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Semi-Markov reward models under enlarged filtrations with applications to disability insurance

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ABSTRACT

This paper proposes a discrete-time non-homogeneous Semi-Markov reward model designed to describe the evolution of health states for disability insurance applications. The model incorporates the duration dependence of transitions and allows for the inclusion of anticipative information through an enlarged filtration framework (ELF), which provides insights into the insured's potential future health trajectory. Within this setting, we derive recursive evolution equations for the expected discounted rewards and show how these can be used for the computation of premiums. The theoretical framework is complemented by an empirical application based on gender- and age-specific transition probabilities, highlighting how anticipative information reduces uncertainty in disability transitions and significantly affects actuarial valuations. The proposed model thus contributes to a better understanding of longevity and disability risks in multi-state disability insurance contracts.

1. Introduction

The increasing importance of disability insurance in modern economies has underscored the need for increasingly accurate and robust risk modeling [1]. In many developed economies, the number of individuals receiving public disability insurance benefits has increased substantially over the past decades, making it one of the largest components of social protection systems in OECD countries [2,3].

Markovian and semi-Markovian models have found widespread application in a variety of finance and insurance problems, offering a rigorous and flexible framework for modeling state transitions and timing uncertainty Yuen et al. [4], Janssen [5], Cortés et al. [6], Carravetta et al. [7], D'Amico et al. [8], Janssen and Manca [9]. In insurance mathematics, the evolution of an individual's health status can be effectively represented through multi-state stochastic models; see, e.g., [10]. These frameworks capture the random transitions between discrete health conditions, such as active, non-self-sufficient, and death based on non-parametric [11] or parametric [12] forms. Building on this perspective, the paper in [13] develops an algorithmic methodology for evaluating non-homogeneous semi-Markov reward processes, providing a flexible tool for the quantitative analysis of insurance systems that depend on time-varying transition dynamics. This approach is particularly well-suited to capture the temporal evolution of health and the associated insurance cash flows. A further contribution in this field is provided by [14], who develop a stochastic framework for disability insurance based on a discrete-time non-homogeneous semi-Markov process. By incorporating the backward recurrence time, their model enables a

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more detailed analysis of disability duration and transition dynamics. Moreover, through the use of semi-Markov reward processes, the authors derive equations for both prospective and retrospective reserves. Furthermore, the authors in [1] introduce a discrete-time trivariate Markov renewal reward model that explicitly accounts for the dependence between the durations spent in successive states. Recent contributions includes the computation od integral and differential equations for the moments of multi-states models in health insurance [15] and the testing of the non-Markov assumption on an up-to-date large American dataset, finding insights on the non-Markovian character of disability evolution [16].

Taken together, these Markovian and semi-Markovian frameworks provide the foundation for computing fair insurance premiums, which serve as proxies for future insurance risk, and for analyzing their statistical properties [17].

Recent developments in medical science and biostatistics have significantly improved access to insurance for individuals previously affected by severe diseases. Building on [18] and [19], semi-Markov models calibrated on Belgian Cancer Registry data demonstrate the ability of this framework to capture the dynamic and path-dependent evolution of health states. Furthermore, in this line of research, [20] and [21] extend multi-state insurance models to address systematic risks in mortality, disability, and long-term care (LTC). By introducing de-risking options and swaps linked to transition intensities, they develop optimization-based strategies that minimize portfolio exposure under cost or conditional Value-at-Risk (CVaR) constraints. Their findings highlight the effectiveness of market-based instruments in mitigating longevity and disability risks faced by insurers.

In line with this research branch, the present paper applies a discrete-time non-homogeneous semi-Markov process within a life insurance framework to model the evolution of disability. This approach provides a realistic representation of health-state dynamics over time by capturing the dependence on both age and duration and by incorporating transition probabilities that vary with the time an individual spends in a given state.

Building upon this framework, we further extend the model by introducing the possibility to include additional sources of information beyond those typically available in standard insurance settings. This is achieved through the enlargement of filtration, which provides a rigorous mathematical structure to incorporate anticipative information and to analyze its impact on the stochastic evolution of the system. The concept of filtration enlargement plays a key role in describing situations where additional information about future events becomes available to an observer, and it is largely used in finance. This procedure was originally applied to financial problems, such as [22–24], where the authors hypothesize that agents in a financial market may have access to anticipative knowledge that alters the probabilistic structure of the model and decompose the underlying asset's price process into two components, where the relatively smaller one captures the private information available to the informed agent.

Here, we introduce the enlargement of filtration within an insurance framework characterized by a semi-Markov process for modelling disability evolution. We focus on initial enlargements, where the additional information is revealed to the informed agent from the outset. Through this framework, we are able to evaluate the rewards of the semi-Markov process, thereby determining the fair premiums and assessing the additional benefits gained by the insurance company from having access to this extra information. Importantly, we can rigorously account for additional information about the future evolution of an insured person's health, leading to refined modelling of survival probabilities and more accurate valuation of life insurance liabilities.

A further important development is the explicit consideration of longevity risk, arising from the systematic decline in mortality and the deviation of observed deaths from expected values [25–27]. To this end, we incorporate this aspect by conditioning on the insured individual's survival beyond a given age and analyzing how this additional information affects the fair premiums and the valuation of the insurance contract.

Finally, we provide an empirical application based on the technical bases in [28], using annual transition probabilities among three health states (active, non-self-sufficient, dead) for males and females, covering ages 20–95 and years 2013–2043. We report results in terms of annual and total fair premiums computed for different scenarios in which we vary the initial age, gender, discount rate, contract duration, and impose alternative conditions on health-state transitions and longevity. The results show that premiums are highly sensitive to the imposed longevity and transition conditions, which alter the expected time spent in the non-self-sufficient state and the timing of benefit payments. Furthermore, gender and contract duration combine with entry age and discounting to shape both levels and slopes: longer terms and stricter survival requirements raise premiums, gender gaps persist across ages, and discounting shifts all curves downward while preserving the qualitative ordering across scenarios.

The structure of the paper is as follows. In Section 2, we introduce the discrete-time non-homogeneous Semi-Markov process to model the disability evolution over time. Section 3 presents the initial enlargement of filtration to incorporate information about the future evolution of the disease into a Semi-Markov life insurance model, together with the evolution equations for computing the expected rewards. In Section 4, we quantify longevity risk by applying the theory of enlargement of filtration, conditioning the rewards on the event that the policyholder survives to a certain age. In Section 5, we describe the input data and present the empirical application, along with a discussion of the obtained results. Finally, Section 6 summarizes and concludes the work.

2. Discrete-time non-homogeneous semi-Markov processes in life-insurance

The evolution of a disability over time can be described using two random variables (r.v.s) defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. These variables operate jointly to model the process. The first random variable,

$$J_n : \Omega \to I = \{1, 2, \dots, D\},\ n \in \mathbb{N},$$

represents the health state of the individual at the n-th transition. The second random variable,

$$T_n : \Omega \to \mathbb{N}, n \in \mathbb{N},$$

denotes the time at which the n-th transition occurs.

In our framework, the elements of the set I represent different levels of disability. State 1 corresponds to the absence of disability, while the subsequent states describe progressively severe disability conditions, culminating in the policyholder's death, represented by state D. Time is modeled as a discrete variable and, depending on data availability and the specific application, may be measured on a yearly, monthly, or even finer scale.

The evolution of the random variables introduced above must be specified through a set of assumptions that define the structure of the model. In this context, we introduce the following assumption:

Assumption 1. (J_n, T_n) is a non-homogeneous Markov Renewal Process (MRP), i.e. for all times $s_0 < \ldots < s_n < t$ and states $i_0, \ldots, i_n, j$ we have

$$\mathbb{P}\left[J_{n+1} = j,\ T_{n+1} \le t |\ J_n = i_n, \ldots, J_0 = i_0\, T_n = s_n, \ldots, T_0 = s_0\right] = \mathbb{P}\left[J_{n+1} = j,\ T_{n+1} \le t |\ J_n = i_n,\ T_n = s_n\right]. \tag{1}$$

This hypothesis is supported by previous investigations in the field; see, e.g., [11,14,29]. Clearly, this assumption can be weakened considering more general dependence structures between the joint process, (J_n, T_n) such as a trivariate renewal process ([1]), a second-order semi-Markov chain in state and duration ([30]) or deep path-dependent specification as provisioned by indexed semi-Markov models; see, e.g., [31,32]. Anyway, for the scope of this study, it is not relevant to advance more sophisticated assumptions of **A1**, which is already very general, as our scope is to better consider the uncertainty related to this base framework. Naturally, our conclusions may translate directly to more complex dynamics.

The probabilities in formula (1) form the kernel $\mathbf{Q} = [Q_{ij}(s,t)]$ associated with the MRP, whose elements are defined in the following way:

$$Q_{ij}(s,t) = \mathbb{P}\left[J_{n+1} = j,\ T_{n+1} \le t |\ J_n = i,\ T_n = s\right].$$

If time represents the age of the person, the element $Q_{ij}(s,t)$ denotes the probability he/she will enter the disability state j with the next transition within age t given that he/she entered state i at the current age s.

The marginal probabilities of the MRP can be obtained in a simple way. First, we compute the probability of the J_n random variable. It results that

$$p_{ij}(s) = \mathbb{P}\left[J_{n+1} = j |\ J_n = i,\ T_n = s\right] = \lim_{t\to\infty} Q_{ij}(s,t) \quad i,j \in I \quad s,t \in \mathbb{N}.$$

Matrix $\mathbf{P}(s) = \left[p_{ij}(s)\right]$ is the transition probability matrix of the embedded non-homogeneous Markov chain $\{J_n\}_{n\in\mathbb{N}}$. It gives the probability of making the next transition in state j for a s-aged person with a disability degree i independently of the time in which this transition occurs. Moreover, the marginal distribution of the process T_n can be evaluated according to

$$H_i(s,t) = \mathbb{P}\left[T_{n+1} \le t \mid J_n = i,\ T_n = s\right] = \sum_{j\in I} Q_{ij}(s,t).$$

Hence, $H_i(s,t)$ is the probability that the person will leave state i from age s within age t and $\overline{H}_i(s,t) = 1 - H_i(s,t)$.

Let $N(t) = \sup\{n \in \mathbb{N}\ :\ T_n \le t\}$ be the number of transitions up to time t; then the semi-Markov process $Z(t)$ can be defined as $Z(t) = J_{N(t)}$ denoting the state occupied by the process at time t, i.e., the health state of the person at age t. The transition probability is defined as:

$$\phi_{ij}(s,t) = \mathbb{P}\left[Z(t) = j \mid Z(s) = i,\ T_{N(s)} = s\right].$$

They are obtained by solving the following evolution equations:

$$\phi_{ij}(s,t) = d_{ij}(s,t) + \sum_{\beta\in I} \sum_{\upsilon=s+1}^{t} q_{i\beta}(s,\upsilon)\phi_{\beta j}(\upsilon,t), \tag{2}$$

where $d_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t))$ and

$$\begin{aligned} q_{ij}(s,t) &= \mathbb{P}\left[J_{n+1} = j,\ T_{n+1} = t \mid J_n = i,\ T_n = s\right] \\ &= \begin{cases} Q_{ij}(s,t) - Q_{ij}(s,t-1) & \text{if } t > s, \\ 0 & \text{if } t = s. \end{cases} \end{aligned}$$

δ_{ij} denotes the Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

In a disability insurance model, $d_{ij}(s,t)$ denotes the probability that the policyholder experiences no change of state between ages s and t. This term contributes to $\phi_{ij}(s,t)$ only when $i = j$. In the second component of (2), $q_{i\beta}(s,\upsilon)$ represents the probability that the policyholder transitions into state β exactly at age υ given that he/she entered in state i at age s. After this transition, the individual may reach state j at age t through one of the possible trajectories originating from state β at age υ, bringing it into state j at age t. Well-established numerical algorithms exist for solving Eq. (2); see, for instance, [9].

In disability modeling, transition probabilities are often found to depend on the duration of time an individual remains in a given state. To capture this effect, duration-dependent transition probabilities can be introduced. Before doing so, however, we define an auxiliary process to facilitate the analysis.

Let $B(t) = t - T_{N(t)}$ be the backward recurrence time process (see, e.g., [9,33]). This process denotes the time since the occurrence of the last transition. In [11], the following probabilities were defined:

$$
\begin{aligned}
{}^{b}\phi_{ij}(l,s;t) &= \mathbb{P}[Z(t)=j \mid Z(s)=i,\ B(s)=s-l] \\
&= \mathbb{P}\big[J_{N(t)}=j \mid J_{N(s)}=i,\ T_{N(s)}=l,\ T_{N(s)+1}>s\big], \qquad (3)\\
\phi^{b}_{ij}(s;l',t) &= \mathbb{P}\big[Z(t)=j,\ B(t)=t-l' \mid Z(s)=i,\ B(s)=0\big] \\
&= \mathbb{P}\big[J_{N(t)}=j,\ T_{N(t)}=l',\ T_{N(t)+1}>t \mid J_{N(s)}=i,\ T_{N(s)}=s\big], \qquad (4)\\
{}^{b}\phi^{b}_{ij}(l,s;l',t) &= \mathbb{P}\big[Z(t)=j,\ B(t)=t-l' \mid Z(s)=i,\ B(s)=s-l\big] \\
&= \mathbb{P}\big[J_{N(t)}=j, T_{N(t)}=l', T_{N(t)+1}>t \mid J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}>s\big]. \qquad (5)
\end{aligned}
$$

The transition probabilities (3)–(5) show the dependence of the chance to do specific transitions w. r. t. the duration of permanence in the initial state i and in the final state j.

