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# A robust least squares based approach to min-max model predictive control

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## SUMMARY

This paper deals with the Model Predictive Control (MPC) of Linear, Time-Invariant Discrete-time Polytopic (LTIDP) systems. The twofold aim is to simplify the treatment of complex issues like stability and feasibility analysis of MPC in the presence of parametric uncertainty as well as to reduce the complexity of the relative optimization procedure. The new approach is based on a two Degrees Of Freedom (2DOF) control scheme where the output  $r(k)$  of the feedforward Input Estimator (IE) is used as input forcing the closed-loop system  $\Sigma_f$ .  $\Sigma_f$  is the feedback connection of an LTIDP plant  $\Sigma_p$  with an LTI feedback controller  $\Sigma_g$ . Both cases of plants with measurable and unmeasurable state are considered. The task of  $\Sigma_g$  is to guarantee the quadratic stability of  $\Sigma_f$ , as well as the fulfillment of hard constraints on some physical variables for any input  $r(k)$  satisfying an "a priori" determined admissibility condition. The input  $r(k)$  is computed by the feedforward IE through the on-line minimization of a worst case finite-horizon quadratic cost functional and is applied to  $\Sigma_f$  according to the usual receding horizon strategy. The on line constrained optimization problem is here simplified reducing the number of the involved constraints and decision variables. This is obtained modeling  $r(k)$  as a B-spline function, which is known to admit a parsimonious parametric representation. This allows us to reformulate the minimization of the worst case cost functional as a box-constrained Robust Least Squares (RLS) estimation problem which can be efficiently solved using Second Order Cone Programming (SOCP). Copyright © 0000 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

In the case of plants affected by parametric uncertainty and/or unknown bounded disturbances, a common approach to robust MPC is its formulation in terms of a min-max constrained optimization problem. The methods derived from this approach can be classified into two main categories: open and closed-loop Min-Max MPC (MMMPC).

Open-loop MMMPC is based on the minimization of the "worst-case" objective functional using a single control sequence [1]-[5]. The inconvenience of this approach is the conservatism due to the open-loop nature of the optimization problem: the optimum is searched as a single control sequence for all possible uncertainties. An improvement is represented by the closed-loop min-max approach where the "worst-case" objective functional is minimized with respect to a sequence of feedback control policies [6]-[8].

Both approaches inherit in a considerably increased way the major issues of MPC for exactly known plants: more complicated stability and feasibility conditions and, especially, much more computationally demanding procedures for the numerical solution of the on line optimization problem. In fact the MMMPC requires minimizing the worst case of a cost functional which is computed as the maximum with respect to all the possible uncertainties over the prediction horizon.

To reduce the computational burden, an approximate solution of the MMMPC is obtained in [9] where an upper bound of the worst case cost functional is minimized using LMI techniques. To further improve computational efficiency, different kinds of cheap upper bound of the worst case with relative minimization procedures have been proposed in [10]-[14]. A different approach exploits the results presented in [15]-[16] where it is shown that a piece-wise affine state-feedback control law can be explicitly precomputed off-line through multi parametric programming. Such results have been extended to MMMPC with  $\ell_1$  or  $\ell_\infty$  norm functional in [17]-[18] and in [19]-[20] with quadratic cost functional. However, explicit formulations of MMMPC require the partition of

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the state space into polyhedral regions whose number is growing in a combinatorial explosion. To reduce the corresponding searching time a search tree structure has been proposed in [16].

The twofold purpose of this paper is: 1) to propose a novel MMMPC strategy characterized by greatly simplified stability and feasibility analysis, 2) to reduce the complexity of the on line constrained optimization procedure.

The basic point of the alternative approach proposed here is the adoption of an MPC strategy in a 2DOF control scheme to exploit the advantages of feedback prediction and of the degrees of freedom introduced by the feedforward IE.

In practice, the present MMMPC works according to the following two-step procedure:

Step 1. Given an LTIDP plant  $\Sigma_p$ , a LTI feedback controller  $\Sigma_g$  is designed to guarantee the quadratic stability of the closed-loop system  $\Sigma_f$  and the fulfillment of hard constraints on some physical variables in correspondence of any admissible input ( i.e.  $\|r(k)\|_2^2 < \gamma, \forall k > 0$ , for a suitably computed  $\gamma$ ) forcing  $\Sigma_f$ .

Step 2. An admissible input sequence  $r(k)$  is applied to  $\Sigma_f$  according to a receding horizon control strategy. This sequence is computed searching for the minimum of a "worst case" quadratic cost functional over each prediction interval in the linear space generated by B-spline functions of a fixed degree. This second step is executed by the feedforward IE.

Decomposing the MMMPC problem in the above two distinct steps entails the following remarkable advantages:

- 1) The internal stability of  $\Sigma_f$  and the admissibility condition on  $r(k)$  assure both the uniform boundedness of any internal variable of the 2DOF control scheme and the fulfillment of all constraints at any time instant. Hence, stability and recursive feasibility of the adopted MMMPC strategy are guaranteed in advance without imposing terminal state constraints and/or using a contraction approach for a suitable tuning of the parameters defining the cost functional.
- 2) If  $\Sigma_g$  also contains an internal model of the desired (and admissible) reference  $y_d(k)$  to be tracked, an exact asymptotic tracking can be directly achieved even in the case of plant-model mismatch, [21].

This greatly simplifies the alternative solutions to the tracking problem where additional decision variables relative to the dynamics of  $y_d(k)$  are introduced in the optimization procedure. A drawback of these methods is that they increase the number of decision variables involved in the optimization problem (see e.g. [22] and references therein). The internal model also yields a  $\Sigma_f$  with a diagonal static gain matrix, so that it guarantees the noticeable advantage of an exact static decoupling, [23].

3) Modeling  $r(k)$  as a B-spline decreases the number of decision variables because these functions admit a parsimonious parametric representation and belong to the convex hull defined by the relative control points, [24]. This property allows the transfer of any amplitude constraint defined on a B-spline function to its control points. As a consequence (see Section 5), the constrained minimization of the cost functional can be formulated as a box-constrained RLS estimation problem with only box constraints on the unknowns (the control points defining the admissible B-spline function  $r(k)$ ). Approaching this problem by SOCP allows the application of numerically efficient primal-dual interior-point methods ([25],[26]).

The paper is organized in the following way. Some mathematical preliminaries are recalled in Section 2, the control problem is defined in Section 3. The design of the stabilizing controller  $\Sigma_g$  is shown in Section 4 considering both cases of measurable and unmeasurable state; the on line estimation of  $r(k)$ , produced by the feedforward IE, is explained in Section 5. Some issues on the tracking problem are discussed in Section 6; two numerical examples are reported in Section 7, some concluding remarks are given in Section 8.

## 2. MATHEMATICAL BACKGROUND

### 2.1. B-spline functions [24]

Analytic scalar B-splines functions are defined in the following way:

$$s(v) = \sum_{i=1}^{\ell} c_i B_{i,d}(v), \quad v \in [\hat{v}_1, \hat{v}_{\ell+d+1}] \subseteq \mathbb{R}, \quad (1)$$

where the  $c_i$ 's are real numbers representing the control points of  $s(v)$ , the integer  $d$  is the degree of the spline, the  $(\hat{v}_i)_{i=1}^{\ell+d+1}$  are the non decreasing knot points and the  $B_{i,d}(v)$  are given by the Cox-de Boor recursion formula

$$B_{i,d}(v) = \frac{v - \hat{v}_i}{\hat{v}_{i+d} - \hat{v}_i} B_{i,d-1}(v) + \frac{\hat{v}_{i+1+d} - v}{\hat{v}_{i+1+d} - \hat{v}_{i+1}} B_{i+1,d-1}(v), \quad d \geq 1, \quad (2)$$

with  $B_{i,0}(v) = 1$  if  $\hat{v}_i \leq v < \hat{v}_{i+1}$ , otherwise 0.

In (2) possible division by zero are resolved by the convention that "anything divided by zero is zero".

*Convex Hull Property.* Any value assumed by  $s(v)$ ,  $\forall v \in [\hat{v}_j, \hat{v}_{j+1}]$ ,  $j > d$ , lies in the convex hull of its  $d + 1$  control points  $c_{j-d}, \dots, c_j$ .  $\triangle$

Identifying the parameter  $v$  of (1) with the time instant  $t$ , the sampled B-spline  $s(k T_c)$  is obtained by direct uniform sampling of the corresponding analytic B-spline.

The discrete B-spline  $s(k)$  (omitting the explicit dependence on  $T_c$ ) can be used to represent a scalar discrete time signal. Defining

$$\mathbf{c} \triangleq [c_1 \cdots c_\ell]^T, \quad \mathbf{B}_d(k) \triangleq [B_{1,d}(k) \cdots B_{\ell,d}(k)], \quad (3)$$

where each  $B_{i,d}(k)$  is obtained by (2) setting  $v = k$  and  $\hat{v}_i = \hat{k}_i$ ,  $i = 1, \dots, d + \ell + 1$ , the sampled B-spline  $s(k)$  can be represented as

$$s(k) = \mathbf{B}_d(k) \mathbf{c}, \quad k \in [\hat{k}_1, \hat{k}_{\ell+d+1}]. \quad (4)$$

For a  $q$ -component vector  $\mathbf{s}(k) = [s_1(k), \dots, s_q(k)]^T$ , a compact B-splines representation can be used

$$\mathbf{s}(k) = \bar{\mathbf{B}}_d(k) \bar{\mathbf{c}}, \quad k \in [\hat{k}_1, \hat{k}_{\ell+d+1}], \quad (5)$$

where

$$\bar{\mathbf{c}} \triangleq [\mathbf{c}_1^T, \dots, \mathbf{c}_q^T]^T, \quad \bar{\mathbf{B}}_d(k) \triangleq \text{diag} [\mathbf{B}_d(k), \mathbf{B}_d(k), \dots, \mathbf{B}_d(k)]. \quad (6)$$

Each  $\mathbf{c}_i \triangleq [c_{i,1}, \dots, c_{i,\ell}]^T$ ,  $i = 1, \dots, q$ , is defined as in (3). The dimensions of  $\bar{\mathbf{c}}$  are  $(q\ell \times 1)$ . The dimensions of the block diagonal matrix  $\bar{\mathbf{B}}_d(k)$  are  $(q \times q\ell)$ .

