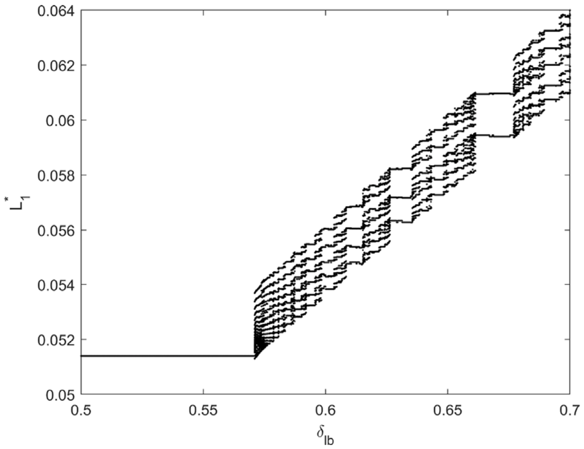

Fig. 3. 1−dimensional bifurcation diagram w.r.t. δ_{lb} for $a = 0.1331$, $b = 0.03$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 30$, $\beta = 0.1$, $K = 0.09$, $\gamma_{lb} = 0.02$.

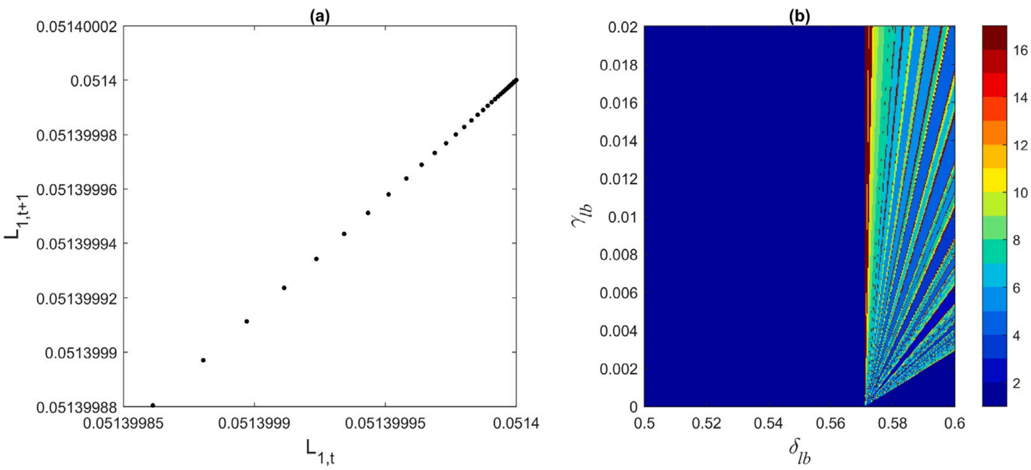

Fig. 4. On the left it is depicted the attractor for $a = 0.1331$, $b = 0.03$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 30$, $\beta = 0.1$, $K = 0.09$, $\gamma_{lb} = 0.02$, $\delta_{lb} = 0.5$. On the right, the corresponding 2−dimensional bifurcation diagram in the plane $(\delta_{lb}, \gamma_{lb})$.

Table 1
Real fixed points belonging to different regions.

	$\mathbb{E}_{i1}$	$\mathbb{E}_{i2}$	$\mathbb{E}_{i3}$	$\mathbb{E}_{i4}$
$\mathbb{E}_{1j}$	$\left(\frac{a-\Lambda}{3b},\frac{a-\Lambda}{3b}\right)$			
$\mathbb{E}_{2j}$	$\left(0,\frac{a-\Lambda}{2b}\right)$			
$\mathbb{E}_{3j}$	$(0,0)$	$\left(\frac{a-\Lambda+r_A\gamma_{lb}}{2b},0\right)$	$\left(0,\frac{a-\Lambda+r_A\gamma_{lb}}{2b}\right)$	$\left(\frac{a-\Lambda+r_A\gamma_{lb}}{3b},\frac{a-\Lambda+r_A\gamma_{lb}}{3b}\right)$
$\mathbb{E}_{4j}$	$\left(\frac{a-\Lambda}{2b},0\right)$			