Here, we just remember that the formulas for computing these transition probabilities and the numerical algorithms are presented in [11]. The usefulness of deriving such evolution equations lies in their role in the recursive evaluation of transition probabilities over time. In particular, they allow the computation of occupation probabilities and expected rewards, which are fundamental for determining actuarial quantities, such as expected benefits and fair premiums (see [34]).

3. Semi-Markov reward process with extended information

In this section, we introduce the core problem addressed by this study, which is the quantification of the benefit that auxiliary information about some events in the dynamic of the policyholder's health status can bring to the actuarial evaluation. In Section 3.1 we describe the enlargement of the filtration idea in relation to our semi-Markov model for the considered disability insurance problem. In Section 3.2, we exploit the influence of the increasing information at the evaluation time s on the basic actuarial quantities of interest, which are expressed in terms of the expected value of the reward process and the fair premiums evaluation.

3.1. Initial enlargement of filtration for a semi-Markov model of life-insurance

A primary objective of the insurers is to set premiums that reflect the expected medical costs of policyholders. Thus, exploiting any possible information about the evolution of an insured person's health is crucial, as it enables more accurate modelling of disability evolution and fairer valuation of life insurance contracts.

By employing the technique of enlargement of filtrations, we demonstrate how to incorporate information about the future evolution of the disease into a Semi-Markov life insurance model.

Nonetheless, the departing point is the definition of the semi-Markov reward process. As already pointed out, we consider a filtered probability space, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathcal{F}$ is the sigma algebra and $\mathbb{F}=\{\mathcal{F}_t, t\geq 0\}$ is the augmented natural filtration generated by the process $(J_{N(t)}, T_{N(t)},\ t\in\mathbb{N})$ which is a $\mathbb{F}$-Markov renewal process.

According to the theory of semi-Markov reward processes, we assume that there are permanence and impulse rewards. They are payments associated with the occupancy in a state and with the transition from one state to another one, respectively. Thus, if the policyholder's health status at time u is i and the time elapsed since the last state change is l units of time, there is a payment given by $\psi_i(l,u)$. Moreover, if at any time t_n there is transition from state i to state j which has occurred after a waiting time equal to x, there is a payment at time t_n given by $\gamma_{i,j}(x,t_n)$. As usual, the payments are discounted at the evaluation epoch s according to a deterministic discount factor $\{v(s,u), u\geq s\}$. The choice of a deterministic discount factor is only for simplicity and can be replaced with stochastic models without any incremental difficulty, just hypothesizing the independence between the interest rate structure and the health-state evolution of the policyholder as per standard practice. Another option is to replace the deterministic discount factors with the expected discounted values derived from a stochastic interest rate model; a strategy to produce forecasts of expected interest rates based on the Vasicek framework is presented in [35].

As the states and transition times are random variables, the permanence and transition rewards are random. The main interest is in the process that computes the total discounted rewards accumulated over a given time interval $(s,t]$. This process is formally defined according to the next relation:

$$
\xi(s;t) := \sum_{u\geq s+1}^{t} \psi_{J_{N(u)}}(u-T_{N(u)},u)\, v(s,u) + \sum_{\tau=1}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})\cdot v(s,T_{N(s)+\tau}), \qquad (6)
$$

where $X_{N(s)+\tau} := T_{N(s)+\tau}-T_{N(s)+\tau-1}$. The first term on the right-hand side of Eq. (6) represents the summation of the discounted rewards at any time u, given that the system has been in the state $J_{N(u)}$ since $u-T_{N(u)}$ periods. Since the model is referenced to the initial time s, the contributions at the future time u are discounted back s using the weight $v(s,u)$, reflecting the time value or relevance of events that occur u when evaluated from the starting point s. The second term on the right-hand side of Eq. (6) gives the summation of the transition rewards paid at the times of transitions $T_{N(s)+\tau}$ and depends on the starting state $J_{N(s)+\tau-1}$, the arrival state $J_{N(s)+\tau}$, the length of the waiting time $X_{N(s)+\tau}$, and the transition time $T_{N(s)+\tau}$.

The analysis of the process (6) proceeds through the computation of its moments (see, e.g., [13]). In this work, we focus only on the expected value, since the results for higher-order moments can be derived by combining the methods we present here with the general approaches developed in [13,36].

Let us consider the random variable

$$
\mathbb{E}[\xi(s,t)|\mathcal{F}_s] = \mathbb{E}[\xi(s,t)|J_{N(s)},T_{N(s)}] =: V_{J_{N(s)}}(T_{N(s)},s;t). \qquad (7)
$$

Now, assuming that $J_{N(s)} = i$ and $T_{N(s)} = s$, according to the results given by [13] we have that the first-order moment of (6) satisfies the next recurrent-type equation

$$\begin{aligned} V_i(s,s;t) &= (1-H_i(s,t)) \cdot \sum_{u=s+1}^{t} \psi_i(u-s,u)v(s,u) \\ &+ \sum_{k\in I}\sum_{\theta=s+1}^{t} q_{i,k}(s,\theta)\Bigg[\sum_{u=s+1}^{\theta-1}\psi_i(u-s,u)v(s,u) + \gamma_{i,k}(\theta-s,\theta)v(s,\theta) + V_k(\theta,\theta;t)v(s,\theta)\Bigg]. \end{aligned} \tag{8}$$

At the current time s, when the actuarial evaluations are done, we assume that the insurer possesses additional information expressed by a filtration $\mathbb{G}$ larger than filtration $\mathbb{F}$, that is, $\mathbb{F} \subseteq \mathbb{G} = \{\mathcal{G}_t,\ t \in \mathbb{N}\}$ being

$$\mathcal{G}_s := \bigcap_{t>s}\left(\mathcal{F}_t \vee \sigma(L^{>}_{a,b})\right)$$

and $L^{>}_{a,b} = 1_{\{T_{N(s)+1}\in[a,b],J_{N(s)+1}>J_{N(s)}\}}$.

From an intuitive point of view, we are assuming that the insurer has at time s some imprecise information about the next evolution of the health status of the policyholder. This is expressed by the positioning of the next change of state in the time interval $[a,b]$ and by the direction of the change expressed by the deterioration of the health denoted by the condition $J_{N(s)+1} > J_{N(s)}$.

The considered additional information may become available to the insurer for various reasons. For instance, doctors can often estimate, with some degree of approximation, how a disease might progress based on existing medical knowledge as well as the results of specific clinical tests. Consequently, the insurance company may obtain this information from healthcare professionals. From this point onward, we will refer to the $\mathbb{G}$-insurer and the $\mathbb{F}$-insurer to distinguish between those operating under the enlarged filtration and those who are not.

The incremental information brought by the random variable $L^{>}_{a,b}$ gives to the $\mathbb{G}$-insurer the possibility of determining a different evaluation of the actuarial values expressed in terms of expected benefits and premiums. To quantify this advantage, we define the expected value of the reward process under the enlarged filtration $\mathbb{G}$. Therefore we define

$$\mathbb{E}[\xi(s,t)|\mathcal{G}_s] = \mathbb{E}[\xi(s,t)|J_{N(s)}, T_{N(s)}, L^{>}_{a,b}]. \tag{9}$$

Now we introduce the notation

$$\begin{aligned} V_i(l,s;[a,b],t) &:= \mathbb{E}[\xi(s,t)|\mathcal{G}_s, J_{N(s)} = i, T_{N(s)} = l, L^{>}_{a,b} = 1] \\ &= \mathbb{E}[\xi(s,t)|J_{N(s)} = i, T_{N(s)} = l, J_{N(s)+1} > i, T_{N(s)+1} \in [a,b]], \end{aligned} \tag{10}$$

for the conditional expected value of accumulated rewards for the $\mathbb{G}$-insurer.

3.2. The evolution equation of expected rewards under the $\mathbb{G}$-filtration and fair premiums assessment

The next result gives a representation formula for the expected reward under the enlargement of filtration.

Theorem 1. *Let's consider a discrete-time non-homogeneous semi-Markov process of kernel $\mathbf{Q} = [Q_{i,j}(s,t)]$, a reward structure with permanence rewards $\Psi = \left(\psi_{J_{N(u)}}(u-T_{N(u)},u)\right)$ and transition rewards $\Gamma = \left(\gamma_{J_n,J_{n+1}}(X_{n+1},T_{n+1})\right)$ and a discount factor $\{v(s,u), u \geq s\}$. The first-order moment $V_i(l,s;[a,b],t)$ of the accumulated discounted reward process under the $\mathbb{G}$-filtration satisfies the following relations:*

- *Case 1: for $l < s < t < a < b$, we have*

$$V_i(l,s;[a,b],t) = \sum_{u=s+1}^{t}\psi_i(u-l,u)v(s,u). \tag{11}$$

- *Case 2: for $l < s < a < t < b$, we have that*

$$\begin{aligned} V_i(l,s;[a,b],t) &= \sum_{u=s+1}^{t}\psi_i(u-l,u)v(s,u)\frac{\sum_{k>i}\left(Q_{ik}(l,b)-Q_{ik}(l,t)\right)}{\sum_{k>i}\left(Q_{ik}(l,b)-Q_{ik}(l,a)\right)} \\ &+ \sum_{k>i}\sum_{\theta=a}^{t}\sum_{u=s+1}^{\theta-1}\left(\psi_i(u-l,u)v(s,u)+\gamma_{ik}(\theta-l,\theta)v(s,\theta)\right)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} \\ &+ \sum_{k>i}\sum_{\theta=a}^{t} v(s,\theta)V_k(\theta,\theta;t)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}. \end{aligned} \tag{12}$$

- *Case 3: for $l < s < a < b < t$, we find that*

$$\begin{aligned} V_i(l,s;[a,b],t) &= \sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta-1}\psi_i(u-l,u)v(s,u)\frac{\sum_{k>i}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} \\ &+ \sum_{k>i}\sum_{\theta=a}^{b}\gamma_{ik}(\theta-l,\theta)v(s,\theta)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} \\ &+ \sum_{k>i}\sum_{\theta=a}^{b} v(s,\theta)V_k(\theta,\theta;t)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}. \end{aligned}$$

Proof. See Proof of Theorem 1 in Appendix C. □

Remark 1. Theorem 1 combines the assumptions on the policyholder's health-state dynamics with the reward structure, interest rates, and the additional information described by the initial enlargement of the filtration at time s. It provides the expected value of the accumulated discounted reward process, taking into account the partial knowledge of the next transition time and the direction of transitions. Naturally, if the insurer does not possess such additional information — namely, when the next visited state $j \in I$ and the transition time $T_{N(s)+1} > s$ — one recovers the unconditional expectation as derived, for instance, in [14]. In other words, the $\mathbb{G}$-insurer reverts to being an $\mathbb{F}$-insurer.

Remark 2. It is worth noticing that the choice of the condition $J_{N(s)+1} > J_{N(s)}$ is only a possibility; in fact, a more general approach would consist of considering for every state i a subset of attainable states $A(i) \subseteq I$ according to the incremental information the insurance company possesses. The results of Theorem 1 can be readily adapted by substituting summations of states $k > i$ with the belongingness condition $k \in A(i)$.

The previous result is the key tool used in the determination of the fair premiums for the considered insurance contract. In fact, what is sufficient to do is to properly set the reward structure in such a way as to distinguish between cash inflows and cash outflows. If we adopt the insurer point of view, we collect premiums at any time the policyholder is in the perfect health state, say state 1 in our model. Thus, if we set $\forall s, u \in \mathbb{N}$

$$\psi_i(s,u) = \begin{cases} \Pi & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_{ik}(s,u) = 0 \;\; \forall i, k \tag{13}$$

and plug in these formulas inside the equation established in Theorem 1 we obtain an expected reward function $V_i(l, s; [a,b], t)$ that expresses the conditional expected accumulated discounted premiums payments. It will be denoted by $EP_i(l, s; [a,b], t)$ in the following, and it assumes different forms according to the three different Cases 1-3:

- Case 1: $l < s < t < a < b$
  $EP_1(l, s; [a,b], t) = \Pi \sum_{u=s+1}^{t} v(s,u)$
  $EP_i(l, s; [a,b], t) = 0 \quad \forall i \neq 1$

- Case 2: $l < s < a < t < b$

$$EP_1(l,s,[a,b];t) = \sum_{u=s+1}^{t} \Pi \cdot v(s,u) \frac{\sum_{k>1}(Q_{1k}(l,b) - Q_{1k}(l,t))}{\sum_{k>1}(Q_{1k}(l,b) - Q_{1k}(l,a))} + \sum_{\theta=a}^{t} \sum_{u=s+1}^{\theta-1} \Pi \cdot v(s,u) \frac{\sum_{k>1} q_{1k}(l,\theta)}{\sum_{j>1}(Q_{1j}(l,b) - Q_{1j}(l,a))}$$

$$EP_i(l,s,[a,b];t) = 0 \quad \forall i \neq 1$$

- Case 3: $l < s < a < b < t$
  $EP_1(l, s; [a,b], t) = \sum_{\theta=a}^{b} \sum_{u=s+1}^{\theta-1} \Pi \cdot v(s,u) \frac{\sum_{k>1} q_{1k}(l,\theta)}{\sum_{j>1}(Q_{1j}(l,b) - Q_{1j}(l,a))}$
  $EP_i(l, s; [a,b], t) = 0 \quad \forall i \neq 1$

On the other hand, when the health status of the policyholder is in any other state different from state 1, the insurer pays benefits to the insured person. The calculation of this expected cash outflow can be done again according to Theorem 1 making use of a different reward structure. In this case we define

$$\psi_i(s,u) = \begin{cases} 0 & \text{if } i = 1 \\ b_i(s,u) & \text{if } 1 < i < D \\ 0 & \text{if } i = D \;\; \forall j \end{cases}$$

$$\gamma_{ij}(s,u) = \begin{cases} 0 & \text{if } i = D \;\; \forall j \\ t_{ij}(s,u) & \text{if } i \neq D \end{cases}$$

where $b_i(s,u)$ is the paid benefit for the occupancy of the state i at the time u for a contract signed at the time s and $t_{ij}(s,u)$ is the transition reward paid for the change of state from i to j at time u. Injecting this reward structure into the equations of Theorem 1 gives three representations for $V_i(l, s; [a,b], t)$ that express the expected benefits and will be denoted by $EB_i(l, s; [a,b], t)$.