**Remark 1.** From (4) it is apparent that, once the degree  $d$  and the knot points  $\hat{k}_i$  have been fixed, the

B-spline  $s(k)$ ,  $k \in [\hat{k}_1, \hat{k}_{\ell+d+1}]$ , is completely determined by the corresponding vector  $\mathbf{c}$  of  $\ell$  control points. As, in general,  $\ell \ll k_M$ , where  $k_M$  is the number of sampled instants of  $[\hat{k}_1, \hat{k}_{\ell+d+1}]$ , B-splines are said to admit a parsimonious parametric representation.

## 2.2. SOCP formulation of the RLS problem [25],[26]

Given an overdetermined set of linear equations  $Df \approx g$ , with  $D \in \mathbb{R}^{r \times s}$ ,  $g \in \mathbb{R}^r$ , subject to unknown but bounded errors:  $\|\delta D \ \delta g\|_F \leq \rho$ , ( $\rho > 0$ ), the robust least squares estimate  $\hat{f} \in \mathbb{R}^s$  is the value of  $f$  minimizing

$$\phi(D, g, \rho) \triangleq \min_f \max_{\|\delta D \ \delta g\|_F \leq \rho} \|(D + \delta D)f - (g + \delta g)\|_2, \quad (7)$$

where  $\|\cdot\|_F$  denotes the Frobenius norm.

Assuming  $\rho = 1$ , in [25] it is shown that problem (7) can be formulated as the following SOCP

$$\text{minimize } \lambda \quad \text{subject to } \|(Df - g)\|_2 \leq \lambda - \tau, \|[f^T, 1]^T\|_2 \leq \tau,$$

which can be efficiently solved using interior point methods. Possible constraints on  $f$  of the kind  $f_{min} \leq f \leq f_{max}$ , can be taken into account by imposing all the scalar linear inequalities deriving from the above vector constraint.

The solution of the above SOCP can be directly extended to the case  $\rho \neq 1$ , using the fact:  $\rho\phi(D/\rho, g/\rho, 1) = \phi(D, g, \rho)$ .

## 2.3. System constraints and invariant sets for polytopic systems

Consider the LTIDP system  $\Sigma \equiv (C, A(\alpha), B)$  given by

$$x(k+1) = \left( \sum_{i=1}^l \alpha_i A_i \right) x(k) + Br(k), \quad (8)$$

$$y(k) = Cx(k) \quad (9)$$

where  $x(k) \in \mathbb{R}^n$ ,  $r(k) \in \mathbb{R}^q$ ,  $y(k) \in \mathbb{R}^q$  and the vector  $\alpha = [\alpha_1, \dots, \alpha_l]^T$ , belongs to the unit simplex (denoted by  $\Lambda_l$ ).

An invariant  $\gamma$ -feasible set of  $\Sigma$  is a convex compact set  $\mathcal{X}$  containing the origin, such that, for every

input  $r(k)$ , satisfying the following admissibility condition

$$r^T(k)r(k) \triangleq \|r(k)\|_2^2 \leq \gamma, \quad \forall k > 0, \text{ and for some } \gamma > 0, \quad (10)$$

one has  $x(k) \in \mathcal{X} \Rightarrow A(\alpha)x(k) + Br(k) \in \mathcal{X}, \forall \alpha \in \Lambda_l$ , and the following constraint is satisfied

$$|z_i(k)| = \|z_i(k)\|_2 \leq \bar{z}_i, \quad i = 1, \dots, h, \quad (11)$$

where  $z_i(k)$  is the  $i$ -th element of the  $h$ -vector  $z(k) = C_z x(k)$ , and  $\bar{z}_i$  is the corresponding pre-specified hard constraint. Vector  $z(k)$  defines the constrained state variables corresponding to some suitably chosen  $C_z$ . Here  $\mathcal{X}$  is assumed to be an ellipsoid set defined as  $\mathcal{E}(P, \gamma) = \{x(k) \mid x(k)^T P x(k) \leq \gamma\}$ , where  $P \triangleq Q^{-1}$  is a symmetric positive definite matrix.

### 3. PROBLEM SETUP

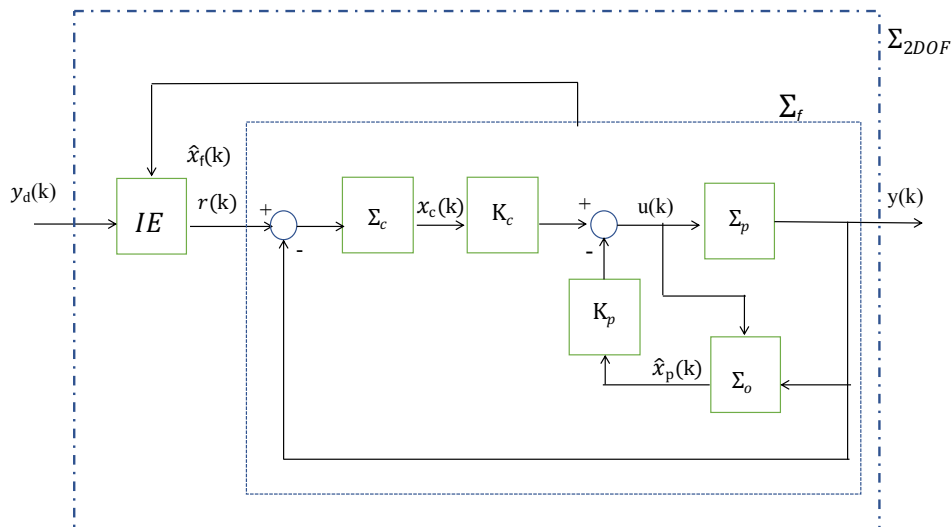


Figure 1. The 2DOF control scheme.

The MMMPC strategy proposed in this paper is realized through the 2DOF control scheme shown in Fig. 1 where:  $y_d(k)$  is the piece-wise constant desired reference to be tracked and  $y(k)$  is the controlled output. The output of the feedforward IE is the input  $r(k) \in \mathbb{R}^q$  forcing  $\Sigma_f$ .

The block  $\Sigma_f$  is the feedback connection of an LTIDP plant  $\Sigma_p \equiv (C_p, A_p(\alpha), B_p)$ ,  $\alpha \in \Lambda_l$ , of the



same kind of (8),(9)

$$x_p(k+1) = \left( \sum_{i=1}^l \alpha_i A_{p_i} \right) x_p(k) + B_p u(k), \quad x_p \in \mathbb{R}^{n_p}, \quad u \in \mathbb{R}^m, \quad \alpha \in \Lambda_l, \quad (12)$$

$$y(k) = C_p x_p(k), \quad y \in \mathbb{R}^q, \quad (13)$$

with a dynamic LTI controller  $\Sigma_g$  which includes the internal model of constant signals  $\Sigma_c$  and a full state observer  $\Sigma_o$  (if  $x_p(k)$  is not measurable).

The state vectors of  $\Sigma_c$ ,  $\Sigma_p$  and  $\Sigma_f$  are denoted by  $x_c(k)$ ,  $x_p(k)$  and  $x_f(k)$  respectively. The vectors  $\hat{x}_p(k)$  and  $\hat{x}_f(k)$  are the estimates of  $x_p(k)$  and  $x_f(k)$ . The control input forcing the LTIDP plant  $\Sigma_p$  is denoted as  $u(k) \in \mathbb{R}^m$ .

In view of an exact asymptotic tracking requirement for constant signals, the following assumptions on  $\Sigma_p$  are made: A1)  $m \geq q$ ; A2)  $\Sigma_p$  has not a transmission zero at  $z = 1$  of  $Z$  plane  $\forall \alpha \in \Lambda_l$ . The explicit expressions of  $\Sigma_c$  and  $\Sigma_o$  will be given in the next section. As  $\Sigma_g$  is LTI and independent of  $\alpha$ , also  $\Sigma_f \equiv (C_f, A_f(\alpha), B_f)$  results to be an LTIDP system of the same kind of  $\Sigma$  given by (8)-(9). The purpose of  $\Sigma_g$  is to guarantee the fulfillment of the following requirements:

- r1) quadratic stability of  $\Sigma_f$ ;
- r2) the existence of an invariant  $\gamma$ -feasible set  $\mathcal{X}$  for  $\Sigma_f$ , such that  $x_f(k) \in \mathcal{X} \Rightarrow A_f(\alpha)x_f(k) + B_f r(k) \in \mathcal{X}, \forall \alpha \in \Lambda_l$ , and constraints like (11) are satisfied by each component of the vector  $z_f(k) = C_{z_f} x_f(k)$ , for any admissible input  $r(k)$  of  $\Sigma_f$  satisfying (10).

Vector  $z_f(k)$  defines the constrained variables corresponding to some suitably defined  $C_{z_f}$ . Asymptotic tracking of a fixed set point  $y_d(k) = y_d$ , can be obtained as a consequence of r1, of assumptions A1) and A2) and of the introduction of  $\Sigma_c$  ([21]), provided that  $r(k)$  converge to  $y_d$ . In the case of a piece-wise constant  $y_d(k)$ , each set point is almost exactly achieved provided it is kept over a sufficiently long time interval.