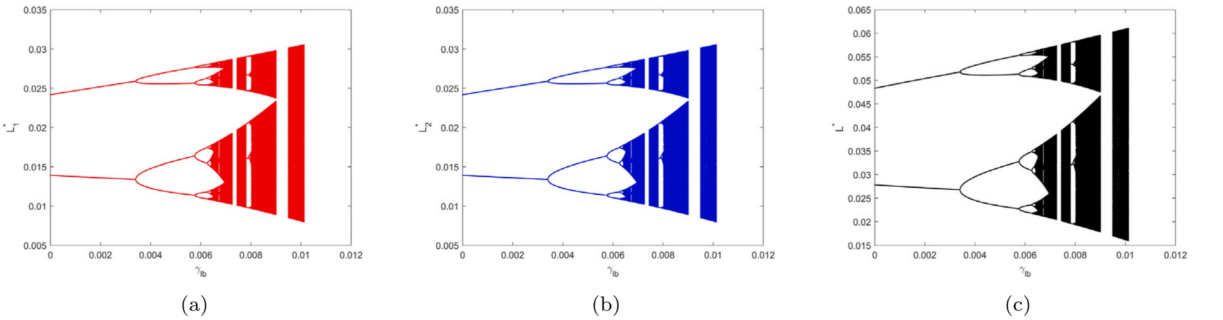

Fig. 5. Bifurcation diagram of the lower disincentive parameter γ_{lb} of (a) $L_{1,t}$, (b) $L_{2,t}$, and (c) $L_t = L_{1,t} + L_{2,t}$. Parameter values: $a = 0.1331$, $b = 0.0755$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 500$, $\beta = 0.1$, $K = 0.09$, $\delta_{lb} = 0.78$, $\Lambda = 0.1285$ and $\gamma_{lb} \in [0; 0.05]$.

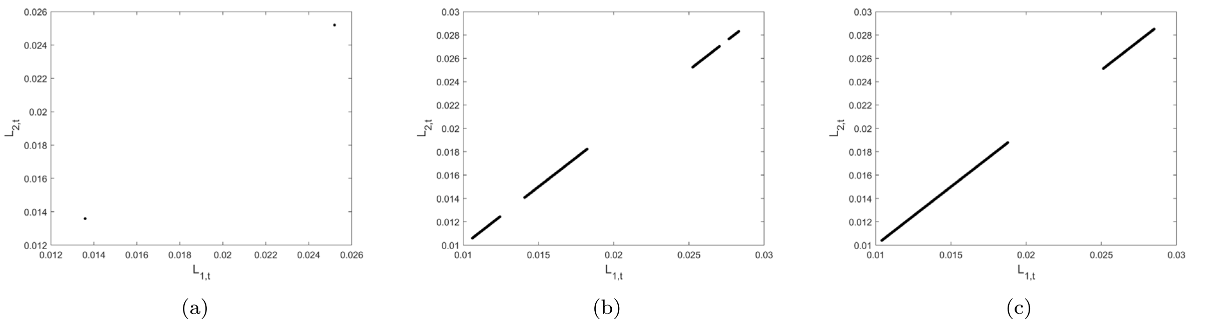

Fig. 6. Attractor of the map in the plane $(L_{1,t}, L_{2,t})$ when $a = 0.1331$, $b = 0.0755$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 500$, $\beta = 0.1$, $K = 0.09$, $\delta_{lb} = 0.78$, $\Lambda = 0.1285$ and $\gamma_{lb} = 0.002$ in (a), $\gamma_{lb} = 0.0067$ in (b), and $\gamma_{lb} = 0.007$ in (c).

is in line with the Italian banks. Although in [2,3] banks are divided into large and small depending on their beliefs and cost functions, the market reaches a better equilibrium if all banks participate in the intermediation process even if all banks are equal.

Based on the outcomes of Corollary 1, in Fig. 5 we report the bifurcation diagrams of the model when we let the lower disincentive parameter to vary in the range $\gamma_{lb} \in [0; 0.05]$. Note that the set of parameter values used in this scenario permits the model to exhibit bifurcations occurring in smooth systems. In line with Remark 1, the equilibrium of the model loses stability via flip bifurcation. From an economic point of view, the situation described in Fig. 5 is useful for three main reasons. First, increasing values of the disincentive parameter mean that the level of intermediation of banks is not appropriate and it needs to foster the lending activity of banks. Second, this scenario is helpful for policy-makers to understand the level of disincentive that banks can tolerate in period of firms' financial distress (where the lending activity is more intense). Third, we are taking into account that banks have a large sensitivity with respect to the environment information (typical when banks are facing a period of financial bust). To this regard, Fig. 5 shows that small increments of the lower disincentive parameter lead to more complicated dynamics. Moreover, the behavior of the banks in the model is symmetric and the loss of stability can amplify the consequences in the whole economy. Fig. 6 confirms the previous facts, indeed, we observe that increasing the value of the lower disincentive parameter the attractor of the system changes its structure from a 2−cycle (Fig. 6(a)) to a four-pieces chaotic attractor (Fig. 6(b)) and to a two-pieces chaotic attractor (Fig. 6(c)). In all these cases, the dynamics of the system is trapped into complex attractors and the path of the loans supplied in the economy is unpredictable.