- Case 1: $l < s < t < a < b$
  $EB_i(l, s; [a,b], t) = \sum_{u=s+1}^{t} b_i(s-l, u)\, v(s,u) \;\; \forall i = 2, \dots, D-1$
  $EB_i(l, s; [a,b], t) = 0 \quad \forall i \in \{1, D\}$

- Case 2: $l < s < a < t < b$
  $EB_1(l, s; [a,b], t) = \sum_{k>1} \sum_{\theta=a}^{t} t_{1k}(\theta - l, \theta)\, v(s,\theta) \frac{q_{1k}(l,\theta)}{\sum_{j>1}(Q_{1j}(l,b) - Q_{1j}(l,a))}$
  $+ \sum_{k=2}^{D-1} \sum_{\theta=a}^{t} v(s,\theta)\, V_k(\theta,\theta;t) \frac{q_{1k}(l,\theta)}{\sum_{j>1}(Q_{1j}(l,b) - Q_{1j}(l,a))}$

$EB_i(l,s;[a,b],t) = \sum_{u=s+1}^{t} b_i(u-l,u)v(s,u)\frac{\sum_{k>i}(Q_{ik}(l,b)-Q_{ik}(l,t))}{\sum_{j>i}(Q_{ij}(l,b)-Q_{ik}(l,a))}$
$+\sum_{\theta=a}^{t}\sum_{u=s+1}^{\theta-1} b_i(u-l,u)v(s,u)\frac{\sum_{k>i} q_{ik}(l,\theta)}{\sum_{j>i}(Q_{ij}(l,b)-Q_{ij}(l,a))}$
$+\sum_{k>i}\sum_{\theta=a}^{t} t_{ik}(\theta-l,\theta)v(s,\theta)\frac{q_{ik}(l,\theta)}{\sum_{j>i}(Q_{ij}(l,b)-Q_{ij}(l,a))}$
$+\sum_{k=i+1}^{D-1}\sum_{\theta=a}^{t} v(s,\theta)\,V_k(\theta,\theta;t)\frac{q_{ik}(l,\theta)}{\sum_{j>i}(Q_{ij}(l,b)-Q_{ij}(l,a))} \qquad \forall i \notin \{1,D\}$

- Case 3: $l < s < a < b < t$
$EB_1(l,s;[a,b],t) = \sum_{k>1}\sum_{\theta=a}^{b} t_{1k}(\theta-l,\theta)v(s,\theta)$
$+V_k(\theta,\theta;t)v(s,\theta)\frac{q_{1k}(l,\theta)}{\sum_{j>1}(Q_{1j}(l,b)-Q_{1j}(l,a))}$
$EB_i(l,s;[a,b],t) = \sum_{k>i}\sum_{\theta=a}^{b}(\sum_{u=s+1}^{\theta-1} b_i(u-l,u)v(s,u) + t_{ik}(\theta-l,\theta)v(s,\theta) + V_k(\theta,\theta;t)v(s,\theta))\frac{q_{ik}(l,\theta)}{\sum_{j>i}(Q_{ij}(l,b)-Q_{ij}(l,a))} \qquad \forall i \notin \{1,D\}$

Once the expected premiums and expected benefits have been obtained case by case, it is possible to get the fair premiums by equating them and solving with respect to the unknown Π.

4. Valuing longevity risk via filtration enlargement

In this section, we aim at quantifying the longevity risk by applying the theory of enlargement of filtration. According to [37], longevity risk can be considered as the risk of systematic deviations from the forecasted mortality trend. This aspect is crucial when dealing with a life annuity product and portfolio ([38]).
In particular, here, we are going to extend the natural filtration generated by the process $(J_{N(t)}, T_{N(t)}, t \in \mathbb{N})$ considering a new filtration $\mathbb{G} = \{\mathcal{G}_t, t \in \mathbb{N}\}$ being

$$\mathcal{G}_s := \bigcap_{t>s}\left(\mathcal{F}_t \vee \sigma(L^{>}_{a,b}) \vee S(x)\right)$$

where $S(x) = 1_{\{J_{N(x)} \neq D\}}$.

From an intuitive point of view, we are assuming that the insurer has at the time s not only some imprecise information about the next evolution of the health status of the policyholder but also knows that at that age x the policyholder will still be alive or possibly he is going to produce a scenario analysis according to the value of x.

Consistent with our earlier notation, we define

$$V_i(l,s;[a,b],x,t) = \mathbb{E}\left[\xi(s,t)\middle| J_{N(s)} = i, T_{N(s)} = l, T_{N(s)+1} \in [a,b], J_{N(s)+1} > i, J_{N(x)} \neq D\right],$$

where D is the policyholder's death state.

For the sake of simplicity, we focus only on the case $l < s < a < b < x \le t$ that is the most interesting, as the information brought by the random variable $L^{>}_{a,b}$ is expected to be temporally antecedent to the condition expressing the longevity aspect. Similar computations can be done for the other cases.

Theorem 2. *Let's consider a discrete-time non-homogeneous semi-Markov process of kernel $\mathbf{Q} = [Q_{i,j}(s,t)]$, a reward structure with permanence rewards $\Psi = \left(\psi_{J_{N(u)}}(u - T_{N(u)}, u)\right)$ and transition rewards $\Gamma = \left(\gamma_{J_n,J_{n+1}}(X_{n+1}, T_{n+1})\right)$ and a discount factor $\{v(s,u), u \ge s\}$. The first-order moment $V_i(l,s;[a,b],x,t)$ of the accumulated discounted reward process under the $\mathbb{G}$-filtration for the case $l < s < a < b < x \le t$ satisfies the following recursive equation:*

$$\begin{aligned} V_i(l,s;[a,b],x,t) = {} & \frac{\sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)\cdot\sum_{k>i}R_k(\theta,x)\cdot q_{ik}(l,\theta)}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)q_{ir}(l,m)} \\ & + \frac{\sum_{k>i}\sum_{\theta=a}^{b}\gamma_{i,k}(\theta-l,\theta)v(s,\theta)R_k(\theta,x)q_{ik}(l,\theta)}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)q_{ir}(l,m)} \\ & + \frac{\sum_{k>i}\sum_{\theta=a}^{b}v(s,\theta)V_k(\theta,\theta;x,t)R_k(\theta,x)q_{ik}(l,\theta)}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)q_{ir}(l,m)}. \end{aligned} \tag{14}$$

where $R_k(\theta,\cdot)$ is the conditional survival function for the policyholder with health status k at the age θ and

$$V_k(\theta,\theta;x,t) = \lim_{b\to\infty} V_k(\theta,\theta;[\theta+1,b),x,t).$$

Proof. See Proof of Theorem 2 in Appendix C. □

We can notice that if we set $\psi_1 = 0$, we only collect the expected future benefits, for which we can establish the fair premiums through the equilibrium condition equating expected premiums and expected benefits. Clearly, the premiums depend on the information conditions and should be denoted by $\Pi_i(l,s;[a,b],x,t)$.
We will use $\Pi(x)$ as a shorthand for $\Pi_i(l,s;[a,b],x,t)$ to simplify notation. When we set l, s, $[a,b]$, and t, and we vary x from s to t, we observe how the reward evolves as a function of the policyholder's survival, thus highlighting the effect of x as a reflection of longevity risk and their impact on the fair premiums $\Pi(x)$.

In order to provide a more detailed view of longevity risk, we complement the previous analysis with a representation linking survival probabilities and premium levels. Let us consider an individual aged s and denote by $S_s(\cdot)$ the corresponding survival function.

To analyse the longevity risk from a probabilistic perspective, for a given confidence level $\alpha \in (0,1)$, we define

$$a_\alpha(s) = \inf\{x > s : S_s(x) \leq 1 - \alpha\}.$$

The value $a_\alpha(s)$ is the smallest remaining lifetime for which the insured individual survives longer than α% of individuals aged s. Therefore, higher values of α correspond to more extreme longevity scenarios.

For the survival age $s + a_\alpha(s)$, we evaluate the premium under the condition that the insured individual is still alive at that age. In particular, we consider the premium conditional on the event $J_{N(s+a_\alpha(s))} \neq D$ and we denote it by $\Pi(s + a_\alpha(s))$. The quantity $s + a_\alpha(s)$ therefore plays the same role as the longevity parameter x in Theorem 2, with the additional advantage that each value is now associated with a probability level α describing the frequency of the corresponding survival scenario. This allows us to study how premiums evolve as a function of the longevity confidence level through the mapping

$$\alpha \longmapsto \Pi(s + a_\alpha(s)).$$

Furthermore, let the expected remaining lifetime of an individual aged s be defined as

$$ELT(s) = \int_s^\infty S_s(x)\, dx.$$

By solving the equation

$$a_\alpha(s) = ELT(s)$$

with respect to α, we obtain a benchmark confidence level α^*. The corresponding premium $\Pi(s + a_{\alpha^*}(s))$ can be interpreted as the premium associated with a longevity scenario in which the policyholder survives beyond the expected remaining lifetime. This value provides a natural reference level for comparison with other longevity scenarios. Therefore, difference between $\Pi(s + a_\alpha(s))$ and $\Pi(s + a_{\alpha^*}(s))$ can be interpreted as the change in the premium associated with a scenario in which the insured survives to age $s + a_\alpha(s)$, occurring with probability $(1 - \alpha)$.

5. Results

In this section, we present the data used in the application and show and discuss the obtained results.

5.1. Data

The data used in this application are taken from the technical bases reported in [28]. In particular, we employ the LTC15L technical basis, which provides a coherent and publicly documented set of annual transition probabilities among the three health states — active, non-self-sufficient, and dead — for both males and females, under the low-scenario assumptions adopted in the Italian market. The dataset covers ages from 20 to 95 years and calendar years from 2013 to 2043, thus allowing us to model both age and calendar-time non-homogeneity in a way that is consistent with the semi-Markov structure introduced in the previous sections.

Figs. 1–3 illustrate the annual transition probabilities by age for both males and females. These transition patterns are consistent with standard LTC experience studies, showing low incidence at young ages, a marked increase after retirement ages, and persistent gender differentials.

The transition from the active to the non-self-sufficient state remains close to zero up to around age 70, after which it increases, reflecting the higher incidence of dependency in old age. Mortality from the active state increases steadily with age and is generally

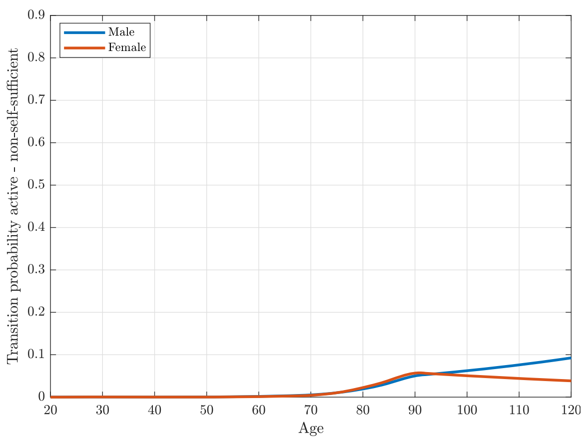


Fig. 1. Annual transition probability from active to non-self-sufficient state as a function of age.

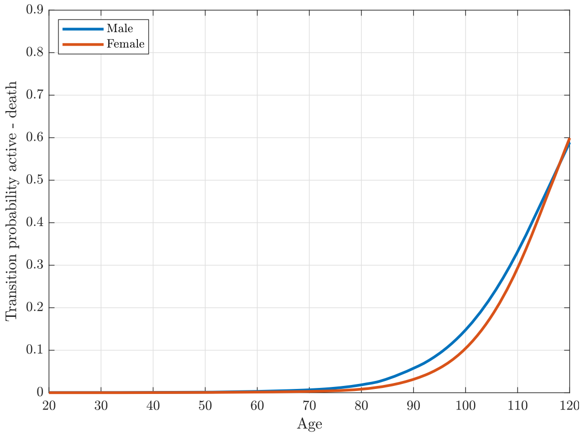


Fig. 2. Annual transition probability from active to death as a function of age.

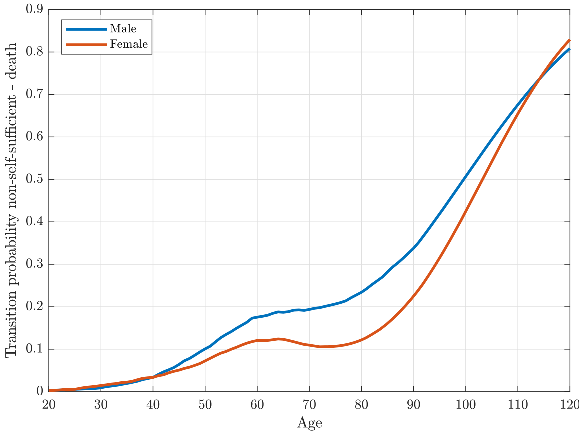


Fig. 3. Annual transition probability from non-self-sufficient to death as a function of age.

higher for males than for females. Finally, the transition from the non-self-sufficient state to death shows a sharp increase at advanced ages. Overall, the figures highlight the increasing intensity of both morbidity and mortality transitions with age and the persistence of gender differentials across all health states.