The inputs of IE are  $y_d(k)$  and  $\hat{x}_f(k)$ . This information is exploited by the IE to compute  $r(k)$  solving the following Min-Max Constrained Optimization Problem (MMCOP) at each  $k = jN_r$ , for some  $N_r > 0, j = 0, 1, 2 \dots$ ,

$$\text{MMCOP:} \quad \min_{[r(k), \dots, r(k+N_y-1)]} \max_{\alpha \in \Lambda_l} J_\alpha,$$

$$\begin{aligned}
J_\alpha &\triangleq \sum_{i=1}^{N_y} e_y^T(k+i|k)Q_y(k)e_y(k+i|k) \\
&\quad + \lambda_1(k) \sum_{i=0}^{N_y-1} e_r^T(k+i|k)Q_r(k)e_r(k+i|k) \\
&\quad + \lambda_2(k) \sum_{i=1}^{N_u} e_u^T(k+i|k)Q_u(k)e_u(k+i|k),
\end{aligned} \tag{14}$$

where  $Q_y(k)$ ,  $Q_r(k)$  and  $Q_u(k)$  are positive definite matrices and

$$N_y \geq N_u, N_y \geq N_r, \lambda_1(k) \geq 0, \lambda_2(k) \geq 0, k \geq 0 \tag{15}$$

$$e_y(k+i|k) \triangleq y_d(k) - y(k+i|k), \tag{16}$$

$$e_r(k+i|k) \triangleq y_d(k) - r(k+i), \tag{17}$$

$$e_u(k+i|k) \triangleq u(k+i|k) - \tilde{u}(k), \tag{18}$$

subject to

$$r_{min} \leq r(k+i) \leq r_{max}, \quad i = 0, \dots, N_y - 1. \tag{19}$$

In the above equations  $\tilde{u}(k)$  is the steady-state value of  $u(k)$  corresponding to a suitably defined nominal plant,  $y(k+i|k)$ ,  $u(k+i|k)$  and  $r(k+i)$  are the predicted output, control effort and B-spline respectively,  $r_{min}$  and  $r_{max}$  are q-vectors computed so as to satisfy (10).

Note that in equations (16),(17), the reference trajectory is evaluated at time instant  $k$  to avoid undesired anticipative effects on  $y(k)$  due to possible set point changes inside the prediction horizon  $N_y$ .

The MMCOP is solved at each time instant  $k = jN_r$  and only the first  $N_r$  samples of the whole sequence  $[r(k), \dots, r(k+N_y-1)]$  are applied to  $\Sigma_f$  according to the receding horizon control policy.

**Remark 2.** The considerations developed in this section clearly show the idea underlying the present approach and the relative advantages of the resulting MMMPC procedure. Designing  $\Sigma_g$  according to r1 guarantees the uniform boundedness of  $x_f(k)$  for any uniformly bounded  $r(k)$ , independently of  $N_y$ ,  $N_r$ ,  $N_u$ ,  $\lambda_1(\cdot)$ ,  $\lambda_2(\cdot)$ ,  $Q_u(\cdot)$ ,  $Q_r(\cdot)$  and  $Q_y(\cdot)$ . This releases the stability issue from the prediction horizon and other tuning parameters. Requirement r2 allows us to transfer any constraint

on  $z_f(k)$  of the kind (11) on a corresponding upper bound  $\gamma$  on  $\|r(k)\|_2^2$ . Namely, unlike the other approaches, the constrained variables coincide with those ones with respect to the optimization problem has to be solved. The bound  $\gamma$  is explicitly taken into account in the MMCOP through the feasible constraints (19). As formally stated in Theorem 2 of Section 5, this implies that the proposed two-step procedure yields an MMMPC strategy with guarantee of internal stability of  $\Sigma_{2DOF}$  and recursive feasibility.

**Remark 3.** The presence of the internal model  $\Sigma_c$  guarantees exact asymptotic tracking if  $r(k)$  exactly converges to the desired set point value. The penalty term  $\lambda_1(k) \sum_{i=0}^{N_y-1} e_r^T(k+i|k) Q_r(k) e_r(k+i|k)$  is useful to speed up such a convergence. This is particularly important in the case of piecewise constant signals  $y_d(k)$  which are not frozen on a fixed set point for a sufficiently long time interval and tracking precision is the dominant criterion.

#### 4. STEP 1: DESIGN OF $\Sigma_g$

The LTI feedback controller  $\Sigma_g$  is designed here using ellipsoidal robust invariant sets because of their closed relation to quadratic Lyapunov functions leading to an LMI-based optimization problem.

##### 4.1. Unmeasurable state

The controller  $\Sigma_g$  includes the internal model of constant signals  $\Sigma_c$ , whose state-space representation is  $x_c(k+1) = A_c x_c(k) + B_c(r(k) - y(k))$ , ( $A_c = B_c = I_q$ ),  $x_c \in \mathbb{R}^{n_c=q}$  and a full state observer  $\Sigma_o$  of the form

$$\hat{x}_p(k+1) = \bar{A}_p \hat{x}_p(k) + B_p u(k) + L(y(k) - C_p \hat{x}_p(k)), \quad \hat{x}_p \in \mathbb{R}^{n_p} \quad (20)$$

where:  $\bar{A}_p \triangleq (\sum_{i=1}^l A_{p_i})/l$  is the assumed nominal dynamical matrix of the plant.

According to Fig. 1, the output  $u(k) \in \mathbb{R}^m$  of  $\Sigma_g$  forcing the polytopic plant  $\Sigma_p$  is given by

$$u(k) = -K_p \hat{x}_p(k) + K_c x_c(k). \quad (21)$$

The state space representation  $(C_f, A_f(\alpha), B_f)$  of the square closed loop system  $\Sigma_f$  with  $x_f \triangleq [\hat{x}_p^T, x_c^T, x_p^T - \hat{x}_p^T]^T \in \mathbb{R}^n$  and  $n \triangleq 2n_p + n_c$  is

$$x_f(k+1) = \begin{bmatrix} \bar{A}_p - B_p K_p & B_p K_c & LC_p \\ -B_c C_p & A_c & -B_c C_p \\ \Delta A_p(\alpha) & \mathbf{0} & A_p(\alpha) - LC_p \end{bmatrix} x_f(k) + \begin{bmatrix} \mathbf{0} \\ B_c \\ \mathbf{0} \end{bmatrix} r(k) \quad (22)$$

$$y(k) = \begin{bmatrix} C_p & \mathbf{0} & C_p \end{bmatrix} x_f(k) \quad (23)$$

where  $\Delta A_p(\alpha) \triangleq A_p(\alpha) - \bar{A}_p$  and  $\mathbf{0}$  denotes the null matrix.

The constrained state is

$$z_f(k) \triangleq [z_u^T(k), z_{x_f}^T(k)]^T, z_u(k) \in \mathbb{R}^{n_u}, z_{x_f}(k) \in \mathbb{R}^{n_{x_f}} \quad (24)$$

where the respective components  $z_{u,r}(k)$  and  $z_{x_f,w}(k)$  ( $r = 1, \dots, n_u$ ,  $w = 1, \dots, n_{x_f}$ ) have to satisfy constraints like (11) for some given  $\bar{z}_{u,r}$  and  $\bar{z}_{x_f,w}$  respectively. Typically  $z_u(k) = C_{z_u} x_f(k) = u(k)$ , so that, by (21),  $n_u = m$  and  $C_{z_u} = \begin{bmatrix} -K_p & K_c & \mathbf{0} \end{bmatrix} \triangleq \hat{K}$  while  $z_{x_f}(k) = C_{z_{x_f}} x_f(k)$  represents any vector of variables linearly depending on the state. For example if  $z_{x_f}(k) = y(k)$  then by (23),  $n_{x_f} = q$  and  $C_{z_{x_f}} = C_f = \begin{bmatrix} C_p & \mathbf{0} & C_p \end{bmatrix}$ .

It is remarked that the above distinction between  $z_u(k)$  and  $z_{x_f}(k)$ , is necessary because, unlike  $z_{x_f}(k)$ ,  $z_u(k) \triangleq u(k)$ , depends on  $x_f(k)$  through of a matrix which is a design parameter. Such a matrix has to be determined imposing the fulfillment of the control specifications.

Once  $\Sigma_c$  has been designed according to the internal model principle, the controller gain matrices are computed as specified beneath.

*Design of the controller gains.*

For any fixed matrix  $L$  of the observer (20), the gain matrix  $\begin{bmatrix} -K_p & K_c & \mathbf{0} \end{bmatrix} \triangleq \hat{K}$  can be computed observing that by (22) the polytopic closed loop dynamical matrix  $A_f(\alpha)$  can be rewritten as  $A_f(\alpha) \triangleq \hat{A}(\alpha) + \hat{B}\hat{K}$ , where

$$\hat{A}(\alpha) = \begin{bmatrix} \bar{A}_p & \mathbf{0} & LC_p \\ -B_c C_p & A_c & -B_c C_p \\ \Delta A_p(\alpha) & \mathbf{0} & A_p(\alpha) - LC_p \end{bmatrix}, \hat{B} = \begin{bmatrix} B_p \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (25)$$

Equations (25) are used to design  $\Sigma_g$  according to the following procedure, which can be devised as a sort of separation principle working for sufficiently small parametric uncertainty:

- i) The observer gain matrix  $L$  is chosen such that  $(A_p(\alpha) - LC_p)$  is quadratically stable  $\forall \alpha \in \Lambda_l$ .
- ii) Once the observer  $\Sigma_o$  has been designed, the gain matrix  $\hat{K}$  is computed as solution of the following problem.