From these facts, we can draw an important conclusion. If banks are more sensitive to the economic environment, and they have homogeneous expectations, in period of financial distress the choice of the policy-maker to increase the lower disincentive

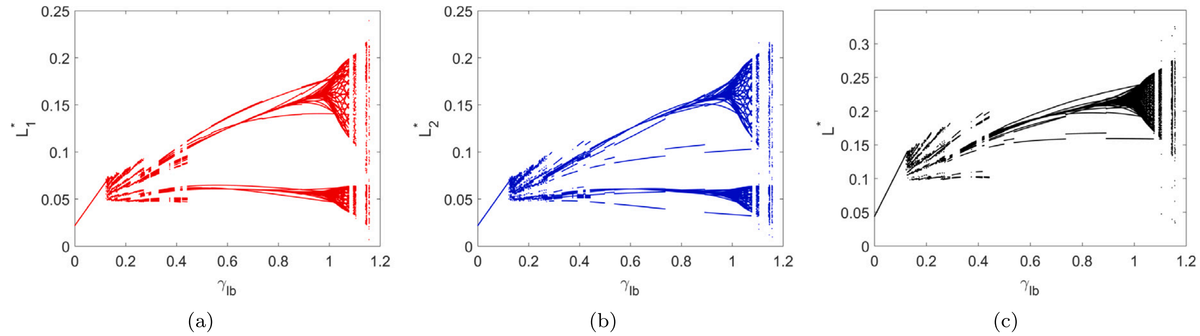

Fig. 7. Bifurcation diagram of the lower disincentive parameter γ_{lb} of (a) $L_{1,t}$, (b) $L_{2,t}$, and (c) $L_t = L_{1,t} + L_{2,t}$. The simulation makes use of $\alpha = 30$.

Table 2
Parameter values for simulations of Figs. 7–9.

Parameter	Value
a	0.133
b	0.07
r_E	0.09
r_A	0.08
κ	0.14
c	0.047
α	Given in each figure's caption
β	0.1
K	0.09
γ_{lb}	0.1 or given in each figure's caption
δ_{lb}	0.78 or given in each figure's caption

parameter could amplify the negative consequences of a contraction of the lending activity in the whole economy. In this regard, it is important the action of the policy-maker to persuade banks to not reduce the intermediation activity, even if it is costly and they have found more profitable opportunity.

The discontinuity structure of our model allows us to study the economic scenario emerging via border-collision bifurcations. The simulations shown in Figs. 7–9 use parameter values as in Table 2 and initial conditions $D_{1,0} = 0.015$, $D_{2,0} = 0.013$, $L_{1,0} = 0.012$, and $L_{2,0} = 0.01$. In all these cases, the first condition of Remark 1 holds. The rationale of this second group of simulations concerns the possibility for the policy-maker to act not only on the lower disincentive parameter (γ_{lb}), but also on the lower bound of LDR parameter (δ_{lb}).

Fig. 7 tells us a different story with respect to what happens in Fig. 5, indeed now the speed of adjustment parameter α is lower than the previous case. As a result, the policy-maker could foster the intermediation activity of banks increasing the lower disincentive parameter. However, when the resistance level of the lower disincentive parameter is violated, we observe a sudden transition from stability to instability.

In Fig. 8 we perform simulations considering larger values of both the speed of adjustment parameter and the lower bound of LDR. In this case, we see that not only the supply of loans increases, but also the possibility for the policy-maker to consider larger value of the lower bound of LDR parameter. These results help us to understand the best choice for sustaining the traditional intermediation activity of banks.