It should be stressed that these technical bases are constructed under the assumption of a stable disability assessment framework and do not explicitly account for future medical or policy shocks. For this reason, in the empirical section we interpret them as a benchmark set of probabilities on which to test the impact of the enlarged filtration, rather than as the only possible LTC experience table.

Fig. 4 shows the probability distributions of death (panels a and c) and transition from the active to the non-self-sufficient state (panels b and d) for males and females, respectively, at four different initial ages (20, 40, 60, and 80 years).
For both genders, the distributions are characterized by a peak occurring around ages 85–95, indicating that the likelihood of transition either to death or to non-self-sufficiency reaches its maximum in advanced ages. As expected, individuals with older initial ages (e.g., 80 years) experience a higher peak, reflecting the shorter expected remaining lifetime and the concentration of transition probabilities in the near term. Comparing genders, females show slightly lower death probabilities (panels a and c) across all initial ages, consistent with their higher longevity.

5.2. Application

In this section, we present the numerical results obtained from the proposed model assuming a zero interest rate ($r = 0\%$). The aim of this empirical exercise is to evaluate how anticipative information, introduced through the enlargement-of-filtration (ELF) framework, affects the actuarial valuation of long-term care (LTC) insurance contracts.

To this end, we compare the classical semi-Markov valuation framework without filtration enlargement (F-insurer) with its ELF-extended counterpart (G-insurer). This comparison provides a natural benchmark for assessing the impact of anticipative information on the valuation of insurance contracts. In particular, we analyze how the incorporation of additional information influences both the distribution of expected rewards and the corresponding fair premium structure.

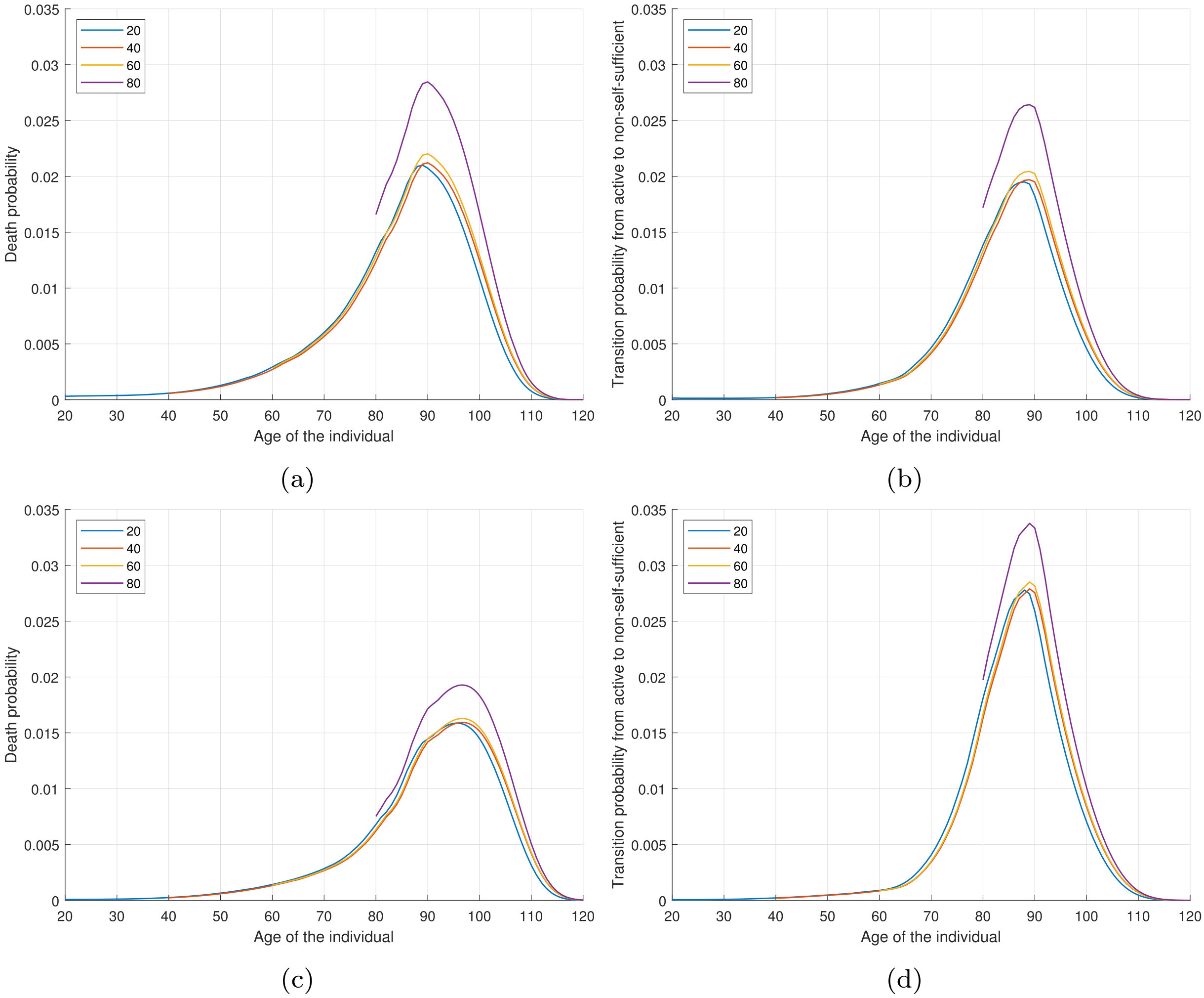


Fig. 4. Death probability distribution for male (a) and female (c) and transition probability distribution from active to non-self-sufficient for male (b) and female (d) at initial ages 20, 40, 60, 80.

The rationale for assuming a zero interest rate is to provide a clean benchmark, where the outcomes are driven solely by actuarial considerations. For comparison, the results obtained with an interest rate $r = 2\%$ are reported in Appendix A. Furthermore, to assess the impact of the constant discounting assumption, we also consider a realistic time-varying term structure of interest rates derived from the risk-free zero-coupon yield curve published by the European Insurance and Occupational Pensions Authority (EIOPA) [39], dated 31 December 2025 and illustrated in Fig. B.1. The corresponding results are presented in Appendix B.

The tables that we consider provide, for each age s and calendar year, the one-year step probabilities of death from active state $p_{13}(s)$, of transition from active to non-self-sufficient $p_{12}(s)$, and death from the non-self-sufficient $p_{23}(s)$. In this application, we select the calendar year 2025. The selected parameter values are representative of typical LTC insurance products in the Italian market and allow for an interpretable comparison across genders and age groups.

The semi-Markov process that is implemented is non-homogeneous and depends both on year t and entry age s. The probabilities $q_{1j}(s,t)$ are obtained as follows:

$$q_{1j}(s,t) = p_{1j}(s) \prod_{k=s}^{t-1} \left(1 - p_{12}(k) - p_{13}(k)\right), \quad j \in \{2,3\}, \quad t \geq s+1. \tag{15}$$

The probabilities $q_{23}(s,t)$ are computed as follows:

$$q_{23}(s,t) = p_{23}(s) \prod_{k=s}^{t-1} \left(1 - p_{23}(k)\right), \quad t \geq s+1. \tag{16}$$

The elements of the kernel $Q_{ij}(s,t)$ of the semi-Markov process are computed as the cumulative sum of elements $q_{ij}(s,t)$ representing the probability of a first transition from state i to state j occurring exactly at age t, conditional on being in state i at age s.

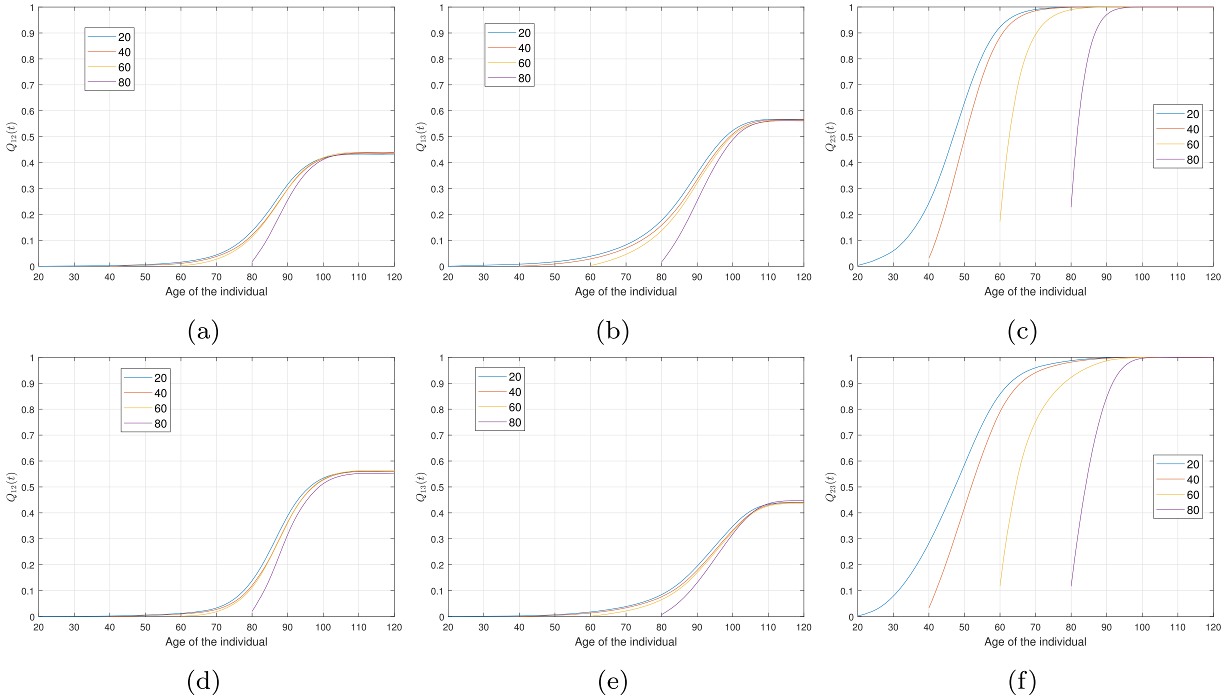


Fig. 5. Elements of kernel characterizing the transitions from active to non-self-sufficient ($Q_{12}(s,t)$), from active to death ($Q_{13}(s,t)$), and from non-self-sufficient to death ($Q_{23}(s,t)$) for male in panels (a), (b), and (c), and for female (d), (e), and (f), respectively, at initial ages $s \in \{20, 40, 60, 80\}$.

Table 1
Annual and total premiums in € by initial age and gender for a whole-life contract with lifetime horizon (t equals the insured's remaining lifetime) and discount rate $r = 0.00$.

		Initial age			
Gender	Type	20	40	60	80
Male	Annual	789.92	869.48	1,606.94	3,875.77
	Total	39,855.90	39,208.68	37,524.57	36,465.58
Female	Annual	654.89	788.53	1,432.98	3,672.23
	Total	54,536.29	51,825.71	44,545.85	44,512.25

For the computation of the fair premiums, we assume $\psi_2 = -10,000€$, $\gamma_{12} = -10,000€$, $\gamma_{13} = -20,000€$, and $\gamma_{23} = -20,000€$. Therefore, for each year in which the insured is non-self-sufficient, the insurance company pays $-10,000€$. In addition, for the transition from the active state to the non-self-sufficient state, a transition payment of $\gamma_{12} = -10,000€$ is provided. Furthermore, for the transition to the death state, the payment is $-20,000€$, independently of the starting state, that is $\gamma_{13} = \gamma_{23} = -20,000€$.

Fig. 5 reports the trends of the elements of kernel $Q_{12}(s,t)$, $Q_{13}(s,t)$, and $Q_{23}(s,t)$ at initial ages $s \in \{20, 40, 60, 80\}$ for male in panels (a), (b), and (c), and for female (d), (e), and (f), respectively.
The transitions $Q_{12}(s,t)$ and $Q_{13}(s,t)$ (from active to non-self-sufficient and from active to death) increase gradually with age and tend to stabilize around older ages. The transition $Q_{12}(s,t)$ grows more steeply, showing the higher probability of death once in state 2. When comparing male and female, male generally display higher cumulative probability of death, while females show higher probabilities of transitioning to non-self-sufficiency.

The results reported in Table 1 refer to the annual and total premiums, by initial age and gender, for a whole-life contract at $r = 0.00\%$.
Annual premiums increase monotonically with the entry age for both males and females, reflecting higher probabilities of transition to states of disability or death at older ages. At the same time, total premiums decrease with age at entry, since the expected number of payments is shorter for late entrants. Comparing genders, female annual premiums are systematically lower than male ones due to lower age-specific risks, while female total premiums are higher at younger and middle ages (20-40) because of their longer life expectancy, which implies more years of contributions. The same qualitative patterns are observed in the discounted case ($r = 2\%$, Table A.1 in Appendix A), although absolute premiums levels are lower because discounting reduces the present value of future payments. The reduction is most pronounced for young entrants in whole-life contracts, whose payments extend far into the future.

Table 2
Annual and total premiums in € by initial age and gender for a contract with duration of $t = 40$ years and $r = 0.00$.

Gender	Type	Initial age 20	40	60	80
Male	Annual	200.19	253.82	1,010.89	3,875.77
	Total	7,981.54	9,274.58	35,344.12	36,465.58
Female	Annual	89.01	144.67	843.35	3,672.23
	Total	3,349.28	7,827.60	45,181.93	44,512.25

Table 3
Annual and total premiums in € by initial age and gender for a whole-life contract with discount rate $r = 0.00$, conditional on a transition from state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years.