**P1** Given the polytopic plant  $(\hat{A}(\alpha), \hat{B})$  in (25), find a matrix  $\hat{K}$  and the maximum invariant  $\gamma$ -feasible set  $\mathcal{X}$  (where also  $\gamma$  is maximized), such that the following conditions are satisfied:

- $\Sigma_f \equiv (C_f, A_f(\alpha), B_f) \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$  is quadratically stable  $\forall \alpha \in \Lambda_l$ ;
- constraints on  $z_f(k)$  are fulfilled for every initial condition  $x_f(0) \in \mathcal{X}$ ,  $\forall \alpha \in \Lambda_l$  and every admissible input  $r(k)$  satisfying (10).

**Remark 4** Since in the augmented state  $x_f$  only the plant state,  $x_p$ , is of interest, instead of maximizing the entire ellipsoid volume only the ellipsoid projection on  $x_p$  subspace is maximized. The projection of  $\mathcal{X}$  onto  $x_p$  is given by  $\mathcal{X}_{x_p} \triangleq \mathcal{E}_{x_p}(P, \gamma) = \{x_p(k) \mid x_p(k)^T (T_{x_p} Q T_{x_p}^T)^{-1} x_p(k) \leq \gamma\}$  with  $T_{x_p}$  defined by  $x_p = T_{x_p} x_f$ .

**Theorem 1** Consider the plant  $(\hat{A}(\alpha), \hat{B})$  in (25) and define  $\eta$  as  $\eta \triangleq \gamma^{-1}$ . Quadratic stability and the invariant  $\gamma$  feasible set  $\mathcal{X}$  (where both  $\mathcal{X}_{x_p}$  and  $\gamma$  are maximized) for  $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$  subject to (11) and forced by any  $r(k)$  satisfying (10), are obtained by solving the following semidefinite programming problem:

$$\text{minimize } (-\log(\det(T_{x_p} Q T_{x_p}^T)) + \eta) \quad \text{subject to:}$$

$$\begin{bmatrix} Q & \mathbf{0} & \beta Q & Q\hat{A}_i^T + Y^T\hat{B}^T \\ \mathbf{0} & \beta I & \mathbf{0} & B_f^T \\ \beta Q & \mathbf{0} & \beta Q & \mathbf{0} \\ \hat{A}_i Q + \hat{B}Y & B_f & \mathbf{0} & Q \end{bmatrix} \geq 0, \quad i = 1, \dots, l \quad (26)$$

$$\begin{bmatrix} Q & Y^T I_r^T \\ I_r Y & \bar{z}_{u,r}^2 \eta \end{bmatrix} \geq 0, \quad r = 1, \dots, m \quad (27)$$

$$\begin{bmatrix} Q & (Q\hat{A}_i^T + Y^T\hat{B}^T)C_{z_{x_f}}^T I_w^T \\ I_w C_{z_{x_f}}(\hat{A}_i Q + \hat{B}Y) & \bar{z}_{x_f,w}^2 \eta - I_w C_{z_{x_f}} B_f B_f^T C_{z_{x_f}}^T I_w^T \end{bmatrix} \geq 0, \quad i = 1, \dots, l, \quad w = 1, \dots, n_{x_f} \quad (28)$$

in the variables  $\eta > 0$ ,  $0 < \beta < 1$ ,  $Q = Q^T = \text{diag}[Q_1, Q_2] \in \mathbb{R}^{n \times n}$ ,  $n = 2n_p + n_c$  and  $Y = [Y_1 \ \mathbf{0}] \in \mathbb{R}^{m \times n}$ ,  $Y_1 \in \mathbb{R}^{m \times (n_p + n_c)}$  and in the vertices

$$\hat{A}_i \triangleq \begin{bmatrix} \bar{A}_p & \mathbf{0} & LC_p \\ -B_c C_p & A_c & -B_c C_p \\ A_{p_i} - \bar{A}_p & \mathbf{0} & A_{p_i} - LC_p \end{bmatrix}$$

and the row vector  $I_r$  ( $I_w$ ) is composed of all null elements save the element 1 in the  $r$ -th ( $w$ -th) position.  $\triangle$

If the set of inequalities admits a solution then the quadratically stabilizing feedback gain  $\hat{K} = YQ^{-1} = [Y_1 Q_1^{-1} \ \mathbf{0}]$  is found. The maximum admissible value  $\gamma = \eta^{-1}$  is found for  $r(k)$  and the invariant  $\gamma$ -feasible set  $\mathcal{X} \equiv \mathcal{E}(P, \gamma)$  with  $P = Q^{-1}$  for  $\Sigma_f$  is obtained.

**Proof of Theorem 1.** For the sake of brevity, the details of the proof are not reported. The theorem can be proved along the lines provided in [27] (see Theorem 1) with some modifications due to: 1) according to Remark 4, a projected ellipsoid is here used, 2) in [27]  $\|r\|_2^2$  is overbounded by 1, here  $\|r\|_2^2$  is overbounded by a scalar  $\gamma$  which is not fixed "a priori" but is maximized including  $\eta = \gamma^{-1}$  in the functional to be minimized; 3) in [27] an euclidean norm bound is imposed to the constrained state  $z(k)$ , here component-wise bounds are more realistically considered.  $\triangle$

**Remark 5** The presence of  $\beta$  makes inequality (26) a BMI, which can be transformed into a LMI through a gridding over the interval  $(0, 1)$  where  $\beta$  takes values.  $\triangle$

#### 4.2. Measurable state

If the state is accessible for measure (namely  $C_p = I_{n_p}$ ), there is no need of an observer. As  $\hat{x}_p(k) = y(k) = x_p(k)$ , the control input  $u(k)$  forcing  $\Sigma_p$  is given by  $u(k) = -K_p x_p(k) + K_c x_c(k)$  and the state space representation  $(C_f, A_f(\alpha), B_f)$  of  $\Sigma_f$  with  $x_f \triangleq [x_p^T, x_c^T]^T \in \mathbb{R}^n$  and  $n \triangleq n_p + n_c$  is

$$x_f(k+1) = \begin{bmatrix} A_p(\alpha) - B_p K_p & B_p K_c \\ -B_c C_p & A_c \end{bmatrix} x_f(k) + \begin{bmatrix} \mathbf{0} \\ B_c \end{bmatrix} r(k) \quad (29)$$

$$y(k) = \begin{bmatrix} C_p & \mathbf{0} \end{bmatrix} x_f(k). \quad (30)$$

The gain matrix  $\begin{bmatrix} -K_p & K_c \end{bmatrix} \triangleq \hat{K}$  is computed observing that the polytopic closed loop dynamical matrix  $A_f(\alpha)$  can be rewritten as  $A_f(\alpha) \triangleq \hat{A}(\alpha) + \hat{B}\hat{K}$ , where

$$\hat{A}(\alpha) = \begin{bmatrix} A_p(\alpha) & \mathbf{0} \\ -B_c C_p & A_c \end{bmatrix}, \hat{B} = \begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix}. \quad (31)$$

Analogously to the case of non-accessible state, the feedback matrix  $\hat{K}$  and the invariant  $\gamma$  feasible set  $\mathcal{X}$  are determined applying the procedure of Theorem 1 with the following variants:

$$\hat{A}_i \triangleq \begin{bmatrix} A_{p_i} & \mathbf{0} \\ -B_c C_p & A_c \end{bmatrix}, C_{z_u} \triangleq \begin{bmatrix} -K_p & K_c \end{bmatrix} = \hat{K}, C_{z_{x_f}} \triangleq \begin{bmatrix} C_p & \mathbf{0} \end{bmatrix} \text{ (if } z_{x_f} \equiv y(k)), B_f = \begin{bmatrix} \mathbf{0} \\ B_c \end{bmatrix},$$

$$Q \triangleq Q_1 \in \mathbb{R}^{n \times n} \text{ and } Y \triangleq Y_1 \in \mathbb{R}^{m \times n}, n = n_p + n_c. \quad \triangle$$

Once  $\Sigma_g$  and  $\mathcal{X}$  are determined (according to 4.1 or 4.2), the idea underlying the preliminary results can be summarized as follows. The stabilizing controller guarantees that,  $\forall x_f(0) \in \mathcal{X}$ ,  $\forall \alpha \in \Lambda_l$ , the fulfillment of all hard constraints (11) is ensured "a priori" provided the obtained quadratically stable  $\Sigma_f$  is forced by an admissible input  $r(k)$ , namely an  $r(k)$  satisfying (10) with  $\gamma = \eta^{-1}$ .

Next step will be determining the trajectory of the admissible input  $r(k)$  driving  $\Sigma_f$ . As detailed in the next section, this step is performed modeling  $r(k)$  as a vector of sampled B-spline functions whose control points are iteratively estimated, and then applying the computed  $r(k)$  according to the receding horizon control strategy defined in Section 3.

## 5. STEP 2: COMPUTATION OF $r(k)$

This section shows how the MMCOP stated in Section 3 can be reformulated as an RLS estimation problem which can be solved using the procedure of Section 2.2. To this purpose the closed loop dynamical matrix  $A_f(\alpha)$  of  $\Sigma_f$  is rewritten as  $A_f(\alpha) \triangleq \bar{A}_f + \Delta A_f(\alpha)$  where  $\bar{A}_f$  is the nominal closed loop dynamical matrix obtained putting  $A_p(\alpha) = \bar{A}_p$  in  $A_f(\alpha)$  and  $\Delta A_f(\alpha) \triangleq A_f(\alpha) - \bar{A}_f$ . Consequently, any term of the kind  $A_f^k(\alpha)$  can be written as  $A_f^k(\alpha) \triangleq \bar{A}_f^k + \Delta A_{f,k}(\alpha)$ , where  $\Delta A_{f,k}(\alpha)$  is a suitably defined matrix.