In Fig. 9, we highlight the consequences of a wrong management of the LDR-RR instruments. Indeed, an excessive increase of the lower bound of LDR alters the stability of the economic system, amplifying the period of financial bust. A further failure of the LDR-RR instruments is represented in Fig. 10 and in Fig. 11. In both the simulations, we depict the attractor of system on the left and the corresponding 2−dimensional bifurcation diagram in the plane (δ_{lb}, γ_{lb}), on the right. We can observe the change in the complex attractor from a four-pieces to one-piece and the related bifurcation structures. These graphs give evidence that persisting with the incorrect use of macroprudential tools can increase the level of instability of the banking system. The key point concerns the role of the policy-maker, who has to tune correctly the level of the parameters of the LDR-RR, in order to sustain the intermediation activity of banks and the well-being of the whole economy. Finally, taking into account our results on the LDR-RR instruments, we would propose a specific program that the regulator could apply in order to better control the activity of banks and, as a consequence, the stability of the financial system. In particular, we saw that an excessive increase of the lower disincentive parameter or of the lower bound of LDR could be detrimental, not only for the stability of the system but also for the survivor of banks. In this respect, the regulator could distinguish between banks of different size according to some specific criteria (for example, relying on the number of branches of a bank) and tune the appropriate parameter. In detail, in order to foster the lending activity, the regulator could increase the value of the lower disincentive parameter for the small banks and, on the other hand, extend the level of reserve requirement for large banks. Indeed, it is true that raising the value of the lower disincentive parameter penalizes small banks more than large

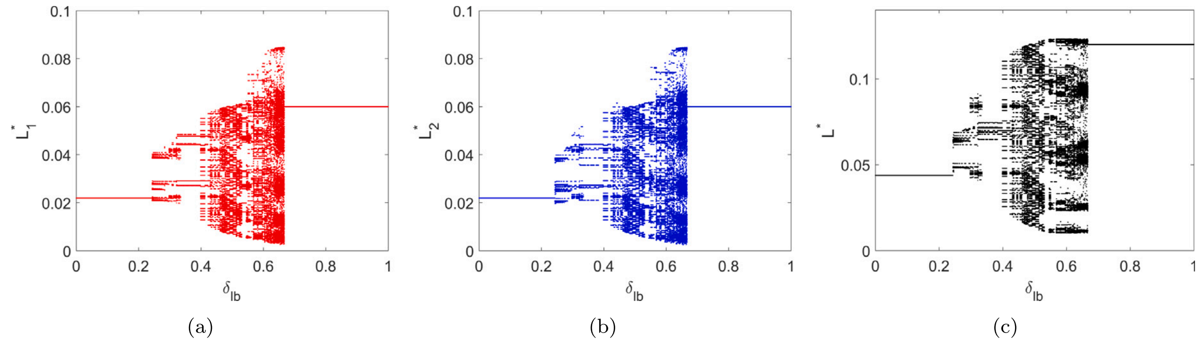

Fig. 8. Bifurcation diagram of the lower bound of LDR parameter δ_{lb} of (a) $L_{1,t}$, (b) $L_{2,t}$, and (c) $L_t = L_{1,t} + L_{2,t}$. The simulation makes use of $\alpha = 110$.

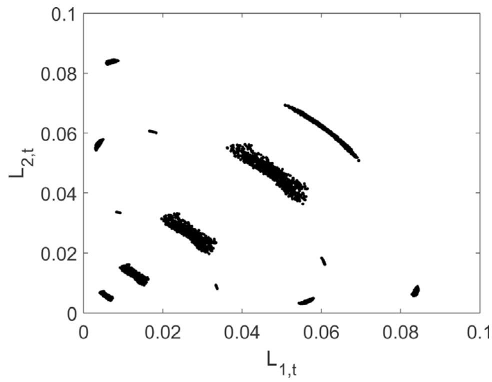

Fig. 9. Chaotic attractor in the $(L_{1,t}, L_{2,t})$−plane when $\alpha = 110$, $\delta_{lb} = 0.64$. The simulation is run on the time interval [501, 5500].