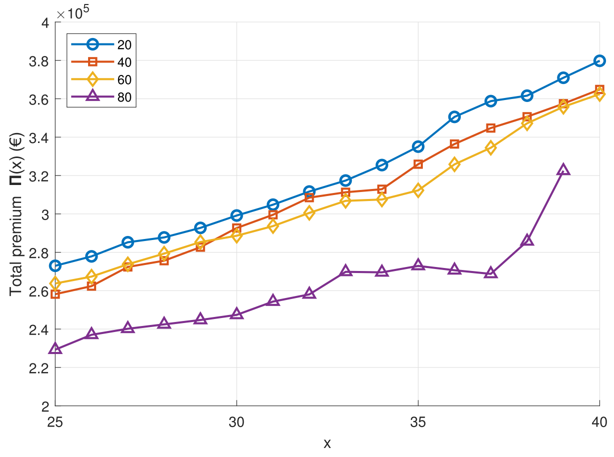
Gender	Type	Initial age 20	40	60	80
Male	Annual	4,244.64	3,060.54	2,833.52	2,603.08
	Total	60,965.96	45,749.32	42,056.77	34,725.95
Female	Annual	6,745.35	4,921.32	4,090.41	3,239.73
	Total	99,426.99	75,074.50	60,408.64	43,696.60

Table 2 shows the case of limited duration ($t = 40$ years).
The annual and total premiums at young ages are substantially lower than in the whole-life case, since much of the lifetime risk lies beyond the contractual horizon. As the entry age increases, premiums converge towards the whole-life figures, because the 40-year window overlaps almost entirely with the residual lifetime. At $r = 2\%$ (Table A.2), the same qualitative structure is preserved, but discounting lowers all premiums. Gender differences remain aligned with the whole-life case: male annual premiums exceed female ones at all ages, but female totals can surpass male totals at advanced ages due to longer expected survival.

As follows, Table 3 considers a whole-life contract under the additional condition that the insured person experiences a transition from the active state to either non-self-sufficiency or death within the interval $[a, b] = [age + 10, age + 20]$.
In this case, except for very old entry ages ($s = 80$), premiums are substantially higher than in the previous scenarios. This is because the transition to either non-self-sufficiency or death is assumed to occur within a time window that is relatively close to the beginning of the contract. For individuals aged $s = 80$, the effect of the enlarged filtration is different. In this case, the additional information guarantees that the insured remains in the healthy state for at least the next 10 years. As a consequence, the expected duration of the non-self-sufficient state is significantly reduced and the probability of observing a transition to death with a single balancing payment (equal to $-20{,}000$ €) becomes more relevant, ultimately leading to lower premiums. For both genders, annual and total premiums decrease with the entry age because the life expectancy decreases. Female annual premiums are lower than male ones, but female totals remain higher at most ages, in line with longer survival. Discounting at $r = 2\%$ (Table A.3) lowers premiums levels substantially, but the relative patterns across ages and genders are unchanged.

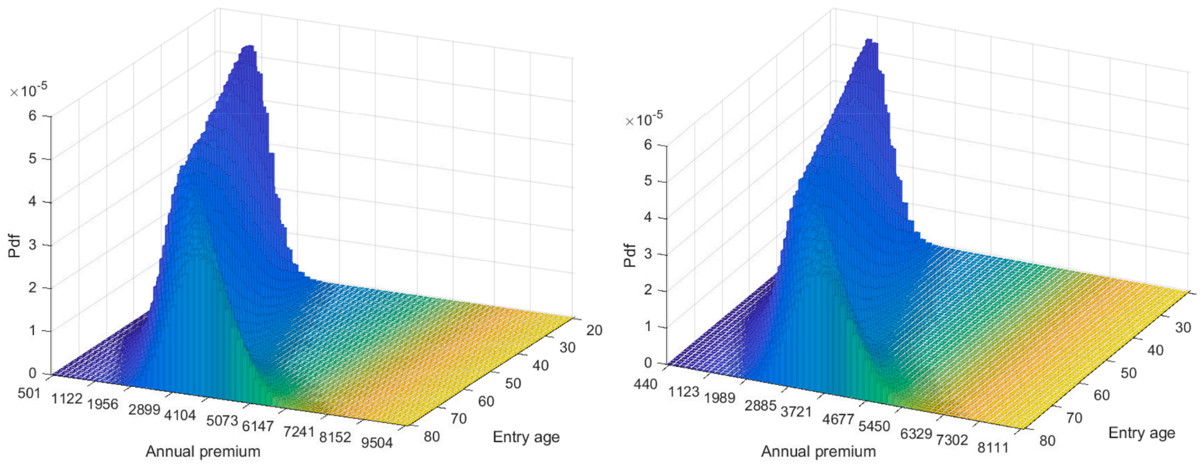
Finally, Fig. 6 illustrates the dependence of premiums on the additional longevity condition that the insured person is still alive at age+x years, with x ranging from 25 to 40 years.
At any fixed x, premiums decrease with the entry age: the highest curve corresponds to age 20, followed by 40 and 60, and the lowest to 80 years. The dependence on x is stronger for older entrants, particularly for larger values of x. Premiums increase as x rises because, conditional on survival to x and on a transition occurring in $[a, b] = [\text{age} + 10, \text{age} + 20]$, the insured is in the non-self-sufficient state by x; hence, the policy is in the benefit-paying rather than the premiums-paying phase, which raises the expected cost of the contract and therefore the fair premiums. Compared with the discounted case in Fig. A.1, discounting shifts all curves downward and attenuates the slope in x while preserving the same ordering by entry age, since the impact of payments made further in the future is reduced by discounting.

In summary, the benchmark analysis at $r = 0\%$ highlights the pure demographic and actuarial drivers of premiums, while the discounted case $r = 2\%$ in Appendix A demonstrates that positive interest rates reduce absolute levels but slightly alter the comparative analysis across entry age, gender, and contract type. Furthermore, the application of a time-varying term structure of interest rates, shown in Appendix B, leaves the qualitative behaviour unchanged, confirming the robustness of the proposed framework. These numerical findings provide the empirical foundation for the following conclusions, where we interpret their actuarial implications in light of the model's anticipative structure, see Tables B.1–B.3 and Fig. B.2. Figs. 7 and 8 provide a distributional representation of premiums that complements the average results reported in Tables 1 and 3.
Fig. 7 shows the empirical densities of annual premiums with lifetime horizon and discount rate $r = 0$. As entry age increases, the distributions shift and become less concentrated, reflecting the reduction in the effective duration of the contract. For younger insured individuals, the annual premium can be interpreted as the average of a larger number n of discounted cash-flow components over a longer time horizon. Intuitively, the variance of such an average decreases proportionally to $1/\sqrt{n}$; this effect can be explained



(a) (b)

Fig. 6. Total premiums in € by initial age and gender for a whole-life contract with discount rate $r = 0.00$, conditional on a transition from state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years and on the additional longevity condition that the insured is alive at age+x years, with $x \in [25, 40]$. Panels: (a) male, (b) female.



(a) (b)

Fig. 7. Empirical distributions of annual premium in € for a whole-life contract, by entry age and gender, with lifetime horizon and discount rate $r = 0$. Panels (a) and (b) refer to male and female, respectively.

by analogy with the Central Limit Theorem for dependent random variables, although a rigorous proof is beyond the scope of this work. Consequently, for lower entry ages, the empirical distributions appear more concentrated around their mean, whereas for higher entry ages, the shorter horizon implies greater dispersion. Male and female distributions display similar shapes. Fig. 8 considers the same representation under the additional conditioning on a transition from the healthy state to either disability or death occurring in the interval $[\text{age} + 10, \text{age} + 20]$. In this case, premiums are substantially higher, and the distributions are shifted to the right, especially for younger entry ages. The conditioning on a transition within a fixed time window reduces the uncertainty surrounding the timing of future events. As a consequence, the width of the empirical distributions decreases, reflecting the lower level of uncertainty embedded in the premium valuation. Fig. 9 summarizes these distributional effects by reporting the standard deviation of the premium distributions for insured males under the two scenarios.

Finally, Fig. 10 provides the empirical distributions of total premium in € and complements the result reported in Fig. 6. In particular, it focuses on the role of longevity by fixing the entry age at 20 years old and allowing the survival age x to vary.
We can notice that as x increases, the distributions progressively shift toward higher values, capturing the increasing cost of longer survival.

Furthermore, we further extend the previous analysis with an additional representation linking survival information to the premium structure. In particular, we investigate how different survival quantiles translate into premium levels through the mapping

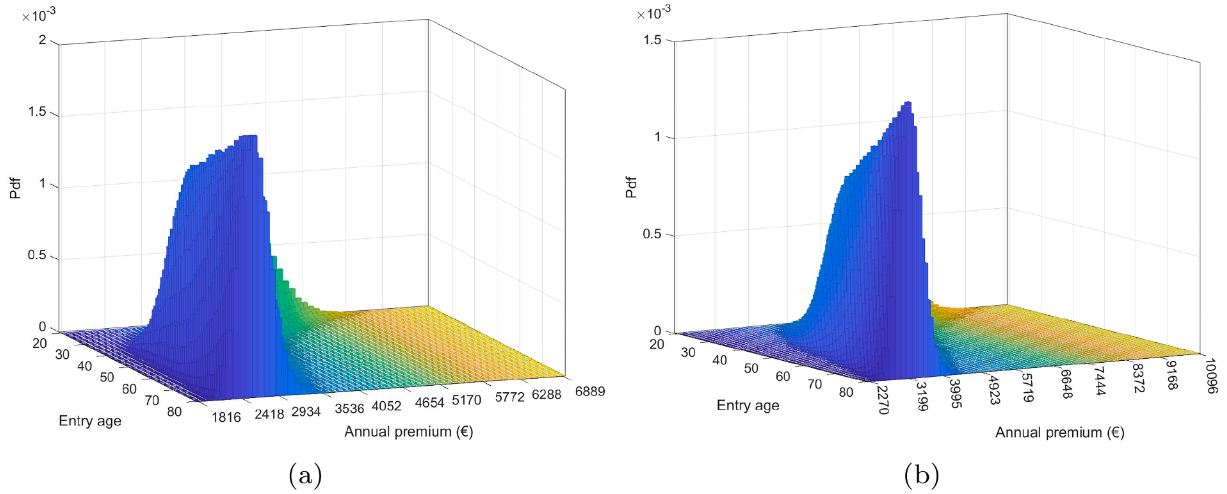


Fig. 8. Empirical distributions of annual premium in € for a whole-life contract, by entry age and gender, with lifetime horizon and discount rate $r = 0$, conditional on a transition from the state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years. Panels (a) and (b) refer to male and female, respectively.

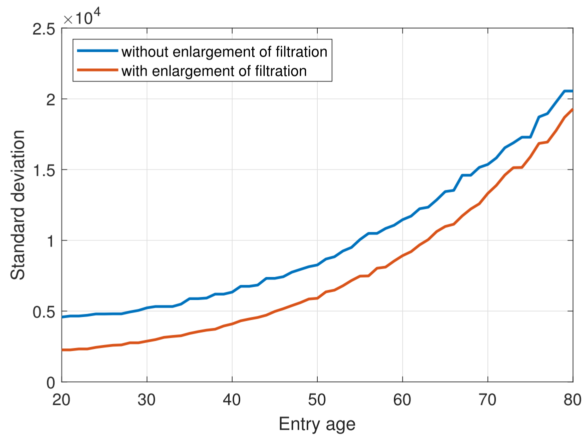


Fig. 9. Standard deviation of the premium distributions by initial age for male insureds under the two considered scenarios: unconditional valuation (Fig. 7), and with enlargement of filtration by conditioning on a transition within the interval $[\text{age} + 10, \text{age} + 20]$ (Fig. 8).

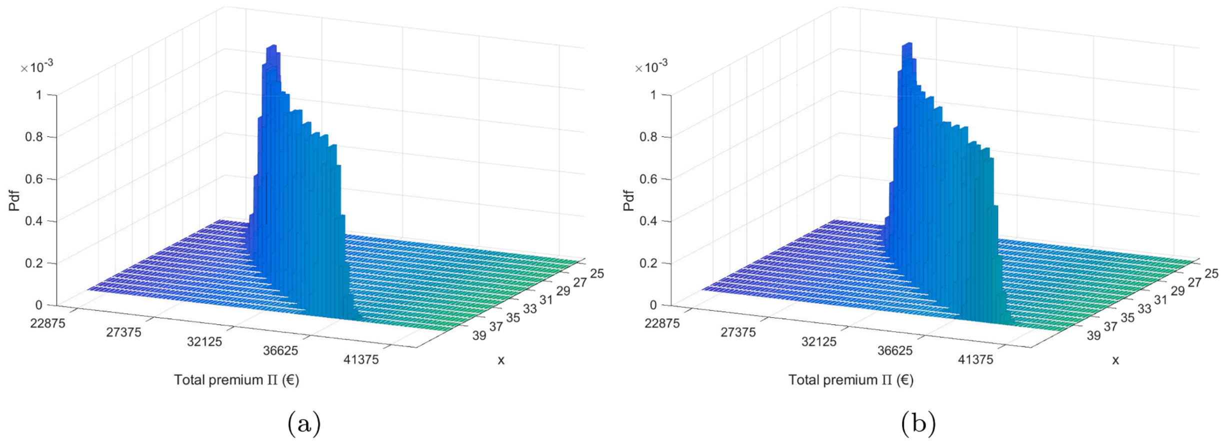


Fig. 10. Empirical distributions of total premium in € for a whole-life contract and a fixed entry age of 20 years old, by x and gender, with lifetime horizon and discount rate $r = 0$, conditional on a transition from the state 1 to state 2 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years and on the additional longevity condition that the insured is alive at age $+ x$ year, with $x \in [25, 40]$. Panels (a) and (b) refer to male and female, respectively.