Expressing the input  $r(k)$  as  $\bar{\mathbf{B}}_d(k)\bar{\mathbf{c}}$  according to (5), and recalling that  $u(k) = C_{z_u}(k)x_f(k)$  and  $\hat{x}_f(k) = \begin{bmatrix} \hat{x}_p^T(k) & \hat{x}_c^T(k) & \mathbf{0}^T \end{bmatrix}^T$  is the current state estimate, the predicted output and control effort are given by

$$y(k+i|k) = C_f A_f^i(\alpha) \hat{x}_f(k) + \sum_{j=k}^{k+i-1} C_f A_f^{k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d(j) \bar{\mathbf{c}}, \quad i = 1, \dots, N_y \quad (32)$$

$$u(k+i|k) = C_{z_u} A_f^i(\alpha) \hat{x}_f(k) + \sum_{j=k}^{k+i-1} C_{z_u} A_f^{k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d(j) \bar{\mathbf{c}}, \quad i = 1, \dots, N_u \quad (33)$$

Explicit dependence on  $\alpha$  is omitted in the most part of vectors and matrices defined in the following for simplicity of notation and because such dependence is clear from the context.

By (32),(33) and  $r(k+i) = \bar{\mathbf{B}}_d(k+i)\bar{\mathbf{c}}$ ,  $e_y(k+i|k)$ ,  $e_r(k+i|k)$  and  $e_u(k+i|k)$  given by (16)-(18) respectively, can be rewritten as

$$e_y(k+i|k) = g_y(k+i|k) + \delta g_y(k+i|k) - (D_y(k+i|k) + \delta D_y(k+i|k))f, \quad (34)$$

$$e_r(k+i|k) = g_r(k+i|k) - D_r(k+i|k)f \quad (35)$$

$$e_u(k+i|k) = g_u(k+i|k) + \delta g_u(k+i|k) + (D_u(k+i|k) + \delta D_u(k+i|k))f, \quad (36)$$

where



$$\begin{aligned}
g_y(k+i|k) &\triangleq y_d(k) - C_f \bar{A}_f^i \hat{x}_f(k) \quad , \quad \delta g_y(k+i|k) \triangleq -C_f \Delta A_{f,i}(\alpha) \hat{x}_f(k), \\
D_y(k+i|k) &\triangleq \sum_{j=k}^{k+i-1} C_f \bar{A}_f^{k+i-j-1} B_f \bar{\mathbf{B}}_d(j) \quad , \quad \delta D_y(k+i|k) \triangleq \sum_{j=k}^{k+i-1} C_f \Delta A_{f,k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d(j) \\
g_r(k+i|k) &\triangleq y_d(k) \quad , \quad D_r(k+i|k) \triangleq \bar{\mathbf{B}}_d(k+i), \\
g_u(k+i|k) &\triangleq C_{z_u} \bar{A}_f^i \hat{x}_f(k) - \tilde{u}(k) \quad , \quad \delta g_u(k+i|k) \triangleq C_{z_u} \Delta A_{f,i}(\alpha) \hat{x}_f(k), \\
D_u(k+i|k) &\triangleq \sum_{j=k}^{k+i-1} C_{z_u} \bar{A}_f^{k+i-j-1} B_f \bar{\mathbf{B}}_d(j) \quad , \quad \delta D_u(k+i|k) \triangleq \sum_{j=k}^{k+i-1} C_{z_u} \Delta A_{f,k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d(j) \\
f &\triangleq \bar{c}.
\end{aligned}$$

Define the following vectors  $e \triangleq [e_y^T \ e_r^T \ e_u^T]^T$ ,  $g \triangleq [g_y^T \ g_r^T \ g_u^T]^T$ ,  $\delta g \triangleq [\delta g_y^T \ \mathbf{0}^T \ \delta g_u^T]^T$  and matrices

$$D \triangleq \begin{bmatrix} D_y \\ D_r \\ -D_u \end{bmatrix}, \quad \delta D \triangleq \begin{bmatrix} \delta D_y \\ \mathbf{0} \\ -\delta D_u \end{bmatrix}, \quad Q_e \triangleq \text{diag}[Q_y, Q_r, Q_u] \text{ where:}$$

$$\begin{aligned}
e_y &\triangleq \begin{bmatrix} e_y^T(k+1|k) & \cdots & e_y^T(k+N_y|k) \end{bmatrix}^T, \quad g_y \triangleq \begin{bmatrix} g_y^T(k+1|k) & \cdots & g_y^T(k+N_y|k) \end{bmatrix}^T, \\
\delta g_y &\triangleq \begin{bmatrix} \delta g_y^T(k+1|k) & \cdots & \delta g_y^T(k+N_y|k) \end{bmatrix}^T, \quad Q_y \triangleq \text{diag}\{Q_y(k)\}, \quad i = 1, \dots, N_y, \\
Q_r &\triangleq \text{diag}\{\lambda_1(k)Q_r(k)\}, \quad i = 0, \dots, N_y - 1, \quad Q_u \triangleq \text{diag}\{\lambda_2(k)Q_u(k)\}, \quad i = 1, \dots, N_u, \\
D_y &\triangleq \begin{bmatrix} D_y(k+1|k) \\ \cdot \\ \cdot \\ D_y(k+N_y|k) \end{bmatrix}, \quad \delta D_y \triangleq \begin{bmatrix} \delta D_y(k+1|k) \\ \cdot \\ \cdot \\ \delta D_y(k+N_y|k) \end{bmatrix}.
\end{aligned}$$

An analogous definition applies to vectors  $e_r$ ,  $e_u$ ,  $g_r$ ,  $g_u$ ,  $\delta g_u$  and matrices  $D_r$ ,  $D_u$ ,  $\delta D_u$ .

From the above definitions, it is evident that only  $\delta g$  and  $\delta D$  are depending on  $\alpha$ . This dependence is now explicitly reintroduced to better clarify the formulation of the MMCOP as an RLS estimation problem.

Exploiting the above defined vectors and matrices, the  $2qN_y + mN_u$  scalar equations (34),(36) can be expressed in the compact form  $e(\alpha) = (g + \delta g(\alpha)) - (D + \delta D(\alpha))f$  and functional (14) can be written as  $J_\alpha \triangleq J(e'(\alpha)) = e'^T(\alpha)e'(\alpha)$ , where  $e'(\alpha) \triangleq Q_e^{1/2}e(\alpha)$ . Also defining  $g' +$

$\delta g'(\alpha) \triangleq Q_e^{1/2}(g + \delta g(\alpha))$  and  $D' + \delta D'(\alpha) \triangleq Q_e^{1/2}(D + \delta D(\alpha))$ , it is evident that the MMCOP is equivalent to the constrained minimization of  $\|e'(\alpha)\|_2^2$ . Hence the MMCOP can be formulated as the following box-constrained RLS problem

$$\min_f \max_{\|\delta D'(\alpha) \delta g'(\alpha)\|_F \leq \rho} \|(D' + \delta D'(\alpha))f - (g' + \delta g'(\alpha))\|_2 \quad (37)$$

$$\text{subject to} \quad f_{min} \leq f \leq f_{max}. \quad (38)$$

The bounds  $f_{min}$  and  $f_{max}$  relative to the vector  $\bar{c} \triangleq f$  of control points are determined on the basis of condition (19) (and hence (10)).

At each  $k = jN_r$ , the bound  $\rho$  such that  $\|\delta D'(\alpha) \delta g'(\alpha)\|_F \leq \rho$  is computed by performing a gridding on the parameter vector  $\alpha \in \Lambda_l$ . Next, the parameter vector  $\bar{c} \triangleq f$  of control points is estimated through an SOCP as explained in Section 2.2. The corresponding B-spline input  $r(k)$  results to be known over  $[k, k + N_y)$ , but only the first  $N_r$  samples are applied to  $\Sigma_f$  according to the receding horizon control strategy.

Feasibility of the MMMPC strategy and stability of  $\Sigma_{2DOF}$  can be now formally stated in the following theorem.

**Theorem 2.** Assume that the problem P1 stated in Section 4 is solvable and that the input  $r(k)$  of  $\Sigma_f$  is computed as the solution of the box-constrained RLS problem (37),(38), then the resulting 2-step MMMPC strategy explained in the above sections is recursively feasible and yields an asymptotically internally stable  $\Sigma_{2DOF}$ .

**Proof of Theorem 2.** Recursive feasibility is a direct consequence of computing  $r(k)$  as the solution of an optimization problem where the feasible box-constraints (38) are imposed on a vector of variables which is the same one with respect to the optimization problem has to be solved. Moreover, by Theorem 1, the fulfillment of (38) directly implies that also the components of  $z_f(k)$  satisfy constraints like (11). Internal asymptotic stability of the resulting overall control system  $\Sigma_{2DOF}$  is a direct consequence of the internal asymptotic stability of  $\Sigma_f$  and of the uniform boundedness of  $r(k)$  resulting from (38).