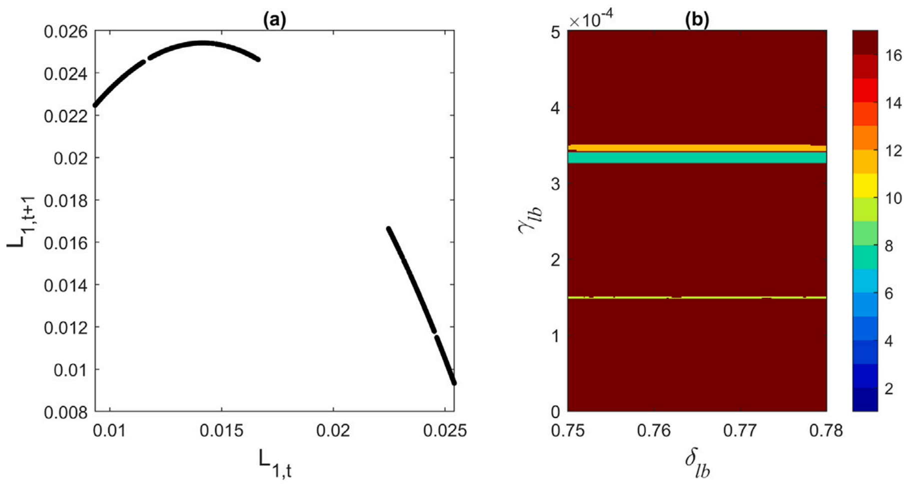

Fig. 10. On the left it is depicted the attractor for $a = 0.1331$, $b = 0.0755$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 560$, $\beta = 0.1$, $K = 0.09$, $\gamma_{lb} = 0.008$, $\delta_{lb} = 0.78$. On the right, the corresponding 2−dimensional bifurcation diagram in the plane $(\delta_{lb}, \gamma_{lb})$.

ones, due to their size. Thus, a common strategy that does not take into account the size of banks could penalize one category more than the other, or it could not have effects.

5. Conclusions and further developments

The dynamics of our model is driven by a two-dimensional discontinuous dynamical model, defined on four different regions, developed in order to analyze the role of the macroprudential policy instrument, the LDR-RR, on the stability of the economic system.

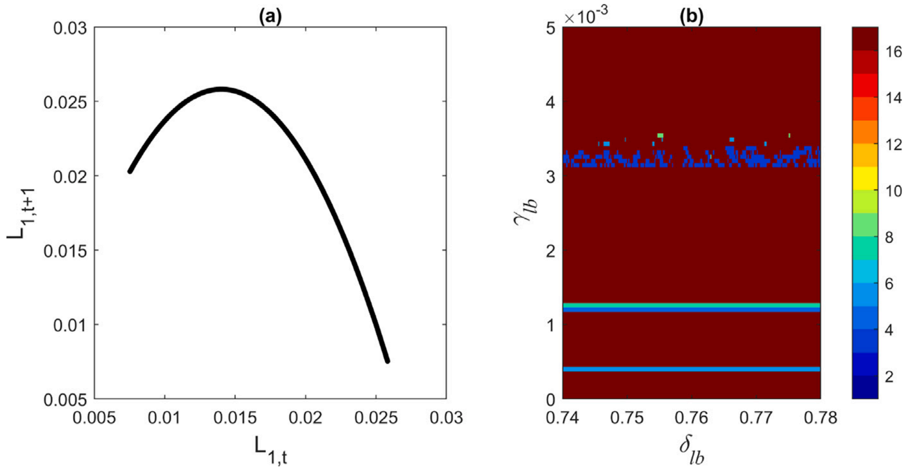

Fig. 11. On the left it is depicted the attractor for $a = 0.1331$, $b = 0.0755$, $r_E = 0.09$, $r_A = 0.0802$, $\kappa = 0.13$, $c = 0.047$, $\alpha = 580$, $\beta = 0.1$, $K = 0.09$, $\gamma_{lb} = 0.002$, $\delta_{lb} = 0.78$. On the right, the corresponding 2−dimensional bifurcation diagram in the plane $(\delta_{lb}, \gamma_{lb})$.

In particular, the final map has been studied analytically, demonstrating that it is able to exhibit border-collision bifurcations, beyond the standard bifurcations of smooth maps. Conditions on the parameters allow us to study the main economic scenarios emerging via bifurcation analysis. Several notable economic results have been obtained. In particular, the role of the lower disincentive parameter and the lower bound of the LDR play a key role in the lending activity of banks. However, an excessive increase of these parameters leads to period of instability of the economic system, emerging via border-collision bifurcations. In this regard, the role of the policy maker is crucial because she must tune correctly these parameters in order to guarantee the stability and the growth of the whole economy.

This model is also suitable for further extensions. In particular, we intend to study in depth the virtual fixed points owned by our model, following e.g. the contribute of [14]). Anyway, our system is characterized by a rich variety of dynamic scenarios and a unique study cannot investigate all of them in detail. For this reason, our framework opens interesting developments.

Finally, in this paper we focus on the role of two important parameters: the lower disincentive parameter and the lower bound of the LDR. In the future, we will also consider the contribute of the upper bound and incentive CAR.

CRediT authorship contribution statement

Moch. Fandi Ansori: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Serena Brianzoni:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Giovanni Campisi:** Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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