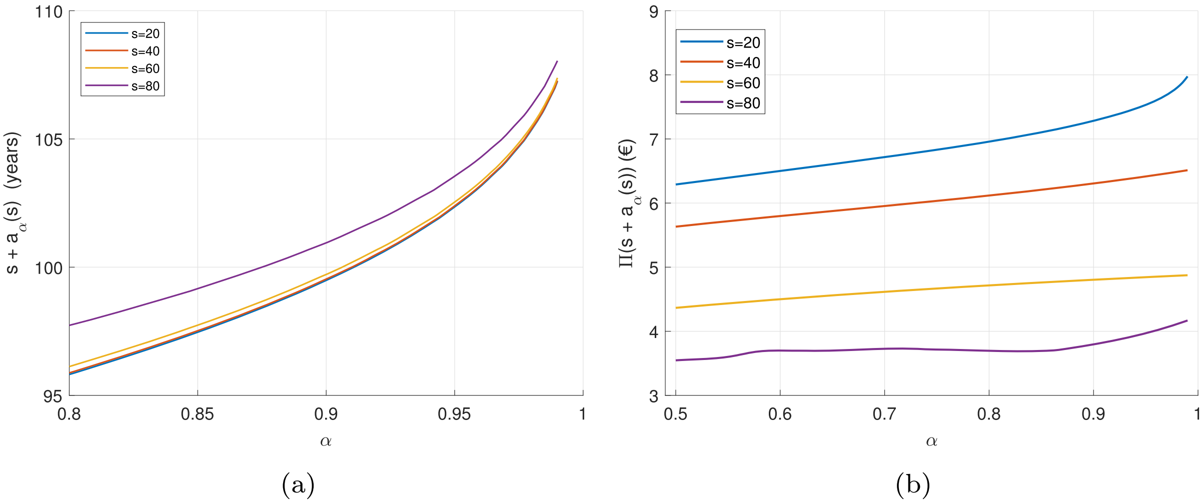


Fig. 11. Survival age and premium mapping for male. Panel (a): $a_\alpha(s)$ as a function of α for entry ages $s \in \{20, 40, 60, 80\}$. Panel (b): $\Pi\big(a_\alpha(s)\big)$ [$\cdot 10^4$ €] for $\alpha \in [0.50, 0.99]$.

Table 4
Mapping of longevity and total premium (€): benchmark α^*, implied survival age $s + a_{\alpha^*}(s) = s + ELT(s)$, and total premium $\Pi(s + a_{\alpha^*}(s))$.

s	$s + a_{\alpha^*}(s)$	α^*	$\Pi(s + a_{\alpha^*}(s))$
20	79.60	0.52	63,000
40	82.10	0.54	56,000
60	84.20	0.56	44,000
80	88.00	0.60	36,000

between survival ages and total premiums. We present the analysis for insured males, while analogous results for females follow the same patterns. In this regard, Fig. 11 illustrates the mapping between the survival function and the premium levels for insured males. In particular, Fig. 11(a) reports the survival age $s + a_\alpha(s)$ as a function of the confidence level α for entry ages $s \in \{20, 40, 60, 80\}$. In Fig. 11(b), we evaluate the total premium as a function of the corresponding survival age, i.e., $\Pi(s + a_\alpha(s))$.
For each initial age, the function $s + a_\alpha(s)$ represents the age corresponding to the α-quantile of the survival distribution. As expected, $s + a_\alpha(s)$ is increasing in α, since higher confidence levels correspond to more extreme longevity scenarios. Moreover, for a fixed α, the survival age is higher for older entry ages, reflecting the conditional nature of survival probabilities. Furthermore, the total premium increases with α for all entry ages. This behaviour is consistent with the actuarial interpretation of α as a longevity risk level: higher values of α correspond to more demanding survival scenarios, which increase the expected cost of the contract. At the same time, for any fixed α, total premiums are higher for younger entry ages, since the remaining contract horizon is longer and the potential duration of benefit payments is larger. Conversely, when the policyholder enters the contract at older ages, the remaining lifetime horizon is shorter, and the total premium decreases accordingly.

Table 4 reports the benchmark confidence level α^*, defined such that $a_{\alpha^*}(s) = ELT(s)$, where $ELT(s)$ denotes the expected remaining lifetime of an individual aged s. The table also reports the corresponding survival age $s + a_{\alpha^*}(s)$ and the total premium evaluated at that age, $\Pi(s + a_{\alpha^*}(s))$.
The benchmark survival age $s + a_{\alpha^*}(s)$ increases with the entry age, reflecting the conditional survival perspective. At the same time, the associated total premium $\Pi(s + a_{\alpha^*}(s))$ decreases as the entry age increases, because the contract is expected to last for a shorter period. Overall, this representation provides a complementary interpretation of the premium structure by linking survival risk levels directly to the economic cost of the contract.

6. Conclusions

This paper has proposed a discrete-time non-homogeneous semi-Markov reward framework extended through an enlarged filtration mechanism to incorporate anticipative information in the evaluation of disability insurance contracts. The approach allows for the explicit modeling of duration dependence and for the conditional assessment of expected discounted rewards under different information settings. Theoretical developments have been complemented by an empirical application based on LTC transition probabilities by age and gender, demonstrating that the inclusion of anticipative information systematically reduces the uncertainty of transition outcomes and stabilizes the corresponding premiums.

From a regulatory and actuarial standpoint, the ELF extension acts as a formal mechanism to incorporate forward-looking knowledge—such as medical assessments or early indicators of dependency—into the pricing framework. This leads to a more stable evaluation framework and can inform the design of innovative disability insurance products where premiums dynamically adjust to updated health information, in line with the growing emphasis on data-driven risk assessment under frameworks, such as IFRS 17 and Solvency II.

Future research may focus on embedding the ELF structure within continuous-time or stochastic mortality frameworks, integrating medical or behavioural indicators as early signals of dependency, and exploring AI-based estimation techniques for transition intensities. Such extensions would further strengthen the practical relevance of the model and its adaptability to evolving regulatory and data environments.

Data availability

The authors do not have permission to share data.

Declaration of competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Results for r = 2%

Table A.1
Annual and total premiums in € by initial age and gender for a whole-life contract with lifetime horizon (t equals the insured's remaining lifetime) and discount rate $r = 0.02$.

		Initial age			
Gender	Type	20	40	60	80
Male	Annual	300.77	349.78	919.65	2,737.49
	Total	11,062.36	11,639.26	15,439.87	21,676.35
Female	Annual	261.72	327.28	797.61	2,410.65
	Total	13,556.65	14,953.55	18,210.47	25,796.86

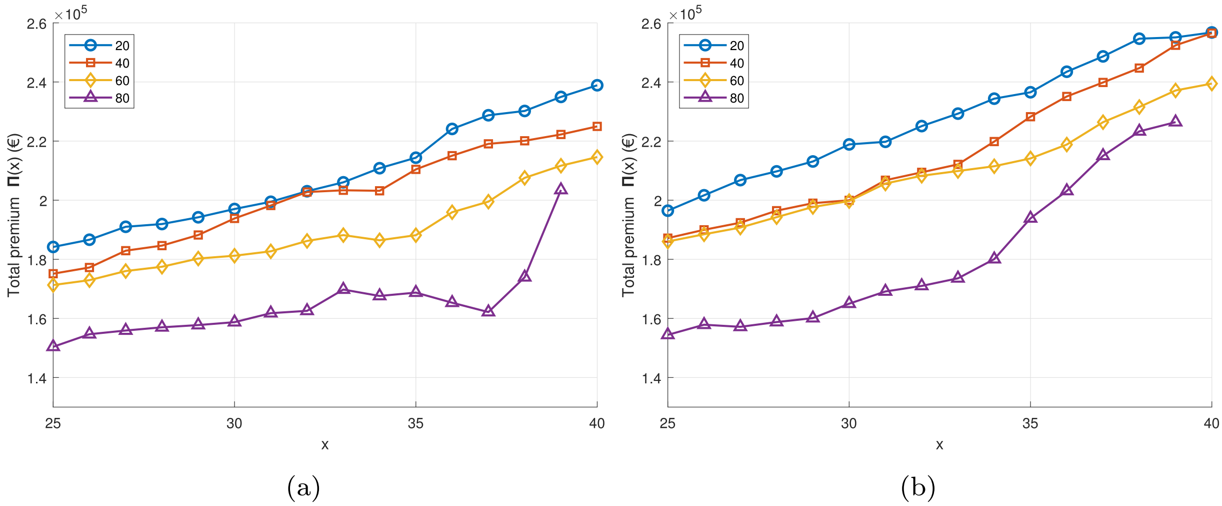


Fig. A.1. Total premiums in € by initial age and gender for a whole-life contract with discount rate $r = 0.02$, conditional on a transition from state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years and on the additional longevity condition that the insured is alive at age+x years, with $x \in [25, 40]$. Panels: (a) male, (b) female.

Table A.2
Annual and total premiums in € by initial age and gender for a contract with duration of $t = 40$ years and $r = 0.02$.

Gender	Type	Initial age 20	40	60	80
Male	Annual	138.03	174.71	704.25	2,737.49
	Total	4,154.04	5,103.67	19,118.61	21,676.35
Female	Annual	56.35	94.26	539.74	2,410.65
	Total	1,753.93	3,204.68	21,061.74	25,796.86

Table A.3
Annual and total premiums in € by initial age and gender for a whole-life contract with discount rate $r = 0.02$, conditional on a transition from state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years.

Gender	Type	Initial age 20	40	60	80
Male	Annual	2,835.75	2,135.04	1,961.89	1,931.51
	Total	40,753.46	31,930.84	29,119.86	25,785.23
Female	Annual	3,143.30	2,731.89	2,379.26	2,192.65
	Total	61,681.41	50,251.13	40,985.04	32,103.17

Appendix B. Results for time-varying discount rate

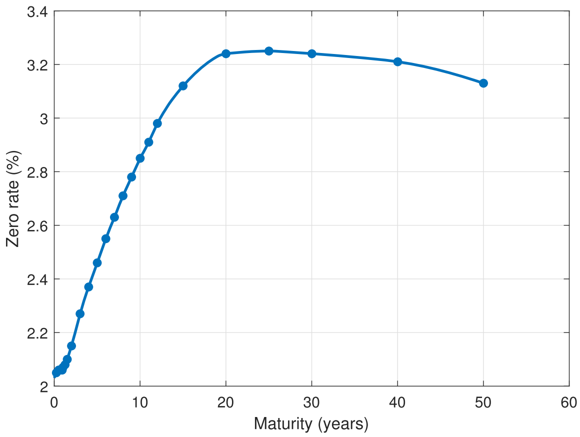


Fig. B.1. Zero-coupon curve on 31/12/2025 used in the time-varying discounting scenario.

Table B.1
Annual and total premiums in € by initial age and gender for a whole-life contract with lifetime horizon (t equals the insured's remaining lifetime) and time-varying discount rate.

Gender	Type	Initial age 20	40	60	80
Male	Annual	268.41	317.66	872.78	2,699.32
	Total	9,120.55	9,842.46	14,021.65	21,123.41
Female	Annual	234.19	296.94	759.48	2,366.27
	Total	11,188.30	12,440.15	16,505.53	25,150.12

Table B.2
Annual and total premiums in € by initial age and gender for a contract with duration of $t = 40$ years and time-varying discount rate.

		Initial age			
Gender	Type	20	40	60	80
Male	Annual	124.94	159.44	672.80	2,699.32
	Total	3,000.33	4,155.57	18,980.69	21,123.41
Female	Annual	50.99	86.18	516.16	2,366.27
	Total	2,202.84	2,842.61	20,020.40	25,150.12

Table B.3
Annual and total premiums in € by initial age and gender for a whole-life contract with time-varying discount rate, conditional on a transition from state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years.

		Initial age			
Gender	Type	20	40	60	80
Male	Annual	2,640.55	1,982.44	1,826.37	1,742.18
	Total	38,420.76	29,884.15	27,116.42	24,608.53
Female	Annual	2,918.33	2,471.06	2,183.52	2,041.27
	Total	58,330.48	47,215.39	38,642.77	31,504.86

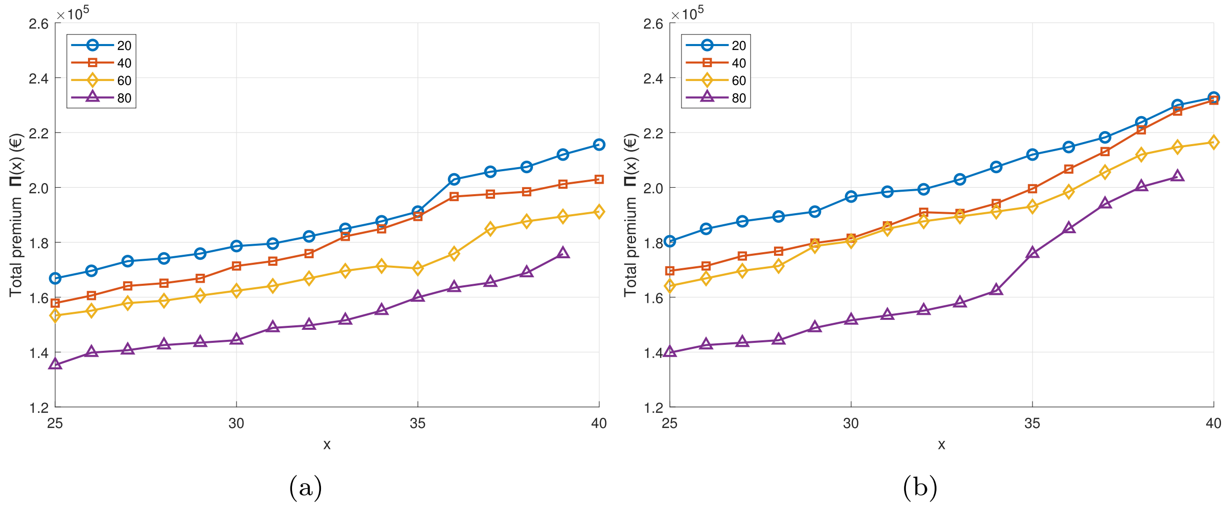


Fig. B.2. Total premiums in € by initial age and gender for a whole-life contract with time-varying discount rate, conditional on a transition from state 1 to state 2 or 3 occurring in the interval $[a, b] = [\text{age} + 10, \text{age} + 20]$ years and on the additional longevity condition that the insured is alive at age+x years, with $x \in [25, 40]$. Panels: (a) male, (b) female.