**Remark 6** Some comments on the claimed simplification of the constrained optimization problem

involved in the new MMMPC strategy are in order. The B-spline parametrization of  $r(k)$  allowed us to formulate the constrained minimization of the worst-case cost as the RLS estimation problem of a parameter vector  $f$ . This problem can be solved through an SOCP for which numerically efficient primal-dual interior point methods can be used (see e.g. [25],[26], and references therein). The vector  $f$  to be estimated is composed of  $q\ell$  elements, where  $q$  is the dimension of  $r(k)$  and  $\ell$  is the number of control points of each scalar B-spline function composing  $r(k)$ . The well known approximation properties of B-splines allow choosing a value  $\ell \ll N_y$ , thus obtaining a greatly reduced number of decision variables with respect to  $qN_y$ , as required by the actual MMMPC methods. Moreover, as shown in Section 3, all constraints on  $z_f(k)$  can be transferred on the surely feasible interval type inequalities (19), whose number is  $qN_y$ . Nevertheless, by the convexity property of B-splines, these constraints must only concern the control points, so that their number reduces to  $q\ell$ . Following the usual approaches, the constraints to be satisfied (provided they are fulfillable) would be  $n_u N_u + n_{x_f} N_y$  where  $n_u$  and  $n_{x_f}$  are the dimensions of  $z_u(k)$  and  $z_{x_f}(k)$  respectively. It is recalled that  $z_u(k) \triangleq u(k)$  and hence  $n_u = m$ . Moreover, if  $z_{x_f}(k) \equiv y(k)$  then  $n_{x_f} = q$ .

Also the numerical minimization of functional (14) is simplified. Its formulation as an RLS problem as in (37), reduces the problem of evaluating the worst case cost for all the possible parametric uncertainties to the computation of the upper bound  $\rho$ . The most part of calculations related to the gridding procedure on  $\alpha \in \Lambda_l$  for determining  $\rho$  can be executed off-line because the only term of matrix  $[\delta D'(\alpha) \delta g'(\alpha)]$  depending on the current value of  $\hat{x}_f(k)$  is  $\delta g'(\alpha)$ . Approaching the RLS estimation problem as a SOCP allows the application of efficient primal-dual interior-point methods. Theoretical analysis shows that in the worst case the number of iterations required to solve an SOCP grows at most as the square root of the problem size, while numerical experiments indicate that the typical number of iterations ranges between 5 and 50, almost independent of the problem size [26].

## 6. SOME ISSUES ON THE TRACKING PROBLEM

Without any loss of generality, assume that the closed loop system  $\Sigma_f$  is characterized by a unitary feedback gain. Then, the internal model  $\Sigma_c$  and the stability of  $\Sigma_f$ ,  $\forall \alpha \in \Lambda_l$ , guarantee  $W(z, \alpha)_{z=1} = I_q$ ,  $\forall \alpha \in \Lambda_l$ , where  $W(z, \alpha)$  denotes the input-output transfer matrix of  $\Sigma_f$ .

So, for any fixed, desired set points vector  $y_d$ , the corresponding desired steady-state vector  $\tilde{r}$  of the input  $r(k)$  is  $\tilde{r} = y_d$ . Hence, if the LMIs stated in Theorem 1 are satisfied for some  $\eta$ , then a robust feasible solution to the exact steady-state tracking problem is obtained when  $r(k)$  has converged to  $\tilde{r}$ , provided that  $\|y_d\|_2^2 \leq \gamma = 1/\eta$ . If this condition is not satisfied the so called unfeasible reference problem arises. With reference to certain plants, this problem has been widely discussed in the MPC literature, see e.g. [28]-[33] and references therein. The commonly adopted solution consists of introducing in the cost functional a term penalizing the distance between the actually required (but unfeasible) reference and a fictitious (but feasible) reference. A different approach has been proposed in [34] where, using an extra optimization variable, the terminal set is moved to an arbitrary set point. The difficulty of extending the previous methods to uncertain plants has been noticed in [35], [36], where an alternative procedure based on a iterative steady-state target optimizer is proposed.

The present approach offers a direct robust solution to the unfeasibility problem: it is enough to define a different (but feasible) vector of set points  $y'_d$  optimally approximating  $y_d$  in the euclidean norm sense, under the constraint  $\|y'_d\|_2^2 \leq \gamma$ . Such a vector  $y'_d$  is the solution of the following (off-line) constrained optimization problem

$$\min_{y'_d} J(e_d) = \min_{y'_d} \|W_{e_d}^{1/2} e_d\|_2^2 = \min_{y'_d} e_d^T W_{e_d} e_d \quad (39)$$

$$\text{subject to} \quad : \|y'_d\|_2^2 \leq \gamma \quad (40)$$

with  $e_d = (y_d - y'_d)$  and  $W_{e_d} = \text{diag}[w_1, \dots, w_q]$ .

The  $w_i$ 's are chosen to define a prioritizing policy on the basis of the possibly different degrees of tracking error tolerated for different output components. Problem (39)-(40) can be reformulated as

the following SOCP:

$$\begin{aligned} & \text{minimize } \lambda_d \\ & \text{subject to: } \|y'_d\|_2^2 \leq \gamma \text{ and } \|W_{e_d}^{1/2} e_d\|_2^2 \leq \lambda_d. \end{aligned}$$

In this way the controlled output of  $\Sigma_f$  is asymptotically and robustly driven to the closest admissible steady -state value  $y'_d$ , solution of (39)-(40).

## 7. NUMERICAL RESULTS

To show the effectiveness of the proposed method two examples taken from the literature are studied. In the first one the feasible reference tracking problem for an uncertain SISO plant with unmeasurable state is considered. The second one concerns the unfeasible reference tracking problem for an uncertain MIMO plant under the assumption of measurable state. All the computations were performed on a *MacBookPro11,4; 2,2 GHz; 4 Core; 16GB RAM*, using the Yalmip toolbox [38] in Matlab. The Matlab functions *cpitime* and *tic-toc* were used to determine the CPU time and the execution time spent to compute the output response of the closed loop system.

### 7.1. Example 1

The first example concerns the angular positioning system of a rotating antenna at the origin of the plane driven by an electric motor [37]. The MPC stabilization problem of this plant under the assumption of an accessible state vector has been considered in [9]. Here, the proposed MMMPC procedure is applied to the tracking problem of a desired piece-wise constant reference signal under the more realistic assumption of an unmeasurable state vector.

Denoting by  $\theta$  (*rad*) and  $\dot{\theta}$  (*rad s<sup>-1</sup>*), respectively, the angular position and the angular velocity of the antenna, and by setting  $x_p \triangleq [\theta, \dot{\theta}]^T$ , the following discretized time equations are obtained from their continuous time counterparts using a sampling time  $T_c$  of 0.1s and Euler's first-order approximation for the derivative

$$x_p(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\omega \end{bmatrix} x_p(k) + \begin{bmatrix} 0 \\ 0.1\kappa \end{bmatrix} u(k) \quad (41)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_p(k), \quad (42)$$

where  $\kappa = 0.787 \text{ rad}^{-1}V^{-1}$ ,  $u \in \mathbb{R}$ ,  $y \in \mathbb{R}$ . The parameter  $\omega$  is proportional to the coefficient of viscous friction in the rotating parts of the antenna and is assumed to be constant but unknown over the range  $0.1s^{-1} \leq \omega \leq 10s^{-1}$ . Consequently, the dynamical matrix of  $\Sigma_p$  belongs to the following polytopic matrix family

$$A_p(\alpha) = \sum_{i=1}^2 \alpha_i A_{p_i} = \alpha_1 \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad \alpha \in \Lambda_2.$$

The control problem consists of using the input voltage ( $\mu V$ ) to the motor to rotate the antenna so that it points in the direction of an object in the plane whose angular position is denoted by  $y_d(k)(rad)$ . The desired reference  $y_d(k)$  to be tracked is the following piece-wise constant signal:  $y_d(k) = y_{d_1} = 0.1(rad)$ ,  $0 \leq k < 300(30s)$  and  $y_d(k) = y_{d_2} = -0.1(rad)$ ,  $300 \leq k \leq 600(60s)$ .

The control effort is required to satisfy the constraint:  $|u(k)| \leq 2V$ ,  $k > 0$ . Hence, according to (24), the constrained state  $z_u(k)$  is assumed to be given by the control effort  $u(k)$  with bound  $\bar{z}_{u,1} = 2$ .

The first step of the whole procedure is to design an output feedback controller  $\Sigma_g$ . According to the procedure described in section 4.1, the observer gain  $L$  of  $\Sigma_o$  is first computed. The gain matrix  $L = [1.2076, 1.0387]$  is found. Considering the pair  $(\hat{A}(\alpha), \hat{B})$  given by (25), the feedback gain  $\hat{K} = [-K_p \ K_c \ 0]$  and the invariant  $\gamma$ -feasible set  $\mathcal{X}$  for  $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$  are determined solving the semidefinite programming problem defined by (26)-(28). As the constraints only concern  $u(k)$ , by (24) one has  $z(k) = z_u(k)$  and the set of inequalities (28) concerning  $z_{x_f}$  are not considered. According to Remark 5, (26) has been transformed in an LMI executing a search line for  $\beta \in (0, 1)$ . As explained in Section 6, a robust feasible solution to the almost exact steady-state tracking problem for  $y_d(k)$  exists if the set of LMIs (26)-(27) admits solution for  $\gamma = 1/\eta \geq \max\{\|y_{d_1}\|_2^2, \|y_{d_2}\|_2^2\} = 0.01$ , and each set point is kept over a sufficiently long time

interval.

For  $\beta = 0.01$ , the feedback gain  $\hat{K} = [-8.0219, -7.8762, 0.0499, 0, 0]$  is found. The invariant  $\gamma$ -

$$\text{feasible set } \mathcal{X} \equiv \mathcal{E}(P, \gamma) \text{ with } P = Q^{-1} = \begin{bmatrix} 1.1971 & 0.7379 & -0.0075 & 0 & 0 \\ 0.7379 & 1.3586 & -0.0046 & 0 & 0 \\ -0.0075 & -0.0046 & 0.0001 & 0 & 0 \\ 0 & 0 & 0 & 31.5090 & -2.3552 \\ 0 & 0 & 0 & -2.3552 & 0.4772 \end{bmatrix}$$

and  $\gamma = \eta^{-1} = 0.0632$  is obtained for the resulting closed loop system  $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$ .