Appendix C. Proofs

Proof of Theorem 1. The following information set is introduced for notational convenience:

$$\mathcal{A}_{i,l}^{(s)}[a, b] = \{J_{N(s)} = i, T_{N(s)} = l, J_{N(s)+1} > i, T_{N(s)+1} \in [a, b]\}.$$

Proof. Let us consider the three cases separately.

- Case 1: According to the definition of the accumulated reward process we have

$$V_i(l, s; [a, b], t) = \mathbb{E}\left[\sum_{u=s+1}^{t} \psi_{J_{N(u)}}(u - T_{N(u)}, u)\, v(s, u)\right] + \mathbb{E}\left[\sum_{s=1}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1}, J_{N(s)+\tau}}(X_{N(s)+\tau}, T_{N(s)+\tau}) v(s, T_{N(s)+\tau})\right]. \tag{C.1}$$

  We know that $T_{N(s)+1} \in [a, b]$, and $s < t < a$, this implies that $N(t) = N(s)$ and therefore that no transition reward can occur; that is

$$\sum_{\tau=1}^{0} \gamma_{J_{N(s)+\tau-1}, J_{N(s)+\tau}}(X_{N(s)+\tau}, T_{N(s)+\tau}) v(s, T_{N(s)+\tau}) = 0.$$

Furthermore, since $T_{N(s)+1} \in [a,b]$ with $a > t$, we have that $J_{N(u)} = i\ \forall u \in [s+1,t] \Rightarrow V_i(l,s;[a,b],t) = \sum_{u\geq s+1}^{t} \psi_i(u-l,u)v(s,u)$.

- Case 2: For $l < s < a < t < b$, we know that $T_{N(s)+1} \in [a,b]$ and $t \in [a,b]$. Hence, we have to consider two subcases according to whether $T_{N(s)+1} \in [a,t]$ or $T_{N(s)+1} \in [t+1,b]$. This allows the following decomposition of the reward function:

$$V_i(l,s;[a,b],t) = \mathbb{E}\Bigg[\sum_{u=s+1}^{t} \psi_{J_{N(u)}}(u-T_{N(u)},u)v(s,u)1_{\{T_{N(s)+1}\in[t+1,b]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg] \tag{C.2}$$

$$+\,\mathbb{E}\Bigg[\sum_{u\geq s+1}^{t} \psi_{J_{N(u)}}(u-T_{N(u)},u)v(s,u)1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg] \tag{C.3}$$

$$+\,\mathbb{E}\Bigg[\sum_{\tau=1}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})1_{\{T_{N(s)+1}\in[t+1,b]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg] \tag{C.4}$$

$$+\,\mathbb{E}\Bigg[\sum_{\tau=1}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg] \tag{C.5}$$

At this point, we start by analyzing the expected values (C.2) and (C.4). We know that $T_{N(s)+1} \in [t+1,b] \Rightarrow J_{N(u)} = i\ \forall u \in [s+1,t]$ and $N(t) = N(s)$. This makes the expected value (C.4) equal to zero, as the upper limit of the summation is null. The expected value in (C.2) becomes

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{u=s+1}^{t} \psi_{J_{N(u)}}(u-T_{N(u)},u)v(s,u)1_{\{T_{N(s)+1}\in[t+1,b]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]\\
&= \sum_{u=s+1}^{t} \psi_i(u-l),u)v(s,u)\cdot \mathbb{P}(T_{N(s)+1}\in[t+1,b]|\mathcal{A}_{i,l}^{(s)}[a,b])\\
&= \sum_{u=s+1}^{t} \psi_i(u-l,u)v(s,u)\cdot \frac{\mathbb{P}(T_{N(s)+1}\in[t+1,b], J_{N(s)+1}>i|J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}\in[a,b])}{\mathbb{P}(J_{N(s)+1}>i|J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}\in[a,b])}\\
&= \sum_{u=s+1}^{t} \psi_i(u-l,u)v(s,u)\cdot \frac{\sum_{k>i}\sum_{\theta=t+1}^{b}\mathbb{P}(J_{N(s)+1}=k, T_{N(s)+1}=\theta|J_{N(s)}=i, T_{N(s)}=l)}{\sum_{k>i}\sum_{\theta=a}^{b}\mathbb{P}(J_{N(s)+1}=k, T_{N(s)+1}=\theta|J_{N(s)}=i, T_{N(s)}=l)}\\
&= \sum_{u=s+1}^{t} \psi_i(u-l,u)v(s,u)\cdot \frac{\sum_{k>i}\sum_{\theta=t+1}^{b} q_{ik}(l,\theta)}{\sum_{k>i}\sum_{\theta=a}^{b} q_{ik}(l,\theta)}.
\end{aligned} \tag{C.6}$$

Since we know that $\sum_{\theta=t+1}^{b} q_{ik}(l,\theta) = Q_{ik}(l,b) - Q_{ik}(l,t)$ and $\sum_{\theta=a}^{b} q_{ik}(l,\theta) = Q_{ik}(l,b) - Q_{ik}(l,a)$, previous equation becomes equal to

$$\sum_{u\geq s+1}^{t} \psi_i(u-l,u)v(s,u)\cdot \frac{\sum_{k>i} Q_{ik}(l,b) - Q_{ik}(l,t)}{\sum_{k>i} Q_{ik}(l,b) - Q_{ik}(l,a)}. \tag{C.7}$$

Now, let us consider the expected values (C.3) and (C.5). We notice that when $T_{N(s)+1} \in [a,t]$, we decompose the expectation by considering also the next visited state $J_{N(s)+1}$ as follows:

$$\mathbb{E}\Bigg[\sum_{u=s+1}^{t} \psi_{J_{N(u)}}(u-T_{N(u)},u)v(s,u)1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]$$

$$= \mathbb{E}\Bigg[\sum_{u=s+1}^{T_{N(s)+1}} \psi_{J_{N(u)}}(u-l,u)v(s,u)1_{\{T_{N(s)+1}\in[a,t]\}} \tag{C.8}$$

$$+ \sum_{u=T_{N(s)+1}}^{t} \psi_{N(s)+1}(u-T_{N(s)+1},u)v(s,u)1_{\{T_{N(s)+1}\in[a,t]\}} \tag{C.9}$$

$$+ \gamma_{i,J_{N(s)+1}}(X_{N(s)+1},T_{N(s)+1})v(s,T_{N(s)+1})1_{\{T_{N(s)+1}\in[a,t]\}} \tag{C.10}$$

$$+ \sum_{\tau=2}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]. \tag{C.11}$$

Now, to compute previous expectations, we need to express the following probability:

$$\begin{aligned}
&\mathbb{P}(J_{N(s)+1}=k, T_{N(s)+1}=\theta|\mathcal{A}_{i,l}^{(s)}[a,b])\\
&= \frac{\mathbb{P}(J_{N(s)+1}=k, T_{N(s)+1}=\theta, J_{N(s)+1}>i|J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}\in[a,b])}{\mathbb{P}(J_{N(s)+1}>i|J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}\in[a,b])}\\
&= \frac{1_{\{k>i\}}\cdot\mathbb{P}(J_{N(s)+1}=k, T_{N(s)+1}=\theta|J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}\in[a,b])}{\mathbb{P}(J_{N(s)+1}>i|J_{N(s)}=i, T_{N(s)}=l, T_{N(s)+1}\in[a,b])}
\end{aligned}$$

$$= \frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}q_{ik}(l,\theta)}{\sum_{j>i}\sum_{\theta=a}^{b} q_{ij}(l,\theta)} = \frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}. \tag{C.12}$$

Indeed, we can compute the expectations (C.8) and (C.10). It follows that

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{u=s+1}^{T_{N(s)+1}} \psi_{J_{N(u)}}(u-l,u)v(s,u)1_{\{T_{N(s)+1}\in[a,t]\}}\\
&+\gamma_{i,J_{N(s)+1}}(X_{N(s)+1},T_{N(s)+1})v(s,T_{N(s)+1})1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg]\\
&=\sum_{k\in I}\sum_{\theta\geq s+1}\left[\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)1_{\{\theta\in[a,t]\}}+\gamma_{i,k}(\theta-l,\theta)v(s,\theta)1_{\{\theta\in[a,t]\}}\right]\\
&\cdot\frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}\cdot q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}\\
&=\sum_{k>i}\sum_{\theta=a}^{t}\left(\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)+\gamma_{i,k}(\theta-l,\theta)v(s,\theta)\right)\\
&\cdot\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}.
\end{aligned}\tag{C.13}$$

Now, it remains to compute the expectations (C.9) and (C.11). To this end, we make use of the consistency property of the discount factor structure, i.e. the fact that

$$v(s,u)=v(s,t)v(t,u),\ \ \forall t\in(s,u),$$

In this way, we obtain that

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{u=T_{N(s)+1}}^{t}\psi_{J_{N(s)+1}}(u-T_{N(s)+1},u)v(s,u)1_{\{T_{N(s)+1}\in[a,t]\}}\\
&+\sum_{\tau=2}^{N(t)-N(s)}\gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]\\
&=\mathbb{E}\Bigg[\sum_{u=T_{N(s)+1}}^{t}\psi_{J_{N(s)+1}}(u-T_{N(s)+1},u)v(s,T_{N(s)+1})v(T_{N(s)+1},u)\cdot 1_{\{T_{N(s)+1}\in[a,t]\}}\\
&+\sum_{\tau=2}^{N(t)-N(s)}\gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+1})v(T_{N(s)+1},T_{N(s)+\tau})\\
&\cdot 1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg].
\end{aligned}\tag{C.14}$$

At this point, we change the variable in the second summation by setting $m=\tau-1$, and by applying the definition of the reward function we obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[\mathbb{E}\Big[\sum_{u=T_{N(s)+1}+1}^{t}\psi_{J_{N(s)+1}}(u-T_{N(s)+1},u)v(s,T_{N(s)+1})v(T_{N(s)+1},u)\cdot 1_{\{T_{N(s)+1}\in[a,t]\}}\\
&+\sum_{m=1}^{N(t)-(N(s)+1)}\gamma_{J_{N(s)+1+(m-1)},J_{N(s)+1+m}}(X_{N(s)+1+m},T_{N(s)+1+m})\\
&\cdot v(s,T_{N(s)+1})v(T_{N(s)+1},T_{N(s)+1+m})1_{\{T_{N(s)+1}\in[a,t]\}}\Big|\mathcal{A}_{i,l}^{(s)}[a,b],J_{N(s)+1},T_{N(s)+1}\Big]\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]\\
&=\mathbb{E}\Bigg[v(s,T_{N(s)+1})\cdot V_{J_{N(s)+1}}(T_{N(s)+1},T_{N(s)+1},t)\cdot 1_{\{T_{N(s)+1}\in[a,t]\}}\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]\\
&=\sum_{k\in I}\sum_{\theta\geq s+1}v(s,\theta)\cdot V_k(\theta,\theta,t)1_{\{\theta\in[a,t]\}}\cdot\frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}
\end{aligned}\tag{C.15}$$

Therefore, for case 2 ($l<s<a<t<b$), we obtain that

$$\begin{aligned}
V_{i,i^>}(l,s,[a,b];t)=&\sum_{u\geq s+1}^{t}\psi_i(u-l,u)v(s,u)\frac{\sum_{k>i}\left(Q_{ik}(l,b)-Q_{ik}(l,t)\right)}{\sum_{k>i}\left(Q_{ik}(l,b)-Q_{ik}(l,a)\right)}\\
&+\sum_{k>i}\sum_{\theta=a}^{t}\sum_{u=s+1}^{\theta}\left(\psi_i(i-l,u)v(s,t)+\gamma_{ik}(\theta-l,\theta)v(s,\theta)\right)
\end{aligned}$$

$$\cdot \frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} + \sum_{k>i}\sum_{\theta=a}^{t} v(s,t)V_k(\theta,t)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}. \tag{C.16}$$

- Case 3: for $l < s < a < b < t$, by definition, we obtain:

$$\begin{aligned} V_i(l,s;[a,b],t) = {} & \mathbb{E}\left[\sum_{u=s+1}^{t} \psi_{J_{N(u)}}(u-T_{N(u)},u)v(s,u)\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right] \\ & + \mathbb{E}\left[\sum_{\tau=1}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right]. \end{aligned} \tag{C.17}$$

Let us consider the next visited random state $J_{N(s)+1} > i$ and random time of the next transition $T_{N(s)+1} \in [a,b]$ with $b < t$. Indeed, we have

$$\begin{aligned} V_i(l,s;[a,b],t) = {} & \mathbb{E}\left[\sum_{u=s+1}^{T_{N(s)+1}} \psi_i(u-l,u)v(s,u)\right. \\ & \left. + \sum_{u=T_{N(s)+1}+1}^{t} \psi_{J_{N(u)}}(u-T_{N(s)+1},u)v(s,u)\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right] \qquad \text{(C.18)} \\ & + \mathbb{E}\left[\gamma_{i,J_{N(s)+1}}(X_{N(s)+1},T_{N(s)+1})v(s,T_{N(s)+1})\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right] \qquad \text{(C.19)} \\ & + \mathbb{E}\left[\sum_{\tau=2}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right]. \qquad \text{(C.20)} \end{aligned}$$

Now, we can compute the first part of the expectation (C.18), as follows:

$$\begin{aligned} & \mathbb{E}\left[\sum_{u=s+1}^{T_{N(s)+1}} \psi_i(u-l,u)v(s,u)\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right] \\ & = \sum_{k\in I}\sum_{\theta\ge s+1}\sum_{u=s+1}^{\theta} \psi_i(u-l,u)v(s,u)\cdot\mathbb{P}\left(J_{N(s)+1}=k,T_{N(s)+1}=\theta\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right) \\ & = \sum_{k\in I}\sum_{\theta\ge s+1}\sum_{u=s+1}^{\theta} \psi_i(u-l,u)v(s,u)\cdot\frac{1_{\{k>1\}}1_{\{\theta\in[a,b]\}}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} \\ & = \sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta} \psi_i(u-l,u)v(s,u)\frac{\sum_{k>i}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)}. \end{aligned} \tag{C.21}$$

Then, we can move to the computation of the expectation (C.19), as follows:

$$\begin{aligned} & \mathbb{E}\left[\gamma_{i,J_{N(s)+1}}(X_{N(s)+1},T_{N(s)+1})v(s,T_{N(s)+1})\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right] \\ & = \sum_{k\in I}\sum_{\theta\ge s+1}\gamma_{i,k}(\theta-l,\theta)v(s,\theta)\frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} \\ & = \sum_{k>i}\sum_{\theta=a}^{b}\gamma_{i,k}(\theta-l,\theta)v(s,\theta)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b)-Q_{ij}(l,a)\right)} \end{aligned} \tag{C.22}$$

It remains to compute the following expectations:

$$\begin{aligned} & \mathbb{E}\left[\sum_{u=T_{N(s)+1}+1}^{t} \psi_{J_{N(u)}}(u-T_{N(s)+1},u)v(s,u)\right. \\ & \left. + \sum_{\tau=2}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})\middle|\mathcal{A}_{i,l}^{(s)}[a,b]\right]. \end{aligned} \tag{C.23}$$

Following the same approach previously used, we change the variable in (C.23) by setting $m = \tau - 1$, and we apply again the consistency property of the discount factor to obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[\mathbb{E}\Big[\sum_{u=T_{N(s)+1}+1}^{t} \psi_{J_{N(u)}}(u - T_{N(s)+1}, u)v(s, T_{N(s)+1})v(T_{N(s)+1}, u)\\
&+ \sum_{m=1}^{N(t)-\left(N(s)+1\right)} \gamma_{J_{N(s)+1+(m-1)}, J_{N(s)+1+m}}(X_{N(s)+1+m}, T_{N(s)+1+m})\\
&v(s, T_{N(s)+1})v(T_{N(s)+1}, T_{N(s)+1+m})\Big|\mathcal{A}_{i,l}^{(s)}[a,b], J_{N(s)+1}, T_{N(s)+1}\Big]\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]\\
&= \mathbb{E}\Bigg[v(s, T_{N(s)+1}) \cdot V_{J_{N(s)+1}}(T_{N(s)+1}, T_{N(s)+1}, t)\Bigg|\mathcal{A}_{i,l}^{(s)}[a,b]\Bigg]\\
&= \sum_{k\in I}\sum_{\theta\geq s+1} v(s,\theta)\cdot V_k(\theta,\theta,t)\cdot \frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b) - Q_{ij}(l,a)\right)}\\
&= \sum_{k>i}\sum_{\theta=a}^{b} v(s,\theta)V_k(\theta,\theta,t)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b) - Q_{ij}(l,a)\right)}.
\end{aligned}$$

Indeed, for Case 3, we find that

$$\begin{aligned}
V_i(l, s; [a,b], t) = &\sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta} \psi_i(u-l,u)v(s,u)\frac{\sum_{k>i} q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b) - Q_{ij}(l,a)\right)}\\
&+ \sum_{k>i}\sum_{\theta=a}^{b}\gamma_{ik}(\theta-l,\theta)v(s,\theta)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b) - Q_{ij}(l,a)\right)}\\
&+ \sum_{k>i}\sum_{\theta=a}^{b} v(s,\theta)V_k(\theta,\theta,t)\frac{q_{ik}(l,\theta)}{\sum_{j>i}\left(Q_{ij}(l,b) - Q_{ij}(l,a)\right)}.
\end{aligned}$$

□

Proof of Theorem 2. The following information set is introduced for notational convenience:

$$\mathcal{A}_{i,l,x}^{(s)}[a,b] = \mathcal{A}_{i,l}^{(s)}[a,b] \cup \{J_{N(x)} \neq D\}.$$

Proof. Using the definition of the accumulated discounted reward process $\xi(s,t)$ we have that

$$\begin{aligned}
V_i(l, s; [a,b], x, t) = &\mathbb{E}\Bigg[\sum_{u=s+1}^{t} \psi_{J_{N(u)}}(u - T_{N(u)})v(s,u)\Bigg|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Bigg]\\
&+ \mathbb{E}\Bigg[\sum_{\tau=1}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1}, J_{N(s)+\tau}}(X_{N(s)+\tau}, T_{N(s)+\tau})v(s, T_{N(s)+\tau})\Bigg|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Bigg].
\end{aligned}$$

Let consider the next visited state and time of transition $(J_{N(s)+1}, T_{N(s)+1})$. Since $T_{N(s)+1} \in [a,b]$ and $b < t$ we have that

$$\begin{aligned}
V_i(l, s; [a,b], x, t) = &\mathbb{E}\Bigg[\sum_{u=s+1}^{T_{N(s)+1}} \psi_i(u-l,u)v(s,u)\\
&+ \sum_{u=T_{N(s)+1}+1}^{t} \psi_{J_{N(u)}}(u - T_{N(s)+1}, u)v(s,u)\Bigg|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Bigg]\\
&+ \mathbb{E}\Bigg[\gamma_{i,J_{N(s)+1}}(X_{N(s)+1}, T_{N(s)+1})v(s, T_{N(s)+1})\Bigg|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Bigg]\\
&+ \mathbb{E}\Bigg[\sum_{\tau=2}^{N(t)-N(s)} \gamma_{J_{N(s)+\tau-1}, J_{N(s)+\tau}}(X_{N(s)+\tau}, T_{N(s)+\tau})v(s, T_{N(s)+\tau})\Bigg|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Bigg].
\end{aligned} \tag{C.24}$$

At this point, denoting by $\mathcal{A}_{i,l}^{(s)} = \{J_{N(s)} = i, T_{N(s)} = l, T_{N(s)+1} > s\}$, we calculate the three expectations in formula (C.24), as follows:

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{u=s+1}^{T_{N(s)+1}} \psi_i(u-l,u)v(s,u)\Bigg|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Bigg]\\
&= \sum_{k\in I}\sum_{\theta\geq s+1}\sum_{u=s+1}^{\theta} \psi_i(u-l,u)v(s,u)\mathbb{P}\left(J_{N(s)+1} = k, T_{N(s)+1} = \theta\Big|\mathcal{A}_{i,l,x}^{(s)}[a,b]\right)
\end{aligned}$$

$$= \sum_{k\in I}\sum_{\theta\geq s+1}\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)$$
$$\cdot\frac{\mathbb{P}\big(J_{N(x)}\neq D, T_{N(s)+1}\in[a,b], J_{N(s)+1}>i, J_{N(s)+1}=k, T_{N(s)+1}=\theta\big|\mathcal{A}_{i,l}^{(s)}\big)}{\mathbb{P}\big(J_{N(x)}\neq D, T_{N(s)+1}\in[a,b], J_{N(s)+1}>i\big|\mathcal{A}_{i,l}^{(s)}\big)}$$
$$= \sum_{k\in I}\sum_{\theta\geq s+1}\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)$$
$$\cdot\frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}\mathbb{P}\big(J_{N(x)}\neq D, J_{N(s)+1}=k, T_{N(s)+1}=\theta\big|\mathcal{A}_{i,l}^{(s)}\big)}{\sum_{m\geq s+1}\sum_{r\in E}1_{\{m\in[a,b]\}}1_{\{r>i\}}\cdot\mathbb{P}\big(J_{N(x)}\neq D, J_{N(s)+1}=r, T_{N(s)+1}=m\big|\mathcal{A}_{i,l}^{(s)}\big)}$$
$$= \sum_{k>i}\sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)\mathbb{P}\big(J_{N(x)}\neq D\big|J_{N(s)+1}=k, T_{N(s)+1}=\theta\big)$$
$$\cdot\,\mathbb{P}\big(J_{N(s)+1}=k, T_{N(s)+1}=\theta\big|\mathcal{A}_{i,l}^{(s)}\big)$$
$$\cdot\frac{1}{\sum_{m=a}^{b}\sum_{r>i}\mathbb{P}\big(J_{N(x)}\neq D\big|J_{N(s)+1}=r, T_{N(s)+1}=m\big)\mathbb{P}\big(J_{N(s)+1}=r, T_{N(s)+1}=m\big|\mathcal{A}_{i,l}^{(s)}\big)}$$
$$= \frac{\sum_{k>i}\sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)\cdot R_k(\theta,x)\cdot\frac{q_{ik}(l,\theta)}{\overline{\overline{H}}_i(l,s)}}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)\cdot\frac{q_{ir}(l,m)}{\overline{\overline{H}}_i(l,s)}}$$
$$= \frac{\sum_{\theta=a}^{b}\sum_{u=s+1}^{\theta}\psi_i(u-l,u)v(s,u)\cdot\sum_{k>i}R_k(\theta,x)\cdot q_{ik}(l,\theta)}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)q_{ir}(l,m)}. \tag{C.25}$$

Then, we can proceed to the computation of the next expected value, as follows:

$$\mathbb{E}\Big[\gamma_{i,J_{N(s)+1}}(X_{N(s)+1},T_{N(s)+1})v(s,T_{N(s)+1})\Big|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Big]$$
$$= \sum_{k\in I}\sum_{\theta\geq s+1}\gamma_{i,k}(\theta-l,\theta)v(s,\theta)\mathbb{P}\big(J_{N(s)+1}=k, T_{N(s)+1}=\theta\big|\mathcal{A}_{i,l}^{(s)}, J_{N(x)}\neq D\big). \tag{C.26}$$

Since the last probability has already been computed, we obtain that (C.26) is equal to

$$= \sum_{k\in I}\sum_{\theta\geq s+1}\gamma_{i,k}(\theta-l,\theta)v(s,\theta)\cdot\frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}R_k(\theta,x)\frac{q_{ik}(l,\theta)}{\overline{\overline{H}}_i(l,s)}}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)\frac{q_{ir}(l,m)}{\overline{\overline{H}}_i(l,s)}}$$
$$= \frac{\sum_{k>i}\sum_{\theta=a}^{b}\gamma_{i,k}(\theta-l,\theta)v(s,\theta)R_k(\theta,x)q_{ik}(l,\theta)}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)q_{ir}(l,m)}. \tag{C.27}$$

Now, we can compute the last expected value in formula (C.24) by changing the variable with $m=\tau-1$ and using the consistency property of the discount factor structure. The result is the following:

$$\mathbb{E}\Big[\sum_{u=T_{N(s)+1}+1}^{t}\psi_{J_{N(u)}}(u-T_{N(s)+1},u)v(s,u)$$
$$+\sum_{\tau=2}^{N(t)-N(s)}\gamma_{J_{N(s)+\tau-1},J_{N(s)+\tau}}(X_{N(s)+\tau},T_{N(s)+\tau})v(s,T_{N(s)+\tau})\Big|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Big]$$
$$= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{u=T_{N(s)+1}+1}^{t}\psi_{J_{N(u)}}(u-T_{N(s)+1},u)v(s,T_{N(s)+1})v(T_{N(s)+1},u)$$
$$+\sum_{m=1}^{N(t)-(N(s)+1)}\gamma_{J_{N(s)+1+(m-1)},J_{N(s)+1+m}}(X_{N(s)+1+m},T_{N(s)+1+m})$$
$$v(s,T_{N(s)+1})v(T_{N(s)+1},T_{N(s)+1+m})\Big|\mathcal{A}_{i,l,x}^{(s)}[a,b],J_{N(s)+1},T_{N(s)+1}\Big|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Big]\Big]$$
$$= \mathbb{E}\Big[v(s,T_{N(s)+1})V_{J_{N(s)+1}}(T_{N(s)+1},T_{N(s)+1};x,t)\Big|\mathcal{A}_{i,l,x}^{(s)}[a,b]\Big]$$
$$= \sum_{k\in I}\sum_{\theta\geq s+1}v(s,\theta)V_k(\theta,\theta;x,t)\mathbb{P}\big(J_{N(s)+1}=k, T_{N(s)+1}=\theta\big|\mathcal{A}_{i,l,x}^{(s)}[a,b]\big)$$

$$= \sum_{k\in I}\sum_{\theta\geq s+1} \upsilon(s,\theta)V_k(\theta,\theta;x,t)\frac{1_{\{k>i\}}1_{\{\theta\in[a,b]\}}R_k(\theta,x)\frac{q_{ik}(l,\theta)}{\overline{H}_i(l,s)}}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)\frac{q_{ir}(l,m)}{\overline{H}_i(l,s)}}$$

$$= \frac{\sum_{k>i}\sum_{\theta=a}^{b}\upsilon(s,\theta)\tilde{V}_k(\theta;x,t)R_k(\theta,x)q_{ik}(l,\theta)}{\sum_{m=a}^{b}\sum_{r>i}R_r(m,x)q_{ir}(l,m)}.$$

□

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