The second step is to determine the trajectory of the input  $r(k)$ , subject to (10) with  $\gamma = 0.0633$ , and optimally driving the output transition between the two consecutive set points of the given switching sequence. This step is performed modeling  $r(k) \in R$ , as a sampled B-spline function. The control points defining the B-spline  $r(k)$  over a moving prediction horizon are iteratively estimated by the SOCP as explained in Section 2.2. At each  $k = jN_r$ ,  $j = 1, 2, \dots$  and  $N_r = 10$ , the bound  $\rho$  such that  $\|\delta D'(\alpha) - \delta g'(\alpha)\|_F \leq \rho$  is computed by performing a gridding for  $\alpha \in \Lambda_2$ . The obtained sequence of  $\rho$  ranges in the interval  $[0, 0.2292]$ . The computed  $r(k)$  is applied according to the usual receding horizon control strategy.

The following parameters are chosen:  $d = 1$  (order of B-spline),  $\ell = 3$  (number of control points of the scalar B spline over each prediction horizon  $N_y$ ),  $5 \triangleq \ell + d + 1$  (number of knot points  $\hat{k}_i$  over each  $N_y$ ) and  $N_y = 40$ . All the weight matrices are set to identity matrix. An S-shaped membership function is chosen for  $\lambda_1(k)$  for the following motivations. In correspondence of the transient response following any set point change, a null initial value of  $\lambda_1(k)$  allows  $r(k)$  to freely vary over all the admissible range. After the transition period has elapsed,  $\lambda_1(k)$  should tend to a suitable positive value  $\bar{\lambda}$  to speed up the convergence of  $r(k)$  to the desired set point value. In this case the value  $\bar{\lambda} = 1$  has been chosen. A null  $\lambda_2(k)$ , has been fixed  $\forall k \geq 0$ , because the feedback controller has been designed to guarantee that, for any  $r(k)$  satisfying (10), the control effort  $u(k) \triangleq z_u(k)$  obey constraint (11). The vector  $f \triangleq \bar{c} = \mathbf{c}_1$  of decision variables to be determined at each  $k = jN_r$

is composed by  $\ell = 3$  control points. As  $\gamma = 0.0632$  and  $r(k)$  is a scalar, the bounds of inequalities (38) are  $|f_{min}| = f_{max} = \sqrt{\gamma} = 0.2565$ .

The simulation has been performed starting from  $x_f(0) = [\hat{x}_p^T(0), x_c^T(0), x_p^T(0) - \hat{x}_p^T(0)] = [-0.05, 0, 0, 0, 0.001]^T \in \mathcal{X}$  and choosing  $A_p(\bar{\alpha}) \triangleq \bar{\alpha}_1 A_1 + \bar{\alpha}_2 A_2 = 0.2A_1 + 0.8A_2$ .

The simulation has been stopped at  $k = 600$  samples (60s). The obtained input  $r(k)$  is depicted in figure 2. The actual controlled output of  $\Sigma_f$  yielded by  $r(k)$  is given in figure 3 (solid line). The behavior of the constrained control effort is shown in figure 4.

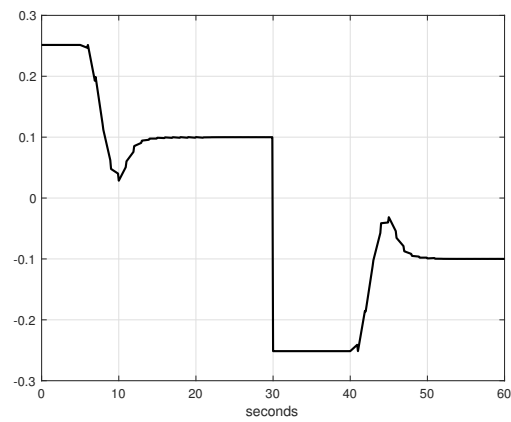


Figure 2. *Example 1.* The computed scalar B-spline input function  $r(k)$ .

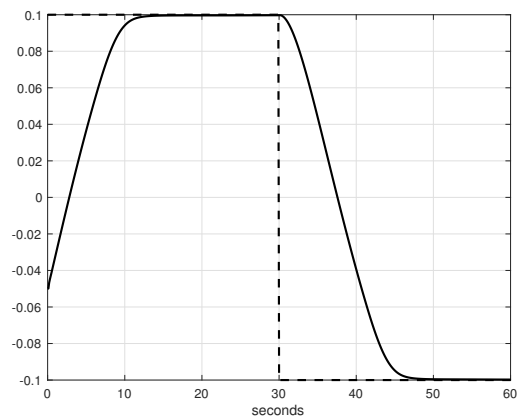


Figure 3. *Example 1.* The desired reference signal (dashed line) and the actual controlled output (solid line).

As for the computational complexity, the following considerations hold:



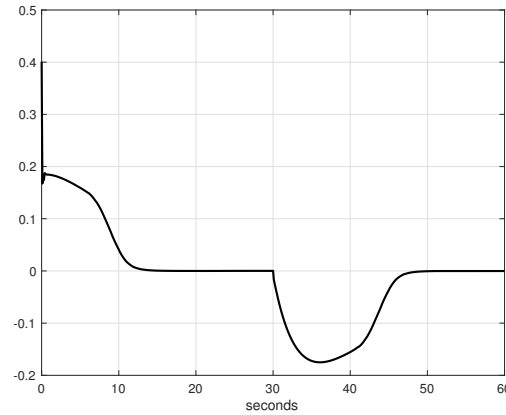


Figure 4. *Example 1.* The trajectory of the constrained control effort.

- the proposed MMCOP is solved at each time instant  $k = jN_r$  where  $j = 0, 1, \dots$  and  $N_r$  is a positive integer while the on-line optimization procedure in [9] must be solved at each time instant  $k \geq 0$ . In this case the value  $N_r = 10$  has been chosen.
- According to Remark 6, at each  $k = jN_r$  the number of scalar decision variables (control points of  $r(k)$ ) involved in the proposed on-line optimization procedure is  $q\ell = 3 \ll qN_y = 40$  and the total number of interval type scalar inequalities (38) to be imposed is  $q\ell = 3 \ll qN_y = 40$ . Both numbers of decision variables and inequalities are independent of the process state dimension. In [9], at each  $k \geq 0$  the number of Linear Matrix Inequalities (LMIs) to be solved is  $(L + 2) = 4$  (namely (20), (21) defined at the vertices  $l = 1, \dots, L$  of the uncertainty polytope and (33)) with respect to two matrix decision variables ( $Q \in \mathbb{R}^{n_p \times n_p}$  and  $Y \in \mathbb{R}^{m \times n_p}$ ) with  $n_p = 2$  and  $m = 1$ ) and a positive scalar decision variable ( $\gamma \in \mathbb{R}$ ). Hence the total number of scalar decision variables increases with  $n_p$  and  $m$ .

Table I shows the euclidean norm of the tracking error over the whole simulation:  $e_t \triangleq [e_t^T(0), \dots, e_t^T(600)]^T$ , with  $e_t(\cdot) \triangleq y_d(\cdot) - y(\cdot)$ . The same table also reports the CPU time and execution time required to compute the closed loop response for different values of the prediction horizon  $N_y$  and  $N_r = 10$ .

$N_y$	CPU time (s)	Execution time (s)	$\ e_t\ _2$
20	29.87	16.2530	1.7743
30	35.00	18.0163	1.7070
40	45.04	20.9938	1.7004

Table I. Example 1. The CPU time, the execution time and the tracking performance measure for different values of the prediction horizon  $N_y$  and  $N_r = 10$ .

## 7.2. Example 2

Consider the following MIMO discrete time system  $\Sigma_p$  inspired by [39]

$$x_p(k+1) = \begin{bmatrix} 0 & 1 \\ -1 + \delta & -1 \end{bmatrix} x_p(k) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(k) \quad (43)$$

$$y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_p(k) \quad (44)$$

where  $\delta \in [-0.1, 0.1]$  is an unknown parameter. Unlike [39], the dynamical matrix of  $\Sigma_p$  belongs to the following polytopic matrix family

$$A_p(\alpha) = \sum_{i=1}^2 \alpha_i A_{p_i} = \alpha_1 \begin{bmatrix} 0 & 1 \\ -1.1 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ -0.9 & -1 \end{bmatrix}, \alpha \in \Lambda_2.$$

A tracking control problem subject to the control input constraint  $|u_i(k)| \leq 5, i = 1, 2$ , is considered. Hence, according to (24), the constrained state  $z_u(k)$  is assumed to be given by the control effort  $u(k)$ , with bound  $\bar{z}_{u,r} = 5, r = 1, 2$ . The desired piece-wise constant reference to be tracked is the following piece-wise constant signal

$$y_d(k) = \begin{cases} y_{d_1} = [1 \ 1]^T & 0 \leq k < 100 \\ y_{d_2} = [2 \ 1]^T & 100 \leq k \leq 200 \end{cases} \quad (45)$$

The first step is to design a state feedback controller  $\Sigma_g$ , following the procedure described in section 4.2. Considering the pair  $(\hat{A}(\alpha), \hat{B})$  given by (31), the feedback gain  $\hat{K} = [-K_p \ K_c]$  and the invariant  $\gamma$ -feasible set  $\mathcal{X}$  for  $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$  are determined by solving the

semidefinite programming problem defined by (26)-(28). Since the constraints only concern  $u(k)$ , by (24) one has  $z(k) = z_u(k)$  and the set of inequalities (28) concerning  $z_{x_f}$  are not considered. According to Remark 5, (26) has been transformed in an LMI executing a search line for  $\beta \in (0, 1)$ . As explained in section 6, a robust feasible solution to the almost exact steady-state tracking problem exists if the set of LMIs (26)-(27) admits solution for

$$\gamma = 1/\eta \geq \max\{\|y_{d_1}\|_2^2, \|y_{d_2}\|_2^2\} = \max\{2, 5\} = 5, \quad (46)$$

and each set point is kept over a sufficiently long time interval. Performing a search line for  $\beta \in (0, 1)$  with a gridding step  $\delta_\beta = 0.2$  and giving up the maximization of  $\mathcal{X}_{x_p}$  the following sequence of pairs of values  $(\beta, \gamma_\beta)$ :  $\{(0.2, 4.7985), (0.4, 0.4, 4.7984), (0.6, 4.51), (0.8, 2.41)\}$  are obtained. By (46) it follows that that exact steady-state tracking can be only obtained for  $y_{d_1}$  and the unfeasible reference problem for  $y_{d_2}$  arises. As mentioned in Section 6 this problem can be directly solved replacing  $y_{d_2}$  in (45) with a different (but feasible) optimal approximation  $y'_{d_2}$  satisfying the constraint  $\|y'_{d_2}\|_2^2 \leq \gamma_\beta = 4.7985$  (obtained for  $\beta = 0.2$ ). Such new vector  $y'_{d_2}$  is obtained as the solution of the off-line constrained optimization problem defined by (39)-(40). Assuming to assign a maximum priority to the first component of  $y_{d_2}$ , the weight matrix  $W_{e_d} = \text{diag}[1, 10^{-4}]$  is chosen and the new vector  $y'_{d_2} = [2, 0.89]^T$  is found. For  $\beta = 0.2$ , the obtained stabilizing feedback gain is

$$\hat{K} = \begin{bmatrix} 0 & -0.2 & 0.22 & 0.4000 \\ -0.0709 & -0.2813 & 0.2501 & -0.1808 \end{bmatrix}.$$

The matrix  $P$  defining the invariant  $\gamma$ -feasible set  $\mathcal{X} \equiv \mathcal{E}(P, \gamma_\beta) = \mathcal{E}(P, 4.7985)$  for  $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$  results to be

$$P = \begin{bmatrix} 0.2956 & 0.1307 & -0.1002 & 0.0063 \\ 0.1307 & 0.1673 & -0.0410 & -0.0394 \\ -0.1002 & -0.0410 & 0.0680 & -0.0029 \\ 0.0063 & -0.0394 & -0.0029 & 0.0470 \end{bmatrix}.$$

Next step is to determine the trajectory of the input  $r(k)$ , subject to (10) with  $\gamma = 4.7985$ , optimally

driving the output transition between the two consecutive set points vectors. This step is performed modeling  $r(k) \in R^q$ , as a vector of  $q = 2$  sampled B-splines. The  $q\ell$  control points vector  $f = \bar{c} = \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T \end{bmatrix}^T$  defining  $r(k)$  over a moving prediction horizon are iteratively estimated by the SOCP as explained in Section 2.2. At each  $k = jN_r$ ,  $j = 1, 2, \dots$  and  $N_r = 10$ , the bound  $\rho$  such that  $\|\delta D'(\alpha) - \delta g'(\alpha)\|_F \leq \rho$  is computed by performing a gridding for  $\alpha \in \Lambda_2$ . The obtained sequence of  $\rho$  ranges in the interval  $[0, 0.2632]$ . The computed  $r(k)$  is applied according to the usual receding horizon control strategy.

The following parameters are set:  $d = 1$  (order of each B-spline),  $\ell = 3$  (number of control points for each B-spline over each prediction horizon  $N_y$ ),  $5 \triangleq \ell + d + 1$  (number of knot points  $\hat{k}_i$  over each  $N_y$ ),  $N_y = 30$ . All the weight matrices are set to identity matrix. Arguing as in example 1, the tuning parameters  $\lambda_i$ ,  $i = 1, 2$ , are chosen in the following way:  $\lambda_1(k)$  is a S-shaped membership function with  $\bar{\lambda} = 50$  and  $\lambda_2(k) = 0, \forall k \geq 0$ .

Let  $\bar{A}_p = (A_1 + A_2)/2$  be the nominal plant, the two steady state values  $\tilde{u}_1$  and  $\tilde{u}_2$  of  $u(k)$  corresponding to  $y_{d_1}$  and  $y'_{d_2}$  are  $\tilde{u}_1 = [3 \ 0]^T$  and  $\tilde{u}_2 = [3.78 \ 1.11]^T$  respectively.

The simulation has been performed starting from null initial conditions and choosing  $A_p(\bar{\alpha}) \triangleq \bar{\alpha}_1 A_1 + \bar{\alpha}_2 A_2 = 0.2A_1 + 0.8A_2$ .

The obtained input  $r(k)$  is depicted in figure 5. The actual controlled output of  $\Sigma_f$  yielded by  $r(k)$  is given in figure 6. The behavior of the constrained control effort is shown in figure 7.

Analogously to Example 1, Table II shows the euclidean norm of the tracking error over the whole simulation:  $e_t \triangleq [e_t^T(0), \dots, e_t^T(200)]^T$ , with  $e_t(\cdot) \triangleq y_d(\cdot) - y(\cdot)$ . The same table also reports the CPU time and execution time required to compute the closed loop response for different values of the prediction horizon  $N_y$  and  $N_r = 10$ .

## 8. CONCLUSIONS

The advantage of using a 2DOF control scheme to deal with the MMMPC consists in the possibility of decomposing the problem in two distinct steps: the first one is the off-line design of a feedback controller which stabilizes the uncertain plant and guarantees in advance the fulfillment of hard

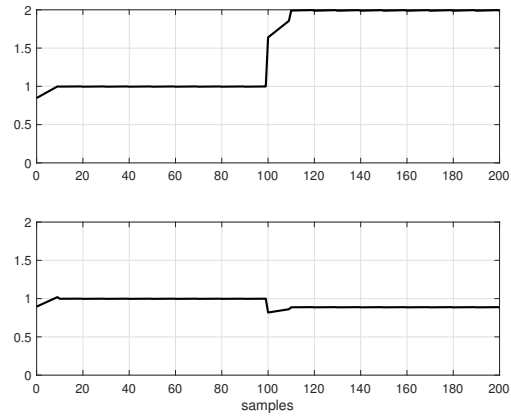


Figure 5. *Example 2*. The two components of the computed B-spline input function  $r(k)$ .

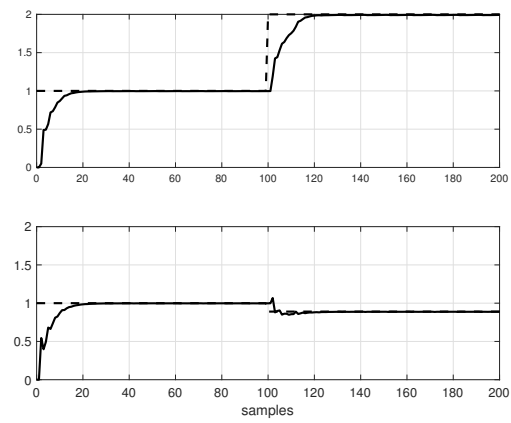


Figure 6. *Example 2*. The feasible desired reference signal (dashed line) and the actual controlled output (solid line).

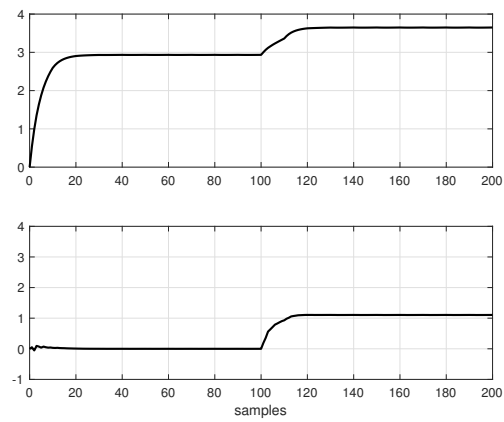


Figure 7. *Example 2*. The trajectory of the constrained control effort  $u(k)$ .

$N_y$	CPU time (s)	Execution time (s)	$\ e_t\ _2$
20	39.66	15.47	1.3466
30	66.72	21.44	1.2688
40	75.6	24.85	1.1960

Table II. Example 2. The CPU time, the execution time and the tracking performance measure for different values of the prediction horizon  $N_y$  and  $N_r = 10$ .

constraints for any input  $r(k)$  satisfying the admissibility condition; the second step consists in the on-line computation of the input  $r(k)$  forcing the stable closed-loop system. Modeling  $r(k)$  as a B-spline decreases the number of decision variables and of hard constraints. It also allows the formulation of the constrained optimization of the quadratic cost functional as a much simpler robust estimation problem with box-constraints on the unknowns (the control points of  $r(k)$ ). Computing the box constraints is a straightforward consequence of the admissibility condition established off line at step 1 and of the membership of B-splines to the convex hull defined by their control points. The robust estimation problem can be formulated as a SOCP, for which numerically efficient interior point methods exist. Finally, it is worth mentioning that the proposed MMMPC can be easily extended to the tracking problem for the more general class of signals generated as the free output response of unstable linear systems. Endowing  $\Sigma_g$  of the opportune internal model, this extension can be performed without increasing the number of decision variables.

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